Endogenous Business Cycles : Character Traits and Income Taxation

髙田, 富士雄

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Endogenous Business Cycles: Character Traits and Income Taxation

気質と税制が引き起こす内生的景気循環

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神戸大学大学院経済学研究科
総合経済政策専攻
指導教員名　入谷純
高田富士雄

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神戸大学大学院経済学研究科
総合経済政策専攻
高田富士雄

Preface

Business cycles are an important topic in economics. In order to describe how the economic variables move, we need to construct a dynamic system, which is often established in terms of a simultaneous equations system.

A simultaneous equations system consists of variables and parameters. Naturally, the values of the variables are determined subject to the values of the parameters. This means that if the values of the parameters change, those of the solutions do as well. And when we extend our scope to dynamic simultaneous equations systems such as differential equations systems, we can easily imagine that this applies to the movements of variables, not only to the values at equilibria. In other words, the movements in a system also depend on the values of parameters. Of course, this includes cyclical movements.

In this context, economists have established a theory called endogenous business cycles, in which a pioneering work is Grandmont's "On Endogenous Competitive Business Cycles" (1985), in *Econometrica.* Since the publication of this work, this theory has become a hot issue in economics, based on the conventional bifurcation theory in mathematics. We are interested in this hot issue.

We will also take up this issue, but we do so based on the notion that a dynamic system can be described in terms of a linear simultaneous differential equations system. In this situation, the values of the ingredients of a Jacobian matrix at fixed points matter, because they determine the eigenvalues of the Jacobian matrix, which control the time paths of the economic variables. And the values of the ingredients also depend on the values of the parameters. As a result, for the values of some parameters, the fixed point can be a sink. In this situation, the time paths of economic variables can not be determined. This is called indeterminacy, and it can cause business cycles.

We are also interested in the possibility that this indeterminacy can occur due to taxation policy, which we deliberately formulate.

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I am also grateful to Professor Jeffrey Herrick, for advising me about English and checking my English in this thesis.

Of course, I accept full responsibility for any possible errors.

Contents

Chapter 1

Perspectives on Economic Fluctuations

Economic fluctuations, especially business cycles, are commonplace, in fact inevitable in our capitalistic economy. We can divide them into two categories, exogenous business cycles, and endogenous business cycles. And the latter can be derived in terms of bifurcation, which implies a variety of movements in an economy, and in terms of indeterminacy, which implies the indeterminacy of time paths in an economy.

1.1 Two business cycles

In general, we should be careful about what is determined and what is not within a model in analyses of economic behaviour. Indeed, the distinction when we think one economic variable is given exogenously and another is determined endogenously is a matter of viewpoint or emphasis, or of goals in the analysis. For example, some, such as Keynes (1978), consider supply of the labour force as given exogenously, while others, such as Lucas (1969), consider it as determined endogenously. And naturally the former approach establishes a model with the supply of the labour force as a given, and the latter establishes one in which the supply of the labour force is determined.

This is applicable to modeling business cycle theories.

In theoretical perspectives of business cycles, we can consider the following two categories:

In the exogenous theory, exogenous shocks to an economy, such as sunspots, changes in technologies, unexpected changes in the supply of money based on finding new gold, and an upper limitation to the labour force, can trigger changes in an economy, and afterwards the influences of an initial shock can lead to business cycles.

In the endogenous theory, distinctions in economic factors built into an economic system, such as the shapes of utility functions, the shapes of production functions, or taxation on economic agents, can cause business cycles. These kinds of business cycles can occur without any shocks from the outside and derive from the rational behaviour of economic agents, which seemingly implies the existence of micro-foundations of economic behaviour by the agents.

In short, in these models, economic variables are naturally determined within a system, but their values also naturally depend on the values of the parameters which are included in the model. That is, the determination of the variables depends on the values of the parameters. This implies that the values of the variables can change according to the parameters, and as a result, economic movements, including business cycles, can change due to changes in the parameters.

In this dissertation, we intend to discuss endogenous business cycles. However, before we begin to do so intensively, we will briefly consider concrete theories of exogenous business cycles and two aspects of endogenous business cycles, bifurcation and indeterminacy.

1.2 Examples of exogenous business cycles

Okishio (1968) established his theory based on a Keynesian perspective to describe economic fluctuations. It assumes an investment function of the Harrod-Okishio type, which strongly reflects a Keynesian view about our capitalistic economy. In Keynesian terms, first of all, we must consider what is left after consumption is deducted from the gross national product. Keynes (1978) calls this investment. Investment behavior is the most important driving force in our capitalistic economy,[1] because of its volatility.

[1] However, the recent literature on business cycles, such as Lucas (1972), Grandmont (1985), Benhabib and Farmer (1994) and Utaka (2003), ignores the importance of this and as a result omits the role of positively-driven investment in their models.

An investment function of the Harrod-Okishio type, which explains the determination of investment in terms of volume, can be expressed as

$$g(t) = \tilde{g}(t-1)[\delta(t-1)]^{\beta}. \tag{1.1}$$

Here, $g(t) = I(t)/K(t-1)$, which means the ratio of capital accumulation, which has been planned in the t period. $I(t)$ means planned demand for investment , $K(t)$ capital stock, respectively in the t period. Moreover, $\tilde{g}(t-1)$ means the ratio of capital accumulation which has been realized. And $\delta(t-1)$ means utilization in the $t-1$ period. Additionally, $\beta > 0$ means the coefficient of resolution for investment in terms of volume.

(1.1) means that the ratio of capital accumulation in the t period is determined by increasing (or decreasing) that in the $t-1$ period, based on the difference between the utilization in the $t-1$ period and 1. This investment function features the necessity for investment even in deep depressions as long as the rate of profit is positive.

Based on this investment function, it becomes clear that an economy can reach a boom and encounter a shortage in the labour force. In this situation, it is assumed that firms raise wages to compete for the labour force. Thereafter, how this boom curves downward and turns to a depression becomes a big issue. As formulated, the description that *firms raise wages to compete for the labour force* is an expression of the initial trigger, caused by an exogenous shock, the upper limitation of the labour supply. Therefore, economists who accept these premises should be devoted to elucidating how this initial raising of wages influences other economic ingredients in the economy, and how this influence is transmitted.[2]

On the other hand, Lucas (1972) offers a monetary business cycle theory from the Monetary School, which, in contrast to the Keynesian model, lacks investment functions and unemployment. The monetarists, such as Freedman, Barro, Sargent and Lucas, argue that the main driving force causing business cycles is not investment but the supply of money.

In addition, they distinguish two cases in terms of the effects of the money supply on a real economy, involving real interest rates, production and employment, as follows:

In one case, economic agents can anticipate the change in the money supply, and in the other one they can not anticipate it. In the former case, since economic agents can naturally change their behaviour according to their anticipations, a change in the money supply does not affect the real economy, and the influence is only on price levels. In short, the neutrality of money holds. Of course, this situation does not show any economic fluctuations.

Conversely, in the latter case, since agents make decisions given unexpected circumstances, their decisions are not rational. So, in this case, a change in the money supply can also naturally affect the real economy. In other words, the neutrality of money does not hold, and this case can show

[2]See Takata (2004) for more details about this mechanism.

business cycles caused by outside shocks, called exogenous business cycles.

With the controversial results stated above, Lucas (1972) establishes a framework, as we can easily guess, that needs modeling, since economic agents have imperfect information about the money supply. For this purpose, he assumes an overlapping generations model consisting of the young and the old. Thus the economy consists of two isolated islands, with the whole population constant, $2N$, half young. The young can not move to the other island when they are young, nor can the old, even in order to balance the money which they hold. Since the population on one island can not trade with that on the other island, they can not know market prices on the other island.

Lucas also assumes that the number of the young on the first island is $\frac{\theta}{2}N$, while that on the second island $(1 - \frac{\theta}{2})N$, and additionally θ means a stochastic variable, defined as $0 < \theta < 2$.

Moreover, the young decide their current consumption, labour supply and savings, in terms of the money which will be used for the purchase of goods, by maximizing the present value of their lifetime utilities. The old only consume goods in exchange for the money which they saved in their young period. And the government intervenes in this situation by transferring money to the old.

On these assumptions, exogenous shocks appear in terms of change in monetary holdings by the old and in the population of the young, that is,

(θ, x), in which it follows that $m' = xm$, where m means the initial monetary holdings when the young become old, and $(x-1)\ m$ is the amount of money transferred to the old by the government. Additionally, x is also defined as a stochastic variable and changes as time passes.

Lucas (1972) then offers the following:

$$p = m \cdot \varphi(\frac{x}{\theta}), \tag{1.2}$$

which determines an equilibrium price of outputs, p. And the function φ has an elasticity which is between 0 and 1.
Here, the demand for outputs means the real monetary level of holdings by the old, while the supply of outputs is the result of the population of the young multiplied by the amount of supplies each holds, which depends on market prices. It is obvious that observation data about p convey information about x/θ, but not about x.

When the young aim to maximize the present value of their lifetime utilities, they need to know the prices in an entire economy, but according to the above assumptions they can not. In other words, the young can not know prices on the other island, nor the present value of money xmN. In this situation, Lucas assumes that agents anticipate on the basis of a rational expectation principle. For example, when they find prices rising, they can not understand why this has occurred, i.e., whether due to a change in the population, θ, or to a change in the aggregate money supply, x. As a result, this change in the aggregate money supply affects the real economy, and can cause business cycles.

We can specifically interpret this mechanism as follows:

Because agents can only have imperfect information about relative prices, they react to changes in relative prices even if changes in the aggregate money supply alone affect price levels. This kind of adjustment in both demand and supply, corresponding to changes in relative prices, causes, first, substitution effects, and, second, income effects. The former refers to the fact that if the ratio between the price in a market and the anticipated price level is high, for example, agents take it as an increase in the opportunity cost of leisure, and consequently reduce leisure, work more and reduce consumption. The latter leads to an increase in real income, which causes increases both in consumption and in leisure.[3]

Some other studies on real business cycles, for instance, Kydland and Prescott (1982), are in the same vein as Lucas (1972).

1.3 Examples of endogenous business cycles

In contrast to exogenous business cycles, we can consider another theory, which claims that business cycles can occur without outside shocks.

[3]As for the relationship between consumption and labour supply, and substitution effects and income effects, see Lucas and Rapping (1969).

1.3.1 Bifurcations

Let us consider, as a good example, the following famous recurrence relation in terms of a quadratic relationship:

$$x_{t+1} = rx_t(1 - x_t), \tag{1.3}$$

where r means a positive constant, t a positive integer, and x a variable.

It is well known that the relationship expressed by (1.3) can be depicted in arbitrary forms, according to the values of r, the initial conditions of x and the number of iterations.[4] At this stage, we should pay sharp attention to the role of r as a parameter, focusing on the movements of x in recurrence equation (1.3).

First, if we consider the map formulated one time, from (1.3) we obtain the Jacobian, J, as follows:

$$J = \frac{dx_{t+1}}{dx_t} = r(1 - 2\ x_0),$$

which implies that if $-1 < J(x_0) < 1$, this map is stable at x_0.

Second, if we consider fixed points on the map, since $x_{t+1} = x_t$, the

[4] Recurrence equation (1.3) can not be solved in general, except for some small numbers of r. Wolfram (2002) offers a general solution for the exceptions of r, as follows:

$$x_t = \frac{1}{2}[1 - f(r^t\ f^{-1}(1 - 2\ x_0))],$$

where f means a function, and f^{-1} its inverse function.
For a detailed discussion, see Wolfram (2002).

following hold:

$$\begin{aligned} g(x) &= rx(1-x) \\ &= x, \end{aligned}$$

which leads to the following conclusion:

$$\begin{aligned} x[1 - rx(1-x)] &= x(1 - r + rx) \\ &= rx[x - (1 - r^{-1})] \\ &= 0. \end{aligned}$$

Consequently, we obtain two fixed points as

$$\begin{cases} x_1^{(1)} = 0, \\ x_2^{(1)} = 1 - r^{-1}. \end{cases}$$

Based on the above conclusion, we know that r must be greater than 1 for one fixed point to exist positively, and that to be stable, $-1 < r(1-2\,x_0) < 1$ must be satisfied. The second condition shows that the stability at a certain point depends on the values of both parameters and variables.

Now, we need to focus on the case of $r > 3$ on a map with two formulations. In this case, the map becomes unstable, and a pitchfork bifurcation appears, in which two stable orbits with two periods exist, corresponding to two stable fixed points of $g^2(x)$. The fixed points of $g^2(x)$ must satisfy $x_{t+2} = x_t$.

When we specifically transform the relationship, we obtain the following:

$$\begin{aligned} x_{t+2} &= rx_{t+1}(1 - x_{t+1}) \\ &= r[rx_t(1 - x_t)][1 - rx_t(1 - x_t)] \\ &= r^2\, x_t(1 - x_t)(1 - rx_t + rx_t^2) \\ &= x_t, \end{aligned}$$

which can be further transformed as follows:[5]

$$x[r^2(1 - x)(1 - rx + rx^2) - 1] = 0,$$

and which can finally be expressed as

$$-r^3x[x - (1 - r^{-1})][x^2 - (1 + r^{-1})x + r^{-1}(1 + r^{-1})] = 0. \qquad (1.4)$$

(1.4) shows the existence of four fixed points on this map. However, the values in a circle with two periods, meaning that variable x has only two values, in other words, that x goes back and forth between the two values, must satisfy the following:

$$x^2 - (1 + r^{-1})x + r^{-1}(1 + r^{-1}) = 0,$$

[5] We omit subscripts below, for convenience.

from which we obtain the following solutions:

$$x_{\pm}^{(2)} = \frac{1}{2}[(1+r)^{-1} \pm r^{-1}\sqrt{(r-3)(r+1)}].$$

Accordingly, we can see that when $r = 3$, a cycle with two periods begins, because $x_{\pm}^{(2)}$ must be real.

Based on the above analysis, we can see that when the value of parameter r increases, the behaviour of x changes, from monotonous to circular. This rule is general, since the periodicity of 2^t increases as r does. And it is well known that when r reaches approximately 3.599, t reaches infinity, and the periodicity is called chaos.

In the above explanation, we describe how periodicity appears based on changes in parameters, but this description is confined to a special case in which $x_{t+1} = r\ x_t(1 - x_t)$, that is, to a logistic map. So, we intend to extend our scope to more general cases.

First, consider the following dynamic system:

$$x_{t+1} = F(x_t; \alpha), \tag{1.5}$$

where $F: X \times \Omega \to X$, $(x, \alpha) \mapsto F(x, \alpha)$, $X \subset R^n$, and $\Omega \subset R^m$. Additionally, R^n is a euclidian n-space.

All fixed points in (1.5) must satisfy the following equation system:

$$\begin{aligned} G(x;\alpha) &= F(x;\alpha) - x \\ &= 0. \end{aligned} \tag{1.6}$$

At this stage, we assume that (x^o, α^o) is a solution in (1.6), that is, with respect to $\alpha = \alpha^o$, x^o is the value in an equilibrium. In what follows, assume that α is a value slightly different from α^o.

Now we consider the following:

$$|D_x G(x^o; \alpha^o)| \neq 0, \tag{1.7}$$

which means that the value of the Jacobian matrix of G is not null.

If bifurcations involving the phenomena of appearance or the vanishing of equilibria occur, (1.7) and the implicit function theorem do not hold.

We respectively define λ_f and λ_g as follows:

$$|D_x F(x^o; \alpha^o) - \lambda_f I| = 0,$$

and

$$|D_x G(x^o; \alpha^o) - \lambda_g I| = 0,$$

which obviously show the eigenvalues of $D_x F(x^o; \alpha^o)$ and $D_x G(x^o; \alpha^o)$, respectively. Additionally, we denote the unit matrix as I.

As a result, we obtain the following:

$$\lambda_g = \lambda_f - 1,$$

and

$$\begin{aligned} |D_x G(x^o; \alpha^o)| &= \prod_i \lambda_g^i \\ &= \prod_i (\lambda_f^i - 1). \end{aligned}$$

Focus on the two eigenvalues. Now assume that all eigenvalues of $D_x F(x)$ are real. In this situation, if at least one of the eigenvalues of λ_f^i equals 1, then $D_x G(x)$ vanishes, because $|D_x\ G(x^o; \alpha^o)| = 0$. Therefore, the implicit function theorem with respect to function $G(x)$ does no hold, which leads to the fact that (1.5) is non-hyperbolic. This is applicable to the case of imaginary eigenvalues.

From what we have discussed so far, we can easily imagine that the eigenvalues of the Jacobian matrix can affect movements of our dynamic system, and that the values of the eigenvalues can be influenced by the values of the parameters in the dynamic system, which are symbolized as α in this example.

In fact, as we have seen in a special case in terms of a logistic map, in mathematics, the relationships between the values of parameters and some kind of bifurcations have been rigorously investigated; among them, the flip bifurcation is useful in describing fluctuations in our economy.

This case has -1 as an eigenvalue, in which, more strictly speaking, it follows that $D_x F(x^o; \alpha^o)$ *has a single real eigenvalue on the boundary of the unit circle with value* -1.[6]

Additionally, note that in this case, the implicit function theorem obviously holds.

Conversely, if the eigenvalues include 1, this case is called a fold bifurcation, in which the implicit function theorem naturally does not hold.

Let us return to (1.5). Moreover, assume that $m = 1$, the eigenvalues of $D_x F(x^o, \alpha^o)$ are less than 1 in terms of absolute values, and that as an exception, a real eigenvalue $\lambda(\alpha^o) = -1$ on the boundary of a unit circle exists.

On these assumptions, and additionally if $\frac{d\lambda(\alpha^o)}{d\alpha} \neq 0$, the equilibrium loses stability when this α crosses the bifurcation point α^o. Consequently, in a small neighborhood of (x^o, α^o), the system has a periodic 2 orbit, on either side of bifurcation point α^o.

This theory was originally explored in mathematics.[7] However, we can

[6]This expression is cited from Azariadis (1993). Besides, see Medio (1992) and Puu (2003) for more details.

[7]For example, Neuman suggested in the 1940s that the logistic map $x_{t+1} = 4x_t(1 - x_t)$ can generate random numbers. The research history of this logistic map is offered by Wolfram (2002), in a sketchy fashion.

easily see that it can be utilized in order to describe economic fluctuations, especially how economic movements change from monotonous trends to cyclical ones, without any shocks from outside.

In other words, if we can establish a dynamic system, which includes parameters such as the degree of greed for consumption, the degree of time preference, indications of technology and tax rates, we can easily imagine from the above mathematical discussions that the movements of this dynamic system can be affected by the values of such parameters. The reason is that the values of variables in the system are determined under parametric preconditions. In this framework, changes in the values of the parameters cause changes in the values of the variables, which are supposed to be equilibria values, and the pattern of movements in economic fluctuations can be changed, for example, to bifurcations or to chaos.

This kind of theory is called an endogenous business cycle theory, and it has been prevalent since the publication of Grandmont (1985). Grandmont (1985) succeeds in explaining how business cycles with two periods can occur without shocks from outside, under the dubious condition that individuals reduce their savings in a circumstance with no investment opportunities, when real interest rates increase. Apart from this dubious but necessary condition, Grandmont's conclusion sharply contrasts with that of Lucas (1972), which asserts that business cycles never occur without shocks from outside.

And we basically discuss, in this dissertation, this bifurcation theory from the viewpoint of business cycles in the framework of an overlapping generations model, as Grandmont (1985) did.

1.3.2 Indeterminacy

Let us consider a model, in which a representative individual lives to infinity, and chooses labour supply, consumption and savings. Of course, the standards of these choices are based on the maximization of the present value of utilities over time, under budget constraints over time. In this context, over time naturally means infinity. This model originates in Ramsey (1928), in which a central planning authority is assumed, but we extend that model into a market model with production and a government.

Our model, which is basically consistent with that in literature such as Benhabib and Farmer (1994), Farmer (1999), and Schmitt-Grohé and Uribe (1997), can deal with time paths of economic variables. This shows that we can know how our economy moves, how some conditions cause fluctuations, and others do not, in terms of a non-linear simultaneous differential equations system.

In general, the system can be expressed as follows:[8]

$$\begin{cases} \dot{K}_t = \phi(K_t, C_t), \\ \dot{C}_t = \psi(K_t, C_t). \end{cases} \tag{1.8}$$

[8]See Benhabib and Farmer (1994).

Here, K means capital stock and C consumption. Functions, ϕ and ψ are not necessarily linear with respect to K_t and C_t. Therefore, in order to identify economic movements, we are forced to linearize our system. In this situation, in general, the values of ingredients of a Jacobian matrix at steady states, if those states exist, matter, because they determine the eigenvalues of the Jacobian matrix, which control the time paths of economic variables in our system.

Moreover, whether the steady state is unique or not is also a big issue.

In addition, as can easily be imagined, the values of ingredients of the Jacobian matrix at a steady state are affected by the parameters in our system. Since our dynamic system consists of both variables and parameters, just as in the case of previous models about bifurcations, in fact, the behaviour of the system can be affected by changes in parameters. The reason is that the values of variables at a steady state are those at a fixed point, which is determined by certain parametric values.

Among changes in economic behaviour, first of all, we should shed light on indeterminacy.

For simplicity, consider that a unique steady state exists and all eigenvalues of the Jacobian matrix at this steady state are real and negative. In other words, the fixed point is assumed to be a sink. In general, in light of (1.8), our system apparently consists of two ingredients, K_t and C_t, and concrete time paths of both variables are respectively controlled by the values of the ingredients and additional initial conditions. Here, we can consider an initial condition of K_t as given historically, and one of the values of C_t as a given by a transversality condition, which, in this context, is derived from the economic assumption that a representative agent entirely consumes savings over time.

In this fashion, the time paths of K_t and of C_t can be concretely determined. However, when all eigenvalues of the Jacobian matrix at the steady state are negative, one possibility, all trajectories of the variables finally converge on this steady state, regardless of the value of C_0, which in turn leads to the indeterminacy of C, which shows the indeterminacy of the system. Of course, in this situation, the transversality condition holds, because the steady state satisfies that condition.

Moreover, this indeterminacy can lead to sunspot fluctuations, i.e., belief-driven fluctuations or fluctuations based on animal spirits, in a Keynesian context. And these kinds of fluctuations can be business cycles, that is, endogenous business cycles.

Second, there is a possibility of multiple equilibria in (1.8). In this case, multiple trajectories with different features can coexist in an economy, which we can also call indeterminacy.

Apparently, these phenomena depend a lot on the values of the parameters in a system such as (1.8). So it is important to identify the relationship among these values, especially in terms of economics. Benhabib and Farmer (1994) lists externality and monopolistic competition as ingredients inducing indeterminacy, and points out that this indeterminacy based on externality in production can occur if the shape of a demand curve for the labour force is steeper than that of a labour supply curve.[9] However, this condition is dubious.

On the other hand, Schmitt-Grohé and Uribe (1997) discusses this indeterminacy issue from another viewpoint, that is, income taxation, which is a product of artificial policy, different from naturally produced economic reflections such as markets, externality, and monopolistic competition. And they assert that regressive taxation in terms of a balanced budget rule can cause indeterminacy, and this view is followed up in Guo and Harrison (2001), Utaka (2003) and Mino (2004). Basically, we intend to explore the possibilities of the occurrence of indeterminacy and of stability caused by income taxation under a balanced budget rule, as in these articles. However, Schmitt-Grohé and Uribe (1997) has serious logical flaws, because in their model, the labour supply by a household can not be derived. We will demonstrate this in chapter 3. And therefore we can not accept their model directly.

1.4 Outline of the dissertation

In chapter 2, we deal with a bifurcation, explaining how business cycles with two periods appear, as individual character changes. We discuss our theory based on the assumption of an overlapping generations model, and of

[9] See Benhabib (1998) and Farmer (1999), for details on the same lines as in our discussions here.

a stationary expectation principle.

In what follows, specifically from chapter 3 through chapter 5, we focus on the influence of income taxation on movements in an economy, in which a balanced budget rule with fixed tax rates is assumed. And in chapter 3, we analyze how labour income taxation affects movements in an economy, especially in light of business cycles. This is based on a framework of infinite horizons, as originally described by Schmitt-Grohé and Uribe (1997), but we modify it and establish a new model because that article has flaws. Thereafter, we direct our attention to the influence of capital income taxes on the movements in an economy, in the same model framework and in the same light as in chapter 3, in chapter 4. Because, contrary to our expectations, this results in affirming that under both forms of taxation, the economy converges to a saddle point in the long term, in chapter 5, we compare which form of taxation is preferable, in light of consumption in relation to labour. Then, chapter 6 is devoted to offering a model framework to determine how indeterminacy can occur under a balanced budget rule, in which tax rates are changeable. Finally, chapter 7 is devoted to a brief conclusion.

Chapter 2

Endogenous Business Cycles: The Influence of Character Traits

This chapter deals with how business cycles can occur, in light of character traits which influence how individuals choose and behave in an economy. We assume an overlapping generations model and utility with additive separability with respect to time. In this situation, individuals choose current consumption and current savings, which leads to consumption in the future. And the result of this choice is dependent of both the shape of its utility function and an evaluation about utility in the future.

Acknowledging that this dependency is rooted in a common characteristic of individuals, we conclude that in an economy comprised of individuals with profligate characteristics, like Aesopian grasshoppers, endogenous business cycles with two periods can occur, and in other cases they do not.

2.1 Introduction

This chapter deals with how the character traits of individuals, which determine the shape of utility functions and subjective time preference, affect economic movements. We rely upon the overlapping generations model, which Samuelson (1958) established and which has since prevailed in discussing movements of economies.

We summarize several models of business cycles and their problems below.

Capital is a driving force in economic movements. Since we can consider that capital is transformed and accumulated from savings, first of all, we should focus on the movements of savings. In this model, the young generation chooses its savings under given budget constraints, in order to maximize the present value of utilities, which include the utility that it will be able to obtain when it becomes old.

In this situation, the young need to add the utility in their old age to that in their young age. In fact, on the assumption of this additive separability of utility with respect to time, Diamond (1965), Grandmont (1985) and Utaka (2003) establish their own theories.

This assumption, however, necessarily involves a problem. How can we determine discount rates, which are utilized in order to reassess utility in the future, as present value? The future is uncertain, since we can not know what will happen. Mas-Colell (1995) focuses on uncertainty about longevity, and Diamond (1965) and Utaka (2003) take the discount rate into account but do not say what determines the degree of the discount rate. Grandmont (1985), however, ignores this problem and assumes that the discount rate is a unity, which means equal assessment of both present and future utility.

In this chapter, we assume the existence of a discount rate, as Mas-Colell (1995) did. Furthermore, we consider that the degree of this discount rate is derived from the characteristics of individuals, who are naturally careful about future events. But in the face of equal future events, some will value the present more than others do.

On the other hand, the shape of a utility function can affect the determination of savings. Furthermore, the way individuals behave in the face of changes in consumption determines the shape of a utility function. In other words, individuals are greedy in some cases and not in other cases. Diamond (1965), Grandmont (1985), Reichlin (1986), and Utaka (2003) introduce the Arrow-Pratt standard as a measure of how much the utility changes, in the face of this kind of consumption change.[1] In this chapter, we consider elasticity of marginal utility to consumption as the measure.

However, there is a critical point. The way individuals behave in the face of changes in consumption is associated with their behavior over time. More

[1]Grandmont (1985) assumes that individuals behave differently when they are young and old, given the existence of this change. If we accept this, we deal with the characteristics only of the young here.

concretely speaking, if individuals behave prudently (or greedily) in the face of changes in consumption, they must similarly behave carefully (or boldly) in the face of future events.

In this chapter, in light of the above points, we will demonstrate that these differences of character can cause fluctuations, in which business cycles are included.

When the young generation decides how much to save, it has to presume the future interest rate, which will determine the returns from savings it will receive when it becomes old. Diamond (1965), Granmont (1985), and Utaka (2003) all assume perfect foresight, which means that the anticipated interest rate equals the real market interest rate in every period. This assumption, in fact, leads to the fact that anticipations about future events surely match real events, and therefore it is kind of artificial and unrealistic.

Conversely, we suppose that the young generation anticipates interest rates on the basis of stationary expectation. In the above papers, the young generation determines savings with reference to the market interest rate in their old age *in advance*, while in this chapter, the determination is seen as taking place in relation to the currently prevailing rate.

Based on the argument so far, we conclude that in an economy composed of individuals with optimistic outlooks, business cycles with two periods can occur, while in other cases they do not. In other words, if we cite one of

Aesop's Fables, business cycles can occur in a society not with ants but with grasshoppers. In this situation, it becomes clear that a change of income does not influence savings very much, but changes of interest rate (the market interest rate for the young by definition) can substantially determine savings. Additionally, we derive the necessary condition for business cycles with two periods when the share of capital income is less than a certain level.[2]

The structure of this chapter is as follows: In section 2.2, we present our basic framework. We establish a fundamental equation, which controls our economy, in section 2.3. Then, after concrete specification of a utility and a production function, we explore some conditions for business cycles with two periods, in section 2.4. Finally, in section 2.5, we examine the economic situations of the business cycles and briefly discuss a policy to guide us toward a better economy.

2.2 Model framework

Assume the following: Our economy consisits of two sectors, individuals and corporations. Individuals consist of two generations, namely the young and the old. This means that individuals live in two periods. The young generation offers labour, receives wages in exchange for it, and loans its savings to corporations in the same period. The old generation receives

[2]Reichlin (1986) asserts, in light of production technology, that the elasticity of substitution between capital and labour is small, as a necessary condition for business cycles. However, in our model, it is assumed to be a unity.

its savings with interest from corporations to which it loaned when it was young. The old generation retires from our model after it entirely consumes its savings with interest.

2.2.1 Individuals

We define our notations as follows: w_t means a wage in the t period which an individual in the young generation receives in exchange for offering labour. S_t means savings in the t period; e_t^1 consumption by a young individual in the t period; e_{t+1}^2 consumption by an old individual in the $t+1$ period; r_{t+1} a real rental rate of capital in the $t+1$ period (we later call this a real interest rate). A young individual designs budget constraints over a lifetime as

$$w_t = e_t^1 + s_t, \tag{2.1}$$

$$(1 + r_{t+1})s_t = e_{t+1}^2. \tag{2.2}$$

Under these constraints, a young individual maximizes utility over lifetime R as

$$R = U(w_t - s_t) + \beta U[(1 + r_{t+1})s_t]. \tag{2.3}$$

Here, β means a discount rate over time, and we assume that $0 < \beta < 1$. Additionally, U means a utility function featuring $U' > 0$ and $U'' < 0$.

(2.3) means that a young individual maximizes the present value of utilities. In this situation, in the t period, the individual is forced to presume

r_{t+1}. We postulate that the individual presumes it on the basis of stationary expectation. In this context, (2.3) is modified as

$$R = U(w_t - s_t) + \beta U[(1 + r_t)s_t]. \tag{2.4}$$

In the t period, both w_t and r_t are predetermined. Therefore, the young maximize (2.4) with respect to s_t.

In order to verify this, let us analyze corporations, as a preparation.

2.2.2 Corporations

The savings borrowed from the young generation in the t period are transformed into capital in the $t+1$ period. In this situation, corporations produce output, utilizing both this capital and the labour offered by the young generation.

Outputs can be used as both capital and consumer goods, with perfect substitution. Production technology is under a constant return to scale, and therefore corporations have no profit. We can assume a production function as

$$y_t = f(k_t), \quad f' > 0, \quad f'' < 0. \tag{2.5}$$

We define our notations as follows: $y_t = Y_t/L_t$ and $k_t = K_t/L_t$. Here, Y_t means output in the t period, L_t labour supply in the t period, and K_t capital stock in the t period, respectively.

We obtain the following relationships, since corporations maximize their profit.

$$w_t = f(k_t) - k_t f'(k_t), \tag{2.6}$$

$$r_t = f'(k_t). \tag{2.7}$$

(2.6) means a demand schedule for labour. (2.7) means a demand schedule for capital. In this context, the whole savings in a society in the t period turn into capital in the next period. In other words, the capital in the t period is determined by the savings in the $t-1$ period. Furthermore, we assume that the supply of labour is exogenous, and that the labour market is fully competitive. So, this labour supply entirely meets the labour demand. Therefore, k_t is predetermined in the t period. In this situation, both w_t and r_t are given (see (2.6) and (2.7)).

Furthermore, when we eliminate k_t from (2.6) and (2.7), we obtain a factor-price frontier, which Samuelson (1962) originally introduced.

$$w_t = \phi(r_t), \tag{2.8}$$

which satisfies

$$\phi'(r_t) < 0. \tag{2.9}$$

2.2.3 Savings determination

Let us focus on (2.3) again; we can see that a young individual maximizes utility in (2.3) with regard to savings, because r_{t+1} is presumed to be equal to r_t. Based on the necessary and sufficient conditions of maximization with respect to s_t, the following holds:

$$U'(w_t - s_t) = \beta U'[(1 + r_t)s_t](1 + r_t). \tag{2.10}$$

(2.10) means that the young choose current savings, s_t, so that the marginal utility of consumption in the t period equals the present value of that supposed to be gained in the $t + 1$ period.

At this stage, let us focus on (2.7), which indicates the equilibrium condition in the capital market, in light of a given k_t. Assuming an economy with a positive rate of increment of the labour supply, we can obtain the following relationship between individual savings and the capital-labour ratio in the whole economy:

$$k_{t+1} = \frac{s_t}{1 + n}. \tag{2.11}$$

Additionally, n means the rate of increment of the labour supply, and is positive and constant.

From (2.7) and (2.11), equilibrium in the capital market is expressed as

$$r_{t+1} = f'(k_{t+1}) = f'(\frac{s_t}{1 + n}). \tag{2.12}$$

Based on (2.12), we obtain the following relationships:

$$s_t = \varphi(r_{t+1}), \tag{2.13}$$

which satisfies

$$\frac{ds_t}{dr_{t+1}} = \frac{1+n}{f''} < 0. \tag{2.14}$$

We need to explain (2.13) from the economic viewpoint. When a young individual determines s_t, this savings turns into capital in the next period, and the capital is utilized in production. In competitive markets, the capital yields a real rental rate equal to the marginal products of capital. This means that s_t determines r_{t+1}. (2.13) describes this mechanism.

2.3 A fundamental equation

2.3.1 The equation

Based on what we have analyzed so far, we obtain the following fundamental equation which controls the movements of our economy:

$$U'[\phi(r_t) - \varphi(r_{t+1})] = \beta U'[(1+r_t)\varphi(r_{t+1})](1+r_t). \tag{2.15}$$

(2.15) determines r_{t+1}, if r_t is given.

In order to verify this, we define the following function:

$$F(r_{t+1}) = U'[\phi(r_t) - \varphi(r_{t+1})] - \beta U'[(1 + r_t)\varphi(r_{t+1})](1 + r_t). \tag{2.16}$$

First of all, we demonstrate that (2.16) has a unique solution.

We assume Inada conditions in the production technology designed by (2.5), as follows:

$$\lim_{k_t \to \infty} f'(k_t) = 0. \tag{2.17}$$

$$\lim_{k_t \to 0} f'(k_t) = \infty. \tag{2.18}$$

Under the above assumptions, the following correspondingly hold:

$$\lim_{r_{t+1} \to \infty} \varphi(r_{t+1}) = 0. \tag{2.19}$$

$$\lim_{r_{t+1} \to 0} \varphi(r_{t+1}) = \infty. \tag{2.20}$$

At this stage, we can easily understand the following:

$$\lim_{r_{t+1} \to \infty} F(r_{t+1}) < 0. \tag{2.21}$$

$$\lim_{r_{t+1} \to 0} F(r_{t+1}) > 0. \tag{2.22}$$

Therefore, solutions can exist.

Secondly, we demonstrate uniqueness.

In order to verify this, it is enough to show that the function F is monotonous with respect to r_{t+1}.

Assuming that $r^1_{t+1} > r^2_{t+1}$, we can obtain the following:

$$F(r^1_{t+1}) - F(r^2_{t+1}) = U'[\phi(r_t) - \varphi(r^1_{t+1})] - U'[\phi(r_t) - \varphi(r^2_{t+1})] \\ - \beta(1+r_t)[U'[(1+r_t)\varphi(r^1_{t+1})] - U'[(1+r_t)\varphi(r^2_{t+1})]]. \quad (2.23)$$

Here, under the above assumption, since $\phi(r_t) - \varphi(r^1_{t+1}) > \phi(r_t) - \varphi(r^2_{t+1})$ holds, $U'[\phi(r_t) - \varphi(r^1_{t+1})] < U'[\phi(r_t) - \varphi(r^2_{t+1})]$ does. In a similar fashion, since $(1+r_t)\varphi(r^1_{t+1}) < (1+r_t)\varphi(r^2_{t+1})$ holds, $U'[(1+r_t)\varphi(r^1_{t+1})] > U'[(1+r_t)\varphi(r^2_{t+1})]$ does.

From the above analysis, we finally obtain the reqired result, that is, $F(r^1_{t+1}) < F(r^2_{t+1})$. The above relationship means that F is a monotonously decreasing function with respect to r_{t+1}.

2.3.2 Non-cyclical movements

At this stage, differentiating the fundamental equation (2.15) totally, we obtain the following:

$$\begin{aligned}[U''[\phi(r_t) - \varphi(r_{t+1})]\phi'(r_t) - \beta U'[(1+r_t)\varphi(r_{t+1})] \\ - \beta(1+r_t)U''[(1+r_t)\varphi(r_{t+1})]\varphi(r_{t+1})]dr_t \\ = [U''[\phi(r_t) - \varphi(r_{t+1})]\varphi'(r_{t+1}) \\ + \beta(1+r_t)^2U''[(1+r_t)\varphi(r_{t+1})]\varphi'(r_{t+1})]dr_{t+1}.\end{aligned} \tag{2.24}$$

Now, we respectively define two parts of (2.24) as

$$F_1 = U''[\phi(r_t) - \varphi(r_{t+1})]\phi'(r_t) - \beta U'[(1+r_t)\varphi(r_{t+1})] - \beta(1+r_t)U''[(1+r_t)\varphi(r_{t+1})]\varphi(r_{t+1}). \tag{2.25}$$

$$F_2 = U''[\phi(r_t) - \varphi(r_{t+1})]\varphi'(r_{t+1}) + \beta(1+r_t)^2U''[(1+r_t)\varphi(r_{t+1})]\varphi'(r_{t+1}). \tag{2.26}$$

In this situation, when we consider $F(r_t, r_{t+1}) = 0$, (2.24) can be expressed in this way:

$$F_1 dr_t - F_2 dr_{t+1} = 0. \tag{2.27}$$

On the other hand, the following holds:

$$\frac{\partial F}{\partial r_t} dr_t + \frac{\partial F}{\partial r_{t+1}} dr_{t+1} = 0. \tag{2.28}$$

We can easily verify the following relationship:

$$-\frac{\partial F}{\partial r_{t+1}} = F_2 > 0. \tag{2.29}$$

Because the above relationship holds in general (of course in a steady state, where $r_t = r_{t+1}$), function F can be expressed by an implicit theorem as $r_{t+1} = G(r_t)$.

Since $dr_{t+1}/dr_t = -\frac{\partial F}{\partial r_t}/\frac{\partial F}{\partial r_{t+1}}$ holds, at this stage we can investigate the movement of r_t.

Based on what we have seen so far, we understand that if F_1 is positive, our system can not have any circulating solution. Furthermore, it seems that the conditions controlling the way our system circulates are intensively related to the shape of an individual's utility function.

In order to verify this, we introduce the elasticity of marginal utility in relation to consumption, ρ,[3] which determines the shape of a utility function.

We can define ρ as

$$\rho = -\frac{U''[(1+r_t)\varphi(r_{t+1})](1+r_t)\varphi(r_{t+1})}{U'[(1+r_t)\varphi(r_{t+1})]}. \tag{2.30}$$

[3]In uncertainty, this concept equals the Arrow-Pratt measure.

Under this definition, we can express F_1 as

$$F_1 = U''[\phi(r_t) - \varphi(r_{t+1})]\phi'(r_t) - \beta(1-\rho)U'[(1+r_t)\varphi(r_{t+1})]. \tag{2.31}$$

(2.31) means that $F_1 > 0$, if $\rho \geqslant 1$. In this situation, $dr_{t+1}/dr_t = F_1/F_2 > 0$ holds, for all the regions of r defined. This implies that cyclical fluctuations never occur, if individuals do not have a greedy enough tendency in relation to consumption. This proposal seems, on appearance, to have no relationship with a careless tendency about future events, which is denoted as β. However, the two are positively associated.[4]

2.4 Occurrence of endogenous business cycles

2.4.1 A further specification of utility and production

Because we aim to investigate why endogenous business cycles occur, we assume $0 < \rho < 1$, below. In order to express (2.15) concretely, at this stage, we respectively specify the utility function and the production function below.

First, we assume there is constant elasticity of marginal utility in relation to consumption. This leads to an assumption that their utility functions are

[4]We will discuss this later.

determined in order to satisfy the following relationship:

$$\rho = -\frac{U''(e)e}{U'(e)}. \tag{2.32}$$

Here, e means consumption by an individual, and ρ is constant. When we solve (2.32) with respect to e, we obtain the following, which shows our utility function.[5]

$$U = \frac{c}{1-\rho}e^{1-\rho}. \tag{2.33}$$

Additionally, $0 < \rho < 1$, and c is positive and constant.

Secondly, we concretely assume a Cobb–Douglas type production function based on (2.5) as

$$y_t = f(k_t) = k_t^{\alpha}, \quad 0 < \alpha < 1. \tag{2.34}$$

(2.34) obviously satisfies the Inada conditions. In this situation, the following relationships are derived, based on (2.6), (2.7), (2.12) and (2.13):

$$s_t = \varphi(r_{t+1}) = (1+n)\alpha^{\frac{1}{1-\alpha}} \, r_{t+1}^{-\frac{1}{1-\alpha}}, \tag{2.35}$$

[5]A proof is as follows: If $U' = y$, we can establish the following differential equation:

$$\frac{dy}{de}e + \rho y = 0.$$

When we transform the above equation, we obtain the following relationship:

$$\int \frac{1}{y}dy + \rho \int \frac{1}{e}d = c'.$$

Solving the above equation, we obtain $ye^{\rho} = c$. Then, when we solve this differential equation with respect to e, supposing an integration constant as null, we finally obtain the requisite formula (2.33).

$$w_t = \phi(r_t) = (1-\alpha)\alpha^{\frac{\alpha}{1-\alpha}} r_t^{-\frac{\alpha}{1-\alpha}}. \tag{2.36}$$

Furthermore, the fundamental equation, (2.15), is modified based on (2.33) as follows:

$$\frac{(1+r_t)\varphi(r_{t+1})}{\phi(r_t)-\varphi(r_{t+1})} = [\beta(1+r_t)]^{\frac{1}{\rho}}. \tag{2.37}$$

Based on (2.35) through (2.37), (2.15) leads to the following:

$$r_{t+1} = G(r_t) = a\, r_t^{\alpha}[(1+r_t)^{-\frac{1-\rho}{\rho}} + b]^{1-\alpha}, \tag{2.38}$$

where a and b are, respectively, defined as

$$a = [\frac{b(1-\alpha)}{(1+n)\alpha}]^{-(1-\alpha)}, \tag{2.39}$$

$$b = \beta^{\frac{1}{\rho}}. \tag{2.40}$$

Now, we will investigate how function $G(r)$ behaves. In this procedure, for simplicity, we ignore the denotation of time below.

By a simple manipulation of (2.38), we obtain the following relationship:

$$\frac{dG}{dr} = ar^{\alpha-1}[(1+r)^{-\frac{1-\rho}{\rho}} + b]^{-\alpha}(\zeta_1 - \zeta_2), \tag{2.41}$$

where $\zeta_1(r)$ and $\zeta_2(r)$ are respectively defined as

$$\zeta_1(r) = \alpha[(1+r)^{-\frac{1-\rho}{\rho}} + b], \tag{2.42}$$

$$\zeta_2(r) = (1-\alpha)r^{\frac{1-\rho}{\rho}}(1+r)^{-\frac{1-\rho}{\rho}-1}. \tag{2.43}$$

The function $\zeta_1(r)$ has properties as follows:

$$\frac{d\zeta_1}{dr} = -\alpha\frac{1-\rho}{\rho}(1+r)^{-\frac{1}{\rho}} < 0, \tag{2.44}$$

$$\zeta_1(0) = \alpha(1+b),$$

$$\zeta_1(\infty) = \alpha b.$$

(2.44) and the subsequent two relationships show how $\zeta_1(r)$ behaves, as follows: It starts at $\alpha(1+b)$, decreases gradually, and then approaches αb, as r increases.

On the other hand, the following holds for the function $\zeta_2(r)$:

$$\frac{d\zeta_2}{dr} = (1-\alpha)\frac{1-\rho}{\rho}(1+r)^{-\frac{1-\rho}{\rho}-2}(1-\frac{1-\rho}{\rho}r), \tag{2.45}$$

$$\zeta_2(0) = 0,$$

$$\zeta_2(\infty) = 0.$$

(2.45) and the subsequent two relationships imply that $\zeta_{2(r)}$ behaves like a curve, which starts at 0, reaches a maximum at $r = \rho/(1-\rho)$, and then approaches null, as r increases.

Furthermore, at this stage, we will confirm the unique existence of a steady state. In a steady state, $r_t = r_{t+1} = r^*$ must be satisfied, where

$t = 1, 2, \ldots$. In this circumstance, r^* is determined by (2.38). Let us consider this more concretely. With $h(r^*)$, the function derived from(2.38) equals null, and the interest rate in a steady state is uniquely determined.

The following hold:

$$h(r^*) = r^* - a^{\frac{1}{1-\alpha}}(1 + r^*)^{-\frac{1-\rho}{\rho}} - ba^{\frac{1}{1-\alpha}}, \tag{2.46}$$

$$h'(r^*) = 1 + a^{\frac{1}{1-\alpha}}\frac{1-\rho}{\rho}(1 + r^*)^{-\frac{1-\rho}{\rho}-1} > 0, \tag{2.47}$$

$$h(0) = -a^{\frac{1}{1-\alpha}}(1 + b) < 0. \tag{2.48}$$

It is obvious that function h is monotonous. Additionally, in light of the three relationships from (2.46) through (2.48), we can easily obtain the conclusion required.

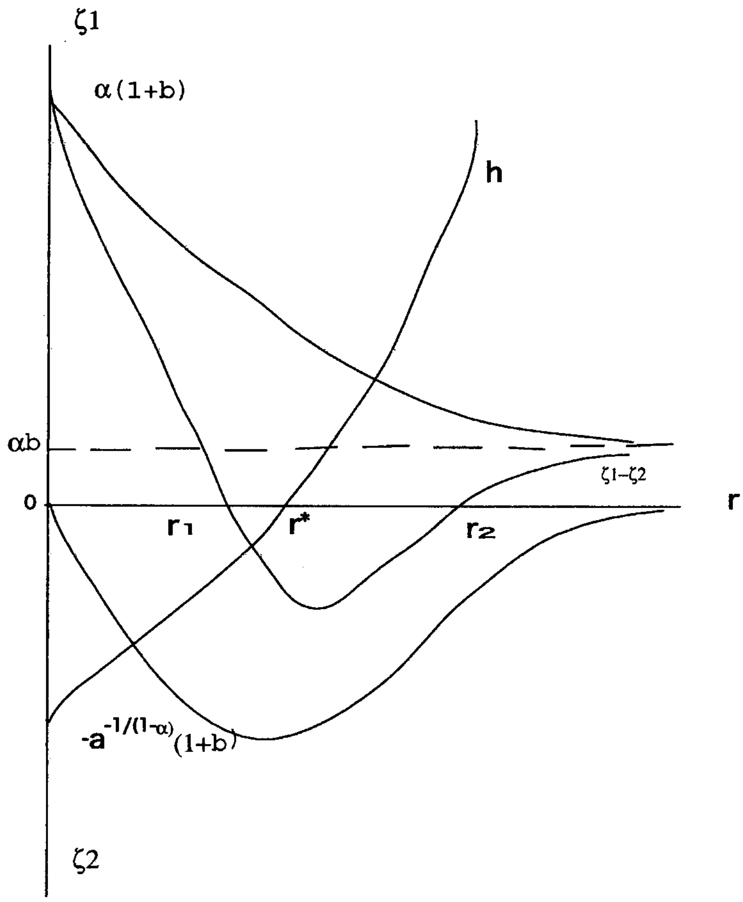

Figure 2.1

Based on the argument so far, it is obvious that the relationships depicted in Figure 2.1 should hold, if endogenous business cycles can occur. Therefore, it is our next aim to look for some conditions for the relationships.

Let us investigate what Figure 2.1 shows. It clearly conveys information about the shape of $G(r)$. As r increases, $G(r)$ increases to point r_1, decreases to point r_2, and then again increases over r_2. However, as r continues to increase and approaches infinity, $G(r)$ also approaches infinity, while the rate of increase in $G(r)$ is less than a unity. This implies that $G(r)$ is necessarily below the forty-five degree line, when r is sufficiently large.

The reason is as follows: In this circumstance, $\zeta_1(r) - \zeta_2(r)$ approaches a constant αb, while $r^{\alpha-1}$ and $(1+r)^{\frac{-(1-\rho)}{\rho}}$ approach null. This finally implies that dG/dr approaches null. On the other hand, the rate of increase in the forty-five degree line is clearly constant, a unity. Therefore, $G(r)$ is necessarily under the line when r is large enough.

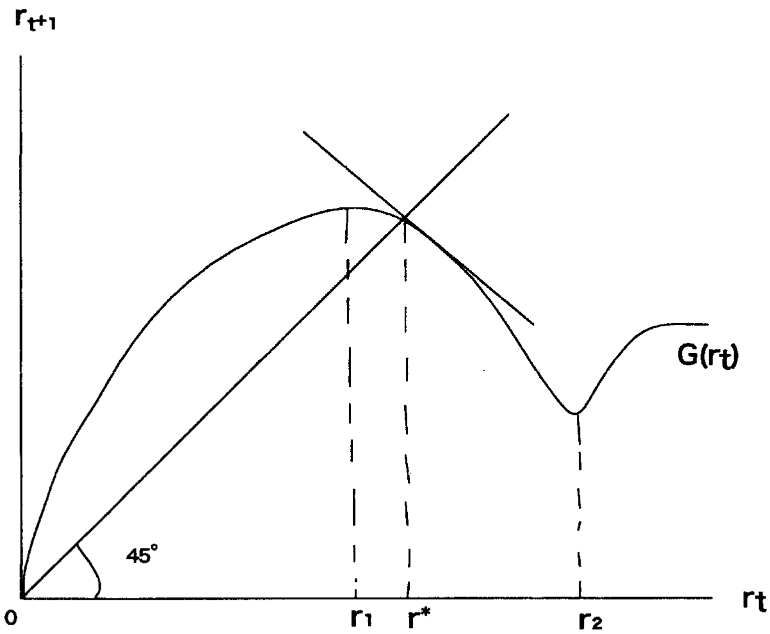

Figure 2.2

What Figure 2.1 shows is seen from another perspective in Figure 2.2. Based on Figure 2.2, it is obvious that endogenous business cycles necessarily occur, if the gradient of $G(r)$ in a steady state is less than -1. In other words, a gradient of less than -1 is a sufficient condition for endogenous business cycles. Additionally, it evidently holds from (2.38) that both $G(r)$ and the second map of $G(r)$, i.e. $G^2(r)$, are continuous with respect to r.

The reason is as follows:

In general, endogenous business cycles with two periods means that both $r_{t+1} = G(r_t)$ and $r_t = G(r_{t+1})$ hold. These lead to the following relation:

$$r_i = G^2(r_i), \tag{2.49}$$

where $i = 1, 2, \ldots$.

(2.49) implies that r_i can have four fixed points on G^2, two of which, p_1 and p_2, are stable, and our economy goes back and forth between these fixed points, p_1 and p_2. In other words, if $G(r)$ is as seen in Figure 2.2, then $G^2(r)$ must be as seen in Figure 2.3, in light of the relationships proved below.

Figure 2.3 shows the above situations.

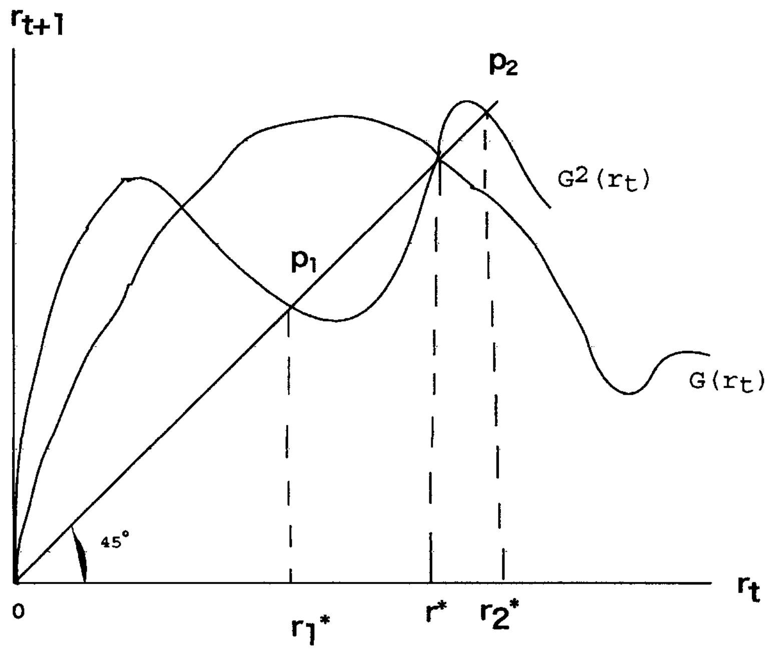

Figure 2.3

At this stage, we will demonstrate why, right of r_2^*, curve G^2 stays under forty-five degrees. From (2.38), we obtain the following:

$$\begin{aligned} G^2(r_t) &= aG(r_t)^\alpha[(1+G(r_t))^{-\frac{1-\rho}{\rho}}+b]^{1-\alpha} \\ &= a\ [a\ r_t^\alpha[(1+r_t)^{-\frac{1-\rho}{\rho}}+b]^{1-\alpha}]^\alpha \\ &\times [(1+[a\ r_t^\alpha[(1+r_t)^{-\frac{1-\rho}{\rho}}+b]^{1-\alpha}]^{-\frac{1-\rho}{\rho}})+b]^{1-\alpha}, \end{aligned}$$

which can be reformulated

$$\frac{G^2(r_t)}{r_t} = aG(r_t)^\alpha[(1+G(r_t))^{-\frac{1-\rho}{\rho}} + b]^{1-\alpha} \times \left(\frac{1}{r_t}\right)$$
$$= a\ [a\ r_t^\alpha[(1+r_t)^{-\frac{1-\rho}{\rho}} + b]^{1-\alpha}]^\alpha$$
$$\times\ [(1 + [a\ r_t^\alpha[(1+r_t)^{-\frac{1-\rho}{\rho}} + b]^{1-\alpha}]^{-\frac{1-\rho}{\rho}}) + b]^{1-\alpha} \times \left(\frac{1}{r_t}\right)^{1-\alpha}. \quad (2.50)$$

(2.50) evidently means that $\lim_{r_t \to \infty} G^2(r_t)/r = 0$, and this implies our anticipated result.

Second, we can confirm important properties between $G(r)$ and $G^2(r)$, as follows[6]

Theorem (1)

If $\bar{x}$ is a fixed point in G, $\bar{x}$ is also a fixed point in G^2.

Proof

By definition, $G(\overline{x})=\overline{x}$. Therefore, $G^2(\overline{x}) = G[G(\overline{x})] = G(\overline{x}) = \overline{x}$.

Theorem (2)

If $\overline{x}$ is a fixed point in G, $\dfrac{dG^2(\overline{x})}{dx} = [\dfrac{dG(\overline{x})}{dx}]^2$.

Proof

By definition, the following obviously hold:

[6] Azariadis (1993) gives similar proofs.

$$\frac{dG^2(\overline{x})}{dx} = \frac{dG[G(\overline{x})]}{dx} = G'[G(\overline{x})]G'(\overline{x}) = [G'(\overline{x})]^2.$$

We summarize the above logic as follows:

Based on the above relationships, we can understand Figure 2.3, that is, the local relationship between $G(r_t)$ and $G^2(r_t)$. When $G(r_t)$ can be shown as in Figure 2.2, it obviously follows from *Theorem (2)* that $G^2(r_t)$ is larger than $G(r_t)$, as r_t increases from null. And it follows from *Theorem (1)* that at r^*, $G^2(r_t)$ intersects with $G(r_t)$ and the gradient of $G^2(r^*)$ is larger than unity. This also follows from *Theorem (2)* and the assumption that the gradient of $G(r_t)$ in a steady state is less than −1. This implies that $G^2(r_t)$ is smaller than $G(r_t)$, as r_t increases through r^*. In light of the demonstration that in large r_t, $G^2(r_t)$ stays under the forty-five degree line, we eventually obtain Figure 2.3, which necessarily shows the existence of both p_1 and p_2.

Here, we can specifically show both r_1^* and r_2^* corresponding to p_1 and p_2, respectively.

2.4.2 Sufficient conditions

It is evident from Figure 2.3 that $\frac{dr_{t+1}}{dr_t} |_{r^*} = \frac{F_1}{F_2} < -1$ is a sufficient condition for endogenous business cycles, and this condition requires the following, at $r_t = r^*$:

$$F_1 + F_2 < 0. \tag{2.51}$$

Here, we investigate the values of both F_1 and F_2, at $r_t = r^*$ respectively.

We obtained the following relationship from (2.31):

$$F_1 = U''[\phi(r^*) - \varphi(r^*)]\phi'(r^*) - \beta(1-\rho)U'[(1+r^*)\varphi(r^*)]. \tag{2.52}$$

Futhermore, the following relationships hold:

$$\phi'(r^*) = -k^*. \tag{2.53}$$

$$U'[\phi(r^*) - \varphi(r^*)] = \beta U'[(1+r^*)\varphi(r^*)](1+r^*). \tag{2.54}$$

Taking the above relationships into account, we can transform F_1 as

$$\begin{aligned} F_1 &= -k^* U''[\phi(r^*) - \varphi(r^*)] - \beta(1-\rho)U'[(1+r^*)\varphi(r^*)] \\ &= -k^* U''[\phi(r^*) - \varphi(r^*)] - (1-\rho)\frac{U'[\phi(r^*) - \varphi(r^*)]}{1+r^*} \\ &= U'[\phi(r^*) - \varphi(r^*)][-\frac{U''(\phi(r^*) - \varphi(r^*))}{U'(\phi(r^*) - \varphi(r^*))}k^* - \frac{1-\rho}{1+r^*}]. \end{aligned} \tag{2.55}$$

Here, because by the definition of ρ, the following holds

$$-\frac{U''[\phi(r^*) - \varphi(r^*)]}{U'[\phi(r^*) - \varphi(r^*)]} = \frac{\rho}{\phi(r^*) - \varphi(r^*)}, \tag{2.56}$$

as a result, F_1 can be expressed as

$$F_1 = U'[\phi(r^*) - \varphi(r^*)](\frac{\rho k^*}{\phi(r^*) - \varphi(r^*)} - \frac{1-\rho}{1+r^*})$$
$$= c(e^1)^{-\rho}(\frac{\rho k^*}{\phi(r^*) - \varphi(r^*)} - \frac{1-\rho}{1+r^*}). \tag{2.57}$$

On the other hand, based on (2.26), we can express F_2 as

$$\begin{aligned} F_2 = & -\frac{\rho\, U'[\phi(r^*) - \varphi(r^*)]}{\phi(r^*) - \varphi(r^*)}\varphi'(r^*) \\ & - \beta(1+r^*)^2 \frac{\rho\, \varphi'(r^*) U'[(1+r^*)\varphi(r^*)]}{(1+r^*)\varphi(r^*)} \\ = & -\rho\, U'[\phi(r^*) - \varphi(r^*]\left[\frac{\varphi'(r^*)}{\phi(r^*) - \varphi(r^*)} + \frac{\varphi'(r^*)}{\varphi(r^*)}\right] \\ = & -\rho\, c(e^1)^{-\rho}\varphi'(r^*)\left(\frac{1}{e^1} + \frac{1}{\varphi(r^*)}\right) > 0. \end{aligned} \tag{2.58}$$

Here, we define $e^1 = \phi(r^*) - \varphi(r^*)$.

Furthermore, define $\psi = F_1 + F_2$, and it finally leads to the following:

$$\psi = c(e^1)^{-\rho}\frac{\rho}{1+r^*}(\triangle - \frac{1-\rho}{\rho}). \tag{2.59}$$

In addition, noticing $s'(r^*)/s(r^*) = -1/[(1-\alpha)r^*]$, we obtain the following relatonship:

$$\triangle = \frac{e^2}{e^1}(\frac{1}{1+n} + \frac{1}{(1-\alpha)r^*}) + \frac{1}{1-\alpha}(1+\frac{1}{r^*}). \tag{2.60}$$

Here, we denote $e^2 = (1+r^*)\varphi(r^*)$.

Our aim is to investigate how an individual's character traits affect the movements of our economy. In other words, it is our goal to find both ρ and β which make ψ negative.

2.4.3 ρ for cyclical movements

From what we have analyzed so far,[7] cyclical movements apparently occur, if F_1 is negative in a steady state. Because this is a necessary condition for endogenous business cycles, first of all, we begin our analysis here.

Based on (2.57), the following formula has to hold:

$$\frac{k^*(1+r^*)}{\phi(r^*) - \varphi(r^*)} < \frac{1-\rho}{\rho}. \tag{2.61}$$

Since $k^* = \varphi(r^*)/(1+n)$, so (2.61) eventually leads to the following :

$$\frac{(1+r^*)\varphi(r^*)}{\phi(r^*) - \varphi(r^*)}\frac{1}{1+n} = \frac{e^2(r^*)}{e^1(r^*)}\frac{1}{1+n} < \frac{1-\rho}{\rho}. \tag{2.62}$$

[7]See Figure 2.2.

(2.62) requires that for cyclical fluctuations to occur, the ratio of consumption by the old generation to that by the young generation must be lower than a certain rate, $(1-\rho)/\rho$.

On the other hand, the fundamental equation, (2.15), leads to the following:

$$\frac{(1+r^*)\varphi(r^*)}{\phi(r^*)-\varphi(r^*)} = [\beta(1+r^*)]^{\frac{1}{\rho}}. \tag{2.63}$$

(2.63) is the equation to determine the rate of interest in a steady state, when two parameters, β and ρ, which respectively show the degree of carelessness about future events and greed for consumption by an individual, are given.

At this stage, as a preparation for later analyses, we need to investigate the movement of r^* induced by ρ. First, we need to consider how this movement is depicted when only ρ moves, while β remains unchanged.

Here, we define the ratio of wages to savings in a steady state as

$$\eta(r^*) = \frac{\phi(r^*)}{\varphi(r^*)}. \tag{2.64}$$

In this situation, when we transform (2.63) as

$$\frac{1+r^*}{\eta(r^*)-1} = [\beta(1+r^*)]^{\frac{1}{\rho}}, \tag{2.65}$$

and put (2.65) into a logarithmic form, we obtain

$$\log(1+r^*) - \log[\eta(r^*) - 1] = \frac{1}{\rho}\log[\beta(1+r^*)].$$

Therefore, the following holds:

$$\frac{\partial r^*}{\partial \rho} = \frac{\log(\frac{1+r^*}{\eta(r^*)-1})}{\frac{1-\rho}{1+r^*} + \frac{\eta'(r^*)\rho}{\eta(r^*)-1}}. \tag{2.66}$$

Here, noticing that $\eta'(r^*) = \eta(r^*)/r^*$ and $(1+r^*)/(\eta(r^*)-1) = e^2/e^1$, we eventually obtain the following relationship:

$$\frac{\partial r^*}{\partial \rho} = \frac{\log(\frac{e^2}{e^1})}{\frac{1-\rho}{1+r^*} + \frac{\rho}{r^*}\left(1 + \frac{\frac{e^2}{e^1}}{1+r^*}\right)}. \tag{2.67}$$

In addition, we have used the following relationship:

$$\eta(r^*) = (\frac{e^1}{e^2})(1+r^*) + 1. \tag{2.68}$$

In (2.67), the denominator is evidently positive, because $0 < \rho < 1$. Therefore, it becomes apparent that the sign of $\partial r^*/\partial \rho$ is not determined uniquely, but done, instead, corresponding to $\left(\frac{e^2}{e^1}\right) \lesseqgtr$.

From the above analysis so far, we obtain the following conclusions:

(i) In the case of $\frac{e^2}{e^1} < 1, \quad \frac{\partial r^*}{\partial \rho} < 0.$

(ii) In the case of $\frac{e^2}{e^1} = 1, \quad \frac{\partial r^*}{\partial \rho} = 0.$

(iii) In the case of $\frac{e^2}{e^1} > 1, \quad \frac{\partial r^*}{\partial \rho} > 0.$

Furthermore, in preparation for later arguments, we need to investigate the relationship between e^2/e^1 and the interest rate r^* in a steady state. Based on (2.1), (2.2), (2.35), and (2.36) we obtain the following two relationships:

$$\begin{aligned} e^1 &= \phi(r^*) - \varphi(r^*) \\ &= \alpha^{\frac{1}{1-\alpha}}(r^*)^{-\frac{1}{1-\alpha}}[\frac{1-\alpha}{\alpha}r^* - (1+n)], \end{aligned} \tag{2.69}$$

and

$$\begin{aligned} e^2 &= (1+r^*)\varphi(r^*) \\ &= (1+r^*)(1+n)\alpha^{\frac{1}{1-\alpha}}(r^*)^{-\frac{1}{1-\alpha}}. \end{aligned} \tag{2.70}$$

Therefore, in a steady state, the following relationship holds:

$$\frac{e^2}{e^1} = \frac{(1+n)r^* + (1+n)}{(\frac{1-\alpha}{\alpha})r^* - (1+n)}, \tag{2.71}$$

which satisfies

$$\frac{d(\frac{e^2}{e^1})}{dr^*} = -\frac{(1+n)[(1+n)+\frac{1-\alpha}{\alpha}]}{[\frac{1-\alpha}{\alpha}r^* - (1+n)]^2} < 0. \tag{2.72}$$

Figure 2.4 shows (2.71).

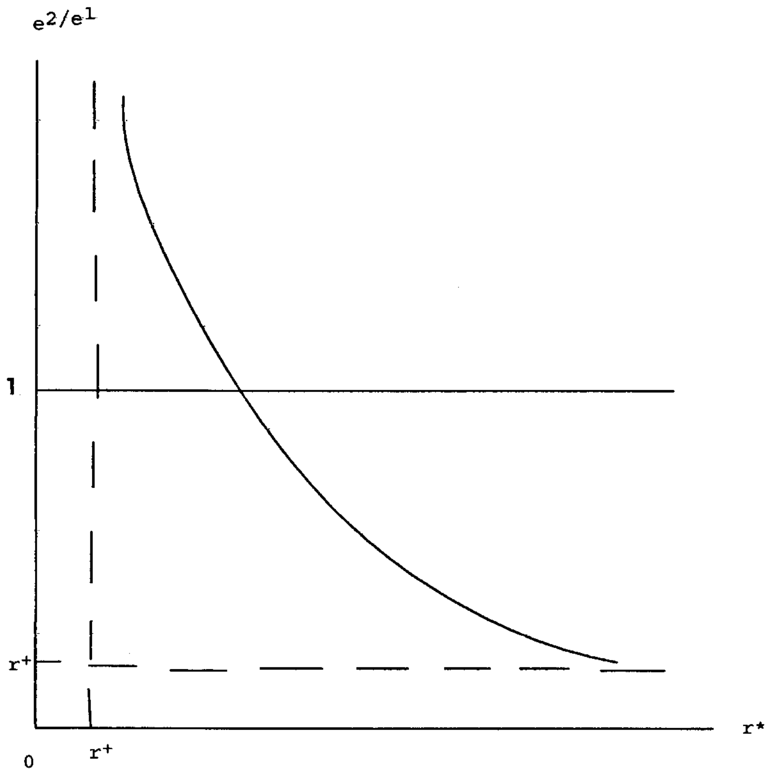

Figure 2.4

Figure 2.4 depicts a negative relation between e^2/e^1 and r^*, showing that there are two asymptotic lines, r^+ , to which both e^2/e^1 and r^* can ultimately approach.

Additionally, in general, $r^+ = [\alpha(1+n)]/(1-\alpha)$ can move from null through infinity, as α moves from null through 1. However, in this chapter, we assume $r^+ < 1$, because otherwise our economy has no cyclical movements, as is later shown. This condition refers to $\alpha < 1/(2+n)$.

Based on the arguments so far, the direction of r^*, when ρ moves, depends obviously on whether e^2/e^1 is larger than 1 or not. And (2.72) shows that the magnitude of e^2/e^1 is unproportional to that of r^*.

At this stage, we should remember key formulation (2.37), which can be expressed as

$$\frac{e^2}{e^1} = [\beta(1+r^*)]^{\frac{1}{\rho}}. \tag{2.73}$$

So, whether e^2/e^1 is larger than 1 or not depends obviously on whether the right-hand side of (2.73) is so or not. This determines whether $\beta(1+r^*)$ is larger than 1 or not. We know the relationship between ρ and r^*.

2.4.4 β for cyclical movements

Similarly, we need to analyze the movement between β and r^*.

We begin with (2.63). Transforming it, we obtained (2.65).

$$\frac{1+r^*}{\eta(r^*)-1} = [\beta(1+r^*)]^{\frac{1}{\rho}}.$$

The above equation can be expressed in terms of a logarithmic form as follows:

$$\log(1+r^*) - \log(\eta(r^*)-1) = \frac{1}{\rho}[\log\beta + \log(1+r^*)]. \tag{2.74}$$

Assuming that ρ is constant, we differentiate the above equation totally, and eventually obtain the following:

$$\frac{\partial r^*}{\partial \beta} = -\frac{1}{\beta}\frac{1+r^*}{1-\rho+\frac{\eta\rho}{r^*}(\frac{e^2}{e^1})} < 0. \tag{2.75}$$

2.4.5 The relationships between r^* and both ρ and β

It is obvious from (2.73) that the interest rate in a steady state, r^*, is determined to satisfy the following relationship:

$$\frac{e^2(r^*)}{e^1(r^*)} = [\beta(1+r^*)]^{\frac{1}{\rho}}. \tag{2.76}$$

(2.76) means that r^* is correspondingly determined when parameters β and ρ are given.

It may seem that r^* is determined when an arbitrary combination of β and ρ is given, but this is not true. β and ρ respectively indicate the degree of carelessness about future events and greed for consumption by individuals. Therefore, for example, individuals who are highly careless about future events are also highly greedy for consumption. In this light, we can say that if β is large, ρ is also large, and that if β is small, ρ is also small.

Additionally, (2.76) means that the steady interest rate satisfying $e^2/e^1 =$

1, that is, r^{**}, is determined independently of the value of ρ. In this situation, the value of β , that is, β^*, is determined by the following:

$$\beta^* = \frac{1}{1 + r^{**}}. \tag{2.77}$$

Based on what we have analyzed so far, it becomes obvious that the movements of ρ are not related to those of r^* uniquely, so we will analyze our economy in two cases.

First, in the case of $e^2/e^1 < 1$, (2.74) leads to the following relationship:

$$\left(\frac{1}{1+r^*} - \frac{\frac{d\eta}{dr^*}}{\eta - 1} - \frac{1}{\rho}\frac{1}{1+r^*}\right)dr^* = -\frac{d\rho}{\rho}\log\frac{e^2}{e^1} + \frac{1}{\rho}\frac{d\beta}{\beta} \tag{2.78}$$

Here, noticing $d\eta/dr^* = \eta/r^*$, we can transform the contents in the parentheses on the left hand, depicted as A, as follows:

$$A = \frac{1}{1+r^*}\frac{\rho - 1}{\rho} - \frac{\eta}{r^*(\eta - 1)} < 0, \tag{2.79}$$

because $\rho < 1$ and $\eta > 1$.

At this stage, we assume the movement of β and ρ as

$$\frac{d\beta}{\beta} = \frac{d\rho}{\rho} < 0. \tag{2.80}$$

On this assumption, the right-hand side of (2.78), which is expressed as B, is transformed as

$$B = -\frac{1}{\rho}\frac{d\beta}{\beta}[\log\beta(1+r^*) - 1] < 0. \tag{2.81}$$

In the above logic, we used $\log\beta(1+r^*) < 0$, because we assume that $e^2/e^1 < 1$, which naturally satisfies $\beta(1+r^*) < 1$.

In light of what we have analyzed above, we can conclude $dr^* > 0$.

We will discuss this logic in more detail. First, we assume certain values of ρ and β, that guarantee r^* under which $e^2/e^1 < 1$ holds. Second, we assume lower values of ρ and β than those, and then larger r^* holds, which results in less e^2/e^1. Of course this new e^2/e^1 remains less than unity, which also leads to $\beta\ (1+r^*) < 1$.

This means that the interest rate in a steady state becomes larger when both β and ρ are smaller. In other words, in this situation, it becomes clear that $e^2(r^*)/e^1(r^*)$ is smaller than 1 and decreases.

2.4.6 Endogenous business cycles and both ρ and β

Let us return to (2.59). We need to look for the conditions under which ψ becomes negative. The above relationship means that if both β and ρ are small enough, $e^2(r^*)/e^1(r^*)$ decreases, and $1/[(1-\alpha)r^*]$ and $(1+\frac{1}{r^*})$ similarly decrease. Furthermore it is obvious from (2.60) that this leads to the decrement of $\triangle$. On the other hand, in this situation, $(1-\rho)/\rho \to \infty$, as $\rho \to 0$. Therefore, $\psi < 0$ necessarily holds, and endogenous business cycles with two periods occur.

We can establish a significant theorem based on the above logic:

Theorem

If ρ and β are both small enough, endogenous business cycles with two periods occur.

Second, in the case of $\frac{e^2}{e^1} > 1$.

Because $e^2/e^1 > 1$ occurs when both ρ and β are large enough, contrary to (2.80), we need to investigate the movements of our economy on the following assumption:

$$\frac{d\beta}{\beta} = \frac{d\rho}{\rho} > 0. \tag{2.82}$$

In this case, the sign of the right-hand side in (2.78) is not certain, and therefore the movement of e^2/e^1 is also unsure. However, it is obvious that $e^2/e^1 > 1$ holds, if only both ρ and β are large enough.

We demonstrate this below.

Since $e^2/e^1 > 1$ is equal to $\beta[1 + r^*(\beta, \rho)] > 1$, we focus on $\beta[1 + r^*(\beta, \rho)]$.

We define the above relationship as

$$H = \beta(1 + r^*(\beta, \rho)). \tag{2.83}$$

Differentiating (2.83) totally, we obtain the following:

$$dH = \frac{\partial H}{\partial \beta} d\beta + \frac{\partial H}{\partial \rho} d\rho. \tag{2.84}$$

Here, the following holds:

$$\frac{\partial H}{\partial \beta} = 1 + r^*(\beta, \rho) - \beta \frac{1}{\beta} \frac{1 + r^*}{1 - \rho + \frac{\eta \rho}{r^*}(\frac{e^2}{e^1})}. \tag{2.85}$$

Transforming the above relationship, we obtain the following:

$$\frac{\partial H}{\partial \beta} = \frac{\rho(1 + r^*)(\phi(r^*) + r^* \varphi(r^*))}{r^*(1 - \rho)[\phi(r^*) - \varphi(r^*)] + \rho\, \phi(r^*)(1 + r^*)} > 0, \tag{2.86}$$

because $\rho < 1$.

On the other hand, the following holds:

$$\frac{\partial H}{\partial \rho} = \beta \frac{\partial r^*}{\partial \rho} > 0, \tag{2.87}$$

because $e^2/e^1 > 1$ and because of (iii) on page 53.

In these situations, $dH > 0$ holds and eventually $e^2/e^1 > 1$ obviously holds, as both ρ and β become larger. On the other hand, as ρ approaches unity, $(1-\rho)/\rho$ decreases sufficiently and to null, $F_1 < 0$, which is a necessary condition for cyclical fluctuations, is not satisfied.[8] Therefore, no business cycles evidently occur.

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[8] See (2.62).

2.5 Conclusion

Based on what we have carefully analyzed so far, we can conclude as follows: In an economy consisting of individuals with strongly careless and greedy characteristics about future events and consumption, which means small β and ρ, endogenous business cycles can occur, but in other cases they can not.

How do these characteristics influence the formation of savings? We know that our whole economy is fluctuating through the movements of k, which means the intensity of capital to labour. Since the capital in the current period results from the savings in the last period, and the labour supply is supposed to be exogenous, it is important for analyzing our fluctuation to know how capital moves. Furthermore, this leads to investigating how savings move. In short, at this stage, we can analyze the relationship between the formation of capital and the degree of careless and greedy character traits in terms of that between the formation of savings and the degree of careless and greedy character traits.

In our economy, based on (2.37), savings are determined on the assumption of stationary expectation regarding r, as

$$s_t = \frac{\beta^{\frac{1}{\rho}}(1+r_t)^{\frac{1-\rho}{\rho}} w_t}{1+\beta^{\frac{1}{\rho}}(1+r_t)^{\frac{1-\rho}{\rho}}}. \tag{2.88}$$

(2.88) means savings by individuals. If an arbitrary combination of (w_t, r_t), that is, an arbitrary k_t, is given, s_t is determined by (2.88) on the basis of stationary expectation with regard to r.

It is obvious that for the functions $G(r)$ and $G^2(r)$, (2.88) must necessarily hold, and that in this light, s_t is determined as two fixed points, p_1 and p_2 on $G^2(r)$.

Based on (2.88), we obtain the following:

$$\frac{\partial s_t}{\partial w_t} = \frac{\beta^{\frac{1}{\rho}}(1+r_t)^{\frac{1-\rho}{\rho}}}{1+\beta^{\frac{1}{\rho}}(1+r_t)^{\frac{1-\rho}{\rho}}} > 0. \tag{2.89}$$

$$\frac{\partial s_t}{\partial r_t} = w_t \frac{\frac{1-\rho}{\rho}\beta^{\frac{1}{\rho}}(1+r_t)^{\frac{1-\rho}{\rho}-1}}{[1+\beta^{\frac{1}{\rho}}(1+r_t)^{\frac{1-\rho}{\rho}}]^2}. \tag{2.90}$$

(2.89) means that individuals increase their current savings when they can earn large enough current wages, regardless of the degree of ρ. However, the sign of (2.90) depends on the degree of ρ. If $\rho > 1$, the sign is negative. That is, savings move in the opposite direction to the interest rate. However, if $\rho < 1$, both move in the same direction, and then, the sign of (2.90) is positive. This conclusion is in sharp contrast with Grandmont (1985). And this means that the substitution effect on savings caused by a change in the interest rate is larger than the income effect.

Furthermore, if ρ is sufficiently small, it becomes clear that only a change in the (anticipated) interest rate can affect savings, but a change in wages can not affect savings very much.

We can verify this as follows:

(2.89) can be transformed as

$$\frac{\partial s_t}{\partial w_t} = \frac{[\beta(1+r_t)]^{\frac{1}{\rho}}}{1 + r_t + [\beta(1+r_t)]^{\frac{1}{\rho}}}. \tag{2.91}$$

Suppose that ρ approaches null. This necessarily entails that β also approaches null. Then, $\frac{\partial s_t}{\partial w_t} \to 0$, because $\beta(1+r_t) < 1$, as $r_t \to \infty$.

The above relationship shows that w_t has little influence on the determination of s_t.

From the above conclusion, we can consider real savings to be determined only by the interest rate. Then, the following chain of logical relationships obviously result, among individual savings, the intensity of capital to labour, and the interest rate.

Initially, we assume $ds_t < 0$. In this situation, $dk_{t+1} < 0$, because $k_{t+1} = \dfrac{s_t}{1+n}$. Then, $dr_{t+1} > 0$ holds. In this situation, $ds_{t+1} > 0$.[9] In the same fashion, similar processes are repeated thereafter, that is, $dk_{t+2} > 0$ $\to dr_{t+2} < 0 \to ds_{t+2} < 0 \to \dots$.

In short, the above circulating movements show our endogenous business cycle with two periods.

Additionally, under our endogenous business cycle, the following obvi-

[9]Note that $\partial s_t / \partial r_t > 0$, because $\rho < 1$.

ously hold:

$$\frac{ds_{t+1}}{ds_t} = -1 \qquad \frac{dr_{t+1}}{dr_t} = -1 \qquad \frac{dk_{t+1}}{dk_t} = -1$$
$$\frac{dy_{t+1}}{dy_t} = -1 \qquad \frac{dw_{t+1}}{dw_t} = -1$$

$$t = 1, 2, \ldots.$$

In this chapter, we can conclude that if economic fundamentals remain unchanged, and if the characteristics of individuals can change, our economy can behave differently. Therefore, if we can, we should consider modifying certain of our character traits, not seeing them as inherent. In this context, if we are allowed to figure that business cycles are not good because they can cause a smaller amount of consumption per person than in stationary economies,[10] our policy should be devoted to encouraging people toward significant prudence. In this sense, Aesop's moral, that it is better to be a prudent ant than an imprudent grasshopper, still holds true.

[10]See Lucas (2003)

Chapter 3

How Do Fixed Labour Income Taxes Affect an Economy?

This chapter deals with whether taxation on labour income can change economic movements or not. Schmitt-Grohé and Uribe (1997) discussed this problem, based on the infinite horizon model which Ramsey (1928) originally established. They affirm that economic behaviour can change from a steady tendency to indeterminacy, if labour income tax rates change, and that it cannot cause endogenous fluctuations if labour income tax rates are fixed.

However, they are inaccurate in the manipulation of the optimal control theory and therefore fail to derive the labour supply, which a household should offer. We establish a new utility function instead of that in the above paper and explore their affirmation about fixed tax rates in detail, because in their paper it is stated without any proof. As a result, we conclude that changes in labour income tax can not cause instability in an economy, and conversely that the economy converges to a steady state. In this sense, chapter 3 offers a proof about their affirmation and follows them.

3.1 Introduction

In this chapter, we deal with whether an economy has a tendency towards stabilization or not, a conventional controversy, from the viewpoint of endogenous business cycles.

Endogenous business cycle theory shows us that if only parameters (not fundamentals) change, such as, for example, the elasticity of marginal utility in relation to consumption determining the shape of a utility function, the behaviour of an economy can differ drastically. In this situation, behavioral differences can range from a steady state to intense fluctuations, the occurrence of business cycles, indeterminacy or chaos.

As one factor behind the parameters in the above sense, we are interested in taxation. That is, we want to examine whether a change in taxation by governments might cause changes in economic movements, i.e., from a steady-state tendency to fluctuations, business cycles, indeterminacy or chaos.

Schmitt-Grohé and Uribe (1997) tries to answer this question on the basis of an infinite horizon model and balanced budget rules, resulting in an affirmation that a certain range of tax rates on labour income can cause intense economic fluctuations and indeterminacy, while other tax rates do not.

Apparently, not a few economists agree, and this theory seems to be widely accepted; we can list Utaka (2003) as a good example.[1]

However, we doubt their theory, especially from the viewpoint of their mathematical manipulations. Their model framework is fundamentally based on Ramsey (1928), but does not deal just with planned economies as Ramsey (1928) did, instead extending the scope to equilibria in markets. In this framework, they naturally try to deduce the supply of labour forces within their system,[2] but in vain, because their manipulation of the optimal control theory is not accurate, as explained in detail below.

In light of these arguments, we aim to prove why their logic is incorrect and instead offer a new model. In this chapter, we deal with labour income taxation, in which the tax rates on labour income are fixed and, correspondingly, government expenditure is endogenous, under the balanced budget rule. As a result, based on our new model, we conclude that this taxation on labour income can not cause any instability, but the economy tends to converge to a steady state. However, we also obtain a conclusion that taxation has other negative impacts on an economy, namely it reduces

[1]Utaka (2003) says: "Schmitt-Grohé and Uribe (1997) show that distortionary taxes can cause instability in the neoclassical growth model. In their model, government expenditures are constant and the tax rate is endogenous. Distortionary taxes make the labour demand curve upward sloping, which generates local determinacy". From this statement, Utaka (2003) seems to accept their theory.

[2]The neoclassical growth models like Solow (1956) often assume the supply of labour as determined exogenously, but they conversely determine it endogenously.

the scale of the economy, while leaving efficiency unchanged. More concretely speaking, the more labour income tax rates increase, the more sharply the economy decreases. Furthermore, leaving efficiency unchanged means that factor prices remain unchanged, before and after the taxation is introduced. This implies the existence of a maximum to government revenue financed by labour income tax, and we prove this.

The structure of this chapter is as follows:
In section 3.2, we point out some mistakes in Schmitt-Grohé and Uribe (1997). In what follows, in section 3.3, we refute Schmitt-Grohé and Uribe (1997), and subsequently, in section 3.4, offer our new model, which roughly follows the model in Schmitt-Grohé and Uribe (1997), except for an assumption about a utility function. Then we establish a system of dynamics in order to identify the time paths of our economy, in section 3.5. Furthermore, we demonstrate the existence and characteristics of an optimal path, in section 3.6. And, in section 3.7, we introduce a system of taxation on labour income and analyze its impacts on an economy, in particular focusing on movements. Finally, section 3.8 offers a conclusion.

3.2 Existing problems

The framework by Schmitt-Grohé and Uribe (1997) is as follows:
An economy consists of three sectors, namely households, firms and govern-

ments. These three sectors are each assumed to be represented by an agent.

First, let us consider a representative household. It possesses capital and loans it to a firm in exchange for rental fees, and additionally offers its labour to a firm. The population is constant, the rate of participation in the labour market is also assumed to be constant, and therefore a representative household determines only how many hours to work and how much capital to offer. This assumption is the same as that in classical economics.

The utility of a household at continuous time t, U_t, is defined as

$$U_t = \log C_t - AH_t. \tag{3.1}$$

Here, C_t means consumption, H_t working hours, and A a positive constant, respectively.

On the other hand, a household is assumed to live forever and to maximize the present value of its lifetime utility, which is expressed as

$$\int_0^\infty e^{-\rho t}(\log C_t - AH_t)dt. \tag{3.2}$$

Here, ρ means a subjective discount rate of utility.

Because a household determines its consumption and savings at every instant, under its budget constraints, and because saving becomes an increment of capital, the following holds:

$$\dot{K}_t = (u_t - \delta)K_t + (1 - \tau_t)w_t H_t - C_t. \tag{3.3}$$

Here, u_t means a rental price of capital, w_t a wage rate, δ a depreciation rate, $\dot{K}_t$ net investment, K_t capital stock, and τ_t labour income tax rate.

In short, the household maximizes (3.2) under (3.3), in which the household takes τ_t as a given. This problem can be solved by the optimal control theory.

The authors rely on a Hamiltonian function, R, for this problem as

$$R(\mu_t, C_t, H_t, K_t) = e^{-\rho t}(\log C_t - AH_t) + \mu_t[(u_t - \delta)K_t + (1 - \tau_t)w_t H_t - C_t]. \tag{3.4}$$

Additionally, μ_t means an adjoint variable.

The necessary conditions for the optimal control theory require the following:

$$\frac{\partial R}{\partial C_t} = e^{-\rho t}\frac{1}{C_t} - \mu_t = 0, \tag{3.5}$$

$$\frac{\partial R}{\partial H_t} = -e^{-\rho t}A + \mu_t(1 - \tau_t)w_t = 0, \tag{3.6}$$

$$\frac{d\mu_t}{dt} = -\frac{\partial R}{\partial K_t} = -\mu_t(u_t - \delta). \tag{3.7}$$

Based on (3.5), the following holds:

$$\mu_t = \frac{e^{-\rho t}}{C_t}, \tag{3.8}$$

which yields

$$\frac{\dot{\mu}_t}{\mu_t} = -\rho - \frac{\dot{C}_t}{C_t}. \tag{3.9}$$

Based on their argument so far, the following key relationship is obviously deduced:

$$\frac{\dot{C}_t}{C_t} = u_t - \rho - \delta. \tag{3.10}$$

Second, the authors introduce a production function F, called a Cobb-Douglas type, as

$$F(K_t, H_t) = K_t^{\alpha} H_t^{1-\alpha}. \tag{3.11}$$

Furthermore, they assume that the markets for labour, capital, and output are fully competitive, and that a firm aims to maximize its profit.

As a result, the following relationships hold:

$$w_t = \frac{\partial F}{\partial H} = F_H(K_t, H_t), \tag{3.12}$$

and

$$u_t = \frac{\partial F}{\partial K} = F_K(K_t, H_t). \tag{3.13}$$

Here, an output price is assumed to be 1.

We can consider the two relationships above as demand schedules for labour force and capital, respectively.

On the other hand, Walras' law requires the following:

$$C_t + G + \dot{K}_t + \delta K_t = F(K_t, H_t). \tag{3.14}$$

Here, G means government spending, which equals labour income taxes. As a consequence, the authors affirm that their economic system can be described by the following four equations:

$$\begin{cases} A = \dfrac{1}{C_t}(1-\tau_t)F_H(K_t, H_t), \\ -\dfrac{\dot{C}_t}{C_t} = \rho + \delta - F_K(K_t, H_t), \\ \dot{K}_t = F(K_t, H_t) - C_t - G - \delta K_t, \\ G = \tau_t F_H(K_t, H_t) H_t. \end{cases} \tag{3.15}$$

3.3 Refutations

The formulas in (3.15) can be arranged as follows:

$$\begin{cases} \dot{K}_t = u_t\left(\dfrac{K_t}{H_t}\right) K_t + (1-\tau_t) w_t \left(\dfrac{K_t}{H_t}\right) H_t - C_t - \delta K_t, \\ \dfrac{\dot{C}_t}{C_t} = u_t\left(\dfrac{K_t}{H_t}\right) - \delta - \rho. \end{cases} \tag{3.16}$$

Because both u_t and w_t are equilibrium prices in the markets, we explore how u_t is determined.

The demand for capital is derived from maximization of profit by a firm and u_t is equal to the marginal product of capital, $\partial F/\partial K_t$. Since the production function is assumed as in (3.11), it leads to $\alpha(\frac{H_t}{K_t})^{1-\alpha}$. In this situation, K_t is predetermined, so u_t is logically determined, if H_t is given.

The same discussion is applicable to how w_t is determined.

The demand for labour is also derived from a profit maximization policy by a firm, and w_t is equal to the marginal product of labour.

But how is the supply of labour determined ? Because the household is supposed to offer labour force, the supply of labour should be determined through the household's optimal principle, that is, by (3.5) through (3.7), which means a process of maximizing the Hamiltonian function. However, H_t does not appear in (3.5) through (3.7), and therefore the system composed by (3.5) through (3.7) cannot determine variables.

At this stage, we should return to (3.4). In (3.4), when we collect items concerning H_t, we obtain in the following:

$$[-Ae^{-\rho t} + \mu_t(1 - \tau_t)w_t]H_t.$$

The above relationship obviously shows that the Hamiltonian function R is linear with respect to H_t. In this situation, the optimal solution with respect to H_t is not an inner-point solution, but an end-point one. Nonetheless, the authors take the inner-point solution as the optimal one.

In other words, if $[-Ae^{-\rho t}+\mu_t(1-\tau_t)w_t] > 0$, the supply of labour force is determined at its maximum, if a maximum exists. Conversely, if $[-Ae^{-\rho t} +$

$\mu_t(1-\tau_t)w_t] < 0$, it is null. This implies that the authors actually postulate an economy that does not satisfy optimal conditions, that is, maximization of the present value of the lifetime utility of a household. This postulation is a serious error. More specifically, the assumption about the utility function $U_t = \log C_t - AH_t$ causes this mistake.

3.4 A new model

We define a new utility function as follows:

$$U_t = \log C_t + A \log (L - H_t). \tag{3.17}$$

Here, L means maximum hours for leisure which a household can utilize and is a positive constant.

In other words, we assume that a household can enjoy both consumption and leisure. It follows that we ignore depreciation for simplicity, because it does not affect our theory in the context below, and other assumptions in the original paper by Schmitt-Grohé and Uribe (1997) are roughly used in our analyses. But to maintain uniformity of symbols we denote a subjective discount rate of utility as γ instead of ρ, below.

In this circumstance, the problem of the choices made by a household is modified as

$$\max_{C_t, Ht} \int_0^\infty e^{-\gamma t}[\log C_t + A \log (L - H_t)]dt, \tag{3.18}$$

subject to

$$\dot{K}_t = u_t K_t + (1 - \tau_t) w_t H_t - C_t. \tag{3.19}$$

The household takes factor prices u_t and w_t as given, and chooses consumption and the supply of labour force. In other words, in this situation, the factor prices mean supply prices in the factor markets.

The Hamiltonian function, R, is defined as

$$R = e^{-\gamma t}[\log C_t + A \log (L - H_t)] \\ + \mu_t [u_t K_t + (1 - \tau_t) w_t H_t - C_t]. \tag{3.20}$$

Concerning control variables, the following have to hold:

$$\frac{\partial R}{\partial C_t} = e^{-\gamma t} \frac{1}{C_t} - \mu_t = 0,$$

which yields

$$\mu_t = \frac{e^{-\gamma t}}{C_t}. \tag{3.21}$$

$$\frac{\partial R}{\partial H_t} = e^{-\gamma t} A \frac{-1}{L - H_t} + \mu_t (1 - \tau_t) w_t = 0,$$

which yields

$$H_t = L - \frac{A C_t}{(1 - \tau_t) w_t}. \tag{3.22}$$

(3.22) means there is a supply curve of labour force, which was lacking in the original paper.

Furthermore, the following must hold:

$$\frac{d\mu_t}{dt} = -\frac{\partial R}{\partial K_t} = -\mu_t u_t,$$

which yields

$$\frac{\dot{\mu}_t}{\mu_t} = -u_t. \tag{3.23}$$

Considering both (3.21) and (3.23), we obtain the following relationship, called an Euler equation:

$$\frac{\dot{C}_t}{C_t} = u_t - \gamma. \tag{3.24}$$

The Euler equation plays a key role in our future analyses, showing how a household chooses consumption.

In addition, since function R is concave with respect to C_t, H_t, μ_t and K_t, the sufficient conditions for the objective function are satisfied.[3]

3.5 Dynamics

3.5.1 Optimization of a firm

Because a firm is supposed to aim to maximize its profit, and a production function is assumed by (3.11), the following hold as a result:

$$w_t = (1 - \alpha)k_t^{\alpha}, \tag{3.25}$$

[3]See Koyama (1995)

$$u_t = \alpha k_t^{-(1-\alpha)}. \tag{3.26}$$

Here, $k_t \stackrel{\text{def}}{=} \frac{K_t}{H_t}$, which means the capital-labour ratio. Additionally, note that in this situation, a firm takes factor prices u_t and w_t as given. In other words, these mean demand prices for the factors.

So far, the factor prices are arbitrary, that is, the ones which a household and a firm are respectively anticipating. These can be called demand prices or supply prices.

However, at this stage, we assume equilibrium prices in competitive markets. In short, we do not assume a command economy as Ramsey (1928) originally supposed, but a market economy. The reason is as follows:

The prices w_t and u_t in (3.22) and (3.24) indicate the ones anticipated by a household. On the other hand, the prices w_t and u_t in (3.25) and (3.26) indicate the ones anticipated by a firm. Equilibrium prices mean the ones at which the former match the latter. We can say that the equilibria hold at every instant on the assumption of perfect foresight.
As a preparation for later discussions, we should derive a relationship. (3.25) holds for arbitrary time, and this means that we can consider (3.25) as an identity with respect to time. Therefore, we can differentiate (3.25) with respect to time, so the following holds:

$$\dot{w}_t = \alpha(1-\alpha)k_t^{\alpha-1}\dot{k}_t. \tag{3.27}$$

3.5.2 Dynamics of the system

By definition, since $\dot{k}_t = \frac{\dot{K}_t}{H_t} - \frac{K_t}{H_t}\frac{\dot{H}_t}{H_t}$ holds, by substituting (3.19) into the above definition, we obtain the following:

$$\dot{k}_t = u_t \frac{K_t}{H_t} + (1 - \tau_t) w_t - \frac{C_t}{H_t} - k_t \frac{\dot{H}_t}{H_t}. \tag{3.28}$$

At this stage, differentiating (3.22) with respect to t, we obtain the following,

$$\dot{H}_t = -\frac{A}{1 - \tau_t} \frac{\dot{C}_t w_t - C_t \dot{w}_t}{w_t^2},$$

from which the following holds:

$$\frac{\dot{H}_t}{H_t} = \frac{-A\left(\frac{\dot{C}_t}{C_t} - \frac{\dot{w}_t}{w_t}\right)}{L(1 - \tau_t)\frac{w_t}{C_t} - A}. \tag{3.29}$$

The above relationship indicates a rate of change in the labour supply. As a consequence, a rate of change in the capital-labour ratio is expressed as

$$\dot{k}_t = u_t k_t + (1-\tau_t) w_t - \frac{(1 - \tau_t) w_t C_t}{(1 - \tau_t) w_t L - A C_t} + A k_t \frac{\frac{\dot{C}_t}{C_t} - \frac{\dot{w}_t}{w_t}}{L(1 - \tau_t)\frac{w_t}{C_t} - A}. \tag{3.30}$$

Here, we respectively define the following:

$$\begin{cases} w_t = \eta(k_t), \\ \eta(k_t) \stackrel{\text{def}}{=} (1-\alpha)k_t^{\alpha}, \end{cases} \tag{3.31}$$

and

$$\begin{cases} u_t = \phi(k_t), \\ \phi(k_t) \stackrel{\text{def}}{=} \alpha k_t^{\alpha-1}. \end{cases} \tag{3.32}$$

Therefore, in light of equilibrium prices, we can transform (3.30) into the following:

$$\begin{aligned} \dot{k}_t = \phi(k_t)k_t + (1-\tau_t)\eta(k_t) - \frac{(1-\tau_t)\eta(k_t)C_t}{(1-\tau_t)\eta(k_t)L - AC_t} \\ + \frac{Ak_t[\frac{\dot{C}_t}{C_t} - \frac{\dot{\eta}(k_t)}{\eta(k_t)}]C_t}{L(1-\tau_t)\eta(k_t) - AC_t}, \end{aligned}$$

which leads to

$$\begin{aligned} \dot{k}_t = \phi(k_t)k_t + (1-\tau_t)\eta(k_t) - \frac{(1-\tau_t)\eta(k_t)C_t}{(1-\tau_t)\eta(k_t)L - AC_t} \\ + \frac{Ak_t[-\rho + \phi(k_t) - \frac{\alpha \dot{k}_t}{k_t}]C_t}{L(1-\tau_t)\eta(k_t) - AC_t}. \end{aligned} \tag{3.33}$$

Furthermore, transforming (3.33), we obtain the following:

$$\begin{aligned}\dot{k}_t[L(1-\tau_t)\eta(k_t) - AC_t + A\alpha C_t] &= [L(1-\tau_t)\eta(k_t) - AC_t]\phi(k_t)k_t \\ &\quad + [L(1-\tau_t)\eta(k_t) - AC_t](1-\tau_t)\eta(k_t) \\ &\quad - (1-\tau_t)\eta(k_t)C_t + AC_t[-\rho + \phi(k_t)]k_t.\end{aligned} \tag{3.34}$$

Yet, $H_t \geqq 0$ must hold, so it follows from (3.22) that $L(1-\tau_t)\eta(k_t)-AC_t \geqq 0$. Therefore, $L(1-\tau_t)\eta(k_t) - A(1-\alpha)C_t > 0$ holds.

As a result, our economic system is expressed by the following:

$$\begin{aligned}\dot{k}_t &= \frac{[L(1-\tau_t)\eta(k_t) - AC_t][\phi(k_t)k_t + (1-\tau_t)\eta(k_t)]}{L(1-\tau_t)\eta(k_t) - A(1-\alpha)C_t} \\ &\quad - \frac{(1-\tau_t)\eta(k_t)C_t - AC_t[-\gamma + \phi(k_t)]k_t}{L(1-\tau_t)\eta(k_t) - A(1-\alpha)C_t},\end{aligned} \tag{3.35}$$

$$\dot{C}_t = [\phi(k_t) - \gamma]C_t. \tag{3.36}$$

Considering (3.31) and (3.32), we can transform the above system into the following simultaneous equations system with respect to k_t and C_t.

$$\begin{aligned}\dot{k}_t &= \frac{L(1-\tau_t)(1-\alpha)[\alpha + (1-\tau_t)(1-\alpha)]k_t^{2\alpha}}{(1-\alpha)[L(1-\tau_t)k_t^{\alpha} - AC_t]} \\ &\quad - \frac{(A+1)(1-\tau_t)(1-\alpha)C_t k_t^{\alpha} + A\gamma C_t k_t}{(1-\alpha)[L(1-\tau_t)k_t^{\alpha} - AC_t]},\end{aligned} \tag{3.37}$$

$$\dot{C}_t = (\alpha k_t^{-(1-\alpha)} - \gamma)C_t. \tag{3.38}$$

3.5.3 A steady state

Here, we confirm the existence of a steady state in the above simultaneous equations system, and define $k = k^*$, $C_t = C^*$, and $\tau_t = \tau$, for any t.

In this situation, when we place $\dot{C}_t = 0$ in (3.38), we obtain the capital-labour ratio in the steady state, k^*, as

$$k^* = \left(\frac{\alpha}{\gamma}\right)^{\frac{1}{1-\alpha}}. \tag{3.39}$$

The above relationship means that the capital-labour ratio at the steady state is not affected by labour tax rates. We can call this property a *Neutral Theorem* of capital intensity on tax rates.

Similarly, when we place $\dot{k}_t = 0$ in (3.37), the following holds:

$$[(A+1)(1-\tau)(1-\alpha)(k^*)^{\alpha-1} + A\gamma]C^* =$$
$$L(1-\tau)(1-\alpha)[\alpha + (1-\tau)(1-\alpha)](k^*)^{2\alpha-1}.$$

Eventually, we obtain the consumption in the steady state, C^*, as

$$C^* = \frac{L(1-\tau)(1-\alpha)\left(\frac{\gamma}{\alpha}\right)^{\frac{\alpha}{\alpha-1}+1}[\alpha + (1-\tau)(1-\alpha)]}{(A+1)(1-\tau)(1-\alpha)\left(\frac{\gamma}{\alpha}\right) + A\gamma}. \tag{3.40}$$

Since both the denominator and the numerator are positive, obviously $C^* > 0$.

Now, we can depict the global movements of our economy, which have been mathematically shown in (3.37) and (3.38). It is convenient for the identification of the movements to utilize a phase diagram on (k, C) axes. Let us subsequently analyze the directions in time of C_t and k_t, respectively.

3.5.4 The movements of C_t

(3.38), an Euler equation, shows the time paths of C_t. Based on (3.38) we obtain Figure 3.1.

The $dC/dt = 0$ locus is expressed by a vertical line at k^*. Consumption is increasing to the left of the locus, where $k_t < k^*$, because in this situation $u_t > \gamma$, so a household can feel more comfortable about consumption in future. Conversely, it is decreasing to the right of the locus, where $k_t > k^*$, because in $u_t < \gamma$, a household feels less comfortable about consumption in future. Vertical arrows show these directions. Of course, at the locus consumption remains unchanged.

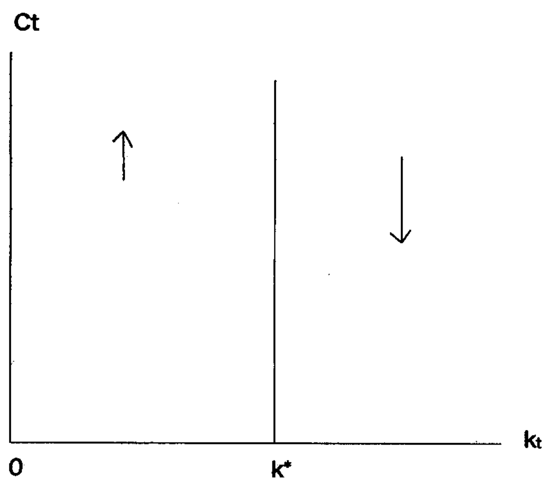

Figure 3.1

3.5.5 The movements of k_t

We define a and b as

$$\begin{cases} a = L(1-\tau_t)(1-\alpha)[\alpha + (1-\tau_t)(1-\alpha)], \\ b = (A+1)(1-\tau_t)(1-\alpha). \end{cases}$$

Using these definitions, we can express (3.37) as

$$\dot{k}_t = \frac{ak_t^{2\alpha} - bC_t k_t^{\alpha} - \gamma A C_t k_t}{(1-\alpha)[L(1-\tau_t)k_t^{\alpha} - AC_t]}. \tag{3.41}$$

In this situation, first, we explore the $dk/dt = 0$ locus. So, the following must hold:

$$k_t = 0,$$

or

$$ak_t^{2\alpha-1} - bC_t k_t^{\alpha-1} - \gamma A C_t = 0.$$

We should focus on the latter relationship, in order to identify the $dk/dt = 0$ locus in general, and therefore transform it as

$$C_t = \frac{ak_t^{2\alpha-1}}{bk_t^{\alpha-1} + \gamma A}. \tag{3.42}$$

Additionally, $bk_t^{\alpha-1} + \gamma A > 0$.

We need to find the shape of (3.42), and therefore interpret (3.42) as a function, in which k_t works as an independent variable and C_t as a dependent one. Following this idea, we differentiate C_t with respect to k_t, in order to identify the (k_t, C_t) relationships. (This is applicable below.)

We obtain the following:

$$\frac{dC_t}{dk_t} = \frac{ak_t^{2(\alpha-1)}[b\alpha k_t^{\alpha-1} + (2\alpha-1)\gamma A]}{(bk_t^{\alpha-1}+\gamma A)^2}. \tag{3.43}$$

Here, based on the numerator in (3.43), we define the following:

$$f(k_t) = b\alpha k_t^{\alpha-1} + (2\alpha-1)\gamma A, \tag{3.44}$$

which has properties such as

$$\begin{cases} f' = b\alpha(\alpha-1)k_t^{\alpha-2} < 0, \\ f'' = b\alpha(\alpha-1)(\alpha-2)k_t^{\alpha-3} > 0, \\ f(0) = \infty, \\ \lim_{k_t\to\infty} f(k_t) = (2\alpha-1)\gamma A. \end{cases} \tag{3.45}$$

Based on what we have analyzed so far, we can describe the shape of (3.44) as follows:

First, in the case of $\alpha \geqq \frac{1}{2}$, $\frac{dC_t}{dk_t} > 0$ and $\frac{d^2C_t}{dk_t^2} < 0$. (The proof is offered in ***Appendix*** 1)

Additionally, because of $2\alpha - 1 \geqq 0$, the following hold:[4]

$$\begin{cases} C_t(0) = 0, \\ \lim_{k_t \to \infty} C(k_t) = \infty. \end{cases}$$

Second, in the case of $0 < \alpha < \frac{1}{2}$, the situation is divided into the following two categories:
In the case of $0 < k_t \leqq \left(\frac{1-2\alpha}{\alpha}\right)^{\frac{1}{\alpha-1}} \left(\frac{\gamma A}{b}\right)^{\frac{1}{\alpha-1}}$, $\frac{dC_t}{dk_t} \geqq 0$, and conversely, in the case of $k_t > \left(\frac{1-2\alpha}{\alpha}\right)^{\frac{1}{\alpha-1}} \left(\frac{\gamma A}{b}\right)^{\frac{1}{\alpha-1}}$, $\frac{dC_t}{dk_t} < 0$ hold, respectively.

In addition, the following hold:[5]

$$\begin{cases} C_t(0) = 0, \\ \lim_{k_t \to \infty} C_t(k_t) = 0. \end{cases}$$

Based on what we have analyzed so far, we can depict two phase diagrams as follows:

First, in the case of $\frac{1}{2} \leqq \alpha < 1$.

---

[4]Note that $C_t = \dfrac{a k_t^{2\alpha-1}}{\dfrac{b}{k_t^{1-\alpha}} + \gamma A}$.

[5]We can transform (3.42) into $C_t = \dfrac{a}{\dfrac{b}{k_t^{\alpha}} + \gamma A k_t^{-(2\alpha-1)}}$.

In this circumstance, when $k_t \to 0$, $\frac{b}{k_t} \to \infty$ and $\gamma A k_t^{-(2\alpha-1)} \to 0$. Therefore, when $k_t \to 0$, $C_t(k_t) \to 0$.
In the same fashion, when $k_t \to \infty$, $C_t(k_t) \to 0$.

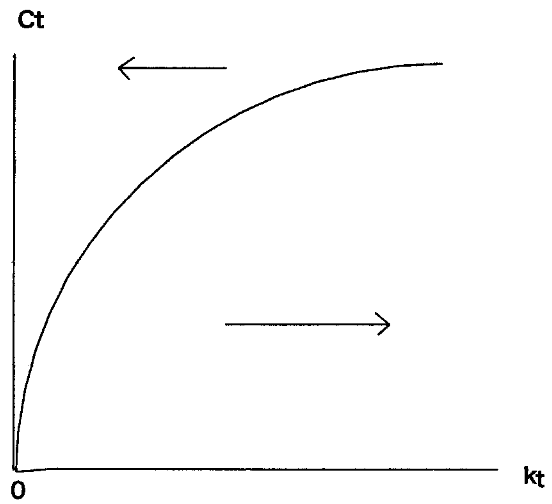

Figure 3.2

The locus $dk/dt = 0$ starts from the origin and increases as k_t increases. Anywhere above the $dk/dt = 0$ locus, the capital-labour ratio decreases, because consumption is too large to leave the capital-labour ratio unchanged. Conversely, anywhere below the $dk/dt = 0$ locus, it increases, because consumption is too small to leave it unchanged. Of course, on the $dk/dt = 0$ locus, the ratio remains unchanged. The horizontal arrows in Figure 3.2 demonstrate these directions of motion.

Second, in the case of $0 < \alpha < \frac{1}{2}$.

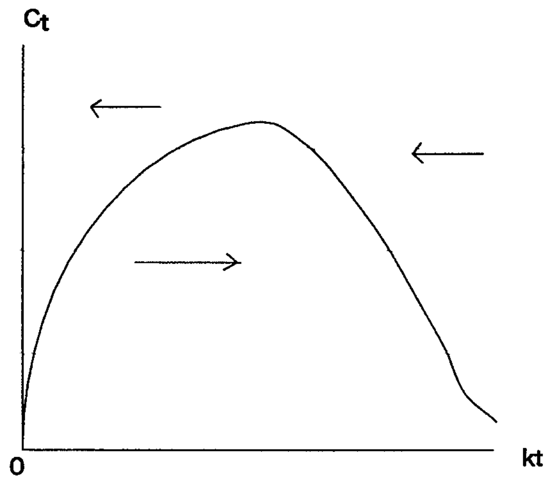

Figure 3.3

The locus $dk/dt = 0$ also starts at the origin, increases until the point $k_t = \left(\frac{1-2\alpha}{\alpha}\right)^{\frac{1}{\alpha-1}} \left(\frac{\gamma A}{b}\right)^{\frac{1}{\alpha-1}}$, and thereafter decreases and approaches null. The horizontal arrows in Figure 3.3 show the same movements as those in Figure 3.2.

3.6 The existence and characteristics of an optimal path

Based on the above argument, we can finally identify the global time paths, utilizing phase diagrams in Figure 3.1, 3.2, and 3.3, drawn on (k, C) axes.

We obtain Figure 3.4 in the case of $\frac{1}{2} \leqq \alpha < 1$, and Figure 3.5 in the case of $0 < \alpha < \frac{1}{2}$, respectively.[6]

[6]The discussion below holds under the condition that $H > 0$, which naturally leads to

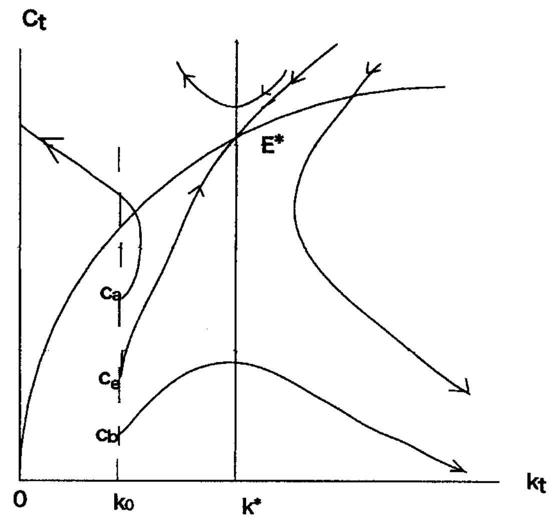

Figure 3.4

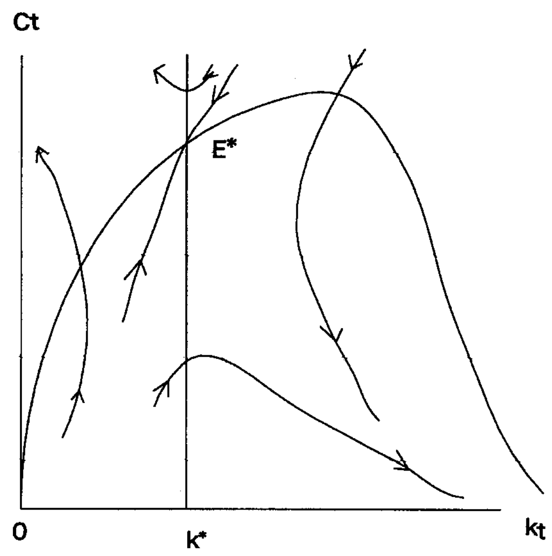

Figure 3.5

In what follows, we consider only the case of Figure 3.4, because the logic of

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$C_t < \frac{L(1-\alpha)(1-\tau_t)}{A} k_t^{\alpha}$. In this sense, in fact, the trajectories with increasing C_t do not exist.

Figure 3.4 is exactly applicable to that of Figure 3.5.

Point E^* means a steady state, where both k and C remain unchanged, and it refers to a situation where consumption, capital stock and the amount of labour in production all are still kept constant. This noticeably means no investment. In short, E^* indicates a classical equilibrium.

In our economy, initial capital stock, K_0, is historically given. So, if initial consumption C_0 is determined, time paths of all economic variables such as K_t, C_t, H_t, u_t and w_t are determined.
We can depict possible time paths in Figure 3.4 on the (k, C) axes.

Time paths can be divided into three categories.
First, suppose that K_0 is given. In this circumstance, in our system, H_0 and C_0 are not uniquely determined in general. Therefore, in addition, suppose H_0 is given and then k_0 is determined. Second, postulate that the corresponding C_0 can be in three points like C_b, C_e and C_a, as in Figure 3.4.

When an economy starts from C_b, both k_t and C_t increase, and after k_t reaches k^*, k_t continues to increase, while C_t asymptotically decreases and approaches the horizontal axis.[7]
And when it starts from C_a, both k_t and C_t increase to some extent, but afterwards k_t declines and finally reaches the vertical axis, while C_t continues to increase.[8]

If the economy starts from C_e, it converges to point E^*.

[7]See the proof in Appendix 2.
[8]See the proof in Appendix 3.

Furthermore, there can be any number of paths on the (k, C) axes and we know that these paths on the (k, C) axes can not intersect each other. (See Koyama (1995))

In this situation, what condition concretely determines our time path uniquely?

Returning to the dynamic budget constraint, (3.19), we can notice that it just shows how much a household allots to savings or consumption with its disposable income in every instance, and that this budget constraint does not necessarily mean its lifetime budget constraint, that is, its intertemporal budget constraint.

In short, we need to convert it into a lifetime budget constraint, under which the lifetime utility of a household should be maximized. It is not a dynamic budget constraint but an intertemporal one that, in our context, we have to consider as a real budget constraint for consumption and leisure.

However, on the assumption that no household has debt, and in addition to the dynamic budget constraint, if a transversality condition holds, which requires that all capital at a final end, that is, at infinity, be entirely consumed, the required intertemporal budget constraint is satisfied.

The proof is as follows:

From (3.14) we have obtained the dynamic budget constraint as

$$\dot{K}_t = u_t K_t + (1 - \tau_t) w_t H_t - C_t.$$

When we integrate the above relationship from time null through infinity, the following holds:[9]

$$\int_0^\infty \dot{K}_t dt + \int_0^\infty C_t \, dt = \int_0^\infty u_t K_t \, dt + \int_0^\infty (1-\tau_t) w_t H_t \, dt, \tag{3.46}$$

and (3.46) should be transformed as follows, when we evaluate (3.46) at time null:

$$\int_0^\infty C_t \, exp\,[-\int_0^t u_\nu \, d\nu]\, dt + \int_0^\infty \dot{K}_t \, exp\,[-\int_0^t u_\nu \, d\nu]\, dt = \int_0^\infty (1-\tau_t) w_t H_t \, exp\,[-\int_0^t u_\nu \, d\nu]\, dt + \int_0^\infty u_t \, K_t \, exp\,[-\int_0^t u_\nu \, d\nu]\, dt. \tag{3.47}$$

Then, the second term on the left side in (3.47) can be transformed as

$$\int_0^\infty \dot{K}_t \, exp\,[-\int_0^t u_\nu \, d\nu]\, dt = [K_t \, exp\,[-\int_0^t u_\nu \, d\nu]]_0^\infty + \int_0^\infty K_t \, u_t \, exp[-\int_0^t u_\nu \, d\nu]\, dt, \tag{3.48}$$

which yields

$$\int_0^\infty \dot{K}_t \, exp\,[-\int_0^t u_\nu \, d\nu]\, dt = -K_0 + \int_0^\infty u_t K_t \, exp[-\int_0^t u_\nu \, d\nu]\, dt. \tag{3.49}$$

---

[9]Notice that τ_t is supposed to be constant.

Note that the transversaliy condition which we require means the following:

$$K_\infty \; exp[-\int_0^\infty \; u_\nu \; d\nu] = 0.$$

As a result, in light of the above relationship, we can transform (3.47) as

$$\int_0^\infty C_t \; exp \; [-\int_0^t u_\nu \; d\nu] \; dt - K_0 + \int_0^\infty u_t \; K_t \; exp \; [-\int_0^t u_\nu \; d\nu] \; dt =$$
$$\int_0^\infty (1-\tau_t) w_t H_t \; exp \; [-\int_0^t u_\nu \; d\nu] \; dt + \int_0^\infty u_t \; K_t \; exp \; [-\int_0^t u_\nu \; d\nu] \; dt,$$

which finally yields

$$\int_0^\infty C_t \; exp[-\int_0^t \; u_\nu \; d\nu] \; dt = K_0 + \int_0^\infty (1-\tau_t) w_t H_t \; exp[-\int_0^t \; u_\nu \; d\nu] \; dt. \tag{3.50}$$

The above relationship is the very condition we explore as a lifetime budget constraint.

(3.50) means that the present value of the consumption enjoyed by a household over infinity is equal to initial capital plus the present value of disposable labour income over infinity. This is what an intertemporal budget constraint actually means.

The transversality condition can be also expressed as follows:

$$\lim_{t \to \infty} \mu_t k_t = 0, \tag{3.51}$$

which in this context, yields

$$\lim_{t\to\infty} \frac{k_t}{C_t} e^{-\gamma t} = 0. \tag{3.52}$$

(3.51) means that the present value of the marginal utility of consumption at infinity in terms of capital stock, which can be transformed into consumption at perfect substitution, must be null. In this sense, the consumption at infinity means that of capital stock at that time. In other words, this context means that a household consumes its capital entirely when it dies.[10]

Now focus a saddle point E^*. The saddle path that converges to E^* obviously satisfies (3.52). Furthermore, because trajectories can not intersect each other, this trajectory is the only path which our economy allows, in light of our intertemporal budget constraint.

Since E^* is a saddle point, our economy converges to E^* in the long term. In this situation, since both $\dot{k} = 0$ and $\dot{C} = 0$ hold, the following is true:

$$\dot{K}_t = u_t K_t + (1 - \tau_t) w_t H_t - C_t = 0.$$

At E^*, since no capital is accumulated, capital stock remains constant, K^*. So, the private sector does not save and consumes all its disposal income. This phenomenon is drastically contrary to that deduced by the neoclassical theories, like Solow (1956).

[10]If our economy is supposed to be controlled by immortal genes, an intertemporal budget constraint is not necessary. So, our story will change drastically. In this sense, the assumption that at infinity a household necessarily dies is crucial.

On the other hand, the labour in production processes also remains constant, H^*, and the scale of production remains constant, $(K^*)^\alpha(H^*)^{1-\alpha}$. These outputs are entirely consumed by both the private sector and the government.

3.7 Tax and economic movements

Next, we explore how changes in the tax rate imposed on labour income affect our economy.

3.7.1 No tax

Initially, as a preparation for later analyses, we consider an economy without taxation. This means our analyses are deduced as a special case in which $\tau = 0$ is applicable.

At a saddle point, which is denoted by $\tilde{E}$, we obtain the following conclusions:

Based on (3.39) and (3.40), we obtain the following:

$$\tilde{k} = \left(\frac{\alpha}{\gamma}\right)^{\frac{1}{1-\alpha}}, \tag{3.53}$$

and

$$\tilde{C} = \frac{L(1-\alpha)\left(\frac{\gamma}{\alpha}\right)^{\frac{\alpha}{\alpha-1}}}{(A+1)(1-\alpha)+A\alpha}. \tag{3.54}$$

Apparently, (3.53) implies that the capital-labour ratio at the steady state is case indifferent, whether with labour income taxes or without (See (3.39)). This is derived from the fact that an Euler equation is not affected by the taxes imposed on labour income.

It follows that the capital-labour ratio given by (3.53) determines both labour and capital factor prices, which are correspondingly the same when a labour income tax is imposed.

That is, based on (3,26) and (3.25), the following hold:

$$\tilde{u} = \gamma, \tag{3.55}$$

and

$$\tilde{w} = (1-\alpha)\left(\frac{\alpha}{\gamma}\right)^{\frac{\alpha}{1-\alpha}}. \tag{3.56}$$

In this situation, labour supply and capital stock are respectively determined as

$$\begin{aligned}\tilde{H} &= L - \frac{A\tilde{C}}{\tilde{w}} \\ &= \frac{L(1-\alpha)}{(A+1)(1-\alpha)+A\alpha},\end{aligned} \tag{3.57}$$

and

$$\tilde{K} = \tilde{k}\tilde{H} = \frac{L(1-\alpha)\left(\frac{\alpha}{\gamma}\right)^{\frac{1}{1-\alpha}}}{(A+1)(1-\alpha)+A\alpha}. \tag{3.58}$$

Furthermore, outputs and outputs in relation to labour, $\tilde{Y}/\tilde{H}$, are respectively expressed as

$$\begin{aligned}\tilde{Y} &= (\tilde{K})^{\alpha}(\tilde{H})^{1-\alpha} \\ &= \frac{L(1-\alpha)\left(\frac{\gamma}{\alpha}\right)^{\frac{\alpha}{\alpha-1}}}{(A+1)(1-\alpha)+A\alpha} \\ &= \tilde{C},\end{aligned} \tag{3.59}$$

and

$$\frac{\tilde{Y}}{\tilde{H}} = (\tilde{k})^{\alpha}. \tag{3.60}$$

(3.59) shows that all products are consumed, while savings do not exist and therefore no capital accumulation exists.

In addition, consumption in relation to labour is demonstrated as

$$\frac{\tilde{C}}{\tilde{H}} = (\tilde{k})^{\alpha}. \tag{3.61}$$

3.7.2 Taxation on labour income

Suppose that the government imposes a tax at a certain rate on labour income and a household considers this taxation will continue forever.

In this situation, the household again plans its optimal scheme under this new budget constraint, in which its disposable labour income is lower by the amount of labour income tax.

It obviously follows from the Euler equation that the capital-labour ratio is not affected by this taxation and remains $\tilde{k}$, unchanged in an equilibrium in the long term.

However, the $\dot{k}_t = 0$ locus shifts downwards, as follows: Based on (3.42), we obtain the following:

$$\frac{\partial C_t}{\partial \tau_t} = \frac{\frac{\partial a}{\partial \tau_t} k_t^{2\alpha-1}(bk_t^{\alpha-1} + \gamma A) - ak_t^{2\alpha-1}\frac{\partial b}{\partial \tau_t} k_t^{\alpha-1}}{(bk_t^{\alpha-1} + \gamma A)^2}. \tag{3.62}$$

Here, we denote the numerator in (3.62) as D, and obtain the following relationship:

$$D = (\frac{\partial a}{\partial \tau_t} b - a\frac{\partial b}{\partial \tau_t})k_t^{3\alpha-2} + \gamma A \frac{\partial a}{\partial \tau_t} k_t^{2\alpha-1},$$

where the following relationships hold:

$$\begin{aligned}\frac{\partial a}{\partial \tau_t} &= -L(1-\alpha)[\alpha + (1-\tau_t)(1-\alpha)] - L(1-\tau_t)(1-\alpha)^2 \\ &= -L(1-\alpha)[\alpha + 2(1-\tau_t)(1-\alpha)] < 0,\end{aligned}$$

and

$$\frac{\partial b}{\partial \tau_t} = -(A+1)(1-\alpha) < 0.$$

As a result, we can show as below that the sign of the parentheses in D is negative:

$$\frac{\partial a}{\partial \tau_t}b - a\frac{\partial b}{\partial \tau_t} = -(A+1)(1-\alpha)(1-\tau_t)L[\alpha+2(1-\tau_t)(1-\alpha)](1-\alpha)$$
$$+ L(1-\tau_t)(1-\alpha)[\alpha+(1-\tau_t)(1-\alpha)](A+1)(1-\alpha)$$
$$= -(A+1)(1-\alpha)^3L(1-\tau_t)^2 < 0.$$

Eventually, we obtain the following:

$$\frac{\partial C_t}{\partial \tau_t} = -\frac{(A+1)(1-\alpha)^3L(1-\tau_t)^2k_t^{3\alpha-2}}{(bk_t^{\alpha-1}+\gamma A)^2} - \frac{\gamma AL(1-\alpha)[\alpha+2(1-\tau_t)(1-\alpha)]k_t^{2\alpha-1}}{(bk_t^{\alpha-1}+\gamma A)^2} < 0. \quad (3.63)$$

Therefore, we can obtain the following conclusions:

The taxation on labour income makes the curve shift downwards, which satisfies $\dot{k}_t = 0$, while the curve which satisfies $\dot{C}_t = 0$ remains unchanged, on the (k, C) axes.

At this stage, we can explore how our economy behaves differently before and after the taxation is introduced. In order to do so, Figure 3.6 is convenient.

$\tilde{E}$ shows the saddle point before the taxation and E^* the one after the taxation, respectively.

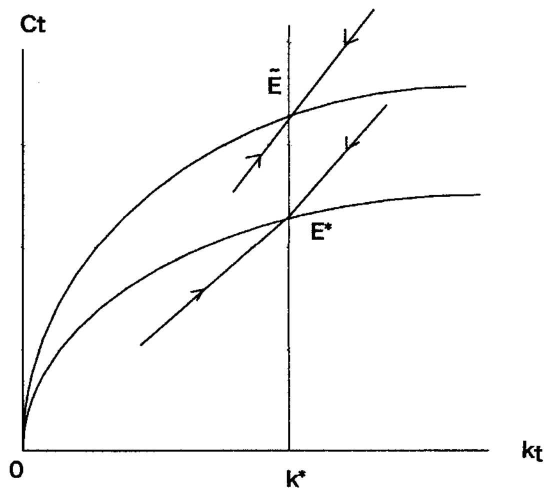

Figure 3.6

Based on what we have analyzed so far, we know that our economy moves along a saddle path to converge on the saddle point, if the intertemporal budget constraint is satisfied, and we assume this condition.

First, we assume that our economy starts at a certain point on the saddle path before the taxation, under an initial condition of state variable, K_0, and then eventually converges to $\tilde{E}$.
Second, suppose that when the economy stays at $\tilde{E}$, a government imposes a tax on labour income. In this situation, a household maximizes its utility under the revised intertemporal budget constraint, and we can obtain a new saddle path and a correspondingly new saddle point E^*. Since the revised budget constraint is satisfied, an initial point is sure on the new saddle path, and our economy converges to E^*.

The way in which the initial point on the new saddle path is determined is a crucial issue here. We can offer a solution to this inquiry as follows:

First, concerning C, from Euler equation (3.24), the following holds:

$$C_t = C_0 \; exp[\int_0^t (u_\nu - \gamma) \; d\nu]. \tag{3.64}$$

Here, C_0 means initial consumption.

On the other hand, an intertemporal budget constraint is given from (3.50) as

$$\int_0^\infty C_t \; exp[-\int_0^t u_\nu \; d\nu] \; dt = \tilde{K} + h_0. \tag{3.65}$$

Here, $\tilde{K}$ means the initial capital corresponding to point $\tilde{E}$. Additionally, h_0, the present value of labour income after taxation to be earned from 0 through infinity, which is constant, is given as

$$h_0 = \int_0^\infty (1-\tau) w_t H_t \; exp[-\int_0^t u_\nu \; d\nu] \; dt.$$

In light of both the Euler equation and the intertemporal budget constraint, we substitute (3.64) into (3.65) and obtain the following:

$$\int_0^\infty C_0 \; exp[\int_0^t (u_\nu - \gamma) \; d\nu] \; exp[-\int_0^t u_\nu \; d\nu] \; dt = \tilde{K} + h_0, \tag{3.66}$$

in which the left side of (3.66) results in $\frac{C_0}{\gamma}$.

As a consequence, we obtain the required conclusions as follows:[11]

[11]The following expression is possible:

$$C_0 = C^* \; exp[-\int_0^\infty (u_\nu - \gamma) \; d\nu].$$

$$C_0 = \gamma\ (\tilde{K} + h_0). \tag{3.67}$$

Furthermore, the following holds in the labour market:

$$\frac{1}{1-\tau}\frac{AC_0}{L-H_0} = (1-\alpha)(\frac{\tilde{K}}{H_0})^{1-\alpha}. \tag{3.68}$$

Since (3.67) determines C_0, H_0 is determined by (3.68). It becomes clear that our economy starts from the point determined by the above conditions, which consist of both an initial condition of a state variable, $\tilde{K}$, and initial values of control variables, C_0 and H_0.

Because this economy moves along a saddle path corresponding to the taxation and eventually converges to E^*, we can conclude that this taxation, that is, the labour income tax, does not make our economy unstable.

However, we can easily guess some other disturbances may occur due to this taxation, so we explore these possibilities.

In the equilibrium denoted by E^*, the values of the variables are determined as follows:

The two factor prices are the same as those in the equilibrium denoted by $\tilde{E}$ and are expressed as

$$u^* = \tilde{u} = \gamma.$$

$$w^* = \tilde{w} = (1-\alpha)\left(\frac{\alpha}{\gamma}\right)^{\frac{\alpha}{1-\alpha}}.$$

The quantities in the equilibrium denoted by E^* are respectively determined as follows:

In the determination of H, the following must hold:

$$\begin{aligned} w^* &= \frac{AC^*}{1-\tau}\frac{1}{L-H^*} \\ &= (1-\alpha)(\frac{K^*}{H^*})^\alpha \\ &= (1-\alpha)\left(\frac{\alpha}{\gamma}\right)^{\frac{\alpha}{1-\alpha}}. \end{aligned}$$

When we substitute C^*, that is, (3.40), into the above, we obtain the following:

$$H^* = \frac{L(1-\tau)(1-\alpha)}{(A+1)(1-\tau)(1-\alpha)+A\alpha}. \tag{3.69}$$

K^* is determined as

$$\begin{aligned} K^* &= k^* H^* \\ &= \frac{L(1-\tau)(1-\alpha)\left(\frac{\alpha}{\gamma}\right)^{\frac{1}{1-\alpha}}}{(A+1)(1-\tau)(1-\alpha)+A\alpha}. \end{aligned} \tag{3.70}$$

In the determination of Y^*,

$$\begin{aligned} Y^* &= (K^*)^\alpha (H^*)^{1-\alpha} \\ &= \frac{L(1-\tau)(1-\alpha)\left(\frac{\alpha}{\gamma}\right)^{\frac{\alpha}{1-\alpha}}}{(A+1)(1-\tau)(1-\alpha)+A\alpha}. \end{aligned} \tag{3.71}$$

In the determination of G,

$$G = \tau w^* H^*$$
$$= \frac{\tau(1-\alpha)^2 \left(\frac{\alpha}{\gamma}\right)^{\frac{\alpha}{1-\alpha}} L(1-\tau)}{(A+1)(1-\tau)(1-\alpha) + A\alpha}. \quad (3.72)$$

From the above analyses, we can verify that the whole demand $C^* + G$ equals the whole output Y^*.

Concerning consumption in relation to labour, the following relationships hold:

$$\frac{C^*}{H^*} = [\alpha + (1-\tau)(1-\alpha)] \left(\frac{\alpha}{\gamma}\right)^{\frac{\alpha}{1-\alpha}} < \left(\frac{\alpha}{\gamma}\right)^{\frac{\alpha}{1-\alpha}}. \quad (3.73)$$

In addition, notice that $\alpha + (1-\tau)(1-\alpha) < 1.$[12]

3.7.3 Disturbances caused by the taxation

Here, we subsequently examine how the taxation causes many disturbances in our economy.

Concerning the supply of labour, comparing the difference in labour supply between the case of non-taxation and that of taxation, we obtain the

[12] We offer a proof here.
Consider a function $g(\alpha) = \alpha + (1-\tau)(1-\alpha)$.
In this situation, the following hold:

$$\begin{cases} g'(\alpha) = \tau > 0, \\ g(0) = 1 - \tau > 0, \\ g(1) = 1. \end{cases}$$

Based on the above relationships, we obviously obtain the required result.

following result:

$$\frac{H^*}{\tilde{H}} = \frac{(1-\tau)[(A+1)(1-\alpha)+A\alpha]}{(1-\tau)(A+1)(1-\alpha)+A\alpha} < 1. \qquad (3.74)$$

In short, labour supply is reduced.

Concerning the capital stock, we obtain the following relationship:

$$\frac{K^*}{\tilde{K}} = \frac{(1-\tau)[(A+1)(1-\alpha)+A\alpha]}{(1-\tau)(A+1)(1-\alpha)+A\alpha} < 1. \qquad (3.75)$$

In other words, capital stock is reduced.

Concerning the scale of production, the following holds:

$$\frac{Y^*}{\tilde{Y}} = \frac{(1-\tau)[(A+1)(1-\alpha)+A\alpha]}{(1-\tau)(A+1)(1-\alpha)+A\alpha} < 1. \qquad (3.76)$$

Based on (3.74), (3.75), and (3.76), we know that the reduction rates for labour, capital stock and production scale are the same.

Concerning consumption, the following holds:

$$\frac{C^*}{\tilde{C}} = \frac{(1-\tau)[(A+1)(1-\alpha)+A\alpha]}{(1-\tau)(A+1)(1-\alpha)+A\alpha}[\alpha+(1-\tau)(1-\alpha)] < 1. \qquad (3.77)$$

Based on the proof in footnote 10, because $\alpha+(1-\tau)(1-\alpha) < 1$, the reduction rate of consumption in the private sector is obviously larger than that of labour, capital stock and production.

Concerning consumption per labour, the reduction is as follows:

$$\frac{\left(\frac{C^*}{H^*}\right)}{\left(\frac{\tilde{C}}{\tilde{H}}\right)} = \alpha + (1-\tau)(1-\alpha) < 1. \tag{3.78}$$

3.7.4 Tax rates and government revenue

We need to compare our economies in steady states.

When the tax rate on labour income is τ, the revenue of a government, which supposedly equals government spending, is given by (3.72) as

$$G = \frac{\tau(1-\alpha)^2 \left(\frac{\alpha}{\gamma}\right)^{\frac{\alpha}{1-\alpha}} L(1-\tau)}{(A+1)(1-\tau)(1-\alpha)+A\alpha}. \tag{3.79}$$

From the above analyses, we know that labour is reduced when the tax rate increases. On the other hand, since the real wage rate remains unchanged, this clearly leads to a reduction of the tax base.

Therefore, we wonder if the revenue necessarily increases when the tax rate does. In this sub-section, we deal with this issue.

Based on (3.79), after some calculations, we obtain the following relationship:

$$\frac{\partial G}{\partial \tau} = \frac{L(1-\alpha)^2 \left(\frac{\alpha}{\gamma}\right)^{\frac{\alpha}{1-\alpha}} [(A+1)(1-\alpha)(1-\tau)^2 + A\alpha(1-2\tau)]}{[(A+1)(1-\alpha)(1-\tau) + A\alpha]^2}. \quad (3.80)$$

Let us analyze the signs of the numerator in (3.80), and denote a part of it as

$$g(\tau) = (A+1)(1-\alpha)(1-\tau)^2 + A\alpha(1-2\tau), \quad (3.81)$$

which has the following properties:

$$\begin{cases} g'(\tau) = -2(A+1)(1-\alpha)(1-\tau) - 2A\alpha, \\ g(0) = (A+1)(1-\alpha) + A\alpha > 0, \\ g(1) = -A\alpha < 0. \end{cases} \quad (3.82)$$

Consider $g(\tau) = 0$, and we obtain the following quadratic equation:

$$(A+1)(1-\alpha)\tau^2 - 2[A + (1-\alpha)]\tau + A + (1-\alpha) = 0. \quad (3.83)$$

Here, if we consider the signs of χ, we obtain the following:

$$\begin{aligned} \chi &= [A + (1-\alpha)]^2 - (A+1)(1-\alpha)(A+1-\alpha) \\ &= [A + (1-\alpha)]A\alpha > 0. \end{aligned}$$

The above relationship shows that quadratic equation (3.83) has two real roots, and we denote these real roots as τ_1 and τ_2, respectively.

Additionally, the following relationships hold:

$$\begin{cases} \tau_1 + \tau_2 = \dfrac{2[A + (1-\alpha)]}{(A+1)(1-\alpha)} > 0, \\ \tau_1 \cdot \tau_2 = \dfrac{A + (1-\alpha)}{(A+1)(1-\alpha)} > 0. \end{cases}$$

The above relationships show that the two roots are positive and the larger root is greater than one.

We can verify this as follows: In light of $g' = 0$ in (3.82), we obtain the following:

$$(A+1)(1-\alpha)(1-\tau) + A\alpha = 0.$$

When we solve the above equation with respect to τ, we eventually obtain the following:

$$\tau^+ = 1 + \frac{A\alpha}{(A+1)(1-\alpha)}.$$

Based on the arguments so far, we can depict g as in Figure 3.7.

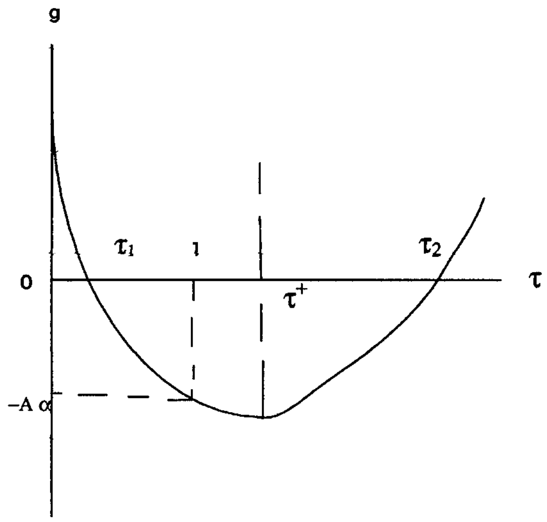

Figure 3.7

Now, we can respectively modify (3.80) and (3.79) as

$$\frac{\partial G}{\partial \tau} = \frac{L(1-\alpha)^2 \left(\frac{\alpha}{\gamma}\right)^{\frac{\alpha}{1-\alpha}} g(\tau)}{[(A+1)(1-\alpha)(1-\tau)+A\alpha]^2}, \tag{3.84}$$

$$G = \frac{\tau(1-\alpha)^2 \left(\frac{\alpha}{\gamma}\right)^{\frac{\alpha}{1-\alpha}} L}{(A+1)(1-\alpha)+\frac{A\alpha}{1-\tau}}. \tag{3.85}$$

From (3.79), we know the following:

$$\begin{cases} G(0) = 0, \\ G(1) = 0. \end{cases}$$

Finally, we obtain Figure 3.8, which implies the existence of an optimal tax rate from the viewpoint of a government.[13]

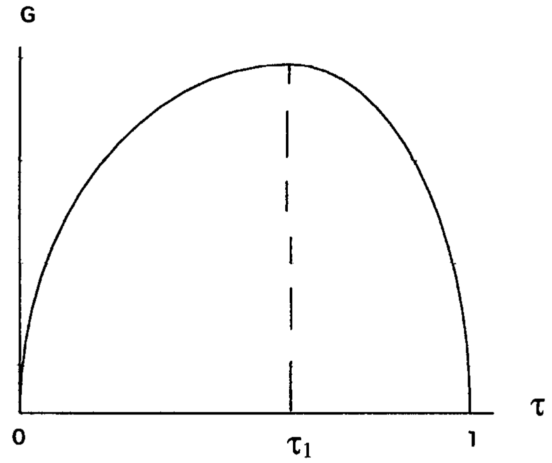

Figure 3.8

τ_1 is the optimal tax rate, at which at least government revenue is maximized.

Additionally,

$$\tau_1 = \frac{A+1-\alpha-\sqrt{(A+1-\alpha)A\alpha}}{(A+1)(1-\alpha)}.$$

3.8 Conclusion

As in the conclusion in Schmitt-Grohé and Uribe (1997), we have seen that a labour income tax does not affect economic movements, which are stable and necessarily converge to a saddle point, when the labour income tax rates are fixed. Of course, neither business cycles nor indeterminacy occur.

In addition, contrary to the notion of neoclassical equilibrium, in which an

[13]This may be called a Laffer curve.

economy is growing, our economy tends to converge to a steady state. In this situation, all economic variables are constant, as in the classical equilibrium.

However, the higher the labour income tax rate is, the more the economy deteriorates, regardless of our assumption of a balanced budget rule. This assumption means that a government spends and consumes the amount equal to the labour income tax levied.

This deterioration means a reduction in production, resulting in a reduction of inputs, that is, a scaled-down economy, but, conversely, production efficiency is not affected, which is typically shown by the neutrality regarding taxation of an Euler equation. At the saddle point, the capital-labour ratio remains unchanged before and after the taxation is introduced or the tax rates are changed.

This can be deduced as follows:

First, when a certain labour income tax is introduced, a household actually procures a smaller amount of labour income than before and anticipates as a result that similar taxation will continue in future.

Second, under this anticipation, in other words, in light of this anticipated reduction in disposable income, a household maximizes its utilities again. As a result, at a new saddle point, as we have already analyzed, consumption, capital stock, the amount of labour force hired and production are reduced.

However, as an Euler equation suggests, the capital-labour ratio remains unchanged. In short, the capital-labour ratio is eventually independent (neutral) of the change of tax rates imposed on labour income.

Because this leads to constant factor prices, that is, because both wages and rental rates remain unchanged, the production efficiency shown by labour productivity remains unchanged as well.

It becomes clear that real interest rates are not influenced by a change in the labour income tax rates. And since we can consider a real interest rate as a ratio of the price of present goods to that of future goods, the consistency of real interest rates naturally implies no business cycles.

Conversely speaking, we should consider another situation, in which the real interest rates can be changed by taxation on capital income, instead of on labour income.

We intend to deal with this interesting issue in the next chapter.

3.9 Appendix

3.9.1 Appendix 1

We omit the character t showing time, below.

The following hold:

$$\frac{dC}{dk} = \frac{ak^{2(\alpha-1)}[b\alpha k^{\alpha-1} + (2\alpha - 1)\gamma A]}{(bk^{\alpha-1} + A\gamma)^2} > 0,$$

$$\frac{d^2C}{dk^2} = \frac{numerator}{(bk^{\alpha-1} + A\gamma)^4}.$$

Additionally, the following relation holds:

$$\begin{aligned} numerator = & [a2(\alpha-1)k^{2\alpha-3}(b\alpha k^{\alpha-1}+(2\alpha-1)\gamma A) \\ & + ak^{2(\alpha-1)}b\alpha(\alpha-1)k^{\alpha-2}](bk^{\alpha-1}+\gamma A)^2 \\ & - ak^{2(\alpha-1)}[b\alpha k^{\alpha-1}+(2\alpha-1)\gamma A] \times 2(bk^{\alpha-1}+\gamma A)b(\alpha-1)k^{\alpha-2}. \end{aligned} \tag{3.86}$$

In (3.86), the contents in brackets can be transformed as

$$[\quad] = ak^{2(\alpha-1)}[3\alpha(\alpha-1)bk^{\alpha-2}+2(\alpha-1)(2\alpha-1)\gamma Ak^{-1}]. \tag{3.87}$$

In this situation, (3.86) is further transformed as

$$\begin{aligned} numerator = & (bk^{\alpha-1}+\gamma A)^2 ak^{2(\alpha-1)}(\alpha-1)k^{-1}(3\alpha bk^{\alpha-1}+2(2\alpha-1)\gamma A) \\ & - 2ak^{2(\alpha-1)}(b\alpha k^{\alpha-1}+(2\alpha-1)\gamma A)b(\alpha-1)k^{\alpha-2}(bk^{\alpha-1}+\gamma A) \\ = & (bk^{\alpha-1}+\gamma A)ak^{2(\alpha-1)}[(bk^{\alpha-1}+\gamma A)(\alpha-1)k^{-1}(3\alpha bk^{\alpha-1} \\ & + 2(2\alpha-1)\gamma A) - 2(b\alpha k^{\alpha-1}+(2\alpha-1)\gamma A)b(\alpha-1)k^{\alpha-2}], \end{aligned} \tag{3.88}$$

and the brackets in (3.88) can be modified as

$$[\quad] = (\alpha-1)[ab^2k^{2\alpha-3}+3\alpha\ b\gamma A\ k^{\alpha-2}+2(2\alpha-1)\gamma^2A^2k^{-1}. \tag{3.89}$$

Therefore, eventually we obtain the following:

$$\begin{aligned} numerator = &(bk^{\alpha-1} + \gamma A)ak^{2(\alpha-1)}(\alpha - 1) \\ &\times [\alpha b^2 k^{2\alpha-3} + 3\alpha\, b\gamma A\; k^{\alpha-2} + 2(2\alpha - 1)\gamma^2 A^2 k^{-1}], \end{aligned} \tag{3.90}$$

which is obviously negative, if $\alpha \geqq \frac{1}{2}$.

As a result, we obtain the required result,

$$\frac{d^2C}{dk^2} < 0.$$

3.9.2 Appendix 2

In this situation, the following hold:

$$\begin{cases} \dot{k}_t > 0, \\ \dot{C}_t < 0. \end{cases}$$

Then, the trajectory asymptotically approaches a horizontal axis.

We can verify this as follows:

Based on an Euler equation, (3.24), the movement of consumption is expressed in the following way:

$$\frac{\dot{C}_t}{C_t} = \alpha k_t^{\alpha-1} - \gamma, \tag{3.91}$$

which can be transformed as follows, and which means consumption at arbitrary time:

$$C_t = C_0 \; exp[\int_0^t (\alpha k_\nu^{\alpha-1} - \gamma) d\nu \;]. \tag{3.92}$$

Here, C_0 means the initial condition with respect to C_t and additionally $\alpha k_t^{\alpha-1} - \gamma < 0$ holds.

In this circumstance, when t approaches infinity, C_t approaches null, while k_t approaches infinity.

From the argument so far, we can conclude that our required condition is satisfied.

3.9.3 Appendix 3

In contrast to ***Appendix* 2**, the following hold:

$$\begin{cases} \dot{k}_t < 0, \\ \dot{C}_t > 0. \end{cases}$$

In this situation, this trajectory cuts a vertical axis at a finite time, because (3.92) holds under the condition of $\alpha k_t^{\alpha-1} - \gamma > 0$, and C_t increases. On the other hand, k_t approaches null.

Therefore, we can obtain the required conclusion.

Chapter 4

How Do Capital Income Taxes Affect an Economy ?

This chapter deals with whether taxation on capital income under fixed tax rates can change economic movements, namely can cause business cycles or not. We rely on the infinite horizon model which was originally offered by Schmitt-Grohé and Uribe (1997), and we modify the model mainly concerning their utility function because the model has logical flaws, as pointed out in the last chapter.

After our analysis, we conclude that there is a saddle point with a saddle path in our economy, and that therefore this economy with its taxation on capital income finally converges to a steady state.

4.1 Introduction

In *How Do Fixed Labour Income Taxes Affect an Economy ?*, we concluded that the capital-labour ratio in a long-term equilibrium is independent (neutral) of the change in tax rates imposed on labour income, like in the case of lump-sum taxation. Based on this conclusion, we know that the interest rates in a long-term equilibrium are also independent of the change in labour income tax rates.

It follows that there is no possibility of business cycles, which are caused by a change in labour income tax rates.[1]

On the other hand, is there not any possibility for changes in taxation imposed on capital income to cause economic fluctuations, in which business cycles are included ? Because the existence of strong effects on savings from changes of interest rates is a requirement for business cycles, and because a capital income tax can affect current savings, which are consumed in future, the taxation may induce changes of interest rates in our economy, and as a result, eventually cause business cycles. Therefore, we need to explore how capital income taxation affects our economy, especially focusing on how savings are formed.

That is our prime aim in this chapter.

We again rely upon the previous model, which is a modified version of the Schmitt-Grohé and Uribe (1997) model.

[1] Benhabib and Nishimura (1983) asserts, in light of no taxation, that no limit cycles occur, on the assumption of one kind of capital goods.

Our conclusion in this chapter is that there is no possibility for business cycles to occur due to taxation on capital income. This becomes obvious in light of the existence of a unique saddle point accompanying a unique saddle path in our economy. This conclusion is similar to that in the case of labour income taxation. However, in the long-term, a capital income tax decreases labour efficiency, while a labour income tax does not.

The overall structure of this chapter is as follows:

In section 4.2, we establish our model, featuring an endogenous labour supply, which moves positively with the real wage rate. In addition, an Euler equation associated with the capital income tax rate is deduced. Subsequently, we deduce a system of dynamics describing our economy's movements, in section 4.3. In what follows, in section 4.4, we demonstrate the existence of a saddle point with a saddle path, where our economy converges in the long term, and analyze disturbances caused by capital income taxes. Section 4.5 is devoted to conclusions.

4.2 The model

A household, at instant null, maximizes the present value of its utilities to be obtained to infinity, as

$$\max_{C_t, H_t} \int_0^\infty e^{-\gamma t}[\log C_t + A \log (L - H_t)]dt. \tag{4.1}$$

The above formula means that a household chooses how much to consume and how long to work, every instant null through infinity. Of course, because it makes the above decisions at null, and because a household can not know anything beyond a null, it is forced to rely on anticipation. In this context, we assume, as in the case of labour income taxation, that the anticipation is based on a perfect foresight principle. In short, according to our assumption, its anticipation with reference to u_t and w_t is all realized in markets.[2]

On the other hand, a household determines how much to save from its disposable income (after tax extraction). In this chapter, since we consider capital income taxation, and since we can consider that current savings lead to an increment of capital in future, the following holds:

$$\dot{K}_t = u_t(1 - \tau_t)K_t + w_t H_t - C_t, \tag{4.2}$$

where τ_t means a capital income tax rate at an instant t, and is postulated to be constant.

On the assumption above, a household maximizes its present value of utilities as in (4.1) subject to (4.2). Again, in this situation, a household takes u_t and w_t as given.

[2] As noted in chapter 3, every market is assumed to be equilibrated.

The Hamiltonian function, R, is defined as

$$R = e^{-\gamma t}[\log C_t + A \log (L - H_t)] + \mu_t[u_t(1-\tau_t)K_t + w_t H_t - C_t]. \tag{4.3}$$

Concerning control variables, the following are necessary:

$$\frac{\partial R}{\partial C_t} = e^{-\gamma t}\frac{1}{C_t} - \mu_t = 0,$$

which yields

$$\mu_t = \frac{e^{-\gamma t}}{C_t}. \tag{4.4}$$

(4.4) means that an adjoined variable, μ_t, equals the present value of the marginal utility of consumption at an instant t.

$$\frac{\partial R}{\partial H_t} = e^{-\gamma t}A\frac{-1}{L - H_t} + \mu_t w_t = 0,$$

which yields

$$H_t = L - \frac{AC_t}{w_t}, \tag{4.5}$$

which can be furthermore transformed as

$$w_t = \frac{AC_t}{L - H_t}. \tag{4.6}$$

(4.6) shows a supply curve of the labour force, which is obviously independent of tax rates, contrary to the case of labour income taxation.

Furthermore, the following must hold:

$$\frac{d\mu_t}{dt} = -\frac{\partial R}{\partial K_t} = -\mu_t(1-\tau_t)u_t,$$

which yields

$$\frac{\dot{\mu}_t}{\mu_t} = -u_t(1-\tau_t). \tag{4.7}$$

Based on both (4.4) and (4.7), we obtain an Euler equation :

$$\frac{\dot{C}_t}{C_t} = u_t(1-\tau_t) - \gamma. \tag{4.8}$$

This Euler equation shows how a household divides its current disposable income into current consumption and future consumption, which will be realized through current savings.

In addition, we can know from (4.8) that this choice is influenced by the tax rates imposed on capital income, contrary to the case of labour income taxation.

Moreover, since function R is obviously concave with respect to C_t, H_t, μ_t and K_t, the sufficient conditions for maximizing the objective function are satisfied, in light of mathematics. However, since we assume that only one kind of output exists, and that outputs have perfect substitution, that is, they can be entirely used both as consumption goods and capital goods, it is finally better for a household to consume its capital entirely. In other words, from this economic viewpoint, we must also consider the transversality conditions mentioned above.[3]

4.3 Dynamics

4.3.1 Optimization of a corporation

We assume that only one corporation exists in our economy, which can be called a representative corporation. This representative corporation is supposed to aim to maximize its profit at every instant.

It produces outputs utilizing both capital and a labour force, which are hired in exchange for payment in the form of rental fees or wages, at every instant. And then the corporation sells its outputs in a market.

The Cobb-Douglas type production function of the corporation is expressed in the following way:

$$F(K_t, H_t) = K_t^{\alpha} H_t^{1-\alpha}. \tag{4.9}$$

All markets concerning outputs, capital and labour are supposed to be fully competitive. In this situation, the demands for capital and a labour force, which are needed by the corporation, are respectively determined in relation to maximizing its profit, as follows:

$$w_t = (1-\alpha) k_t^{\alpha}, \tag{4.10}$$

[3]We later deal with this big issue in more detail.

$$u_t = \alpha k_t^{-(1-\alpha)}. \tag{4.11}$$

Besides, factor prices, both u_t and w_t, are anticipated values, respectively.

4.3.2 Dynamics of the system

First, as a preparation for later discussions, we derive a rate of change in the labour supply, below.

(4.5) holds for arbitrary time, and this means that we can consider (4.5) as an identify with respect to time.

So, we can differentiate (4.5) with respect to time and as a consequence obtain the following:

$$\dot{H}_t = -A\frac{\dot{C}_t w_t - C_t \dot{w}_t}{w_t^2},$$

from which the following holds:

$$\frac{\dot{H}_t}{H_t} = \frac{-A\left(\dfrac{\dot{C}_t}{C_t} - \dfrac{\dot{w}_t}{w_t}\right)}{L\left(\dfrac{w_t}{C_t}\right) - A}. \tag{4.12}$$

The above relationship indicates the rate of change in the labour supply with respect to time, which we need to know.

Second, we investigate movements of a capital-labour ratio $k_t = \dfrac{K_t}{H_t}$. By definition, the following holds:

$$\dot{k}_t = \frac{\dot{K}_t}{H_t} - \frac{K_t}{H_t}\frac{\dot{H}_t}{H_t}.$$

Substituting both (4.2) and (4.12) into the above relationship, we obtain the following:

$$\dot{k}_t = (1-\tau_t)u_t\frac{K_t}{H_t} + w_t - \frac{C_t}{H_t} - k_t\frac{\dot{H}_t}{H_t}, \tag{4.13}$$

which yields

$$\dot{k}_t = (1-\tau_t)u_t k_t + w_t - \frac{C_t}{H_t} - k_t\frac{A\left(\dfrac{\dot{C}_t}{C_t} - \dfrac{\dot{w}_t}{w_t}\right)}{A - L\left(\dfrac{w_t}{C_t}\right)}. \tag{4.14}$$

Additionally, as a preparation for further calculation concerning (4.14), we can establish the following relationship:

$$\begin{aligned}\frac{C_t}{H_t} &= C_t\frac{w_t}{Lw_t - AC_t}\\ &= \frac{C_t}{L - A\left(\dfrac{C_t}{w_t}\right)}.\end{aligned} \tag{4.15}$$

Based on (4.15), (4.14) can be transformed as

$$\dot{k}_t = (1-\tau_t)u_t k_t + w_t - \frac{C_t}{L - A\left(\frac{C_t}{w_t}\right)} - k_t\frac{A\left(\dfrac{\dot{C}_t}{C_t} - \dfrac{\dot{w}_t}{w_t}\right)}{A - L\left(\dfrac{w_t}{C_t}\right)}. \tag{4.16}$$

Since we discuss movements of equilibrium prices and equilibrium volumes in the markets, we define the following respectively, as in the previous chapter:

$$\begin{cases} w_t = \eta(k_t), \\ \eta(k_t) \stackrel{\text{def}}{=} (1-\alpha)k_t^{\alpha}, \end{cases} \tag{4.17}$$

and

$$\begin{cases} u_t = \phi(k_t), \\ \phi(k_t) \stackrel{\text{def}}{=} \alpha k_t^{\alpha-1}. \end{cases} \tag{4.18}$$

Here, in light of equilibrium in the markets, we can transform (4.16) into the following:

$$\begin{aligned} \dot{k}_t &= (1-\tau_t)\phi(k_t)k_t + \eta(k_t) - \frac{\eta(k_t)C_t}{\eta(k_t)L - AC_t} \\ &+ \frac{Ak_t[\frac{\dot{C}_t}{C_t} - \frac{\dot{\eta(k_t)}}{\eta(k_t)}]C_t}{L\eta(k_t) - AC_t}, \end{aligned}$$

which leads to

$$\begin{aligned} \dot{k}_t &= (1-\tau_t)\phi(k_t)k_t + \eta(k_t) - \frac{\eta(k_t)C_t}{\eta(k_t)L - AC_t} \\ &+ \frac{Ak_t[-\rho + (1-\tau_t)\phi(k_t) - \frac{\alpha\dot{k}_t}{k_t}]C_t}{L\eta(k_t) - AC_t}. \end{aligned} \tag{4.19}$$

Additionally, in the above procedure we utilized the following:

$$\frac{\dot{w}_t}{w_t} = \alpha \frac{\dot{k}_t}{k_t}, \tag{4.20}$$

and

$$\frac{\dot{C}_t}{C_t} = (1 - \tau_t)\phi(k_t) - \gamma. \tag{4.21}$$

By the definition of both (4.17) and (4.18), transforming (4.19) further, we obtain the following:

$$\begin{aligned} \dot{k}_t[A(1-\alpha)C_t - L(1-\alpha)k_t^{\alpha}] = \\ C_t k_t[(1-\alpha)(A+1)k_t^{\alpha-1} + A\gamma] \\ - L(1-\alpha)[(1-\tau_t)\alpha + (1-\alpha)]k_t^{2\alpha}. \end{aligned} \tag{4.22}$$

At this stage, when we focus on an equilibrium in the labour market, we can notice that $AC_t - L\ k_t^{\alpha}\ <\ 0$.

We can verify this as follows:

Let us return to the labour supply (4.5). In (4.5), naturally $H_t \geqq 0$ holds. So, it leads to $L\ w_t^s - A\ C_t \geqq 0$. In this context, we should notice that w_t^s means a supply price with which a household is satisfied in deciding its labour supply.

On the other hand, a demand price is denoted as $w_t^d = (1 - \alpha)k_t^{\alpha}$, with which a corporation is satisfied in deciding its demand. And in the labour market, since the demand price equals the supply price, the following must eventually hold:

$$L(1-\alpha)k_t^{\alpha} - AC_t \geqq 0,$$

which leads to the following expression:

$$A\ C_t - Lk_t^{\alpha} + L\alpha k_t^{\alpha} \leqq 0. \tag{4.23}$$

In (4.23), since $L\alpha k_t^{\alpha} > 0$, the following obviously holds:

$$A\ C_t - L\ k_t^{\alpha} < \ 0. \tag{4.24}$$

Based on the above analyses, we obtain the following key relationship:

$$\begin{aligned} \dot{k}_t = & \frac{L(1-\alpha)[(1-\tau_t)\alpha + (1-\alpha)]k_t^{2\alpha}}{(1-\alpha)[L\ k_t^{\alpha} - AC_t]} \\ & - \frac{(A+1)(1-\alpha)C_t k_t^{\alpha} + A\ \gamma C_t k_t}{(1-\alpha)[L\ k_t^{\alpha} - AC_t]}. \end{aligned} \tag{4.25}$$

In addition, we have to consider another key relationship, that is, an Euler equation, expressed as

$$\dot{C}_t = [(1-\tau_t)\alpha k_t^{-(1-\alpha)} - \gamma]C_t. \tag{4.26}$$

Both (4.25) and (4.26) constitute a simultaneous differential equations system with respect to both k_t and C_t. In other words, this implies that the solutions in the above system show the economic movements with respect to time, which we are investigating.

4.3.3 A steady state

Here, we can confirm the existence of a steady state, which will later be obvious as a saddle point.

As in our previous notations, we respectively define symbols as $k_t = k^{**}$, $C_t = C^{**}$, and $\tau_t = \tau$, for any t.

At first, when we place $\dot{C}_t = 0$ in (4.26), we obtain the steady value of a capital-labour ratio, k^{**}, as

$$k^{**} = \left(\frac{\alpha}{\gamma}\right)^{\frac{1}{1-\alpha}} (1-\tau)^{\frac{1}{1-\alpha}}. \tag{4.27}$$

Similarly, placing $\dot{k}_t = 0$ in (4.25), we obtain the following:

$$C^{**}\ k^{**}[(1-\alpha)(A+1)\ (k^{**})^{\alpha-1} + A\ \gamma] = L(1-\alpha)[(1-\tau)\ \alpha + (1-\alpha)]\ (k^{**})^{2\alpha},$$

where, assuming $k_t \neq 0$, we obtain the following:

$$C^{**} = \frac{L\ (1-\alpha)[(1-\tau)\ \alpha + (1-\alpha)]\ (k^{**})^{2\alpha-1}}{(1-\alpha)(A+1)\ (k^{**})^{\alpha-1} + A\ \gamma},$$

which yields

$$C^{**} = \frac{L\ (1-\alpha)[(1-\tau)\alpha + (1-\alpha)]\left(\frac{\alpha}{\gamma}\right)^{\frac{\alpha}{1-\alpha}}\ (1-\tau)^{\frac{\alpha}{1-\alpha}}}{(A+1)(1-\alpha) +\ A\alpha(1-\tau)}. \tag{4.28}$$

In this transformation , we utilized (4.27). Obviously, $C^{**} > 0$, which shows the steady state consumption in our economy.

At this stage, we can identify the global movement of our economy, which is shown by both (4.25) and (4.26) mathematically.

Let us subsequently analyze the directions in time of C_t and k_t respectively, on a phase diagram of (k, C) axes, as in the last chapter.

4.3.4 The movements of C_t

First, let us analyze the movements of C_t, focusing on an Euler equation, (4.26).

Based on (4.26), we can easily obtain Figure 4.1.

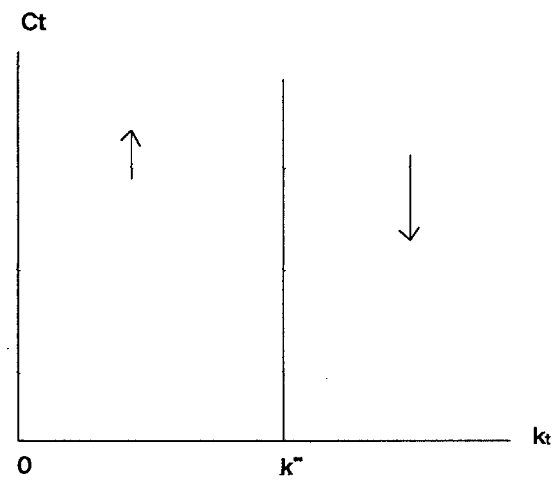

Figure 4.1

The $dC/dt = 0$ locus is shown by a vertical line at k^{**}, which is affected by tax rates imposed on capital income. This result is diametrically contrary to the case in labour income taxation, as discussed in the previous chapter.

Naturally, when the tax rates are higher, the consumption in the future is lower. Therefore, it is apparent from (4.27) that k^{**}, which satisfies $dC/dt = 0$ at this newly higher tax rate, must be smaller, as higher interest rates are realized.

Because consumption in the future is a part of the outcome of current savings, the regions increasing future consumption, which are denoted in the area showed by the upward arrow, are smaller towards the point of origin, when the tax rates are higher. This means that capital income taxation has incentives towards smaller current savings, which leads to smaller consumption in the future.

Of course, as time passes, consumption is decreasing to the right of the locus, where $k_t > k^{**}$, because a household feels uncomfortable about consumption in the future, due to $(1 - \tau_t)u_t < \gamma$.

4.3.5 The movement of k_t

Based on (4.25) and in light of $A\ C_t - L\ k_t^{\alpha} < 0$, we can judge the following:

$$\begin{cases} \dot{k}_t \gtreqless 0 \leftrightarrows \textit{numerator in } (4.25) \gtreqless 0, \\ \dot{k}_t < 0 \leftrightarrows \textit{numerator in } (4.25) < 0. \end{cases}$$

Now, we begin with analyzing the case in which *the numerator of* (4.25) = 0. The following from (4.25) holds:

$$C_t\ k_t[(1-\alpha)(A+1)k_t^{\alpha-1} + A\gamma] = L(1-\alpha)[(1-\tau_t)\alpha + (1-\alpha)]k_t^{2\alpha}, \quad (4.29)$$

and here we assume $k_t \neq 0$.

In this situation, since $(1-\alpha)(A+1)k_t^{\alpha-1} + A\gamma > 0$, the following eventually holds:

$$C_t = \frac{L(1-\alpha)[(1-\tau_t)\alpha + (1-\alpha)]k_t^{\alpha}}{(A+1)(1-\alpha) + A\gamma\ k_t^{1-\alpha}}. \tag{4.30}$$

In light of both (4.25) and (4.30), we can obtain the following conclusions:

$$\begin{cases} C_t > right\ side\ of\ (4.30) \leftrightarrows numerator\ in\ (4.25) < 0 \leftrightarrows \dot{k}_t < 0, \\ C_t < right\ side\ of\ (4.30) \leftrightarrows numerator\ in\ (4.25) > 0 \leftrightarrows \dot{k}_t > 0, \\ C_t = right\ side\ of\ (4.30) \leftrightarrows numerator\ in\ (4.25) = 0 \leftrightarrows \dot{k}_t = 0. \end{cases}$$

From what we have analyzed so far, we need to know how C_t behaves as k_t moves. So, in this situation, we focus on (4.30), which shows the $dk/dt = 0$ locus, and define the following function with an independenct variable in terms of k_t:

$$f(k_t) = \frac{L(1-\alpha)[(1-\tau_t)\alpha + (1-\alpha)]k_t^{2\alpha-1}}{(A+1)(1-\alpha)k_t^{\alpha-1} + \gamma A}. \tag{4.31}$$

Based on (4.31), the following relationship holds:

$$\frac{df}{dk_t} = \frac{numerator}{[(A+1)(1-\alpha)k_t^{\alpha-1} + \gamma A]^2},$$

where

$$\begin{aligned} numerator = {} & L(1-\alpha)[(1-\tau_t)\alpha + (1-\alpha)](2\alpha-1)k_t^{2\alpha-2}[(A+1)(1-\alpha)k_t^{\alpha-1} + \gamma A] \\ & - L(1-\alpha)[(1-\tau_t)\alpha + (1-\alpha)]k_t^{2\alpha-1}(A+1)(1-\alpha)(\alpha-1)k_t^{\alpha-2}, \end{aligned}$$

which eventually yields

$$numerator = L(1-\alpha)[(1-\tau_t)\alpha+(1-\alpha)]k_t^{2\alpha-2}[(2\alpha-1)\gamma A+\alpha(1-\alpha)(A+1)k_t^{\alpha-1}].$$

As a result, we can obtain the following:

$$\frac{df}{dk_t} = \frac{L(1-\alpha)[(1-\tau_t)\alpha+(1-\alpha)]k_t^{2\alpha-2}[(2\alpha-1)\gamma A+\alpha(1-\alpha)(A+1)k_t^{\alpha-1}]}{[(A+1)(1-\alpha)k_t^{\alpha-1}+\gamma A]^2}. \tag{4.32}$$

In this circumstance, we focus on the bracket [ ] of the numerator of (4.32), and define its function as

$$\varphi(k_t) = \alpha(1-\alpha)(A+1)k_t^{\alpha-1} + (2\alpha-1)\gamma\ A. \tag{4.33}$$

We can know the following properties concerning $\varphi(k_t)$, through some simple mathematical manipulations:

$$\begin{cases} \varphi'(k_t) = -\alpha(1-\alpha)^2(A+1)k_t^{\alpha-2} < 0, \\ \varphi(0) = \infty, \\ \lim_{k_t\to\infty}\varphi(k_t) = (2\alpha-1)\gamma A \gtrless 0. \end{cases}$$

Additionally, the following relationship holds:

$$\frac{df}{dk_t} = \frac{L(1-\alpha)[(1-\tau_t)\alpha+(1-\alpha)]k_t^{2(\alpha-1)}\varphi(k_t)}{[(A+1)(1-\alpha)k_t^{\alpha-1}+A\gamma]^2}, \tag{4.34}$$

which is eventually transformed as follows:

$$\frac{df}{dk_t} = \frac{L(1-\alpha)[(1-\tau_t)\alpha + (1-\alpha)][\frac{\alpha(1-\alpha)(A+1)}{k_t^{1-\alpha}} + (2\alpha-1)A\gamma]}{[(A+1)(1-\alpha) + A\ \gamma k_t^{1-\alpha)}]^2}. \quad (4.35)$$

Based on (4.35), the following obviously holds:

$$\lim_{k_t \to \infty} \frac{df}{dk_t} = 0. \quad (4.36)$$

As a result, it follows that df/dk_t starts at infinity and decreases to a certain negative value (in the case of $0 < \alpha < 1/2$), to a certain positive value (in the case of $1/2 < \alpha < 1$), or to null (in the case of $\alpha = 1/2$), as k_t increases from null to infinity.

In short, in the case of $0 < \alpha < 1/2$, df/dk_t finally approaches $(2\alpha - 1)A\gamma$ that stays negative, while in the case of $\frac{1}{2} \leqq \alpha < 1$, it stays positive.

In light of (4.34) and (4.35), we can obtain two categories of df/dk_t, which are shown as Figure 4.2 and Figure 4.3, respectively.

In the case of $0 < \alpha < \frac{1}{2}$:

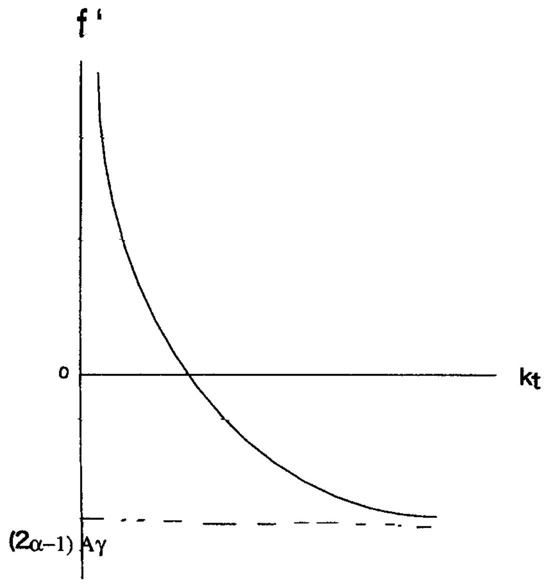

Figure 4.2

In the case of $\frac{1}{2} \leqq \alpha < 1$:

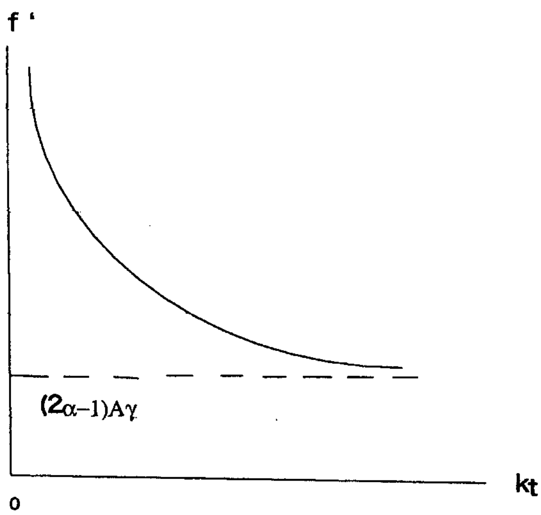

Figure 4.3

On the other hand, we can transform (4.31) into the following:

$$f(k_t) = \frac{L(1-\alpha)[(1-\tau_t)\alpha + (1-\alpha)]k_t^{\alpha}}{(A+1)(1-\alpha) + A\gamma\, k_t^{1-\alpha}}. \tag{4.37}$$

From what we have analyzed so far, we can conclude in the following way:

(a) in the case of $0 < \alpha < 1/2$,

$$\begin{cases} \lim_{k_t \to \infty} f(k_t) = 0, \\ f(0) = 0. \end{cases}$$

(b) in the case of $\frac{1}{2} \leqq \alpha < 1$,

$$\begin{cases} \lim_{k_t \to \infty} f(k_t) = \infty, \\ f(0) = 0. \end{cases}$$

Eventually, we can depict the following two relationships for (k_t, C_t), which form phase diagrams:[4]

In the case of $0 < \alpha < \frac{1}{2}$:

[4] Additionally, we later offer a proof concerning $f'' < 0$ in the case of $\frac{1}{2} \leqq \alpha < 1$, in the Appendix.

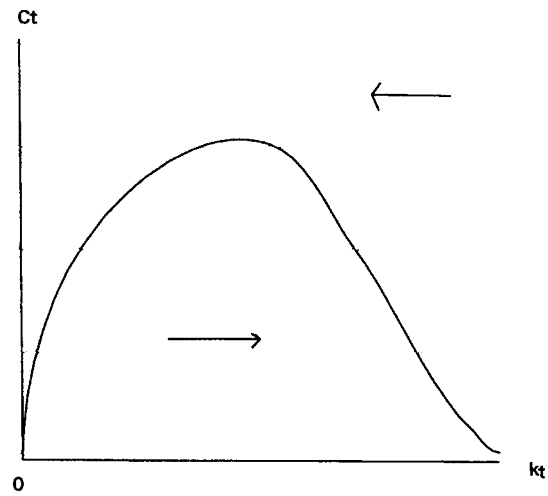

Figure 4.4

The locus $dk/dt = 0$ starts at the origin, increases until the point $k_t = [-\frac{\alpha(1-\alpha)(A+1)}{(2\alpha-1)\gamma A}]^{\frac{1}{1-\alpha}}$, and afterward decreases and approaches null, as k_t increases.

Of course, anywhere above the $dk/dt = 0$ locus, the capital-labour ratio decreases. Conversely, anywhere below that locus, it increases. Furthermore, on the $dk/dt = 0$ locus, k_t naturally remains unchanged.

The horizontal arrows demonstrate these directions of motion.

In the case of $\frac{1}{2} \leqq \alpha < 1$:

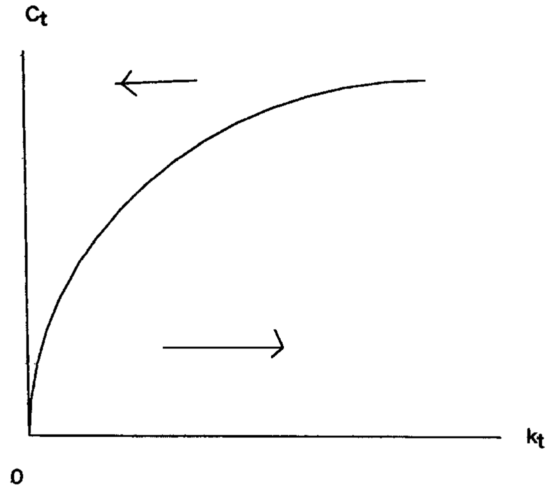

Figure 4.5

The locus $dk/dt = 0$ starts at the origin and increases, as k_t increases.
As in the case of $0 < \alpha < \frac{1}{2}$, anywhere above the $dk/dt = 0$ locus, the ratio decreases, anywhere below it, the ratio increases, and on the $dk/dt = 0$ locus, the ratio remains unchanged.
Furthermore, the horizontal arrows similarly show the directions of motion.

4.4 The optimal path under taxation

Here, we can confirm the existence of a globally stable optimal path, on the postulation of capital income taxation.

Let us focus on (4.37), which shows the $dk/dt = 0$ locus when capital income tax is imposed at a rate τ.

When we differentiate (4.37) with respect to τ, we obtain the following relationship:

$$\frac{\partial f}{\partial \tau_t} = -\frac{L\alpha(1-\alpha)k_t^{\alpha}}{(A+1)(1-\alpha)+A\gamma k_t^{1-\alpha}} < 0. \tag{4.38}$$

(4.38) implies that C_t in response to an arbitrary k_t decreases, as τ_t increases, in other words that the $\dot{k}_t = 0$ locus shifts downwards as τ_t increases.

This means that this taxation reduces the disposable income of a household, and that this requires less consumption in order to keep the same capital-labour ratio after the taxation as before it.

On the other hand, as for the $\dot{C}_t = 0$ locus, based on (4.27), the following relationship holds:

$$\frac{\partial k^{**}}{\partial \tau_t} = -\frac{1}{1-\alpha}\left(\frac{\alpha}{\gamma}\right)^{\frac{1}{1-\alpha}}(1-\tau_t)^{\frac{\alpha}{1-\alpha}} < 0. \tag{4.39}$$

(4.39) implies that the $\dot{C}_t = 0$ locus shifts to the left, as τ_t increases. This means that the taxation on capital income makes a household feel uncomfortable about consumption in future, and this requires higher rental fees in order to keep the same consumption in future as at present.

Based on the above analyses, we can describe the following relationship on a phase diagram with $(k_t,\, C_t)$ axes.

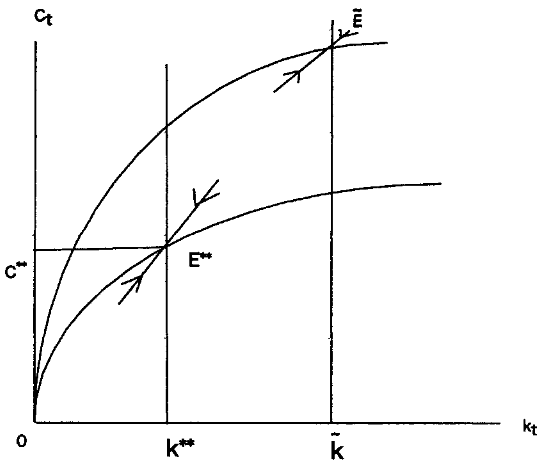

Figure 4.6

In this situation, there is a crucial point concerning a budget constraint on a household, which we have learned. We have deduced a phase diagram in Figure 4.6, under the condition expressed by (4.2).

However, this condition does not mean a lifetime budget constraint on a household, because a household can entirely consume its capital stock at infinity. In other words, we must deduce a phase diagram under a lifetime budget constraint, by considering this transversality condition.

Without the tranversality condition, we can not obtain any condition in which a household can attain its maximum utility.[5]

[5]In light of the transversality condition, we can establish a lifetime budget constraint

In this context, we can obtain a saddle path and a saddle point as in Figure 4.6. The existence of both a saddle path and a saddle point indicates the global stability of our economy, as in the case of labour income taxation. In short, it is clear that capital income taxation does not make our economy unstable.

Here, E^{**} means the long-term equilibrium of our economy, a saddle point, when capital income tax is imposed at a rate τ. Besides, k^{**} means the long-term capital-labour ratio under the tax rate τ.

Of course, $\tilde{E}$ means the long-term equilibrium of our economy and $\tilde{k}$ the long-term capital-labour ratio, respectively, when capital income tax is not imposed.

Incidentally, we can interpret the transversality condition from the household utilities viewpoint, as follows: Since a household has no desire to retain its capital at infinity, at that time the marginal utility of consumption of a household, which should be measured by capital stock at infinity, must be null.

We can express this as

$$\lim_{t \to \infty} \mu_t k_t = 0, \tag{4.40}$$

---

as

$$\int_0^\infty C_t \; exp[-\int_0^t u_\nu d\nu] \; dt = K_0 + \int_0^\infty (w_t H_t - \tau u_t K_t) \; exp[-\int_0^t u_\nu d\nu] \; dt.$$

The above relationship means that the present value of the consumption enjoyed by a household over infinity equals the value of initial capital plus the present value of disposable income over infinity, which is labour income minus the capital income tax levied. We have omitted a mathematical proof, because this proof is in the same vein as that in the last chapter, that is, in the case of labour income taxation.

which eventually yields

$$\lim_{t\to\infty} \frac{k_t}{C_t} e^{-\gamma t} = 0. \tag{4.41}$$

(4.41) means that the path through point E^{**} satisfies the tranversality condition.

We have learned that our economy goes along the corresponding saddle path, and eventually converges to the corresponding saddle point E^{**}, when capital income tax is imposed.
In this circumstance, when the tax is newly introduced in our economy, what initial conditions are determined? In short, when the tax is newly imposed, where our economy is placed on the saddle path is a new issue.

Let us discuss this issue.

First, we need to determine the initial position C_0 of consumption.

We have an Euler equation, (4.8), which describes a time path of consumption by a household, when a capital income tax is imposed at a rate τ, as

$$\frac{\dot{C}_t}{C_t} = u_t \ (1 - \tau_t) - \gamma.$$

Solving the above Euler equation with respect to time, we obtain the following:

$$C_t = C_0 \ exp[\int_0^t [(1 - \tau) \ u_\nu - \gamma] \ d\nu. \tag{4.42}$$

On the other hand, we have determined the lifetime budget constraint of a household, as follows:[6]

$$\int_0^\infty C_t \; exp[-\int_0^t u_\nu d\nu] \; dt = \tilde{K} + h_1, \tag{4.43}$$

where

$$h_1 = \int_0^\infty (w_t H_t - \tau u_t K_t) \; exp[-\int_0^t u_\nu d\nu] \; dt,$$

and $\tilde{K}$ means an initial capital stock when the tax is imposed, and is given.

Since (4.42) must be compatible with (4.43), we substitute (4.42) into (4.43).

Then, the following eventually holds:

$$C_0 \int_0^\infty exp[-\int_0^t (\tau \; u_\nu + \gamma) \; d\nu] \; dt = \tilde{K} + h_1. \tag{4.44}$$

Because both $\tilde{K}$ and h_1 are determined, C_0 is determined in order to satisfy (4.44).

On the other hand, the following holds in the labour market:

$$\frac{AC_0}{L - H_0} = (1-\alpha)\left(\frac{\tilde{K}}{H_0}\right)^{1-\alpha}. \tag{4.45}$$

In this situation, (4.45) determines the initial condition of H, H_0.

[6]See footnote 4.

4.4.1 Taxation on capital income

Suppose that the government imposes capital income tax at a certain rate, and that a household considers this taxation will continue forever. Then, a household replans its optimal scheme, taking this capital income tax into account, which lowers its disposable income.

In this situation, we know that our economy inevitably converges to a steady state, but now we need to specify both the value of respective economic variables in the long-term equilibrium and how these values move as the tax rates change.

As for factor prices, we can obtain the following:

$$u^{**} = \alpha(k^{**})^{\alpha-1},$$

which finally leads to

$$u^{**} = \frac{\gamma}{1-\tau}. \tag{4.46}$$

Based on (4.46), the following palpably holds:

$$\frac{\partial u^{**}}{\partial \tau} = \frac{\gamma}{(1-\tau)^2} > 0. \tag{4.47}$$

(4.47) means that capital prices increase, when capital income tax rates do, and vice versa.

The economic reason is as follows: once τ increases and a household is convinced that this situation will continue forever, it replans its optimal scheme. Then, because of this reduced disposable income, the household requires higher reward fees from capital stock in order to keep $\dot{C}_t = 0$, the end result of our economy. This conclusion is diametrically different from the one in the case of labour income tax, in the previous chapter.

Next, as for factor prices, the following obviously holds:

$$w^{**} = (1-\alpha)\,[\frac{(1-\tau)\alpha}{\gamma}]^{\frac{\alpha}{1-\alpha}}, \tag{4.48}$$

which features

$$\frac{\partial w^{**}}{\partial \tau} = -\frac{\alpha^2}{\gamma}[\frac{(1-\tau)\alpha}{\gamma}]^{\frac{2\alpha-1}{1-\alpha}} < 0. \tag{4.49}$$

(4.49) means that the equilibrium wage rate in the long term decreases when the tax rate increases, and vice versa, because the capital-labour ratio decreases.

From what we have seen so far, a tax increase on capital income causes less capital-labour intensity, and as a result, our economy converges to one with higher interest rates and lower wages.

Furthermore, other variables are subsequently determined as follows:

In the labour market, the following obviously holds:

$$w^{**} = \frac{A\,C^{**}}{L-H^{**}},$$

which yields

$$H^{**} = \frac{L\,w^{**} - A\,C^{**}}{w^{**}}, \tag{4.50}$$

and which can be further transformed as

$$H^{**} = \frac{L(1-\alpha)}{(A+1)(1-\alpha)+A\alpha(1-\tau)}. \tag{4.51}$$

In the above mathematical procedure, we utilize both (4.28) and (4.48). From (4.51), we obtain the following:

$$\frac{\partial H^{**}}{\partial \tau} = \frac{L\,(1-\alpha)A\alpha}{[(A+1)(1-\alpha)+A\alpha(1-\tau)]^2} > 0, \tag{4.52}$$

which means that the labour supply increases when the tax rates increase, because in this situation, wage rates become lower, while interest rates increase. Of course, the contrary phenomenon appears when the tax rates decrease. And these results sharply contrast with that in the case of labour income taxation, as shown in (3.74).

On the other hand, the capital stock is determined as

$$\begin{aligned} K^{**} &= k^{**}\,H^{**} \\ &= \frac{L(1-\alpha)\,[\frac{(1-\tau)\alpha}{\gamma}]^{\frac{1}{1-\alpha}}}{(A+1)(1-\alpha)+A\alpha(1-\tau)}. \end{aligned} \tag{4.53}$$

Based on a clue from (4.53), we need to investigate how K^{**} changes when the tax rates change. When we differentiate K^{**} with respect to τ, we obtain the following:

$$\frac{\partial K^{**}}{\partial \tau} = \frac{\psi}{[(A+1)(1-\alpha)+A\alpha(1-\tau)]^2},$$

where

$$\psi = L(1-\alpha)\frac{1}{1-\alpha}[\frac{(1-\tau)\alpha}{\gamma}]^{\frac{1}{1-\alpha}-1}(-\frac{\alpha}{\gamma})[(A+1)(1-\alpha)+A\alpha(1-\tau)] + L(1-\alpha)[\frac{(1-\tau)\alpha}{\gamma}]^{\frac{1}{1-\alpha}} A\alpha,$$

which leads to

$$\psi = L[\frac{(1-\tau)\alpha}{\gamma}]^{\frac{1}{1-\alpha}}[-\frac{\alpha}{\gamma}[\frac{(1-\tau)\alpha}{\gamma}]^{-1}[(A+1)(1-\alpha) + A\alpha(1-\tau)] + (1-\alpha)A\alpha],$$

and which is finally transformed as

$$\psi = -L[\frac{(1-\tau)\alpha}{\gamma}]^{\frac{1}{1-\alpha}}[\frac{(A+1)(1-\alpha)}{1-\tau} + A\alpha^2] < 0.$$

Eventually, we obtain the relationship between K^{**} and τ as

$$\frac{\partial K^{**}}{\partial \tau} = -\frac{L[\frac{(1-\tau)\alpha}{\gamma}]^{\frac{1}{1-\alpha}}[\frac{(A+1)(1-\alpha)}{1-\tau} + A\alpha^2]}{[(A+1)(1-\alpha)+A\alpha(1-\tau)]^2} < 0. \quad (4.54)$$

From (4.54), we can understand that the amount of capital in production decreases as the tax rates increase, and vise versa.

Next, we will explore the equilibrium production in the long term.

The following hold:

$$
\begin{aligned}
Y^{**} &= (K^{**})^{\alpha}(H^{**})^{1-\alpha} \\
&= \frac{L(1-\alpha)[\frac{(1-\tau)\alpha}{\gamma}]^{\frac{\alpha}{1-\alpha}}}{(A+1)(1-\alpha)+A\alpha(1-\tau)}.
\end{aligned} \tag{4.55}
$$

When the tax rates increase, K^{**} decreases, while H^{**} increases. So on appearance, we can not identify whether production is larger or not, before and after the tax rates increase. We need a more detailed analysis of this important issue.

When we differentiate (4.55) with respect to τ, the following holds:

$$
\frac{\partial Y^{**}}{\partial \tau} = \frac{\xi(\tau)}{[(A+1)(1-\alpha)+A\alpha(1-\tau)]^2},
$$

where

$$
\begin{aligned}
\xi(\tau) = -L(1-\alpha)\frac{\alpha}{1-\alpha}[\frac{(1-\tau)\alpha}{\gamma}]^{\frac{\alpha}{1-\alpha}-1}\left(\frac{\alpha}{\gamma}\right)[(A+1)(1-\alpha)+A\alpha(1-\tau)] \\
+ L(1-\alpha)A\alpha[\frac{(1-\tau)\alpha}{\gamma}]^{\frac{\alpha}{1-\alpha}},
\end{aligned}
$$

and which can be transformed in the following way:

$$
\begin{aligned}
\xi(\tau) = -L[\frac{(1-\tau)\alpha}{\gamma}]^{\frac{\alpha}{1-\alpha}}[[\frac{\alpha(1-\alpha)}{1-\tau} \\
+ \alpha(2\alpha-1)]A + \frac{\alpha(1-\alpha)}{1-\tau}]].
\end{aligned} \tag{4.56}
$$

Now, we define the function with respect to A, as follows:

$$\Phi(A) = \frac{1}{1-\tau}[\alpha(1-\alpha) + \alpha(2\alpha-1)(1-\tau)]\ A + \frac{\alpha(1-\alpha)}{1-\tau}. \qquad (4.57)$$

In the case of $2\alpha - 1 \geqq 0$, $\Phi(A) > 0$ obviously holds, which leads to $\xi(\tau) < 0$. Therefore we consider only the case of $\alpha < \frac{1}{2}$.

Besides, we define the contents in the brackets on the right side of (4.57), as

$$h(\tau) = -\alpha(2\alpha-1)\tau + \alpha(1-\alpha) + \alpha(2\alpha-1),$$

which has the properties such as

$$\begin{cases} h(\tau) = -\alpha(2\alpha-1)\tau + \alpha^2, \\ h(0) = \alpha^2, \\ h(1) = \alpha(1-\alpha) > 0, \\ h'(\tau) = -\alpha(2\alpha-1) > 0. \end{cases}$$

Based on the above analysis, we can say $h(\tau) > 0$, for $0 \leqq \tau \leqq 1$.
As a result, the value of the contents in the brackets on the right side of (4.57) is positive. This conclusion leads to $\xi(\tau) < 0$, for $0 < \alpha < 1$. And this means $\partial Y^{**}/\partial\tau < 0$.

We can express this situation in the following way:

$$\frac{\partial Y^{**}}{\partial \tau} = -\frac{L[\frac{(1-\tau)\alpha}{\gamma}]^{\frac{\alpha}{1-\alpha}}\Phi(A)}{[(A+1)(1-\alpha) + A\alpha(1-\tau)]^2} < 0.$$

In short, a tax increase on capital income causes a decrease of the production scale in the long-term equilibrium.

Here, government spending, which equals tax revenue, is determined as

$$\begin{aligned} G^{**} &= \tau u^{**} K^{**} \\ &= \frac{L(1-\alpha)\,\gamma\left(\frac{\alpha}{\gamma}\right)^{\frac{1}{1-\alpha}} \tau(1-\tau)^{\frac{\alpha}{1-\alpha}}}{(A+1)(1-\alpha)+A\alpha(1-\tau)}. \end{aligned} \tag{4.58}$$

Based on both (4.58) and (4.28), we can easily confirm that $C^{**} + G^{**} = Y^{**}$, that is, all output is entirely alloted to consumption and government spending.

Now, we should investigate how C^{**} changes as τ moves.

Differentiating (4.28) with respect to τ, we obtain the following:

$$\frac{\partial C^{**}}{\partial \tau} = \frac{numerator}{[(A+1)(1-\alpha)+A\alpha(1-\tau)]^2},$$

where the following holds in addition:

$$\begin{aligned} numerator = {} & L(1-\alpha)\left(\frac{\alpha}{\gamma}\right)^{\frac{\alpha}{1-\alpha}} [-\alpha(1-\tau)^{\frac{\alpha}{1-\alpha}} \\ & - [(1-\tau)\alpha+(1-\alpha)]\frac{\alpha}{1-\alpha}(1-\tau)^{\frac{\alpha}{1-\alpha}-1}]][(A+1)(1-\alpha)+A\alpha(1-\tau)] \\ & + L(1-\alpha)\left(\frac{\alpha}{\gamma}\right)^{\frac{\alpha}{1-\alpha}} [(1-\tau)\alpha+(1-\alpha)](1-\tau)^{\frac{\alpha}{1-\alpha}} A\,\alpha, \end{aligned}$$

which leads to

$$numerator = L(1-\alpha)\left(\frac{\alpha}{\gamma}\right)^{\frac{\alpha}{1-\alpha}}(1-\tau)^{\frac{\alpha}{1-\alpha}}\times$$
$$[-\alpha - [(1-\tau)\alpha + (1-\alpha)]\frac{\alpha}{1-\alpha}\frac{1}{1-\tau}][(A+1)(1-\alpha) + A\alpha(1-\tau)]$$
$$+ A\alpha[(1-\tau)\alpha + (1-\alpha)]].$$

Moreover, after some mathematical manipulations, the contents in the brackets [ ] in the above relationship can be transformed as

$$[\quad] = -[\alpha(1+2A\alpha) + \frac{A\alpha^2(1-\tau)}{1-\alpha} + \frac{(A+1)\alpha(1-\alpha)}{1-\tau}] < 0.$$

As a consequence, we obtain a conclusion as follows:

$$\frac{\partial C^{**}}{\partial \tau} = -\frac{L(1-\alpha)\left(\frac{\alpha}{\gamma}\right)^{\frac{\alpha}{1-\alpha}}(1-\tau)^{\frac{\alpha}{1-\alpha}}[\alpha(1+2A\alpha) + \frac{A\alpha^3(1-\tau)}{1-\alpha} + \frac{(A+1)\alpha(1-\alpha)}{1-\tau}]}{[(A+1)(1-\alpha) + A\alpha(1-\tau)]^2} < 0. \tag{4.59}$$

In other words, the consumption in the long-term equilibrium decreases as τ increases.

Finally, let us explore the movement of consumption in relation to labour. Because we have already obtained the values of both C^{**} and H^{**}, we can easily calculate this, as follows:

$$\frac{C^{**}}{H^{**}} = [(1-\tau)\alpha + (1-\alpha)]\left(\frac{\alpha}{\gamma}\right)^{\frac{\alpha}{1-\alpha}}(1-\tau)^{\frac{\alpha}{1-\alpha}}. \tag{4.60}$$

Differentiating (4.60) with respect to τ, we obtain the following:

$$\frac{\partial(\frac{C^{**}}{H^{**}})}{\partial\tau} = \left(\frac{\alpha}{\gamma}\right)^{\frac{\alpha}{1-\alpha}} [-\alpha(1-\tau)^{\frac{\alpha}{1-\alpha}}$$
$$-[(1-\tau)\alpha + (1-\alpha)]\frac{\alpha}{1-\alpha}(1-\tau)^{\frac{\alpha}{1-\alpha}-1}],$$

which after some mathematical manipulations yields

$$\frac{\partial(\frac{C^{**}}{H^{**}})}{\partial\tau} = -\alpha\left(\frac{\alpha}{\gamma}\right)^{\frac{\alpha}{1-\alpha}}(1-\tau)^{\frac{\alpha}{1-\alpha}}(\frac{1}{1-\tau}+\frac{1}{1-\alpha}) < 0. \qquad (4.61)$$

(4.61) implies that the tax increase on capital income makes consumption in relation to labour in the long-term equilibrium decrease, as we found that on labour income does in the previous chapter.

4.4.2 Disturbances caused by capital income taxation

We have seen how taxation on capital income affects several economic variables in our economy, but here, in addition, we explore how much these variables are affected concretely.

We need to compare two values in the long-term equilibrium, that is, the value with capital income tax and that with no tax, for respective economic variables. We denote the values with no tax in the long-term equilibrium with a tilde below.

We can obtain the following results concerning factor prices, respectively:

$$\frac{u^{**}}{\tilde{u}} = \frac{1}{1-\tau} > 1, \tag{4.62}$$

and

$$\frac{w^{**}}{\tilde{w}} = (1-\tau)^{\frac{\alpha}{1-\alpha}} < 1. \tag{4.63}$$

The above outcomes result from the following phenomenon:

$$\frac{k^{**}}{\tilde{k}} = (1-\tau)^{\frac{1}{1-\alpha}} < 1. \tag{4.64}$$

Moreover, we obtain the following relationship concerning consumption:

$$\frac{C^{**}}{\tilde{C}} = \frac{(A+1)(1-\alpha)+A\alpha}{(A+1)(1-\alpha)+A\alpha(1-\tau)}[(1-\tau)\alpha+(1-\alpha)](1-\tau)^{\frac{\alpha}{1-\alpha}}. \tag{4.65}$$

In addition, the following obviously hold:

$$\frac{(A+1)(1-\alpha)+A\alpha}{(A+1)(1-\alpha)+A\alpha(1-\tau)} > 1,$$

$$(1-\tau)\alpha+(1-\alpha) < 1,$$

and

$$(1-\tau)^{\frac{\alpha}{1-\alpha}} < 1.$$

So, on appearance, the degree of $C^{**}/\tilde{C}$ may be greater than 1 or not.

However, we have found that $\partial C^{**}/\partial\tau < 0$ from (4.59), and this implies $C^{**}/\tilde{C} < 1$.

Concerning factors H and K, we obtain the following:

$$\frac{H^{**}}{\tilde{H}} = \frac{(A+1)(1-\alpha)+A\alpha}{(A+1)(1-\alpha)+A\alpha(1-\tau)} > 1. \tag{4.66}$$

And

$$\frac{K^{**}}{\tilde{K}} = \frac{(A+1)(1-\alpha)+A\alpha}{(A+1)(1-\alpha)+A\alpha(1-\tau)}(1-\tau)^{\frac{1}{1-\alpha}} < 1, \tag{4.67}$$

because of $\partial K^{**}/\partial\tau < 0$ from (4.54).

We can interpret this situation as follows:

As τ increases, the first term on the right side of (4.67),

$$\frac{(A+1)(1-\alpha)+A\alpha}{(A+1)(1-\alpha)+A\alpha(1-\tau)},$$

equals the rate of increment in H, and is larger than 1, while the second term, $(1-\tau)^{\frac{1}{1-\alpha}}$, is less than 1. In this situation, since the degree of the latter is larger than that of the former, $K^{**}/\tilde{K}$ becomes less than 1.

As for production, we obtain the following conclusion:

$$\frac{Y^{**}}{\tilde{Y}} = \frac{(A+1)(1-\alpha)+A\alpha}{(A+1)(1-\alpha)+A\alpha(1-\tau)}(1-\tau)^{\frac{\alpha}{1-\alpha}} < 1, \tag{4.68}$$

because of $\partial Y^{**}/\partial\tau < 0$, which we have seen.

Based on logic similar to that used for $K^{**}/\tilde{K}$, we can verify that $Y^{**}/\tilde{Y}$

becomes less than 1.

Besides, based on both (4.67) and (4.68), $K^{**}/\tilde{K} = (1-\tau)Y^{**}/\tilde{Y}$ holds. This implies that the rate of decrease in K is larger than that in Y by $(1-\tau)$, and this is independent of α. In other words, when tax is imposed, K decreases largely, while H increases, in the long term. However, Y has a lower rate of decrease than that in K.

From what we have analyzed so far, we can conclude that the taxation on capital income causes an increase in labour, a decrease in capital stock, and as a result, a reduction in production or the scale of an economy. These outcomes are contrary to those in the case of taxation on labour income, in which labour, capital stock and production all decrease in the same proportion.

Finally, concerning consumption in relation to labour, we can easily verify the following:

$$\frac{\frac{C^{**}}{H^{**}}}{\frac{\tilde{C}}{\tilde{H}}} = [(1-\tau)\alpha + (1-\alpha)](1-\tau)^{\frac{\alpha}{1-\alpha}} < 1. \tag{4.69}$$

In other words, the taxation on capital income causes a reduction in consumption in relation to labour, while productivity in relation to labour decreases.

4.4.3 Tax rates and government revenue

We need to analyze the relationship between tax rates and government revenue, which comes from taxation only on capital income.

As deduced in terms of (4.58), when the tax rate is τ, government revenue, $G^{**} = \tau u^{**}K^{**}$, is determined as

$$G^{**} = \frac{L(1-\alpha)\gamma\left(\frac{\alpha}{\gamma}\right)^{\frac{1}{1-\alpha}}\tau(1-\tau)^{\frac{\alpha}{1-\alpha}}}{(A+1)(1-\alpha)+A\alpha(1-\tau)}. \tag{4.70}$$

From the above definition of G^{**}, we know that the tax base $u^{**}K^{**}$ may tend to increase or decrease, as the tax rates increase. If the tax base can decrease, as the tax rates increase, the government revenue levied by capital income tax can have a maximum at a certain tax rate. We deal with this issue in what follows.

Differentiating (4.70) partially with respect to τ, we obtain the following:

$$\frac{\partial G^{**}}{\partial \tau} = \frac{numerator}{[(A+1)(1-\alpha)+A\alpha(1-\tau)]^2},$$

where

$$numerator = L(1-\alpha)\ \gamma\ \left(\frac{\alpha}{\gamma}\right)^{\frac{1}{1-\alpha}} [((1-\tau)^{\frac{\alpha}{1-\alpha}} - \frac{\alpha}{1-\alpha}\tau(1-\tau)^{\frac{\alpha}{1-\alpha}-1})\times$$
$$((A+1)(1-\alpha)+A\alpha(1-\tau)) + \tau(1-\tau)^{\frac{\alpha}{1-\alpha}}\ A\ \alpha]^1, \tag{4.71}$$

which is transformed by defining the bracket $[\quad]^1$ as below.

$$[\quad]^1 = (1-\tau)^{\frac{\alpha}{1-\alpha}}[(1-\frac{\alpha\ \tau}{1-\alpha}(1-\tau)^{-1})((A+1)(1-\alpha)+A\alpha(1-\tau))+\tau A\alpha].$$

This can be further expressed as

$$[\quad]^1 = (1-\tau)^{\frac{\alpha}{1-\alpha}}[\quad]^2.$$

Here, it follows

$$[\quad]^2 = \frac{1-\alpha-\tau}{(1-\alpha)(1-\tau)}[(A+1)(1-\alpha)+A\alpha(1-\tau)]+\tau A\alpha,$$

and this can be transformed as

$$[\quad]^2 = \frac{1}{(1-\alpha)(1-\tau)}[(1-\alpha-\tau)(A+1)(1-\alpha)+(1-\alpha-\tau)A\alpha(1-\tau)$$
$$+\tau A\alpha(1-\alpha)(1-\tau)].$$

Additionally, the contents of the bracket [ ] on the right side in the above relationship, which is defined as Ψ, can be finally transformed as

$$\Psi(\tau) = A\alpha^2\tau^2 - (A+1-\alpha)\,\tau + (1-\alpha)(A+1-\alpha). \tag{4.72}$$

From what we have analyzed so far, we can eventually express the *numerator* in the following way:

$$numerator = L(1-\alpha)\gamma\left(\frac{\alpha}{\gamma}\right)^{\frac{1}{1-\alpha}}(1-\tau)^{\frac{\alpha}{1-\alpha}}\,\frac{\Psi(\tau)}{(1-\alpha)(1-\tau)}. \tag{4.73}$$

Since the sign of the *numerator* obviously depends only on $\Psi(\tau)$, in what follows, we focus only on $\Psi(\tau)$.

We consider (4.72) as a function with respect to τ, and investigate how function $\Psi(\tau)$ behaves as τ moves.

Now, we consider (4.72) as an equation, and $\Psi(\tau) = 0$. In short, we consider the following quadratic equation with respect to τ:

$$A\alpha^2\tau^2 - (A+1-\alpha)\tau + (1-\alpha)(A+1-\alpha) = 0. \tag{4.74}$$

First of all, we investigate whether (4.74) has real roots or not.

The sign of a discriminant for (4.74) depends on that of the following function $g(A)$:[7]

$$g(A) = [1 - 4\alpha^2\ (1-\alpha)]A + (1-\alpha). \tag{4.75}$$

[7]The discriminant for a quadratic equation can be derived, in general, as follows: Consider the following quadratic form:

$$P(x) = x'Ex.$$

Here, x means a vector with 2×1, and E a symmetric matrix with 2×2, which is specifically defined as

$$E = \begin{pmatrix} a & b \\ b & c \end{pmatrix}.$$

In addition, we define x as $x' = [y, 1]'$. In this situation, the quadratic form $P(x)$ can be expressed as

$$x'Ex = ay^2 + 2by + c.$$

If the above quadratic form is positive, the principal minor determinants of E must all be positive, as both necessary and sufficient conditions. In short, the following must hold:

$$\begin{cases} a > 0, \\ ac - b^2 > 0. \end{cases}$$

Of course, if the above conditions are satisfied, the quadratic equation does not have any real roots.

Of course, if $g(A) \geqq 0$, the sign of the discriminant is positive or null, and conversely if $g(A) < 0$, it is negative.

Here, our aim is to show $g(A) > 0$.

Based on (4.75), we obtain the following:

$$\frac{dg}{dA} = 1 - 4\alpha^2(1 - \alpha).$$

In this situation, we define the above relationship as a function with respect to α, and then the following holds:

$$\Omega(\alpha) = 4\alpha^3 - 4\alpha^2 + 1. \tag{4.76}$$

After some mathematical manipulations, we can verify some characteristics concerning $\Omega(\alpha)$, as follows:

$$\frac{d\Omega}{d\alpha} \begin{cases} \leqq 0 & \text{in the case of } 0 < \alpha \leqq \dfrac{2}{3}, \\ > 0 & \text{in the case of } \dfrac{2}{3} < \alpha < 1. \end{cases} \tag{4.77}$$

And

$$\begin{cases} \Omega(\frac{2}{3}) = \dfrac{11}{27}, \\ \Omega(0) = 1, \\ \Omega(1) = 1. \end{cases}$$

From the above analyses, obviously $\Omega(\alpha) > 0$.

Then it becomes clear that $g(A) = \Omega(\alpha)A + (1-\alpha) > 0$, in other words, that the discriminant is positive. Therefore, $\Psi(\tau) = 0$ has two real roots.

We respectively denote these roots as τ_1 and τ_2, and assume that $\tau_1 < \tau_2$, which then have properties such as

$$\begin{cases} \tau_1 + \tau_2 = \dfrac{A+1-\alpha}{A\alpha^2} > 0, \\ \tau_1 \cdot \tau_2 = \dfrac{(1-\alpha)(A+1-\alpha)}{A\alpha^2} > 0. \end{cases}$$

The above relationships clearly show that both roots are positive.

Next, differentiating (4.72) with respect to τ, we obtain the following:

$$\Psi'(\tau) = 2A\alpha^2\tau - (A+1-\alpha). \tag{4.78}$$

Denote τ as τ^*, which satisfies $\Psi'(\tau) = 0$ and $\tau^* = \frac{A+1-\alpha}{2A\alpha^2}$. In this situation, we can obtain the following:

$$\Psi(\tau^*) = -\frac{(A+1-\alpha)^2}{4A\alpha^2}g(A) < 0. \tag{4.79}$$

Additionally, $\Psi(\tau)$ has properties such as

$$\begin{cases} \Psi(0) = (1-\alpha)(A+1-\alpha) > 0, \\ \Psi(1) = -\alpha(1-\alpha)(1+A) < 0. \end{cases}$$

Moreover, let us examine which of the two, $\tau*$ or 1, is larger.

Now, we consider τ^* as a function of A and define it as follows:

$$Q(A) = \frac{A + (1-\alpha)}{2A\alpha^2},$$

which yields

$$Q(A) = \frac{1 + \dfrac{1-\alpha}{A}}{2\alpha^2}. \tag{4.80}$$

From (4.80) we can identify the following:

$$Q(A) \begin{cases} \rightarrow \dfrac{1}{2\alpha^2} & \text{in the case of } A \rightarrow \infty, \\ \rightarrow \infty & \text{in the case of } A \rightarrow 0. \end{cases} \tag{4.81}$$

In addition, $Q(A)$ is a decreasing function with respect to A, and we can verify this as

$$\frac{dQ}{dA} = -\frac{1-\alpha}{2\alpha^2 A^2} < 0. \tag{4.82}$$

Based on what we have discussed so far, we can conclude that τ^* may be larger than 1 in some cases, or less than 1 in other cases. Therefore, for $0 < \tau < 1$, $\Psi(\tau)$ changes from $(1-\alpha)(A+1-\alpha) > 0$ to $-\alpha(1-\alpha)(1+A) < 0$, as τ increases.

Returning to $\partial G^{**}/\partial\tau$, we obtain the following:

$$\frac{\partial G^{**}}{\partial \tau} = \frac{L(1-\alpha)\gamma\left(\dfrac{\alpha}{\gamma}\right)^{\frac{1}{1-\alpha}}(1-\tau)^{\frac{\alpha}{1-\alpha}}\dfrac{\Psi(\tau)}{(1-\alpha)(1-\tau)}}{[(A+1)(1-\alpha)+A\alpha(1-\tau)]^2}. \tag{4.83}$$

Furthermore, G^{**} has properties such as

$$\begin{cases} G^{**}(0) = 0, \\ G^{**}(1) = 0. \end{cases}$$

From what we have analyzed so far, we obtain the following relationship, depicted in Figure 4.7.

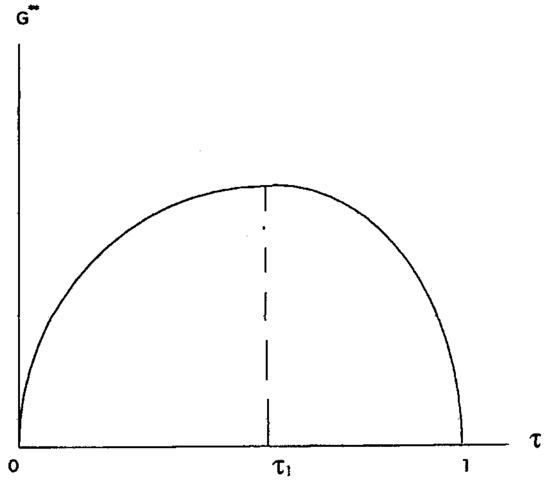

Figure 4.7

Here, τ_1 is the lower root in $\Psi(\tau)$ (see (4.72)), which is explicitly expressed as

$$\tau_1 = \frac{A+1-\alpha-\sqrt{(A+1-\alpha)[A+1-\alpha-4A\alpha^2(1-\alpha)]}}{2A\alpha^2}. \tag{4.84}$$

τ_1 means an optimal tax rate, at which government revenue is maximized.

4.5 Conclusions

We have dealt with how our economy moves when capital income taxes are introduced. And we conclude that a mechanism towards stabilization exists in our economy, as in the case of labour income taxes. This is derived from the demonstration of the existence of a unique saddle point, which necessarily accompanies a corresponding unique saddle path.

We should notice that in the process of this demonstration, the condition $AC_t - Lk_t^{\alpha} < 0$, which is derived from the equilibrium condition in the labour market, is crucial. Capital income taxes reduce the disposable income of a household by the amount of the capital income tax levied on the assumption of a balanced-budget rule, and this influences our economy through the following two processes. In the first, the $\dot{C}_t = 0$ locus is shifted to the point of origin. The capital income tax reduces the real rental rates which a household in fact receives, as Euler equation (4.21) suggests. This implies that a household feels uncomfortable about future consumption at a certain level of rental rates, where it felt comfortable before. This means that a household chooses savings at this rental rate before the taxation on capital income, while it chooses current consumption at a similar rental rate after the taxation. It follows that higher rental rates are necessary for $\dot{C}_t = 0$; in other words, the taxation causes lower capital-labour ratios. However, this increase in rental rates associated with lower capital-labour ratios does not correspond with an increase in savings, but rather a decrease. In the second process, the $\dot{k}_t = 0$ locus is shifted downwards. This implies that a household chooses small consumption after the taxation on capital income, because this taxation reduces the disposable income of a household. In other words, in order to keep $\dot{k}_t = 0$, smaller consumption is correspondingly needed.

Additionally, suppose that once the capital income taxation is introduced, $\dot{C}_t$ becomes negative, which shows that the consumption in future, that is, current savings, decreases. This leads to a decrease in the capital-labour ratios, which causes higher rental ratios, but since the increase in the higher rental ratios is reduced by the capital income tax, $\dot{C}_t$ remains negative, and this tendency continues until $(1-\tau)u$ equals γ. There is no cycle derived from the mechanism by which this higher rental rates is reversed by the increase in k, which leads to cycles. Therefore, no circulating movements occur under this taxation.

We have analyzed how our economies respectively move, when both kinds of taxation are introduced. So far, we know that fluctuations never occur, but we do not know which form of taxation leads to better economic performance. We will deal with this interesting topic in the next chapter.

4.6 Appendix

Here, we establish a proof $f'' < 0$ in the case of $\frac{1}{2} \leqq \alpha < 1$.

We begin with (4.34), and for simplicity omit the subscript t in both k_t and τ_t, below.

In this situation, we can express (4.34) in more detail as

$$\frac{df}{dk} = \frac{bk^{2(\alpha-1)}[(2\alpha-1)\gamma A + \alpha(1-\alpha)(A+1)k^{\alpha-1}]}{[(A+1)(1-\alpha)k^{\alpha-1}+\gamma A]^2}. \tag{4.85}$$

Additionally we define b below:

$$b = L(1-\alpha)[(1-\tau)\alpha + (1-\alpha)].$$

Differentiating (4.85) with respect to k, we obtain the following:

$$\frac{d^2 f}{dk^2} = \frac{numerator}{[(A+1)(1-\alpha)k^{\alpha-1}+\gamma A]^4}. \tag{4.86}$$

Here,

$$\begin{aligned} numerator = {} & [b\ 2(\alpha-1)\ k^{2\alpha-3}((2\alpha-1)\ \gamma\ A + \alpha(1-\alpha)(A+1)k^{\alpha-1}) \\ & + b\ k^{2\alpha-2}\alpha(1-\alpha)(A+1)(\alpha-1)k^{\alpha-2}][(A+1)(1-\alpha)k^{\alpha-1}+\gamma A]^2 \\ & - b\ k^{2\alpha-2}[(2\alpha-1)\gamma A + \alpha(1-\alpha)(A+1)k^{\alpha-1} \times 2[(A+1)(1-\alpha)k^{\alpha-1} \\ & + \gamma A](A+1)(1-\alpha)(\alpha-1)k^{\alpha-2}], \end{aligned} \tag{4.87}$$

which is eventually transformed in the following way:

$$\begin{aligned} numerator &= 2\ b((A+1)(1-\alpha)\ k^{\alpha-1} + \gamma\ A)(1-\alpha) \\ &\times [-((A+1)(1-\alpha)\ k^{\alpha-1} + \gamma\ A)\ k^{2\alpha-3}((2\alpha-1)\gamma\ A \\ &+ \alpha(1-\alpha)(A+1)\ k^{\alpha-1}) \\ &- ((A+1)((1-\alpha)\ k^{\alpha-1} + \gamma\ A)\alpha(1-\alpha)(A+1)\ k^{3\alpha-4} \\ &+ (1-\alpha)(A+1)((2\alpha-1)\gamma\ A + \alpha(1-\alpha)(A+1)k^{\alpha-1})\ k^{3\alpha-4}]. \end{aligned} \tag{4.88}$$

When we describe the *contents* of the bracket [ ] in (4.88) more clearly, we can obtain the following :

$$\begin{aligned} contents &= -((A+1)((1-\alpha)\ k^{\alpha-1} + \gamma\ A)((2\alpha-1)\gamma\ A + \alpha(1-\alpha)(A+1)\ k^{\alpha-1})\ k^{2\alpha-3} \\ &+ k^{3\alpha-4}(1-\alpha)(A+1)[(2\alpha-1)\ \gamma A + \alpha(1-\alpha)(A+1)\ k^{\alpha-1} \\ &- \alpha((A+1)(1-\alpha)\ k^{\alpha-1} + \gamma\ A)]. \end{aligned} \tag{4.89}$$

Since the contents of the bracket [ ] on the right side of (4.89) lead to $-(1-\alpha)\gamma\ A$, (4.89) can be further transformed as follows:

$$\begin{aligned} contents &= -((A+1)((1-\alpha)\ k^{\alpha-1} + \gamma\ A)((2\alpha-1)\gamma\ A + \alpha(1-\alpha)(A+1)\ k^{\alpha-1})\ k^{2\alpha-3} \\ &- k^{3\alpha-4}(1-\alpha)(A+1)(1-\alpha)\ \gamma\ A. \end{aligned} \tag{4.90}$$

When we summarize (4.90), focusing on the order of k, the following holds:

$$\begin{aligned} contents &= -2\ \alpha(1-\alpha)\gamma\ A(A+1)\ k^{3\alpha-4} - \alpha(1-\alpha)^2(A+1)^2 k^{4\alpha-5} \\ &\quad - (2\alpha-1)\gamma^2\ A^2\ k^{2\alpha-3}, \end{aligned} \tag{4.91}$$

which is finally expressed in the following way:

$$\begin{aligned} contents &= -[2\alpha(1-\alpha)\gamma\ A(A+1)\ k^{\alpha-1} + \alpha(1-\alpha)^2(A+1)^2\ k^{2\alpha-2} \\ &\quad + (2\alpha-1)\gamma^2\ A^2]\ k^{2\alpha-3} < 0. \end{aligned} \tag{4.92}$$

Based on what we have analyzed so far, we can obtain a conclusion:

$$\begin{aligned} numerator &= -2\ b[(A+1)(1-\alpha)\ k^{\alpha-1} + \gamma\ A](1-\alpha) \\ &\quad \times [2\alpha(1-\alpha)\ \gamma\ A(A+1)\ k^{\alpha-1} + \alpha(1-\alpha)^2(A+1)^2\ k^{2\alpha-2} \\ &\quad + (2\alpha-1)\gamma^2\ A^2]\ k^{2\alpha-3} < 0. \end{aligned} \tag{4.93}$$

(4.93) obviously means $d^2 f/dk^2 < 0$, which we require.

Chapter 5

A Comparison of Labour vs. Capital Income Taxes

This chapter deals with a conventional but hot issue regarding which form of income taxation, labour or capital, governments should adopt, in light of the maximization of social welfare. Social welfare in this context means consumption in relation to labour. We eventually obtain a simple and clear conclusion: that labour income taxation unconditionally leads to better economic performance in terms of consumption in relation to labour.

5.1 Introduction

We have previously derived the existence of an internal mechanism in our economy which leads towards stabilization, and this appears as a unique saddle point. Therefore, our economy eventually converges to this saddle point in the long term. This is applicable to both labour and capital income taxation.

In this situation, which form of taxation yields better economic performance? This is the main topic in this chapter.

Our model framework assumes that governments only levy taxes and the corresponding spending does not benefit a household. This is implied in the assumption of utility functions, in which government spending is not included as an ingredient. In this situation, it would be palpably best for a household to have no taxation and no government spending, but we assume that government spending is necessary for some reasons. So we can consider the levy is reasonable. Then we explore which form of taxation is preferable.

We concretely assume the following:

First, we select consumption in relation to labour as an indication of economic performance, that is, of social welfare, as Diamond (1965) did. Chamley (1981) selects the utility of a household as an indication of this, but the utility shows the economic performance only from the viewpoint of a household.

Second, Chamley (1981) deals with changes in utilities along a saddle path as well as in terms of long-term equilibrium, before and after capital income taxes are introduced. However, we deal with the changes in values of economic variables only in terms of long-term equilibrium.

Third, we analyze whether consumption in relation to labour is larger under one form of taxation than the other. Similar government spending is assumed, whether labour or capital taxes are introduced. Of course, government plays an important role and its role has two sides, positive or negative, from the viewpoint of the private sector. We intend, throughout this dissertation, to focus on only the negative points. In this sense, in our situation, it is natural to assume similar government spending, under both forms of taxation. So, we can judge in this postulation that the form of taxation in which consumption in relation to labour is larger is better.

In other words, we assume a comparison as follows:

In light of constant government spending, first of all, governments determine their corresponding tax rate. Then, under this tax rate, they evaluate consumption in relation to labour. This procedure is carried out under both forms of taxation. Finally, governments compare the two forms of consumption in relation to labour.

Based on the above framework, we finally obtain a simple and clear conclusion: that consumption in relation to labour is greater under labour than capital income taxes, regardless of the level of α and of government spending. Therefore, we deduce that labour income taxation is preferable as a policy, in other words, that capital income tax rates should be null.

The overall structure of this chapter is as follows:

In section 5.2, we deduce the situations in which consumption in relation to labour are larger, at arbitrary tax rates. As a result of this analysis, we find that in the case of comparatively small α, consumption in relation to labour under capital income taxes can be larger than under labour income taxes, and that in other cases of α, the opposite phenomenon occurs. Subsequently, we investigate how governments determine their tax rates under both forms of taxation, in section 5.3. In what follows, in section 5.4, we compare both forms of consumption in relation to labour. And section 5.5 is devoted to conclusions.

5.2 Consumption in relation to labour

In what follows, we intend first to investigate the relationship between both forms of consumption in relation to labour, on the assumption that τ^* equals τ^{**}, and second to return to the above discussions, on the assumption that $\tau^* \neq \tau^{**}$. Here, τ^* means labour income tax rates, τ^{**} capital income tax rates.

In a sense, we intend to solve a problem to maximize an objective function, under a constraint. Of course, in our context, this objective function means consumption in relation to labour, and the constraint is constant taxes levied, that is, constant government spending.

As a preparation for solving this problem, first of all, we investigate the relationship between both forms of consumption in relation to labour.

We have obtained the following:

$$\frac{C^*}{H^*} = [\alpha + (1-\alpha)(1-\tau)] \left(\frac{\alpha}{\gamma}\right)^{\frac{\alpha}{1-\alpha}}, \tag{5.1}$$

and

$$\frac{C^{**}}{H^{**}} = [(1-\tau)\alpha + (1-\alpha)]\left(\frac{\alpha}{\gamma}\right)^{\frac{\alpha}{1-\alpha}}(1-\tau)^{\frac{\alpha}{1-\alpha}}. \tag{5.2}$$

Here, C^{*}/H^{*} means the consumption in relation to labour attained under labour income taxes, and C^{**}/H^{**} that attained under capital income taxes.

At this stage, because we need to know which is larger, we consider the ratio of the latter divided by the former.

Then, we obtain the following:

$$\frac{\frac{C^{**}}{H^{**}}}{\frac{C^{*}}{H^{*}}} = \frac{[(1-\tau)\alpha + (1-\alpha)](1-\tau)^{\frac{\alpha}{1-\alpha}}}{\alpha + (1-\alpha)(1-\tau)}. \tag{5.3}$$

Here, based on (5.3) we define a function $\phi(t)$ as

$$\phi(t) = \frac{(t\ \alpha + 1 - \alpha)\ t^{\frac{\alpha}{1-\alpha}}}{\alpha + (1-\alpha)\ t}. \tag{5.4}$$

Additionally, we define $t = 1 - \tau$, and $0 < t < 1$.

Our aim is to investigate how $\phi(t)$ behaves, as t moves.

Differentiating (5.4) with respect to t, we obtain the following:

$$\phi'(t) = \frac{numerator}{[\alpha + (1-\alpha)t]^2}. \tag{5.5}$$

The *numerator* is defined as

$$
\begin{aligned}
numerator &= [\alpha\ t^{\frac{\alpha}{1-\alpha}} + (t\ \alpha + 1 - \alpha)\frac{\alpha}{1-\alpha}\ t^{\frac{\alpha}{1-\alpha}-1}][\alpha + (1-\alpha)\ t] \\
&- [t\ \alpha + (1-\alpha)]\ t^{\frac{\alpha}{1-\alpha}}(1-\alpha) \\
&= t^{\frac{\alpha}{1-\alpha}}[(\alpha + (t\alpha + 1 - \alpha)\frac{\alpha}{1-\alpha}t^{-1})(\alpha + (1-\alpha)t) \\
&- (1-\alpha)(t\ \alpha + (1-\alpha))]. \qquad (5.6)
\end{aligned}
$$

Here, the contents in the brackets [ ] on the right side of (5.6) can eventually be transformed as

$$[\quad] = \frac{1}{1-\alpha}(2\alpha^3 - 4\alpha^2 + 4\alpha - 1) + \frac{\alpha^2}{t} + \alpha^2\ t. \qquad (5.7)$$

As a result, we can express (5.5) in more detail as

$$\phi'(t) = \frac{t^{\frac{\alpha}{1-\alpha}}[\frac{1}{1-\alpha}(2\alpha^3 - 4\alpha^2 + 4\alpha - 1) + \alpha^2(t + \frac{1}{t})]}{[\alpha + (1-\alpha)t]^2}. \qquad (5.8)$$

At this stage, we define a function, h, as follows:

$$h(\alpha) = \frac{1}{1-\alpha}(2\alpha^3 - 4\alpha^2 + 4\alpha - 1),$$

and under this definition we can express (5.8) as

$$\phi'(t) = \frac{t^{\frac{\alpha}{1-\alpha}} \left[h(\alpha) + \alpha^2\, t + \alpha\, t^{-1}\right]}{[\frac{\alpha}{t} + (1-\alpha)]^2 t^2}$$
$$= \frac{h(\alpha)\, t^{\frac{\alpha}{1-\alpha}-2} + \alpha^2\, t^{\frac{\alpha}{1-\alpha}-1} + \alpha^2\, t^{\frac{\alpha}{1-\alpha}-3}}{(\frac{\alpha}{t} + 1 - \alpha)^2}, \tag{5.9}$$

which can be further transformed as

$$\phi'(t) = \frac{t^{\frac{\alpha}{1-\alpha}-3}\, (\alpha^2\, t^2 + h(\alpha)\, t + \alpha^2)}{(\frac{\alpha}{t} + 1 - \alpha)^2}. \tag{5.10}$$

Based on (5.10), we explore the conditions in which the numerator of (5.10) is null. We can easily describe these conditions as

$$\begin{cases} t^{\frac{\alpha}{1-\alpha}-3} = 0, \quad or \\ \alpha^2\, t^2 + h(\alpha)\, t + \alpha^2 = 0. \end{cases}$$

Then, let us examine the latter condition,

$$\alpha^2\, t^2 + h(\alpha)\, t + \alpha^2 = 0. \tag{5.11}$$

Since we can interpret (5.11) as a quadratic equation with respect to t, we denote two roots as t_1 and t_2. Additionally, we denote the discriminant for (5.11) as D.

Then, we obtain the following:

$$D = h(\alpha)^2 - 4\alpha^4$$
$$= [h(\alpha) + 2\alpha^2][h(\alpha) - 2\alpha^2], \tag{5.12}$$

and

$$\begin{cases} t_1 + t_2 = -\dfrac{h(\alpha)}{\alpha^2}, \\ t_1 \cdot t_2 = 1. \end{cases}$$

At this stage, we examine two cases, those in which (5.11) has real roots or imaginary ones. We can specifically transform (5.12) as

$$D = \frac{1}{(1-\alpha)^2}(4\alpha^3 - 6\alpha^2 + 4\alpha - 1)(-2\alpha^2 + 4\alpha - 1). \tag{5.13}$$

Additionally, we can utilize the definition of $h(\alpha)$ and find the following relationships:

$$h(\alpha) + 2\alpha^2 = \frac{1}{1-\alpha}(-2\alpha^2 + 4\alpha - 1),$$

and

$$h(\alpha) - 2\alpha^2 = \frac{1}{1-\alpha}(4\alpha^3 - 6\alpha^2 + 4\alpha - 1).$$

First, we define a function with respect to α, $f_1(\alpha)$, as follows:

$$f_1(\alpha) = 4\alpha^3 - 6\alpha^2 + 4\alpha - 1, \tag{5.14}$$

which has the following features:

$$\begin{cases} f_1(0) = -1, \\ f_1(1) = 1, \\ f_1(\frac{1}{2}) = 0. \end{cases}$$

Furthermore, differentiating (5.14) with respect to α, we obtain the following relationship:

$$f_1'(\alpha) = 4(3\alpha^2 - 3\alpha + 1), \tag{5.15}$$

which has the following features:

$$\begin{cases} f_1'(1) = -1, \\ f_1'(\frac{1}{2}) = \frac{1}{4}. \end{cases}$$

On the other hand, we can consider (5.15) = 0 as an equation. So, in this situation, the following hold:

$$\begin{cases} \alpha_1 + \alpha_2 = 1, \\ \alpha_1 \cdot \alpha_2 = \frac{1}{3}, \\ D_1 = -3. \end{cases}$$

In addition, $\alpha_i, (i = 1, 2)$ means the roots of equation (5.15) = 0, and D_1 the discriminant for an equation (5.15)=0, respectively.

Based on the above analyses, we know that function f_1 is an increasing one with respect to α, and that it intersects a horizontal line at $\alpha = \frac{1}{2}$.

Next, let us examine the following function:

$$f_2(\alpha) = -2\alpha^2 + 4\alpha - 1, \tag{5.16}$$

which has the following features:

$$\begin{cases} f_2(0) = -1, \\ f_2(1) = 1. \end{cases}$$

We can describe (5.16) in Figure 5.1, which shows the shape of the quadratic function with respect to α.

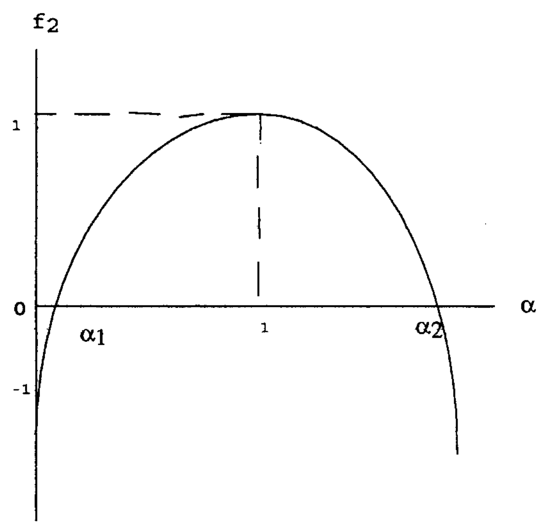

Figure 5.1

Here, $\alpha_1 = 1 - \frac{\sqrt{2}}{2}$ and $\alpha_2 = 1 + \frac{\sqrt{2}}{2}$.

Now, in light of (5.13), D can be expressed as

$$D = \frac{1}{(1-\alpha)^2}\, f_1(\alpha)\cdot f_2(\alpha), \tag{5.17}$$

which leads to the relationship described by Figure 5.2.

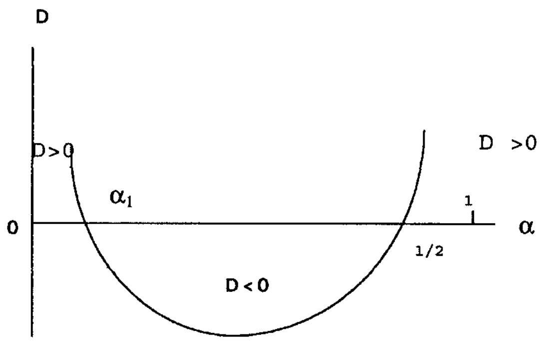

Figure 5.2

Based on Figure 5.2, we can judge a corresponding relationship between α and the sign of D, as follows:

$$D\begin{cases} <0 & \text{in the case of } \alpha_1 < \alpha < \dfrac{1}{2}, \\ \geqq 0 & \text{in the case of } 0 < \alpha \leqq \alpha_1, \text{ or } \dfrac{1}{2} \leqq \alpha < 1. \end{cases} \tag{5.18}$$

From the arguments so far, it becomes clear that quadratic equation (5.11) has imaginary roots in the case of $1 - \dfrac{\sqrt{2}}{2} < \alpha < \dfrac{1}{2}$ and real roots in the case of $0 < \alpha \leqq 1 - \dfrac{\sqrt{2}}{2}$, or $\dfrac{1}{2} \leqq \alpha < 1$.

Let us return to (5.10).

First, in the case of $1 - \dfrac{\sqrt{2}}{2} < \alpha < \dfrac{1}{2}$, since $\alpha^2\, t^2 + h(\alpha)\, t + \alpha^2 > 0$ and $t^{\frac{\alpha}{1-\alpha}-3} > 0$, as a consequence, $\phi'(t) > 0$. In other words, $\phi(t)$ is a monotonously increasing function with respect to t. In addition, notice that $0 < t < 1$.

Second, in the case of $0 < \alpha \leqq 1 - \frac{\sqrt{2}}{2}$, we obtain $h(\alpha) < 0$, because the function with respect to α monotonously increases from -1 through infinity and intersects a horizontal line at approximately $\alpha = 0.352$, as α moves from null through 1.

In this situation, we obtain the following:

$$\begin{cases} t_1 + t_2 = -\dfrac{h(\alpha)}{\alpha^2} > 0, \\ t_1 \cdot t_2 = 1. \end{cases}$$

Since $D > 0$, $\phi'(t)$ has two positive real roots, in which one, t_1, is less than 1 and the other, t_2, is larger than 1, therefore, $\phi(t)$ is monotonously increasing, with respect to $0 < t \leqq t_1$, while decreasing with respect to $t_1 < t < 1$.

Third, in the case of $\frac{1}{2} \leqq \alpha < 1$, we obtain $h(\alpha) > 0$. According to the above logic, the following holds:

$$\begin{cases} t_1 + t_2 = -\dfrac{h(\alpha)}{\alpha^2} < 0, \\ t_1 \cdot t_2 = 1. \end{cases}$$

Noticing $D > 0$, we conclude that $\phi'(t)$ has two negative real roots. This conclusion leads to the corollary that $\phi(t)$ is monotonously increasing with respect to $0 < t < 1$.

Based on the arguments so far, we obtain the following two relationships, which are depicted in Figure 5.3 and Figure 5.4, respectively:

In the case of $1 - \frac{\sqrt{2}}{2} < \alpha < 1$:

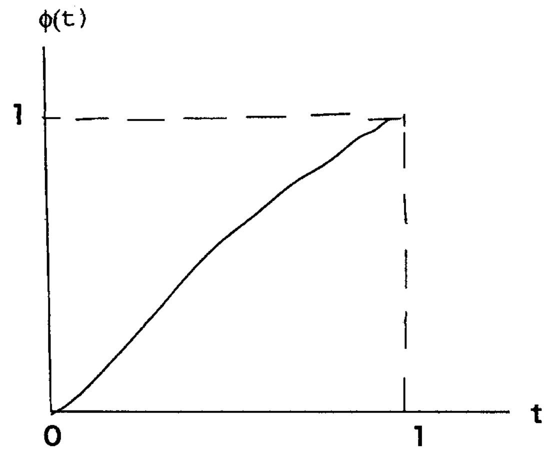

Figure 5.3

In the case of $0 < \alpha \leqq 1 - \frac{\sqrt{2}}{2}$:

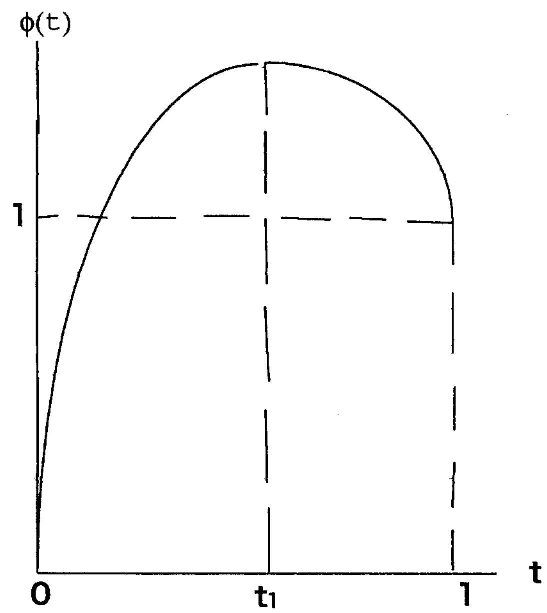

Figure 5.4

From the above conclusions, we again consider the function derived from $\phi(\tau)$, which is described as $\varphi(\tau)$, and we respectively obtain Figure 5.5 and Figure 5.6, as follows:

In the case of $1 - \frac{\sqrt{2}}{2} < \alpha < 1$:

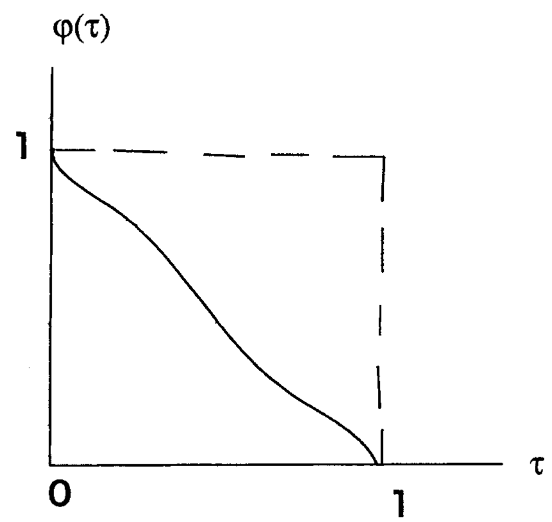

Figure 5.5

With respect to $0 < \tau < 1$, $\varphi(\tau)$ is less than 1. This shows that consumption in relation to labour under labour income taxation is larger than that under capital income taxation, with respect to total tax rates.

On the other hand, in the case of $0 < \alpha \leqq 1 - \frac{\sqrt{2}}{2}$, Figure 5.6 holds:

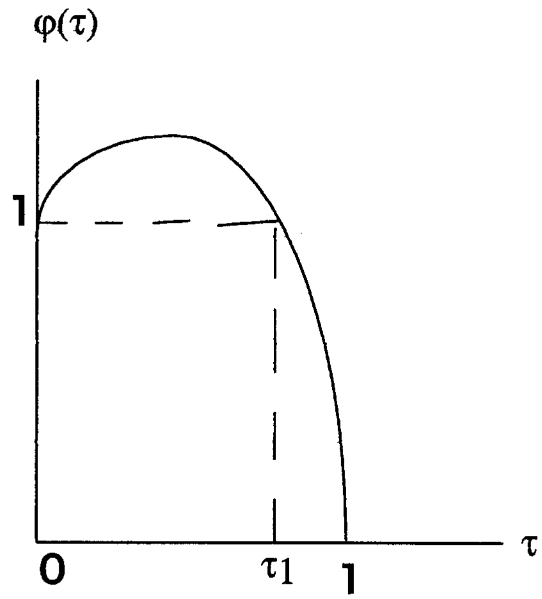

Figure 5.6

Contrary to the case depicted in Figure 5.5, consumption in relation to labour under labour income taxation is smaller than that under capital income taxation, with respect to a comparatively small τ, that is, $0 < \tau < \tau_1$, but with respect to a comparatively large one, that is, $\tau_1 < \tau < 1$, the relationship is reversed. Additionally, $\tau_1 = 1 - t_1$.

5.3 Determination of tax rates

In this section, we explore how tax rates are determined by governments. We postulate that governments naturally determine them in order to maximize the indication C/H, on the assumption that their tax revenue is a given.

5.3.1 Government revenue

Since we need to judge which tax rate under both forms of taxation is greater, first of all, we investigate how the relationship between the tax revenue under both forms of taxation moves, as τ does.

We have determined the following:

$$G^* = \frac{L(1-\alpha)^2 \left(\frac{\alpha}{\gamma}\right)^{\frac{\alpha}{1-\alpha}} \tau(1-\tau)}{(A+1)(1-\alpha)(1-\tau) + A\ \alpha}, \tag{5.19}$$

and

$$G^{**} = \frac{L(1-\alpha)\ \gamma \left(\frac{\alpha}{\gamma}\right)^{\frac{1}{1-\alpha}} \tau(1-\tau)^{\frac{\alpha}{1-\alpha}}}{(A+1)(1-\alpha) + A\ \alpha(1-\tau)}. \tag{5.20}$$

Because our aim here is to know whether government revenue is larger or smaller in (5.19) and (5.20), at certain tax rates, we consider the difference between the two, as follows:

$$\begin{aligned} G^* - G^{**} &= \frac{L(1-\alpha)^2 \left(\frac{\alpha}{\gamma}\right)^{\frac{\alpha}{1-\alpha}} \tau(1-\tau)}{(A+1)(1-\alpha)(1-\tau) + A\alpha} - \frac{L(1-\alpha)\gamma \left(\frac{\alpha}{\gamma}\right)^{\frac{1}{1-\alpha}} \tau(1-\tau)^{\frac{\alpha}{1-\alpha}}}{(A+1)(1-\alpha) + A\alpha(1-\tau)} \\ &= L(1-\alpha)\left(\frac{\alpha}{\gamma}\right)^{\frac{\alpha}{1-\alpha}} \tau(1-\tau)\times \\ &\quad [\frac{1-\alpha}{(A+1)(1-\alpha)(1-\tau) + A\alpha} - \frac{\gamma\left(\frac{\alpha}{\gamma}\right)(1-\tau)^{\frac{2\alpha-1}{1-\alpha}}}{(A+1)(1-\alpha) + A\alpha(1-\tau)}]. \end{aligned} \tag{5.21}$$

Here, we can transform [ ] = χ as

$$\chi = \frac{numerator}{[(A+1)(1-\alpha)(1-\tau)+A\alpha][(A+1)(1-\alpha)+A\alpha(1-\tau)]}. \tag{5.22}$$

Additionally, the following holds:

$$\begin{aligned} numerator &= (1-\alpha)[(A+1)(1-\alpha)+A\alpha(1-\tau)] - \alpha(1-\tau)^{\frac{2\alpha-1}{1-\alpha}} \times \\ &[(A+1)(1-\alpha)(1-\tau)+A\alpha] \\ &= (A+1)(1-\alpha)^2 + A\alpha(1-\tau)(1-\alpha) - \alpha(1-\alpha)(A+1)(1-\tau)^{1+\frac{2\alpha-1}{1-\alpha}} \\ &- A\alpha^2(1-\tau)^{\frac{2\alpha-1}{1-\alpha}} \\ &= -A\alpha^2(1-\tau)^{\frac{\alpha}{1-\alpha}-1} - \alpha(1-\alpha)(A+1)(1-\tau)^{\frac{\alpha}{1-\alpha}} \\ &+ A\alpha(1-\alpha)(1-\tau) + (A+1)(1-\alpha)^2. \end{aligned} \tag{5.23}$$

From what we have analyzed so far, since it has become clear that whether G^* or G^{**} is larger or smaller depends on the sign of the *numerator*, we explore only the *numerator* intensively.

Based on (5.23), we define a function as follows and focus on it:

$$\begin{aligned} f(t) = -A\alpha^2 t^{\frac{2\alpha-1}{1-\alpha}} &- \alpha(1-\alpha)(A+1)t^{\frac{\alpha}{1-\alpha}} \\ &+ A\alpha(1-\alpha)t + (A+1)(1-\alpha)^2 \end{aligned} \tag{5.24}$$

In addition, $t = 1-\tau$ and $0 < t < 1$.

Eventually, we can express (5.21) as

$$G^* - G^{**} = L(1-\alpha)\left(\frac{\alpha}{\gamma}\right)^{\frac{\alpha}{1-\alpha}}(1-t)\ t[-A\alpha^2 t^{\frac{\alpha}{1-\alpha}-1}$$
$$-\alpha(1-\alpha)(A+1)t^{\frac{\alpha}{1-\alpha}}$$
$$+A\alpha(1-\alpha)t + (A+1)(1-\alpha)^2]/[(A+1)(1-\alpha)\ t$$
$$+A\ \alpha][(A+1)(1-\alpha) + A\ \alpha\ t], \tag{5.25}$$

which is further transformed as

$$G^* - G^{**} = \frac{B(1-t)tf(t)}{[(A+1)(1-\alpha)\ t + A\ \alpha][(A+1)(1-\alpha) + A\ \alpha\ t]}. \tag{5.26}$$

Here, we define $B = L(1-\alpha)\left(\frac{\alpha}{\gamma}\right)^{\frac{\alpha}{1-\alpha}}$ and $\psi(t) = B\ (1-t)\ t\ f(t)$.

Now, returning to (5.24), let us investigate the movements of function $f(t)$.

Differentiating (5.24) with respect to t, we obtain the following:

$$f'(t) = -\alpha^2\ t^{\frac{\alpha}{1-\alpha}-2}[(A+1)\ t + \frac{2\alpha-1}{1-\alpha}\ A] + A\alpha(1-\alpha). \tag{5.27}$$

At this stage, in preparation for later discussions, we define as follows:

$$y = t^{\frac{\alpha}{1-\alpha}-2},$$

which has the following properties:

$$y(0)\begin{cases} = 0 & \text{in the case of } \dfrac{2}{3} \leqq \alpha < 1, \\ \infty & \text{in the case of } 0 < \alpha < \dfrac{2}{3}. \end{cases}$$

In further preparation, we need to note the following relationships:

$$\begin{cases} f(0) = (A+1)(1-\alpha)^2 > 0, \\ f(1) = (1-2\alpha)(A+1-\alpha). \end{cases} \tag{5.28}$$

Our aim here is to explore how $f(t)$ behaves as t moves. So, in order to examine how $f(t)$ moves, we need to analyze (5.27).

In this context, we need to divide our analyses into some cases.

First, in the case of $\alpha \geqq \frac{2}{3}$, when we rearrange (5.27) in terms of $y(t)$, we can establish the following:

$$f'(t) = -\alpha^2\, y(t)[(A+1)\, t + \frac{2\alpha-1}{1-\alpha}A] + A\, \alpha(1-\alpha). \tag{5.29}$$

In this situation, notice that we assume $\alpha \geqq \frac{2}{3}$. As a result, $0 \leqq y(t) \leqq 1$. Then, we obtain the following:

$$\begin{aligned} f'(0) &= -\alpha^2\, y(0)\, \frac{2\alpha-1}{1-\alpha}A + A\, \alpha(1-\alpha) \\ &= A\, \alpha(1-\alpha) > 0, \end{aligned} \tag{5.30}$$

because $y(0) = 0$.

On the other hand, the following holds:

$$\begin{aligned} f'(1) &= -\alpha^2 y(1)[(A+1) + \frac{2\alpha - 1}{1-\alpha}A] + A\,\alpha(1-\alpha) \\ &= -\alpha^2(1 + \frac{\alpha}{1-\alpha}A) + A\alpha(1-\alpha), \end{aligned}$$

which is finally transformed as

$$f'(1) = \alpha(-\alpha + \frac{1-2\alpha}{1-\alpha}A) < 0. \tag{5.31}$$

In the above procedure, we utilize $y(1) = 1$ and $1 - 2\alpha < 0$.[1]

Returning to (5.28), we easily identify $f(1) < 0$, because we assume $\alpha \geqq \frac{2}{3}$, which satisfies $1 - 2\alpha < 0$.[2]

[1]Additionally, we can conclude that $f'(t)$ is a decreasing function with respect to t, and offer a proof as follows:

$$\begin{aligned} f''(t) &= -\alpha^2[((A+1)t + \frac{2\alpha-1}{1-\alpha}\ A)\ y'(t) + y(t)(A+1)] \\ &= -\alpha^2[((A+1)t + \frac{2\alpha-1}{1-\alpha}\ A)(\frac{\alpha}{1-\alpha} - 2)\ t^{\frac{\alpha}{1-\alpha}-3} + (A+1)\ t^{\frac{\alpha}{1-\alpha}-2}]. \end{aligned}$$

In this context, the following holds:

$$\begin{cases} (A+1)\ t + \frac{2\alpha-1}{1-\alpha}A \geqq 0, \\ \frac{\alpha}{1-\alpha} - 2 \geqq 0. \end{cases}$$

Obviously, $f''(t) < 0$.

[2]In general, $f(0) > f(1)$ holds, and here we offer a proof:

$$\begin{aligned} f(0) - f(1) &= (A+1)(1-\alpha)^2 - (1-2\alpha)(A+1-\alpha) \\ &= \alpha^2 A + \alpha(1-\alpha) > 0. \end{aligned}$$

From the above analyses, we know that $f(t)$ starts at $(A+1)(1-\alpha)^2$, increases until a certain t, and then declines to a negative $(1-2\alpha)(A+1-\alpha)$, as t increases.

Now, we should return to the movements of the numerator, shown in (5.26) as

$$\psi(t) = B(1-t)tf(t).$$

We can easily describe the shape of $\psi(t)$ as in Figure 5.7:

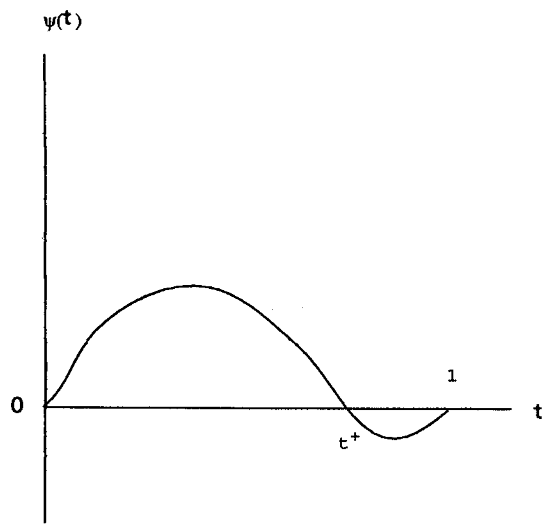

Figure 5.7

Figure 5.7 shows that $G^* \geqq G^{**}$, when $0 < t \leqq t^+$, and that, conversely, $G^* < G^{**}$, when $t^+ < t < 1$. This means that $G^* \geqq G^{**}$, when $\tau^+ \leqq \tau < 1$, and $G^* < G^{**}$, when $0 < \tau < \tau^+$. Additionally, $\tau^+ = 1 - t^+$.

Second, in the case of $\frac{1}{2} < \alpha < \frac{2}{3}$, consider (5.29) again. In this case, because $\alpha < \frac{2}{3}$, $1 \leqq y(t) < \infty$.

Noticing that, we obtain the following:

$$\begin{aligned} f'(0) &= -\alpha^2\, y(0)\, \frac{2\alpha - 1}{1-\alpha} A + A\alpha(1-\alpha) \\ &= -\infty, \end{aligned}$$

and

$$\begin{aligned} f'(1) &= -\alpha^2\, y(1)\, (A + 1 + \frac{2\alpha - 1}{1-\alpha} A) + A\alpha(1-\alpha) \\ &= \alpha(-\alpha + \frac{1-2\alpha}{1-\alpha} A) < 0. \end{aligned}$$

In this context, $\alpha > \frac{1}{2}$ naturally holds.

The above relationship means $f'(t) < 0$, because $f'(t)$ is continuous with respect to t.

Furthermore, we easily obtain the following:

$$\begin{cases} f(0) = (A+1)(1-\alpha)^2 > 0, \\ f(1) = (1-2\alpha)(A+1-\alpha) < 0. \end{cases}$$

In what follows, we can obtain the same conclusion based on similar logic to the case of $\frac{2}{3} \leqq \alpha < 1$.

In other words, $G^* < G^{**}$ holds, when τ is comparatively small, but $G^* \geqq G^{**}$, when τ is comparatively large.

Third, in the case of $0 < \alpha \leqq \frac{1}{2}$, we obtain the following relationships:

$$\begin{cases} f(0) = (A+1)(1-\alpha)^2 > 0, \\ f(1) = (1-2\alpha)(A+1-\alpha) \geqq 0. \end{cases}$$

On the other hand, we have established $f(0) > f(1)$.[3] Therefore, obviously $f(t) > 0$ holds.

This leads to $G^* > G^{**}$, for any case in which $0 < \tau < 1$.

5.3.2 Tax rates

We assume that governments choose between labour and capital income taxation, whichever yields higher consumption in relation to labour, when government spending is a given. In this situation, how is the tax rate determined? Based on the discussions so far, we can respectively describe two curves, stating a relation between government revenue and tax rates, as follows:

In the case of $0 < \alpha \leqq \frac{1}{2}$:

[3]See footnote 2.

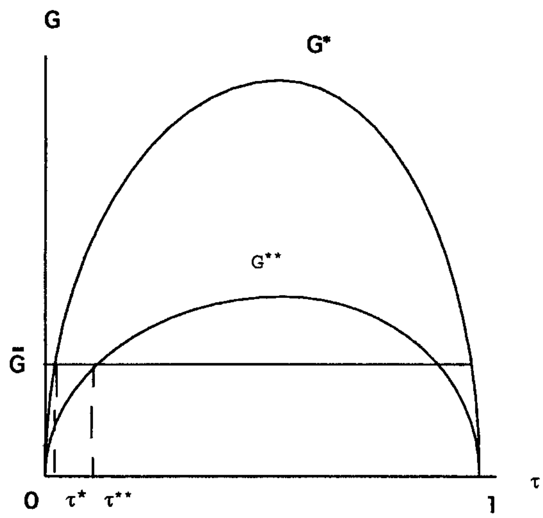

Figure 5.8

In the case of $\frac{1}{2} < \alpha < 1$:

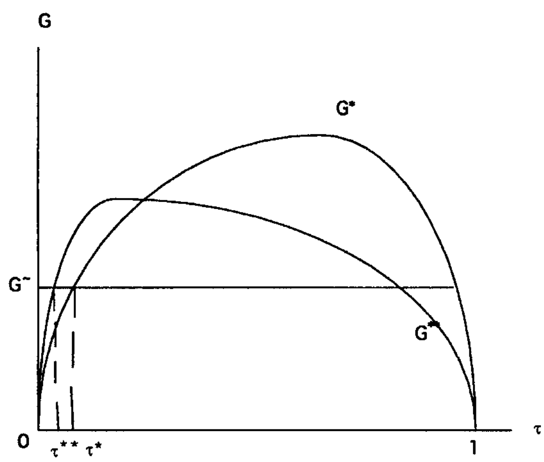

Figure 5.9

In light of constant G, governments can determine their tax rates in two categories, one a lower rate on a larger tax base, the other a higher rate on a smallerr tax base. However, in this situation, they naturally choose the former, because it absolutely yields more consumption in relation to labour than the latter does.

In other words, when our economy is in the circumstance described in Figure 5.8, for example, the government should choose labour income taxation with τ^*, if C^*/H^* is larger than C^{**}/H^{**}. Conversely, if C^*/H^* is smaller than C^{**}/H^{**}, it should choose capital income taxation with τ^{**}.

5.4 Comparisons

From the preparations so far, we have seen how consumption in relation to labour and tax rates moves, as τ does.

Based on the above discussions and preparations, we can affirm a key relationship, which we call the K.R. condition, as follows:

If the following conditions are satisfied, labour income taxation is inevitably preferable. Of course, we assume that tax revenue is given at a certain level.

$$\begin{cases} C.1 \quad \dfrac{C^*}{H^*} > \dfrac{C^{**}}{H^{**}} \; holds, \; for \; any \; case \; in \; which \; 0 < \tau < 1, \\ C.2 \quad \tau^* < \tau^{**} \quad is \;\; satisfied. \end{cases}$$

Additionally, the first condition concerning the location of the two curves is derived from the first preparation,[4] and the second one concerning the determination of tax rates is derived from the second preparation.

[4]See $\varphi(\tau)$ in Figure 5.5 and Figure 5.6.

More specifically, we can depict our situation in Figure 5.10, as follows:

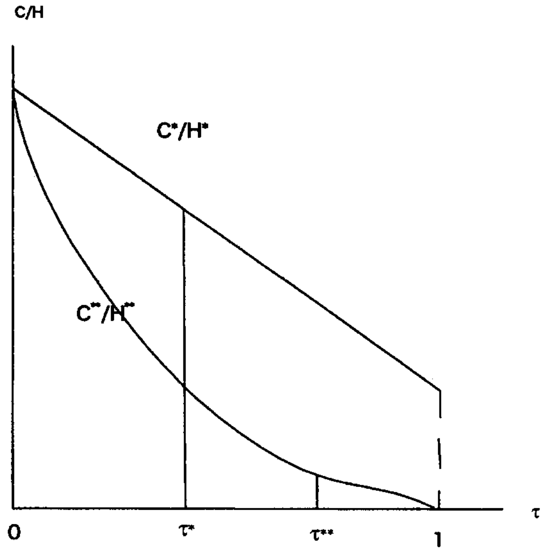

Figure 5.10

We can understand from Figure 5.10 that governments naturally choose labour income taxation, because the labour income tax rate is lower than the capital income tax rate, and because the corresponding consumption in relation to labour under labour income taxation is inevitably larger than that under capital income taxation.

Furthermore, it becomes clear from both Figure 5.5 and Figure 5.8 that this occurs when $1 - \frac{\sqrt{2}}{2} < \alpha \leqq \frac{1}{2}$.

Based on what we have analyzed above, it is clear that the relationship between consumption and labour under our two forms of taxation is strongly associated with the value of α, and moreover that this is applicable to the determination of tax rates under the two forms of taxation.

So, we proceed to analyze which form of taxation yields more consumption in relation to labour, as α changes. In this process, we simulate when necessary.

Here, we assume parameters in these simulations as follows:

$L = 48$; $A = 1$; and $\gamma = 0.1$.

First, we deal with the case of $0 < \alpha \leqq \frac{1}{2}$.

In the case of $\alpha = 0.1$:

G	τ^*	τ^{**}	C^*/H^*	C^{**}/H^{**}
0.5	0.0244664	0.223516	0.97798	0.950551
0.8	0.0391786	0.362858	0.964739	0.916636
1.1	0.0539164	0.509612	0.951475	0.876798
1.4	0.0686812	0.672348	0.938187	0.824004

In the case of $\alpha = 0.2$:

G	τ^*	τ^{**}	C^*/H^*	C^{**}/H^{**}
0.5	0.024705	0.100049	1.1657	1.1351
1	0.0495567	0.203921	1.14206	1.07749
1.5	0.0745686	0.313461	1.11827	1.01463
2	0.0997558	0.432289	1.0943	0.943014
2.5	0.125136	0.569888	1.07016	0.853292

Moreover, in the cases of $\alpha = 0.3$ through $\alpha = 0.5$, since apparently the K.R. condition holds, consumption in relation to labour under labour income taxation is obviously larger than that under capital income taxation.

In the case of $\alpha = 0.6$:

G	τ^*	τ^{**}	C^*/H^*	C^{**}/H^{**}
5	0.063829	0.0433663	14.3217	13.3935
10	0.132127	0.0917768	13.9202	12.0204
15	0.206845	0.147646	13.4809	10.5408
20	0.291892	0.216263	12.981	8.87407
25	0.397912	0.315872	12.3577	6.74021

In the case of $\alpha = 0.7$:

G	τ^*	τ^{**}	C^*/H^*	C^{**}/H^{**}
20	0.0666788	0.0290145	91.8586	85.7328
40	0.139641	0.0617362	89.8069	77.292
60	0.222269	0.0998465	87.4834	68.2078
80	0.322737	0.146835	84.6582	58.059

In the case of $\alpha = 0.8$:

G	τ^*	τ^{**}	C^*/H^*	C^{**}/H^{**}
500	0.0807627	0.0204332	4029.84	3709.7
1000	0.17402	0.0443907	3953.44	3294.41
1200	0.21692	0.0553677	3918.3	3116.99
1500	0.291732	0.0739777	3857.01	2833.68
1800	0.393416	0.0963664	3773.71	2520.5

In the case of $\alpha = 0.9$:

G	τ^*	τ^{**}	C^*/H^*	C^{**}/H^{**}
1×10^7	0.0623713	0.00695892	3.85004×10^8	3.61542×10^8
2×10^7	0.133175	0.0148578	3.82261×10^8	3.34061×10^8
3×10^7	0.21791	0.0240676	3.78978×10^8	3.04404×10^8
4×10^7	0.333455	0.0352714	3.74502×10^8	2.71531×10^8

We understand from the above analyses that C^*/H^* is larger than C^{**}/H^{**}, on all the levels of G given. This is applicable even when $\tau^* > \tau^{**}$ holds due to high α, like in the cases of $\alpha = 0.6$ through $\alpha = 0.9$.

Based on the results so far, we can conclude that regardless of the degree of α, which shows the ratio of capital income distribution in outputs, or eventually the degree of labour productivity, labour income taxation is preferable.

5.5 Conclusions

Now we can clearly answer the controversial question of which form of taxation is better, labour or capital income taxation.

Neither form of taxation under a balanced budged rule causes instability in our economy, but they entail a key difference concerning labour productivity.

More concretely speaking, the amount of labour in production processes decreases after labour income taxes are introduced, while it conversely increases after capital income taxes are introduced. This sharp contrast causes a difference concerning consumption in relation to labour, which is assumed as an indication of social welfare.

As a result, under capital income taxes this is smaller than that under labour income taxes.

And this holds for $0 < \alpha < 1$ and for all levels of G assumed. Furthermore, this leads to a simple and clear conclusion: that labour income taxation is unconditionally better than capital income taxation. Therefore, capital income tax rates should be null.

Chapter 6

How Do Unfixed Labour Income Taxes Affect an Economy?

This chapter deals with the possibility of indeterminacy of equilibria, which can be caused by labour income taxation.

We establish a model, which describes the movements of an economy, on the assumption of an infinite horizon and a balanced-budget rule. We demonstrate that this rule in fact causes regressive taxes; therefore, multiple equilibria can exist in an economy and in some cases these equilibria can be sinks.

6.1 Introduction

We can divide a balanced-budget rule into two categories, one in which the government entirely spends the taxes levied, after imposing them on household income in terms of a fixed tax rate; the other in which the government initially determines its spending and afterwards changes tax rates in order to attain revenue matched to its spending plans.

In terms of economic policy, it should be obvious that the former is better than the latter, because stability of taxation, including tax rates, has long been considered desirable.

In this chapter, however, we deal with the latter issue, in a direct refutation of the logic in Schmitt-Grohé and Uribe (1997).

A balanced-budget rule in the above sense means that the government raises tax rates when tax bases decrease and it cuts tax rates when tax bases increase. This ensures, in other words, regressive taxation. However, in general, it is a realistic policy for the government, unlike the private sector, to initially determine its spending in order to support its activities, and afterwards to determine the tax rates.

We focus on the relationship between this taxation and stability in an economy, as in Schmitt-Grohé and Uribe (1997).

As a result, we conclude that this regressive taxation system can have multiple steady states and can cause indeterminacy, which can lead to sunspot fluctuations. This conclusion itself is affirmed by Schmitt-Grohé and Uribe

(1997), but we have demonstrated in the third chapter that their procedures were wrong, and will demonstrate that based on our procedures, indeterminacy can be derived. In general, it has been shown that the existence of increasing returns or externality in production cause indeterminacy.[1] However, we demonstrate that regressive taxation on labour income can play a similar part in the occurrence of indeterminacy.[2]

The constitution of this chapter is as follows:

Section 6.2 deals with basic models, in which the behaviour of the government, a household, and a firm is described. In section 6.3, we derive the movements of three fundamental economic variables, C_t, k_t, and τ_t. In what follows, we establish a linear system, which can be more easily manipulated, and show an imaginary example, in section 6.4. Finally, section 6.5 is devoted to a conclusion.

6.2 The model

We roughly assume a structural model similar to the one in *How Do Fixed Labour Income Taxes Affect an Economy ?*, in the third chapter, but change the assumption concerning the determination of labour income tax rates, τ_t. In that chapter, we assumed those tax rates are a given and a constant.

[1]See Benhabib, J. and Farmer, R. F. (1994).

[2]Guo and Harrison (2001) deals with issues parallel to ours, on the assumption of externality and regressive or progressive taxes, and in the framework of a two-sector model. The authors derive the same conclusion as we do and moreover specify conditions which cause indeterminancy.

In contrast, in this chapter, we assume the government imposes a tax rate on the labour income of a household in order to satisfy the following:

$$G = \tau_t \; w_t \; H_t. \tag{6.1}$$

Here, G means government spending and is a given.

6.2.1 Optimization of a household

In this situation, a household considers this tax rate, τ_t, as a given and solves the following problem:

$$\max_{C_t, Ht} \int_0^{\infty} e^{-\gamma t}[\log C_t + A \log (L - H_t)]dt, \tag{6.2}$$

subject to

$$\dot{K}_t = u_t K_t + (1 - \tau_t) w_t H_t - C_t. \tag{6.3}$$

Additionally, we assume that the household takes factor prices u_t and w_t as given, and anticipates them on the basis of perfect foresight. Furthermore, the price of outputs is assumed to be 1.

We can describe the Hamiltonian function, R, as

$$R = e^{-\gamma t}[\log C_t + A \log (L - H_t)] \\ + \mu_t[u_t K_t + (1 - \tau_t) w_t H_t - C_t]. \tag{6.4}$$

Of course, μ_t means an adjoint variable.

With respect to control variables C_t and H_t, the following must hold:

$$\frac{\partial R}{\partial C_t} = e^{-\gamma t}\frac{1}{C_t} - \mu_t = 0,$$

which leads to

$$\mu_t = \frac{e^{-\gamma t}}{C_t}, \tag{6.5}$$

and

$$\frac{\partial R}{\partial H_t} = e^{-\gamma t}A\frac{-1}{L - H_t} + \mu_t(1 - \tau_t)w_t = 0,$$

which yields

$$H_t = L - \frac{AC_t}{(1 - \tau_t)w_t}. \tag{6.6}$$

(6.6) shows the supply curve of the labour force which a household offers.

Moreover, an adjoint equation must hold as follows:

$$\frac{d\mu_t}{dt} = -\frac{\partial R}{\partial K_t} = -\mu_t u_t, \tag{6.7}$$

which yields

$$\frac{\dot{\mu}_t}{\mu_t} = -u_t. \tag{6.8}$$

Based on both (6.5) and (6.8), we finally obtain an Euler equation:

$$\frac{\dot{C}_t}{C_t} = u_t - \gamma. \tag{6.9}$$

6.2.2 Optimization of a firm

We assume that a representative firm aims to maximize its profit under the production function called the Cobb-Douglas type, and this production function is explicitly expressed as

$$F(K_t, H_t) = K_t^{\alpha} H_t^{1-\alpha}, \quad \text{and} \quad 0 < \alpha < 1, \tag{6.10}$$

which can be transformed as

$$y_t = k_t^{\alpha}. \tag{6.11}$$

Here, $y_t = F(K_t, H_t)/H_t$ and $k_t = K_t/H_t$. Of course, the former shows the productivity of labour, and the latter the capital intensity of labour.

We assume that the markets of output, labour and capital are fully competitive, and furthermore that in these markets equilibria hold.
As a result, the following hold:

$$w_t = (1 - \alpha)k_t^{\alpha}, \tag{6.12}$$

$$u_t = \alpha k_t^{-(1-\alpha)}. \tag{6.13}$$

6.3 Dynamics

6.3.1 The movements of k_t

Based on both the definition of k_t and (6.3), we obtain the following:

$$\dot{k}_t = u_t \frac{K_t}{H_t} + (1 - \tau_t) w_t - \frac{C_t}{H_t} - k_t \frac{\dot{H}_t}{H_t}. \tag{6.14}$$

At this stage, to prepare for later discussions, we need specifically to explore $\dot{H}_t / H_t$.

We return to (6.6), which is derived from maximization of the present value of utilizations of a representative household.

Since (6.6) holds for arbitrary time, we can differentiate both sides of (6.6) with respect to time. And we should notice that τ_t is also differentiable with respect to time in this context, contrary to the assumption of fixed tax rates in the third chapter.

As a result, we obtain the following:

$$\dot{H}_t = -A \frac{\dot{C}_t (1 - \tau_t) w_t - C_t [-w_t \dot{\tau}_t + (1 - \tau_t) \dot{w}_t]}{[(1 - \tau_t) w_t]^2}, \tag{6.15}$$

which, based on (6.6), yields

$$\frac{\dot{H}_t}{H_t} = -A \frac{\frac{\dot{C}_t}{C_t}(1 - \tau_t) + \dot{\tau}_t - (1 - \tau_t) \frac{\dot{w}_t}{w_t}}{(1 - \tau_t)[L(1 - \tau_t) \frac{w_t}{C_t} - A]}. \tag{6.16}$$

Moreover, we can transform (6.16) as follows:

$$\frac{\dot{H}_t}{H_t} = -A\frac{\frac{\dot{C}_t}{C_t}\frac{1-\tau_t}{\tau_t} + \frac{\dot{\tau}_t}{\tau_t} - \frac{1-\tau_t}{\tau_t}\frac{\dot{w}_t}{w_t}}{\frac{1-\tau_t}{\tau_t}[L(1-\tau_t)\frac{w_t}{C_t} - A]}. \tag{6.17}$$

Here, since government revenue G is supposed to be constant in every instance, as in (6.1), the following holds:

$$\frac{\dot{\tau}_t}{\tau_t} = -\frac{\dot{w}_t}{w_t} - \frac{\dot{H}_t}{H_t}. \tag{6.18}$$

Substituting (6.18) into (6.17), we obtain the following:

$$\frac{\dot{H}_t}{H_t} = -A\frac{\frac{\dot{C}_t}{C_t}\frac{1-\tau_t}{\tau_t} - \frac{\dot{w}_t}{w_t} - \frac{\dot{H}_t}{H_t} - \frac{1-\tau_t}{\tau_t}\frac{\dot{w}_t}{w_t}}{\frac{1-\tau_t}{\tau_t}[L(1-\tau_t)\frac{w_t}{C_t} - A]}, \tag{6.19}$$

from which we can eventually obtain the following relationship, showing movements of the labour supply:

$$\frac{\dot{H}_t}{H_t} = \frac{-A[\frac{\dot{C}_t}{C_t}(1-\tau_t) - \frac{\dot{w}_t}{w_t}]}{L(1-\tau_t)^2\frac{w_t}{C_t} - A}. \tag{6.20}$$

At this stage, we can express (6.14), which shows the rate of change in the capital intensity of labour, as follows:

$$\dot{k}_t = u_t\ k_t + (1-\tau_t)\ w_t - \frac{(1-\tau_t)w_t\ C_t}{L(1-\tau_t)w_t - A\ C_t} + \frac{A\ k_t[\ \frac{\dot{C}_t}{C_t}(1-\tau_t) - \frac{\dot{w}_t}{w_t}]}{L(1-\tau_t)^2\ \frac{w_t}{C_t} - A}. \tag{6.21}$$

Here, we respectively define the following:

$$\begin{cases} w_t = \eta(k_t), \\ \eta(k_t) \overset{\text{def}}{=} (1-\alpha)k_t^{\alpha}, \end{cases} \tag{6.22}$$

and

$$\begin{cases} u_t = \phi(k_t), \\ \phi(k_t) \overset{\text{def}}{=} \alpha k_t^{\alpha-1}. \end{cases} \tag{6.23}$$

From what we have discussed so far, we can express (6.21) in terms of equilibrium prices as

$$\dot{k}_t = \phi(k_t)\ k_t + (1-\tau_t)\ \eta(k_t) - \frac{(1-\tau_t)\ \eta(k_t)\ C_t}{L(1-\tau_t)\ \eta(k_t) - A\ C_t} + \frac{A\ k_t\ [\ \frac{\dot{C}_t}{C_t}\ (1-\tau_t) - \frac{\dot{\eta(k_t)}}{\eta(k_t)}]}{L\ (1-\tau_t)^2\ \frac{\eta(k_t)}{C_t} - A}. \tag{6.24}$$

Subsequently, we can derive the following relationship from (6.22):

$$\frac{\dot{\eta(k_t)}}{\eta(k_t)} = \alpha \, \frac{\dot{k_t}}{k_t}. \tag{6.25}$$

Now, substituting (6.9), (6.22), (6.23), and (6.25) into (6.24), we can transform (6.24) as follows:

$$\begin{aligned}\dot{k_t} = \alpha \, k_t^{\alpha-1} \, k_t + (1-\tau_t)(1-\alpha) \, k_t^{\alpha} - \frac{(1-\tau_t)(1-\alpha) \, k_t^{\alpha} \, C_t}{L(1-\tau_t)(1-\alpha) \, k_t^{\alpha} - A \, C_t} \\ + \frac{A \, k_t[\,(1-\tau_t)(\alpha \, k_t^{\alpha-1} - \gamma) - \alpha \, \frac{\dot{k_t}}{k_t}]}{L(1-\tau_t)^2 \, \frac{(1-\alpha) \, k_t^{\alpha}}{C_t} - A},\end{aligned} \tag{6.26}$$

which can further be transformed as follows:

$$\begin{aligned}\dot{k_t} = [\alpha + (1-\alpha)(1-\tau_t)] \, k_t^{\alpha} - \frac{(1-\tau_t)(1-\alpha) \, k_t^{\alpha} \, C_t}{L(1-\tau_t)(1-\alpha) \, k_t^{\alpha} - A \, C_t} \\ + \frac{A[(1-\tau_t)(\alpha \, k_t^{\alpha} - \gamma \, k_t) - \alpha \, \dot{k_t}] \, C_t}{L(1-\tau_t)^2(1-\alpha) \, k_t^{\alpha} - A \, C_t}.\end{aligned} \tag{6.27}$$

Multiplying both sides in (6.27) by both $L(1-\tau_t)(1-\alpha) \, k_t^{\alpha} - A \, C_t$ and $L(1-\tau_t)^2(1-\alpha) \, k_t^{\alpha} - A \, C_t$, and arranging the results, we eventually obtain the following:

$$\begin{aligned}[L(1-\tau_t)^2(1-\alpha) \, k_t^{\alpha} - (1-\alpha)A \, C_t][L(1-\alpha) \\ \times (1-\tau_t) \, k_t^{\alpha} - A \, C_t] \, \dot{k_t} =\end{aligned}$$

$$
\begin{aligned}
&[\alpha + (1-\alpha)(1-\tau_t)][L(1-\tau_t)^2(1-\alpha)\ k_t^\alpha - A\ C_t] \\
&\times [L(1-\alpha)(1-\tau_t)k_t^\alpha - A\ C_t]\ k_t^\alpha \\
&- (1-\alpha)(1-\tau_t)C_t[L(1-\tau_t)^2(1-\alpha)\ k_t^\alpha - A\ C_t]\ k_t^\alpha \\
&+ A[L(1-\alpha)(1-\tau_t)\ k_t^\alpha - A\ C_t](1-\tau_t)(\alpha\ k_t^\alpha - \gamma\ k_t)\ C_t. \qquad (6.28)
\end{aligned}
$$

In this situation, we can rearrange *the right side* in (6.28), as follows:

$$
\begin{aligned}
\textit{the right side} = k_t^\alpha\ &[(\alpha + (1-\alpha)(1-\tau_t))(L(1-\tau_t)^2(1-\alpha)\ k_t^\alpha \\
&- A\ C_t)(L(1-\alpha)(1-\tau_t)k_t^\alpha - A\ C_t) \\
&- (1-\alpha)(1-\tau_t)\ C_t(L(1-\tau_t)^2(1-\alpha)\ k_t^\alpha - A\ C_t) \\
&+ A\ \alpha(1-\tau_t)\ C_t(L(1-\alpha)(1-\tau_t)k_t^\alpha - A\ C_t)] \\
&- A\ \gamma\ k_t\ C_t\ (1-\tau_t)(L(1-\alpha)(1-\tau_t)\ k_t^\alpha - A\ C_t),
\end{aligned}
$$

and in terms of the above expression, $\dot{k}_t$ can be described as

$$
\dot{k}_t = \frac{\textit{the right side}}{[L(1-\tau_t)^2(1-\alpha)\ k_t^\alpha - (1-\alpha)A\ C_t][L(1-\alpha)(1-\tau_t)\ k_t^\alpha - A\ C_t]}. \qquad (6.29)
$$

Here, first of all, we need to analyze the bracket [ ] in *the right side*, and for convenience denote symbols as

$$
\begin{cases}
\theta_1 = \alpha + (1-\alpha)(1-\tau_t), \\
\theta_2 = L(1-\alpha)(1-\tau_t).
\end{cases} \qquad (6.30)
$$

Under these definitions, the bracket [ ] is expressed as

$$
\begin{aligned}
[\quad] = \theta_1[\theta_2\ (1-\tau_t)\ k_t^{\alpha} - A\ C_t][\theta_2\ k_t^{\alpha} - A\ C_t] \\
- \frac{\theta_2}{L}\ C_t[\theta_2\ (1-\tau_t)\ k_t^{\alpha} - A\ C_t] \\
+ A\ \alpha(1-\tau_t)C_t(\theta_2\ k_t^{\alpha} - A\ C_t),
\end{aligned} \tag{6.31}
$$

which eventually yields

$$
\begin{aligned}
[\quad] = \theta_1\ \theta_2^2\ (1-\tau_t)\ k_t^{2\alpha} + [-A\ C_t\ \theta_1\ \theta_2(2-\tau_t) \\
- \frac{\theta_2^2}{L}\ C_t(1-\tau_t) + A\ \alpha\ C_t(1-\tau_t)\ \theta_2]\ k_t^{\alpha} \\
+ A\ C_t^2\ [\ \frac{\theta_2}{L}\ + A\ \theta_1 - A\ \alpha(1-\tau_t)].
\end{aligned} \tag{6.32}
$$

Based on (6.32), *the right side* can be described as

$$
\begin{aligned}
\textit{the right side} = \theta_1\ \theta_2^2\ (1-\tau_t)k_t^{3\alpha} + C_t\ \theta_2[-A\ \theta_1(2-\tau_t) - \frac{\theta_2}{L}(1-\tau_t) \\
+ A\ \alpha(1-\tau_t)]k_t^{2\alpha} \\
+ A\ C_t^2\ [\frac{\theta_2}{L} + A\ \theta_1 - A\ \alpha(1-\tau_t)]\ k_t^{\alpha} \\
- A\ \gamma\ C_t\ (1-\tau_t)\ k_t\ (\theta_2\ k_t^{\alpha} - A\ C_t),
\end{aligned} \tag{6.33}
$$

which is then transformed as follows:

$$
\begin{aligned}
\textit{the right side} = & (\alpha + (1-\alpha)(1-\tau_t))\ L^2\ (1-\alpha)^2\ (1-\tau_t)^3\ k_t^{3\alpha} \\
& - A\ \gamma\ L(1-\alpha)(1-\tau_t)^2\ C_t\ k_t^{\alpha+1} \\
& + L(1-\alpha)(1-\tau_t)[-A\ (\alpha + (1-\alpha)(1-\tau_t))(2-\tau_t) \\
& + A\ \alpha(1-\tau_t) - (1-\alpha)(1-\tau_t)^2]\ C_t\ k_t^{2\alpha} \\
& + A[A(\alpha + (1-\alpha)(1-\tau_t)) + (1-\alpha)(1-\tau_t) \\
& - A\ \alpha(1-\tau_t)]\ C_t^2\ k_t^{\alpha} + A^2\ \gamma\ C_t^2\ (1-\tau_t)\ k_t. \qquad (6.34)
\end{aligned}
$$

In light of (6.28), we can finally express $\dot{k}_t$, as follows:

$$
\dot{k}_t = \frac{\textit{the right side}}{[L(1-\tau_t)^2(1-\alpha)k_t^{\alpha} - (1-\alpha)\ A\ C_t][L(1-\alpha)(1-\tau_t)\ k_t^{\alpha} - A\ C_t]}. \qquad (6.35)
$$

6.3.2 The movements of τ_t

From (6.18), the following holds:

$$\begin{aligned}\frac{\dot{\tau}_t}{\tau_t} &= -\frac{\dot{w}_t}{w_t} - \frac{\dot{H}_t}{H_t} \\ &= -\frac{\dot{w}_t}{w_t} + \frac{A[\frac{\dot{C}_t}{C_t}(1-\tau_t) - \frac{\dot{w}_t}{w_t}]}{L(1-\tau_t)^2 \frac{w_t}{C_t} - A} \\ &= \frac{-\frac{\dot{w}_t}{w_t}[L(1-\tau_t)^2 \frac{w_t}{C_t} - A] + A[\frac{\dot{C}_t}{C_t}(1-\tau_t) - \frac{\dot{w}_t}{w_t}]}{L(1-\tau_t)^2 \frac{w_t}{C_t} - A}.\end{aligned} \tag{6.36}$$

Here, let us focus on *the numerator* in (6.36):

$$\textit{the numerator} = -\frac{\dot{w}_t}{w_t} L(1-\tau_t)^2 \left(\frac{w_t}{C_t}\right) + A(1-\tau_t)\frac{\dot{C}_t}{C_t}, \tag{6.37}$$

which as a consequence, can be transformed as follows:

$$\textit{the numerator} = -\alpha \frac{\dot{k}_t}{k_t} L(1-\tau_t)^2 \left(\frac{w_t}{C_t}\right) + A(1-\tau_t)\frac{\dot{C}_t}{C_t}. \tag{6.38}$$

Now, we can formulate a rate of change in τ_t, as follows:

$$\frac{\dot{\tau}_t}{\tau_t} = \frac{-\alpha \frac{\dot{k}_t}{k_t} L(1-\tau_t)^2 \left(\frac{w_t}{C_t}\right) + A(1-\tau_t)\frac{\dot{C}_t}{C_t}}{L(1-\tau_t)^2 \left(\frac{w_t}{C_t}\right) - A}, \tag{6.39}$$

which eventually leads to

$$\frac{\dot{\tau}_t}{\tau_t} = \frac{(1-\tau_t)[-\alpha\ L(1-\tau_t)\ w_t\ \dot{k}_t + A\ \dot{C}_t\ k_t]}{L(1-\tau_t)^2\ w_t\ k_t - A\ k_t\ C_t}. \tag{6.40}$$

At this stage, in light of (6.22), (6.23), and (6.9), we finally obtain the formula describing the movements of τ_t, as follows:

$$\dot{\tau}_t = \frac{\tau_t(1-\tau_t)[\ -\alpha\ L(1-\tau_t)(1-\alpha)\ k_t^{\alpha}\ \dot{k}_t + A\ C_t\ k_t(\alpha\ k_t^{\alpha-1} - \gamma)]}{L(1-\tau_t)^2(1-\alpha)\ k_t^{\alpha+1} - A\ k_t\ C_t}. \tag{6.41}$$

Because (6.35) determines $\dot{k}_t$, (6.41) apparently determines $\dot{\tau}_t$.

Additionally, in the process from (6.40) to (6.42), we have utilized (6.42), cited below.

6.3.3 The movements of C_t

Based on both (6.9) and (6.23), we obtain the formula in terms of both C_t and k_t which describes the movements of C_t, as follows:

$$\dot{C}_t = (\alpha\ k_t^{\alpha-1} - \gamma)\ C_t. \tag{6.42}$$

6.4 Time paths

(6.35), (6.41), and (6.42) respectively are differential equations with three unknown variables, k_t, C_t and τ_t,[3] and moreover we consider that they consist

[3]Note that (6.42) does not contain τ_t.

of a simultaneous differential equations system.

Since the system is not linear with respect to the unknown variables, however, we can not directly solve this simultaneous differential equations system.

So, first of all, we intend to linearize and analyze the system approximately.[4]

6.4.1 The existence of steady states

As a preparation for later discussions, we need to confirm the existence and the possibility of the non-uniqueness of steady states.

First, based on (6.42), we can easily confirm the unique value of k in a steady state, as below.

When we place $\dot{C}_t = 0$ in (6.42), we obtain the value, k^*, as follows:

$$k^* = \left(\frac{\alpha}{\gamma}\right)^{\frac{1}{1-\alpha}}. \tag{6.43}$$

(6.43) means that k^* is determined only by parameters α and γ, in the simultaneous differential equations system. This relatioship has been called a *neutrality theorem* of capital intensity on tax rates, τ_t, as established in (3.39).

Second, when we place $\dot{k}_t = 0$ and omit $k^* = 0$ in (6.35), the following

[4]For mathematical details, see Hartman (1982).

must hold:

$$
\begin{aligned}
&(\alpha + (1-\alpha)(1-\tau^*))L^2(1-\alpha)^2(1-\tau^*)^3(k^*)^{3\alpha-1} \\
&\qquad + L(1-\alpha)C^*(1-\tau^*)[-A(\alpha + (1-\alpha)(1-\tau^*))(2-\tau^*) \\
&\qquad - (1-\alpha)(1-\tau^*)^2 \\
&\qquad + A\ \alpha(1-\tau^*)](k^*)^{2\alpha-1} + A\ (C^*)^2[A(\alpha + (1-\alpha)(1-\tau^*)) \\
&\qquad + (1-\alpha)(1-\tau^*) \\
&\qquad - A\ \alpha(1-\tau^*)](k^*)^{\alpha-1} + A^2\ \gamma\ (1-\tau^*)(C^*)^2 \\
&\qquad = A\ \gamma\ L(1-\alpha)(1-\tau^*)^2\ C^*(k^*)^{\alpha}. \qquad (6.44)
\end{aligned}
$$

On the other hand, based on (6.1), the following must hold:

$$\overline{G} = \tau^*\ w^* H^*. \qquad (6.45)$$

$\overline{G}$ is arbitrary government spending and a given, while H^* means the value of the labour supply in a steady state, which is determined by (6.6), and can be expressed as

$$H^* = L - \frac{A\ C^*}{(1-\tau^*)w^*}.$$

In this situation, since k^* is determined by (6.43), w^* is also determined by (6.43).

In light of the above relationship, we can easily deduce the following:

$$\overline{G} = \tau^* \, w^* \, (L - \frac{A \, C^*}{(1-\tau^*) \, w^*}) \tag{6.46}$$

Our simultaneous equations system consists of (6.35), (6.41), and (6.42). However, if (6.42) and (6.35) hold, (6.41) correspondingly does. This relationship is obvious from (6.42). In other words, the system is not linear independent, but $\dot{\tau}_t$ is linear dependent on (6.35) and (6.42). However, even in this situation, the government budget constraint showed by (6.46) must hold.

Following the above logic, both C^* and τ^* are simultaneously solved by (6.44) and (6.46). In this context, since C^* and τ^* are determined, H^* is also determined. Besides, K^* has been determined by (6.43).

In the above processes, C^* and τ^*, simultaneously solved by (6.44) and (6.46), are not necessarily unique.

We can confirm this as follows:

From (6.46) we obtain the following:

$$C^* = \frac{(1-\tau^*)(w^* \; L \; \tau^* - \overline{G})}{A \; \tau^*}. \tag{6.47}$$

When we substitute (6.47) into (6.44) and transform the results, we finally obtain the following:

$$
\begin{aligned}
&\gamma\ L(1-\alpha)\ (k^*)^{\alpha}\ (w^*\ L\ \tau^* - \overline{G})\ \tau^*(1-\tau^*)^3 \\
&\quad = \alpha\ (1-\alpha)^2\ L^2\ (k^*)^{3\alpha-1}\ (\tau^*)^2\ (1-\tau^*)^3 + (1-\alpha)^3\ L^2\ (k^*)^{3\alpha-1}\ (\tau^*)^2\ (1-\tau^*)^4 \\
&\quad - \alpha(1-\alpha)\ L\ (k^*)^{2\alpha-1}\ (w^*\ L\ \tau^* - \overline{G})(\tau^*)(2-\tau^*)(1-\tau^*)^2 \\
&\quad - (1-\alpha)^2 L\ (k^*)^{2\alpha-1}\ (w^*\ L\ \tau^* - \overline{G})(\tau^*)(2-\tau^*)(1-\tau^*)^3 \\
&\quad - \frac{(1-\alpha)^2\ L}{A}\ (k^*)^{2\alpha-1}\ (w^*\ L\ \tau^* - \overline{G})\ (\tau^*)\ (1-\tau^*)^4 \\
&\quad + \alpha\ (1-\alpha)\ L\ (k^*)^{2\alpha-1}(w^*\ L\ \tau^* - \overline{G})\ (\tau^*)(1-\tau^*)^3 \\
&\quad + (k^*)^{\alpha-1}\ (w^*\ L\ \tau^* - \overline{G})^2\ (1-\tau^*)^2(\alpha + (1-\alpha)(1-\tau^*)) \\
&\quad + \frac{1-\alpha}{A}\ (k^*)^{\alpha-1}(w^*\ L\ \tau^* - \overline{G})^2\ (1-\tau^*)^3 \\
&\quad - (w^*\ L\ \tau^* - \overline{G})^2\ (1-\tau^*)^3\ (\alpha\ (k^*)^{\alpha-1} - \gamma), \qquad (6.48)
\end{aligned}
$$

which shows that the above relationship forms an equation with six orders, with respect to τ^*.

Accordingly, as a solution, we obtain six coupled formulations for both C^* and τ^* from the simultaneous system concerning both. Among them, we have possibilities that there might exist multiple solutions, that is, multiple equilibria in our economy. This situation can be called indeterminacy, which specifically means the existence of multiple time paths or multiple equilibria in an economy.

Naturally, some solutions from (6.48) might not be associated with real solutions from an economic viewpoint, namely those which do not satisfy

Warlas' law.[5] Therefore, we need to examine whether the solutions from (6.48) satisfy this law or not. If two solutions, for example, satisfy the law, our economy can have two independent trajectories.

Now, assume that two solutions meet the criteria. This means that based on (6.48), we obtain two τ^*, which we denote τ^1 and τ^2, respectively. Furthermore, based on (6.47), C^*, that is, C^1 and C^2 are correspondingly determined. However, it is obvious that k^* remains unchanged in the two cases, because of (6.43). This means that in the steady states in the two cases, the capital intensity of labour and the corresponding equilibrium prices are similar, but the scale of two inputs, K^* and H^*, is different, if they exist.

Naturally, if $\overline{G}$ changes, both C^* and τ^* correspondingly do, while k^* is not affected by this change.

6.4.2 Linearization

At this stage, we can define three functions, which respectively govern the rates of change in C_t, k_t and τ_t, as follows:

$$\dot{C}_t = f_1(C_t, k_t), \tag{6.49}$$

$$\dot{k}_t = f_2(C_t, k_t, \tau_t), \tag{6.50}$$

[5] In our situation, the law can be described as

$$(K^*)^\alpha \ (H^*)^{1-\alpha} = C^* + \overline{G}.$$

and

$$\dot{\tau}_t = f_3(C_t, k_t, \tau_t). \tag{6.51}$$

Apparently, (6.49) through (6.51) constitute a simultaneous differential equations system. However, since this system is not linear, we have difficulties in terms of calculus to identify concretely how our economy moves. Therefore, we utilize linear approximation around (C^*, k^*, τ^*).

In other words, we expand the above three functions into Taylor series around (C^*, k^*, τ^*), ignore terms with more than second orders, and consider only the linearized relationships left. It is well known that under this linearization the solutions are topologically identical with the original solutions.

In sequence, we explore the values of elements of the Jacobian matrix.

First, concerning f_1, based on (6.42) we obtain the following:

$$\begin{aligned} f_{11} &= \frac{\partial f_1}{\partial C_t} \\ &= \alpha \ (k^*)^{\alpha-1} - \gamma \\ &= 0, \end{aligned} \tag{6.52}$$

and

$$\begin{aligned} f_{12} &= \frac{\partial f_1}{\partial k_t} \\ &= \alpha \ (\alpha - 1) \ (k^*)^{\alpha-2} \ C^* < \ 0. \end{aligned} \tag{6.53}$$

Second, concerning f_2, based on (6.35), we can define the following:

$$
\begin{aligned}
f_2(C_t, k_t, \tau_t) = & [(\alpha + (1-\alpha)(1-\tau_t))\ L^2(1-\alpha)^2(1-\tau_t)^3 k_t^{3\alpha} \\
& - A\ \gamma\ L(1-\alpha)(1-\tau_t)^2\ C_t\ k_t^{\alpha+1} \\
& + L(1-\alpha)(1-\tau_t)(-A(\alpha + (1-\alpha)(1-\tau_t))(2-\tau_t) \\
& + A\ \alpha(1-\tau_t) - (1-\alpha)(1-\tau_t)^2)\ C_t k_t^{2\alpha} \\
& + A\ C_t^2(A(\alpha + (1-\alpha)(1-\tau_t)) + (1-\alpha)(1-\tau_t) \\
& - A\ \alpha(1-\tau_t))\ k_t^{\alpha} + A^2\ \gamma\ C_t^2\ (1-\tau_t)\ k_t] \\
& \times [L(1-\tau_t)^2(1-\alpha)k_t^{\alpha} - (1-\alpha)\ A\ C_t]^{-1} \\
& \times [L(1-\tau_t)(1-\alpha)k_t^{\alpha} -\ A\ C_t]^{-1}. \qquad (6.54)
\end{aligned}
$$

Under the definition of (6.54) we can derive the following:

$$
\begin{aligned}
f_{21} = & \frac{\partial f_2}{\partial C_t} \\
= & [L(1-\alpha)(1-\tau^*)[-A\ (\alpha + (1-\alpha)(1-\tau^*))(2-\tau^*) - (1-\alpha)(1-\tau^*)^2 \\
& + A\ \alpha(1-\tau^*)]\ (k^*)^{2\alpha} + 2\ A\ C^*[A\ (\alpha + (1-\alpha)(1-\tau^*)) \\
& + (1-\alpha)(1-\tau^*) - A\ \alpha(1-\tau^*)]\ (k^*)^{\alpha} - A\ \gamma\ L(1-\alpha)(1-\tau^*)^2(k^*)^{\alpha+1} \\
& + 2A^2\ \gamma\ k^*\ (1-\tau^*)\ C^*] \\
& \times [L(1-\tau^*)^2(1-\alpha)\ (k^*)^{\alpha} - (1-\alpha)A\ C^*]^{-1} \\
& \times [L(1-\alpha)(1-\tau^*)\ (k^*)^{\alpha} - A\ C^*]^{-1}. \qquad (6.55)
\end{aligned}
$$

Additionally, notice that *the numerator* in (6.35) is null, because we are dealing with the values at steady states.

Similarly, we obtain the following relationship:

$$\begin{aligned}
f_{22} &= \frac{\partial f_2}{\partial k_t} \\
&= [3\alpha(\alpha + (1-\alpha)(1-\tau^*))\ L^2(1-\alpha)^2(1-\tau^*)^3(k^*)^{3\alpha-1} \\
&+ 2\alpha\ L(1-\alpha)\ C^*(1-\tau^*)[-A\ (\alpha + (1-\alpha)(1-\tau^*))(2-\tau^*) \\
&- (1-\alpha)(1-\tau^*)^2 + A\ \alpha(1-\tau^*)](k^*)^{2\alpha-1} \\
&+ A\ \alpha\ (C^*)^2[A\ (\alpha + (1-\alpha)(1-\tau^*)) + (1-\alpha)(1-\tau^*) \\
&- A\ \alpha(1-\tau^*)]\ (k^*)^{\alpha-1} - A\ \gamma L(1-\alpha)(1+\alpha)(1-\tau^*)^2\ C^*\ (k^*)^{\alpha} \\
&+ A^2\ \gamma\ (C^*)^2\ (1-\tau^*)] \\
&\times [L(1-\tau^*)(1-\alpha)(k^*)^{\alpha} - A\ C^*]^{-1} \\
&\times\ [L(1-\tau^*)^2(1-\alpha)(k^*)^{\alpha} - (1-\alpha)A\ C^*]^{-1}. \qquad (6.56)
\end{aligned}$$

Before an analysis of the influence on $\dot{\tau}_t$ by τ, for convenience, we define $1-\tau_t = \theta_t$ and θ^* as the value of θ in steady states. Under these definitions, we can denote f_2 as

$$\begin{aligned}
f_2 = {} & [(\alpha + (1-\alpha)\ \theta_t)L^2(1-\alpha)^2\ k_t^{3\alpha}\ \theta_t^3 \\
& - A\ \gamma\ L(1-\alpha)\ C_t\ k_t^{\alpha+1}\ \theta_t^2 + L(1-\alpha)\theta_t[-A(\alpha + (1-\alpha)\ \theta)(1+\theta_t) \\
& + A\ \alpha\ \theta_t - (1-\alpha)\ \theta_t^2]\ C_t\ k_t^{2\alpha} \\
& + A[A\ (\alpha + (1-\alpha)\ \theta_t) + (1-\alpha)\ \theta_t - A\ \alpha\ \theta_t]\ C_t^2\ k_t^{\alpha} + A^2\ \gamma\ C_t^2\ \theta_t\ k_t] \\
& \times [L(1-\alpha)k_t^{\alpha}\ \theta_t^2 - (1-\alpha)\ A\ C_t]^{-1} \\
& \times [L(1-\alpha)k_t^{\alpha}\ \theta_t - A\ C_t]^{-1}.
\end{aligned} \tag{6.57}$$

In this situation, we obtain the following:

$$
\begin{aligned}
f_{23} &= \frac{\partial f_2}{\partial \tau_t} \\
&= -\frac{\partial f_2}{\partial \theta_t} \\
&= -[L^2(1-\alpha)^2(k^*)^{3\alpha}((1-\alpha)(\theta^*)^3 \\
&+ (\alpha + (1-\alpha)\theta^*)3(\theta^*)^2) \\
&+ L(1-\alpha)C^*(k^*)^{2\alpha}[-A(\alpha + (1-\alpha)\ \theta^*)(\theta^* + 1) \\
&- (1-\alpha)(\theta^*)^2 + A\ \alpha\ \theta^*) + \theta^*(-A(1-\alpha)(\theta^* + 1) \\
&- A(\alpha + (1-\alpha)\theta^*) - 2(1-\alpha)\theta^* + A\ \alpha)] \\
&+ A\ (C^*)^2\ (k^*)^{\alpha}(A(1-\alpha) + (1-\alpha) - A\ \alpha) \\
&- 2\ \theta^*\ A\ \gamma\ L(1-\alpha)\ C^*(k^*)^{\alpha+1} + A^2\ \gamma\ (C^*)^2\ k^*] \\
&\times [L(1-\alpha)(k^*)^{\alpha}(\theta^*)^2 - (1-\alpha)A\ C^*]^{-1} \\
&\times [L(1-\alpha)(k^*)^{\alpha}\theta^* - A\ C^*]^{-1},
\end{aligned}
\tag{6.58}
$$

which finally leads to the following:

$$\begin{aligned} f_{23} = & -[L^2(1-\alpha)^2(k^*)^{3\alpha}((1-\alpha)(\theta^*)^3 \\ & + 3(\alpha + (1-\alpha)\ \theta^*)\ (\theta^*)^2) - L(1-\alpha)\ C^*\ (k^*)^{2\alpha}[A(1-\alpha)\ \theta^*(3\theta^* + 2) \\ & + 3(1-\alpha)\ (\theta^*)^2 + A\ \alpha] \\ & + A\ (C^*)^2\ (k^*)^\alpha((1-\alpha)(A+1) - A\ \alpha) \\ & - 2\ A\ \gamma\ L\ (1-\alpha)\ C^*\ \theta^*\ (k^*)^{\alpha+1} + A^2\ \gamma\ (C^*)^2\ k^*] \\ & \times [L(1-\alpha)(k^*)^\alpha(\theta^*)^2 - (1-\alpha)A\ C^*]^{-1} \\ & \times [L(1-\alpha)(k^*)^\alpha\theta^* - A\ C^*]^{-1}. \end{aligned} \tag{6.59}$$

Third, based on (6.41) and (6.51) we can express f_3 as

$$\begin{aligned} f_3 = & [\tau_t(1-\tau_t)\ (-\alpha(1-\alpha)\ L(1-\tau_t)k_t^\alpha\ f_2(C_t, k_t, \tau_t) \\ & + A\ k_t\ C_t\ f_1(C_t, k_t))][L(1-\alpha)(1-\tau_t)^2\ k_t^{\alpha+1} - A\ k_t\ C_t]^{-1}. \end{aligned} \tag{6.60}$$

In this situation, we obtain the following results:

$$\begin{aligned} f_{31} = & \frac{\partial f_3}{\partial C_t} \\ = & [\tau^*(1-\tau^*)(-\alpha(1-\alpha)\ L(1-\tau^*)\ (k^*)^\alpha \frac{\partial f_2}{\partial C_t} \\ & + A\ k^*\ C^*\ \frac{\partial f_1}{\partial C_t} + A\ k^*\ f_1)] \\ & \times [L(1-\alpha)(1-\tau^*)^2\ (k^*)^{\alpha+1} - A\ k^*\ C^*]^{-1}. \end{aligned} \tag{6.61}$$

In the procedure to formulate (6.61), we utilized $f_2(C^*, k^*, \tau^*) = 0$ and $f_1(C^*, k^*) = 0$.

Moreover, since $f_{11} = \dfrac{\partial f_1}{\partial C_t} = 0$ here, we can eventually transform (6.61) as follows:

$$\begin{aligned} f_{31} &= \tau^*(1-\tau^*)^2(-\alpha(1-\alpha))\ L\ (k^*)^\alpha\ f_{21} \\ &\times [L(1-\alpha)(1-\tau^*)^2\ (k^*)^{\alpha+1} - A\ k^*\ C^*]^{-1}. \end{aligned} \tag{6.62}$$

In a similar fashion, we obtain the following relationship:

$$\begin{aligned} f_{32} &= \frac{\partial f_3}{\partial k_t} \\ &= [\tau^*(1-\tau^*)^2(-\alpha(1-\alpha)\ L\ (k^*)^\alpha\ f_{22} + A\ k^*\ C^*\ f_{12})] \\ &\times [L(1-\alpha)(1-\tau^*)^2\ (k^*)^{\alpha+1} - A\ k^*\ C^*]^{-1}. \end{aligned} \tag{6.63}$$

At this stage, we denote f_3 in terms of θ_t and obtain the following:

$$\begin{aligned} f_3 =& [(1-\theta_t)\theta_t\ (-\alpha\ (1-\alpha)\ L\ k_t^\alpha\ \theta_t\ f_2(C_t, k_t, \theta_t) \\ &+ A\ k_t\ C_t\ f_1(C_t, k_t))] \\ &\times [L(1-\alpha)\ k_t^{\alpha+1}\theta_t^2 - A\ C_t\ k_t]^{-1}. \end{aligned}$$

Then, from (6.63), we can establish the following:

$$\begin{aligned} f_{33} &= \frac{\partial f_3}{\partial \tau_t} \\ &= -\frac{\partial f_3}{\partial \theta_t} \\ &= (1-\theta^*)\ (\theta^*)^2 \alpha\ (1-\alpha)\ L\ (k^*)^\alpha\ f_{23} \\ &\times [L(1-\alpha)(k^*)^{\alpha+1}\ (\theta^*)^2 - A\ C^*\ k^*]^{-1}, \end{aligned} \tag{6.64}$$

because both $f_2(C^*, k^*, \theta^*) = 0$ and $f_1(C^*, k^*) = 0$.

Based on what we have analyzed so far, we can establish a linear system, which approximately depicts our economy, as follows:

$$\begin{pmatrix} C_t \dot{-} C^* \\ k_t \dot{-} k^* \\ \tau_t \dot{-} \tau^* \end{pmatrix} = \begin{pmatrix} 0 & f_{12} & 0 \\ f_{21} & f_{22} & f_{23} \\ f_{31} & f_{32} & f_{33} \end{pmatrix} \begin{pmatrix} C_t - C^* \\ k_t - k^* \\ \tau_t - \tau^* \end{pmatrix} \tag{6.65}$$

6.4.3 A linear system

We define the matrix in (6.65) as B.

In what follows, we explore both eigenvalues and corresponding eigenvectors of B.

First, we can explore eigenvalues concretely by solving the following characteristic equation:

$$\left| \lambda\ I - B \right| = 0,$$

where I means a 3×3 unit matrix , which we can solve with respect to eigenvalue λ.

When we express the above characteristic equation more concretely, the following holds:

$$\begin{vmatrix} \lambda & -f_{12} & 0 \\ -f_{21} & \lambda - f_{22} & -f_{23} \\ -f_{31} & -f_{32} & \lambda - f_{33} \end{vmatrix} = 0,$$

which can be further transformed as follows:

$$\lambda^3 - (f_{22} + f_{33})\ \lambda^2 + (f_{22}\ f_{33} - f_{23}f_{32} - f_{12}f_{21})\lambda$$
$$+ f_{12}(f_{21}f_{33} - f_{23}f_{31}) = 0. \qquad (6.66)$$

(6.66) determines the eigenvalues of B, λ_i. In addition, $i = 1, 2, 3$.

Second, we explore characteristic vectors by solving the following equation:

$$\left(\lambda_i\ I - B\right) h^i = 0,$$

where h^i means a characteristic vector corresponding to λ_i.

We can express the above equation more concretely as

$$\begin{pmatrix} \lambda_i & -f_{12} & 0 \\ -f_{21} & \lambda_i - f_{22} & -f_{23} \\ -f_{31} & -f_{32} & \lambda_i - f_{33} \end{pmatrix} \begin{pmatrix} h_1^i \\ h_2^i \\ h_3^i \end{pmatrix} = 0, \qquad (6.67)$$

which can be expressed as

$$\begin{cases} \lambda_i\, h_1^i - f_{12}\, h_2^i = 0, \\ -f_{21}\, h_1^i + (\lambda_i - f_{22})h_2^i - f_{23}\, h_3^i = 0, \\ -f_{31}\, h_1^i - f_{32}\, h_2^i + (\lambda_i - f_{33})h_3^i = 0. \end{cases} \tag{6.68}$$

h_1^i, h_2^i and h_3^i are respectively determined by (6.68).

Consequently, we can express the movements of our economy as

$$\begin{aligned} \begin{pmatrix} C_t - C^* \\ k_t - k^* \\ \tau_t - \tau^* \end{pmatrix} &= c_1\, h^1 e^{\lambda_1\, t} + c_2\, h^2 e^{\lambda_2\, t} + c_3\, h^3 e^{\lambda_3\, t} \\ &= c_1 \begin{pmatrix} h_1^1 \\ h_2^1 \\ h_3^1 \end{pmatrix} e^{\lambda_1\, t} + c_2 \begin{pmatrix} h_1^2 \\ h_2^2 \\ h_3^2 \end{pmatrix} e^{\lambda_2\, t} + c_3 \begin{pmatrix} h_1^3 \\ h_2^3 \\ h_3^3 \end{pmatrix} e^{\lambda_3\, t}. \end{aligned} \tag{6.69}$$

Additionally, c_i means a constant determined by initial conditions.

6.4.4 An example

Based on what we have investigated so far, it is obvious that (6.69) shows the time paths of our economy, and we have seen the possible existence of multiple equlibria in our economy, that is, of indeterminacy.[6] Here, we need to specify how our economy moves under this indeterminacy.

[6]Note that the existence of multiple equlibria can be proven without a postulation of linear approximation and for the whole range of $0 < \tau < 1$.

Since our system in (6.69) is highly complicated, we rely on simulation in order to identify it.

Let us suppose that $L = 48$; $\alpha = 0.1$; $A = 1$; $\gamma = 0.1$.

Moreover, we postulate that a government chooses to impose labour income taxation through the determination of tax rates, in order to finance its spending, and $\overline{G} = 1$.

Under these assumptions, $k^* = 1$ is determined by (6.43). Of course, w^* is known because k^* has been determined, and it is actually 0.9. In this situation, a government imposes labour tax rates in order to satisfy condition (6.46). In this circumstance, C^* and τ^* are determined to satisfy both (6.44) and (6.46). In fact, from (6.44) and (6.46) we obtain six couples of C^* and τ^*, but only one couple, $C^* = 21.6753$ and $\tau^* = 0.0490008$, satisfies the Walras' law. In these processes $H^* = 22.6753$ is determined, and afterward $K^* = 22.6753$ is also determined by (6.43)

Next, we need to explore how our economy moves in this case.

First, in the case of $C^* = 21.6753$ and $\tau^* = 0.0490008$, we obtain the ingredients of a Jacobian matrix as follows:

f_{12}	-1.95078	f_{31}	0.000913613
f_{21}	-0.0830103	f_{32}	-0.108663
f_{22}	0.0850923	f_{33}	-0.0197942
f_{23}	-1.79848		

Then, based on matrix B, we obtain three eigenvalues of B, $\lambda_1 = 0.641019$, $\lambda_2 = -0.557792$, and $\lambda_3 = -0.0179293$.

We can judge from the above eigenvalues that the equilibrium with $C^* = 21.6753$ and $\tau^* = 0.0490008$ is a saddle.

If there are other values for the parameter α, such as $\lambda_1 < 0$ and λ_2 and λ_3 stay negative, this equilibrium is a sink, which means indeterminacy.

Under these conditions, there can exist numerous time paths toward a steady state, which start with arbitrary capital stocks, and we can not identify a specific time path, even if a transversality condition holds. Therefore, this indeterminacy is considered to be one of the necessary conditions for sunspot fluctuations, in which business cycles can be included.

6.5 Concluding remarks

We need to explore, from an economic viewpoint, why the system in (6.69) has three negative eigenvalues, or two negative parts with double imaginary roots and a negative eigenvalue. This will show us why the indeterminacy occurs.

And we have to explain why this indeterminacy matters, if it occurs.

Now, let us consider the situation with no taxation in a steady state that a representative household faces. Suppose that for some reasons, the household expects higher rental rates on capital in future. Then, it cuts consumption today, while it increases its saving today, which will become an increase in capital stock in future.

But, conversely, this increment of capital usually causes lower rental rates on capital because of the law of decreasing returns to scale. This means that the optimism of a household is not realized.

However, in this situation, if regressive taxation (our balanced-budget rule is included in this category) is introduced, the theory can be reversed. Under enough regressive taxation, both after-tax labour income and the rate of return on capital can be increased. Through this mechanism, the initial expectation about an increase of rental rates on capital by a household can be self-fulfilled. Of course, this is a kind of sunspot fluctuation, in which we can explain how business cycles can occur. And for this fluctuation to occur, indeterminacy is one of the preconditions.

We are forced to rely on simulation throughout, in order to investigate what concrete conditions in our economy can cause this indeterminacy. This interesting research will be carried out in a subsequent article.

Chapter 7

Conclusion

We have established our theory based on an implicit common context, namely that economic outcomes by an individual agent directly lead to aggregate ones. This philosophy is simple and prevalent, because of its micro-foundations. Macroeconomics since Keynes (1978) has been criticized by neoclassical economists such as Lucas, due to its lack of micro-foundations.

With regard to this, our theory overcomes such criticism on appearance, because our logical stance is in the same line as that prevailing currently in economics, that of the neoclassical economists.

However, there might be room for controversies about this big issue.[1]

The *Zeitgeist*, as discussed by Hegel, is a term which implies that many authors, including economists, have a tendency to be influenced by prevailing ideas in their days. In light of the *Zeitgeist*, in this sense, this dissertation

[1] See Yoshikawa (2003).

might not also be able to avoid being seen as a product of our era.

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