



Supersymmetry in quantum mechanics and higher dimensional gauge theories

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Doctoral Dissertation
(博士論文)

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and
higher dimensional gauge theories

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Graduate School of Science and Technology, Kobe University

January 2006

Doctoral Dissertation

**Supersymmetry in quantum mechanics
and
higher dimensional gauge theories**

(量子力学的超対称性と高次元ゲージ理論)

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January 2006

This doctoral thesis is dedicated to my lover, Hitomi.

Abstract

In this doctoral thesis, we study quantum-mechanical supersymmetry with point singularities and its application to higher dimensional gauge theories. We give a detailed analysis in quantum-mechanical supersymmetry on a circle with point singularities, and the study has a physical application to higher dimensional gauge theories. We show that a quantum-mechanical $N = 2$ supersymmetry is always hidden in any gauge invariant theories with extra dimensions, and the physical application to higher dimensional gauge theories is discussed. We propose the criteria of obtaining a possible set of boundary conditions in a supersymmetry point of view, and construct a toy model of GUT.

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Contents

1	Introduction	9
2	$N = 2$ Supersymmetric quantum mechanics	19
2.1	Supersymmetry in quantum mechanics	19
2.2	Supersymmetry algebra and differential forms	22
3	Supersymmetry in quantum mechanics with point singularities	29
3.1	Quantum mechanics with a point singularity	29
3.2	Point singularities and discrete transformations	32
3.3	$N = 2n + 1$ supersymmetry	37
3.3.1	$N = 2n + 1$ supercharges	37

3.3.2	Connection conditions compatible with supersymmetry	39
3.3.3	Degeneracy of the spectrum	46
3.4	Reduction of the supersymmetry	60
3.4.1	General discussion	60
3.4.2	$N = 2n$ supersymmetry	61
4	Supersymmetry and the higher dimensional gauge theories	79
4.1	Supersymmetric structure in the higher dimensional gauge theories	79
4.2	Supersymmetry on the action	85
5	Extra dimensions with boundaries	89
5.1	Boundary conditions and supersymmetry	89
5.2	A model with a large mass hierarchy	96
5.2.1	A mechanism to obtain a large mass hierarchy	97
5.2.2	A toy model of GUT	101

6 Summary	109
A HODGE DECOMPOSITION THEOREM	113

Chapter 1

Introduction

Higher dimensional theories have recently attracted enormous attention because hierarchy problem can be solved by geometry of extra dimensions or symmetry of the higher dimensions [1]-[46].

The standard model of elementary particles has been very successful in describing observed phenomena. Theoretically, however, a serious problem arises when it is regarded as the effective low energy theory of a certain more fundamental theory. We think that the standard model is valid up to some energy scale Λ , and at Λ the theory is replaced by more fundamental theory. For instance, if we assume that the fundamental theory is grand unified theory (GUT), which is invented to unify the electroweak force and the strong force, it has two hierarchical mass scales which are the weak scale $M_W \sim 10^2[GeV]$, which is the symmetry breaking scale of the electroweak theory, and the GUT scale (M_{GUT}), which is determined by the scale where the electroweak and strong coupling constants are unified. The ratio of these masses, for instance, is $\frac{M_{GUT}}{M_W} \sim 10^{13}$ for $SU(5)$ GUT. The hierarchy problem is the problem of how to maintain the hierarchy of these mass scales. If we abandon the GUT, the

hierarchy problem still remains. There is the hierarchy between the Planck scale $M_{Pl} \sim 10^{18}[GeV]$, where gravity becomes as strong as the gauge interactions, and the weak scale M_W . The hierarchy problem is serious in the sector of the Higgs mass. In the standard model, the quadratic divergence arises in the quantum correction to the Higgs mass-squared. It is possible to adjust the bare mass of the Higgs, so that the renormalized Higgs mass remains to be the weak scale. It, however, will require a fine tuning of the bare parameter at the precision $\sim 10^{-26}$ in a $SU(5)$ GUT scenario. This problem at the quantum correction is known as that of “quadratic divergence”. Many physicists think that solving the hierarchy problem would be a clue of new physics beyond the standard model.

An interesting possibility to solve the hierarchy problem at the classical level has been suggested by Shaposhnikov and Tinyakov[1]. They considered a gauge invariant theory with a noncompact extra dimension with a weight function and showed that massless modes disappear from the theory, while massive modes appear in such a way that one is much lighter than the others. In ref. [2, 3], we extend the idea of them, and study the possibility of solving the hierarchy problem in a $SU(5)$ GUT scenario, based on 5D gauge invariant theories on orbifold with nontrivial boundary conditions at fixed points. We present a toy model in which a large mass hierarchy of the spectrum naturally arises without a fine tuning. Hidden supersymmetry in higher dimensional gauge theories plays a crucial role to obtain the hierarchy structure.

Arkani-Hamed *et al.* have solved the hierarchy problem between M_W and M_{Pl} in the brane world scenario[4]. The standard model particles are confined to a $(3 + 1)$ -dimensional subspace, or “D3-brane”, in the higher dimensions. However, this confinement will not work for gravity, which necessarily propagates in all dimensions as it is the dynamics of spacetime itself. In higher dimensional theory, the effective four-dimensional Planck scale M_{Pl} is related

to the fundamental $(4 + n)$ -dimensional Planck scale $M_{Pl}^{(4+n)}$ as,

$$M_{Pl}^2 = \left(M_{Pl}^{(4+n)}\right)^2 R^n, \quad (1.1)$$

where R^n is the volume of the compact extra space. Since gravity is tested only down to the size of a millimeter or so, R could be as large as fraction of a millimeter. If we take $R \sim 0.1[mm]$ and $n = 2$, the fundamental Planck scale $M_{Pl}^{(4+n)}$ in the theory becomes of the same order as the weak scale. In their model, fundamental scale is only M_W , and the original hierarchy problem does not exist. This scenario, however, produces a new hierarchy, namely that between the compactification scale and the weak scale. They do not consider the reaction of gravity to the presence of the branes themselves, either.

More attractive model which does not have the problem mentioned above has been suggested by Randall and Sundrum[5]. They explicitly constructed a solution to the Einstein's equations in 5 spacetime dimensions, one of which is compactified on S^1/Z_2 , with two D3-branes located at the fixed points of S^1/Z_2 . The vacuum metric solution is given by

$$ds^2 = e^{-2k|y|} \eta_{\mu\nu} dx^\mu dx^\nu + dy^2, \quad (1.2)$$

where $e^{-2k|y|}$ is called warp factor, and $\eta_{\mu\nu}$ is the 4d Minkowski metric with $\eta_{\mu\nu} = \text{diag}(-1, 1, 1, 1)$. The 5d-dimensional coordinates are denoted by $x^M = (x^\mu, y)$, where y is the extra dimensional coordinate. The geometry of the bulk 5d space-time is Anti-de Sitter spacetime, i.e. a space-time with a negative cosmological constant. We note that even though the extra dimension is curved, the brane itself remains static and flat, that is it preserves 4d Poincaré invariance. In that scenario, all mass scales except the scale of gravity are reduced to the weak scale by the warp factor. The exponential factor is the source of the large hierarchy between the observed Planck scale and the weak scale. With this scenario, various attempts have been made to construct realistic models[6]-[11].

The Gauge-Higgs unification scenario attracts attentions from the view point of solving the hierarchy problem[12]-[24]. In that scenario, the Higgs field is identified as the zero mode of the extra dimensional components of the gauge field. Since the gauge symmetry in the higher dimension protects the Higgs mass from quadratic divergences, the hierarchy problem at quantum level is solved. In the Gauge-Higgs unification scenario, since particles of different spin in four dimensions can be unified within a higher dimensional theory, this scenario is also attractive in the sense of unification of particles with different spin. Recently, Hasegawa, Lim and Maru have suggested the solution to the hierarchy problem in the Gravity-Gauge-Higgs scenario, in which four dimensional bosonic fields with all possible integer spins, graviton, gauge boson and Higgs are unified in a framework of a gravity theory with extra dimensions[25, 26]. The Higgs is identified with the extra space component of the metric tensor. General coordinate transformation invariance protects the Higgs mass from quadratic divergence in this model.

A recent new proposal for solving the hierarchy problem is the Higgsless scenario[2, 3],[27]-[38]. In extra dimensions, it is indeed possible to break gauge symmetries via boundary conditions without any light scalar appearing in the theory. The scattering amplitude of longitudinal mode of massive gauge bosons $W^\pm, Z$ can be unitarized by the gauge boson resonances rather than by the Higgs field.

It is phenomenologically important to obtain the mass spectrum in a four dimensional point of view when we consider such higher dimensional theories. We note that the problem of how to obtain the 4d mass spectrum can be replaced by the eigenvalue problem in quantum mechanics. Let us consider a scalar field in the $(4 + 1)$ -dimensional space-time. The $(4 + 1)$ -dimensional coordinates are denoted by $x^M = (x^\mu, y)$, where $x^\mu (\mu = 0, 1, 2, 3)$ are the 4-dimensional coordinates and y is the extra space coordinate. The action we

consider is

$$\begin{aligned}
S &= \int d^4x \int dy \sqrt{-g(y)} \left(-|\partial_M \Phi(x, y)|^2 - M^2 |\Phi(x, y)|^2 \right) \\
&= \int d^4x \int dy \sqrt{-g(y)} \left(-\Phi^*(x, y) (-\partial_\mu \partial^\mu) \Phi(x, y) \right. \\
&\quad \left. - \Phi^*(x, y) (-\Delta_y + M^2) \Phi(x, y) \right), \quad (1.3)
\end{aligned}$$

where Δ_y is an extra dimensional laplacian. One can expand the field $\Phi(x, y)$ in an appropriate complete set with respect to the fifth coordinate,

$$\Phi(x, y) = \sum_n \phi_n(x) f_n(y), \quad (1.4)$$

where $f_n(y)$ satisfies

$$(-\Delta_y + M^2) f_n(y) = m_n^2 f_n(y). \quad (1.5)$$

Here m_n is a mass of the 4d scalar field $\phi_n(x)$, the problem we have to solve is the eigenvalue problem(1.5). Integrating with respect to the extra space coordinate y , the action becomes

$$S = \int d^4x \sum_n \left(-\phi_n^*(x, y) (-\partial_\mu \partial^\mu) \phi_n(x, y) - m_n^2 |\phi_n(x)|^2 \right). \quad (1.6)$$

We can obtain the 4d mass spectrum $\{m_n\}$, if the eigenvalue problem in quantum mechanics is solved.

Note that the mass operator is given by a square of a certain hermitian operator when we consider the gauge invariant theory with extra dimensions[39]. Let us consider a $(4+1)$ dimensional Abelian gauge theory on a curved space-time,

$$S = \int d^4x \int dy \sqrt{-g(y)} \left(-\frac{1}{4} g^{MM'} g^{NN'} F_{M'N'}(x, y) F_{MN}(x, y) \right), \quad (1.7)$$

where

$$F_{MN}(x, y) = \partial_M A_N(x, y) - \partial_N A_M(x, y), \quad (1.8)$$

and the metric is assumed to be of the form with a wight function $\Delta(y)$

$$ds^2 = \Delta(y) \left(\eta_{\mu\nu} dx^\mu dx^\nu + dy^2 \right). \quad (1.9)$$

The mode expansions of the gauge fields are given by

$$A_\mu(x, y) = \sum_n A_\mu^{(n)}(x) f_n(y), \quad (1.10)$$

$$A_5(x, y) = \sum_n A_5^{(n)}(x) g_n(y), \quad (1.11)$$

where $f_n(y)$, $g_n(y)$ satisfy

$$\begin{pmatrix} -\frac{1}{\Delta(y)} \partial_y \Delta(y) \partial_y & 0 \\ 0 & -\partial_y \frac{1}{\Delta(y)} \partial_y \Delta(y) \end{pmatrix} \begin{pmatrix} f_n(y) \\ g_n(y) \end{pmatrix} = m_n^2 \begin{pmatrix} f_n(y) \\ g_n(y) \end{pmatrix}, \quad (1.12)$$

where m_n^2 is a mass-squared of the 4d gauge field $A_\mu^{(n)}$. This mass operator can be written as

$$H = Q^2 = \begin{pmatrix} -\frac{1}{\Delta(y)} \partial_y \Delta(y) \partial_y & 0 \\ 0 & -\partial_y \frac{1}{\Delta(y)} \partial_y \Delta(y) \end{pmatrix}, \quad (1.13)$$

$$Q = \begin{pmatrix} 0 & -\frac{1}{\Delta(y)} \partial_y \Delta(y) \\ \partial_y & 0 \end{pmatrix}. \quad (1.14)$$

This eigenvalue problem is known as that of the supersymmetric quantum mechanics, where supercharge is denoted by Q . In this thesis, we first study the supersymmetric quantum mechanics in order to discuss the 4d mass spectrum in the higher dimensional gauge theories.

There exist some evidence for the existence of some kind of supersymmetry in 4d mass spectrum of higher dimensional gauge theories: If a 4d gauge symmetry is not broken, a massless 4d gauge field appears because 4d gauge invariance guarantees the gauge field to be massless. This may be explained from a supersymmetry point of view because supersymmetry ensures that the ground state has zero energy and the zero energy state is interpreted as the massless gauge field. The second evidence is that in a 5d gauge theory with

a single extra dimension, every massive mode of $A_5^{(n)}$ (a massive KK mode of the gauge field in the direction of the extra dimension) can be absorbed into the longitudinal mode of $A_\mu^{(n)}$ (a massive KK mode of the 4d gauge field) by gauge transformations. This fact implies that there should exist a one-to-one correspondence between $A_5^{(n)}$ and $A_\mu^{(n)}$. The correspondence may be interpreted as supersymmetry between a “bosonic” state and a “fermionic” one, although both modes are bosonic. The last evidence is that a massless mode $A_5^{(0)}$ (if exists) cannot be gauged-away and it appears as a physical state, in contrast to the massive modes $A_5^{(n)}$, which can be gauged-away and hence are unphysical modes. This observation is again consistent with supersymmetry because zero energy states do not form any supermultiplets between bosonic and fermionic states.

Another appealing and well-known scenario to solve the hierarchy problem is to invoke supersymmetry in quantum field theories. Then, it may be natural to ask a question whether these two kinds of theories, gauge theories with extra dimensions and supersymmetric theories, have some relation. The immediate answer is negative, since supersymmetry necessarily needs fermionic degree of freedom, while the cancellation mechanism of the quadratic divergence in the higher dimensional gauge theory does not necessitate it. However, as mentioned above, these two theories are related through quantum mechanical supersymmetry.

Now, we note that many attractive models with extra dimensions include singularities. Orbifolds[40]-[45], on which phenomenologically interesting models have been proposed, are one of such example because orbifolds have fixed points, which are singularities on manifolds. In most studies, it has been tacitly assumed that the field configurations are smooth at fixed points of the orbifolds. However, it is natural to allow field configurations to be discontinuous at fixed points. Another example of the theory including singularities

is a brane world scenario[4, 5], [46]-[49]. For instance, a D3-brane located at a point on the extra dimension may be regarded as a singularity in the 4d point of view. Singularities, indeed, leads to more realistic and attractive models[2, 3, 46].

Now, we pay attention to singularities and their treatment. A point singularity in quantum mechanics is parameterized by the group $U(2)$ and the parameters characterize connection conditions between a wavefunction and its derivative at the singularity[50]-[52]. The variety of the connection conditions leads to various interesting physical phenomena, such as duality[53, 54], the Berry phase[55, 56] and scale anomaly[57] and supersymmetry[58]-[62]. We would like to emphasize that the study of supersymmetry in one-dimensional quantum mechanics with point singularities not only is interesting in a mathematical physics point of view but also has a physical application to higher dimensional gauge theories. $N = 1$ or $N = 2$ supersymmetry in the model of a free particle on a line $\mathbf{R}$ or an interval $[-l, l]$ with a point singularity was discussed in [58]. This work was extended to $N = 4$ supersymmetry in the model on a pair of lines or intervals each having a point singularity[60]. In this thesis, we extend the previous study to the N -extended supersymmetry on a circle with 2^n point singularities[59]-[62].

The plan of this thesis is as follows. In chapter 2, we give a brief review of quantum mechanical supersymmetry and discuss how to treat the singularities in quantum mechanics. In chapter 3, we give a review of singularity in quantum mechanics, and the $N = 2$ supercharges are explicitly constructed in terms of differential forms. After that, we study N -extended supersymmetry in one-dimensional quantum mechanics on a circle with point singularities in detail. For any integer n , $N = 2n + 1$ supercharges are explicitly constructed in terms of discrete transformations, and a class of singularities compatible with supersymmetry is clarified. In our formulation, the supersymmetry can be re-

duced to M -extended supersymmetry for any integer $M < N$. The degeneracy of the spectrum and spontaneous supersymmetry breaking are also studied in this chapter. The analysis discussed here can be applied to the study of higher dimensional gauge theories. In chapter 4, we show that a quantum mechanical $N = 2$ supersymmetric structure is always hidden in the 4d mass spectrum of any gauge invariant theories with extra dimensions. We also show that the higher dimensional gauge theory on the whole possesses the $N = 2$ quantum mechanical supersymmetry and the supersymmetry is a part of the higher dimensional gauge symmetry. In chapter 5, we extend the previous analysis to extra dimensions with boundaries, and propose the criteria of obtaining a possible set of boundary conditions in a supersymmetry point of view. We construct a toy model of GUT using the boundary condition, and demonstrate that it possesses gauge symmetry breaking with a large mass hierarchy with the help of the previous analysis. Chapter 6 is devoted to summary.

Chapter 2

$N = 2$ Supersymmetric quantum mechanics

2.1 Supersymmetry in quantum mechanics

In this section, we give a brief review of the $N = 2$ supersymmetric quantum mechanics which is called the Witten model.

The supersymmetry algebra in quantum mechanics is given by

$$\{Q_i, Q_j\} = \delta_{ij}H, \tag{2.1}$$

$$Q_i^\dagger = Q_i, \quad (i, j = 1, 2, \dots, N), \tag{2.2}$$

where H is the Hamiltonian and Q_i ($i = 1, 2, \dots, N$) are called supercharges. Here we consider the supersymmetry algebra with $N = 2$. We will see later that the $N = 2$ supersymmetry appears in 4d mass spectrum of any gauge invariant theories with extra dimensions, and any gauge invariant theories with

extra dimensions possess the $N = 2$ supersymmetry, i.e. the action is invariant under the transformations whose generators are the $N = 2$ supercharges.

Let us introduce a two-component wave function as

$$\psi(x) = \begin{pmatrix} \phi_1(x) \\ \phi_2(x) \end{pmatrix}. \quad (2.3)$$

The $N = 2$ supercharges which act on the two-component wave function can be written by

$$Q_1 = \frac{1}{2} \left[\sigma_1 i \frac{d}{dx} + \sigma_2 W'(x) \right], \quad (2.4)$$

$$Q_2 = \frac{1}{2} \left[\sigma_2 i \frac{d}{dx} - \sigma_1 W'(x) \right], \quad (2.5)$$

where $W'(x) = \frac{d}{dx}W(x)$ and $W(x)$ is called superpotential. This model is called the Witten model[63]. The σ_i ($i = 1, 2, 3$) stands for the Pauli matrices,

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \quad (2.6)$$

The Hamiltonian is given by

$$H = \frac{1}{2} \left[-\frac{d^2}{dx^2} + (W'(x))^2 - \sigma_3 W''(x) \right]. \quad (2.7)$$

They form the $N = 2$ supersymmetry algebra

$$\{Q_i, Q_j\} = 2\delta_{ij}H, \quad (2.8)$$

$$Q_i^\dagger = Q_i, \quad i, j = 1, 2. \quad (2.9)$$

We can introduce the “fermion” number operator as

$$(-1)^F = \sigma_3, \quad (2.10)$$

which satisfies

$$\{Q_i, (-1)^F\} = 0, \quad i = 1, 2, \quad (2.11)$$

$$\left((-1)^F\right)^2 = I_2, \quad (2.12)$$

where I_M denotes an $M \times M$ unit matrix. We call states with $(-1)^F = +1(-1)$ “bosonic” (“fermionic”) ones, that is, the “bosonic” and “fermionic” states $\psi_+(x), \psi_-(x)$ are given by

$$\psi_+(x) = \begin{pmatrix} \phi_1(x) \\ 0 \end{pmatrix}, \quad \psi_-(x) = \begin{pmatrix} 0 \\ \phi_2(x) \end{pmatrix}. \quad (2.13)$$

Since

$$[(-1)^F, H] = 0, \quad (2.14)$$

we can introduce simultaneous eigenfunctions of H and $(-1)^F$ such that

$$H\psi_{E;\pm} = E\psi_{E;\pm}, \quad (2.15)$$

$$(-1)^F \psi_{E;\pm} = \pm \psi_{E;\pm}. \quad (2.16)$$

The important properties of this model are as follows:

(i) Since the Hamiltonian is given by the square of each supercharge which is hermitian, the energy is positive or zero,

$$E \geq 0. \quad (2.17)$$

(ii) The supersymmetry algebra implies that

$$H(Q_i \psi_{E;\pm}) = E(Q_i \psi_{E;\pm}), \quad (2.18)$$

$$(-1)^F (Q_i \psi_{E;\pm}) = \mp (Q_i \psi_{E;\pm}). \quad (2.19)$$

Noting that $Q_1 = -iQ_2(-1)^F$, we have the relation

$$Q_1 \psi_{E;\pm}(x) = \mp i Q_2 \psi_{E;\pm} \propto \psi_{E;\mp}(x) \quad (2.20)$$

when $E > 0$. Thus, all the non-zero energy states are degenerate between “bosonic” state $((-1)^F = +1)$ and “fermionic” one $((-1)^F = -1)$.

(iii) The above argument cannot apply for $E = 0$ states because $Q_i\psi_0 = 0$. If there exists a supersymmetrically invariant state then it is the vacuum and supersymmetry is not spontaneously broken.

The equations $Q_i\psi_{0;\pm} = 0$ ($i = 1, 2$) are easily solved and we have formal solutions

$$\psi_{0,+} = \begin{pmatrix} N_+ e^{-W(x)} \\ 0 \end{pmatrix}, \quad \psi_{0,-} = \begin{pmatrix} 0 \\ N_- e^{+W(x)} \end{pmatrix}. \quad (2.21)$$

Here, $N_{\pm}$ denote normalization constants. The solutions can be removed from the functional space because some of them may be non-normalizable¹. We assume that $|W(x)| \rightarrow \infty$ as $|x| \rightarrow \infty$. If $W(\infty) \cdot W(-\infty) < 0$, then the sign of W for $x \rightarrow +\infty$ is opposite from the sign of W for $x \rightarrow -\infty$. In this case, both of the solutions are not normalizable, then the supersymmetry is spontaneously broken. On the other hand, if $W(\infty) \cdot W(-\infty) > 0$, then the sign of W is the same for $x \rightarrow +\infty$ as for $x \rightarrow -\infty$. In this case, one of the solutions is always normalizable and another is not normalizable. Thus, the supersymmetry is not broken. We note that whether the supersymmetry is broken or not only depends on the asymptotic behavior of the superpotential.

2.2 Supersymmetry algebra and differential forms

In this section, we construct the $N = 2$ supersymmetry algebra in terms of differential forms. We will see later that the formulation of differential forms becomes powerful tool.

¹If space is compact, the solutions are always normalizable. Nevertheless, some of them must be removed from the functional space. This occurs due to incompatibility with boundary conditions (connection conditions). This will be described in next chapter.

Let K be a D -dimensional compact Riemannian manifold with a metric g_{ij} ($i, j = 1, 2, \dots, D$). A k form $\omega^{(k)}$ on K is given by

$$\omega^{(k)} = \frac{1}{k!} \omega_{i_1 i_2 \dots i_k} dy^{i_1} \wedge dy^{i_2} \wedge \dots \wedge dy^{i_k}, \quad (2.22)$$

where y^i ($i = 1, 2, \dots, D$) are coordinates on K and $\wedge$ denotes the wedge product. The coefficient $\omega_{i_1 \dots i_k}$ is totally antisymmetric in all k indices. The Hodge star (Poincaré dual) operator on the k form $\omega^{(k)}$ is defined by

$$*\omega^{(k)} = \frac{\sqrt{g}}{k!(D-k)!} \omega_{i_1 \dots i_k} g^{i_1 j_1} \dots g^{i_k j_k} \epsilon_{j_1 \dots j_k j_{k+1} \dots j_D} dy^{j_{k+1}} \wedge \dots \wedge dy^{j_D}, \quad (2.23)$$

where $g = \det g_{ij}$ and $\epsilon_{i_1 i_2 \dots i_D}$ is a totally antisymmetric tensor with $\epsilon_{12 \dots D} = 1$. Repeated applications of $*$ on any k form give

$$**\omega^{(k)} = (-1)^{k(D-k)} \omega^{(k)}. \quad (2.24)$$

The inner product of any two k forms $\omega^{(k)}$ and $\eta^{(k)}$ is defined by

$$\begin{aligned} (\eta^{(k)}, \omega^{(k)})_{\Delta} &\equiv \frac{1}{k!} \int_K d^D y \sqrt{g} \Delta \eta_{i_1 \dots i_k} \omega_{j_1 \dots j_k} g^{i_1 j_1} \dots g^{i_k j_k} \\ &= \int_K \Delta \eta^{(k)} \wedge *\omega^{(k)}. \end{aligned} \quad (2.25)$$

Here, we have introduced a weight function $\Delta(y)$ for later convenience with a property

$$\Delta(y) > 0. \quad (2.26)$$

Then, it follows that

$$(\eta^{(k)}, \omega^{(k)})_{\Delta} = (\omega^{(k)}, \eta^{(k)})_{\Delta}, \quad (2.27)$$

$$(\omega^{(k)}, \omega^{(k)})_{\Delta} \geq 0, \quad (2.28)$$

$$(\eta^{(k)}, *\omega^{(D-k)})_{\Delta} = (-1)^{k(D-k)} (*\eta^{(k)}, \omega^{(D-k)})_{\Delta}. \quad (2.29)$$

The adjoint of the exterior derivative d is defined, with respect to the inner product (2.25), by

$$(\eta^{(k-1)}, d^\dagger \omega^{(k)})_{\Delta} \equiv (d\eta^{(k-1)}, \omega^{(k)})_{\Delta}. \quad (2.30)$$

For compact manifold without a boundary², the action of $d^\dagger$ on a k form $\omega^{(k)}$ turns out to be of the form

$$d^\dagger \omega^{(k)} = -(-1)^{(k-1)D} \Delta^{-1} * d\Delta * \omega^{(k)}. \quad (2.31)$$

For $k = 0$ and 1 , $d^\dagger \omega^{(k)}$ are explicitly written as

$$d^\dagger \omega^{(0)} = 0, \quad (2.32)$$

$$d^\dagger \omega^{(1)} = -\frac{1}{\Delta \sqrt{g}} \partial_j (\Delta \sqrt{g} g^{ij} \omega_i), \quad (2.33)$$

where $\omega^{(1)} = \omega_i dy^i$. The first equation comes from the fact that $d^\dagger$ maps k forms into $k-1$ forms and there is no -1 form. We notice that the nilpotency of $d^\dagger$ still holds irrespective of $\Delta(y)$, i.e.

$$(d^\dagger)^2 = 0. \quad (2.34)$$

We can now construct the $N = 2$ supersymmetry algebra. To this end, we introduce a two-component vector

$$|\Omega^{(k)}\rangle \equiv \begin{pmatrix} \omega^{(k)} \\ \phi^{(k+1)} \end{pmatrix}, \quad (2.35)$$

where the upper (lower) component consists of a k ($k+1$) form. The inner product of two two-component vectors $|\Omega_1^{(k)}\rangle$ and $|\Omega_2^{(k)}\rangle$ is defined by

$$\langle \Omega_2^{(k)} | \Omega_1^{(k)} \rangle \equiv (\omega_2^{(k)}, \omega_1^{(k)})_\Delta + (\phi_2^{(k+1)}, \phi_1^{(k+1)})_\Delta. \quad (2.36)$$

Then, the $N = 2$ supercharges Q_i ($i = 1, 2$) are given by³

$$Q_1 = \begin{pmatrix} 0 & d^\dagger \\ d & 0 \end{pmatrix}, \quad Q_2 = \begin{pmatrix} 0 & -id^\dagger \\ id & 0 \end{pmatrix}. \quad (2.37)$$

²An extension with boundaries will be discussed in chapter 4.

³Here, the inner product (2.25) should be extended for complex forms as

$$(\eta^{(k)}, \omega^{(k)})_\Delta \equiv \frac{1}{k!} \int_K d^D y \sqrt{g} \Delta (\eta_{i_1 \dots i_k})^* \omega_{j_1 \dots j_k} g^{i_1 j_1} \dots g^{i_k j_k},$$

and the relation (2.30) is then replaced by $(\eta^{(k)}, \omega^{(k)})_\Delta = ((\omega^{(k)}, \eta^{(k)})_\Delta)^*$.

We note that the action of Q_i on $|\Omega^{(k)}\rangle$ is well defined. It is easy to show that they form the following $N = 2$ supersymmetry algebra:

$$\{Q_i, Q_j\} = 2\delta_{ij}H, \quad (2.38)$$

$$[Q_i, H] = 0, \quad (2.39)$$

$$[(-1)^F, H] = 0, \quad (2.40)$$

$$\{(-1)^F, Q_i\} = 0, \quad \text{for } i, j = 1, 2, \quad (2.41)$$

where the Hamiltonian H and the operator $(-1)^F$ with F being the “fermion” number operator are defined by

$$H = \begin{pmatrix} d^\dagger d & 0 \\ 0 & dd^\dagger \end{pmatrix}, \quad (2.42)$$

$$(-1)^F = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \quad (2.43)$$

We may call states with $(-1)^F = +1$ (-1) “bosonic” (“fermionic”) ones. All the operators Q_i , H and $(-1)^F$ are hermitian with respect to the inner product (2.36).

To examine the structure of the $N = 2$ supersymmetry algebra, let us consider the following Schrödinger-type equations:

$$H|\Omega_n^{(k)}\rangle = (m_n^{(k)})^2|\Omega_n^{(k)}\rangle. \quad (2.44)$$

Since $H = (Q_1)^2$ and $Q_1^\dagger = Q_1$, the eigenvalues $(m_n^{(k)})^2$ are positive semi-definite, i.e.

$$(m_n^{(k)})^2 \geq 0. \quad (2.45)$$

Since $(-1)^F$ commutes with H , we can have simultaneous eigenfunctions of H and $(-1)^F$ such that

$$H|m_n^{(k)}, \pm\rangle = (m_n^{(k)})^2|m_n^{(k)}, \pm\rangle, \quad (2.46)$$

$$(-1)^F|m_n^{(k)}, \pm\rangle = \pm|m_n^{(k)}, \pm\rangle. \quad (2.47)$$

Since Q_1 commutes (anticommutes) with $H \left((-1)^F \right)$, $Q_1|m_n^{(k)}, \pm\rangle$ have the same (opposite) eigenvalues of $H \left((-1)^F \right)$ as to $|m_n^{(k)}, \pm\rangle$, i.e.

$$H \left(Q_1|m_n^{(k)}, \pm\rangle \right) = (m_n^{(k)})^2 \left(Q_1|m_n^{(k)}, \pm\rangle \right), \quad (2.48)$$

$$(-1)^F \left(Q_1|m_n^{(k)}, \pm\rangle \right) = \mp \left(Q_1|m_n^{(k)}, \pm\rangle \right). \quad (2.49)$$

Actually, the states $|m_n^{(k)}, \pm\rangle$ and $Q_1|m_n^{(k)}, \mp\rangle$ are mutually related, with an appropriate phase convention, as

$$Q_1|m_n^{(k)}, \pm\rangle = m_n^{(k)}|m_n^{(k)}, \mp\rangle. \quad (2.50)$$

Since $|m_n^{(k)}, \pm\rangle$ have the form

$$|m_n^{(k)}, +\rangle = \begin{pmatrix} \omega_n^{(k)} \\ 0 \end{pmatrix}, \quad (2.51)$$

$$|m_n^{(k)}, -\rangle = \begin{pmatrix} 0 \\ \phi_n^{(k+1)} \end{pmatrix}, \quad (2.52)$$

the relations (2.50) are rewritten as

$$d\omega_n^{(k)} = m_n^{(k)}\phi_n^{(k+1)}, \quad (2.53)$$

$$d^\dagger\phi_n^{(k+1)} = m_n^{(k)}\omega_n^{(k)}. \quad (2.54)$$

Therefore, there is a one-to-one correspondence between the eigenstates of $d^\dagger d$ and $dd^\dagger$ for k and $k+1$ forms, respectively, and the eigenvalues are, in general, doubly degenerate except for $m_n^{(k)} = 0$.

In Appendix, we have shown that any k form $A^{(k)}$ can be expanded as

$$A^{(k)} = \sum_{p=1}^{b_k} c_p \eta_p^{(k)} + \sum_{n_k}' a_{n_k} \omega_{n_k}^{(k)} + \sum_{n_{k-1}}' b_{n_{k-1}} \phi_{n_{k-1}}^{(k)}, \quad (2.55)$$

where $\{\eta_p^{(k)}, p = 1, 2, \dots, b_k\}$ is a complete set of the harmonic k forms and $\omega_{n_k}^{(k)}$ and $\phi_{n_{k-1}}^{(k)}$ are eigenfunctions of the equations

$$d^\dagger d \omega_{n_k}^{(k)} = (m_{n_k}^{(k)})^2 \omega_{n_k}^{(k)} \quad \text{for } m_{n_k}^{(k)} \neq 0, \quad (2.56)$$

$$dd^\dagger \phi_{n_{k-1}}^{(k)} = (m_{n_{k-1}}^{(k)})^2 \phi_{n_{k-1}}^{(k)} \quad \text{for } m_{n_{k-1}}^{(k)} \neq 0. \quad (2.57)$$

The summations in eq. (2.55) should be taken over the eigenfunctions with nonzero eigenvalues. Thus, we have found the following structure among a sequence of differential forms:

$$\begin{array}{ccccccc}
\vdots & & & & & & \ddots \\
k+2 \text{ form} & & & & \phi^{(k+2)} & & \\
& & & & \begin{array}{c} \downarrow d \\ \uparrow d^\dagger \end{array} & & \\
k+1 \text{ form} & \begin{array}{c} 0 \\ \uparrow d \\ (\eta^{(k+1)}, \\ \downarrow d^\dagger \\ 0 \end{array} & \begin{array}{c} \omega^{(k+1)}, \\ \downarrow d^\dagger \\ 0 \end{array} & \begin{array}{c} \uparrow d \\ \downarrow d^\dagger \end{array} & \begin{array}{c} 0 \\ \uparrow d \\ \phi^{(k+1)} \end{array} & & \\
& & & & \begin{array}{c} \downarrow d \\ \uparrow d^\dagger \end{array} & & \\
k \text{ form} & & \begin{array}{c} \eta^{(k)}, \\ \downarrow d^\dagger \\ 0 \end{array} & \begin{array}{c} \omega^{(k)}, \\ \downarrow d^\dagger \\ 0 \end{array} & \begin{array}{c} \uparrow d \\ \downarrow d^\dagger \end{array} & \begin{array}{c} 0 \\ \uparrow d \\ \phi^{(k)} \end{array} & \\
& & & & \begin{array}{c} \downarrow d \\ \uparrow d^\dagger \end{array} & & \\
k-1 \text{ form} & & & & & \begin{array}{c} \omega^{(k-1)} \\ \uparrow d \\ \downarrow d^\dagger \end{array} & \\
& & & & & & \ddots
\end{array}$$

Chapter 3

Supersymmetry in quantum mechanics with point singularities

3.1 Quantum mechanics with a point singularity

In this section, we give a brief review of one-dimensional quantum mechanics with a point singularity.

We first consider the model of a free particle on a line $\mathbf{R}$ with a point singularity placed at $x = 0$ [57].

Since the model contains a point singularity, wavefunction will, in general,

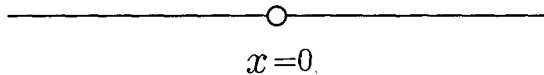


Figure 3.1: Point singularity in one dimension.

be discontinuous there. This thesis deals with the model in which the wavefunction and its derivative are continuous everywhere except for the point singularity. We need to impose appropriate connection conditions there, namely, our model is specified not only by the Hamiltonian but also by the connection conditions. The functional space in our model is required to be squared integrable and to be spanned by eigenfunctions of the Hamiltonian with connection conditions that make the Hamiltonian hermitian.

The Hamiltonian is given by

$$H = -\frac{1}{2} \frac{d^2}{dx^2}, \quad (x \neq 0). \quad (3.1)$$

The discontinuity of the wavefunction at $x = 0$ has to be controlled by the connection conditions that make the Hamiltonian hermitian. The hermiticity condition of the Hamiltonian is

$$\left(\int_{-\infty}^{0-\varepsilon} dx + \int_{0+\varepsilon}^{\infty} dx \right) \psi^*(x) (H\varphi)(x) = \left(\int_{-\infty}^{0-\varepsilon} dx + \int_{0+\varepsilon}^{\infty} dx \right) (H\psi)^*(x) \varphi(x) \quad (3.2)$$

for any wavefunctions $\psi(x), \varphi(x)$. Here ε is an infinitesimal positive constant. We introduce a two-component boundary vector as

$$\Phi_\varphi = \begin{pmatrix} \varphi(0+\varepsilon) \\ \varphi(0-\varepsilon) \end{pmatrix}. \quad (3.3)$$

In terms of this boundary vector, the requirement (3.2) can simply be rewritten as the constraints on the boundary vector

$$\Phi_\psi^\dagger \sigma_3 \Phi_{\varphi'} = (\sigma_3 \Phi_{\psi'})^\dagger \Phi_\varphi, \quad (3.4)$$

where, $\varphi'(x) = \frac{d\varphi(x)}{dx}$ and

$$\Phi_{\varphi'} = \begin{pmatrix} \varphi'(0 + \varepsilon) \\ \varphi'(0 - \varepsilon) \end{pmatrix}. \quad (3.5)$$

It is easy to show that equation (3.4) is equivalent to

$$|\Phi_{\varphi} - iL_0\sigma_3\Phi_{\varphi'}| = |\Phi_{\varphi} + iL_0\sigma_3\Phi_{\varphi'}| \quad (3.6)$$

where L_0 is an arbitrary non-zero constant with the dimension of length. Then, the condition (3.6) can be written as

$$(I_2 - U_2)\Phi_{\varphi} - iL_0(I_2 + U_2)\sigma_3\Phi_{\varphi'} = 0, \quad (3.7)$$

where U_M is a $M \times M$ unitary matrix. Thus, we have found that the connection conditions which make the Hamiltonian hermitian are given by equation (3.7). The singularity is characterized by a 2×2 unitary matrix U_2 .

For instance, when $U_2 = \sigma_1$, the connection conditions become

$$\varphi(0 + \varepsilon) = \varphi(0 - \varepsilon), \quad (3.8)$$

$$\varphi'(0 + \varepsilon) = \varphi'(0 - \varepsilon), \quad (3.9)$$

in other words, this model has no singularity at all.

For the case of

$$U = e^{i\frac{\theta}{2}} \begin{pmatrix} i \sin \frac{\theta}{2} & \cos \frac{\theta}{2} \\ \cos \frac{\theta}{2} & i \sin \frac{\theta}{2} \end{pmatrix}, \quad (3.10)$$

the connection conditions become

$$\varphi(0 + \varepsilon) = \varphi(0 - \varepsilon), \quad (3.11)$$

$$\varphi'(0 + \varepsilon) - \varphi'(0 - \varepsilon) = -\frac{2}{L_0} \tan \frac{\theta}{2} \varphi(0 + \varepsilon), \quad (3.12)$$

The model with the connection conditions (3.11),(3.12) are equivalent to the model which has delta-function potential, instead of the singularity, i.e. the corresponding Hamiltonian is

$$H = -\frac{1}{2} \frac{d^2}{dx^2} - \frac{1}{L_0} \tan \frac{\theta}{2} \delta(x). \quad (3.13)$$

3.2 Point singularities and discrete transformations

In this section, we introduce the discrete transformations which make the wavefunction and/or its derivative discontinuous at some points. They play a crucial role to analyze the model with point singularities on a circle.

We consider the model which is one-dimensional quantum mechanics on a circle S^1 ($-l < x \leq l$) with 2^n point singularities placed at

$$x = l_s \equiv \left(1 - \frac{s}{2^{n-1}}\right) l \quad \text{for } s = 0, 1, \dots, 2^n - 1. \quad (3.14)$$

These point singularities are placed at regular intervals on the circle (Fig. 3.2).

We define discrete transformations on the circle as

$$(\mathcal{P}_k \varphi)(x) = \sum_{s=1}^{2^{k-1}} \Theta \left(x - \left(1 - \frac{s}{2^{k-2}}\right) l \right) \Theta \left(\left(1 - \frac{s-1}{2^{k-2}}\right) l - x \right) \times \varphi \left(-x + \left(2 - \frac{2s-1}{2^{k-1}}\right) l \right), \quad (3.15)$$

$$(\mathcal{R}_k \varphi)(x) = \sum_{s=1}^{2^{k-1}} (-1)^s \left[-\Theta \left(x - \left(1 - \frac{s-1/2}{2^{k-2}}\right) l \right) \Theta \left(\left(1 - \frac{s-1}{2^{k-2}}\right) l - x \right) + \Theta \left(x - \left(1 - \frac{s}{2^{k-2}}\right) l \right) \Theta \left(\left(1 - \frac{s-1/2}{2^{k-2}}\right) l - x \right) \right] \varphi(x), \quad (3.16)$$

$$(\mathcal{Q}_k \varphi)(x) \equiv -i (\mathcal{R}_k \mathcal{P}_k \varphi)(x) \quad (3.17)$$

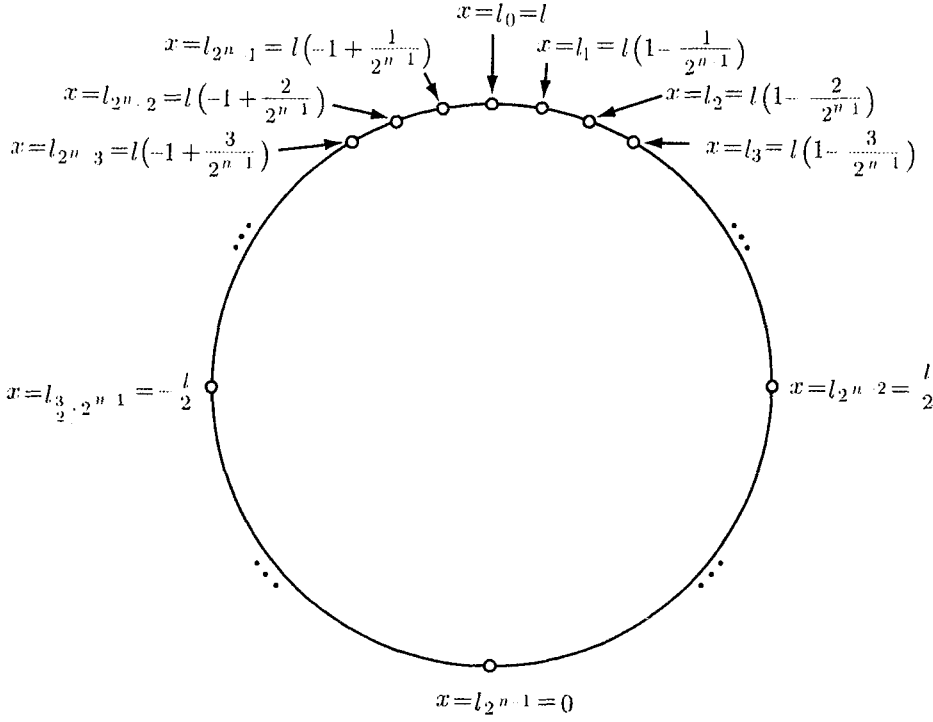


Figure 3.2: The positions of the singularities on a circle $S^1(-l < x \leq l)$

for $k = 1, 2, \dots, n + 1$. Here $\Theta(x)$ is the Heaviside step function defined by

$$\Theta(x) = \begin{cases} 1 & \text{for } 0 < x < l, \\ 0 & \text{for } -l < x < 0. \end{cases} \quad (3.18)$$

The $\mathcal{P}_k$ ($k = 1, 2, \dots, n + 1$) is a kind of the parity transformation. The $\mathcal{P}_1$ is just the familiar parity transformation, $(\mathcal{P}_1\varphi)(x) = \varphi(-x)$. The $\mathcal{R}_k$ ($k = 1, 2, \dots, n + 1$) is a kind of the half-reflection transformation. The action of $\mathcal{P}_k, \mathcal{R}_k$ ($k = 1, 2, 3$) is schematically depicted in Fig. 3.3- Fig. 3.5. In each figure, the upper figures mean the geometrical meaning of $\mathcal{P}_k, \mathcal{R}_k$ ($k = 1, 2, 3$). In the lower figures, the dashed line denotes the original function $\varphi(x)$, and the solid line denotes $(\mathcal{P}_k\varphi)(x), (\mathcal{R}_k\varphi)(x)$ ($k = 1, 2, 3$).

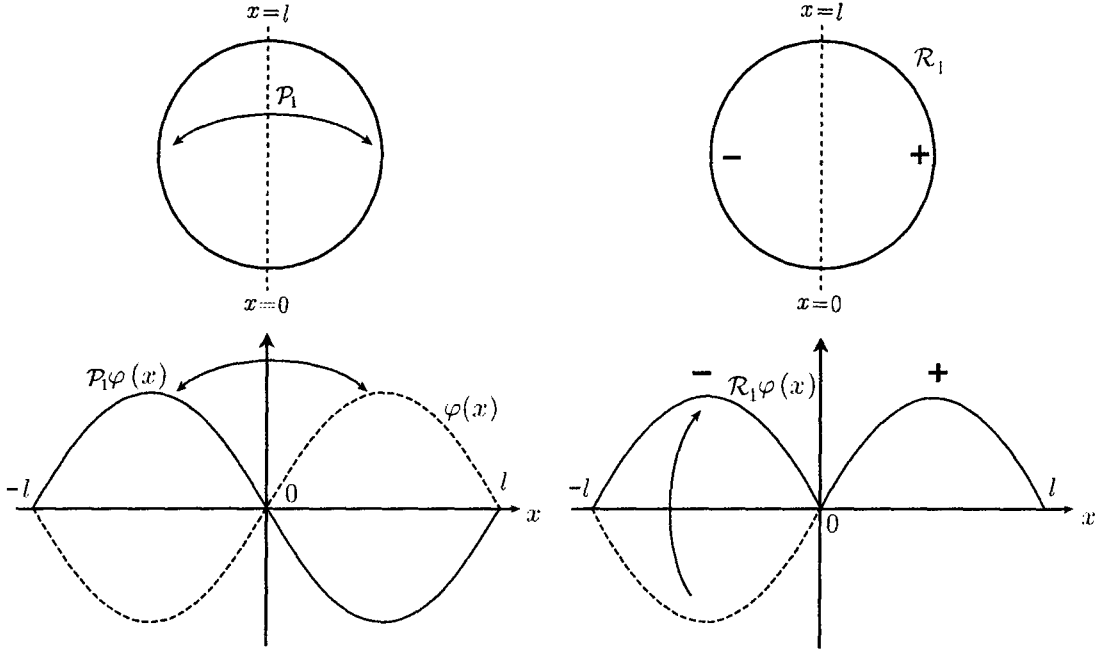


Figure 3.3: The action of $\mathcal{P}_1, \mathcal{R}_1$

The $\mathcal{P}_k, \mathcal{Q}_k, \mathcal{R}_k$ ($k = 1, 2, \dots, n$) and $\mathcal{P}_{n+1}$ produce no new singularities except for the original ones. Note that $\mathcal{Q}_{n+1}$ and $\mathcal{R}_{n+1}$ make wavefunction and/or its derivative discontinuous at $x \neq l_s$ ($s = 1, 2, \dots, 2^n - 1$). For instance, if we consider the case of $n = 1$, there are two singularities located at $x = 0, l$. As seen in fig. 3.3, a set of discrete transformation $\{\mathcal{P}_1, \mathcal{Q}_1, \mathcal{R}_1\}$ does not make wavefunction and its derivative discontinuous at $x \neq 0, l$. We introduce a new set of discrete transformation $\{\mathcal{P}_2, \mathcal{Q}_2, \mathcal{R}_2\}$. Although the $\mathcal{P}_2$ produces no new discontinuity for wavefunction and its derivative at $x \neq 0, l$, the $\mathcal{Q}_2$ and $\mathcal{R}_2$, however, produce discontinuity for wavefunction and/or its derivative at the different points of the original singularities, $x = \pm \frac{l}{2}$ (see fig. 3.4).

A crucial observation is that each set $\{\mathcal{P}_k, \mathcal{Q}_k, \mathcal{R}_k\}$ ($k = 1, 2, \dots, n + 1$) forms an $su(2)$ algebra of spin $1/2$,

$$\mathcal{P}_k \mathcal{Q}_k = -\mathcal{Q}_k \mathcal{P}_k = i\mathcal{R}_k, \quad \mathcal{Q}_k \mathcal{R}_k = -\mathcal{R}_k \mathcal{Q}_k = i\mathcal{P}_k, \quad \mathcal{R}_k \mathcal{P}_k = -\mathcal{P}_k \mathcal{R}_k = i\mathcal{Q}_k,$$

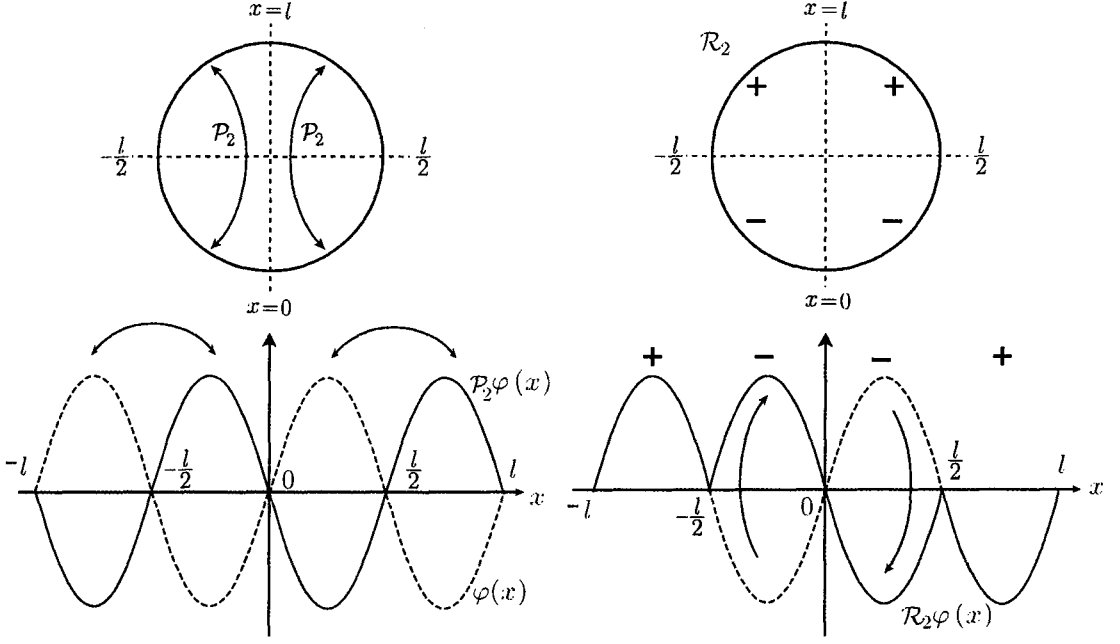


Figure 3.4: The action of $\mathcal{P}_2, \mathcal{R}_2$

$$(\mathcal{P}_k)^2 = (\mathcal{Q}_k)^2 = (\mathcal{R}_k)^2 = 1 \quad (3.19)$$

and that $\mathcal{O}_k = \{\mathcal{P}_k, \mathcal{Q}_k, \mathcal{R}_k\}$ and $\mathcal{O}_{k'} = \{\mathcal{P}_{k'}, \mathcal{Q}_{k'}, \mathcal{R}_{k'}\}$ commute with each other if $k \neq k'$,

$$[\mathcal{O}_k, \mathcal{O}_{k'}] = 0 \quad \text{for } k \neq k'. \quad (3.20)$$

For later use, let us introduce new sets of $su(2)$ generators $\{\mathcal{G}_{\mathcal{P}_k}, \mathcal{G}_{\mathcal{Q}_k}, \mathcal{G}_{\mathcal{R}_k}\}$ ($k = 1, 2, \dots, n$) as

$$\begin{aligned} \mathcal{G}_{\mathcal{P}_k} &= \mathcal{V}^\dagger \mathcal{P}_k \mathcal{V}, & \mathcal{G}_{\mathcal{Q}_k} &= \mathcal{V}^\dagger \mathcal{Q}_k \mathcal{V}, & \mathcal{G}_{\mathcal{R}_k} &= \mathcal{V}^\dagger \mathcal{R}_k \mathcal{V}, \\ & & k &= 1, 2, \dots, n, \end{aligned} \quad (3.21)$$

where

$$\mathcal{V} \equiv \mathcal{V}_1 \mathcal{V}_2 \cdots \mathcal{V}_n, \quad \mathcal{V}_k = e^{i\vec{v}_k \cdot \vec{\mathcal{P}}_k} \in SU(2). \quad (3.22)$$

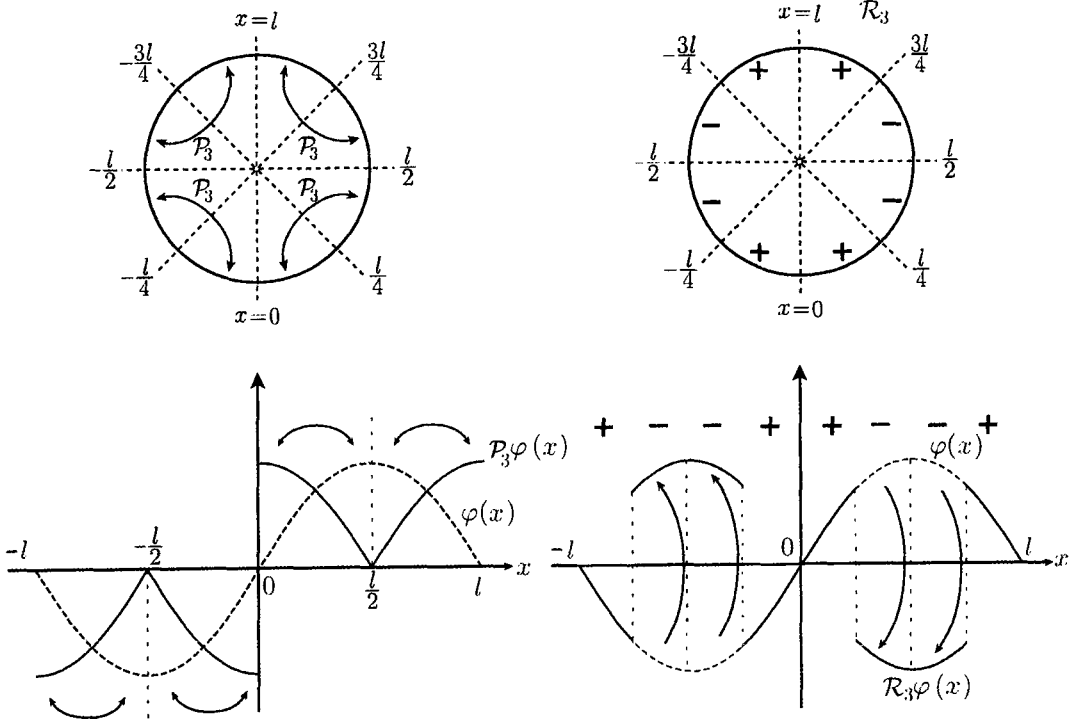


Figure 3.5: The action of $\mathcal{P}_3, \mathcal{R}_3$

Here, $\vec{\mathcal{P}}_k (\vec{v}_k)$ is an abbreviation of $\vec{\mathcal{P}}_k = (\mathcal{P}_k, \mathcal{Q}_k, \mathcal{R}_k)$ ($\vec{v}_k = (v_{\mathcal{P}_k}, v_{\mathcal{Q}_k}, v_{\mathcal{R}_k})$), and $(v_{\mathcal{P}_k}, v_{\mathcal{Q}_k}, v_{\mathcal{R}_k})$ are real parameters. The new $su(2)$ generators $\{\mathcal{G}_{\mathcal{P}_k}, \mathcal{G}_{\mathcal{Q}_k}, \mathcal{G}_{\mathcal{R}_k}\}$ have to be linearly related to $\vec{\mathcal{P}}_k$ as

$$\mathcal{G}_{\mathcal{P}_k} = \vec{e}_{\mathcal{P}_k} \cdot \vec{\mathcal{P}}_k, \quad \mathcal{G}_{\mathcal{Q}_k} = \vec{e}_{\mathcal{Q}_k} \cdot \vec{\mathcal{P}}_k, \quad \mathcal{G}_{\mathcal{R}_k} = \vec{e}_{\mathcal{R}_k} \cdot \vec{\mathcal{P}}_k, \quad k = 1, 2, \dots, n, \quad (3.23)$$

where $\{\vec{e}_{\mathcal{P}_k}, \vec{e}_{\mathcal{Q}_k}, \vec{e}_{\mathcal{R}_k}\}$ are three-dimensional orthogonal unit vectors.

3.3 $N = 2n + 1$ supersymmetry

3.3.1 $N = 2n + 1$ supercharges

Equipped with the discrete transformations given in the previous section, we construct $N = 2n + 1$ supercharges in terms of the $n + 1$ sets of the discrete transformations. There are two types of $N = 2n + 1$ supercharges (type A and type B),

- Type A

$$Q_a^A = \frac{i}{2}\Gamma_a\mathcal{D}^A = \frac{i}{2}\bar{\mathcal{D}}^A\Gamma_a, \quad a = 1, 2, \dots, 2n + 1, \quad (3.24)$$

- Type B

$$Q_a^B = \frac{i}{2}\Gamma_a\mathcal{D}^B = \frac{i}{2}\bar{\mathcal{D}}^B\Gamma_a, \quad a = 1, 2, \dots, 2n + 1, \quad (3.25)$$

where

$$\mathcal{D}^A = \left(\mathcal{R}_1 \cdots \mathcal{R}_{n+1} \frac{d}{dx} \right) + \mathcal{G}_{\mathcal{R}_1} \cdots \mathcal{G}_{\mathcal{R}_n} (\mathcal{R}_1 \cdots \mathcal{R}_{n+1} W'(x)), \quad (3.26)$$

$$\bar{\mathcal{D}}^A = \left(\mathcal{R}_1 \cdots \mathcal{R}_{n+1} \frac{d}{dx} \right) - \mathcal{G}_{\mathcal{R}_1} \cdots \mathcal{G}_{\mathcal{R}_n} (\mathcal{R}_1 \cdots \mathcal{R}_{n+1} W'(x)), \quad (3.27)$$

$$\mathcal{D}^B = \left(\mathcal{R}_1 \cdots \mathcal{R}_{n+1} \frac{d}{dx} \right) - \mathcal{P}_{n+1} (\mathcal{R}_1 \cdots \mathcal{R}_{n+1} V'(x)), \quad (3.28)$$

$$\bar{\mathcal{D}}^B = \left(\mathcal{R}_1 \cdots \mathcal{R}_{n+1} \frac{d}{dx} \right) + \mathcal{P}_{n+1} (\mathcal{R}_1 \cdots \mathcal{R}_{n+1} V'(x)) \quad (3.29)$$

and

$$\begin{aligned} \Gamma_{2k-1} &= \mathcal{G}_{\mathcal{R}_1} \mathcal{G}_{\mathcal{R}_2} \cdots \mathcal{G}_{\mathcal{R}_{k-1}} \mathcal{G}_{\mathcal{P}_k} \mathcal{R}_{n+1}, \\ \Gamma_{2k} &= \mathcal{G}_{\mathcal{R}_1} \mathcal{G}_{\mathcal{R}_2} \cdots \mathcal{G}_{\mathcal{R}_{k-1}} \mathcal{G}_{\mathcal{Q}_k} \mathcal{R}_{n+1}, \quad k = 1, 2, \dots, n, \\ \Gamma_{2n+1} &= \mathcal{Q}_{n+1}. \end{aligned} \quad (3.30)$$

Here, $W'(x) = \frac{d}{dx}W(x)$, $V'(x) = \frac{d}{dx}V(x)$, and $W(x), V(x)$ are superpotentials. We note that $\Gamma_a, \mathcal{D}^{A(B)}, \bar{\mathcal{D}}^{A(B)}$ in the supercharges contain $\mathcal{R}_{n+1}, \mathcal{Q}_{n+1}$ which produce new singularities except for the original singular point $x = l_s (s = 1, 2, \dots, 2^n - 1)$. The combinations of these operators in the supercharges, however, make the $\mathcal{Q}_{n+1}$ and $\mathcal{R}_{n+1}$ vanished due to the $su(2)$ algebra.

We assume that the functions $W'(x), V'(x)$ are continuous and finite valued functions at intervals between singularities and are allowed to have discontinuities at $x = l_s (s = 0, 1, \dots, 2^n - 1)$. In order to construct the $N = 2n + 1$ superalgebra, the functions turn out to be required to

$$\mathcal{P}_k W'(x) = -W'(x) \mathcal{P}_k, \quad k = 1, 2, \dots, n, \quad (3.31)$$

$$\mathcal{P}_{n+1} W'(x) = W'(x) \mathcal{P}_{n+1} \quad (3.32)$$

and

$$\mathcal{P}_i V'(x) = -V'(x) \mathcal{P}_i, \quad i = 1, 2, \dots, n + 1. \quad (3.33)$$

Noting that $\mathcal{R}_1 \cdots \mathcal{R}_{n+1} \frac{d}{dx}$, $\mathcal{R}_1 \cdots \mathcal{R}_{n+1} W'(x)$ and $\mathcal{R}_1 \cdots \mathcal{R}_{n+1} V'(x)$ commute with $\{\mathcal{P}_k, \mathcal{Q}_k, \mathcal{R}_k\} (k = 1, 2, \dots, n + 1)$, we have the relations

$$\{\Gamma_a, \Gamma_b\} = 2\delta_{ab}, \quad (3.34)$$

$$\Gamma_a \mathcal{D}^{A(B)} = \bar{\mathcal{D}}^{A(B)} \Gamma_a, \quad a, b = 1, 2, \dots, 2n + 1. \quad (3.35)$$

It follows that the supercharges form the $N = 2n + 1$ superalgebra;

- Type A

$$\{Q_a^A, Q_b^A\} = H^A \delta_{ab}, \quad a, b = 1, 2, \dots, 2n + 1, \quad (3.36)$$

$$H^A = -\frac{1}{2} \bar{\mathcal{D}}^A \mathcal{D}^A = \frac{1}{2} \left[-\frac{d^2}{dx^2} - \mathcal{G}_{\mathcal{R}_1} \cdots \mathcal{G}_{\mathcal{R}_n} W''(x) + (W'(x))^2 \right], \quad (3.37)$$

- Type B

$$\{Q_a^B, Q_b^B\} = H^B \delta_{ab}, \quad a, b = 1, 2, \dots, 2n+1, \quad (3.38)$$

$$H^B = -\frac{1}{2} \bar{\mathcal{D}}^B \mathcal{D}^B = \frac{1}{2} \left[-\frac{d^2}{dx^2} + \mathcal{P}_{n+1} V''(x) + (V'(x))^2 \right], \quad (3.39)$$

where H^A and H^B are the Hamiltonian in each model.

3.3.2 Connection conditions compatible with supersymmetry

In this section, we clarify the connection conditions compatible with the $N = 2n+1$ supersymmetry. Since the model contains the point singularities, we need to impose appropriate connection conditions there, namely, our model is specified not only by the Hamiltonian but also by the connection conditions. The functional space in our model is required to be squared integrable and to be spanned by eigenfunctions of the Hamiltonian with connection conditions that make the Hamiltonian hermitian.

In order to obtain the appropriate connection conditions, let us introduce a 2^{n+1} -dimensional boundary vector Φ_φ that consists of boundary values of a wavefunction $\varphi(x)$ in the vicinity of the singularities, $\varphi(l_s \pm \epsilon)$ for $s = 0, 1, \dots, 2^n - 1$ with an infinitesimal positive constant ϵ . For later convenience, we arrange $\varphi(l_s \pm \epsilon)$ in such a way that Φ_φ satisfies the relations

$$\Phi_{\mathcal{P}_k \varphi} = (\overbrace{I_2 \otimes \dots \otimes I_2}^k \otimes \sigma_1 \otimes I_2 \otimes \dots \otimes I_2) \Phi_\varphi, \quad (3.40)$$

$$\Phi_{\mathcal{Q}_k \varphi} = (I_2 \otimes \dots \otimes I_2 \otimes \sigma_2 \otimes I_2 \otimes \dots \otimes I_2) \Phi_\varphi, \quad (3.41)$$

$$\Phi_{\mathcal{R}_k \varphi} = (\underbrace{I_2 \otimes \dots \otimes I_2}_{n+1} \otimes \sigma_3 \otimes I_2 \otimes \dots \otimes I_2) \Phi_\varphi. \quad (3.42)$$

The boundary vector can be arranged as

$$\Phi_\varphi = \begin{pmatrix} \varphi(l_0 - \epsilon) \\ \varphi(l_1 + \epsilon) \\ (\mathcal{P}_n \varphi)(l_0 - \epsilon) \\ (\mathcal{P}_n \varphi)(l_1 + \epsilon) \\ (\mathcal{P}_{n-1} \varphi)(l_0 - \epsilon) \\ (\mathcal{P}_{n-1} \varphi)(l_1 + \epsilon) \\ (\mathcal{P}_{n-1} \mathcal{P}_n \varphi)(l_0 - \epsilon) \\ (\mathcal{P}_{n-1} \mathcal{P}_n \varphi)(l_1 + \epsilon) \\ (\mathcal{P}_{n-2} \varphi)(l_0 - \epsilon) \\ (\mathcal{P}_{n-2} \varphi)(l_1 + \epsilon) \\ (\mathcal{P}_{n-2} \mathcal{P}_n \varphi)(l_0 - \epsilon) \\ (\mathcal{P}_{n-2} \mathcal{P}_n \varphi)(l_1 + \epsilon) \\ (\mathcal{P}_{n-2} \mathcal{P}_{n-1} \varphi)(l_0 - \epsilon) \\ (\mathcal{P}_{n-2} \mathcal{P}_{n-1} \varphi)(l_1 + \epsilon) \\ (\mathcal{P}_{n-2} \mathcal{P}_{n-1} \mathcal{P}_n \varphi)(l_0 - \epsilon) \\ (\mathcal{P}_{n-2} \mathcal{P}_{n-1} \varphi)(l_1 + \epsilon) \\ \vdots \\ (\mathcal{P}_2 \mathcal{P}_3 \cdots \mathcal{P}_n \varphi)(l_0 - \epsilon) \\ (\mathcal{P}_2 \mathcal{P}_3 \cdots \mathcal{P}_n \varphi)(l_1 + \epsilon) \\ \dots\dots\dots \\ \mathcal{P}_1 \begin{bmatrix} \text{upper half} \\ \text{components} \end{bmatrix} \end{pmatrix}. \quad (3.43)$$

For instance, Φ_φ for the case of $n = 1$ (two singular points are placed at $x = 0, l$) is arranged as

$$\Phi_\varphi = \begin{pmatrix} \varphi(l - \epsilon) \\ \varphi(0 + \epsilon) \\ \varphi(-l + \epsilon) \\ \varphi(0 - \epsilon) \end{pmatrix} = \begin{pmatrix} \varphi(l - \epsilon) \\ \varphi(0 + \epsilon) \\ (\mathcal{P}_1 \varphi)(l - \epsilon) \\ (\mathcal{P}_1 \varphi)(0 + \epsilon) \end{pmatrix}, \quad (3.44)$$

which obeys the relations (3.40)-(3.42) with $n = 1$,

$$\Phi_{\mathcal{P}_1\varphi} = \begin{pmatrix} (\mathcal{P}_1\varphi)(l-\epsilon) \\ (\mathcal{P}_1\varphi)(0+\epsilon) \\ (\mathcal{P}_1\varphi)(-l+\epsilon) \\ (\mathcal{P}_1\varphi)(0-\epsilon) \end{pmatrix} = \begin{pmatrix} \varphi(-l+\epsilon) \\ \varphi(0-\epsilon) \\ \varphi(l-\epsilon) \\ \varphi(0+\epsilon) \end{pmatrix} = \begin{pmatrix} 0 & I_2 \\ I_2 & 0 \end{pmatrix} \Phi_\varphi = (\sigma_1 \otimes I_2) \Phi_\varphi, \quad (3.45)$$

$$\Phi_{\mathcal{R}_1\varphi} = \begin{pmatrix} (\mathcal{R}_1\varphi)(l-\epsilon) \\ (\mathcal{R}_1\varphi)(0+\epsilon) \\ (\mathcal{R}_1\varphi)(-l+\epsilon) \\ (\mathcal{R}_1\varphi)(0-\epsilon) \end{pmatrix} = \begin{pmatrix} \varphi(l-\epsilon) \\ \varphi(0+\epsilon) \\ -\varphi(-l+\epsilon) \\ -\varphi(0-\epsilon) \end{pmatrix} = \begin{pmatrix} I_2 & 0 \\ 0 & -I_2 \end{pmatrix} \Phi_\varphi = (\sigma_3 \otimes I_2) \Phi_\varphi, \quad (3.46)$$

$$\Phi_{\mathcal{Q}_1\varphi} = \begin{pmatrix} (\mathcal{Q}_1\varphi)(l-\epsilon) \\ (\mathcal{Q}_1\varphi)(0+\epsilon) \\ (\mathcal{Q}_1\varphi)(-l+\epsilon) \\ (\mathcal{Q}_1\varphi)(0-\epsilon) \end{pmatrix} = \begin{pmatrix} -i\varphi(-l+\epsilon) \\ -i\varphi(0-\epsilon) \\ i\varphi(l-\epsilon) \\ i\varphi(0+\epsilon) \end{pmatrix} = \begin{pmatrix} 0 & -iI_2 \\ iI_2 & 0 \end{pmatrix} \Phi_\varphi = (\sigma_2 \otimes I_2) \Phi_\varphi. \quad (3.47)$$

Since our model has the singular points, wavefunction will, in general, be discontinuous there, but the discontinuity has to be controlled by the connection conditions that make the Hamiltonian hermitian. The hermiticity condition is

$$\int_{-l}^l dx \psi^*(x) (H^{A(B)}\varphi)(x) = \int_{-l}^l dx (H^{A(B)}\psi)^*(x) \varphi(x) \quad (3.48)$$

for any wavefunctions $\psi(x), \varphi(x)$, where the integral $\int_{-l}^l dx$ is defined by

$$\int_{-l}^l dx \equiv \sum_{s=1}^{2^n} \int_{l_s+\epsilon}^{l_{s-1}-\epsilon} dx \quad \text{with } l_{2^n} \equiv l_0. \quad (3.49)$$

In terms of the boundary vector, the requirement (3.48) can simply be rewritten as the constraints on the boundary vector

$$\Phi_\psi^\dagger \Phi_{\mathcal{D}^{A(B)}\varphi} = \Phi_{\mathcal{D}^{A(B)}\psi}^\dagger \Phi_\varphi. \quad (3.50)$$

In order to derive it, we have used the relation (3.42) and the formula of integration by parts

$$\int_{-l}^l dx \xi^*(x) \left(\frac{d}{dx} \eta(x) \right) = - \int_{-l}^l dx \left(\frac{d}{dx} \xi(x) \right)^* \eta(x) + \Phi_\xi^\dagger (\sigma_3 \otimes \cdots \otimes \sigma_3) \Phi_\eta, \quad (3.51)$$

where the functions $\xi(x)$ and $\eta(x)$ are assumed to be continuous everywhere except for the singular points. It is easy to show that eq. (3.50) is equivalent to

$$|\Phi_\varphi - iL_0 \Phi_{\mathcal{D}^{A(B)}\varphi}| = |\Phi_\varphi + iL_0 \Phi_{\mathcal{D}^{A(B)}\varphi}|, \quad (3.52)$$

where L_0 is an arbitrary nonzero constant with the dimension of length. Then, the condition (3.52) can be written as

$$(I_{2^{n+1}} - U) \Phi_\varphi + iL_0 (I_{2^{n+1}} + U) \Phi_{\mathcal{D}^{A(B)}\varphi} = 0, \quad (3.53)$$

where U is a $2^{n+1} \times 2^{n+1}$ unitary matrix. Thus, we have found that the connection conditions which make the Hamiltonian hermitian are given by eq. (3.53). The 2^n singularities in our model are characterized by a $2^{n+1} \times 2^{n+1}$ unitary matrix, U .

It is important to notice that the above connection condition does not necessarily guarantee the $N = 2n + 1$ supersymmetry, because the hermiticity of the Hamiltonian does not, in general, ensure that of the supercharges. Moreover, the state, $(Q_a \varphi)(x)$ does not necessarily belong to the same functional space as $\varphi(x)$. This also means that $(Q_a \varphi)(x)$ does not obey the same connection conditions as $\varphi(x)$. In order for any operator $\mathcal{O}$ to be physical, the state $(\mathcal{O}\varphi)(x)$ for any state $\varphi(x)$ has to obey the same connection conditions as $\varphi(x)$. In the following, we call $\mathcal{O}$ a physical operator if $\mathcal{O}$ is hermitian *and* $(\mathcal{O}\varphi)(x)$ satisfies the same connection conditions as $\varphi(x)$.

In the standard argument of supersymmetry, for any energy eigenfunction φ with an energy E , $Q_a \varphi$ is also an energy eigenfunction with the same energy

$E(> 0)$ and then $Q_a\varphi$ is called a supersymmetric partner of φ . In our model, this is true if and only if Q_a is a physical operator, otherwise the state $Q_a\varphi$ should be removed from the functional space, in which any state satisfies the connection conditions (3.53) with a characteristic matrix U . Therefore the $N = 2n + 1$ supersymmetry can be realized in our model, only when all the $N = 2n + 1$ supercharges are physical. We will see below that this requirement severely restricts a class of the connection conditions.

Let $\varphi(x)$ be any wavefunction obeying the connection conditions (3.53), and satisfying Schrödinger equation, $H^{A(B)}\varphi(x) = E\varphi(x)$.

We first require that $(Q_a^{A(B)}\varphi)(x)$ ($a = 1, 2, \dots, 2n + 1$) obey the same connection conditions as $\varphi(x)$,

$$(I_{2n+1} - U) \Phi_{Q_a^{A(B)}\varphi} + iL_0 (I_{2n+1} + U) \Phi_{\mathcal{D}^{A(B)}(Q_a^{A(B)}\varphi)} = 0. \quad (3.54)$$

Here, φ in eq. (3.53) has been replaced by $Q_a^{A(B)}\varphi$. By noting (3.24), (3.25), (3.35), (3.36)-(3.39) and $H^{A(B)}\varphi(x) = E\varphi(x)$, eq. (3.54) leads to

$$(I_{2n+1} - U) \Phi_{\Gamma_a \mathcal{D}^{A(B)}\varphi} - 2iL_0 E (I_{2n+1} + U) \Phi_{\Gamma_a \varphi} = 0. \quad (3.55)$$

The connection conditions must be independent of the energy E , so that we find the eigenvalues of U must be ± 1 , i.e. $U^2 = I_{2n+1}$. Then, (3.55) can be written as

$$(I_{2n+1} - U) \Phi_{\Gamma_a \mathcal{D}^{A(B)}\varphi} = 0, \quad (3.56)$$

$$(I_{2n+1} + U) \Phi_{\Gamma_a \varphi} = 0. \quad (3.57)$$

We note that the case of $U = \pm I_{2n+1}$ turns out to lead to no nontrivial models because all wavefunctions would vanish. It is convenient to introduce γ_a ($a = 1, 2, \dots, 2n + 1$) as

$$\gamma_{2k-1} = \left(\vec{e}_{\mathcal{R}_1} \cdot \vec{\sigma} \otimes \dots \otimes \vec{e}_{\mathcal{R}_{k-1}} \cdot \vec{\sigma} \otimes \vec{e}_{\mathcal{P}_k} \cdot \vec{\sigma} \otimes I_2 \otimes \dots \otimes I_2 \otimes \sigma_3 \right), \quad (3.58)$$

$$\gamma_{2k} = \left(\vec{e}_{\mathcal{R}_1} \cdot \vec{\sigma} \otimes \dots \otimes \vec{e}_{\mathcal{R}_{k-1}} \cdot \vec{\sigma} \otimes \vec{e}_{\mathcal{Q}_k} \cdot \vec{\sigma} \otimes I_2 \otimes \dots \otimes I_2 \otimes \sigma_3 \right) \quad (3.59)$$

$$\gamma_{2n+1} = (I_2 \otimes \dots \otimes I_2 \otimes \sigma_2), \quad k = 1, 2, \dots, n, \quad (3.60)$$

where $\vec{\sigma} = (\sigma_1, \sigma_2, \sigma_3)$. The γ_a for $a = 1, 2, \dots, 2n + 1$ satisfy the same anti-commutation relations as Γ_a ,

$$\{\gamma_a, \gamma_b\} = 2\delta_{ab}. \quad (3.61)$$

Using these, we have the relation

$$\Phi_{\Gamma_a \varphi} = \gamma_a \Phi_\varphi, \quad (3.62)$$

and then, eqs. (3.56), (3.57) become

$$(I_{2^{n+1}} - \gamma_a U \gamma_a) \Phi_{\mathcal{D}^{A(B)} \varphi} = 0, \quad (3.63)$$

$$(I_{2^{n+1}} + \gamma_a U \gamma_a) \Phi_\varphi = 0. \quad (3.64)$$

Eq. (3.53) is reduced to

$$(I_{2^{n+1}} - U) \Phi_\varphi = 0, \quad (3.65)$$

$$(I_{2^{n+1}} + U) \Phi_{\mathcal{D}^{A(B)} \varphi} = 0, \quad (3.66)$$

under the condition of $U^2 = I_{2^{n+1}}$. Since our model contains the 2^n point singularities, we need to impose 2×2^n connection conditions at the singularities to solve the Schrödinger equation. If the number of conditions is larger than 2×2^n , the model would become trivial due to overconstraints. Since the number of the conditions (3.53), which make the Hamiltonian hermitian, is 2×2^n , eqs. (3.63) and (3.64) should produce no new constraints. This implies that the characteristic matrix U has to satisfy

$$\gamma_a U \gamma_a = -U, \quad a = 1, 2, \dots, 2n + 1. \quad (3.67)$$

Thus, we have found the constraints on the characteristic matrix U ,

$$U^\dagger U = I_{2^{n+1}}, \quad (3.68)$$

$$U^2 = I_{2^{n+1}}, \quad (3.69)$$

$$\gamma_a U \gamma_a = -U, \quad a = 1, 2, \dots, 2n + 1 \quad (3.70)$$

which guarantee that the Hamiltonian is hermitian and $(Q_a^{A(B)}\varphi)(x)$ ($a = 1, 2, \dots, 2n+1$) satisfy the same connection conditions as $\varphi(x)$.

Before we discuss the hermiticity of the supercharges, let us determine the form of the characteristic matrix U . An arbitrary $2^{n+1} \times 2^{n+1}$ matrix can be uniquely expanded as

$$\sum_{\alpha_1=0}^3 \cdots \sum_{\alpha_{n+1}=0}^3 C_{\alpha_1, \alpha_2, \dots, \alpha_{n+1}} \left(\tilde{e}_{\alpha_1}^{(1)} \otimes \tilde{e}_{\alpha_2}^{(2)} \otimes \cdots \otimes \tilde{e}_{\alpha_n}^{(n)} \otimes \tilde{e}_{\alpha_{n+1}}^{(n+1)} \right), \quad (3.71)$$

where

$$\begin{aligned} \tilde{e}_{\alpha_k=0}^{(k)} &\equiv I_2, \quad \tilde{e}_{\alpha_k=1}^{(k)} \equiv \vec{e}_{\mathcal{P}_k} \cdot \vec{\sigma}, \quad \tilde{e}_{\alpha_k=2}^{(k)} \equiv \vec{e}_{\mathcal{Q}_k} \cdot \vec{\sigma}, \quad \tilde{e}_{\alpha_k=3}^{(k)} \equiv \vec{e}_{\mathcal{R}_k} \cdot \vec{\sigma}, \\ k &= 1, 2, \dots, n, \end{aligned} \quad (3.72)$$

and

$$\tilde{e}_{\alpha_{n+1}=0}^{(n+1)} \equiv I_2, \quad \tilde{e}_{\alpha_{n+1}=1}^{(n+1)} \equiv \sigma_1, \quad \tilde{e}_{\alpha_{n+1}=2}^{(n+1)} \equiv \sigma_2, \quad \tilde{e}_{\alpha_{n+1}=3}^{(n+1)} \equiv \sigma_3. \quad (3.73)$$

First, we impose the conditions (3.70) on the characteristic matrix U . Then, some coefficients in the expansion vanish except for $C_{\alpha_1=0, \dots, \alpha_n=0, \alpha_{n+1}=1}$, $C_{\alpha_1=3, \dots, \alpha_n=3, \alpha_{n+1}=3}$. Thus, the characteristic matrix U becomes

$$U = b_1 (I_2 \otimes \cdots \otimes I_2 \otimes \sigma_1) + b_2 (\vec{e}_{\mathcal{R}_1} \cdot \vec{\sigma} \otimes \cdots \otimes \vec{e}_{\mathcal{R}_n} \cdot \vec{\sigma} \otimes \sigma_3), \quad (3.74)$$

where

$$b_1 \equiv C_{\alpha_1=0, \dots, \alpha_n=0, \alpha_{n+1}=1}, \quad b_2 \equiv C_{\alpha_1=3, \dots, \alpha_n=3, \alpha_{n+1}=3}. \quad (3.75)$$

The conditions $U^\dagger U = I_{2^{n+1}}$ and $U^2 = I_{2^{n+1}}$ imply $U^\dagger = U$, so that we have

$$b_1, b_2 \in \mathbf{R}, \quad (b_1)^2 + (b_2)^2 = 1. \quad (3.76)$$

Therefore, the connection conditions can be written by

$$(I_{2^{n+1}} - U(b_1, b_2)) \Phi_\varphi = 0, \quad (3.77)$$

$$(I_{2^{n+1}} + U(b_1, b_2)) \Phi_{\mathcal{D}^{A(B)}\varphi} = 0, \quad (3.78)$$

where

$$U(b_1, b_2) = b_1 (I_2 \otimes \cdots \otimes I_2 \otimes \sigma_1) + b_2 (\vec{e}_{\mathcal{R}_1} \cdot \vec{\sigma} \otimes \cdots \otimes \vec{e}_{\mathcal{R}_n} \cdot \vec{\sigma} \otimes \sigma_3) \quad (3.79)$$

with $b_1, b_2 \in \mathbf{R}$ and $(b_1)^2 + (b_2)^2 = 1$.

Now, an important question that remains to be answered is whether the supercharges are hermitian under the connection conditions (3.77) and (3.78) or not. Using (3.51), we have the relation

$$\int_{-l}^l dx \psi^*(x) (Q_a \varphi)(x) = \int_{-l}^l dx (Q_a \psi)^*(x) \varphi(x) + \frac{i}{2} \Phi_\psi^\dagger \gamma_a \Phi_\varphi, \quad a = 1, 2, \dots, 2n+1. \quad (3.80)$$

It is not difficult to show that the surface term in (3.80) vanishes,

$$\Phi_\psi^\dagger \gamma_a \Phi_\varphi = 0, \quad a = 1, 2, \dots, 2n+1, \quad (3.81)$$

if wavefunctions φ and ψ obey the connection conditions (3.77) and (3.78). Thus, the connection conditions (3.77) and (3.78) also make all the supercharges hermitian.

Thus, we have found that the connection conditions, which make not only the Hamiltonian hermitian but also the $2n+1$ supercharges physical, are given by eq. (3.77), (3.78). Then, the $N = 2n+1$ supersymmetry can be realized with the connection conditions (3.77), (3.78).

3.3.3 Degeneracy of the spectrum

In this section, we study the degeneracy of the spectrum in the $N = 2n+1$ supersymmetric models, in particular, vacuum states with the vanishing energy. We consider the two types of the $N = 2n+1$ supersymmetric models

(type A and type B), separately.

(i) Type A model

We first consider the degeneracy in the type A supersymmetric model. The connection conditions compatible with the $N = 2n+1$ supercharges $Q_1^A, \dots, Q_{2n+1}^A$ are given by

$$(I_{2n+1} - U(b_1, b_2))\Phi_\varphi = 0, \quad (3.82)$$

$$(I_{2n+1} + U(b_1, b_2))\Phi_{\mathcal{D}^A\varphi} = 0, \quad (3.83)$$

where

$$U(b_1, b_2) = b_1(I_2 \otimes \dots \otimes I_2 \otimes \sigma_1) + b_2(\vec{e}_{\mathcal{R}_1} \cdot \vec{\sigma} \otimes \dots \otimes \vec{e}_{\mathcal{R}_n} \cdot \vec{\sigma} \otimes \sigma_3) \quad (3.84)$$

with $b_1, b_2 \in \mathbf{R}$ and $(b_1)^2 + (b_2)^2 = 1$. We note that the $\mathcal{G}_{\mathcal{R}_k}$ ($k = 1, 2, \dots, n$) commutes with the Hamiltonian H^A and also with each other. Since they are also physical under the connection conditions (3.82) and (3.83), we can introduce simultaneous eigenfunctions of H^A and $\mathcal{G}_{\mathcal{R}_k}$ such that

$$H^A \varphi_{E; \lambda_1, \dots, \lambda_n}(x) = E \varphi_{E; \lambda_1, \dots, \lambda_n}(x), \quad (3.85)$$

$$\mathcal{G}_{\mathcal{R}_k} \varphi_{E; \lambda_1, \dots, \lambda_n}(x) = \lambda_k \varphi_{E; \lambda_1, \dots, \lambda_n}(x) \quad (3.86)$$

with $\lambda_k = 1$ or -1 for $k = 1, 2, \dots, n$. Since Q_a^A ($a = 1, 2, \dots, 2n+1$) and $\mathcal{G}_{\mathcal{R}_k}$ ($k = 1, 2, \dots, n$) satisfy the relations

$$Q_a^A \mathcal{G}_{\mathcal{R}_k} = \begin{cases} -\mathcal{G}_{\mathcal{R}_k} Q_a^A & \text{for } a = 2k-1 \text{ or } 2k, \\ +\mathcal{G}_{\mathcal{R}_k} Q_a^A & \text{otherwise,} \end{cases} \quad (3.87)$$

the states $Q_{2k-1}^A \varphi_{E; \lambda_1, \dots, \lambda_n}(x)$ and $Q_{2k}^A \varphi_{E; \lambda_1, \dots, \lambda_n}(x)$ should be proportional to $\varphi_{E; \lambda_1, \dots, -\lambda_k, \dots, \lambda_n}(x)$, and the states $Q_{2n+1}^A \varphi_{E; \lambda_1, \dots, \lambda_n}(x)$ should be proportional to $\varphi_{E; \lambda_1, \dots, \lambda_n}(x)$. Noting that $Q_{2k-1}^A = -iQ_{2k}^A \mathcal{G}_{\mathcal{R}_k}$, we have the relations

$$Q_{2k-1}^A \varphi_{E; \lambda_1, \dots, \lambda_n}(x) = -i\lambda_k Q_{2k}^A \varphi_{E; \lambda_1, \dots, \lambda_n}(x) \propto \varphi_{E; \lambda_1, \dots, -\lambda_k, \dots, \lambda_n}(x), \quad (3.88)$$

$$Q_{2n+1}^A \varphi_{E; \lambda_1, \dots, \lambda_n}(x) \propto \varphi_{E; \lambda_1, \dots, \lambda_n}(x), \quad (3.89)$$

when $E > 0$. These imply that the degeneracy of the spectrum for $E > 0$ is given by 2^n . Some examples of the transition for the eigenfunction $\varphi_{E_1; \lambda_1, \dots, \lambda_n}$ by acting the supercharges are depicted in Fig. 3.6. This degeneracy can also

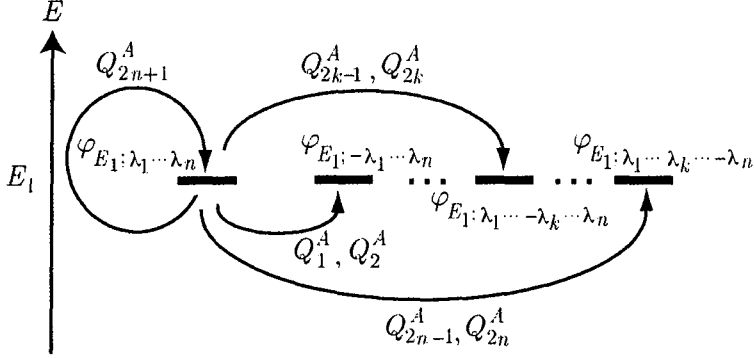


Figure 3.6: Examples of the transition for the eigenfunction $\varphi_{E_1; \lambda_1, \dots, \lambda_n}$ under the action of the supercharges $(Q_1^A, \dots, Q_{2n+1}^A)$

be understood from an algebraic point of view; for fixed nonzero energy E , $Q_a^A/\sqrt{E}$ for $a = 1, 2, \dots, 2n+1$ form the Clifford algebra in $(2n+1)$ -dimensions, and the representation is known as 2^n .

The above argument cannot apply for the case of $E = 0$. This is because any state $\varphi_0(x)$ with vanishing energy must satisfy $Q_a^A \varphi_0(x) = 0$ for $a = 1, 2, \dots, 2n+1$. The equations $Q_a^A \varphi_0(x) = 0$ are equivalent to

$$\mathcal{D}^A \varphi_0(x) = 0. \quad (3.90)$$

The above equation is easily solved, and we have formal solutions as

$$\varphi_{0; \lambda_1, \dots, \lambda_n}(x) = N_{\lambda_1, \dots, \lambda_n} \left[\prod_{k=1}^n \frac{1}{2} (1 + \lambda_k \mathcal{G}_{\mathcal{R}_k}) \right] e^{-\lambda_1 \dots \lambda_n W(x)}. \quad (3.91)$$

Here, $N_{\lambda_1, \dots, \lambda_n}$ denotes normalization constants. By noting the Hamiltonian H^A includes only $W'(x)$ (but not $W(x)$ itself), we have an ambiguity to choose

$W(x)$ to satisfy

$$\mathcal{P}_k W(x) \mathcal{P}_k = W(x), \quad k = 1, 2, \dots, n, \quad (3.92)$$

$$\mathcal{P}_{n+1} W(x) \mathcal{P}_{n+1} = -W(x). \quad (3.93)$$

For a noncompact space, any non-normalizable states would be removed from the functional space. The space is, however, compact in our model, so that these solutions are always normalizable. Nevertheless, some of them must be removed from the functional space. This is because some solutions are incompatible with the connection conditions. Although the zero energy states trivially satisfy the half of the connection conditions (3.83), we need to verify whether the states (3.91) satisfy the remaining connection condition (3.82) or not.

In order to investigate the consistency of the zero energy states with the connection conditions, we use the transformation (3.21), (3.22). This transformation yields

$$\tilde{U}(b_1, b_2) \equiv V U(b_1, b_2) V^\dagger = b_1 (I_2 \otimes \dots \otimes I_2 \otimes \sigma_1) + b_2 (\sigma_3 \otimes \dots \otimes \sigma_3), \quad (3.94)$$

where

$$V \equiv \left(e^{i\vec{v}_1 \cdot \vec{\sigma}} \otimes e^{i\vec{v}_2 \cdot \vec{\sigma}} \otimes \dots \otimes e^{i\vec{v}_n \cdot \vec{\sigma}} \otimes I_2 \right), \quad e^{i\vec{v}_k \cdot \vec{\sigma}} \in SU(2). \quad (3.95)$$

The transformation is a *singular* unitary transformation because it, in general, changes connection conditions of wavefunctions at singular points. The transformation may be regarded as a duality connecting different theories (with different connection conditions). Moreover, the transformation gives us the convenient expression for the connection condition.

For instance, we consider the model which possesses two point singularities placed at $x = 0, l$ ($n = 1$) as an example. The connection condition (3.82) is transformed by (3.95) to

$$(I_4 - \tilde{U}(b_1, b_2)) \Phi_{\vec{\varphi}} = 0, \quad (3.96)$$

where

$$\tilde{U}(b_1, b_2) = b_1(I_2 \otimes \sigma_1) + b_2(\sigma_3 \otimes \sigma_3), \quad \tilde{\varphi} = \mathcal{V}_1 \varphi, \quad \mathcal{V}_1 = e^{i\vec{v}_1 \cdot \vec{P}_1}. \quad (3.97)$$

Explicitly, eq. (3.96) is written as

$$\begin{pmatrix} 1 - b_2 & -b_1 & 0 & 0 \\ -b_1 & 1 + b_2 & 0 & 0 \\ 0 & 0 & 1 + b_2 & -b_1 \\ 0 & 0 & -b_1 & 1 - b_2 \end{pmatrix} \begin{pmatrix} \tilde{\varphi}(l - \epsilon) \\ \tilde{\varphi}(0 + \epsilon) \\ \tilde{\varphi}(-l + \epsilon) \\ \tilde{\varphi}(-0 - \epsilon) \end{pmatrix} = 0. \quad (3.98)$$

We observe that the above connection condition is separated into two independent conditions, depending on the argument of the wavefunction. This suggests that it is useful to assign the eigenvalue λ_1 of the operator $\mathcal{R}_1$ for the wavefunction. To this end, we define the projection operator,

$$\frac{1}{2} (1 + \lambda_1 \mathcal{R}_1). \quad (3.99)$$

By multiplying $(1 + \lambda_1 \mathcal{R}_1)/2$ from the left to eq. (3.96), we obtain the boundary vector as

$$\Phi_{\left(\frac{1+\lambda_1 \mathcal{R}_1}{2}\right) \tilde{\varphi}} = \begin{cases} \begin{pmatrix} \tilde{\varphi}_{\lambda_1=+1}(l - \epsilon) = 0 \\ \tilde{\varphi}_{\lambda_1=+1}(0 + \epsilon) = 0 \\ 0 \\ 0 \end{pmatrix} & \text{for } \lambda_1 = +1, \\ \begin{pmatrix} 0 \\ 0 \\ \tilde{\varphi}_{\lambda_1=-1}(-l + \epsilon) = 0 \\ \tilde{\varphi}_{\lambda_1=-1}(0 - \epsilon) = 0 \end{pmatrix} & \text{for } \lambda_1 = -1. \end{cases} \quad (3.100)$$

Then, the connection conditions (3.96) are reduced to

- for $b_2 = 1$ (and $b_1 = 0$)

$$\tilde{\varphi}_{\lambda_1=+1}(0+\epsilon) = 0, \quad (3.101)$$

$$\tilde{\varphi}_{\lambda_1=-1}(-l+\epsilon) = 0, \quad (3.102)$$

- for $b_2 \neq 1$

$$(1-b_2)\tilde{\varphi}_{\lambda_1=+1}(l-\epsilon) - b_1\tilde{\varphi}_{\lambda_1=+1}(0+\epsilon) = 0, \quad (3.103)$$

$$(1-b_2)\tilde{\varphi}_{\lambda_1=-1}(0-\epsilon) - b_1\tilde{\varphi}_{\lambda_1=-1}(-l+\epsilon) = 0. \quad (3.104)$$

Let us also note that the Schrödinger equations can be written separately in each region of $(0 < x < l)$ and $(-l < x < 0)$,

$$\tilde{H}\tilde{\varphi}_{\lambda_1=+1}(x) = E\tilde{\varphi}_{\lambda_1=+1}(x) \quad (0 < x < l) \quad \text{with (3.101) or (3.103),} \quad (3.105)$$

$$\tilde{H}\tilde{\varphi}_{\lambda_1=-1}(x) = E\tilde{\varphi}_{\lambda_1=-1}(x) \quad (-l < x < 0) \quad \text{with (3.102) or (3.104),} \quad (3.106)$$

where $\tilde{H} \equiv \mathcal{V}H\mathcal{V}^\dagger$.

Similarly, we can generalize the above argument to the case of n . The connection conditions (3.82) can be written as

$$(I_{2^{n+1}} - \tilde{U}(b_1, b_2))\Phi\left(\prod_{k=1}^n \frac{1+\lambda_k\mathcal{R}_k}{2}\right)\tilde{\varphi} = 0 \quad (3.107)$$

where

$$\tilde{\varphi} \equiv \mathcal{V}\varphi \quad \text{with} \quad \mathcal{V} \equiv \mathcal{V}_1\mathcal{V}_2 \cdots \mathcal{V}_n, \quad \mathcal{V}_i = e^{i\vec{v}_i \cdot \vec{\mathcal{P}}_i}. \quad (3.108)$$

In this basis, the boundary vector can simply be written as

$$\Phi\left(\prod_{k=1}^n \frac{1+\lambda_k \mathcal{R}_k}{2}\right) \tilde{\varphi} = \Phi_{\tilde{\varphi}_{\lambda_1, \dots, \lambda_n}} = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ \left(\mathcal{P}_1^{\frac{1-\lambda_1}{2}} \dots \mathcal{P}_n^{\frac{1-\lambda_n}{2}} \tilde{\varphi}_{\lambda_1, \dots, \lambda_n}\right) (l_0 - \epsilon) \\ \left(\mathcal{P}_1^{\frac{1-\lambda_1}{2}} \dots \mathcal{P}_n^{\frac{1-\lambda_n}{2}} \tilde{\varphi}_{\lambda_1, \dots, \lambda_n}\right) (l_1 + \epsilon) \\ 0 \\ \vdots \\ 0 \end{pmatrix}. \quad (3.109)$$

The connection conditions (3.107) are reduced to

- for $b_2 = 1$ (and $b_1 = 0$)

$$\begin{aligned} \left(\mathcal{P}_1^{\frac{1-\lambda_1}{2}} \dots \mathcal{P}_n^{\frac{1-\lambda_n}{2}} \tilde{\varphi}_{\lambda_1, \dots, \lambda_n}\right) (l_0 - \epsilon) &= 0 \\ \text{for } \lambda_1 \dots \lambda_n &= -1, \end{aligned} \quad (3.110)$$

$$\begin{aligned} \left(\mathcal{P}_1^{\frac{1-\lambda_1}{2}} \dots \mathcal{P}_n^{\frac{1-\lambda_n}{2}} \tilde{\varphi}_{\lambda_1, \dots, \lambda_n}\right) (l_1 + \epsilon) &= 0 \\ \text{for } \lambda_1 \dots \lambda_n &= +1, \end{aligned} \quad (3.111)$$

- for $b_2 \neq 1$

$$\begin{aligned} (1 - b_2) \left(\mathcal{P}_1^{\frac{1-\lambda_1}{2}} \dots \mathcal{P}_n^{\frac{1-\lambda_n}{2}} \tilde{\varphi}_{\lambda_1, \dots, \lambda_n}\right) (l_0 - \epsilon) - b_1 \left(\mathcal{P}_1^{\frac{1-\lambda_1}{2}} \dots \mathcal{P}_n^{\frac{1-\lambda_n}{2}} \tilde{\varphi}_{\lambda_1, \dots, \lambda_n}\right) (l_1 + \epsilon) &= 0 \\ \text{for } \lambda_1 \dots \lambda_n &= +1, \end{aligned} \quad (3.112)$$

$$\begin{aligned} (1 - b_2) \left(\mathcal{P}_1^{\frac{1-\lambda_1}{2}} \dots \mathcal{P}_n^{\frac{1-\lambda_n}{2}} \tilde{\varphi}_{\lambda_1, \dots, \lambda_n}\right) (l_1 + \epsilon) - b_1 \left(\mathcal{P}_1^{\frac{1-\lambda_1}{2}} \dots \mathcal{P}_n^{\frac{1-\lambda_n}{2}} \tilde{\varphi}_{\lambda_1, \dots, \lambda_n}\right) (l_0 - \epsilon) &= 0 \\ \text{for } \lambda_1 \dots \lambda_n &= -1. \end{aligned} \quad (3.113)$$

The Schrödinger equations can be separated into 2^n regions of $(l_s < x < l_{s+1})$ ($s = 0, 1, \dots, 2^n - 1$).

Now, we are ready to discuss whether the zero energy states satisfy the connection conditions or not. With the transformations (3.108), the zero energy states (3.91) become

$$\tilde{\varphi}_{0;\lambda_1,\dots,\lambda_n}(x) \equiv \mathcal{V}\varphi_{0;\lambda_1,\dots,\lambda_n}(x) = N_{\lambda_1,\dots,\lambda_n} \left[\prod_{k=1}^n \frac{1}{2}(1 + \lambda_k \mathcal{R}_k) \right] e^{-\lambda_1 \dots \lambda_n W(x)}. \quad (3.114)$$

The states (3.114) do not satisfy the connection conditions (3.110), (3.111) for $b_2 = 1$. Although for general $W(x)$, the connection conditions (3.112), (3.113) are not satisfied by the zero energy states for $b_2 \neq 1$, there is, however, an exceptional case, where $W(x)$ satisfies

$$\sqrt{\frac{1-b_1}{1+b_1}} = e^{2W(l_0-\epsilon)}. \quad (3.115)$$

With this special $W(x)$, all the states (3.114) become supersymmetric vacuum states compatible with the connection conditions.

There is no supersymmetric vacuum state for general $W(x)$. Therefore, the supersymmetry is “spontaneously”¹ broken. While, for the special $W(x)$ satisfying (3.115), we have the supersymmetric vacuum states, and the degeneracy is 2^n .

Let us note that the $\mathcal{P}_{n+1}$ becomes physical when $b_2 = 0$ and commutes with the Hamiltonian H^A only for $W''(x) = 0$. We can introduce the simultaneous eigenfunctions of the Hamiltonian H^A , $\mathcal{G}_{\mathcal{R}_k}$ ($k = 1, 2, \dots, n$) and $\mathcal{P}_{n+1}$,

$$H^A \varphi_{E;\lambda_1,\dots,\lambda_n,\lambda_{n+1}}(x) = E \varphi_{E;\lambda_1,\dots,\lambda_n,\lambda_{n+1}}(x), \quad (3.116)$$

¹If there is no zero energy state, we say that supersymmetry is spontaneously broken by analogy with supersymmetric quantum field theory. Other mechanisms of (spontaneous) supersymmetry breaking due to boundary effects have been found in Refs. [64, 65, 66].

$$\mathcal{G}_{\mathcal{R}_k} \varphi_{E; \lambda_1, \dots, \lambda_n, \lambda_{n+1}}(x) = \lambda_k \varphi_{E; \lambda_1, \dots, \lambda_n, \lambda_{n+1}}(x), \quad (3.117)$$

$$\mathcal{P}_{n+1} \varphi_{E; \lambda_1, \dots, \lambda_n, \lambda_{n+1}}(x) = \lambda_{n+1} \varphi_{E; \lambda_1, \dots, \lambda_n, \lambda_{n+1}}(x). \quad (3.118)$$

The new label λ_{n+1} on the wavefunction suggests the enhancement of the degeneracy for the state with $E > 0$. In order to confirm it, let us define new operator $\hat{Q}_a^A$ ($a = 1, 2, \dots, 2n+1$) as

$$\hat{Q}_a^A \equiv \frac{i}{2} \Gamma_a \left(\mathcal{R}_1 \cdots \mathcal{R}_{n+1} \frac{d}{dx} \right). \quad (3.119)$$

Then, the Hamiltonian H^A can be written as

$$H^A = 2(Q_a^A)^2 = 2(\hat{Q}_a^A)^2 + \frac{1}{2}c^2, \quad (3.120)$$

where $c^2 \equiv (W'(x))^2$ is independent of x for $W''(x) = 0$. The energy is bounded from below as

$$E \geq \frac{1}{2}c^2. \quad (3.121)$$

We can show that $\hat{Q}_a^A$ ($a = 1, 2, \dots, 2n+1$) is physical under the connection conditions (3.82) and (3.83). Since $\hat{Q}_a^A$ and $\mathcal{G}_{\mathcal{R}_k}$ ($k = 1, 2, \dots, n$), $\mathcal{P}_{n+1}$ satisfy

$$\mathcal{P}_{n+1} \hat{Q}_a^A = -\hat{Q}_a^A \mathcal{P}_{n+1} \quad \text{for } a = 1, 2, \dots, 2n+1. \quad (3.122)$$

$$\mathcal{G}_{\mathcal{R}_k} \hat{Q}_a^A = \begin{cases} -\hat{Q}_a^A \mathcal{G}_{\mathcal{R}_k} & \text{for } a = 2k-1, 2k, \\ +\hat{Q}_a^A \mathcal{G}_{\mathcal{R}_k} & \text{otherwise} \end{cases} \quad (3.123)$$

and $\hat{Q}_{2k-1}^A = -i\hat{Q}_{2k}^A \mathcal{G}_{\mathcal{R}_k}$, we have the relations

$$\hat{Q}_{2k-1}^A \varphi_{E; \lambda_1, \dots, \lambda_n, \lambda_{n+1}} = -i\lambda_k \hat{Q}_{2k}^A \varphi_{E; \lambda_1, \dots, \lambda_n, \lambda_{n+1}} \propto \varphi_{E; \lambda_1, \dots, -\lambda_k, \dots, \lambda_n, -\lambda_{n+1}}, \quad (3.124)$$

$$\hat{Q}_{2n+1}^A \varphi_{E; \lambda_1, \dots, \lambda_n, \lambda_{n+1}} \propto \varphi_{E; \lambda_1, \dots, \lambda_n, -\lambda_{n+1}}. \quad (3.125)$$

Thus, in this case, the degeneracy for $E > \frac{1}{2}c^2$ is given by 2^{n+1} . This result can also be obtained from an algebraic point of view; $\hat{Q}_a^A/\sqrt{E}$ ($a = 1, 2, \dots, 2n+1$)

and $\mathcal{P}_{n+1}$ form the Clifford algebra in $(2n+2)$ - dimensions and the representation is known as 2^{n+1} .

Let us discuss the case of $E = \frac{1}{2}c^2$. The state with $E = \frac{1}{2}c^2$ satisfies $\hat{Q}_a^A \varphi_{E=\frac{1}{2}c^2}(x) = 0$ ($a = 1, 2, \dots, 2n+1$), which is equivalent to

$$\frac{d}{dx} \tilde{\varphi}_{E=\frac{1}{2}c^2; \lambda_1, \dots, \lambda_n, \lambda_{n+1}}(x) = 0, \quad (3.126)$$

where we have performed the singular transformation (3.108). The solution satisfying (3.126) always have only $\lambda_{n+1} = 1$ state because the state with $\lambda_{n+1} = -1$ have discontinuity except for the original singularities. The formal solutions for the state with $E = \frac{1}{2}c^2$ are given by

$$\tilde{\varphi}_{E=\frac{1}{2}c^2; \lambda_1, \dots, \lambda_n, \lambda_{n+1}=+1} = N_{\lambda_1, \dots, \lambda_n, \lambda_{n+1}=+1} \left[\prod_{k=1}^n \frac{1}{2} (1 + \lambda_k \mathcal{R}_k) \right]. \quad (3.127)$$

Let us note that the solution (3.127) shows that it takes constant values between the two neighboring singularities.

Let us study the compatibility of the solutions (3.127) with the connection conditions (3.110)-(3.113). For $b_1 = +1$, all the states with $E = \frac{1}{2}c^2$ satisfy the connection conditions (3.112) and (3.113), so that the degeneracy is 2^n , while for $b_1 = -1$, all the states do not satisfy the connection conditions (3.112), (3.113) and are removed from the functional space.

We can introduce the “fermion” number operator as

$$(-1)^F \equiv \mathcal{P}_{n+1} \quad (3.128)$$

for $b_2 = 0$ and $W'(x) = 0$ ($c = 0$). We note that $W'(x) = 0$ ensures²

$$\{Q_a^A, (-1)^F\} = 0. \quad (3.129)$$

²Let us note that when $W''(x) = 0$, $\mathcal{P}_{n+1}$ anticommutes with $\hat{Q}_a^A$. If we regard $\hat{Q}_a^A$ as a supercharge, $\mathcal{P}_{n+1}$ can be taken as the “fermion” number operator.

We call states with $(-1)^F = +1(-1)$ “bosonic” (“fermionic”) ones. Under the relation (3.129), $Q_a^A \varphi_{E;\lambda_1,\dots,\lambda_n}(x)$ have the opposite eigenvalues of $(-1)^F$ as $\varphi_{E;\lambda_1,\dots,\lambda_n}(x)$.

We discuss the Witten index in this model. The Witten index of an operator $\mathcal{O}$ with $\mathcal{O}^2 = 1$ is defined by

$$\Delta_{W;\mathcal{O}} \equiv N_+^{E=0} - N_-^{E=0}, \quad (3.130)$$

where $N_{\pm}^{E=0}$ denotes the number of the zero energy states with $\mathcal{O} = \pm 1$, respectively and all the nonzero energy states between $\mathcal{O} = 1$ and $\mathcal{O} = -1$ are degenerate.

The Witten index of $\mathcal{P}_{n+1}$ for $W'(x) = 0$ ($c = 0$)³ and $b_2 = 0$ is given by

$$\Delta_{W;\mathcal{P}_{n+1}} = 2^n. \quad (3.131)$$

When $W''(x) = 0$, the zero energy states do not exist. However, by shifting the energy, $E \rightarrow \hat{E} = E - \frac{1}{2}c^2$, there is a nonzero Witten index for $\hat{E} = 0$ ($E = \frac{1}{2}c^2$),

$$\Delta_{W;\mathcal{P}_{n+1}} = N_+^{\hat{E}=0} - N_-^{\hat{E}=0} = 2^n. \quad (3.132)$$

We should make a comment on the special case (3.115) in which the zero energy states exist. In this case, the Witten index of $\mathcal{G}_{\mathcal{R}_1} \cdots \mathcal{G}_{\mathcal{R}_n}$ vanishes because $N_+^{E=0}$ and $N_-^{E=0}$ are the same,

$$N_{\mathcal{G}_{\mathcal{R}_1} \cdots \mathcal{G}_{\mathcal{R}_n}=+1}^{E=0} = N_{\mathcal{G}_{\mathcal{R}_1} \cdots \mathcal{G}_{\mathcal{R}_n}=-1}^{E=0} = 2^{n-1}. \quad (3.133)$$

(ii) Type B model

³In this case, Q_a^A is the same as $\hat{Q}_a^A$.

The connection conditions compatible with the $N = 2n + 1$ supercharges $Q_1^B, \dots, Q_{2n+1}^B$ are given by

$$(I_{2n+1} - U(b_1, b_2))\Phi_\varphi = 0, \quad (3.134)$$

$$(I_{2n+1} + U(b_1, b_2))\Phi_{\mathcal{D}^B \varphi} = 0, \quad (3.135)$$

where

$$U(b_1, b_2) = b_1(I_2 \otimes \dots \otimes I_2 \otimes \sigma_1) + b_2(\vec{e}_{\mathcal{R}_1} \cdot \vec{\sigma} \otimes \dots \otimes \vec{e}_{\mathcal{R}_n} \cdot \vec{\sigma} \otimes \sigma_3) \quad (3.136)$$

with $b_1, b_2 \in \mathbf{R}$ and $(b_1)^2 + (b_2)^2 = 1$. We first note that the $\mathcal{G}_{\mathcal{R}_k}$ ($k = 1, 2, \dots, n$) commute with H^B and also with each other. Since they are also physical under the connection conditions (3.134), (3.135), we can introduce simultaneous eigenfunctions,

$$H^B \varphi_{E; \lambda_1, \dots, \lambda_n}(x) = E \varphi_{E; \lambda_1, \dots, \lambda_n}(x), \quad (3.137)$$

$$\mathcal{G}_{\mathcal{R}_k} \varphi_{E; \lambda_1, \dots, \lambda_n}(x) = \lambda_k \varphi_{E; \lambda_1, \dots, \lambda_n}(x) \quad (3.138)$$

with $\lambda_k = 1$ or $\lambda_k = -1$ for $k = 1, 2, \dots, n$. The supercharges Q_a^B ($a = 1, 2, \dots, 2n + 1$) and $\mathcal{G}_{\mathcal{R}_k}$ ($k = 1, 2, \dots, n$) satisfy the relations

$$Q_a^B \mathcal{G}_{\mathcal{R}_k} = \begin{cases} -\mathcal{G}_{\mathcal{R}_k} Q_a^B & \text{for } a = 2k - 1, 2k, \\ +\mathcal{G}_{\mathcal{R}_k} Q_a^B & \text{otherwise.} \end{cases} \quad (3.139)$$

Noting that $Q_{2k-1}^B = -iQ_{2k}^B \mathcal{G}_{\mathcal{R}_k}$, we have the relations

$$Q_{2k-1}^B \varphi_{E; \lambda_1, \dots, \lambda_n}(x) = -i\lambda_k Q_{2k}^B \varphi_{E; \lambda_1, \dots, \lambda_n}(x) \propto \varphi_{E; \lambda_1, \dots, -\lambda_k, \dots, \lambda_n}(x), \quad (3.140)$$

$$Q_{2n+1}^B \varphi_{E; \lambda_1, \dots, \lambda_n}(x) \propto \varphi_{E; \lambda_1, \dots, \lambda_n}(x), \quad (3.141)$$

then the degeneracy of the spectrum is given by 2^n for $E > 0$. This result can also be obtained from an algebraic point of view, that is, for fixed nonzero energy E , $Q_a^B/\sqrt{E}$ for $a = 1, 2, \dots, 2n + 1$ form Clifford algebra in $(2n + 1)$ -dimensions, and the representation is known as 2^n .

The above argument cannot apply for $E = 0$ states because $Q_a^B \varphi_0 = 0$. The equations $Q_a^B \varphi_0(x) = 0$ are easily solved and we have formal solutions as

$$\varphi_{0;\lambda_1,\dots,\lambda_n}(x) = N_{\lambda_1,\dots,\lambda_n} \left[\prod_{k=1}^n \frac{1}{2} (1 + \lambda_k \mathcal{G}_{\mathcal{R}_k}) \right] e^{V(x)}, \quad (3.142)$$

where $N_{\lambda_1,\dots,\lambda_n}$ denotes normalization constants. Since the Hamiltonian H^B includes only $V'(x)$ (but not $V(x)$ itself), we have an ambiguity to choose $V(x)$ to satisfy

$$\mathcal{P}_i V(x) \mathcal{P}_i = V(x), \quad i = 1, 2, \dots, n+1. \quad (3.143)$$

Under the singular unitary transformation (3.108), the zero energy states become

$$\tilde{\varphi}_{0;\lambda_1,\dots,\lambda_n}(x) \equiv \mathcal{V} \varphi_{0;\lambda_1,\dots,\lambda_n}(x) = N_{\lambda_1,\dots,\lambda_n} \left[\prod_{k=1}^n \frac{1}{2} (1 + \lambda_k \mathcal{R}_k) \right] e^{V(x)}. \quad (3.144)$$

The zero energy states trivially satisfy the half of the connection conditions (3.135). Let us discuss the remaining connection conditions (3.134), which are given by eqs. (3.110)-(3.113).

The states (3.144) do not satisfy the connection conditions (3.110), (3.111). On the other hand, the connection conditions (3.112), (3.113) are neither satisfied by the zero energy states. There is, however, an exceptional case

$$b_1 = 1. \quad (3.145)$$

This is because $V(l_0 - \epsilon) = V(l_1 + \epsilon)$.

The $\mathcal{P}_{n+1}$ commutes with $\mathcal{G}_{\mathcal{R}_k}$ ($k = 1, 2, \dots, n$), H^B and becomes physical only for $b_2 = 0$. An accidental degeneracy occurs in this case. We can introduce simultaneous eigenfunctions of the Hamiltonian H^B , $\mathcal{G}_{\mathcal{R}_k}$ ($k = 1, 2, \dots, n$) and $\mathcal{P}_{n+1}$ such that

$$H^B \varphi_{E;\lambda_1,\dots,\lambda_n,\lambda_{n+1}}(x) = E \varphi_{E;\lambda_1,\dots,\lambda_n,\lambda_{n+1}}(x), \quad (3.146)$$

$$\mathcal{G}_{\mathcal{R}_k} \varphi_{E;\lambda_1,\dots,\lambda_n,\lambda_{n+1}}(x) = \lambda_k \varphi_{E;\lambda_1,\dots,\lambda_n,\lambda_{n+1}}(x), \quad (3.147)$$

$$\mathcal{P}_{n+1} \varphi_{E;\lambda_1,\dots,\lambda_n,\lambda_{n+1}}(x) = \lambda_{n+1} \varphi_{E;\lambda_1,\dots,\lambda_n,\lambda_{n+1}}(x). \quad (3.148)$$

The Q_a^B and $\mathcal{P}_{n+1}$ satisfy the relations

$$Q_a^B \mathcal{P}_{n+1} = -\mathcal{P}_{n+1} Q_a^B, \quad a = 1, 2, \dots, 2n+1. \quad (3.149)$$

Eqs. (3.139), (3.140) and (3.149) imply that the degeneracy of the spectrum for $E > 0$ is given by 2^{n+1} .

The zero energy formal solutions are

$$\varphi_{0;\lambda_1,\dots,\lambda_n,\lambda_{n+1}=+1}(x) = N_{\lambda_1,\dots,\lambda_n} \left[\prod_{k=1}^n \frac{1}{2} (1 + \lambda_k \mathcal{G}_{\mathcal{R}_k}) \right] e^{V(x)}. \quad (3.150)$$

Here $\varphi_{0;\lambda_1,\dots,\lambda_n,\lambda_{n+1}=-1}(x)$ do not exist because the states cannot construct without new singularities except for the original singular points $x = l_s$ ($s = 0, 1, \dots, 2^n - 1$). For $b_1 = 1$, all the zero energy states (3.150) become super-symmetric vacuum states compatible with the connection conditions (3.112), (3.113), while, for $b_1 = -1$, all the states are incompatible with the connection conditions (3.112), (3.113).

The $\mathcal{P}_{n+1}$ can be regarded as the “fermion” number operator only if $b_2 = 0$,

$$(-1)^F = \mathcal{P}_{n+1}. \quad (3.151)$$

The operator satisfies

$$\{Q_a^B, (-1)^F\} = 0. \quad (3.152)$$

There is a nonzero Witten index of $\mathcal{P}_{n+1}$ only for $b_1 = 1$ and $b_2 = 0$,

$$\Delta_{W;\mathcal{P}_{n+1}} = 2^n. \quad (3.153)$$

3.4 Reduction of the supersymmetry

3.4.1 General discussion

In this section, we provide a general discussion on a reduction of the supersymmetry by relaxing the connection conditions⁴. So far, we have considered the connection conditions compatible with all the $2n + 1$ supercharges. Here we require that only a subset of the supercharges are physical. To put it the other way around, we allow some of the $2n + 1$ supercharges to become incompatible with connection conditions.

Let us consider the m supercharges that are a subset of the $2n + 1$ supercharges (3.24) ((3.25)). The connection conditions compatible with the m supercharges are given by

$$(I_{2n+1} - U) \Phi_\varphi = 0, \quad (3.154)$$

$$(I_{2n+1} + U) \Phi_{\mathcal{D}^{A(B)}\varphi} = 0, \quad (3.155)$$

where the constraints on the characteristic matrix U are

$$U^\dagger U = I_{2n+1}, \quad (3.156)$$

$$U^2 = I_{2n+1}, \quad (3.157)$$

$$\gamma_i U \gamma_i = -U. \quad (3.158)$$

Here the number of γ_i is m , which is a subset of the γ_a ($a = 1, 2, \dots, 2n + 1$). The difference from the case of the $N = 2n + 1$ supersymmetry is in eq. (3.158), that is, the number of γ_i is smaller than that of the case of the $N = 2n + 1$ supersymmetry. The smaller the number of γ_i in eq. (3.158) becomes, the

⁴Falomir and Pisani have also discussed the reduction of supersymmetry in the presence of singular superpotentials[67].

larger the number of parameters of the characteristic matrix U becomes. This implies that the supercharges not belonging to the subset are not the physical operators.

3.4.2 $N = 2n$ supersymmetry

In this section, we consider some $N = 2n$ supersymmetric models as examples of the reduced supersymmetry. We choose $2n$ supercharges that are subset of the $2n + 1$ supercharges (3.24), (3.25) as

- (a) $Q_1^A, Q_2^A, \dots, Q_{2n}^A,$
- (b) $Q_1^A, Q_2^A, \dots, Q_{2n-1}^A, Q_{2n+1}^A,$
- (c) $Q_1^B, Q_2^B, \dots, Q_{2n}^B,$
- (d) $Q_1^B, Q_2^B, \dots, Q_{2n-1}^B, Q_{2n+1}^B.$

We will obtain the connection conditions compatible with the constraints (3.156)-(3.158) and study the degeneracy of the spectrum and the zero energy states in each model.

(a) $Q_1^A, Q_2^A, \dots, Q_{2n}^A$

In order for the supercharges $Q_1^A, \dots, Q_{2n}^A$ to satisfy the superalgebra, we need only eq. (3.31). And the Hamiltonian is also given by eq. (3.37).

The connection conditions compatible with the $2n$ supercharges $Q_1^A, \dots, Q_{2n}^A$ are

$$(I_{2n+1} - U)\Phi_\varphi = 0, \quad (3.159)$$

$$(I_{2n+1} + U)\Phi_{\mathcal{D}^A\varphi} = 0, \quad (3.160)$$

where

$$U^2 = I_{2n+1}, \quad (3.161)$$

$$U^\dagger U = I_{2n+1}, \quad (3.162)$$

$$\gamma_i U \gamma_i = -U, \quad i = 1, 2, \dots, 2n. \quad (3.163)$$

In order to obtain the expression of the characteristic matrix U , we use the expansion (3.71)-(3.73). Eq. (3.163) implies that the nonvanishing coefficients are only four,

$$a_1 \equiv C_{\alpha_1=0, \alpha_2=0, \dots, \alpha_n=0, \alpha_{n+1}=1}, \quad (3.164)$$

$$a_2 \equiv C_{\alpha_1=0, \alpha_2=0, \dots, \alpha_n=0, \alpha_{n+1}=2}, \quad (3.165)$$

$$a_3 \equiv C_{\alpha_1=3, \alpha_2=3, \dots, \alpha_n=3, \alpha_{n+1}=3}, \quad (3.166)$$

$$a_4 \equiv C_{\alpha_1=3, \alpha_2=3, \dots, \alpha_n=3, \alpha_{n+1}=0}. \quad (3.167)$$

Thus, the characteristic matrix U can be expanded as

$$\begin{aligned} U = & a_1 (I_2 \otimes \dots \otimes I_2 \otimes \sigma_1) + a_2 (I_2 \otimes \dots \otimes I_2 \otimes \sigma_2) \\ & + a_3 (\vec{e}_{\mathcal{R}_1} \cdot \vec{\sigma} \otimes \dots \otimes \vec{e}_{\mathcal{R}_n} \cdot \vec{\sigma} \otimes \sigma_3) + a_4 (\vec{e}_{\mathcal{R}_1} \cdot \vec{\sigma} \otimes \dots \otimes \vec{e}_{\mathcal{R}_n} \cdot \vec{\sigma} \otimes I_2). \end{aligned} \quad (3.168)$$

Let us note the number of parameters in the characteristic matrix U is larger than that in the $N = 2n + 1$ supersymmetric model because of the decrease of the conditions on the U . We further restrict the form of U by imposing the conditions (3.161) and (3.162). Then, we obtain that

$$(a_4)^2 = 1 \quad \text{and} \quad a_1 = a_2 = a_3 = 0, \quad (3.169)$$

or

$$(a_1)^2 + (a_2)^2 + (a_3)^2 = 1 \quad \text{and} \quad a_4 = 0 \quad (3.170)$$

with $a_1, a_2, a_3, a_4 \in \mathbf{R}$. Thus, the characteristic matrix U compatible with the $2n$ supersymmetry is given by

(I) Type I

$$U_I(\pm) = \pm (\vec{e}_{\mathcal{R}_1} \cdot \vec{\sigma} \otimes \cdots \otimes \vec{e}_{\mathcal{R}_n} \cdot \vec{\sigma} \otimes I_2), \quad (3.171)$$

(II) Type II

$$\begin{aligned} U_{II}(a) = & a_1 (I_2 \otimes \cdots \otimes I_2 \otimes \sigma_1) + a_2 (I_2 \otimes \cdots \otimes I_2 \otimes \sigma_2) \\ & + a_3 (\vec{e}_{\mathcal{R}_1} \cdot \vec{\sigma} \otimes \cdots \otimes \vec{e}_{\mathcal{R}_n} \cdot \vec{\sigma} \otimes \sigma_3) \end{aligned} \quad (3.172)$$

with $(a_1)^2 + (a_2)^2 + (a_3)^2 = 1$.

Let us next study the degeneracy of the spectrum. We note that $\mathcal{G}_{\mathcal{R}_k}$ ($k = 1, 2, \dots, n$) commutes with the Hamiltonian H^A and is physical under the connection conditions (3.171), (3.172), so that the state with $E(> 0)$ is labeled by the quantum number λ_k , which is the eigenvalue of $\mathcal{G}_{\mathcal{R}_k}$. Noting that $Q_{2k-1}^A = -iQ_{2k}^A \mathcal{G}_{\mathcal{R}_k}$ and the relations

$$Q_i^A \mathcal{G}_{\mathcal{R}_k} = \begin{cases} -\mathcal{G}_{\mathcal{R}_k} Q_i^A & \text{for } i = 2k - 1 \text{ or } 2k, \\ +\mathcal{G}_{\mathcal{R}_k} Q_i^A & \text{otherwise} \end{cases} \quad (3.173)$$

for $k = 1, 2, \dots, n$, we have the relations (3.88) for $k = 1, 2, \dots, n$. Thus, the degeneracy for $E > 0$ is given by 2^n . This result can also be obtained from an algebraic point of view; for fixed nonzero energy E , $Q_i^A/\sqrt{E}$ for $i = 1, 2, \dots, 2n$ form the Clifford algebra in $2n$ -dimensions, and the representation is known as 2^n .

Let us study the zero energy states. The states, which satisfy eq. (3.90), are formally given by

$$\varphi_{0;\lambda_1,\dots,\lambda_n}(x) = N_{\lambda_1,\dots,\lambda_n} \left[\prod_{k=1}^n \frac{1}{2} (1 + \lambda_k \mathcal{G}_{\mathcal{R}_k}) \right] e^{-\lambda_1 \cdots \lambda_n W(x)} \quad (3.174)$$

with $\lambda_k = 1$ or -1 for $k = 1, 2, \dots, n$. The zero energy states trivially satisfy the half of the connection conditions (3.160). We need to verify whether the

zero energy states (3.174) satisfy the remaining connection conditions (3.159) or not. In order to investigate the consistency of the zero energy states with the connection conditions, let us perform the singular transformation (3.108), under which the zero energy states become

$$\tilde{\varphi}_{0;\lambda_1,\dots,\lambda_n}(x) \equiv \mathcal{V}\varphi_{0;\lambda_1,\dots,\lambda_n}(x) = N_{\lambda_1,\dots,\lambda_n} \left[\prod_{k=1}^n \frac{1}{2} (1 + \lambda_k \mathcal{R}_k) \right] e^{-\lambda_1 \cdots \lambda_n W(x)}. \quad (3.175)$$

According to the singular transformation, the connection conditions (3.159) can be written, depending on the type I, II, as

(I) Type I with $U_I(\pm)$

$$\left(\mathcal{P}_1^{\frac{1-\lambda_1}{2}} \cdots \mathcal{P}_n^{\frac{1-\lambda_n}{2}} \tilde{\varphi}_{\lambda_1,\dots,\lambda_n} \right) (l_0 - \epsilon) = 0, \quad (3.176)$$

$$\left(\mathcal{P}_1^{\frac{1-\lambda_1}{2}} \cdots \mathcal{P}_n^{\frac{1-\lambda_n}{2}} \tilde{\varphi}_{\lambda_1,\dots,\lambda_n} \right) (l_1 + \epsilon) = 0 \quad \text{for } \lambda_1 \cdots \lambda_n = \mp 1, \quad (3.177)$$

where the double sign of $\lambda_1 \cdots \lambda_n$ correspond to the double sign in the $U_I(\pm)$

(II) Type II with $U_{II}(a)$

$$\left(\mathcal{P}_1^{\frac{1-\lambda_1}{2}} \cdots \mathcal{P}_n^{\frac{1-\lambda_n}{2}} \tilde{\varphi}_{\lambda_1,\dots,\lambda_n} \right) (l_1 + \epsilon) = 0 \quad \text{for } a_3 \lambda_1 \cdots \lambda_n = +1, \quad (3.178)$$

$$\begin{aligned} (1 - a_3 \lambda_1 \cdots \lambda_n) \left(\mathcal{P}_1^{\frac{1-\lambda_1}{2}} \cdots \mathcal{P}_n^{\frac{1-\lambda_n}{2}} \tilde{\varphi}_{\lambda_1,\dots,\lambda_n} \right) (l_0 - \epsilon) \\ - (a_1 - i a_2) \left(\mathcal{P}_1^{\frac{1-\lambda_1}{2}} \cdots \mathcal{P}_n^{\frac{1-\lambda_n}{2}} \tilde{\varphi}_{\lambda_1,\dots,\lambda_n} \right) (l_1 + \epsilon) = 0 \\ \text{for otherwise.} \end{aligned} \quad (3.179)$$

Let us first discuss the case of $U_I(+)$. We note that the connection condition (3.159) does not yield nontrivial conditions for the state with $\lambda_1 \cdots \lambda_n = +1$, while the other connection condition (3.160) is automatically satisfied due to (3.90). On the other hand, we must check whether the state with $\lambda_1 \cdots \lambda_n = -1$ satisfies the connection conditions (3.176), (3.177) or not, but it is obviously impossible due to the exponential factor in the solutions (3.175). Hence, the degeneracy is given by 2^{n-1} . The discussion goes the same as above for the case of $U_I(-)$, so that the degeneracy is, again, 2^{n-1} . Thus, the supersymmetry is unbroken. We have seen that once we determine the connection condition $U_I(\pm)$, the states with $\lambda_1 \cdots \lambda_n = \pm 1$ can satisfy the connection conditions.

For the type II connection conditions, the zero energy states (3.175) are found to be inconsistent with the connection conditions (3.178), (3.179), so that there are no vacuum states with zero energy. There is, however, an exception. If the following relations are satisfied

$$\sqrt{\frac{1-a_3}{1+a_3}} = e^{W(l_0-\epsilon)-W(l_1+\epsilon)}, \quad a_1 = \sqrt{1-(a_3)^2}, \quad a_2 = 0, \quad (3.180)$$

all the states (3.175) accidentally become supersymmetric vacuum states compatible with the connection conditions (3.179), hence, the degeneracy is 2^n . Therefore, for the type II connection conditions, supersymmetry is spontaneously broken except for the above case.

If $a_2 = 0$, the supercharge Q_{2n+1}^A become physical. Then, $Q_1^A, \dots, Q_{2n+1}^A$ form the $N = 2n + 1$ superalgebra if we require the conditions (3.32). Therefore, the supersymmetry is enhanced to the $N = 2n + 1$ supersymmetry, and the degeneracy has been studied in the previous section.

We can introduce the “fermion” number operator as

$$(-1)^F \equiv \mathcal{G}_{\mathcal{R}_1} \cdots \mathcal{G}_{\mathcal{R}_n}, \quad (3.181)$$

which is physical under the connection conditions (3.159) and (3.160), and commutes (anticommutes) with the Hamiltonian H^A (all the supercharges Q_i^A ($i = 1, 2, \dots, 2n$)). Let us note that eigenvalues of the fermion number operator are $\lambda_1 \cdots \lambda_n = \pm 1$. The Witten index of $\mathcal{G}_{\mathcal{R}_1} \cdots \mathcal{G}_{\mathcal{R}_n}$ with $U_I(\pm)$ is given by

$$\Delta_{W; \mathcal{G}_{\mathcal{R}_1} \cdots \mathcal{G}_{\mathcal{R}_n}} = \pm 2^{n-1}, \quad (3.182)$$

where the double sign correspond to the double sign in the $U_I(\pm)$.

If $W'(x) = 0$, the $\mathcal{P}_{n+1}$ is physical under the connection condition (3.171). And it (anti)commutes with the (supercharges) H^A . Then, we can introduce the Witten index of $\mathcal{P}_{n+1}$,

$$\Delta_{W; \mathcal{P}_{n+1}} = 2^{n-1}. \quad (3.183)$$

For the type II connection conditions with $U_{II}(a)$, the Witten index of $\mathcal{G}_{\mathcal{R}_1} \cdots \mathcal{G}_{\mathcal{R}_n}$ vanishes because there are no zero energy states. We should make a comment on the special case (3.180), in which the zero energy states exist. The Witten index of $\mathcal{G}_{\mathcal{R}_1} \cdots \mathcal{G}_{\mathcal{R}_n}$ is, however, zero because $N_+^{E=0}$ and $N_-^{E=0}$ are the same,

$$N_{\mathcal{G}_{\mathcal{R}_1} \cdots \mathcal{G}_{\mathcal{R}_n} = +1}^{E=0} = N_{\mathcal{G}_{\mathcal{R}_1} \cdots \mathcal{G}_{\mathcal{R}_n} = -1}^{E=0} = 2^{n-1}. \quad (3.184)$$

(b) $Q_1^A, Q_2^A, \dots, Q_{2n-1}^A, Q_{2n+1}^A$

The $N = 2n$ supercharges $Q_1^A, Q_2^A, \dots, Q_{2n-1}^A, Q_{2n+1}^A$ form the $N = 2n$ superalgebra if the function $W'(x)$ obeys (3.31) and (3.32). The constraints on the characteristic matrix U are

$$U^2 = I_{2n+1}, \quad (3.185)$$

$$U^\dagger U = I_{2n+1}, \quad (3.186)$$

$$\gamma_i U \gamma_i = -U, \quad i = 1, 2, \dots, 2n-1, 2n+1. \quad (3.187)$$

We expand the U in the expression (3.71)-(3.73). Because of the conditions (3.187), nonzero coefficients are only four,

$$\begin{aligned} a_1 &\equiv C_{\alpha_1=0, \alpha_2=0, \dots, \alpha_n=0, \alpha_{n+1}=1}, & a_2 &\equiv C_{\alpha_1=3, \alpha_2=3, \dots, \alpha_{n-1}=3, \alpha_n=2, \alpha_{n+1}=3}, \\ a_3 &\equiv C_{\alpha_1=3, \alpha_2=3, \dots, \alpha_n=3, \alpha_{n+1}=3}, & a_4 &\equiv C_{\alpha_1=0, \alpha_2=0, \dots, \alpha_{n-1}=0, \alpha_n=1, \alpha_{n+1}=1}. \end{aligned} \quad (3.188)$$

The conditions (3.185) and (3.186) imply that

$$(a_4)^2 = 1 \quad \text{and} \quad a_1 = a_2 = a_3 = 0, \quad (3.189)$$

or

$$(a_1)^2 + (a_2)^2 + (a_3)^2 = 1 \quad \text{and} \quad a_4 = 0 \quad (3.190)$$

with $a_1, a_2, a_3, a_4 \in \mathbf{R}$. Thus, the characteristic matrix U compatible with the $2n$ supercharges $Q_1^A, \dots, Q_{2n-1}^A, Q_{2n+1}^A$ are

(I) Type I

$$U_I(\pm) = \pm (I_2 \otimes \dots \otimes I_2 \otimes \vec{e}_{\mathcal{P}_n} \cdot \vec{\sigma} \otimes \sigma_1), \quad (3.191)$$

(II) Type II

$$\begin{aligned} U_{II}(a) &= a_1 (I_2 \otimes \dots \otimes I_2 \otimes \sigma_1) \\ &\quad + a_2 (\vec{e}_{\mathcal{R}_1} \cdot \vec{\sigma} \otimes \dots \otimes \vec{e}_{\mathcal{R}_{n-1}} \cdot \vec{\sigma} \otimes \vec{e}_{\mathcal{Q}_n} \cdot \vec{\sigma} \otimes \sigma_3) \\ &\quad + a_3 (\vec{e}_{\mathcal{R}_1} \cdot \vec{\sigma} \otimes \dots \otimes \vec{e}_{\mathcal{R}_n} \cdot \vec{\sigma} \otimes \sigma_3) \end{aligned} \quad (3.192)$$

with $(a_1)^2 + (a_2)^2 + (a_3)^2 = 1$. With the above characteristic matrix U , the connection conditions are given by

$$(I_{2^{n+1}} - U)\Phi_\varphi = 0, \quad (3.193)$$

$$(I_{2^{n+1}} + U)\Phi_{\mathcal{D}^A\varphi} = 0. \quad (3.194)$$

In this case, $\mathcal{G}_{\mathcal{R}_n}$ is no longer physical under the connection conditions (3.191), (3.192), but instead, $\mathcal{G}_{\mathcal{P}_n}\mathcal{P}_{n+1}$ is physical. The Hamiltonian H^A , $\mathcal{G}_{\mathcal{R}_k}$ ($k = 1, 2, \dots, n-1$) and $\mathcal{G}_{\mathcal{P}_n}\mathcal{P}_{n+1}$ are physical, and they commute with each other. We can introduce simultaneous eigenfunctions of these operators such that

$$H^A\varphi_{E;\lambda_1,\dots,\lambda'_n}(x) = E\varphi_{E;\lambda_1,\dots,\lambda'_n}(x), \quad (3.195)$$

$$\mathcal{G}_{\mathcal{R}_k}\varphi_{E;\lambda_1,\dots,\lambda'_n}(x) = \lambda_k\varphi_{E;\lambda_1,\dots,\lambda'_n}(x), \quad k = 1, 2, \dots, n-1, \quad (3.196)$$

$$(\mathcal{G}_{\mathcal{P}_n}\mathcal{P}_{n+1})\varphi_{E;\lambda_1,\dots,\lambda'_n}(x) = \lambda'_n\varphi_{E;\lambda_1,\dots,\lambda'_n}(x) \quad (3.197)$$

with $\lambda_k = 1$ or -1 for $k = 1, 2, \dots, n-1$ and $\lambda'_n = 1$ or -1 . In this case, we have the relations

$$Q_{2k-1}^A\varphi_{E;\lambda_1,\dots,\lambda'_n}(x) = -i\lambda_k Q_{2k}^A\varphi_{E;\lambda_1,\dots,\lambda'_n}(x) \propto \varphi_{E;\lambda_1,\dots,-\lambda_k,\dots,\lambda'_n}(x), \quad k = 1, 2, \dots, n-1, \quad (3.198)$$

$$Q_{2n-1}^A\varphi_{E;\lambda_1,\dots,\lambda'_n}(x) \propto \varphi_{E;\lambda_1,\dots,-\lambda'_n}(x), \quad (3.199)$$

$$Q_{2n+1}^A\varphi_{E;\lambda_1,\dots,\lambda'_n}(x) \propto \varphi_{E;\lambda_1,\dots,-\lambda'_n}(x), \quad (3.200)$$

because

$$\mathcal{G}_{\mathcal{R}_k}Q_i^A = \begin{cases} -Q_i^A\mathcal{G}_{\mathcal{R}_k} & \text{for } k = 2i-1, 2i, \\ +Q_i^A\mathcal{G}_{\mathcal{R}_k} & \text{otherwise,} \end{cases} \quad (3.201)$$

$$(\mathcal{G}_{\mathcal{P}_n}\mathcal{P}_{n+1})Q_i^A = -Q_i^A(\mathcal{G}_{\mathcal{P}_n}\mathcal{P}_{n+1}) \quad (3.202)$$

for $i = 1, 2, \dots, 2n-1, 2n+1$ and $Q_{2k-1}^A = -iQ_{2k}^A\mathcal{G}_{\mathcal{R}_k}$ for $k = 1, 2, \dots, n-1$. Thus, the degeneracy for $E > 0$ is given by 2^n . This result can also be

obtained from an algebraic point of view; for fixed nonzero energy E , $Q_i^A/\sqrt{E}$ for $i = 1, 2, \dots, 2n-1, 2n+1$ form the Clifford algebra in $2n$ -dimensions, and the representation is known as 2^n .

The above argument cannot apply for the zero energy states. The zero energy states are given by solving

$$\mathcal{D}^A \varphi_0(x) = \left[\left(\mathcal{R}_1 \cdots \mathcal{R}_{n+1} \frac{d}{dx} \right) + \lambda_1 \cdots \lambda_{n-1} \mathcal{G}_{\mathcal{R}_n} (\mathcal{R}_1 \cdots \mathcal{R}_{n+1} W'(x)) \right] \varphi_{0;\lambda_1, \dots, \lambda'_n}(x) = 0. \quad (3.203)$$

The solutions to eq. (3.203) are formally given by

$$\begin{aligned} \varphi_{0;\lambda_1, \dots, \lambda'_n}(x) &= N_{\lambda_1, \dots, \lambda'_n} \left(\prod_{k=1}^{n-1} \frac{1 + \lambda_k \mathcal{G}_{\mathcal{R}_k}}{2} \right) \\ &\times \left\{ \left(\frac{1 + \mathcal{G}_{\mathcal{R}_n}}{2} \right) e^{-\lambda_1 \cdots \lambda_{n-1} W(x)} + \lambda'_n \left(\frac{1 - \mathcal{G}_{\mathcal{R}_n}}{2} \right) e^{\lambda_1 \cdots \lambda_{n-1} W(x)} \right\}, \end{aligned} \quad (3.204)$$

and trivially satisfy the half of the connection conditions (3.194). We need to verify whether the zero energy states satisfy the remaining connection conditions (3.193) or not. Under the singular unitary transformation (3.108), the zero energy states become

$$\begin{aligned} \tilde{\varphi}_{0;\lambda_1, \dots, \lambda'_n}(x) &\equiv \mathcal{V} \varphi_{0;\lambda_1, \dots, \lambda'_n}(x) \\ &= N_{\lambda_1, \dots, \lambda'_n} \left(\prod_{k=1}^{n-1} \frac{1 + \lambda_k \mathcal{R}_k}{2} \right) \\ &\times \left\{ \left(\frac{1 + \mathcal{R}_n}{2} \right) e^{-\lambda_1 \cdots \lambda_{n-1} W(x)} + \lambda'_n \left(\frac{1 - \mathcal{R}_n}{2} \right) e^{\lambda_1 \cdots \lambda_{n-1} W(x)} \right\}, \end{aligned} \quad (3.205)$$

and accordingly the connection conditions, (3.193) can be written as

(I) Type I with $U_I(\pm)$

$$\left(\mathcal{P}_1^{\frac{1-\lambda_1}{2}} \cdots \mathcal{P}_{n-1}^{\frac{1-\lambda_{n-1}}{2}} \tilde{\varphi}_{\lambda_1, \dots, \lambda'_n} \right) (l_0 - \epsilon) = 0, \quad (3.206)$$

$$\left(\mathcal{P}_1^{\frac{1-\lambda_1}{2}} \cdots \mathcal{P}_{n-1}^{\frac{1-\lambda_{n-1}}{2}} \tilde{\varphi}_{\lambda_1, \dots, \lambda'_n} \right) (l_1 + \epsilon) = 0 \quad \text{for } \lambda'_n = \mp 1, \quad (3.207)$$

where the double sign of λ'_n correspond to the double sign in the $U_I(\pm)$,

(II) Type II with $U_{II}(a)$

$$\left(\mathcal{P}_1^{\frac{1-\lambda_1}{2}} \cdots \mathcal{P}_{n-1}^{\frac{1-\lambda_{n-1}}{2}} \tilde{\varphi}_{\lambda_1, \dots, \lambda'_n} \right) (l_1 + \epsilon) = 0 \quad \text{for } a_3 \lambda_1 \cdots \lambda_{n-1} = +1, \quad (3.208)$$

$$\begin{aligned} (1 - a_3 \lambda_1 \cdots \lambda_{n-1}) \left(\mathcal{P}_1^{\frac{1-\lambda_1}{2}} \cdots \mathcal{P}_{n-1}^{\frac{1-\lambda_{n-1}}{2}} \tilde{\varphi}_{\lambda_1, \dots, \lambda'_n} \right) (l_0 - \epsilon) \\ - (a_1 - i a_2 \lambda_1 \cdots \lambda'_n) \left(\mathcal{P}_1^{\frac{1-\lambda_1}{2}} \cdots \mathcal{P}_{n-1}^{\frac{1-\lambda_{n-1}}{2}} \tilde{\varphi}_{\lambda_1, \dots, \lambda'_n} \right) (l_1 + \epsilon) = 0 \\ \text{for otherwise.} \end{aligned} \quad (3.209)$$

We first consider the case of $U_I(+)$ ($U_I(-)$). The zero energy states with $\lambda'_n = +1$ ($\lambda'_n = -1$) can satisfy the connection conditions (3.193), (3.194) because the connection conditions (3.193) do not yield nontrivial conditions for the states with $\lambda'_n = +1$ ($\lambda'_n = -1$), and the connection conditions (3.194) are automatically satisfied due to (3.203). On the other hand, the states with $\lambda'_n = -1$ ($\lambda'_n = +1$) do not satisfy the connection conditions (3.206), (3.207) because of the exponential factor in the solution. Thus, the degeneracy of the zero energy states for the case of $U_I(\pm)$ is given by 2^{n-1} , and the supersymmetry is unbroken.

For the case of $U_{II}(a)$, the zero energy states do not satisfy the connection conditions (3.208), (3.209) with one exception. If the following relations are

satisfied

$$\sqrt{\frac{1-a_3}{1+a_3}} = e^{2W(l_0-\epsilon)}, \quad a_1 = \sqrt{1-(a_3)^2}, \quad a_2 = 0, \quad (3.210)$$

all the states (3.205) are compatible with the connection conditions (3.209), then, the degeneracy is 2^n . Hence, for the type II connection conditions, the supersymmetry is spontaneously broken except for the above case.

Note that if $a_2 = 0$, the supercharge Q_{2n}^A becomes physical. Then, the supersymmetry is enhanced to the $N = 2n + 1$ supersymmetry.

In this model, $\mathcal{G}_{\mathcal{P}_n} \mathcal{P}_{n+1}$ can be regarded as the “fermion” number operator,

$$(-1)^F = \mathcal{G}_{\mathcal{P}_n} \mathcal{P}_{n+1}. \quad (3.211)$$

The Witten indices of $\mathcal{G}_{\mathcal{P}_n} \mathcal{P}_{n+1}$, $\mathcal{P}_{n+1}$ for $U_I(\pm)$ are given by

$$\Delta_{W; \mathcal{G}_{\mathcal{P}_n} \mathcal{P}_{n+1}} = \pm 2^{n-1}, \quad (3.212)$$

$$\Delta_{W; \mathcal{P}_{n+1}} = 2^{n-1}, \quad \text{only for } W'(x) = 0, \quad (3.213)$$

where the double sign in eq. (3.212) correspond to the double sign in the $U_I(\pm)$. For $U_{II}(a)$, the Witten index of $\mathcal{G}_{\mathcal{P}_n} \mathcal{P}_{n+1}$ vanishes because the zero energy states do not exist. Let us comment on the special case (3.210), where we have the zero energy states. We note that the supersymmetry is enhanced to the same $N = 2n + 1$ supersymmetry in the special case (3.210) as that in the previous section.

(c) $Q_1^B, \dots, Q_{2n}^B$

The $N = 2n$ supercharges $Q_1^B, \dots, Q_{2n}^B$ form the $N = 2n$ superalgebra if the $V'(x)$ obeys (3.33). In this case, the constraints on the characteristic matrix U are same with the case of (a). The connection conditions are given by

$$(I_{2n+1} - U)\Phi_\varphi = 0, \quad (3.214)$$

$$(I_{2n+1} + U)\Phi_{\mathcal{D}^B \varphi} = 0, \quad (3.215)$$

where U is given by (3.171) or (3.172).

For $U_I(\pm)$, the $\mathcal{G}_{\mathcal{R}_k}$ ($k = 1, 2, \dots, n$) and $\mathcal{P}_{n+1}$ are physical under the connection conditions (3.214), (3.215) and commute with H^B . We can introduce simultaneous eigenfunctions of H^B , $\mathcal{G}_{\mathcal{R}_k}$ ($k = 1, 2, \dots, n$) and $\mathcal{P}_{n+1}$ such that

$$H^B \varphi_{E; \lambda_1, \dots, \lambda_{n+1}}(x) = E \varphi_{E; \lambda_1, \dots, \lambda_{n+1}}(x), \quad (3.216)$$

$$\mathcal{G}_{\mathcal{R}_k} \varphi_{E; \lambda_1, \dots, \lambda_{n+1}}(x) = \lambda_k \varphi_{E; \lambda_1, \dots, \lambda_{n+1}}(x), \quad k = 1, 2, \dots, n, \quad (3.217)$$

$$\mathcal{P}_{n+1} \varphi_{E; \lambda_1, \dots, \lambda_{n+1}}(x) = \lambda_{n+1} \varphi_{E; \lambda_1, \dots, \lambda_{n+1}}(x). \quad (3.218)$$

Although the labels in φ increase, the degeneracy of the spectrum for $E > 0$ states does not increase in this model. The Q_i^B ($i = 1, 2, \dots, 2n$), $\mathcal{G}_{\mathcal{R}_k}$ ($k = 1, 2, \dots, n$) and $\mathcal{P}_{n+1}$ satisfy

$$\mathcal{P}_{n+1} Q_i^B = -Q_i^B \mathcal{P}_{n+1} \quad \text{for } i = 1, 2, \dots, 2n, \quad (3.219)$$

$$\mathcal{G}_{\mathcal{R}_k} Q_i^B = \begin{cases} -Q_i^B \mathcal{G}_{\mathcal{R}_k} & \text{for } i = 2k - 1, 2k, \\ +Q_i^B \mathcal{G}_{\mathcal{R}_k} & \text{otherwise,} \end{cases} \quad (3.220)$$

The supercharge Q_i^B ($i = 1, 2, \dots, n$) changes not only the sign of λ_i but also the sign of λ_{n+1} , so that no supercharge connects $\lambda_1 \cdots \lambda_{n+1} = +1$ states with $\lambda_1 \cdots \lambda_{n+1} = -1$ ones. Therefore, The degeneracy between the states with $\lambda_1 \cdots \lambda_{n+1} = +1$ and the states with $\lambda_1 \cdots \lambda_{n+1} = -1$ is not guaranteed, so that the 2^{n+1} -fold degeneracy is not ensured.

Since $Q_{2k-1}^B = -iQ_{2k}^B \mathcal{G}_{\mathcal{R}_k}$ for $k = 1, 2, \dots, n$ and

$$Q_{2k-1}^B \varphi_{E; \lambda_1, \dots, \lambda_n}(x) = -i\lambda_k Q_{2k}^B \varphi_{E; \lambda_1, \dots, \lambda_n}(x) \propto \varphi_{E; \lambda_1, \dots, -\lambda_k, \dots, -\lambda_n}(x) \quad (3.221)$$

for $k = 1, 2, \dots, n$, the 2^n -fold degeneracy for $E > 0$ states is ensured. This result can also be obtained from an algebraic point of view; for fixed nonzero energy E , $Q_i^B / \sqrt{E}$ for $i = 1, 2, \dots, 2n$ form the Clifford algebra in $2n$ -dimensions, and the representation is known as 2^n . The $\mathcal{P}_{n+1}$ is regarded as a “fermion”

number operator $(-1)^F$ in the type I connection conditions. For $U_{\text{II}}(a)$, since the $\mathcal{P}_{n+1}$, in general, is not physical, the degeneracy for $E > 0$ is 2^n .

The zero energy states satisfying $Q_i^B \varphi_0(x) = 0$ are formally

$$\varphi_{0;\lambda_1,\dots,\lambda_n}(x) = N_{\lambda_1,\dots,\lambda_n} \left(\prod_{k=1}^n \frac{1 + \lambda_k \mathcal{G}_{\mathcal{R}_k}}{2} \right) e^{V(x)}. \quad (3.222)$$

The zero energy states trivially satisfy the half of the connection conditions (3.215). Under the singular unitary transformation (3.108), the remaining connection conditions (3.214) can be written by eqs. (3.176)-(3.179), and the zero energy states become

$$\tilde{\varphi}_{0;\lambda_1,\dots,\lambda_n}(x) \equiv \mathcal{V} \varphi_{0;\lambda_1,\dots,\lambda_n}(x) = N_{\lambda_1,\dots,\lambda_n} \left(\prod_{k=1}^n \frac{1 + \lambda_k \mathcal{R}_k}{2} \right) e^{V(x)}. \quad (3.223)$$

Let us first discuss the case of $U_{\text{I}}(+)$ ($U_{\text{I}}(-)$). We note that the connection conditions (3.214) do not yield nontrivial conditions for the states with $\lambda_1 \cdots \lambda_n = +1$ ($\lambda_1 \cdots \lambda_n = -1$), while the other connection conditions (3.215) are automatically satisfied due to $\mathcal{D}^B \varphi_0(x) = 0$. On the other hand, we must check whether the states with $\lambda_1 \cdots \lambda_n = -1$ ($\lambda_1 \cdots \lambda_n = +1$) satisfy the connection conditions (3.176), (3.177) or not. It is obviously impossible because of the exponential factor in the solution. Hence the degeneracy for the zero energy states for the case of $U_{\text{I}}(\pm)$ is given by 2^{n-1} , and the supersymmetry is unbroken. We have seen that once we determine the connection conditions $U_{\text{I}}(\pm)$, the state with $\lambda_1 \cdots \lambda_n = \pm 1$ can satisfy the connection conditions.

For the type II connection conditions, the zero energy states (3.223) are found to be inconsistent with the connection conditions (3.178), (3.179), so that there are no vacuum states with zero energy. There is, however, an exception. If the following relation is satisfied

$$a_1 = 1, \quad (3.224)$$

all the states (3.223) accidentally become supersymmetric vacuum states compatible with the connection conditions (3.179), then, the degeneracy is 2^n . Hence, for the type II connection conditions, supersymmetry is spontaneously broken except for the above case. Note that if $a_2 = 0$, the supercharge Q_{2n+1}^B becomes physical, therefore the supersymmetry is enhanced to the $N = 2n + 1$ supersymmetry, which is same as the type B model in the previous section.

The fermion number operators can be defined as

$$(-1)^F = \mathcal{G}_{\mathcal{R}_1} \cdots \mathcal{G}_{\mathcal{R}_n} \quad \text{for } U_I(\pm) \text{ or } U_{II}(a), \quad (3.225)$$

$$(-1)^F = \mathcal{P}_{n+1} \quad \text{for } U_I(\pm) \text{ or } U_{II}(a_1 = \pm 1). \quad (3.226)$$

For $U_I(\pm)$, the Witten indices of $\mathcal{G}_{\mathcal{R}_1} \cdots \mathcal{G}_{\mathcal{R}_n}$ and $\mathcal{P}_{n+1}$ are given by

$$\Delta_{W; \mathcal{G}_{\mathcal{R}_1} \cdots \mathcal{G}_{\mathcal{R}_n}} = \pm 2^{n-1}, \quad (3.227)$$

$$\Delta_{W; \mathcal{P}_{n+1}} = 2^{n-1}, \quad (3.228)$$

where the double sign in eq. (3.227) correspond to the double sign in the $U_I(\pm)$. Let us comment on the special case of $U_{II}(a)$ with $a_1 = 1$, where we have the zero energy states. We note that the supersymmetry is enhanced to the same $N = 2n + 1$ supersymmetry even in the special case of $a_1 = 1$ as that in the previous section.

(d) $Q_1^B, \dots, Q_{2n-1}^B, Q_{2n+1}^B$

The $N = 2n$ supercharges $Q_1^B, \dots, Q_{2n-1}^B, Q_{2n+1}^B$ form the $N = 2n$ superalgebra if the $V'(x)$ are assumed to obey (3.33). The characteristic matrix U is again given by (3.191), (3.192). The connection conditions become

$$(I_{2n+1} - U)\Phi_\varphi = 0, \quad (3.229)$$

$$(I_{2n+1} + U)\Phi_{\mathcal{D}^B \varphi} = 0. \quad (3.230)$$

In this model, $\mathcal{G}_{\mathcal{R}_k}$ ($k = 1, 2, \dots, n-1$) and $\mathcal{G}_{\mathcal{P}_n} \mathcal{P}_{n+1}$ are physical and commute with H^B . Since these operators commute with each other, we can introduce

simultaneous eigenfunctions of H^B , $\mathcal{G}_{\mathcal{R}_k}$ ($k = 1, 2, \dots, n-1$) and $\mathcal{G}_{\mathcal{P}_n}\mathcal{P}_{n+1}$ such that

$$H^B \varphi_{E;\lambda_1, \dots, \lambda'_n}(x) = E \varphi_{E;\lambda_1, \dots, \lambda'_n}(x), \quad (3.231)$$

$$\mathcal{G}_{\mathcal{R}_k} \varphi_{E;\lambda_1, \dots, \lambda'_n}(x) = \lambda_k \varphi_{E;\lambda_1, \dots, \lambda'_n}(x), \quad k = 1, 2, \dots, n-1, \quad (3.232)$$

$$(\mathcal{G}_{\mathcal{P}_n}\mathcal{P}_{n+1}) \varphi_{E;\lambda_1, \dots, \lambda'_n}(x) = \lambda'_n \varphi_{E;\lambda_1, \dots, \lambda'_n}(x). \quad (3.233)$$

Since

$$\mathcal{G}_{\mathcal{R}_k} Q_i^B = \begin{cases} -Q_i^B \mathcal{G}_{\mathcal{R}_k} & \text{for } k = 2i-1, 2i, \\ +Q_i^B \mathcal{G}_{\mathcal{R}_k} & \text{otherwise,} \end{cases} \quad (3.234)$$

$$(\mathcal{G}_{\mathcal{P}_n}\mathcal{P}_{n+1}) Q_i^B = -Q_i^B (\mathcal{G}_{\mathcal{P}_n}\mathcal{P}_{n+1}) \quad (3.235)$$

and $Q_{2k-1}^B = -iQ_{2k}^B \mathcal{G}_{\mathcal{R}_k}$ for $k = 1, 2, \dots, n-1$, we have the relations

$$Q_{2k-1}^B \varphi_{E;\lambda_1, \dots, \lambda'_n}(x) = -i\lambda_k Q_{2k}^B \varphi_{E;\lambda_1, \dots, \lambda'_n}(x) \propto \varphi_{E;\lambda_1, \dots, -\lambda_k, \dots, -\lambda'_n}(x), \quad k = 1, 2, \dots, n-1, \quad (3.236)$$

$$Q_{2n-1}^B \varphi_{E;\lambda_1, \dots, \lambda'_n}(x) \propto \varphi_{E;\lambda_1, \dots, -\lambda'_n}(x), \quad (3.237)$$

$$Q_{2n+1}^B \varphi_{E;\lambda_1, \dots, \lambda'_n}(x) \propto \varphi_{E;\lambda_1, \dots, -\lambda'_n}(x). \quad (3.238)$$

Then, the degeneracy of the spectrum for $E > 0$ is given by 2^n . This result can also be obtained from an algebraic point of view; for fixed nonzero energy E , $Q_i^B/\sqrt{E}$ for $i = 1, 2, \dots, 2n-1, 2n+1$ form the Clifford algebra in $2n$ -dimensions, and the representation is known as 2^n .

The zero energy states are given by solving

$$\mathcal{D}^B \varphi_0(x) = \left[\left(\mathcal{R}_1 \cdots \mathcal{R}_{n+1} \frac{d}{dx} \right) - \mathcal{P}_{n+1} (\mathcal{R}_1 \cdots \mathcal{R}_{n+1} V'(x)) \right] \varphi_{0;\lambda_1, \dots, \lambda'_n}(x) = 0. \quad (3.239)$$

The solutions are formally written by

$$\varphi_{0;\lambda_1, \dots, \lambda'_n}(x) = N_{\lambda_1, \dots, \lambda'_n} \left(\prod_{k=1}^{n-1} \frac{1 + \lambda_k \mathcal{G}_{\mathcal{R}_k}}{2} \right) \left\{ \left(\frac{1 + \mathcal{G}_{\mathcal{R}_n}}{2} \right) + \lambda'_n \left(\frac{1 - \mathcal{G}_{\mathcal{R}_n}}{2} \right) \right\} e^{V(x)}. \quad (3.240)$$

The zero energy states trivially satisfy the half of the connection conditions (3.230). Under the singular unitary transformation (3.108), the remaining connection conditions (3.229) become eqs. (3.206)-(3.209), and accordingly, the zero energy states (3.240) become

$$\begin{aligned}\tilde{\varphi}_{0;\lambda_1,\dots,\lambda'_n}(x) &\equiv \mathcal{V}\varphi_{0;\lambda_1\dots\lambda'_n}(x) \\ &= N_{\lambda_1,\dots,\lambda'_n} \left(\prod_{k=1}^{n-1} \frac{1+\lambda_k\mathcal{R}_k}{2} \right) \left\{ \left(\frac{1+\mathcal{R}_n}{2} \right) + \lambda'_n \left(\frac{1-\mathcal{R}_n}{2} \right) \right\} e^{V(x)}.\end{aligned}\tag{3.241}$$

We first consider the case of $U_I(+)$ ($U_I(-)$). The zero energy states with $\lambda'_n = +1$ ($\lambda'_n = -1$) can satisfy the connection conditions (3.229), (3.230) because the connection conditions (3.229) do not yield nontrivial conditions for the states with $\lambda'_n = +1$ ($\lambda'_n = -1$), and the connection conditions (3.230) are automatically satisfied due to $\mathcal{D}^B\varphi_0(x) = 0$. On the other hand, the states with $\lambda'_n = -1$ ($\lambda'_n = +1$) do not satisfy the connection conditions (3.206), (3.207). Hence, the degeneracy of the zero energy states for the case is given by 2^{n-1} , and the supersymmetry is unbroken.

For the case of $U_{II}(a)$, the zero energy states (3.241) do not satisfy the connection conditions (3.208), (3.209) with one exception. If

$$a_1 = 1,\tag{3.242}$$

all the states (3.241) accidentally become supersymmetric vacuum states compatible with the connection conditions (3.209), then, the degeneracy is 2^n . Thus, for the type II connection conditions, the supersymmetry is spontaneously broken except for the above case.

The fermion number operator can be introduced as

$$(-1)^F = \mathcal{G}_{\mathcal{P}_n} \mathcal{P}_{n+1}.\tag{3.243}$$

For $U_I(\pm)$, the Witten indices of $\mathcal{G}_{\mathcal{P}_n}\mathcal{P}_{n+1}$ and $\mathcal{P}_{n+1}$ are given by

$$\Delta_{W;\mathcal{G}_{\mathcal{P}_n}\mathcal{P}_{n+1}} = \pm 2^{n-1}, \quad (3.244)$$

$$\Delta_{W;\mathcal{P}_{n+1}} = 2^{n-1}, \quad (3.245)$$

where the double sign in eq. (3.244) correspond to the double sign in the $U_I(\pm)$. For $U_{II}(a)$, the Witten index of $\mathcal{G}_{\mathcal{P}_n}\mathcal{P}_{n+1}$ vanishes because the zero energy states do not exist. Let us comment on the special case (3.242), where we have the zero energy states. We note that the supersymmetry is enhanced to the same $N = 2n + 1$ supersymmetry in the special case as that in the previous section.

Chapter 4

Supersymmetry and the higher dimensional gauge theories

4.1 Supersymmetric structure in the higher dimensional gauge theories

In the previous chapters, we study supersymmetric quantum mechanics. In this section, we show that an $N = 2$ supersymmetric structure is hidden in 4d mass spectrum of any gauge invariant theories with compact extra dimensions without a boundary. We should notice, to avoid confusion, that the $N = 2$ supersymmetric structure can be seen for mass eigenfunctions appearing in the mode expansion of gauge fields, rather than a symmetry of the action. We can introduce $N = 2$ supercharges, that make a mass eigenfunction associated with a 4d vector boson transform into a mass eigenfunction associated with a 4d would-be Nambu-Goldstone boson, and vice versa, which can be absorbed into the longitudinal mode of the 4d vector boson.

To this end, we consider a (4+D)-dimensional abelian gauge theory with a weight function. The (4+D)-dimensional metric is assumed to be of the form

$$d\tilde{s}^2 = e^{-\frac{4}{D}W(y)} \left(\eta_{\mu\nu} dx^\mu dx^\nu + g_{ij}(y) dy^i dy^j \right). \quad (4.1)$$

The (4+D)-dimensional coordinates are denoted by $x^M = (x^\mu, y^i)$, where x^μ ($\mu = 0, 1, 2, 3$) are the 4-dimensional coordinates and y^i ($i = 1, 2, \dots, D$) are the extra D-dimensional coordinates. The $\eta_{\mu\nu}$ is a 4d Minkowski metric with $\eta_{\mu\nu} = \text{diag}(-1, 1, 1, 1)$, and $W(y)$ and $g_{ij}(y)$ are assumed to depend only on the coordinates y^i . We note that for $D = 1$ and $W(y) = \frac{1}{2}k|y|$ with $g_{55}(y) = e^{4W(y)}$, the metric reduces to the warped metric discussed by Randall and Sundrum[5]. The action we consider is

$$\begin{aligned} S &= \int d^4x \mathcal{L}_K \\ &= \int d^4x \int d^Dy \sqrt{-G} \tilde{\Delta}(y) \left\{ -\frac{1}{4} G^{MM'} G^{NN'} F_{MN} F_{M'N'} \right\}, \end{aligned} \quad (4.2)$$

where $\tilde{\Delta}(y)$ is a weight function depending on y^i and

$$F_{MN}(x, y) = \partial_M A_N(x, y) - \partial_N A_M(x, y), \quad (4.3)$$

$$G_{MN}(y) = \begin{pmatrix} e^{-\frac{4}{D}W(y)} \eta_{\mu\nu} & 0 \\ 0 & e^{-\frac{4}{D}W(y)} g_{ij}(y) \end{pmatrix}, \quad (4.4)$$

$$G(y) = \det G_{MN}(y). \quad (4.5)$$

For our purpose, it is convenient to rewrite $\mathcal{L}_K$ into the form

$$\mathcal{L}_K = \int d^Dy \sqrt{g} \Delta(y) \left\{ -\frac{1}{4} g^{MM'} g^{NN'} F_{MN} F_{M'N'} \right\}, \quad (4.6)$$

where

$$g_{MN}(y) = \begin{pmatrix} \eta_{\mu\nu} & 0 \\ 0 & g_{ij}(y) \end{pmatrix}, \quad (4.7)$$

$$g(y) = \det g_{ij}(y), \quad (4.8)$$

$$\Delta(y) = \tilde{\Delta}(y) e^{-2W(y)}. \quad (4.9)$$

Thus, the system becomes identical to that of the gauge theory with the metric

$$ds^2 = \eta_{\mu\nu} dx^\mu dx^\nu + g_{ij}(y) dy^i dy^j \quad (4.10)$$

and the weight function $\Delta(y)$.

In a viewpoint of the extra D-dimensions, the 4-dimensional gauge fields $A_\mu(x, y)$ are regarded as 0 forms, while the extra D-dimensional components $A_i(x, y)$ are regarded as a 1 form, so that we may write

$$A^{(1)}(x, y) \equiv A_i(x, y)dy^i. \quad (4.11)$$

Here, the exterior derivative d is defined by $d = dy^i \partial_i$ with respect to the coordinates on the extra dimensions. As was done in the previous section, we introduce the inner product for k forms as

$$(\eta^{(k)}, \omega^{(k)})_\Delta = \frac{1}{k!} \int d^D y \sqrt{g(y)} \Delta(y) \eta_{i_1 \dots i_k}(y) \omega_{j_1 \dots j_k}(y) g^{i_1 j_1}(y) \dots g^{i_k j_k}(y). \quad (4.12)$$

It turns out that $\mathcal{L}_K$ can be written into the form

$$\begin{aligned} \mathcal{L}_K = & -\frac{1}{4} (\partial_\mu A_\nu - \partial_\nu A_\mu, \partial^\mu A^\nu - \partial^\nu A^\mu)_\Delta \\ & -\frac{1}{2} (dA_\mu - \partial_\mu A^{(1)}, dA^\mu - \partial^\mu A^{(1)})_\Delta \\ & -\frac{1}{2} (dA^{(1)}, dA^{(1)})_\Delta. \end{aligned} \quad (4.13)$$

As proved in Appendix, 0 forms $A_\mu(x, y)$ can be expanded as¹

$$\begin{aligned} A_\mu(x, y) &= A_{\mu,0}(x) \eta^{(0)} + \sum'_{n_0} A_{\mu,n_0}(x) \omega_{n_0}^{(0)}(y) \\ &= A_{\mu,0}(x) \eta^{(0)} + \sum'_{n_0} \frac{A_{\mu,n_0}(x)}{m_{n_0}^{(0)}} d^\dagger \phi_{n_0}^{(1)}(y), \end{aligned} \quad (4.14)$$

where

$$\eta^{(0)} = \left(\int d^D y \sqrt{g} \Delta \right)^{-1/2}, \quad (4.15)$$

$$d^\dagger d \omega_{n_0}^{(0)} = (m_{n_0}^{(0)})^2 \omega_{n_0}^{(0)}, \quad (4.16)$$

$$dd^\dagger \phi_{n_0}^{(1)} = (m_{n_0}^{(0)})^2 \phi_{n_0}^{(1)}, \quad \text{for } m_{n_0}^{(0)} \neq 0, \quad (4.17)$$

¹We have used the fact that the 0th Betti number b_0 is equal to 1.

and

$$\omega_{n_0}^{(0)} = \frac{1}{m_{n_0}^{(0)}} d^\dagger \phi_{n_0}^{(1)} \quad \text{or} \quad \phi_{n_0}^{(1)} = \frac{1}{m_{n_0}^{(0)}} d\omega_{n_0}^{(0)}. \quad (4.18)$$

Since $A^{(1)}(x, y) = A_i(x, y) dy^i$ is a 1 form, it can be expanded as

$$\begin{aligned} A^{(1)}(x, y) &= \sum_{p=1}^{b_1} \varphi_p(x) \eta_p^{(1)}(y) + \sum_{n_1}' \Phi_{n_1}(x) \omega_{n_1}^{(1)}(y) + \sum_{n_0}' h_{n_0}(x) \phi_{n_0}^{(1)}(y) \\ &= \sum_{p=1}^{b_1} \varphi_p(x) \eta_p^{(1)}(y) + \sum_{n_1}' \frac{\Phi_{n_1}(x)}{m_{n_1}^{(1)}} d^\dagger \phi_{n_1}^{(2)}(y) + \sum_{n_0}' \frac{h_{n_0}(x)}{m_{n_0}^{(0)}} d\omega_{n_0}^{(0)}(y), \end{aligned} \quad (4.19)$$

where b_1 is the 1st Betti number and

$$d\eta_p^{(1)} = 0 = d^\dagger \eta_p^{(1)}, \quad p = 1, 2, \dots, b_1, \quad (4.20)$$

$$d^\dagger d\omega_{n_1}^{(1)} = (m_{n_1}^{(1)})^2 \omega_{n_1}^{(1)}, \quad (4.21)$$

$$dd^\dagger \phi_{n_1}^{(2)} = (m_{n_1}^{(1)})^2 \phi_{n_1}^{(2)}, \quad \text{for } m_{n_1}^{(1)} \neq 0, \quad (4.22)$$

and

$$\omega_{n_1}^{(1)} = \frac{1}{m_{n_1}^{(1)}} d^\dagger \phi_{n_1}^{(2)} \quad \text{for} \quad \phi_{n_1}^{(2)} = \frac{1}{m_{n_1}^{(1)}} d\omega_{n_1}^{(1)}. \quad (4.23)$$

It is now clear that the $N = 2$ supersymmetric structure discussed in the previous section can be seen in the mode expansion of the gauge fields. The mass eigenfunction $\omega_{n_0}^{(0)}$ and $\phi_{n_0}^{(1)}$ associated with the 4d vector $A_{\mu, n_0}(x)$ and the (would-be Nambu-Goldstone) boson $h_{n_0}(x)$, respectively, obey the $N = 2$ supersymmetric quantum mechanics. They transform into each other by the action of the $N = 2$ supercharges and hence form a supermultiplet, i.e.

$$H \begin{pmatrix} \omega_{n_0}^{(0)} \\ \phi_{n_0}^{(1)} \end{pmatrix} = (m_{n_0}^{(0)})^2 \begin{pmatrix} \omega_{n_0}^{(0)} \\ \phi_{n_0}^{(1)} \end{pmatrix}, \quad (4.24)$$

$$Q_1 \begin{pmatrix} \omega_{n_0}^{(0)} \\ 0 \end{pmatrix} = m_{n_0}^{(0)} \begin{pmatrix} 0 \\ \phi_{n_0}^{(1)} \end{pmatrix}, \quad Q_1 \begin{pmatrix} 0 \\ \phi_{n_0}^{(1)} \end{pmatrix} = m_{n_0}^{(0)} \begin{pmatrix} \omega_{n_0}^{(0)} \\ 0 \end{pmatrix}, \quad (4.25)$$

$$Q_2 \begin{pmatrix} \omega_{n_0}^{(0)} \\ 0 \end{pmatrix} = im_{n_0}^{(0)} \begin{pmatrix} 0 \\ \phi_{n_0}^{(1)} \end{pmatrix}, \quad Q_2 \begin{pmatrix} 0 \\ \phi_{n_0}^{(1)} \end{pmatrix} = -im_{n_0}^{(0)} \begin{pmatrix} \omega_{n_0}^{(0)} \\ 0 \end{pmatrix}, \quad (4.26)$$

where

$$H = (Q_1)^2 = (Q_2)^2 = \begin{pmatrix} d^\dagger d & 0 \\ 0 & dd^\dagger \end{pmatrix}, \quad (4.27)$$

$$Q_1 = \begin{pmatrix} 0 & d^\dagger \\ d & 0 \end{pmatrix}, \quad Q_2 = \begin{pmatrix} 0 & -id^\dagger \\ id & 0 \end{pmatrix}. \quad (4.28)$$

We note that the mass eigenfunction $\omega_{n_0}^{(0)}$ correspond to 4d vector boson $A_{\mu,n_0}(x)$ and hence the mass spectrum of the 4d vector bosons is governed by the above $N = 2$ supersymmetric quantum mechanics.

Inserting the mode expansions (4.14) and (4.19) into the Lagrangian $\mathcal{L}_K$ and using the orthogonal relations of the eigenfunctions, we have

$$\begin{aligned} \mathcal{L}_K = & -\frac{1}{4} (F_{\mu\nu,0}(x))^2 \\ & + \sum'_{n_0} \left[-\frac{1}{4} (F_{\mu\nu,n_0}(x))^2 - \frac{(m_{n_0}^{(0)})^2}{2} \left(A_{\mu,n_0}(x) - \frac{1}{m_{n_0}^{(0)}} \partial_\mu h_{n_0}(x) \right)^2 \right] \\ & - \frac{1}{2} \sum_{p=1}^{b_1} (\partial_\mu \varphi_p(x))^2 \\ & + \sum'_{n_1} \left[-\frac{1}{2} (\partial_\mu \Phi_{n_1}(x))^2 - \frac{(m_{n_1}^{(1)})^2}{2} (\Phi_{n_1}(x))^2 \right], \end{aligned} \quad (4.29)$$

where

$$F_{\mu\nu,n}(x) = \partial_\mu A_{\nu,n}(x) - \partial_\nu A_{\mu,n}(x). \quad (4.30)$$

Therefore, we conclude that in a 4-dimensional point of view, the field contents of the model are given as follows: $A_{\mu,0}$ is a massless gauge field. A_{μ,n_0} are massive vector bosons with mass $m_{n_0}^{(0)}$. h_{n_0} are would-be Nambu-Goldstone bosons and can be absorbed into the longitudinal modes of A_{μ,n_0} . φ_p ($p = 1, 2, \dots, b_1$) are massless scalars and cannot be gauged-away. They could play a role of Higgs fields for non-abelian gauge theories[68]. Φ_{n_1} are massive scalars with mass $m_{n_1}^{(1)}$. The origin of the scalar fields φ_p and Φ_{n_1} are the extra dimensional components of the gauge fields.

We finally make comments on some previous works related to this section. Howe et al.[69] discussed an $N = 2$ worldline supersymmetry for a relativistic spin $N/2$ particle and succeeded to present field equations for massless and massive antisymmetric tensors in arbitrary space-time dimensions. The $N = 2$ worldline supersymmetry seems to have some connections to the $N = 2$ supersymmetry found in this paper, but a direct relation between them is not clear. Shaposhnikov and Tinyakov[1] considered a gauge invariant theory with a noncompact extra dimension with a weight function depending on the extra dimensional coordinate and observed that mass eigenfunctions for $A_\mu(x, y)$ satisfy a similar equation to eq. (4.16). They, however, missed the full supersymmetric structure, since they worked on the gauge $A_5(x, y) = 0$. Although we found an $N = 2$ quantum-mechanical supersymmetry in the spectrum of gauge fields, a similar $N = 2$ supersymmetric structure has already been found in the spectrum of spinor fields. Jackiw and Rebbi[70] showed that a dynamical localization of a chiral fermion can occur in a nontrivial soliton background and that mode functions for left-handed and right-handed spinors are governed by an $N = 2$ supersymmetric quantum mechanics, though they did not use the word “supersymmetry”. Arkani-Hamed and Schmaltz[71] considered a five dimensional fermion coupled to a background scalar that has the dependence on the extra dimensional coordinate and explicitly constructed a Hamiltonian and $N = 2$ supercharges that form an $N = 2$ supersymmetry algebra. The connection between the de Rham cohomology and the quantum mechanical superalgebra was argued by R.P. Malik in the context of noncommutative geometry[72].

4.2 Supersymmetry on the action

In the previous section, we showed that a quantum-mechanical $N = 2$ supersymmetry is hidden in 4d mass spectrum of any gauge invariant theories with extra dimensions, but whether the whole action possesses the quantum-mechanical supersymmetry is not addressed. In this section, we show that the higher dimensional gauge theory possesses the quantum-mechanical supersymmetry and the supersymmetry, namely the property that the action is invariant under the transformations whose generators are the $N = 2$ supercharges, is a part of the higher dimensional gauge symmetry.

Let us consider a 4 + 1-dimensional Abelian gauge theory on a curved space-time. The metric is assumed to be of the form

$$ds^2 = \Delta(y)^2 (\eta_{\mu\nu} dx^\mu dx^\nu + dy^2). \quad (4.31)$$

The action we consider is

$$S = \int d^4x \int dy \sqrt{-g} \left(-\frac{1}{4} g^{MM'} g^{NN'} F_{M'N'} F_{MN} \right) = \int d^4x \int dy \mathcal{L}, \quad (4.32)$$

where

$$g_{MN} = \Delta(y)^2 \begin{pmatrix} \eta_{\mu\nu} & 0 \\ 0 & 1 \end{pmatrix} \equiv \Delta(y)^2 \eta_{MN}, \quad (4.33)$$

$$F_{MN}(x, y) = \partial_M A_N(x, y) - \partial_N A_M(x, y). \quad (4.34)$$

We impose a gauge fixing condition

$$\partial^\mu A_\mu + \frac{1}{\xi \Delta(y)} \partial^y \Delta(y) A_y = 0, \quad (4.35)$$

where ξ denotes an arbitrary constant. The indices are raised and lowered with η_{MN} . We decompose the 4d component of gauge field into the longitudinal mode and transverse one,

$$A_\mu = A_\mu^L + A_\mu^T \quad (4.36)$$

with $\partial^\mu A_\mu^T = 0$, so that the lagrangian can be written as

$$\begin{aligned} \mathcal{L} = & \frac{1}{2\xi} \Delta(A^y, \rho) \begin{pmatrix} \partial_\mu \partial^\mu & 0 \\ 0 & \partial_\mu \partial^\mu \end{pmatrix} \begin{pmatrix} A_y \\ \rho \end{pmatrix} - \frac{1}{2} \Delta(A^y, \rho) H \begin{pmatrix} A_y \\ \rho \end{pmatrix} \\ & + \frac{1}{2} A^{\mu T} (\partial_\nu \partial^\nu + \Delta^{-1} \partial_y \Delta \partial_y) A_\mu^T. \end{aligned} \quad (4.37)$$

where

$$H = \begin{pmatrix} -\partial_y \Delta^{-1} \partial_y \Delta & 0 \\ 0 & -\Delta^{-1} \partial_y \Delta \partial_y \end{pmatrix}, \quad (4.38)$$

$$A_\mu^L \equiv \frac{1}{\sqrt{\partial_\nu \partial^\nu}} \partial_\mu \rho. \quad (4.39)$$

Here, we assumed that the surface terms vanish. Note that the lagrangian (4.37) can also be obtained by adding a gauge fixing term to the original lagrangian

$$\mathcal{L}_{GF} = -\frac{1}{2} \Delta(y) \xi \left(\partial^\mu A_\mu + \frac{1}{\xi \Delta} \partial_y \Delta A^y \right)^2, \quad (4.40)$$

instead of the gauge fixing condition (4.35).

The lagrangian (4.37) is invariant under the supersymmetry transformation

$$\begin{pmatrix} A_y \\ \rho \end{pmatrix} \rightarrow \begin{pmatrix} A'_y \\ \rho' \end{pmatrix} = e^{i\theta Q} \begin{pmatrix} A_y \\ \rho \end{pmatrix}, \quad (4.41)$$

$$Q = \begin{pmatrix} 0 & i\partial_y \\ i\Delta^{-1} \partial_y \Delta & 0 \end{pmatrix} \quad (4.42)$$

with $Q^2 = H$, because of $[Q, H] = 0$. Here, θ is a bosonic parameter and the gauge fixing condition is also invariant under the supersymmetry transformation using the equation of motion.

The infinitesimal supersymmetry transformations are given by

$$\delta A_y = -\theta \sqrt{\xi} \partial_y \rho, \quad (4.43)$$

$$\delta \rho = -\frac{1}{\sqrt{\xi}} \theta \frac{1}{\Delta} \partial_y \Delta A_y. \quad (4.44)$$

These transformations can be written as

$$\delta A_M = -\theta \sqrt{\xi} \partial_M \rho \tag{4.45}$$

by using the gauge fixing condition (4.35). The transformation can be regarded as a part of the 5d gauge transformation. Thus, we can define the supersymmetry transformation of matter fields in any gauge invariant theories with matter fields.

Chapter 5

Extra dimensions with boundaries

5.1 Boundary conditions and supersymmetry

In this section, we extend the previous analysis to extra dimensions with boundaries. In this case, we have to impose boundary conditions at the boundaries. The criteria of obtaining a possible set of boundary conditions are, however, less obvious. For instance, the Dirichlet boundary conditions are used in Higgsless gauge symmetry breaking scenario[2, 3],[27]-[38], but the boundary conditions break 4d gauge symmetries explicitly. Thus, it is not clear whether such boundary conditions lead to consistent gauge theories. Recently, a criterion to select a possible set of boundary conditions has been proposed in ref. [27]. The authors require the boundary conditions to obey the least action principle. Since the requirement does not, however, rely on gauge invariance directly, it is still unclear that such boundary conditions lead

to consistent gauge theories. Since gauge symmetry breaking can occur via boundary conditions, it is important to clarify a class of boundary conditions compatible with higher-dimensional gauge invariance.

In the following, we discuss how to obtain a possible set of boundary conditions compatible with gauge invariance from a supersymmetry point of view. To this end, let us consider a 4+1-dimensional abelian gauge theory on an interval

$$S = \int d^4x \int_0^L dy \sqrt{-g(y)} \left\{ -\frac{1}{4} F_{MN}(x, y) F^{MN}(x, y) \right\} \quad (5.1)$$

with a non-factorizable metric

$$ds^2 = e^{-4W(y)} \eta_{\mu\nu} dx^\mu dx^\nu + g_{55}(y) dy^2. \quad (5.2)$$

The metric reduces to the warped metric discussed by Randall and Sundrum[5] when $g_{55}(y) = 1$ and $W(y) = \frac{1}{2}k|y|$. Another choice of $g_{55}(y) = e^{-4W(y)}$ leads to the model discussed in ref. [2, 3], in which a hierarchical mass spectrum has been observed.

In order to expand the 5d gauge fields $A_\mu(x, y)$ and $A_5(x, y)$ into mass eigenstates, we follow the discussions in Section 3 and consider the supersymmetric Hamiltonian,¹

$$H = Q^2 = \begin{pmatrix} -\frac{1}{\sqrt{g_{55}}} \partial_y \frac{e^{-4W}}{\sqrt{g_{55}}} \partial_y & 0 \\ 0 & -\partial_y \frac{1}{\sqrt{g_{55}}} \partial_y \frac{e^{-4W}}{\sqrt{g_{55}}} \end{pmatrix}, \quad (5.3)$$

$$Q = \begin{pmatrix} 0 & -\frac{1}{\sqrt{g_{55}}} \partial_y \frac{e^{-4W}}{\sqrt{g_{55}}} \\ \partial_y & 0 \end{pmatrix} \quad (5.4)$$

which act on two-component vectors

$$|\Psi\rangle = \begin{pmatrix} f(y) \\ g(y) \end{pmatrix}. \quad (5.5)$$

¹Here, we have represented H and Q in terms of the differential operator ∂_y , instead of d and $d^\dagger$, so that they act on functions rather than forms.

The inner product of two states $|\Psi_1\rangle$ and $|\Psi_2\rangle$ is defined by

$$\langle\Psi_2|\Psi_1\rangle = \int_0^L dy \sqrt{g_{55}(y)} \left\{ f_2(y)f_1(y) + \frac{e^{-4W(y)}}{g_{55}(y)} g_2(y)g_1(y) \right\}. \quad (5.6)$$

To obtain consistent boundary conditions for the functions $f(y)$ and $g(y)$ in $|\Psi\rangle$, we first require that the supercharge Q is hermitian with respect to the inner product (5.6), i.e.

$$\langle\Psi_2|Q\Psi_1\rangle = \langle Q\Psi_2|\Psi_1\rangle. \quad (5.7)$$

It turns out that the functions $f(y)$ and $g(y)$ have to obey one of the following four types of boundary conditions²:

$$\text{i) } g(0) = g(L) = 0, \quad (5.8)$$

$$\text{ii) } f(0) = f(L) = 0, \quad (5.9)$$

$$\text{iii) } g(0) = f(L) = 0, \quad (5.10)$$

$$\text{iv) } f(0) = g(L) = 0. \quad (5.11)$$

We further require that the state $Q|\Psi\rangle$ obeys the same boundary conditions as $|\Psi\rangle$, otherwise Q is not a well defined operator and “bosonic” and “fermionic” states would not form supermultiplets. The requirement leads to

$$\partial_y f(0) = \partial_y f(L) = 0 \quad \text{for i),} \quad (5.12)$$

$$\partial_y \left(\frac{e^{-4W}}{\sqrt{g_{55}}} g \right) (0) = \partial_y \left(\frac{e^{-4W}}{\sqrt{g_{55}}} g \right) (L) = 0 \quad \text{for ii),} \quad (5.13)$$

$$\partial_y f(0) = \partial_y \left(\frac{e^{-4W}}{\sqrt{g_{55}}} g \right) (L) = 0 \quad \text{for iii),} \quad (5.14)$$

$$\partial_y \left(\frac{e^{-4W}}{\sqrt{g_{55}}} g \right) (0) = \partial_y f(L) = 0 \quad \text{for iv).} \quad (5.15)$$

²If we allow $f(L)$ ($g(L)$) to be connected with $f(0)$ ($g(0)$), we have a one parameter family of the boundary conditions[59, 61, 62]:

$$\sin \theta f(0) - \cos \theta f(L) = \cos \theta \left(\frac{e^{-4W}}{\sqrt{g_{55}}} g \right) (0) - \sin \theta \left(\frac{e^{-4W}}{\sqrt{g_{55}}} g \right) (L) = 0.$$

Combining the conditions (5.8)-(5.11) together with (5.12)-(5.15), we have found the four types of boundary conditions compatible with supersymmetry,

$$\text{Type (N, N)} : \begin{cases} \partial_y f(0) = \partial_y f(L) = 0, \\ g(0) = g(L) = 0, \end{cases} \quad (5.16)$$

$$\text{Type (D, D)} : \begin{cases} f(0) = f(L) = 0, \\ \partial_y \left(\frac{e^{-4W}}{\sqrt{g_{55}}} g \right) (0) = \partial_y \left(\frac{e^{-4W}}{\sqrt{g_{55}}} g \right) (L) = 0, \end{cases} \quad (5.17)$$

$$\text{Type (N, D)} : \begin{cases} \partial_y f(0) = f(L) = 0, \\ g(0) = \partial_y \left(\frac{e^{-4W}}{\sqrt{g_{55}}} g \right) (L) = 0, \end{cases} \quad (5.18)$$

$$\text{Type (D, N)} : \begin{cases} f(0) = \partial_y f(L) = 0, \\ \partial_y \left(\frac{e^{-4W}}{\sqrt{g_{55}}} g \right) (0) = g(L) = 0. \end{cases} \quad (5.19)$$

It follows that the above boundary conditions ensure the hermiticity of the Hamiltonian, i.e.

$$\langle \Psi_2 | H \Psi_1 \rangle = \langle H \Psi_2 | \Psi_1 \rangle. \quad (5.20)$$

Therefore, we have succeeded to obtain the consistent set of boundary conditions that ensure the hermiticity of the supercharges and the Hamiltonian and also that the action of the supercharge on $|\Psi\rangle$ is well defined. Since the supersymmetry is a direct consequence of higher-dimensional gauge invariance, our requirements on boundary conditions should be, at least, necessary conditions to preserve it. It turns out that the boundary conditions obtained above are consistent with those in ref. [27], although it is less obvious how the requirement of the least action principle proposed in ref. [27] is connected to gauge invariance. We should emphasize that the supercharge Q is well defined for all the boundary conditions (5.16)-(5.19) and hence that the supersymmetric structure always appears in the spectrum, even though the boundary conditions other than the type (N,N) break 4d gauge symmetries, as we will see below.

From the above analysis, the 5d gauge fields $A_\mu(x, y)$ and $A_5(x, y)$ are

expanded in the mass eigenstates as follows:

$$A_\mu(x, y) = \sum_n A_{\mu,n}(x) f_n(y), \quad (5.21)$$

$$A_5(x, y) = \sum_n h_n(x) g_n(y) \quad (5.22)$$

where $f_n(y)$ and $g_n(y)$ are the eigenstates of the Schrödinger-like equations

$$-\frac{1}{\sqrt{g_{55}}} \partial_y \frac{e^{-4W}}{\sqrt{g_{55}}} \partial_y f_n(y) = m_n^2 f_n(y), \quad (5.23)$$

$$-\partial_y \frac{1}{\sqrt{g_{55}}} \partial_y \frac{e^{-4W}}{\sqrt{g_{55}}} g_n(y) = m_n^2 g_n(y) \quad (5.24)$$

with one of the four types of the boundary conditions (5.16)-(5.19). Since the massless states are especially important in phenomenology, let us investigate the massless states of the equations (5.23) and (5.24). Thanks to supersymmetry, the massless modes would be the solutions to the first order differential equation $Q|\Psi_0\rangle = 0$, i.e.

$$\partial_y f_0(y) = 0, \quad (5.25)$$

$$\partial_y \left(\frac{e^{-4W}}{\sqrt{g_{55}}} g_0(y) \right) = 0. \quad (5.26)$$

The solutions are easily found to be

$$f_0(y) = c, \quad (5.27)$$

$$g_0(y) = c' e^{4W(y)} \sqrt{g_{55}(y)}, \quad (5.28)$$

where c and c' are some constants. We should emphasize that the above solutions do not necessarily imply physical massless states of $A_{\mu,0}(x)$ and $h_0(x)$ in the spectrum. This is because the boundary conditions exclude some or all of them from the physical spectrum. Indeed, $f_0(y)$ ($g_0(y)$) satisfies only the boundary conditions of the type (N,N) (type (D,D)). Thus, a massless vector $A_{\mu,0}(x)$ (a massless scalar $h_0(x)$) appears only for the type (N,N) (type (D,D)) boundary conditions. This implies that the 4d gauge symmetry is broken except for the type (N,N) boundary conditions.

Let us next discuss geometrical meanings of the boundary conditions. To this end, it is convenient to rewrite the equations (5.23) and (5.24) into a familiar form of the $N = 2$ supersymmetric quantum mechanics. Reparametrizing the coordinate y such that

$$ds^2 = e^{-4W(\tilde{y})} \left(\eta_{\mu\nu} dx^\mu dx^\nu + d\tilde{y}^2 \right), \quad (5.29)$$

where $W(\tilde{y}) = W(y(\tilde{y}))$, we can rewrite the equations (5.23) and (5.24) into the form

$$-\bar{\mathcal{D}}_{\tilde{y}} \mathcal{D}_{\tilde{y}} \tilde{f}_n(\tilde{y}) = m_n^2 \tilde{f}_n(\tilde{y}), \quad (5.30)$$

$$-\mathcal{D}_{\tilde{y}} \bar{\mathcal{D}}_{\tilde{y}} \tilde{g}_n(\tilde{y}) = m_n^2 \tilde{g}_n(\tilde{y}), \quad (5.31)$$

where

$$e^{W(\tilde{y})} \tilde{f}_n(\tilde{y}) = f_n(y), \quad (5.32)$$

$$e^{3W(\tilde{y})} \sqrt{g_{55}(\tilde{y})} \tilde{g}_n(\tilde{y}) = g_n(y), \quad (5.33)$$

$$\mathcal{D}_{\tilde{y}} = \partial_{\tilde{y}} + W'(\tilde{y}), \quad (5.34)$$

$$\bar{\mathcal{D}}_{\tilde{y}} = \partial_{\tilde{y}} - W'(\tilde{y}). \quad (5.35)$$

Here, the prime denotes the derivative with respect to $\tilde{y}$. The Hamiltonian and the supercharge can be written, in this basis, as

$$\begin{aligned} \tilde{H} &= \tilde{Q}^2 = \begin{pmatrix} -\bar{\mathcal{D}}_{\tilde{y}} \mathcal{D}_{\tilde{y}} & 0 \\ 0 & -\mathcal{D}_{\tilde{y}} \bar{\mathcal{D}}_{\tilde{y}} \end{pmatrix} \\ &= \begin{pmatrix} -\partial_{\tilde{y}}^2 - W''(\tilde{y}) + (W'(\tilde{y}))^2 & 0 \\ 0 & -\partial_{\tilde{y}}^2 + W''(\tilde{y}) + (W'(\tilde{y}))^2 \end{pmatrix}, \end{aligned} \quad (5.36)$$

$$\tilde{Q} = \begin{pmatrix} 0 & -\bar{\mathcal{D}}_{\tilde{y}} \\ \mathcal{D}_{\tilde{y}} & 0 \end{pmatrix}. \quad (5.37)$$

These expressions are nothing but the $N = 2$ supersymmetric quantum mechanics given by Witten[63], and $W(\tilde{y})$ is called a superpotential. In this basis,

the boundary conditions (5.16)-(5.19) become

$$\text{Type (N, N)} : \begin{cases} \tilde{f}'(\tilde{y}_0) + W'(\tilde{y}_0)\tilde{f}(\tilde{y}_0) = \tilde{f}'(\tilde{y}_L) + W'(\tilde{y}_L)\tilde{f}(\tilde{y}_L) = 0, \\ \tilde{g}(\tilde{y}_0) = \tilde{g}(\tilde{y}_L) = 0, \end{cases} \quad (5.38)$$

$$\text{Type (D, D)} : \begin{cases} \tilde{f}(\tilde{y}_0) = \tilde{f}(\tilde{y}_L) = 0, \\ \tilde{g}'(\tilde{y}_0) - W'(\tilde{y}_0)\tilde{g}(\tilde{y}_0) = \tilde{g}'(\tilde{y}_L) - W'(\tilde{y}_L)\tilde{g}(\tilde{y}_L) = 0, \end{cases} \quad (5.39)$$

$$\text{Type (N, D)} : \begin{cases} \tilde{f}'(\tilde{y}_0) + W'(\tilde{y}_0)\tilde{f}(\tilde{y}_0) = \tilde{f}(\tilde{y}_L) = 0, \\ \tilde{g}(\tilde{y}_0) = \tilde{g}'(\tilde{y}_L) - W'(\tilde{y}_L)\tilde{g}(\tilde{y}_L) = 0, \end{cases} \quad (5.40)$$

$$\text{Type (D, N)} : \begin{cases} \tilde{f}(\tilde{y}_0) = \tilde{f}'(\tilde{y}_L) + W'(\tilde{y}_L)\tilde{f}(\tilde{y}_L) = 0, \\ \tilde{g}'(\tilde{y}_0) - W'(\tilde{y}_0)\tilde{g}(\tilde{y}_0) = \tilde{g}(\tilde{y}_L) = 0, \end{cases} \quad (5.41)$$

where $\tilde{y}_0 = \tilde{y}(y = 0)$ and $\tilde{y}_L = \tilde{y}(y = L)$. For the Dirichlet boundary conditions of $\tilde{f}(\tilde{y}) = 0$ and $\tilde{g}(\tilde{y}) = 0$ at $\tilde{y} = \tilde{y}_0, \tilde{y}_L$, we can interpret them as the existence of rigid walls at the boundaries. For the other boundary conditions of $\tilde{f}'(\tilde{y}) + W'(\tilde{y})\tilde{f}(\tilde{y}) = 0$ and $\tilde{g}'(\tilde{y}) - W'(\tilde{y})\tilde{g}(\tilde{y}) = 0$ at $\tilde{y} = \tilde{y}_0, \tilde{y}_L$, we can also interpret them as the existence of delta function potentials at the boundaries. Since for a delta function potential a localized (bound) state can appear, the low energy spectrum will have interesting properties for a non-trivial function $W(\tilde{y})$ [2, 3, 46].

Before closing this section, it is instructive to investigate 5d gauge invariance in non-abelian gauge theories. Let G and H be a non-abelian gauge group and its subgroup, respectively. We consider the situation that the 4d gauge symmetry G is broken to H via boundary conditions. We denote the generators of G, H and G/H by $\{T^I\}, \{T^a\}$ and $\{T^{\hat{a}}\}$, respectively. The 4d gauge symmetry breaking of $G \rightarrow H$ may be realized by imposing the type (N,N) boundary conditions on the gauge fields $A_M^a(x, y)$, which correspond to the unbroken generators of H , and one of the other boundary conditions on $A_M^{\hat{a}}(x, y)$, which correspond to the broken generators of G/H .

Infinitesimal gauge transformations will be given by

$$\delta A_M^I(x, y) = \partial_M \varepsilon^I(x, y) + g f^{IJK} A_M^J(x, y) \varepsilon^K(x, y), \quad (5.42)$$

where g is the 5d gauge coupling constant and f^{IJK} are the structure constants of G . The boundary conditions for the gauge parameters $\varepsilon^I(x, y)$ should be taken to be the same as $A_\mu^I(x, y)$. This requirement comes from the consistency with the inhomogeneous terms in eq. (5.42). In order for the 5d gauge invariance under the transformations (5.42) to preserve, the homogeneous terms on the right-hand-side of eq. (5.42) have to obey the same boundary conditions as $A_M^I(x, y)$. Then, it turns out that this is the case provided that [83, 84]

$$f^{\hat{a}\hat{b}\hat{c}} = 0. \quad (5.43)$$

Although the above conditions are not, in general, satisfied for arbitrary choice of G and H , they can be realized, for instance, if the Z_2 parity for the generators of G are assigned as

$$\mathcal{P}(T^a) = +T^a, \quad (5.44)$$

$$\mathcal{P}(T^{\hat{a}}) = -T^{\hat{a}}. \quad (5.45)$$

It is interesting to note that this happens for gauge symmetry breaking via Z_2 -orbifolding [40]–[44]. Thus, we have found that the 5d gauge invariance is preserved under the infinitesimal gauge transformations (5.42) with the conditions (5.43), even though the 4d gauge symmetry G is broken to H in the 4d effective theory.

5.2 A model with a large mass hierarchy

In this section, we propose a mechanism of Higgsless gauge symmetry breaking with a large mass hierarchy. Hidden supersymmetry discussed in the previous analysis is an important role in this mechanism.

5.2.1 A mechanism to obtain a large mass hierarchy

Let us consider one-dimensional quantum mechanics on a circle S^1 ($-l < y \leq l$) with two point singularities put at $y = 0$ and l , and start with the following

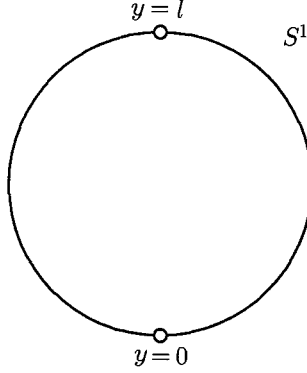


Figure 5.1: a circle $S^1(-l < y \leq l)$ with two point singularities at $y = 0$ and l .

Hamiltonian :

$$H = -\frac{d^2}{dy^2} + (W'(y))^2 - \mathcal{P}W''(y), \quad (5.46)$$

where a prime denotes the derivative with respect to y , and $\mathcal{P}$ is the parity transformation defined by

$$\mathcal{P} : \varphi(y) \rightarrow \mathcal{P}\varphi(y) = \varphi(-y). \quad (5.47)$$

The Hamiltonian forms the $N = 2$ superalgebra [63]

$$\{Q_a, Q_b\} = 2H\delta_{ab}, \quad a, b = 1, 2 \quad (5.48)$$

with the supercharges [59]-[62]

$$Q_1 = -\mathcal{P}\frac{d}{dy} + W'(y), \quad (5.49)$$

$$Q_2 = i\frac{d}{dy} - i\mathcal{P}W'(y), \quad (5.50)$$

provided that $W'(y)$ is an odd function of y , i.e.

$$W'(-y) = -W'(y). \quad (5.51)$$

We should stress that to solve the eigenvalue problem

$$H\varphi_n(y) = m_n^2\varphi_n(y), \quad (5.52)$$

the setting given above is insufficient. Since we put singularities at $y = 0$ and l , there are no reasons that wavefunctions as well as $W'(y)$ are smooth at the singularities, so that we should allow $\varphi_n(y)$ and $W'(y)$ to have discontinuities there. Therefore, to complete the setting of our model, we need to impose BC's at the singularities. For BC's to be consistent with supersymmetry, we require H and $Q_a(a = 1, 2)$ to be hermitian with respect to the inner product

$$\langle f|g \rangle \equiv \int_{-l_+}^{0_-} dy f^*(y)g(y) + \int_{0_+}^{l_-} dy f^*(y)g(y), \quad (5.53)$$

where $0_{\pm} \equiv 0 \pm \varepsilon$ and $\pm l_{\mp} \equiv \pm l \mp \varepsilon$ with an infinitesimal positive constant ε . Possible BC's that make the Hamiltonian hermitian are parameterized, in general, by the group $U(4)$.³ The further requirement that the supercharges are hermitian puts severe restrictions on allowed BC's. In fact, there are only four types of BC's, [59]-[62]⁴ which are found to be consistent with the parity transformation, given by

$$\begin{aligned} \text{Type I BC's} : \quad & \varphi'_+(0_+) + W'(0_+)\varphi_+(0_+) = 0 = \varphi_-(0_+), \\ & \varphi'_+(l_-) + W'(l_-)\varphi_+(l_-) = 0 = \varphi_-(l_-), \end{aligned} \quad (5.54)$$

$$\begin{aligned} \text{Type II BC's} : \quad & \varphi'_+(0_+) + W'(0_+)\varphi_+(0_+) = 0 = \varphi_-(0_+), \\ & \varphi_+(l_-) = 0 = \varphi'_-(l_-) - W'(l_-)\varphi_-(l_-), \end{aligned} \quad (5.55)$$

where $\varphi_+(y)$ ($\varphi_-(y)$) denotes a parity-even (odd) function of y . The other two BC's can be obtained by the replacement of $\varphi_+ \leftrightarrow \varphi_-$ in eqs.(5.54) and

³In the case of a point singularity, allowed BC's are shown to be specified by $U(2)$ [57].

⁴If we allow $\varphi(0_{\pm})$ and $\varphi'(0_{\pm})$ to be connected with $\varphi(\pm l_{\mp})$ and $\varphi'(\pm l_{\mp})$, we have a two parameter family of BC's [59]-[62].

(5.55). It should be noted that the type I BC's reduce to the ordinary BC's for smooth parity eigenfunctions when $W'(0_+) = W'(l_-) = 0$; otherwise, some of the functions $\varphi_{\pm}(y)$ and $\varphi'_{\pm}(y)$ will have discontinuities at $y = 0$ and l . Since the parity operator $\mathcal{P}$ commutes (anticommutes) with H (Q_a),⁵ the energy spectrum is found to be doubly-degenerate between $\varphi_{+,n}$ and $\varphi_{-,n}$ (except for zero energy states). For appropriately normalized eigenfunctions, we will have the relations

$$Q_1 \varphi_{\pm,n}(y) = m_n \varphi_{\mp,n}(y). \quad (5.56)$$

Similar relations hold for Q_2 . The analysis of zero energy solutions needs some care. The fact that the functions $\varphi_{\pm,0}(y) \propto \exp\{\mp W(y)\}$, which are normalizable, satisfy $H\varphi_{\pm,0}(y) = 0$ does not necessarily imply the existence of zero energy solutions. In fact, only $\varphi_{+,0}(y)$ ($\varphi_{-,0}(y)$) is a zero energy solution for the model with the type I BC's (the type I BC's with the replacement of $\varphi_+ \leftrightarrow \varphi_-$), and there are no other zero energy solutions at all. This is because other solutions do not match with the desired BC's, so that they have to be eliminated from the spectrum.

For later use, it is convenient to rewrite the equations (5.52) in a different basis such that

$$-\frac{1}{\Delta_{\pm}(y)} \frac{d}{dy} \left(\Delta_{\pm}(y) \frac{d}{dy} \chi_{\pm,n}(y) \right) = m_n^2 \chi_{\pm,n}(y), \quad (5.57)$$

where

$$\chi_{\pm,n}(y) = e^{\pm W(y)} \varphi_{\pm,n}(y), \quad (5.58)$$

$$\Delta_{\pm}(y) = e^{\mp 2W(y)}. \quad (5.59)$$

In this basis, the BC's (5.54) and (5.55) reduce to

$$\begin{aligned} \text{Type I BC's} : \quad \chi'_+(0_+) = 0 = \chi_-(0_+), \\ \chi'_+(l_-) = 0 = \chi_-(l_-), \end{aligned} \quad (5.60)$$

⁵This implies that $\mathcal{P}$ can be regarded as the “fermion” number operator $(-1)^F$ [63].

$$\begin{aligned}\text{Type II BC's} : \quad & \chi'_+(0_+) = 0 = \chi_-(0_+), \\ & \chi_+(l_-) = 0 = \chi'_-(l_-).\end{aligned}\tag{5.61}$$

It follows that $\chi_{\pm}(y)$ simply obey either the Dirichlet or the Neumann BC at each singular point. The relations (5.56) are replaced by

$$\chi'_{\pm,n}(y) = \pm m_n \Delta_{\mp}(y) \chi_{\mp,n}(y).\tag{5.62}$$

To clarify characteristics of eigenvalues m_n^2 in eq.(5.52) or (5.57), let us examine a simple but still nontrivial example of the superpotential, $W(y) = c|y|$, in some detail. Then, the eigenvalues and the associated eigenfunctions are found to be

Type I model :

$$\begin{aligned}0) \quad & m_0^2 = 0 \\ & \varphi_{+,0}^{\text{type I}}(y) = \sqrt{\frac{c}{1 - e^{-2cl}}} e^{-c|y|},\end{aligned}\tag{5.63}$$

$$\begin{aligned}n) \quad & m_n^2 = c^2 + \left(\frac{n\pi}{l}\right)^2 \quad (n = 1, 2, 3, \dots) \\ & \varphi_{+,n}^{\text{type I}}(y) = \frac{1}{\sqrt{l}} \sin\left(\frac{n\pi}{l}|y| - \theta_n\right),\end{aligned}\tag{5.64}$$

$$\varphi_{-,n}^{\text{type I}}(y) = \frac{1}{\sqrt{l}} \sin\left(\frac{n\pi}{l}y\right),\tag{5.65}$$

$$\text{with } \tan \theta_n = \frac{n\pi}{cl}.$$

Type II model :

$$0) \quad m_0^2 = c^2 - K^2 \text{ with } K = c \tanh(Kl)\tag{5.66}$$

$$\varphi_{+,0}^{\text{type II}}(y) = \begin{cases} \sqrt{\frac{2K}{\sinh(2Kl)-2Kl}} \sinh K(|y| - l) & \text{for } cl > 1, \\ \sqrt{\frac{3}{2l^3}}(|y| - l) & \text{for } cl = 1, \\ \text{non} & \text{for } cl < 1, \end{cases}\tag{5.67}$$

$$\varphi_{-,0}^{\text{type II}}(y) = \begin{cases} \sqrt{\frac{2K}{\sinh(2Kl)-2Kl}} \sinh Ky & \text{for } cl > 1, \\ \sqrt{\frac{3}{2l^3}}y & \text{for } cl = 1, \\ \text{non} & \text{for } cl < 1, \end{cases}$$

(5.68)

$$n) \ m_n^2 = c^2 + k_n^2 \quad (n = 1, 2, 3, \dots) \text{ with } k_n = c \tan k_n l$$

$$\varphi_{+,n}^{\text{type II}}(y) = \sqrt{\frac{2k_n}{2k_n l - \sin(2k_n l)}} \sin k_n(|y| - l), \quad (5.69)$$

$$\varphi_{-,n}^{\text{type II}}(y) = \sqrt{\frac{2k_n}{2k_n l - \sin(2k_n l)}} \sin k_n y. \quad (5.70)$$

The important observations of the above results are summarized as follows: For the type I model, a zero energy solution $\varphi_{+,0}^{\text{type I}}$ appears as a localized state around a singular point, and the higher modes (KK modes) are doubly-degenerate between $\varphi_{\pm,n}^{\text{type I}}$ with eigenvalues $m_n^2 = c^2 + \left(\frac{n\pi}{l}\right)^2$ for $n = 1, 2, 3, \dots$. On the other hand, for the type II model the lowest eigenvalue for $cl \gg 1$ is non-vanishing but

$$m_0^2 \sim 4c^2 e^{-2cl}. \quad (5.71)$$

Thus, the eigenvalue is exponentially suppressed by the factor e^{-2cl} . This turns out to be a key ingredient of our gauge symmetry breaking with a large mass hierarchy, as we will see in the next section. The higher modes have eigenvalues $c^2 + k_n^2$, so that the lowest eigenvalue is much lighter than the others. Thus, the type II model is a realization of the idea given by Shaposhnikov and Tinyakov. It should be noticed that for $c < 0$ there is no state with the mass squared (5.71) and thus that all masses are heavier than $|c|$. In constructing phenomenological models, the lowest modes, which are massless for the type I BC's and are massive but much lighter than the others for the type II BC's, will survive at low energies.

5.2.2 A toy model of GUT

In this section, we have considered a 5D $SU(5)$ gauge theory with a single extra dimension of the orbifold S^1/Z_2 . This orbifold has singularities at fixed points, and thus we can impose nontrivial boundary conditions that make field

configurations discontinuous at fixed points. We can apply the mechanism discussed in previous section since gauge invariant theories possess $N = 2$ supersymmetric structure shown in chapter 4. We present a toy model in which a large mass hierarchy of the spectrum naturally arises without fine tuning.

Let us consider an $SU(5)$ gauge theory with a single extra dimension compactified on the orbifold S^1/Z_2 , where S^1 is a circle with $-l < y \leq l$ and Z_2 is defined by $y \rightarrow -y$.

$$\begin{aligned} S &= \int d^4x \int_{S^1/Z_2} dy \Delta(y) \left\{ -\frac{1}{4} F_{MN}^a F^{aMN} \right\} \\ &= \int d^4x \int_{S^1/Z_2} dy \Delta(y) \left\{ -\frac{1}{4} F_{\mu\nu}^a F^{a\mu\nu} - \frac{1}{2} F_{5\nu}^a F^{a5\nu} \right\}, \end{aligned} \quad (5.72)$$

where $F_{MN}^a = \partial_M A_N^a - \partial_N A_M^a + g_5 f^{abc} A_M^b A_N^c$, and the f^{abc} 's are the structure constants of the gauge group $SU(5)$. The five-dimensional coordinates are denoted by $x^M = (x^\mu, y)$, where four-dimensional Lorentz indices are symbolized by Greek letters, e.g. $\mu, \nu = 0, 1, 2, 3$. The $\Delta(y)$ is some weight function depending on the fifth coordinate y . We assume $\Delta(y)$ to be an even function with the form $\Delta(y) = e^{-2W(y)}$, and we will take $W(y) = M_G|y|$, as an illustrative example, later. Although we will not discuss the origin of the weight function, it may arise, for instance, in a way from compactification on 3-branes with the metric ansatz

$$ds^2 = \Delta(y)^2 (\eta_{\mu\nu} dx^\mu dx^\nu - dy^2), \quad (5.73)$$

⁶ or a dilaton background.[73, 74, 75]

The orbifold projection on the gauge fields is given by

$$A_\mu(x, -y) = P A_\mu(x, y) P^{-1}, \quad (5.74)$$

⁶We note that the metric $ds^2 = e^{-4M_G|y|}(\eta_{\mu\nu} dx^\mu dx^\nu - dy^2)$ resembles the warped metric $ds_{RS}^2 = e^{-2k|y|} \eta_{\mu\nu} dx^\mu dx^\nu - dy^2$, [5] but gives a quite different mass spectrum, as we will see below.

$$A_5(x, -y) = -PA_5(x, y)P^{-1}, \quad (5.75)$$

where $A_M = A_M^a T^a$ (T^a 's are the generators of $SU(5)$), and P is a 5×5 matrix defined by $P = \text{diag}(1, 1, 1, -1, -1)$. The above orbifold projection implies that the gauge fields $A_\mu^a(x, y)$ belonging to the $SU(3) \times SU(2) \times U(1)$ gauge symmetry are even functions of y , while the remaining gauge fields $A_\mu^{\hat{a}}(x, y)$ are odd functions of y . On the other hand, the scalar components of the gauge fields have opposite parity; $A_5^a(x, y)$ ($A_5^{\hat{a}}(x, y)$) are odd (even) functions of y . Thus, one might conclude that the $SU(5)$ gauge symmetry is reduced to $SU(3) \times SU(2) \times U(1)$. This is, however, a hasty conclusion.

The above setting of the orbifold model is not enough to get the mass spectrum in four dimensions. Most of the literature have tacitly assumed that field configurations are smooth at fixed points of orbifolds. It will, however, be natural to allow field configurations to be discontinuous at fixed points since orbifolds are singular there.⁷ Thus, we here impose nontrivial BC's discussed in the previous section at the fixed points $y = 0$ and l .

We would like to have a model that exhibits a desired GUT symmetry breaking, i.e.

$$SU(5) \xrightarrow{M_G} SU(3) \times SU(2) \times U(1) \xrightarrow{M_W} SU(3) \times U(1)_{em}, \quad (5.76)$$

where M_G denotes a GUT scale on the order of 10^{16} GeV and M_W a weak scale on the order of 10^2 GeV. Let $A_M^{SU(3) \times U(1)_{em}}$ be the abbreviation of the gauge fields belonging to the unbroken gauge symmetry $SU(3) \times U(1)_{em}$, and let $A_M^{SU(5)/SU(3) \times SU(2) \times U(1)}$ and $A_M^{SU(2) \times U(1)/U(1)_{em}}$ be the abbreviations of the gauge fields of the broken generators corresponding to $SU(5) \rightarrow SU(3) \times SU(2) \times U(1)$ and $SU(2) \times U(1) \rightarrow U(1)_{em}$, respectively. The lowest modes of $A_\mu^{SU(3) \times U(1)_{em}}$ have to be massless, and those of $A_\mu^{SU(2) \times U(1)/U(1)_{em}}$, which correspond to the $W^\pm$ and Z^0 bosons in the Standard Model, should have masses on

⁷Some attempts [76, 77] to generalize orbifold models with discontinuities at fixed points have been made.

the order of M_W . On the other hand, all the modes of $A_\mu^{SU(5)/SU(3)\times SU(2)\times U(1)}$ are expected to have masses on the order of M_G or heavier. It turns out that the above situation can be realized if the gauge fields of the unbroken symmetry $SU(3) \times U(1)_{em}$ obey the type I BC's and the gauge fields belonging to the broken generators obey the type II BC's. The mode expansions of the gauge fields are given by ⁸

$$A_\mu^{SU(3)\times U(1)_{em}}(x, y) = \sum_{n=0}^{\infty} A_{\mu,n}^{SU(3)\times U(1)_{em}}(x) \chi_{+,n}^{\text{type I}}(y), \quad (5.77)$$

$$A_\mu^{SU(2)\times U(1)/U(1)_{em}}(x, y) = \sum_{n=0}^{\infty} A_{\mu,n}^{SU(2)\times U(1)/U(1)_{em}}(x) \chi_{+,n}^{\text{type II}}(y), \quad (5.78)$$

$$A_\mu^{SU(5)/SU(3)\times SU(2)\times U(1)}(x, y) = \sum_{n=1}^{\infty} A_{\mu,n}^{SU(5)/SU(3)\times SU(2)\times U(1)}(x) \bar{\chi}_{-,n}^{\text{type II}}(y), \quad (5.79)$$

and

$$A_5^{SU(3)\times U(1)_{em}}(x, y) = \sum_{n=1}^{\infty} A_{5,n}^{SU(3)\times U(1)_{em}}(x) e^{2W(y)} \chi_{-,n}^{\text{type I}}(y), \quad (5.80)$$

$$A_5^{SU(2)\times U(1)/U(1)_{em}}(x, y) = \sum_{n=0}^{\infty} A_{5,n}^{SU(2)\times U(1)/U(1)_{em}}(x) e^{2W(y)} \chi_{-,n}^{\text{type II}}(y), \quad (5.81)$$

$$A_5^{SU(5)/SU(3)\times SU(2)\times U(1)}(x, y) = \sum_{n=1}^{\infty} A_{5,n}^{SU(5)/SU(3)\times SU(2)\times U(1)}(x) e^{2W(y)} \bar{\chi}_{+,n}^{\text{type II}}(y), \quad (5.82)$$

where $\chi_{\pm,n}(y)$ are defined through eq.(5.58), and $\bar{\chi}_{\pm,n}(y)$ are also defined through eq.(5.58) with the replacement of $W(y) \rightarrow -W(y)$. It follows that all the massive modes are degenerate between $A_{\mu,n}$ and $A_{5,n}$, and each massive mode of $A_{5,n}$ may be regarded as a supersymmetric partner of the longitudinal mode of $A_{\mu,n}$. Since every $A_{5,n}$ mode is massive, there are no massless 4D scalars in our model.

⁸Since the orbifold S^1/Z_2 is given by modding out S^1 under the parity reflection, the normalization of the eigenfunctions should be multiplied by $\sqrt{2}$.

Table 5.1: Z_2 parity, BC's and mass spectrum: k_n ($\bar{k}_n$) for $n = 1, 2, 3, \dots$ are the solutions to $k_n = M_G \tan k_n l$ ($\bar{k}_n = -M_G \tan \bar{k}_n l$).

4D fields	Z_2 parity	BC's	mass spectrum
$A_{\mu,n}^{SU(3) \times U(1)_{em}}$	+	type I	$m_0 = 0$ $m_n = \sqrt{M_G^2 + (\frac{n\pi}{l})^2}$
$A_{\mu,n}^{SU(2) \times U(1)/U(1)_{em}}$	+	type II	$m_0 \sim 2M_G e^{-M_G l}$ $m_n = \sqrt{M_G^2 + k_n^2}$
$A_{\mu,n}^{SU(5)/SU(3) \times SU(2) \times U(1)}$	-	type II	$m_n = \sqrt{M_G^2 + (\bar{k}_n)^2}$
$A_{5,n}^{SU(3) \times U(1)_{em}}$	-	type I	$m_n = \sqrt{M_G^2 + (\frac{n\pi}{l})^2}$
$A_{5,n}^{SU(2) \times U(1)/U(1)_{em}}$	-	type II	$m_0 \sim 2M_G e^{-M_G l}$ $m_n = \sqrt{M_G^2 + k_n^2}$
$A_{5,n}^{SU(5)/SU(3) \times SU(2) \times U(1)}$	+	type II	$m_n = \sqrt{M_G^2 + (\bar{k}_n)^2}$

If we take $\Delta(y) = e^{-2M_G|y|}$ with $M_G \sim 10^{16}$ GeV and $M_{Gl} \sim 33$, the low energy effective theory consists of $A_{\mu,0}^{SU(3) \times U(1)_{em}}(x)$ and $A_{\mu,0}^{SU(2) \times U(1)/U(1)_{em}}(x)$. The gauge fields $A_{\mu,0}^{SU(3) \times U(1)_{em}}(x)$ remain massless, while $A_{\mu,0}^{SU(2) \times U(1)/U(1)_{em}}(x)$, which correspond to the $W^\pm$ and Z^0 bosons of the Standard Model, acquire the masses on the order of $2M_G e^{-M_{Gl}} \sim 10^2$ GeV. The other modes get masses on the order of M_G and decouple at low energies, as desired.

The extension to orbifolds $S^1/(Z_2 \times \cdots \times Z_2)$ should be straightforward. One-dimensional quantum mechanics on S^1 with 2^N point singularities has been investigated from a supersymmetric point of view. That analysis is equally applicable to orbifold models on $S^1/(Z_2 \times \cdots \times Z_2)$, as considered in this section.

To construct a realistic model, we have to introduce fermions. In our approach, fermions can exist in the bulk. This is because nontrivial BC's at fixed points. Because in our model, low energy modes of gauge fields appear to be localized around fixed points, it is suggested that all the content of the Standard Model, except for the Higgs scalars, is given by localized modes around fixed points. We view the model given in this section as a step toward a realistic theory of GUT symmetry breaking without Higgs scalars. It would be interesting to explore orbifold models along the line discussed here.

We finally comment on the question of unitarity violation in our model. The Higgs scalar in the Standard Model is necessary to ensure the unitarity of massive gauge boson scattering at high energies.[78]-[82] Without it, the scattering amplitude for the longitudinal components of the massive W and Z bosons would increase with energy as $\sim E^2$. It has been shown [27, 85, 86] that higher-dimensional gauge theories maintain unitarity in the sense that the terms in the amplitude that would grow with energy as $\sim E^4$ or $\sim E^2$ cancel, and there massive KK modes play the role of the Higgs scalar to ensure the unitarity. In our model, however, the tree-level unitarity should be violated at

the TeV scale, just as in the case of the Standard Model without a Higgs scalar, because the KK modes that could cancel the terms of the amplitude that increase with energy as $\sim E^2$ are too heavy ($\sim M_G$) to have much influence before unitarity is violated. Therefore, our model becomes a strong coupling theory and is no longer a reliable perturbative framework at energies above the TeV scale. To make reliable calculations based on our scenario, additional low energy degrees of freedom are required in order to unitarize the amplitude.⁹

⁹For instance, a scalar field existing at a boundary may rescue the low energy unitarity. [27]

Chapter 6

Summary

In this thesis, we have studied the quantum mechanical supersymmetry, and its application to the higher dimensional gauge theories.

In section 3, we have studied in full detail the supersymmetry on a circle with 2^n point singularities placed at regular intervals. The two types of $N = 2n + 1$ supercharges (type A and type B) are constructed in terms of the $n + 1$ sets of discrete transformations $\{P_k, Q_k, R_k\}$ ($k = 1, 2, \dots, n + 1$). The singularities can be described by the connection conditions in our formulation. We have found the connection conditions make all the supercharges physical, so that the $N = 2n + 1$ supersymmetry can be realized under the connection conditions. In our analysis, the connection conditions, under which an arbitrary subset of the $N = 2n + 1$ supercharges is physical and others are not physical, can be obtained. Thus, the $N = 2n + 1$ supersymmetry can be reduced to M -extended supersymmetry for any integer $M < N$ due to the connection conditions. We have also studied the degeneracy of the spectrum in particular zero energy state in the $N = 2n + 1$ supersymmetric models and some $N = 2n$ supersymmetric models as examples of the reduction of the su-

persymmetry. The energy eigenfunctions can be labeled by the eigenvalues of the physical operators which commute with the Hamiltonian and each other. The degeneracy of the spectrum for the states with $E > 0$ have been obtained by discussing the transition of the labeled states under the action of the supercharges. The analysis cannot apply for the states with $E = 0$. This is because the zero energy states satisfy $Q\varphi_0 = 0$, that is, the transition does not occur. The zero energy states, however, satisfy the first order differential equation thanks to the supersymmetry, and the formal solutions can be obtained. We have checked whether the formal solutions satisfy the connection conditions or not. The degeneracy of the spectrum and the Witten index are summarized in table 6.1 and table 6.2.

In chapter 4, we have investigated gauge invariant theories with extra dimensions and observed the quantum-mechanical $N = 2$ supersymmetric structure between 4d and extra-space components of gauge fields. The supersymmetric structure manifests itself in their 4d mass spectrum and massless 4d modes are found to be the solutions to the first order differential equation $Q|\Psi_0\rangle = 0$. It is then clear that the massless modes possess distinct analytic properties from other massive modes, which obey the 2nd order differential equations. We also have shown that the higher dimensional gauge theory on the whole possesses the $N = 2$ quantum-mechanical supersymmetry and the supersymmetry is a part of the higher dimensional gauge symmetry. Thus, we can define the supersymmetry transformation of matter fields in any gauge invariant theories with matter fields.

In chapter 5, we have discussed boundary conditions in gauge theories on extra dimensions with boundaries. In a gauge symmetry point of view, it is less obvious to obtain a possible set of boundary conditions consistent with gauge invariance because some of the boundary conditions explicitly break 4d gauge symmetries. On the other hand, in a supersymmetry point of view,

the requirement of 5d gauge invariance is replaced by the conditions that the supercharges are physical (hermitian and the action of the supercharges are well defined on a functional space with definite boundary conditions). The boundary conditions we obtained are consistent with 5d gauge invariance even if some of 4d gauge symmetries are broken via boundary conditions. We present a toy model in which a large mass hierarchy of the spectrum naturally arise without fine tuning. It would be interesting to explore orbifold models along the line discussed here.

Table 6.1: degeneracy of the spectrum in the $N = 2n + 1$ supersymmetric models

Type	parameters	degeneracy		Witten index
Type A	$b_2 = 0$ and $W''(x) = 0$ ($W'(x) \equiv c$)	$E > \frac{1}{2}c^2$	2^{n+1}	$\Delta_{W, \mathcal{P}_{n+1}} = 2^n$ (for $b_1 = 1$)
		$E = \frac{1}{2}c^2$	$\begin{cases} 2^n & \text{for } b_1 = +1 \\ 0 & \text{for } b_1 = -1 \end{cases}$	
	otherwise	$E > 0$	2^n	0
		$E = 0$	$\begin{cases} 2^n & \text{for } * \\ 0 & \text{for otherwise} \end{cases}$	
Type B	$b_2 = 0$	$E > 0$	2^{n+1}	$\Delta_{W, \mathcal{P}_{n+1}} = 2^n$ (for $b_1 = 1$)
		$E = 0$	$\begin{cases} 2^n & \text{for } b_1 = +1 \\ 0 & \text{for } b_1 = -1 \end{cases}$	
	otherwise	$E > 0$	2^n	0
		$E = 0$	0	

where $*$ denotes $\sqrt{\frac{1-b_2}{1+b_2}} = e^{2W(l_0-\epsilon)}$.

Table 6.2: degeneracy of the spectrum in the $N = 2n$ supersymmetric models

Type	U_I or U_{II}	degeneracy		Witten index
a	$U_I(\pm)$	$E > 0$	2^n	$\Delta_{W; \mathcal{G}_{\mathcal{R}_1} \dots \mathcal{G}_{\mathcal{R}_n}} = \pm 2^{n-1}$
		$E = 0$	2^{n-1}	$\Delta_{W; \mathcal{P}_{n+1}} = 2^{n-1}$ for $W'(x) = 0$
	$U_{II}(a)$	$E > 0$	2^n	0
		$E = 0$	$\begin{cases} 2^n & \text{for } * \\ 0 & \text{for otherwise} \end{cases}$	
b	$U_I(\pm)$	$E > 0$	2^n	$\Delta_{W; \mathcal{G}_{\mathcal{P}_n} \mathcal{P}_{n+1}} = \pm 2^{n-1}$
		$E = 0$	2^{n-1}	$\Delta_{W; \mathcal{P}_{n+1}} = 2^{n-1}$ for $W'(x) = 0$
	$U_{II}(a)$	$E > 0$	2^n	0
		$E = 0$	0	
c	$U_I(\pm)$	$E > 0$	2^n	$\Delta_{W; \mathcal{G}_{\mathcal{R}_1} \dots \mathcal{G}_{\mathcal{R}_n}} = \pm 2^{n-1}$
		$E = 0$	2^{n-1}	$\Delta_{W; \mathcal{P}_{n+1}} = 2^{n-1}$
	$U_{II}(a)$	$E > 0$	2^n	0
		$E = 0$	0	
d	$U_I(\pm)$	$E > 0$	2^n	$\Delta_{W; \mathcal{G}_{\mathcal{P}_n} \mathcal{P}_{n+1}} = \pm 2^{n-1}$
		$E = 0$	2^{n-1}	$\Delta_{W; \mathcal{P}_{n+1}} = 2^{n-1}$
	$U_{II}(a)$	$E > 0$	2^n	0
		$E = 0$	0	

where $*$ denotes $\sqrt{\frac{1-a_3}{1+a_3}} = e^{W(l_0-\epsilon)-W(l_1+\epsilon)}$, $a_1 = \sqrt{1-(a_3)^2}$, $a_2 = 0$,

Appendix A

HODGE DECOMPOSITION THEOREM

In this appendix, we give a simple proof of the Hodge decomposition theorem by use of the eigenfunctions of the differential operators $d^\dagger d$ and $dd^\dagger$.

Since $d^\dagger d$ is a hermitian operator, the eigenfunctions will form a complete set. Thus, any k form $A^{(k)}$ can be expanded as

$$A^{(k)} = \omega_0^{(k)} + \sum'_{n_k} a_{n_k} \omega_{n_k}^{(k)}, \quad (\text{A.1})$$

where

$$d\omega_0^{(k)} = 0, \quad (\text{A.2})$$

$$d^\dagger d\omega_0^{(k)} = (m_{n_k}^{(k)})^2 \omega_{n_k}^{(k)} \quad \text{for } m_{n_k}^{(k)} \neq 0. \quad (\text{A.3})$$

Here, $\sum'_{n_k}$ denotes the summation over all eigenstates with $m_{n_k}^{(k)} \neq 0$. Since $dd^\dagger$ is also a hermitian operator, $\omega_0^{(k)}$ can be expanded as

$$\omega_0^{(k)} = \eta_0^{(k)} + \sum'_{n_{k-1}} b_{n_{k-1}} \phi_{n_{k-1}}^{(k)}, \quad (\text{A.4})$$

where

$$d^\dagger \eta_0^{(k)} = d\eta_0^{(k)} = 0, \quad (\text{A.5})$$

$$dd^\dagger \phi_{n_{k-1}}^{(k)} = (m_{n_{k-1}}^{(k-1)})^2 \phi_{n_{k-1}}^{(k)} \quad \text{for } m_{n_{k-1}}^{(k-1)} \neq 0. \quad (\text{A.6})$$

It is convenient to further introduce the eigenfunctions of $dd^\dagger$ for $k+1$ forms and $d^\dagger d$ for $k-1$ forms as

$$dd^\dagger \phi_{n_k}^{(k+1)} = (m_{n_k}^{(k)})^2 \phi_{n_k}^{(k+1)}, \quad (\text{A.7})$$

$$d^\dagger d \omega_{n_{k-1}}^{(k-1)} = (m_{n_{k-1}}^{(k-1)})^2 \omega_{n_{k-1}}^{(k-1)}. \quad (\text{A.8})$$

As shown in Section 2, $\omega_{n_k}^{(k)}$ and $\phi_{n_{k-1}}^{(k)}$ are related to $\phi_{n_k}^{(k+1)}$ and $\omega_{n_{k-1}}^{(k-1)}$ as

$$m_{n_k}^{(k)} \omega_{n_k}^{(k)} = d^\dagger \phi_{n_k}^{(k+1)} \quad \text{or} \quad m_{n_k}^{(k)} \phi_{n_k}^{(k+1)} = d \omega_{n_k}^{(k)}, \quad (\text{A.9})$$

$$m_{n_{k-1}}^{(k-1)} \omega_{n_{k-1}}^{(k-1)} = d^\dagger \phi_{n_{k-1}}^{(k)} \quad \text{or} \quad m_{n_{k-1}}^{(k-1)} \phi_{n_{k-1}}^{(k)} = d \omega_{n_{k-1}}^{(k-1)}. \quad (\text{A.10})$$

A k form satisfying eqs.(A.5) is called a harmonic k form which can be expanded, in terms of a complete set of the harmonic k forms $\{\eta_p^{(k)}, p = 1, 2, \dots, b_k\}$, as

$$\eta_0^{(k)} = \sum_{p=1}^{b_k} c_p \eta_p^{(k)}. \quad (\text{A.11})$$

The integer b_k , which is the number of the independent harmonic k forms, is called the k th Betti number and is known as a topological number of the manifold.

We have thus found that any k form $A^{(k)}$ can be expanded as

$$\begin{aligned} A^{(k)} &= \sum_{p=1}^{b_k} c_p \eta_p^{(k)} + \sum_{n_k}' a_{n_k} \omega_{n_k}^{(k)} + \sum_{n_{k-1}}' b_{n_{k-1}} \phi_{n_{k-1}}^{(k)} \\ &= \sum_{p=1}^{b_k} c_p \eta_p^{(k)} + \sum_{n_k}' \frac{a_{n_k}}{m_{n_k}^{(k)}} d^\dagger \phi_{n_k}^{(k+1)} + \sum_{n_{k-1}}' \frac{b_{n_{k-1}}}{m_{n_{k-1}}^{(k-1)}} d \omega_{n_{k-1}}^{(k-1)}. \end{aligned} \quad (\text{A.12})$$

This implies that any k form has the decomposition of the form

$$A^{(k)} = \eta_0^{(k)} + d^\dagger \alpha^{(k+1)} + d \beta^{(k-1)}. \quad (\text{A.13})$$

This completes a proof of the Hodge decomposition theorem.

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