Structural Change of Dynamical Systems based on Heterarchical Duality

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Doctoral Dissertation

Structural Change of Dynamical Systems Based on Heterarchical Duality

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Preface

Y.-P. Gunji says, "Time is the other name of life." (Gunji 2004): where "time" does not mean a parameter as a master clock for a system but means "flow of structural change" of the system. In this dissertation, statuses of the above "time" and "life" are replaced by that of "emergence" and "robustness", which are explored as the introduction of our view. Although this text has an aspect of article on complex systems science, one must see that philosophical and ontological problems are dealt with in this text based on the viewpoint of *internal measurement* (Matsuno 1989).

This doctoral dissertation consists of five chapters and one appendix.

In Chapter 1, our problem establishment is presented via forms of physics and the problems of emergence, autonomy and robustness in complex systems science.

In Chapter 2, the basic terms of category theory are surveyed, and a dynamical system is expressed via the terms of category.

In Chapter 3 and Chapter 4, our motivation is concretely developed via forms of discrete dynamical systems, such as coupled map systems of logistic maps (Chapter 3) or Henon maps (Chapter 4). Diagrams of category theory is incorporated to reify the discussion in Chapter 1 and to extend each dynamical system. Chapter 3, Chapter 4 and Appendix A of this dissertation are the following articles which were partially added and altered:

- Chapter 3:
  Moto Kamiura and Yukio-Pegio Gunji,
  Robust and Ubiquitous On-Off Intermittency in Active Coupling.
  Physica D **218**(2006)122-130.

- Chapter 4 and Appendix A:
  Moto Kamiura, Kohei Nakajima and Yukio-Pegio Gunji,
  Generative Pointer: Dynamical System with a Fluctuant Parameter Motivated by Origin of Fraction.
  Physca D (submitted).

Finally, deconstructionalistic significance of this dissertation is reaffirmed in Conclusion.

I belonged to nonlinear science laboratory in graduate school of Kobe university from 2002 to 2007. During the five years, although I am grateful to many people who have helped my studies and my life, I write only some people's name here.

I am deeply grateful to Professor Yukio-Pegio Gunji, my direct supervisor. When I studied physics in Toho University, I knew his articles, *Protocomputing and Ontological Measurement*, in a Japanese philosophical journal. Later, I met him with his cratered sofa and strong coffee in his laboratory. Since then, his articles and instructions have made deep impacts on my life.

I am extremely grateful to Professor Tamiki Komatsuzaki for kind instruction. to Professor Yoji Aizawa (Waseda Univ.) and Professor Masayoshi Inoue (Kagoshima Univ.) for kind support to my activity.

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I am deeply grateful to my family.

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Chapter 1

Introduction

In this chapter, our problem establishment is presented via forms of physics and the problems of emergence, autonomy and robustness in complex systems science. In the following chapters, some dynamical systems are extended based on this discussion.

1.1 Emergence, Autonomy and Robustness

A concept of emergence has been spread across various disciplines. It usually means the phenomenon such that advanced and complicated order / new behavior are generated. When the subject of research is dealt with as a system, local interaction between the system components or integration of many autonomous elements is regarded as the causes of the order / behavior. In this sense, emergence is regarded as a property of the particular systems and emergent systems are separated from non-emergent systems.

In the study of multi-agent systems and/or multi-degree of freedom dynamical systems, the emergence is regarded as behavior that is not observed on the constituent parts on the system but is generated on the whole system. In this sense, the distinction of "the part and the whole" means that of the spatial scales. We call it narrowly-defined emergence.

By contrast, we emphasize the aspect of "emergence" depending on observation. In our viewpoint, the emergence is regarded as just generating new things and is not dealt with as the feature of the spatial scale. We call it broadly-defined emergence or just emergence.

Emergence is not a property of the particular systems but is found under the relation between the systems and their environment / observers. For example, suppose a system that outputs a sequence of signals. As the example of the signals, one can take utterances of human beings, electric bits patterns or proteins in cells, etc. These signals are received and interpreted by the observers such as human beings, computers or DNA, etc. If an observer of the

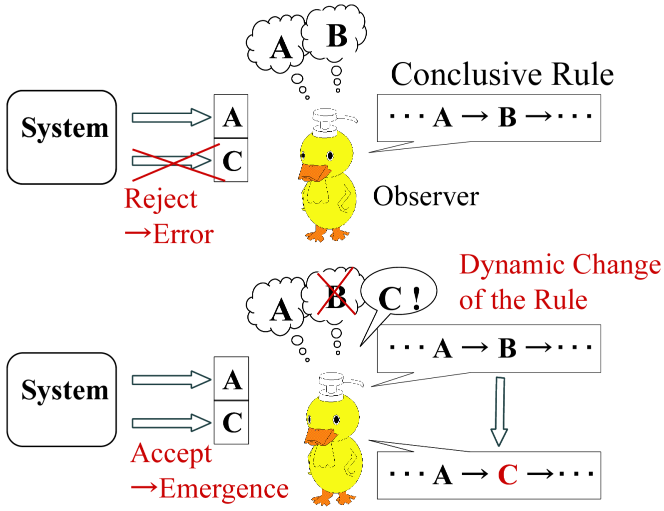


Figure 1.1: Difference between error and emregence. Unexpected outputs are regarded not as the errors but as the emergence when they are accepted by the system observers.

system presumes that the system outputs the signals according to a conclusive rule, deviated signals from the presumed rule are regarded as just errors (i.e. fluctuations in terms of physics). By contrast, If the observer presumes that the system has autonomy, the deviating signals are regarded not as errors but as the consequence of the autonomy. Such unexpected outputs are regarded not as the errors but as the emergence when they are accepted by the system observers (Figure 1.1). Thus emergent phenomena are connected to autonomy via observers.

The autonomy of the system is found by the observer that accepts the change of the rules on the system. The emergence is the unexpected outputs with the change of the rules. Therefore, the autonomy and the emergence are based on the same phenomenon, and correspond to the aspect of the system and the events occurring on the system, respectively. In such a viewpoint, we can no longer divide the system from the observer. [1]

[1]i.e. if the picture of emergence is based on the dynamic change of the presupposed rule, the cause of emergence can not boil down to only system or to only observer. Such a change of rule is the ceaseless process, such that final decision of the rule is postponed, and such that the rule is determined "for the time being".

However, we must note the following with the above: i.e. if a pair of system and observer, [system-observer]system, is simply introduced into the model, one can suppose a meta-observer for the [sysem-observer]system who has more complete knowledge / information (Figure 1.2). Moreover, one can suppose "a set of complete knowledge" as a limit of such a process, [· · · [[system-observer]system-observer]system-· · ·]. This supposition means that

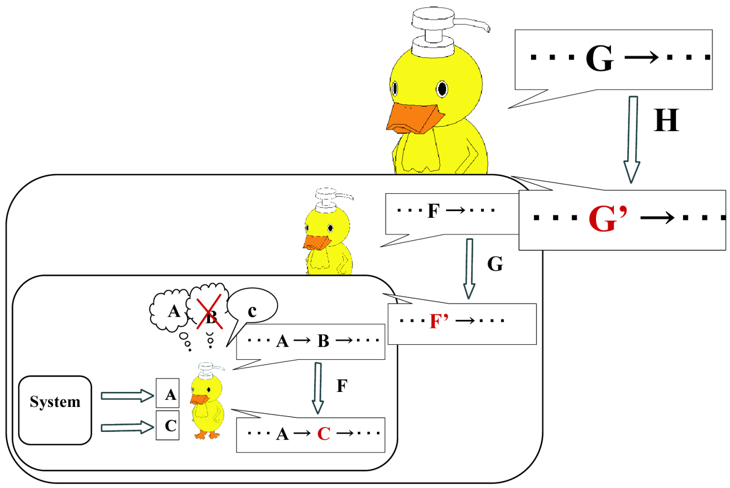


Figure 1.2: (See footnote 1): If a pair of system and observer, [system-observer]system, is simply introduced into the model, one can suppose a meta-observer for the [sysem-observer]system who has more complete knowledge / information. This infinite regress is coincided with the frame problem (McCarthy and Hayes 1969; Dennet 1984; McCarthy 1990).

When mathematical models are constructed from such an aspect, the acceptance of the unexpected outcomes can be formalized as indefinite domains and/or ranges of functions and as dynamic and structural change of the functions. In the chapter thesis, we propse two models based on the two points.

1.2 Indefiniteness in the Form of Physics

Let us paraphrase the above viewpoint in terms of physics.

In conventional science, if one analyzes a physical system, he replaces intensity with extensive quantities. For example, given a Newtonian equation $m\dot{\mathbf{v}} = \mathbf{F}$, one replaces the force (i.e. intensity) $\mathbf{F}$ with an extensive quantity such as $-k\mathbf{x}$ since he cannot directly deal with the intensity. Such replacement is a means to deal with the intensity analytically. If the intensity and the extensive quantity have an one-to-one correspondence between them, you have only to deal with the extensive quantity. Conventional quantitative science is

there is an ultimate rule of the system and that only real observers can not know the rule. If we suppose the set of complete knowledge, there are not "new events" or "emergence" in principle.

In contrast, if we accept the concept of emergence, we must also accept incompleteness / indefiniteness of the systems. In other words, introduction of the concept of system observers means acceptance of incompleteness / indefiniteness of the systems.

based on this viewpoint, and fixed experimental environment is necessary for such an one-to-one correspondence.

Matsuno showed that a conventional viewpoint of physics restricts an understanding of a concept of emergence that is found on complex systems such as biological or economic systems, and he proposed a concept of internal measurement (Matsuno 1989). Internal measurement is explained as a motion that carries on canceling conflicts between particles with local perspective and correspond to a process of transformation from intensity into an extensive quantity. The extensive quantity satisfies each physical law in hindsight. He emphasizes the indefiniteness of the determination of the boundary conditions in the process of the internal measurement. In the following, we amplify the implications.

Generally, a model for a physical system is constructed based on several hypothesis and experimental facts. Specified experimental environment is necessary to keep reproducibility and ability to control the system. And reproducibility and control ability are necessary to fit a mathematical form to the system. When the system is modeled in a mathematical form, we obtain sets of data rechaptering behavior of the system. The model that is empirically constructed is regarded as a real rule to control the system.

Modeling the system in a specified mathematical form needs actively ignoring unexpected influence that is not included in the model. For example, if one hooks his foot on an experimental setup (i.e. unexpected influence), he works over the experiment to obtain correct results. He does not rewrite the model but does the experiment again. Such unexpected influence is the same kind of the frame problem on artificial intelligence (McCarthy and Hayes 1969; Dennet 1984; McCarthy 1990). Ignoring unexpected influence means separating the system from the indefiniteness, and means expressing the model of the system by a closed form. This simple example shows that the system is constructed, is indicated and is described in the open environment.

On the other hand, reproducibility and control ability are incompatible with emergence. If deviation of the data is observed in the system under the reproducibility and the control ability, we conclude that the deviation is derived from fluctuation or improper experimental conditions and the rules that administer the system are invariable; i.e. the system is regarded as a machine that works by the rules. By contrast, if the deviation of the data is regarded as not fluctuation but a result of change of the rules, we find emergence by the system itself.

The change of the rules corresponds to rewriting the model of the system; i.e. the frame problem and the emergence are two sides of a coin. If we suppose the invariable rules and the fluctuation in the system, unexpected change of the system means the frame problem. By contrast, if we suppose the variable rules in the system, the unexpected change means the emergence. The concept of emergence obviously is not properties of particular systems but lies between the system and its observer that assumes the specified rules.

The frame problem and the emergence are two sides of a coin. If we suppose the invariable rules and the fluctuation in the system, unexpected change of the system means the frame problem. By contrast, if we suppose the variable rules in the system, the unexpected change means the emergence. The concept of emergence obviously is not properties of particular systems but lies between the system and its environment that assumes the specified rules.

Our viewpoint can be generalize as heterarchy on the system and the environment. A concept of heterarchy was chaptered by McCulloch (1945). and is related to emergence and robustness (jen 2003; Kamiura and Gunji 2006). A heterarchical system is generally characterized by a hierarchical system with interaction between its layers or with mixture of them.

In the chapter thesis, we studied this problem using category theory and a dynamical system. On a viewpoint of a dynamical system, dynamic change of the rule corresponds to change of the vector field/the time evolution operator of the dynamical system. Thus, when we try to understand a concept of emergence on the dynamical system, one of the most important aim is a formalization of dynamic change of the time evolution operator. Such change of the operator is equal to transformation of a manifold on a phase space.

Chapter 2

Mathematical Preliminary

To discuss the view of the previous chapter more concretely, we use dynamical systems and category theory (Maclane 1971=1997; Awodey 2006). In the present thesis, a dynamical system is expressed in terms of category theory. On the study of complex systems, the viewpoint of dynamical systems has given the outstanding knowledge. Moreover, category theory contributes to bringing out the status of the models and that of the variables.

2.1 Category Theory

In this section, we survey terms of category theory that is useful for an expansion of a dynamical system.

Definition 2.1 (category) A category consists of the following data:

- Objects: $A, B, C, \cdots$
- Arrows: $f, g, h, \cdots$
- For each arrow f there are given objects:

  $$\mathrm{dom}(f), \quad \mathrm{cod}(f)$$

  called the domain and codomain of f. We write:

  $$f : A \to B$$

  to indicate that $A = \mathrm{dom}(f)$ and $B = \mathrm{cod}(f)$.
- Given arrows $f : A \to B$ and $g : B \to C$, i.e. with:

  $$\mathrm{cod}(f) = \mathrm{dom}(g)$$

there is given an arrow:

$$g \circ f : A \to C$$

called the composite of f and g.

- For each object A there is given an arrow:

$$id_A : A \to A$$

called the identity arrow of A.

These data are required to satisfy the following laws:

- Associativity:

$$h \circ (g \circ f) = (h \circ g) \circ f$$

for all $f : A \to B$, $g : B \to C$, $h : C \to D$.

- Unit:

$$f \circ id_A = f = id_B \circ f$$

for all $f : A \to B$.

Definition 2.2 (functor) A functor

$$F : \mathbf{C} \to \mathbf{D}$$

between categories $\mathbf{C}$ and $\mathbf{D}$ is a mapping of objects to objects and arrows to arrows, in such a way that:

- $F(f : A \to B) = F(f) : F(A) \to F(B)$
- $F(g \circ f) = F(g) \circ F(f)$
- $F(id_A) = id_{F(A)}$

Note every category $\mathbf{C}$ has an identity functor $1_{\mathbf{C}} : \mathbf{C} \to \mathbf{C}$.

Definition 2.3 (coproduct) Suppose a category $\mathbf{C}$ and an index set Λ. A coproduct $\coprod C_i$ is defined as an object with arrows $\{\iota_i : C_i \to \coprod C_i\}_{i \in \Lambda}$ such that it satisfies the following condition: given $g_i : C_i \to A$, there is an unique arrow $h : \coprod C_i \to A$ such that $h \circ \iota_i = g_i$ is commutateive for each $i \in \Lambda$. Also h is expressed by $[\iota_i]_{i \in \Lambda} : \coprod C_i \to A$.

Definition 2.4 (coequalizer) A coequalizer of arrows $g, h : A \to B$ in a category $\mathbf{C}$ is defined by an allow $e : B \to X$ such that it satisfies the following two conditions: i) $e \circ g = e \circ h$; ii) For an arbitrary arrow $e' : B \to X'$ that satisfies $e' \circ g = e' \circ h$, there is an unique arrow $k : X \to X'$ such that it satisfies $k \circ e = e'$.

Definition 2.5 (cocone) Given a category $\mathbf{C}$, its diagram $\mathbf{C}'$ and its set of vertexes V, a cocone of $\mathbf{C}'$ is an object X with a family of arrows $\nu = \{\nu_i : C_i \to X\}_{i \in V}$ (expressed by $\nu : \mathbf{C}' \to X$) that satisfies the following: For arbitrary arrows $g : C_i \to C_j$ on $\mathbf{C}$, ν satisfies $\nu_j \circ g = \nu_i$.

Definition 2.6 (colimit) Given a category $\mathbf{C}$ and its diagram $\mathbf{C}'$, a colimit of $\mathbf{C}'$ is a cocone $\mu : \mathbf{C}' \to M$ that satisfies the following: For arbitrary cocone $\nu : \mathbf{C}' \to X$ of $\mathbf{C}'$, there is an unique arrow $k : M \to X$ such that $k \circ \mu = \nu$.

Definition 2.7 (pullback) In any category $\mathbf{C}$, pullback of arrows $g : A \to C$ and $h : B \to C$ is a pair of arrows $p_1 : P \to A$ and $p_2 : P \to B$, that satisfies the following i) and ii): i) $g \circ p_1 = h \circ p_2$, ii) Given any $z_1 : Z \to A$ and $z_2 : Z \to B$ with $g \circ z_1 = h \circ z_2$, there is a unique map $u : Z \to P$ with $z_1 = p_1 \circ u$ and $z_2 = p_2 \circ u$.

Remark 2.8 $A \times_C B$ that is a subobject of $A \times B$ with projections $\pi_1 : A \times B \to A$ and $\pi_2 : A \times B \to B$ is a pullback of $g : A \to C$ and $h : B \to C$, where $A \times_C B = \{(a, b) | a \in A, b \in B, g(a) = h(b)\}$.

Definition 2.9 (slice category) A slice category $\mathbf{C}/X$ of a category $\mathbf{C}$ over an object $X \in \mathbf{C}$ consists of the following objects and arrows: objects are all arrows $\nu_i \in \mathbf{C}$ such that $\mathrm{cod}(\nu_i) = X$, and an arrow g from $\nu_i : C_i \to X$ to $\nu_j : C_j \to X$ is $g : C_i \to C_j$ in $\mathbf{C}$ such that $\nu_j \circ g = \nu_i$.

Remark 2.10 (composition functor) Composition induces a functor. For any slice category $\mathbf{C}/A$ with objects $\{\mu_i : C_i \to A\}$ and arrows $g : C_i \to C_j$, an arrow $k : A \to B$ induces a slice category $\mathbf{C}/B$ with objects $\{\nu_i = k \circ \mu_i : C_i \to B\}$ and arrows $g : C_i \to C_j$ and a functor $K : \mathbf{C}/A \to \mathbf{C}/\mathbf{B}$ such that $K(\mu_i) = k \circ \mu_i = \nu_i$ and $K(g) = g$.

Remark 2.11 (pullback functor) Pullback induces a functor. For $k : A \to B$ in a category $\mathbf{C}$ with pullbacks, there is a functor $K^* : \mathbf{C}/B \to \mathbf{C}/A$ defined by $(\nu_i : C_i \to B) \mapsto (\mu'_i : C_i \times_B A \to A)$ where μ'_i is the pullback of ν_i along k.

2.2 Category of a Dynamical System

A dynamical system can be expressed via the terms of category theory (Louie 1985). We can see the following two facts: a set of trajectories of a dynamical system is the colimit of the category of the dynamical system; Hamiltonian is a cocone if the dynamical system is Hamiltonian system. Therefore, the motion and the energy as micro-level of the system and macro-level of that are connected via a composition functor between the two slice category induced by them.

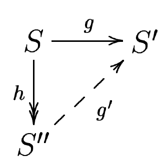
Definition 2.12 (dynamical system and its category) Suppose a topological space D and a continuous map $f : D \times \mathbb{R} \to D$. For each $t \in \mathbb{R}$, a map $f_t : D \to D$ is defined by $f_t(x) = f(x, t)$ $(x \in D)$. If a family of the maps $\{f_t\}_{t \in \mathbb{R}}$ satisfies the following conditions i) and ii), then (D, f) is called a continuous dynamical system on D: i) $f_t \circ f_{t'} = f_{t+t'}$ for all $t, t' \in \mathbb{R}$; ii) $f_0 = id_D$. The map f_t means a time evolution operator of the dynamical system. And for each $x \in D$, $Gx = \{f_t(x)|t \in \mathbb{R}\}$ is called a trajectory or an orbit through x.

The composition i) satisfies associative law, thus we obtain a category of the dynamical system **D** that has the phase space D as its object and the map f_t as its arrow. **D** is obviously a subcategory of **Top**, thus a functor F :**D** $\to$ **Top** is defined by an inclusion mapping.

We use the following lemma to construct a colimit of a dynamical system.

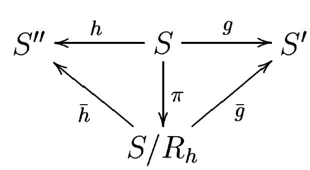
Lemma 2.13 Given a map $g : S \to S'$ and a surjective map $h : S \to S''$, the following two condition are equivalent.

1. For $x, y \in S$, $h(x) = h(y) \Rightarrow g(x) = g(y)$.
2. There is a unique map $g' : S'' \to S'$ such that $g = g' \circ h$.



Proof (2. $\Rightarrow$ 1.) $h(x) = h(y) \Rightarrow g'(h(x)) = g'(h(y)) \Rightarrow g(x) = g(y)$. h is a surjective map, thus an arbitrary element in S'' is expressed by $h(x)$ $(x \in S)$ and its image of g' is $g'(h(x)) = g(x)$. The fact is independent of x, thus g' is unique.

(1. $\Rightarrow$ 2.) h induces a injection $\bar{h} : S/R_h \to S''$ and $h = \bar{h} \circ \pi$ $(\pi : S \to S/R_h$ is a canonical mapping). h is a surjection, thus $\bar{h}$ is a bijection.



By the condition 1., g induces a map $\bar{g} : S/R_h \to S'$ and $g = \bar{g} \circ \pi$. If g' is defined by $g' = \bar{g} \circ \bar{h}^{-1}$, we obtain $g = \bar{g} \circ \pi = \bar{g} \circ \bar{h}^{-1} \circ h = g' \circ h$ (i.e. the condition 2.).

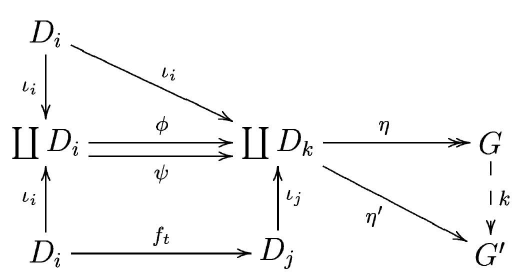
We construct a colimit of a dynamical system $\mathbf{D}$ using the coproduct and the coequalizer in $\mathbf{D}$.

Construction 2.14 (colimit of diagram for a dynamical system) Suppose a category of a Hamiltonian dynamical system $\mathbf{D}$ with constant energy. For arbitrary $t \in [0, \infty) \subset \mathbb{R}$, objects in $\mathbf{D}$ are $\mathrm{dom}(f_t) = D_i$ and $\mathrm{cod}(f_t) = D_j$. An arbitrary vertex of the diagram of $\mathbf{D}$ is represented by D_k. $\mathbf{D}$ is a subcategory of **Top**, thus there are coproducts $\coprod D_i$ and $\coprod D_k$ in **Top**. For canonical injections $\iota_i : D_i \to \coprod D_i$ and $\iota_i : D_i \to \coprod D_k$, there is an unique arrow $\phi : \coprod D_i \to \coprod D_k$ such that $\phi \circ \iota_i = \iota_i$ because of a definition of a coproduct. Again, for $\iota_i : D_i \to \coprod D_i$, $\iota_j : D_j \to \coprod D_k$ and $\iota_j \circ f_t : D_i \to \coprod D_k$, there is an unique arrow $\psi : \coprod D_i \to \coprod D_k$ such that $\psi \circ \iota_i = \iota_j \circ f_t$. ϕ and ψ stand for $\phi = [\iota_i]_{i \in \{\mathrm{dom}(f_t)\}}$ and $\psi = [\iota_j \circ f_t]_{j \in \{\mathrm{cod}(f_t)\}, t \in \mathbb{R}}$.

Moreover, a quotient space $\coprod D_k / \sim$ is induced from the equivalence relation $x_t \sim x_{t+s} \Leftrightarrow {}^{\exists}s \in \mathbb{R}(x_{t+s} = f_s(x_t))$. Therefore, we obtain $G := \coprod D_k / \sim \ni Gx_t := \{x_t | x_t = f_t(x_0), x_0 \in D_0, t \in [0, \infty) \subset \mathbb{R}\}$. A surjection $\eta : \coprod D_k \to G; x_t \mapsto Gx_t$ satisfies $\eta \circ \phi = \eta \circ \psi$.

Given a cocone G' with $\nu : \mathbf{D} \to G'$, it induces an unique arrow $\eta' : \coprod D_k \to G'$ such that $\eta' \circ \iota_i = \nu$, $\rho \circ \iota_i = \nu$ and $\rho = \eta' \circ \phi = \eta' \circ \psi$ because of the definition of coproduct. And there is an unique arrow $k : G \to G'$ because of Lemma 2.11 (note that η is a surjection). Therefore, η is a coequalizer of ϕ and ψ and G with $\mu = \eta \circ \iota$ is a colimit of $\mathbf{D}$.

The above facts are expressed by the following diagram:



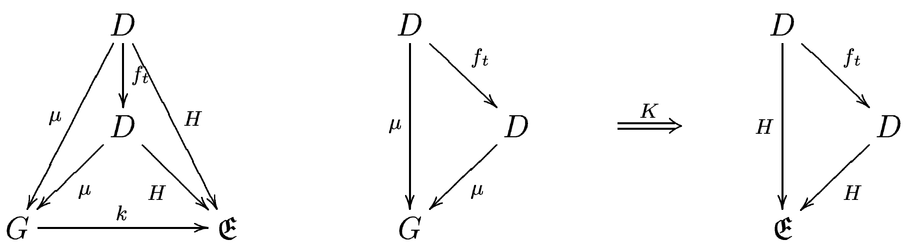
(2.1)

The colimit G corresponds to a set of trajectories.

We show static structure between a micro-level layer (i.e. a set of the vectors on the phase space) and a macro-level layer (i.e. the energy conservation law) in a Hamiltonian dynamical system.

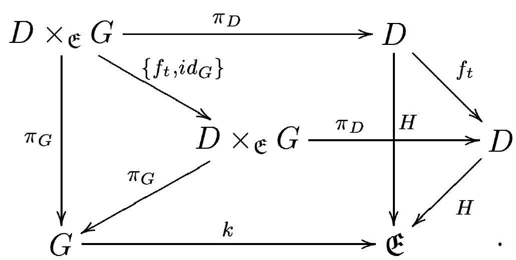
Suppose a Hamiltonian of a n-dimensional system with constant energy $H(\mathbf{p},\mathbf{q}) = E$. A category of the Hamiltonian dynamical system $\mathbf{D}$ consists of objects $D \ni (\mathbf{p},\mathbf{q})$ and allows $\{f_t = e^{-\mathcal{L}t} : D \to D\}$ where $\mathcal{L} = \sum_{i=0}^{n-1}\left(\frac{\partial H}{\partial q_i}\frac{\partial}{\partial p_i} - \frac{\partial H}{\partial p_i}\frac{\partial}{\partial q_i}\right)$ is a Liouville operator. We obtain a colimit G of a diagram $\mathbf{D}'$ of $\mathbf{D}$ by the construction 2.12. If $H(\mathbf{p},\mathbf{q}) = E$ then $H(f_t(\mathbf{p},\mathbf{q})) = E$ for arbitrary t, thus a set of energy values $\mathfrak{E} = \{E | E \in [0,\infty) \subset \mathbb{R}\}$ with arrows $\{H : D \to \mathfrak{E}\}$ is a cocone of $\mathbf{D}'$.

Two slice categories $\mathbf{D}/G$ and $\mathbf{D}/\mathfrak{E}$ are induced from $\mathbf{D}$, G and $\mathfrak{E}$. A composition functor $K : \mathbf{D}/G \to \mathbf{D}/\mathfrak{E}$ is induced from $k : G \to \mathfrak{E}$ that uniquely exists.



(2.2)

And the pullback of H along k induces a functor $K^* : \mathbf{D}/\mathfrak{E} \to \mathbf{D}/G$. it is expressed by the following diagram:



(2.3)

Thus a consistency between the micro-level layer and the macro-level layer in the Hamiltonian dynamical system is expressed by $K : \mathbf{D}/G \to \mathbf{D}/\mathfrak{E}$ and $K^* : \mathbf{D}/\mathfrak{E} \to \mathbf{D}/G$.

Chapter 3

Active Coupling

On-off intermittency is an aperiodic switching between synchronized chaos and non-synchronized chaos, and is observed near the edge of chaos on a bifurcation diagram of coupling strength. Therefore, its occurrence needs to tune parameters of the system sensitively. In this chapter, a concept of *observational heterarchy* is reviewed with a simple model and characters of on-off intermittency derived from *active coupling* systems (ACS) are investigated. Generally, on-off intermittency results from fluctuation of the local transverse Lyapunov exponent. ACS can retain such a character of the Lyapunov exponent by the time evolution of the maps, so on-off intermittency occurs on wider parameter regions. These are concrete approaches to the notion of robustness that is regarded as persistence different from stability on dynamical systems.

3.1 Introduction

Intermittency is observed in various nonlinear dynamical systems. Three types of intermittency concerning the transition between periodic oscillation and aperiodic it are distinguished and are called Type I, II, III intermittemcy (Pomeau and Manneville, 1980). Crisis-induced intermittency caused by the collision of the chaotic attractor is studied by Grebogi et al (1982). Moreover, intermittency found out by Yamada and Fujisaka (1985;1986) is observed on a coupled map system composed of chaotic oscillators and it is called type B intermittency or intermittency caused by chaotic modulation. The same kind of phenomena called on-off intermittency by Platt et al.(1993), and the statistic features are discussed by Heagy et al.(1994). On-off intermittency is an aperiodic switching between synchronized chaos (laminar phase) and non-synchronized chaos (burst phase). On-off intermittency occurs near the edge of chaos on a bifurcation diagram of coupling strength, is applied to a chaos neural network, and is related to information processing[21].

In a modeling of a complex system, such as life, economics, etc., it is necessary to tune parameters for relevant behavior of the model. Complex behavior that is neither stable nor chaotic is characterized by class IV cellular automata or dynamical systems at the edge of chaos, for example, and it appears at phase transition points of parameters of their systems. Since the parameter region leading to such behavior is generally narrow, the following question arises: Is such complex behavior exceptional event in nature? Are the characters of this world fragile things based on delicate parameters? For these problems, importance of robustness that is a different notion from stability of a dynamical system is addressed recently (Jen, 2003). The concepts of trajectory stability and structural stability are known in a dynamical system. In contrast, robustness that is not defined strictly is regarded as feature persistence against dynamical change of the system structures, and as a concept that is relevant to a notion of heterarchy presented by McCulloch (1945). Generally, a heterarchical system is characterized by a hierarchical system with interaction between various layers.

The generalized heterarchy concept is called *observational heterarchy* (Gunji and Kamiura, 2004; 2006). This point of view is consistent with a notion of internal measurement (Matsuno, 1985; Gunji et al,). If heterarchy is conceptually extended to observational heterarchy, heterarchical structures are found not only in particular systems but also in the systems with observation ubiquitously.

Through the application of our method based on the notion of observational heterarchy to a coupled map system, the maps of the coupling system can also evolve and the parameter region giving the intermittency is extended. In other words, the intermittency occurs against the dynamical change of the maps and disturbance of the parameter. Therefore, we think that our system based on observational heterarchy gives a concrete example of robustness.

Technical process of this chapter is the following: First, two heterarchical layers are arranged mathematically and observational heterarchy is expressed as a pair of these layers. Secondly, the form of observational heterarchy is applied to a coupled map system composed of nonlinear oscillators for appraisal of our concept and it is called *active coupling* system (ACS) to distinguish from the conventional coupled map systems. The system applied observational heterarchy has time evolution of the maps. Thirdly, Statistical features of on-off intermittency given by ACS are estimated.

3.2 Mathematical Form of Observational Heterarchy

3.2.1 Formalization of Heterarchical Layers by Category Theory

The concept of heterarchy is proposed in a context of the topology of nervous nets that shows what value is not magnitude of any one kind (McCulloch, 1945). Generally, a heterarchical system consists of two (or plural) layers and interaction between them. In hierarchical system, one layer depends on the other layer. By contrast, heterarchical system can switch the dependence relations of each layer (Figure 3.1 (a)).

Now we survey the heterarchy concept through an example (de May, 1982). Suppose a hypothetical program reading a document (Figure 3.1 (b)). A document is resolved into sentences, phrases, words and letters for reading it. A sentence is composed of letters, so the sentence transaction layer depends on the letter transaction layer. If this hierarchical structure is fixed, the transaction is stopped when one letter cannot be deciphered. In contrast, a heterarchical system can read the document by making up for the letter through using the contexts. The proper switching between these layers produces robustness of the transaction.

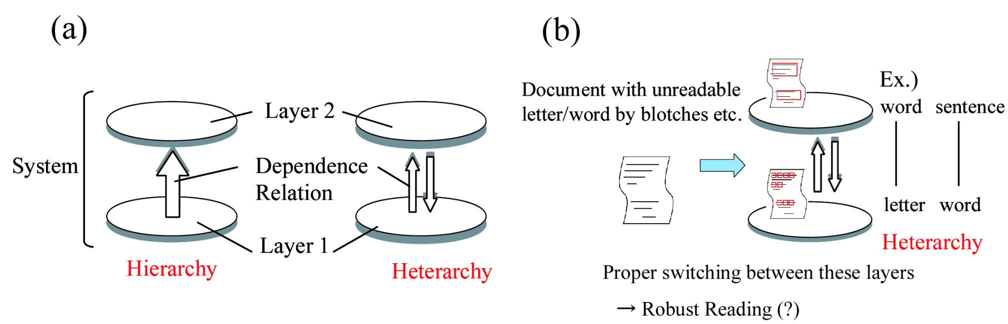


Figure 3.1: (a) Hierarchicy and heterarchy. (b) An example of heterarchical system reading a document (de May, 1982). If this hierarchical structure is fixed, the transaction is stopped when one letter cannot be deciphered. In contrast, a heterarchical system can read the document by making up for the letter through using the contexts. The proper switching between these layers produces robustness of the transaction.

Such an example is lucid and useful. In fact, however, the distinction between these layers is fuzzy and the meaning or usage of a word is opened to dynamical change. It is necessary to broaden our horizons not only for the switching between solid layers but also for the dynamical change of the layers themselves derived from interaction between them. In the following, we expand the concept of heterarcy and propose observational heterarchy. For the expansion of the concept and the application to coupled map systems, mathematical forms in terms of category theory are introduced.

Definition 2.1 (slice category) Given a category $\mathbf{C}$, a slice category $\mathbf{C}/A$ is defined by the following; an object is an arrow $\psi : C \to A$ in $\mathbf{C}$ satisfying $\mathrm{cod}(\psi) = A$ and an arrow is an arrow $i : C' \to C$ satisfying $\psi \circ i = \psi'$, with $\psi' : C' \to A$.

A slice category is one of the categories containing arrows as its objects. Since a system can be generally regarded as a set of functions, a slice category is fit for a description of such a system. In this chapter, observing a system is defined as constructing an equivalence between two slice category.

Construction 2.2 For an arbitrary slice category $\mathbf{C}/A$, we can construct the subcategory $\mathbf{D}_A \subseteq \mathbf{C}/A$ with epic arrows $\psi : C \to A$ as its objects. And an arrow $f : A \to B$ induces a slice category $\mathbf{D}_{B'} \subseteq \mathbf{C}/B'$ with $\phi = f \circ \psi : C \to B'$ as its object where $B' = \mathrm{Im}(f) \subseteq B$ and a mapping $F : \mathbf{D}_A \to \mathbf{D}_{B'}; \psi \mapsto f \circ \psi$. Then F maps an arrow $i \in \mathbf{D}_A$ such that $\psi \circ i = \psi'$ to $i \in \mathbf{D}_{B'}$ such that $\phi \circ i = \phi'$. Since one can check the preservation of identity arrows and composition, F is a functor.

It is trivial that $F : \mathbf{D}_A \to \mathbf{D}_{B'}$ is an onto mapping. Moreover, if f is bijective, F is an one-to-one mapping ($F(\psi) = F(\psi') \Leftrightarrow f \circ \psi = f \circ \psi' \Rightarrow \psi = \psi'$). Then we can construct a mapping $F^{-1} : \mathbf{D}_B \to \mathbf{D}_A; \phi \mapsto f^{-1} \circ \phi$ that is a functor as well as F. In this case, F shows the equivalence $\mathbf{D}_A \cong \mathbf{D}_B$.

For a category $\mathbf{C}$ and an arrow $f \in \mathbf{C}$ we introduce $L1 \equiv \langle \mathbf{C}, f \rangle$ and $L2 \equiv \langle \mathbf{D}_A, \mathbf{D}_{B'}, F \rangle$ as hierarchical/heterarchical layers. If F is an one-to-one and onto mapping, then $L2$ satisfies the form of ovservation $\mathbf{D}_A \cong \mathbf{D}_B$. These layers, $L1$ and $L2$, are given as the mathematical framework of observational heterarchy.

Lemma 2.3 A mapping $R : \mathrm{Hom}_{L1}(A, B) \to \mathrm{Hom}_{L2}(\mathbf{D}_A, \mathbf{D}_{B'}); f \mapsto F$ can be defined and is an one-to-one mapping.

Proof Suppose $R(f) = F; \psi \mapsto f \circ \psi$ and $R(f') = F'; \psi \mapsto f' \circ \psi$. Then,

$$\begin{aligned}
&\forall f, f' \in \mathrm{Hom}_{L1}(A, B)(R(f) = R(f')) \\
&\Leftrightarrow \forall f, f' \in \mathrm{Hom}_{L1}(A, B)(F = F') \\
&\Leftrightarrow \forall f, f' \in \mathrm{Hom}_{L1}(A, B)\forall\psi \in |\mathbf{D}_A|(F(\psi) = F'(\psi)) \\
&\Leftrightarrow \forall f, f' \in \mathrm{Hom}_{L1}(A, B)\forall\psi \in |\mathbf{D}_A|(f \circ \psi = f' \circ \psi) \\
&\Rightarrow \forall f, f' \in \mathrm{Hom}_{L1}(A, B)(f = f')
\end{aligned} \tag{3.1}$$

since ψ is epic. Therefor R is an one-to-one mapping.

Arrows and functors are generally separated as operations differing in logical status. Such a distinction is represented as $F(\psi)$ and $f \circ \psi$, for example. On *L1* and *L2*, f is separated from F on the surface. But f is necessarily confused with F when a concrete operation of F is executed. The confusion of the functor F with the arrow f is expressed by the following;

$$F(\psi) = f \circ \psi \tag{3.2}$$

where the left hand and the righgt hand of the equation show the operations of *L2* and *L1*. And if $f : A \to B$ is bijection, we obtain

$$F^{-1}(\phi) = f^{-1} \circ \phi \tag{3.3}$$

and such a confusion is consistent.

In the expression of hierarchical/heterarchical layers, *L1* and *L2*, we can paraphrase the previous example about a document reading: if f is not bijective, the transaction is switched from F on *L2* that requires an equivalence between $\mathbf{D}_A$ and $\mathbf{D}_B$ to f on *L1*. For the situation, we define a *pseudo-inverse map* f^* instead of f^{-1}.

Definition 2.4 (pseudo-inverse map) Given a map $g : A \to B$, the inverse image of $y \in \mathrm{Im}(g) = B' \subseteq B$ is induced from g: $g^{-1}(\{y\}) = \{x | x \in A, g(x) = y \in B'\}$. One can pick up an element $x_y \in g^{-1}(\{y\})$ for all of $y \in B'$. Then, a map

$$g^* : B' \to A; y \mapsto x_y$$

is induced. It is called *pseudo-inverse map* of g and expressed by g^* (cf. Figure 3.2).

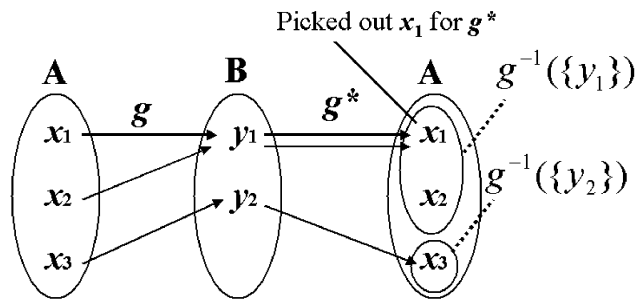


Figure 3.2: An example of pseudo-inverse map g^* of a map g. The value of $g^*(y_1)$ can arbitrarily select from the inverse image $g^{-1}(\{y_1\}) = \{x_1, x_2\}$ of g. The value of $g^*(y_1)$ selected from $\{x_1, x_2\}$ randomly, time evolution of ACS is defined.

g^* satisfies the following properties: g^* is not unique for g; $gg^* = id_{B'}$ and $g^*g \neq id_A$; By a proper restriction $g|A' = g \circ j$ where j is an injection $j : A' \to A$ and $A' \subseteq A$, $g|A' \circ g^* = id_{B'}$ and $g^* \circ g|A' = id_{A'}$; If g is a bijection, g^* results the inverse map $g^{-1} : B \to A$; If g is a continuous function, A is a set of continuous real number and $g(A)$ has any local minimum and local maximum in $\{x | x \in A, g^{-1}(\{\min(g(A))\}) < x < g^{-1}(\{\max(g(A))\})\}$, g^* is necessarily discontinuous.

In the following section, we express dynamical change of the heterarchcal layers by defining an equivalence-like operation derived from a pseudo-inverse map.

3.2.2 Internal Perspective and Observational Heterarchy

As one of the important topic in the science of complex systems, there is a concept of an agent that have a perspective from the inside of a system[**?**][28]. Since the agent with the internal perspective cannot look out over the whole system, disturbance beyond its expectations is inevitable on their observation. But the system features persist against the disturbance. Therefore, the concepts of heterarchy and internal measurement share the same motivations. It is necessary to express such disturbance or indetermination on the framework of observational heterarchy.

Suppose the layers *L1* and *L2* in the previous section. We express observation of the internal perspective by the following equivalence-like operations $\tilde{F}$ and $\tilde{F}^*$: Given an arrow $\tilde{f} : A \to B$ that is not monic but epic and is per-

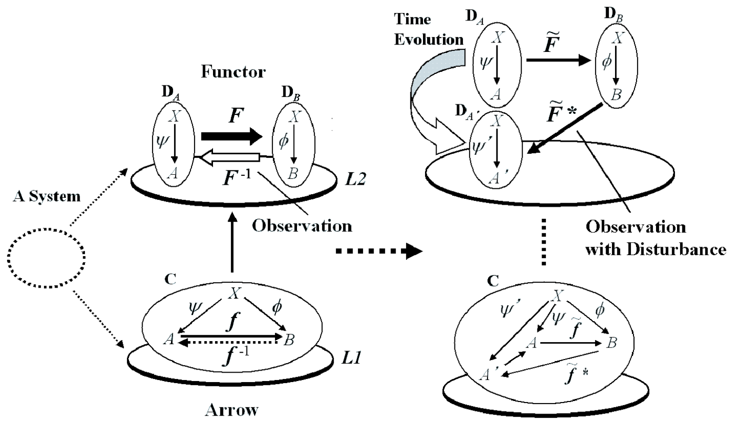


Figure 3.3: The concept of observational heterarchy based on the form of category theory. Each layer, *L1* or *L2*, is constructed by a commutative diagram with arrows or functors and observation in the system is expressed by an equivalence between $\mathbf{D}_A$ and $\mathbf{D}_B$. If there is disturbance on the observation, the corresponding functor is regarded as an operator for time evolution of *L2*.

mitted approximation $\tilde{f} \cong f$ where $f : A \to B$ is a bijection, a pseudo-inverse map $\tilde{f}^* : B \to A$ is induced. $\tilde{f}^*$ is not unique for $\tilde{f}$ and there is the family of the arrows $\{\tilde{f}^*_{(t)}\}_{t \in N}$ satisfying $\tilde{f}\tilde{f}^*_{(t)} = id_B$ and $\tilde{f}^*_{(t)}\tilde{f} \neq id_A$ where N is a set of infinite natural numbers. Then functors $\tilde{F} : \mathbf{D}_A \to \mathbf{D}_B; \psi \mapsto \tilde{f} \circ \psi$ and $\tilde{F}^*_{(t)} : \mathbf{D}_B \to \mathbf{D}_{A(t)}; \phi \mapsto \tilde{f}^*_{(t)} \circ \phi$ can be constructed where $A_{(t)} = \mathrm{Im}(\tilde{f}^*_{(t)}) \subseteq A$. The facts that $\tilde{f}$ is not monic and there are infinite $\tilde{f}^*$ parallel what observation of the internal perspective involves the disturbance and the indetermination.

If $\tilde{f} = f$ is a bijection, then $\tilde{F} = F : \mathbf{D}_A \to \mathbf{D}_B; \psi \mapsto f \circ \psi$ and $\tilde{F}^*_{(t)} = F^{-1} : \mathbf{D}_B \to \mathbf{D}_A; \phi \mapsto f^{-1} \circ \phi$. Hence $\tilde{F}$ and $\tilde{F}^*_{(t)}$ show the equivalence $\mathbf{D}_A \cong \mathbf{D}_B$ in the approximation $\tilde{f} \cong f$. On the other hand, if we strictly deal with $\tilde{f}$ that is not monic, then the identity functor $F^{-1} \circ F = I_{\mathbf{D}_A}$ is replaced with $\tilde{F}^*_{(t)} \circ \tilde{F} : \mathbf{D}_A \to \mathbf{D}_{A(t)}$ that can be regarded as an operator of time evolution. Therefore, the ovservation of the agent with the internal perspective makes the agent itself change (i.e. the heterarchical layer *L2* changes). In other word, the external observation does not influence the system but the internal observation derives the dynamical change of the system.

And the previous fact is inconsistent with the demand that the operation F on *L2* must be an equivalence of slice categories, but

$$\tilde{F}^*_{(t)} \circ \tilde{F}(\psi) = \tilde{f}^*_{(t)} \circ \tilde{f} \circ \psi \tag{3.4}$$

as well as Eq.(3.2)-(3.3) and the operation on the right hand of Eq.(3.2) is consistent on *L1*. Apparently the aspect of internal observation is paradoxical, although actual operations never deadlock over the observation. i.e. the stipulated rule is broken on *L2* but not on *L1*.

Regarding $t \in N$ as discrete time, we can construct the time evolution of ψ such that; $\phi_{(1)} = \tilde{f}\psi_{(0)}$; $\psi_{(1)} = \tilde{f}^*_{(1)}\phi_{(1)} = \tilde{f}^*_{(1)}\tilde{f}\psi_{(0)}$; $\phi_{(2)} = \tilde{f}\psi_{(1)} = \tilde{f}\tilde{f}^*_{(1)}\phi_{(1)} = \phi_{(1)}$; $\psi_{(2)} = \tilde{f}^*_{(2)}\phi_{(2)} = \tilde{f}^*_{(2)}\phi_{(1)} = \tilde{f}^*_{(2)}\tilde{f}\psi_{(0)}$.

Note $\tilde{f}$ and ϕ is invariant over the time. In this chapter, $\tilde{f}^*_{(t)}$ is constructed by picking up an element of $\tilde{f}^{-1}(\{y\}) = \{x | x \in A, \tilde{f}(x) = y \in B\}$ randomly. The N of $\{\tilde{f}^*_{(t)}\}_{t \in N}$ can be infinite natural number formally but it is finite natural number in the experiments. And in previous paper[**?**][17], the time evolution of ψ and ϕ is formalized by a diagram of a pullback functor. The present model is simplified by the use of a composition instead of the pullback, but it is based on the same concept.

As a result, we obtain a system that keeps changing via two heterarchical layers and confusing them in the present section (cf. Figure 3.3).

3.3 Application to Coupled Map System

3.3.1 Active Coupling System

In this section, observational heterarchy is applied to a coupled map system for an appraisal of the concept.

Suppose $\mathbf{C}/A \supseteq \mathbf{D}_A \ni \psi : [0.0, 1.0] \to [0.0, 1.0]$, $\mathbf{C}/B \supseteq \mathbf{D}_B \ni \phi : [0.0, 1.0] \to [0.0, c]$ as $\phi(x) = f \circ \psi(x)$ and $f : [0.0, 1.0] \to [0.0, c]$ where $c \in [0.0, 1.0]$ is a constant. Selecting the bijection $f(z) = cz$ for f, we compose a conventional coupled map system such as the following;

$$\begin{pmatrix} x_{t+1} \\ y_{t+1} \end{pmatrix} = \begin{pmatrix} (1-c)\psi(x_t) + \phi(y_t) \\ (1-c)\psi(y_t) + \phi(x_t) \end{pmatrix} \tag{3.5}$$

where x_t and y_t are states at time t and c is utilized as the coupling strength.

Eq.(3.5) is modified based on the concept of observational heterarchy such as the following;

$$\begin{pmatrix} x_{t+1} \\ y_{t+1} \end{pmatrix} = \begin{pmatrix} (1-c)\psi_{(t)}(x_t) + \phi_{(t)}(y_t) \\ (1-c)\psi_{(t)}(y_t) + \phi_{(t)}(x_t) \end{pmatrix} \tag{5.a}$$

$$\phi_{(t+1)} := \tilde{f}\psi_{(t)} = \tilde{f}\psi_{(0)} \tag{5.b}$$

$$\psi_{(t+1)} := \tilde{f}^*_{(t+1)}\phi_{(t+1)} = \tilde{f}^*_{(t+1)}\tilde{f}\psi_{(0)} \tag{5.c}$$

where the second and third equations reveal the time evolution of maps and $\tilde{f} : [0.0, 1.0] \to [0.0, c]$. Eq.(5.a-c) are called Active Coupling system (ACS)

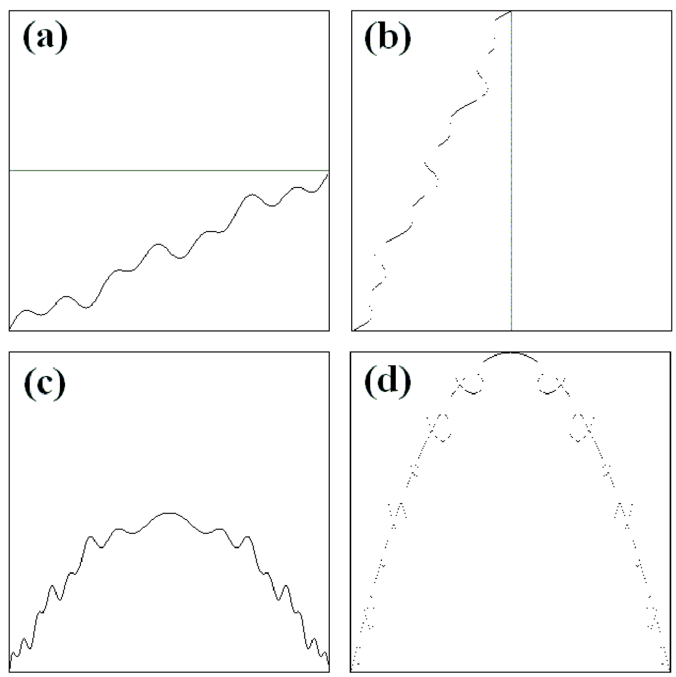


Figure 3.4: The graphs of the maps applied to ACS. Each square is the coordinate of $(x, y) \in [0,1] \times [0,1]$. **(a)** $y = \tilde{f}(x) = cx + 0.1c\sin(15\pi x)\cos(4\pi x)$. The horizontal broken line expresses $y = c$. **(b)** $y = \tilde{f}^*_{(t)}(x)$. The vertical broken line expresses $y = \frac{1}{c}$. **(c)** $y = \phi^{(t)}(x) = \phi^{(0)}(x)$. **(d)** $y = \psi^{(t)}(x)$.

so as to distinguish it from the conventional coupling system (Eq.(3.5)). If $\tilde{f}(z) \cong f(z) = cz$, then ACS results in Eq.(3.5).

The following are applied to each system in this chapter;

$$\psi_{(0)}(x) = \psi(x) = 4x(1-x) \tag{3.7}$$

$$\tilde{f}(z) = \begin{cases} -h(z) & (h(z) < 0) \\ h(z) & (0 \leq h(z) \leq c) \\ 2c - h(z) & (c < h(z)) \end{cases} \tag{3.8}$$

where we experimented on the following $h(x)$;

$$h(z) = cz + 0.1c\sin(10\pi z). \tag{3.9}$$

The pseudo-inverse map $\tilde{f}^*$ for these $\tilde{f}$ becomes discontinuous map and cannot be determined uniquely. The graphs of $\tilde{f}$, $\tilde{f}^*_{(t)}$, $\psi_{(t)}$ and $\phi_{(t)} = \phi^{(0)}$ are shown in Figure 3.4 . Time evolution of the map $\psi_{(t)}$ is derived from random selections of $\tilde{f}^*_{(t)}$ from $\{\tilde{f}^*_{(t)}\}_{t \in I}$. For making $\tilde{f}^*_{(t)}$, we divided $\mathrm{dom}(\tilde{f}) = [0.0, 1.0]$ into 10^2, 10^3 or 10^4 intervals and $y = \tilde{f}^*_{(t)}(x)$ was constructed by linear approximation in each interval. If the wavelength of the disturbance in $\tilde{f}$ is long enough for the intervals, every calculation is not influenced in kind.

And in addition, we obtain the following ACS composed of asymmetrical maps without inconsistency for every previous definition;

$$\begin{pmatrix} x_{t+1} \\ y_{t+1} \end{pmatrix} = \begin{pmatrix} (1-c)\tilde{f}^*_{(j)}\tilde{f}\psi(x_t) + \tilde{f}\psi(y_t) \\ (1-c)\tilde{f}^*_{(j')}\tilde{f}\psi(y_t) + \tilde{f}\psi(x_t) \end{pmatrix} \tag{3.10}$$

where $\{j | j \in I\}$ and $\{j' | j' \in I' \cong I, I' \subseteq I\}$. In the present experiments, $\tilde{f}^*_{(j')}$ is fitted into $\tilde{f}^*_{(j)}$ in the probability of 98% when $|x_t - y_t| < 10^{-4}$ and else $\tilde{f}^*_{(j)} = \tilde{f}^*_{(j')}$.

Active coupling is different from the conventional coupling in terms of the two parts, $\tilde{f}$ and $\tilde{f}^*_{(t)}$. For comparison, we conducted a control experiment as,

$$\begin{pmatrix} x_{t+1} \\ y_{t+1} \end{pmatrix} = \begin{pmatrix} (1-c)\psi(x_t) + \tilde{f}\psi(y_t) \\ (1-c)\psi(y_t) + \tilde{f}\psi(x_t) \end{pmatrix}. \tag{3.11}$$

3.3.2 Lyapunov Analysis

On-off intermittency generally results from fluctuation of the local transverse Lyapunov exponent, that can be also analized in the ACS as well as the conventional coupling system (Eq.(3.5)). In Eq.(3.5) or Eq.(5), the synchronized state $x^0_t \equiv x_t = y_t$ in time t is expressed by the following;

$$\begin{pmatrix} x^0_{t+1} \\ x^0_{t+1} \end{pmatrix} = \begin{pmatrix} (1-c)\psi(x^0_t) + \phi(x^0_t) \\ (1-c)\psi(x^0_t) + \phi(x^0_t) \end{pmatrix}. \tag{3.12}$$

Therefore, $\delta x_t \equiv x_t - x_t^0$ satisfies the following;

$$\delta \mathbf{x}_{t+1} \equiv \begin{pmatrix} \delta x_{t+1} \\ \delta y_{t+1} \end{pmatrix} = \begin{pmatrix} (1-c)\{\psi(x_t) - \psi(x_t^0)\} + \{\phi(y_t) - \phi(x_t^0)\} \\ (1-c)\{\psi(y_t) - \psi(x_t^0)\} + \{\phi(x_t) - \phi(x_t^0)\} \end{pmatrix}. \tag{3.13}$$

If δx_t is small, the linear approximation $\psi(x_t) - \psi(x_t^0) = \psi(x_t^0 + \delta x_t) - \psi(x_t^0) \approx \psi'(x_t^0)\delta x_t$ is permitted, and then one obtains

$$\begin{aligned} \delta \mathbf{x}_{t+1} &= \begin{pmatrix} (1-c)\psi'(x_t^0)\delta x_t + \phi'(x_t^0)\delta y_t \\ (1-c)\psi'(x_t^0)\delta y_t + \phi'(x_t^0)\delta x_t \end{pmatrix} \\ &= \begin{pmatrix} (1-c)\psi'(x_t^0) & \phi'(x_t^0) \\ \phi'(x_t^0) & (1-c)\psi'(x_t^0) \end{pmatrix} \begin{pmatrix} \delta x_t \\ \delta y_t \end{pmatrix} \\ &\equiv K(x_t^0)\delta \mathbf{x}_t. \end{aligned} \tag{3.14}$$

The transformation of the bases $(\mathbf{e}_x, \mathbf{e}_y) \mapsto (\mathbf{e}_{||}, \mathbf{e}_\perp)$ induces the following;

$$\delta \mathbf{x}_t = \delta x_t \mathbf{e}_x + \delta y_t \mathbf{e}_y = v_{||}(t)\mathbf{e}_{||} + v_\perp(t)\mathbf{e}_\perp \tag{3.15}$$

$$v_\perp(t+1)\mathbf{e}_\perp = K(x_t^0)v_\perp(t)\mathbf{e}_\perp = \{(1-c)\psi'(x_t^0) - \phi'(x_t^0)\}v_\perp(t)\mathbf{e}_\perp \tag{3.16}$$

where

$$(\mathbf{e}_x, \mathbf{e}_y) = \left(\begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right), (\mathbf{e}_{||}, \mathbf{e}_\perp) = \left(\begin{pmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{pmatrix}, \begin{pmatrix} \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} \end{pmatrix} \right). \tag{3.17}$$

Therefore, the local transverse Lyapunov exponent is

$$\Lambda_\perp(t) = \ln \frac{|v_\perp(t+1)|}{|v_\perp(t)|} = \ln \left| (1-c)\psi'(x_t^0) - \phi'(x_t^0) \right| \tag{3.18}$$

and the transverse Lyapunov exponent is

$$\lambda_\perp(t) = \lim_{t\to\infty} \frac{1}{t} \sum_{t=0}^{t-1} \ln \frac{|v_\perp(t+1)|}{|v_\perp(t)|} = \lim_{t\to\infty} \frac{1}{t} \sum_{t=0}^{t-1} \ln \left| (1-c)\psi'(x_t^0) - \phi'(x_t^0) \right|. \tag{3.19}$$

If the maps have time evolution,

$$\begin{aligned} \lambda_\perp(t) &= \lim_{t\to\infty} \frac{1}{t} \sum_{t=0}^{t-1} \ln \left| (1-c)\psi'_{(t)}(x_t^0) - \phi'_{(t)}(x_t^0) \right| \\ &= \lim_{t\to\infty} \frac{1}{t} \sum_{t=0}^{t-1} \ln \left| (1-c)\frac{d}{dx}(f^*_{(t)} f \psi(x_t^0)) - \frac{d}{dx}(f\psi(x_t^0)) \right| \\ &= \lim_{t\to\infty} \frac{1}{t} \sum_{t=0}^{t-1} \ln \left| (1-c)\frac{df^*_{(t)}(q_t^0)}{dq}\frac{df(p_t^0)}{dp}\frac{d\psi(x_t^0)}{dx} - \frac{df(p_t^0)}{dp}\frac{d\psi(x_t^0)}{dx} \right| \\ &= \lim_{t\to\infty} \frac{1}{t} \sum_{t=0}^{t-1} \ln \left| (1-c)s_t(q_t^0) - 1 \right| \left| \phi'(x_t^0) \right| \end{aligned} \tag{3.20}$$

where $p = \psi(x)$, $q = f(p)$, $\phi'(x) = \frac{df(p)}{dp}\frac{d\psi(x)}{dx}$, $s_t(q) = \frac{df^*_{(t)}(q)}{dq}$, and note $\phi_{(t)} = \phi_{(0)}$.

For Eq.(3.10) the local transverse Lyapunov exponent is

$$\lambda_\perp(t) = \ln \frac{\left|(1-c)\left(s_j\delta x_t - s_{j'}\delta y_t\right) - (\delta x_t - \delta y_t)\right| |\phi'(x_t^0)|}{|\delta x_t - \delta y_t|} \tag{3.21}$$

where $s_j = \frac{df^*_{(j)}(q_t^0)}{dq}$. δx_t and δy_t are entangled in $\lambda_\perp(t)$ because of the difference of $f^*_{(j)}$ and $f^*_{(j')}$.

3.4 Results

Eq.(3.9) is used as $h(z)$ in the following results but we obtain the qualitatively same results in statistics when $0.1c\sin(10\pi z)$ is replaced with $0.15c\sin(10\pi z)$, $0.1c\sin(12\pi z)$ or $0.1c\sin(15\pi z)\cos(4\pi z)$.

Figure 3.5 show bifurcation diagrams with coupling strength c. Each result of the conventional coupling (Eq.(3.5)), ACS with the symmetrical $\tilde{f}^*$ (Eq.(4)), ACS with the asymmetrical $\tilde{f}^*_{(j)}$ and $\tilde{f}^*_{(j')}$ (Eq.(3.10)) and the control experiment (Eq.(3.11)) are shown in (a)-(d) where $h(x) = cx + 0.1c\sin(10\pi x)$, and the values of $x_t - y_t$ with $4000 \leq t \leq 4200$ were uesed to construct the diagrams. In conventional coupling, the edge of chaos appear just in $c = 0.25$ and $c = 0.75$ and on-off intermittency is observed only near the critical parameters. By contrast, the intermittency of ACS is observed in the wide parameter region, i.e. in $0.0 < c < 1.2$ and $3.5 < c < 7.8$ of the symmetrical ACS, and in $0.0 < c < 0.8$ of the asymmetrical ACS.

And, although the control experiment shows the feature of systems depends on the form of $\tilde{f}$ (i.e.$h(x)$) in the region of large c, ACS keep that feature against the replacement of $\tilde{f}$ by such as the previous $h(z)$.

Figure 3.6 show frequency of alternations from burst phase to laminar phase in $10000 \leq t \leq 20000$ where the correspondence of (a)-(d) is the same as in the Figure 3.5. The state is regarded as burst phase if $|x_t - y_t| > \delta$, and the state is regarded as laminar phase if $|x_t - y_t| \leq \delta$ where $\delta = 10^{-4}$ is a reasonable threshold. The alternations is counted if $|x_{t-1} - y_{t-1}| > \delta$ and $|x_t - y_t| \leq \delta$ are satisfied. Although the differance of chaotic state and intermittent state is not distinctive in the bifurcation diagrams, Figure 3.6 show that the parameter regions with on-off intermittency is clearly extended in ACS.

Figure 3.7 shows transverse Lyapunov exponent in each model where the correspondence of (a)-(d) is the same as in the Figure 3.5. It is checked that the values of the Lyapunov exponent is saturated by 1000 steps, therefore these diagrams show the values in $t = 2000$. On-off intermittency is observed near the transition point since the intermittency results from fluctuation of the local

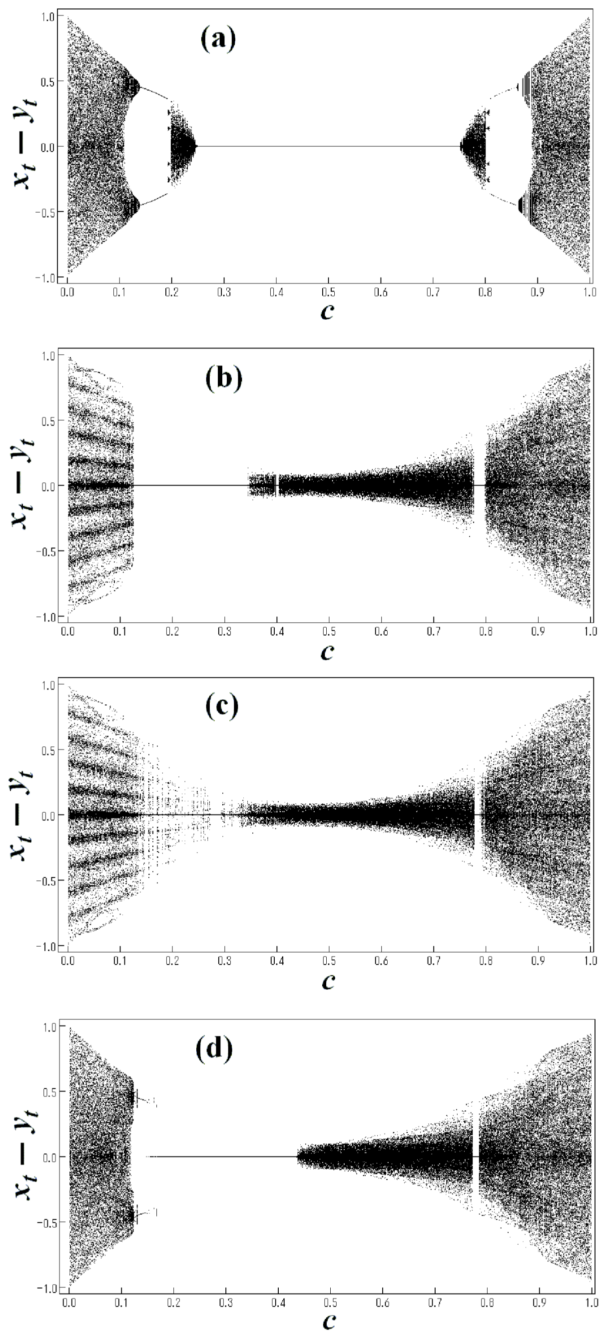


Figure 3.5: The bifurcation diagrams of the values of $x_t - y_t$ in $4000 \leq t \leq 4200$. $h(x) = cx + 0.1c\sin(10\pi x)$ is used in the following (b)-(d). **(a)** Conventional coupling, Eq.(4). **(b)** Active coupling with symmetrical $\tilde{f}^*$, Eq.(4). **(c)** Active coupling with asymmetrical $\tilde{f}^*_{(j)}$ and $\tilde{f}^*_{(j')}$, Eq.(9). **(d)** Control experiment, Eq.(10).

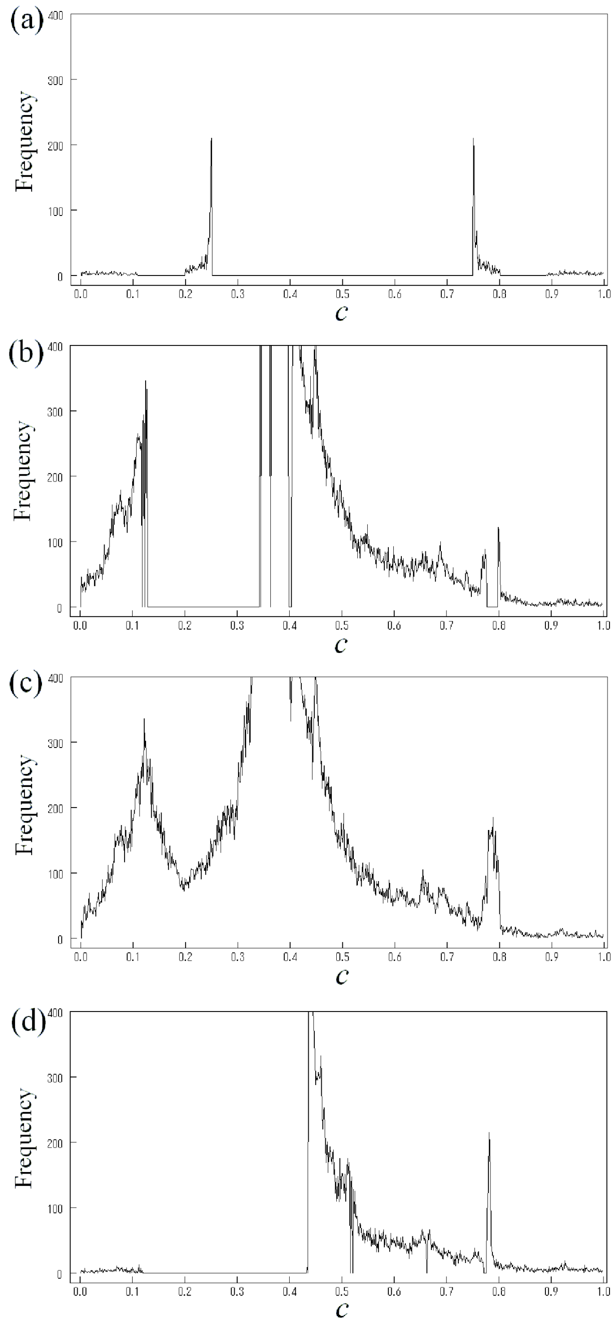


Figure 3.6: Frequency of alternation from burst phase to laminar phase in $10000 \leq t \leq 20000$. The parameter regions with on-off intermittency is clearly extended in ACS. The correspondence of (a)-(d) is the same as in the Figure 3.5.

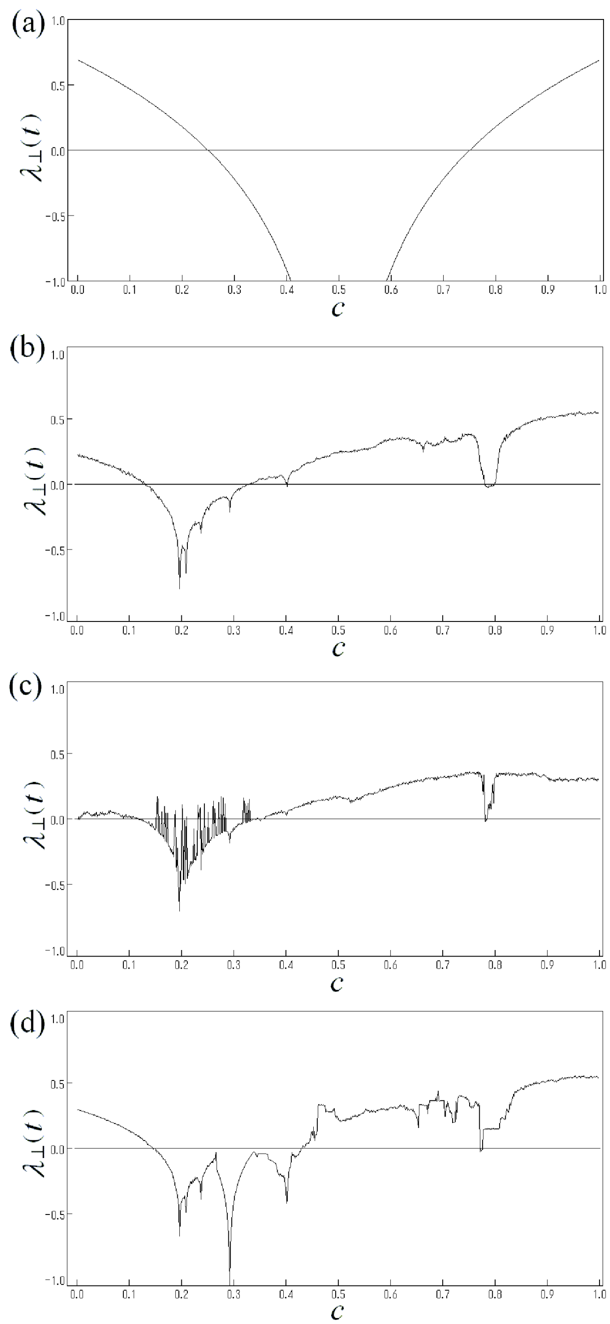


Figure 3.7: Transverse Lyapunov exponent in $t = 2000$. The correspondence of (a)-(d) is the same as in the Figure 3.5.

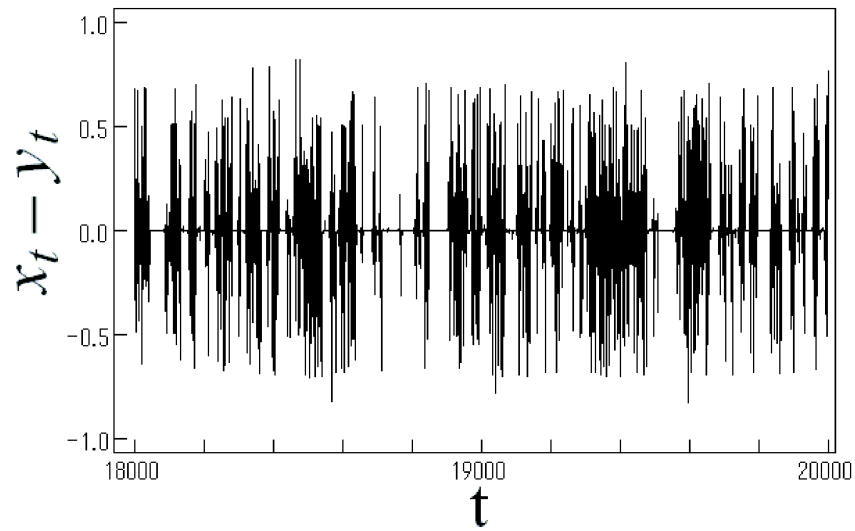


Figure 3.8: A typical on-off intermittency of time series of $x_t - y_t$ in $18000 \leq t \leq 20000$ in an Asymmetrical ACS (Eq.(9)) with $c = 0.10$ where $h(x) = cx + 0.1c\sin(10\pi x)$.

transverse Lyapunov exponent near zero value. We can find the correlation with their bifurcation diagrams.

Figure 3.8 is a typical on-off intermittency of time series of $x_t - y_t$ in $18000 \leq t \leq 20000$ in an Asymmetrical ACS (Eq.(3.10)) with $c = 0.10$ where $h(x) = cx + 0.1c\sin(10\pi x)$.

It is known that on-off intermittency satisfies (i) $\Pr(\tau) \propto \tau^{-\frac{3}{2}}$ and (ii) $\langle\tau\rangle \propto \epsilon^{-1}$ as its statistical properties[20] where τ is the continuous time of laminar phases, $\Pr(\tau)$ is its probability, ϵ is deviation from critical onset parameter and $\langle\tau\rangle$ is the mean of the laminar continuous time. In ACS, power laws similar to (i) are observed but they do not always satisfy $-\frac{3}{2}$ of the slope. For example, ACS with symmetrical $\tilde{f}^*$, $h(z) = cz + 0.1c\sin(10\pi z)$ and $c = 0.1$ shows the slope $-\frac{3}{2}$ on $1 \leq \tau \leq 30$ and -3 on $30 \leq \tau \leq 100$, and ACS with asymmetrical $\tilde{f}^*$, $h(z) = cz + 0.1c\sin(15\pi z)\cos(4\pi z)$ and $c = 0.15$ shows the slope -1.1 on $1 \leq \tau \leq 100$. Figure 3.9 show graphs of the power laws and the data of time series for 2×10^5 steps by 100 times were used to construct the distribution. And the law (ii) is not satisfied in ACS.

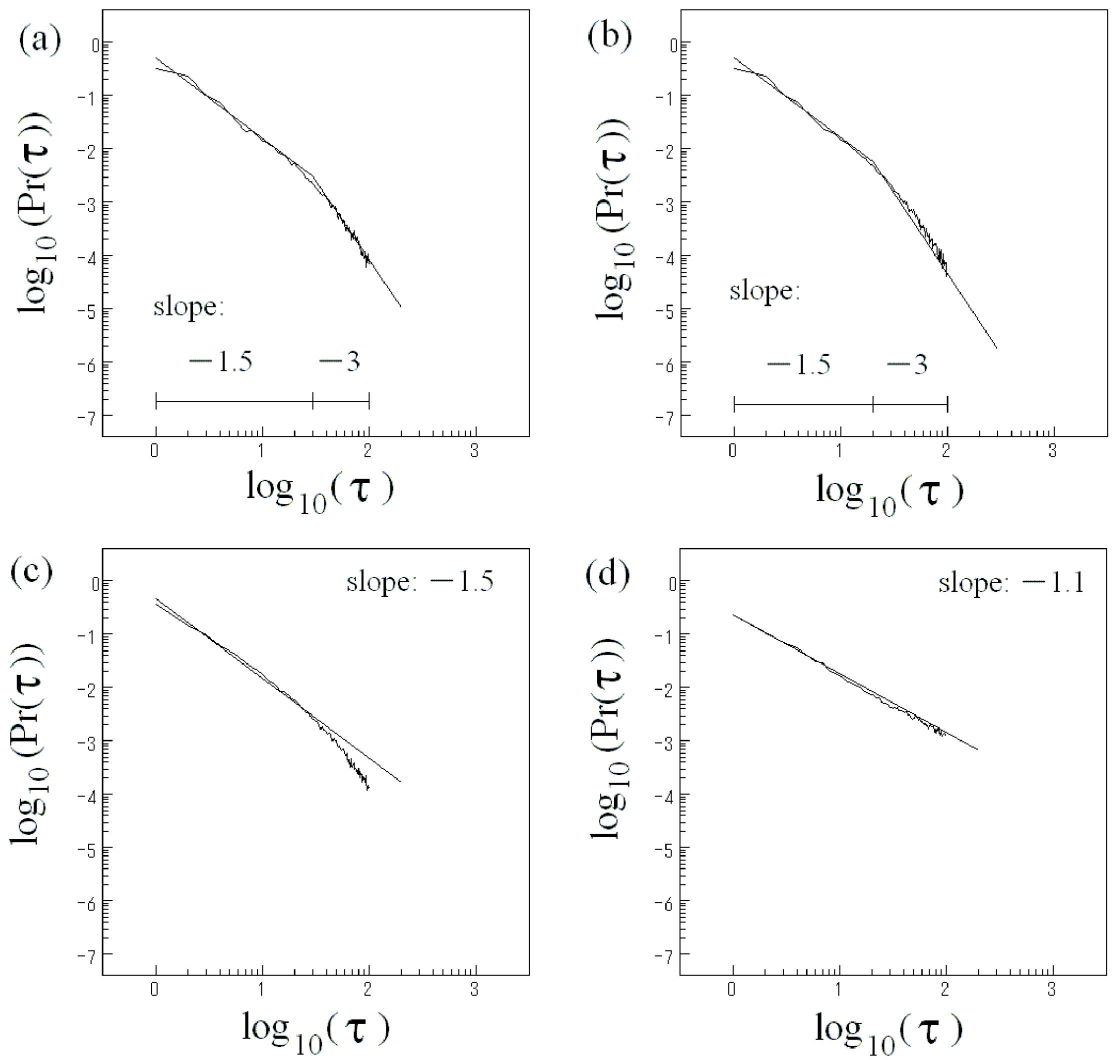


Figure 3.9: Power laws of $\Pr(\tau)$. The data of time series for 2×10^5 steps by 100 times were used to construct the distribution. **(a)** ACS with symmetrical $\tilde{f}^*$, $h(z) = cz + 0.1c\sin(10\pi z)$ and $c = 0.1$. The lines show the slope $-\frac{3}{2}$ on $1 \leq \tau \leq 30$ and -3 on $30 \leq \tau \leq 100$. **(b)** ACS with asymmetrical $\tilde{f}^*$, $h(z) = cz + 0.1c\sin(10\pi z)$ and $c = 0.1$. The lines show the slope $-\frac{3}{2}$ on $1 \leq \tau \leq 20$ and -3 on $20 \leq \tau \leq 100$. **(c)** ACS with symmetrical $\tilde{f}^*$, $h(z) = cz + 0.1c\sin(15\pi z)\cos(4\pi z)$ and $c = 0.1$. The line shows the slope $-\frac{3}{2}$. **(d)** ACS with asymmetrical $\tilde{f}^*$, $h(z) = cz + 0.1c\sin(15\pi z)\cos(4\pi z)$ and $c = 0.15$. The line shows the slope -1.1.

3.5 Conclusion

This world as a complex system is neither chaotic nor static absolutely but with mingled them. To express such an aspect, intermittency and the system bringing it may be useful. However, this phenomenon is fragile against the parameter change as the perturbation from the outside of the system since it is necessary to set the parameters of the system sensitively for its occurrence. The structural stability is known as a persistence against the perturbation and it is particular feature with each dynamical system. But, in science of complex systems, it is important to understand the coexistence of the flexible change of the system structures and the feature persistence of the system, and such an aspect is regarded as robustness. On the other hand, heterarchy concept is related to the robustness and the flexibility. In this chapter, heterarchy is extended to observational heterarchy that is a structure of the system modeling internal perspective. The mathematical formalization of this concept leads to the dynamical change of maps. Moreover, such maps with structural time evolution was applied to coupled map system. As a result, on-off intermittency occurs in wide parameter region of the system. It means the coexistence of the changeable map and the occurrence of intermittency against the parameter shift. The result shows ubiquitous intermittency featuring not stability but robustness.

An essence of observational heterarchy is coexistence of the layers that is different each other but reveal the identical system. Supposing an external perspective for the system enables their layers to be consistent, but it does not permit time evolution of the system structure. In our sense, an internal perspective is expressed by confusion of their different layers. If a static picture of the system is required, the expression is just a paradox or an inconsistency. However, the confusion induces evolvability of the system by modifying a part of the condition. Mathematically, it is expressed by the use of an non-commutative operator as a commutative operator in approximation. We think such a method is helpful in the study for complex systems that inherits structural evolvability.

Chapter 4

Generative Pointer

The concept of a *generative pointer* suggests an extended subobject classifier with *local identity* (i.e. an incomplete indication or identification), and its introduction is motivated by a generalization of the emergence process of rational numbers. Generative pointers induce heterarchical structure in dynamical systems. Generally, although intermittency observed in time series of dynamical systems is sensitive with respect to parameter shifts, ubiquitous on-off intermittency is observed for a wide range of parameter values when a generative pointer is applied to a Henon map, which implies robustness to parameter shifts. The concept of structural stability or trajectory stability of dynamical systems is based on the specific invariable map, but robustness, in contrast with stability, is based on the dynamic change of the map. Thus, in our view, the robustness does not conflict with the emergence, but coincides with it.

4.1 Introduction

4.1.1 Emergence, Autonomy and Robustness

General system theory reveals that not only physical substances but also structures of phenomena with common aspects can be the subject of research (von Bertalanffy 1968). This viewpoint is one of the origins of the concept of dynamical systems and complex systems, etc. System theory expresses the concept of emergence as hierarchy (Checkland 1981).

It is important to study the concepts of autonomy, evolution, emergence and robustness for complex systems such as biological systems or economic systems. Evolutionary models and/or emergence phenomena have been studied actively within the discipline of dynamical systems (ex. Nicolis and Prigogine 1989). In this regard, the mechanism of each dynamical system is characterized by the topology of the phase space (Rosen 1970). These models presuppose the invariance of the global structure (i.e. the vector field) describing the local motion (i.e. the 1-parameter transformation group).

In contrast, Conrad, Matsuno and Gunji et al. have emphasized the co-operative and interactive hierarchy between the local motion and the global structure, and have discussed concepts such as evolutionary computation (Conrad 1999) and internal measurement (Matsuno 1985; Gunji et al. 1997). These concepts make use of heterarchical structures (McCulloch 1945). Heterarchy is an extended hierarchy with dynamic interaction between the layers, and is described as the meta-structure carrying the robustness of systems (Jen 2003). It is important to formalize the system with a local perspective and the dynamic changes in system structure more explicitly. In this chapter, the framework of dynamical systems is extended, and the hierarchical structure of the system and its environment is introduced into the models.

In our view, emergence is not a property of particular systems but is intrinsic to the relation between the systems and their environment / observers. For example, assume a system that outputs a sequence of signals. As examples of signals, one can take human communicative utterances, electric bit patterns in computers, proteins in cells, etc. These signals are received and interpreted by the observers, such as human beings, computers or DNA, etc. If an observer of the system presumes that the system outputs the signals according to a particular rule, signals deviating from the presumed rule are regarded as mere errors (i.e. fluctuations in terms of physics). In contrast, if the observer presumes that the system is autonomous, the deviating signals are regarded not as errors but as outcomes of the autonomy. Such unexpected outputs are regarded not as errors but as emergence when they are perceived by the observers of the system. Thus, emergent phenomena are connected to autonomy through the observers.

The autonomy of the system is discovered by the observer who is monitoring the changes in the rules of the system. The emergence is the unexpected outcomes generated by changes in those rules. Therefore, the autonomy and the emergence are based on the same phenomenon, and correspond to the aspects of the system and the events occurring in the system, respectively. In this light, we can no longer divide the system from the viewpoint of the observer. When a mathematical model is constructed on the basis of such aspects, the acceptance of the unexpected outcomes can be formalized as indefinite domains and/or ranges of functions and as structural changes of the functions.

We have studied the general formalization of the dynamic complementary relation using the framework of information theory (Gunji et al. 2003; 2006) and/or dynamical system theory (Gunji and Kamiura 2004a,b; 2006; Kamiura and Gunji 2006a,b). For example, we studied a coupled map system based on a heterarchical structure (i.e. Active Coupling System; ACS) (Kamiura and Gunij 2006c). On-off intermittency was discovered by Fujisaka and Yamada (1985; 1986), has been named (Platt et al. 1993), and its statistical features have been explored (Heagy et al. 1994). Moreover, it has been applied to chaotic neural networks and been related to information processing (Inoue et al. 1991). Using the category theory (MacLane 1971=1997; Louie 1985; Awodey 2006), we define the concept of heterarchy as an extended isomorphism between two comma categories in order to discuss the issue of robustness mathematically. ACS has non-monotonic and time evolutional maps derived from the extended functors. Consequently, the values of the transverse Lyapunov exponent of ACS converge to zero despite the fluctuations in the coupling parameter. In other words, on-off intermittency is a critical feature of the specific coupling parameter in Fujisaka-Yamada systems, which occurs ubiquitously for all parameter values. We can evaluate this as the robustness of the phenomenon. Moreover, if this scheme is applied to 1-dimensional logistic map, the map also shows on-off intermittemcy (Nakajima and Shinkai submitted).

In this chapter, the origin of rational numbers is shown as a hypothetical process by using the equivalence between the category of pointed sets and that of partial maps. The forms of pointed sets and partial maps are connected to indicating states of a dynamical system in phase space. Indicating a singleton (or a state in phase space) is formalized by a *generative pointer*, which is an extension of the concept of subobject classifiers. Generative pointers induce structural changes in dynamical systems.

When generating positive rational numbers from natural numbers, we need to note the maps that become an identity only on a subset when they are composed. In the composition, the difference set of the subset with the identity must map to a base point. The outside region (i.e. the difference set) indicated by the base point is used for fractions.

The concept of subobject classifiers is a formalization of the process of specifying a point or a subset, and is extended to generative pointers. Indicating the outside region by a base point is connected to indicating a point by the generative pointer. This concept is applied to the phase space of the dynamical system, and, consequently, dynamic change of the parameter of the system is induced. In this chapter, the Henon map is extended by virtue of generative pointers. We call the system the *extended Henon map based on Generative Pointers* (HGP) system.

In the sense of conventional dynamical systems, the equation system gives the vector field (i.e. the global structure of the phase space). When a state is given in phase space, the state has time evolution that is consistent with the global structure. Since the global vector field determines the local time evolution, the one-directional dependence relation from global structure to local motion is hierarchical. In contrast, in HGP, the global structure is virtual and temporal since the equation is inevitably constructed locally. Dynamic changes in the global structure are induced by indicating the local state, and one step of the time evolution is defined in the global phase space. We can find a heterarchical structural dependency between the global structure and the local motion. As a result, HGP shows intermittency for a wide range of parameter values, as well as ACS.

Generally, intermittency is observed at a critical point between a fixed point (or periodic solution) and a chaotic solution, and hence it is sensitive with respect to parameter shifts. On the contrary, in HGP, the on-off intermittency occurs in for a wide range of parameter regions, which determines the coexistence of the changeable map and the occurrence of intermittency against parameter shifts. The result shows ubiquitous intermittency featuring robustness instead of stability.

4.1.2 From Natural Numbers to Rational Numbers

One of the most fundamental examples of emergence can be found in the relation between the division of natural numbers $\mathbb{N}$ and positive rational numbers $\mathbb{Q}^+$. The operation, such that a natural number is multiplied by 3, is expressed by the map $f_3 : \mathbb{N} \to \mathbb{N} : n \mapsto 3n$. There is no problem if division, as the inverse of multiplication, is defined by $g_3 : \mathrm{Im}(f_3) \to \mathbb{N} : 3n \mapsto n$. However, a new sign expressing "a fraction" is required when the domain of the map g_3 is defined not by $\mathrm{Im}(f_3)$ but by $\mathbb{N}$. For example, when g_3 is applied to $2 \in \mathbb{N} \backslash \mathrm{Im}(f_3)$, a "new number", 2/3, is required since $g_3(2) \notin \mathbb{N}$, where $A \backslash B = \{x \in A | x \notin B\}$ is the difference set of B in A. Moreover, if we regard this concept as something which did not exist a priori but has been generated historically, we can see the emergence of positive rational numbers $\mathbb{Q}^+$ when arbitrary natural numbers are substituted for the top and bottom blanks ($\square$) of the division sign "$\frac{\square}{\square}$".

Needless to say, rational numbers are constructed as an equivalence class of ordered pairs of integers (a, b) with $b \neq 0$, and are induced by the equivalence relation $(a, b) \sim (c, d) \iff ad = bc$. We need to consider the relationship between the emergence on the division sign and such a formal construction of the rational numbers.

To discuss this issue, we used an equivalence of categories, $\mathbf{Par} \cong \mathbf{Sets}_*$, based on functors, $F : \mathbf{Par} \to \mathbf{Sets}_*$ and $G : \mathbf{Sets}_* \to \mathbf{Par}$, where $\mathbf{Par}$ is a category of partial maps and $\mathbf{Sets}_*$ is a category of pointed sets (cf. Appendix A). Assume sets A, B, its partial set $U \subset A$, a map $f : A \to B$ and its partial map $|f| : U \to B$. $\mathbf{Par}$ has the sets as objects and the partial maps as arrows. In the partial map $|f|$, a difference set $A \backslash U$ (i.e. the outside of the partial set U) is hidden or is painted out by black. By the functor F, such "being painted out" is connected to the map $f_* = F(|f|)$, which is an arrow in $\mathbf{Sets}_*$ and $f_*(A \backslash U) = \{*\}$.

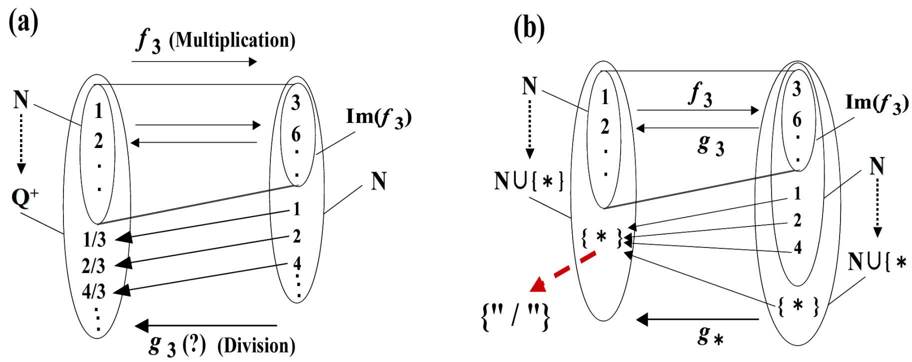


Figure 4.1: (a) The operation such that a natural number is multiplied by 3 is expressed by the map $f_3 : \mathbb{N} \to \mathbb{N} : n \mapsto 3n$. There is no problem if division, as the inverse of multiplication, is defined by $g_3 : \mathrm{Im}(f_3) \to \mathbb{N} : 3n \mapsto n$. However, a new sign expressing a "fraction" is required when the domain of the map g_3 is defined not by $\mathrm{Im}(f_3)$ but by $\mathbb{N}$. (b) The replacement of the point $*$ with the fraction sign "$\frac{\square}{\square}$" reveals the replacement of the operand $(*)$ with the operator $(\frac{\square}{\square})$. The point $*$ is an element on the pointed set $\mathbb{N}_*$ but "$\frac{\square}{\square}$" is a map in $\mathrm{Hom}(\mathbb{N} \times \mathbb{N}, \mathbb{Q}^+)$.

Applying $\mathrm{Im}(f_3)$, $\mathbb{N}$ and $\mathbb{Q}^+$ to the above U, A and B, we can regard the map g_3 as the arrow $g_{3*} : \mathbb{N}_* \to \mathbb{N}_*$ in **Sets**$_*$:

$$g_{3*}(x) = \begin{cases} g_3(x) & (\text{if } x \in \mathrm{Im}(f_3) = \{3n | n \in \mathbb{N}\} \subset \mathbb{N}) \\ * & (\text{if } x \in \mathbb{N}_* \backslash \mathrm{Im}(f_3) = \{3n+1 | n \in \mathbb{N}\} \cup \{3n+2 | n \in \mathbb{N}\}) \end{cases}$$

In this sense, by hiding or painting out the fractions by the point $*$, we can deal with only $\mathrm{Im}(f_3) = \{3n | n \in \mathbb{N}\}$ as the domain of g_3 (Figure 4.1).

Now, we use the expressions such as "hide" or "paint out" since we know the concepts of division and rational numbers. However, if we focus on the emergence of fractions, we can suppose that the fractions are not hidden by the point $*$ but the concept of the region of the numbers is extended from natural numbers to rational numbers by replacing the sign "$*$" with the sign "$\frac{\square}{\square}$" (Figure 4.1 (b)). In other words, the replacement of the point $*$ with the fraction sign "$\frac{\square}{\square}$" reveals the replacement of the operand $(*)$ with the operator $(\frac{\square}{\square})$. The point $*$ is an element on the pointed set $\mathbb{N}_*$ but "$\frac{\square}{\square}$" is a map in $\mathrm{Hom}(\mathbb{N} \times \mathbb{N}, \mathbb{Q}^+)$.

Although we do not insist that the process of the replacement of the signs is a historical fact, we consider that it is effective to model the emergent process through such mixture of operands and operators.

At the end of this section, we introduce a term, *Local Identity*, so as to use it in the following section.

Definition 1 (*Local Identity*) For topological spaces A and B such that the measure $M(A \cap B) \neq 0$ (i.e. $A \cap B$ is neither empty nor singleton), and for the subspace U such that $U \subset A \cap B$ and the measure $M(U) \neq 0$, if a map $Lid_U : A \to B$ is defined by

$$Lid_U(x) = \begin{cases} x & (\text{if } x \in U) \\ f(x) & (\text{if } x \in A \backslash U), \end{cases}$$

then we call the map Lid_U *local identity* on U, where $f : A\backslash U \to B$ is an arbitrary map.

Again, assume the multiplication map $f_3 : \mathbb{N} \to \mathbb{N}$. It naturally induces the map $f_{3*} : \mathbb{N}_* \to \mathbb{N}_*$ which is defined by

$$f_{3*}(x) = \begin{cases} f_3(x) & (\text{if } x \in \mathbb{N}) \\ * & (\text{if } x \in \{*\}). \end{cases}$$

We obtain the following compositions of f_{3*} and g_{3*}:

$$g_{3*} \circ f_{3*}(x) = \begin{cases} x & (\text{if } x \in \mathbb{N}) \\ * & (\text{if } x \in \{*\}) \end{cases}$$

and

$$f_{3*} \circ g_{3*}(x) = \begin{cases} x & (\text{if } x \in \text{Im}(f_3)) \\ * & (\text{if } x \in \mathbb{N}_* \backslash \text{Im}(f_3)). \end{cases}$$

The maps $g_{3*} \circ f_{3*}$ and $f_{3*} \circ g_{3*}$ are local identities.

4.2 Generative Pointers and their Application in Dynamical Systems

In order to strengthen the above viewpoint of emergence, we introduce the concept of a *generative pointer*. In connection with dynamical systems, this concept describes structural changes of the time evolution operator of the system.

4.2.1 Parameter with Time Evolution

Suppose a dynamical system $(D, \varphi_{\Delta t})$ on discrete time where D is the phase space of the system and $\{\varphi_{\Delta t}\}_{\Delta t \in \mathbb{Z}}$ is the 1-parameter transformation group.

Conventionally, the time evolution of the system is expressed by $x_{t+\Delta t} = \varphi_{\Delta t}(x_t)$ or $\varphi_{\Delta t} : D \to D : x_t \mapsto x_{t+\Delta t}$, i.e. the time evolution operator $\varphi_{\Delta t}$ is independent of the time t. In addition, dynamical systems can have some parameters other than the time t, such as angular frequency in the case of harmonic oscillators. Furthermore, if the dynamical system has a parameter β, the time evolution is expressed by:

$$x_{t+\Delta t} = \varphi_{\Delta t}(\beta)(x_t) \qquad \text{or} \qquad \varphi_{\Delta t}(\beta) : D \to D : x_t \mapsto x_{t+\Delta t}.$$

On the other hand, we assume a time evolution operator dependent on the time t, i.e. it is expressed by $\varphi^{(t)}_{\Delta t} : D_t \to D_{t+\Delta t} : x_t \mapsto x_{t+\Delta t}$ and it accounts for the structural changes occurring in the system with the passing of time. If the time interval Δt is defined exactly, the operator can be expressed by $\varphi^{(t)}$ without the need for Δt. In particular, if the structural changes in the system are constructed by changes in the parameter β_t, the time evolution is expressed by:

$$x_{t+\Delta t} = \varphi^{(t)}(x_t) = \varphi(\beta_t)(x_t) \qquad \text{or} \qquad \varphi^{(t)} = \varphi(\beta_t) : D_t \to D_{t+\Delta t} : x_t \mapsto x_{t+\Delta t}.$$

Using the above notations, we construct a concrete structural change of the dynamical system.

4.2.2 Characteristic Functions and Subobject Classifiers

In this section, we consider the form of indication of a point (i.e. an element in a set). In set theory, separating a subset $B \subset A$ from the difference set $A \backslash B$ on a set A is formalized by a characteristic function. The characteristic function of a subset B is the two-valued function $m_B : A \to \{0, 1\}$ on A with the following values for any $x \in A$

$$m_B(x) = \begin{cases} 0 & (\text{if } x \in B) \\ 1 & (\text{otherwise}). \end{cases}$$

$\{0\} \subset \{0, 1\}$ represents the simplest non-trivial subset. An arbitrary subset $B \subset A$ can be mapped onto this simplest subset by m_B. This map produces a pullback square:

$$\begin{CD} B @>>> \{0\} \\ @V{p}VV @VV{s}V \\ A @>{m_B}>> \{0,1\}. \end{CD} \tag{4.1}$$

One says that the monomorphism (the typical subset) s is a subobject classifier for the category of sets, **Sets**. If the subset A is a one-point set $\{a\}$, the subobject classifier corresponds to indicating the specific state a.

In general, a subobject classifier in a category is defined in the following way.

Definition 2 (Subobject Classifier) Given a category **C** with a terminal object 1, a subobject classifier $s : 1 \to \Omega$ is a monomorphism with the following property: for each monomorphism $p : B \to A$, there is an unique morphism $m_B : A \to \Omega$ such that the following diagram commutes:

$$\begin{array}{ccc} B & \xrightarrow{\iota} & 1 \\ {\scriptstyle p}\downarrow & & \downarrow{\scriptstyle s} \\ A & \overset{m_B}{\dashrightarrow} & \Omega \end{array}$$

where $\iota : B \to 1$ is uniquely defined.

In **Sets**, Ω is $\{0, 1\}$ since m_B must be uniquely defined.

4.2.3 Pointer

In section 4.1.2, we considered the replacement of the operand with the operator by virtue of an example from the origin of fractions (i.e. dynamic change of the domains and ranges). In this section, we modify the characteristic function to connect the indication of a point to the dynamic change of the domains and ranges.

Again, assume a conventional dynamical system $(D, \varphi(\beta))$. Given the state $x_t \in D$ at the time t, the system and the state induce the following partial map:

$$|\varphi(\beta)| : \{x_t\} \to \{x_{t+1}\} : x_t \mapsto x_{t+1} \tag{4.2}$$

where the singleton $\{x_t\} = X_t$ $(t \in \mathbb{N})$ is a subspace of D.

On the other hand, $\{x_t\}$ induces the natural monomorphism $p_t : \{x_t\} \to D : x_t \mapsto x_t$. Now, given a map $r : \{*\} \to I : * \mapsto r_0$, there is an isomorphism $m_t : D \to I$ such that the following diagram is a pullback:

$$\begin{array}{ccc} X_t & \xrightarrow{\iota} & \{*\} \\ {\scriptstyle p_t}\downarrow & & \downarrow{\scriptstyle r} \\ D & \xrightarrow{m_t} & I \end{array} \qquad \begin{array}{ccc} x_t & \longmapsto & * \\ \downarrow & & \downarrow \\ x_t & \longmapsto & r_0. \end{array} \tag{4.3}$$

where I is an open topological space $(-1, 1)$ and r_0 is such that $-1 < r_0 < 1$. Note that the map m_t is not unique in the above case. In the diagram of the subobject classifier, I is defined in such a way as to uniquely determine m_t, however, in the above diagram (4.3), I is defined by $(-1, 1)$ first, and then m_t is chosen such that the diagram commutes.

Next, we restrict the kind of the maps m_t to get back the uniqueness, i.e. we define the map by:

$$m_t(a_t)(x_t) = \frac{2}{\pi} \arctan a_t x_t \tag{4.4}$$

where a_t is a parameter and it is defined in such a way that the diagram (4.3) commutes, i.e. the uniqueness of the map m_t is replaced by that of the parameter a_t. Note that the above $m_t(a_t) : D \to I$ is a naturally isomorphic, continuous, smooth and odd function (i.e. preserving the positive and negative signs) if D is an open topological space $(-\infty, \infty)$.

Since the $m_t(a_t)$ or m_t is defined as the isomorphism, there is an inverse map m_t^{-1} and the following diagrams commute:

$$\begin{array}{ccc} X_t & \xrightarrow{\iota} & \{*\} \\ {\scriptstyle p_t}\downarrow & & \downarrow{\scriptstyle r} \\ D & \xleftarrow{m_t^{-1}} & I \end{array} \qquad \begin{array}{ccc} x_t & \longmapsto & * \\ \downarrow & & \downarrow \\ x_t & \longleftarrow\!\shortmid & r_0 \end{array} \tag{4.5}$$

Consequently, the inclusion p_t satisfies the composition $m_t^{-1} \circ r \circ \iota = p_t$. We call the map p_t with the above commutative diagram a *pointer*.

Returning to the diagram of the subobject classifier (4.1) on **Sets**, let us consider the significance of the pointer p_t.

In diagram (4.1), the map s is given by:

$$s : \{*\} \to \{0, 1\} : * \mapsto 0$$

where the singleton $\{0\}$ is replaced by $\{*\}$, although this is not essential. Moreover, in (4.1), the characteristic function $m_B : A \to \{0, 1\}$ is determined via s and $p : B \to A$.

Now, suppose we replace the map s by

$$s' : \{*\} \to \{0, 1\} : * \mapsto 1$$

after the above determination of m_B. The pullback for the map s' and m_B is the difference set $A \backslash B$ with the inclusion $p' : A \backslash B \to A$:

$$\begin{array}{ccc} A\backslash B & \longrightarrow & \{*\} \\ {\scriptstyle p'}\downarrow & & \downarrow{\scriptstyle s'} \\ A & \xrightarrow{m_B} & \{0, 1\}, \end{array} \tag{4.6}$$

i.e. the replacement of s by s' induces the indication of $A \backslash B$, which means that the outside of the subset B is chosen.

In addition, in the pointer diagram defined above, if the map

$$r : \{*\} \to I : * \mapsto r_0$$

is replaced by

$$r' : \{*\} \to I : * \mapsto r_1,$$

then a new pullback X'_t for r' and m_t is induced as follows:

$$\begin{array}{ccc} X'_t & \longrightarrow & \{*\} \\ {\scriptstyle p'_t}\downarrow & & \downarrow{\scriptstyle r'} \\ D & \xrightarrow{m_t} & I \end{array} \qquad\qquad \begin{array}{ccc} x'_t & \longmapsto & * \\ \downarrow & & \downarrow \\ x'_t & \longmapsto & r_1. \end{array} \tag{4.7}$$

This means that a new point $X'_t = \{x'_t\}$ on D is chosen by picking up the value r_1 on I. In our construction, m_t is given by (4.4), and since it is an isomorphism, we obtain the value of $\{x'_t\}$:

$$x'_t = m_t^{-1}(r_1) = \frac{1}{a_t} \tan \frac{\pi}{2} r_1. \tag{4.8}$$

By replacing r with r', the pointer p_t is replaced with p'_t, and, furthermore, x'_t (i.e. the outside of $\{x_t\}$) is chosen. The m_t governs the indication of the point on D.

4.2.4 Generative Pointer

Next, we introduce the following map $m_t^\sharp : I \to D$:

$$m_t^\sharp(r) := Dec[m_t^{-1}(r)] \tag{4.9}$$

where Dec is an operation such that $Dec[a]$ is the number of the decimal part of a real number a. In this definition, the map $m_t^\sharp$ has the image $\mathrm{Im}(m_t^\sharp) = I \subset D$.

The maps m_t and $m_t^\sharp$ induce m_{t*} and $m_{t*}^\sharp$, which are morphisms on pointed sets. If $m_{t*}(r) : D_* \to I_*$ and $m_{t*}^\sharp(r) : I_* \to D_*$ are defined by

$$m_{t*}(x) = \begin{cases} m_t(x) & (\text{if } x \in I) \\ * & (\text{if } x \in D_* \backslash I) \end{cases}$$

and

$$m_{t*}^\sharp(r) = \begin{cases} m_t^\sharp(r) & (\text{if } r \in U_{m_t}) \\ * & (\text{if } r \in I_* \backslash U_{m_t}) \end{cases}$$

where $U_{m_t} := m_t(I)$, then we obtain

$$m_{t*}^\sharp \circ m_{t*}(x) = \begin{cases} x & (\text{if } x \in I) \\ * & (\text{if } x \in D_* \backslash I) \end{cases}$$

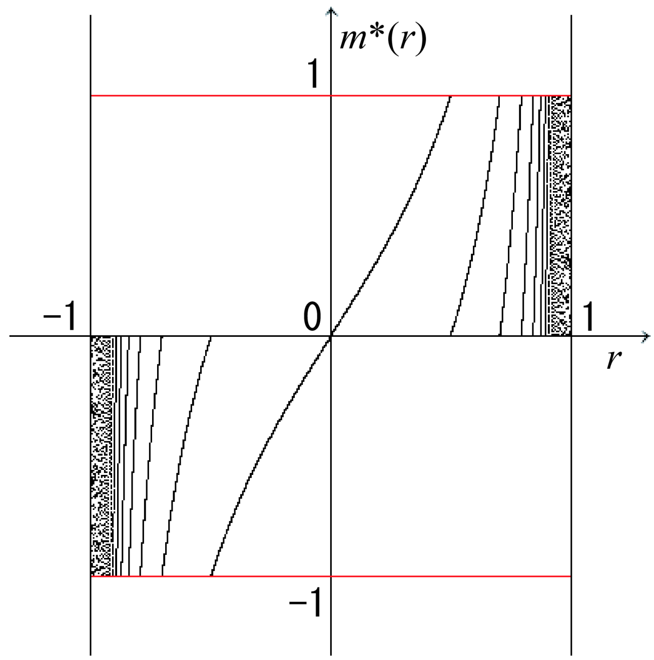


Figure 4.2: The graph of the function $m_t^\sharp$.

and

$$m_{t*} \circ m_{t*}^{\sharp}(r) = \begin{cases} r & (\text{if } r \in U_{m_t}) \\ * & (\text{if } r \in I_* \backslash U_{m_t}). \end{cases}$$

Thus, $m_{t*}^{\sharp} \circ m_{t*}$ and $m_{t*} \circ m_{t*}^{\sharp}$ are local identities as well as the compositions of the multiplication and the division, $g_{3*} \circ f_{3*}$ and $f_{3*} \circ g_{3*}$, in the previous section.

The map $m_t^{\sharp}$ induces the following diagram:

$$\begin{array}{ccc} \widetilde{X}_t & \longrightarrow & \{*\} \\ {\scriptstyle \widetilde{p}_t}\downarrow & & \downarrow{\scriptstyle r'} \\ D & \underset{m_t^{\sharp}}{\longleftarrow} & I \end{array} \qquad\qquad \begin{array}{ccc} \widetilde{x}_t & \longmapsto & * \\ \downarrow & & \downarrow \\ \widetilde{x}_t & \longleftarrow\!\shortmid & r_1 \end{array} \tag{4.10}$$

where $\widetilde{X}_t = \{\widetilde{x}_t\}$ is determined by

$$\widetilde{x}_t = m_t^{\sharp}(r_1),$$

and the map $\widetilde{p}_t : \{\widetilde{x}_t\} \to D : \widetilde{x}_t \mapsto \widetilde{x}_t$ is a natural monomorphism. We call the map $\widetilde{p}_t$ with the above diagram a *generative pointer.*

4.2.5 Dynamic Change of a Parameter

Again, consider the extension of natural numbers to rational numbers. $\mathbb{N}$ is expanded to the Cartesian product $\mathbb{N} \times \mathbb{N}$ and then $\mathbb{N} \times \mathbb{N}$ is mapped onto the equivalence class $\mathbb{N} \times \mathbb{N}/\sim = \mathbb{Q}^+$, where $\sim$ indicates the equivalence relation $(a, b) \sim (c, d) \iff ad = bc$. Emulating this procedure, we construct the time evolution of the parameter β in dynamical systems.

In section 4.2.2, we obtained the partial map on the dynamical system, (4.2):

$$|\varphi(\beta)| : \{x_t\} \to \{x_{t+1}\} : x_t \mapsto x_{t+1}.$$

Given these data and $r_0 \in I$, from definition (4.4) we obtain m_t and m_{t+1} such that the following diagram is commutative:

$$\begin{array}{ccccc} D & & \xrightarrow{\varphi(\beta)} & & D \\ & & \Downarrow{\scriptstyle \text{Restrict}} & & \\ \uparrow & & & & \uparrow \\ X_t & & \xrightarrow{|\varphi(\beta)|} & & X_{t+1} \\ & {\scriptstyle p_t \circ m_t}\searrow & & \swarrow{\scriptstyle p_{t+1} \circ m_{t+1}} & \\ & & I & & \end{array} \qquad \begin{array}{ccccc} x_t & & \xmapsto{|\varphi(\beta)|} & & x_{t+1} \\ & {\scriptstyle p_t \circ m_t}\searrow & & \swarrow{\scriptstyle p_{t+1} \circ m_{t+1}} & \\ & & r_0 & & \end{array}, \tag{4.11}$$

i.e. $r_0 = m_t(x_t)$ and $r_0 = m_{t+1}(x_{t+1})$.

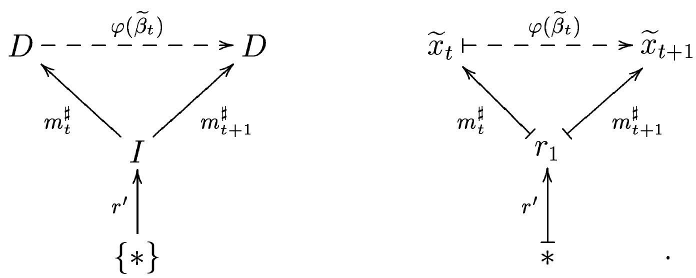
Moreover, given the map $r' : \{*\} \to I : * \mapsto r_1$, "the tentative states" $\widetilde{x}_t$ and $\widetilde{x}_{t+1}$ are defined by the generative pointers:

$$\widetilde{x}_t = m_t^\sharp(r_1) = m_t^\sharp \circ r'(*), \quad \widetilde{x}_{t+1} = m_{t+1}^\sharp(r_1) = m_{t+1}^\sharp \circ r'(*)$$

where $r_1 \in I \backslash \{r_0\}$ is chosen randomly. The parameter β is changed to $\widetilde{\beta}_t$ so as to satisfy the following condition:

$$\widetilde{x}_{t+1} = \varphi(\widetilde{\beta}_t)(\widetilde{x}_t), \tag{4.12}$$

i.e. the new value $\widetilde{\beta}_t$ of the parameter is determined in such a way that the following diagrams commute:



(4.13)

Note that the top triangle diagrams must commute only on $r_1 \in I$, while $m_{t+1}^\sharp(r) \neq \varphi(\widetilde{\beta}_t) \circ m_t^\sharp(r)$ is accepted on $r \in I \backslash \{r_1\}$.

Using $\widetilde{\beta}_t$, the time evolution of the parameter is defined by

$$\beta_{t+1} = \beta_0 + \delta \widetilde{\beta}_t$$

where β_t is the value of the parameter at time t, and $\delta \ll \beta_0$ is a constant number. Consequently, we obtain the dynamic change of the parameter through this procedure.

4.2.6 Application to Henon Map

Let us experiment with the generative pointer on a concrete dynamical system. Suppose a 2-dimensional dynamical system, $\Psi : D \times D \to D \times D : \langle x_t, y_t \rangle \mapsto \langle x_{t+1}, y_{t+1} \rangle$ where $D = (-\infty, \infty)$. Now we choose a Henon map for this system:

$$\begin{pmatrix} x_{t+1} \\ y_{t+1} \end{pmatrix} = \Psi \begin{pmatrix} x_t \\ y_t \end{pmatrix} = \begin{pmatrix} 1 - \alpha x_t^2 + \beta y_t \\ x_t \end{pmatrix}. \tag{4.14}$$

Specifically, in a system with time evolution of the parameter β, it is expressed by

$$\begin{pmatrix} x_{t+1} \\ y_{t+1} \end{pmatrix} = \Psi \begin{pmatrix} x_t \\ y_t \end{pmatrix} = \begin{pmatrix} 1 - \alpha x_t^2 + \beta_t y_t \\ x_t \end{pmatrix}. \tag{4.15}$$

The Henon map induces a binary operation $\psi(\beta_t) : D \times D \to D : \langle x_t, y_t \rangle \mapsto x_{t+1}$, that is,

$$x_{t+1} = \psi(\beta_t)(x_t, y_t) = 1 - \alpha x_t^2 + \beta_t y_t \tag{4.16}$$

where β_t in $\psi(\beta_t)$ is explicitly expressed in a way that allows it to be emphasized as the parameter. Moreover, if the value of $y_t = x_{t-1}$ is fixed, an unary operation $\psi_{\beta_t, y_t} : D \to D : x_t \mapsto x_{t+1}$ is induced by the binary operation ψ_{β_t}:

$$x_{t+1} = \psi_{\beta_t, y_t}(x_t). \tag{4.17}$$

Following the generative pointer procedure, let us construct β_t.

[Step 1] Given a random number $r_t \in I = (-1, 1)$, the parameters a_t and a_{t+1} of m_t and m_{t+1} are set so as to satisfy the following equations:

$$m_t(x_t) = \frac{2}{\pi} \arctan a_t x_t = r_t \quad \text{and} \quad m_{t+1}(x_{t+1}) = \frac{2}{\pi} \arctan a_{t+1} x_{t+1} = r_t, \tag{4.18}$$

i.e. a_t and a_{t+1} are given by

$$a_t = \frac{1}{x_t} \tan \frac{\pi}{2} r_t \quad \text{and} \quad a_{t+1} = \frac{1}{x_{t+1}} \tan \frac{\pi}{2} r_t. \tag{4.19}$$

[Step 2] For the above a_t, a_{t+1} and the random number $r'_t \in I \backslash \{r_t\}$, $\widetilde{x}_t$ and $\widetilde{x}_{t+1}$ are defined as

$$\widetilde{x}_t = m_t^\sharp(r'_t) = \frac{1}{a_t} \tan \frac{\pi}{2} r'_t \quad \text{and} \quad \widetilde{x}_{t+1} = m_{t+1}^\sharp(r'_t) = \frac{1}{a_{t+1}} \tan \frac{\pi}{2} r'_t. \tag{4.20}$$

For these two values, $\widetilde{\beta}_t$ is chosen such that the following condition is satisfied:

$$\widetilde{x}_{t+1} = \psi_{\widetilde{\beta}, \widetilde{y}_t}(\widetilde{x}_t) \iff \widetilde{x}_{t+1} = 1 - \alpha \widetilde{x}_t^2 + \widetilde{\beta}_t \widetilde{y}_t \tag{4.21}$$

where $\widetilde{y}_t = \widetilde{x}_{t-1} = m_{t-1}^\sharp(r'_{t-1}) = \frac{1}{a_{t-1}} \tan \frac{\pi}{2} r'_{t-1}$ is the value given in the previous step in this calculation process. Consiquently, under the condition that $\widetilde{y}_t \neq 0$, $\widetilde{\beta}_t$ is given by

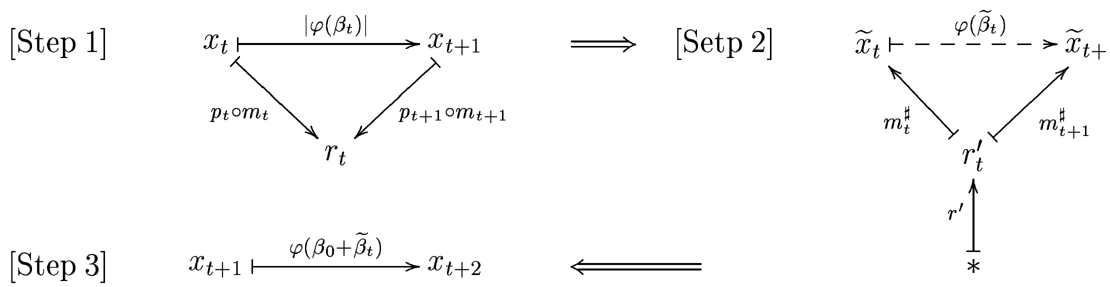
$$\widetilde{\beta}_t := \frac{\widetilde{x}_{t+1} + \alpha \widetilde{x}_t^2 - 1}{\widetilde{y}_t}. \tag{4.22}$$

[Step 3] Using the above value of $\widetilde{\beta}_t$, β_{t+1} is defined by

$$\beta_{t+1} := \beta_0 + \delta \widetilde{\beta}_t \tag{4.23}$$

where $\delta \ll \beta_0$ is a small constant. If $\{\widetilde{x}_{t+i}\}_{i \in \{-1,0,1\}} = \{x_{t+i}\}_{i \in \{-1,0,1\}}$, then $\widetilde{\beta}_t = \beta_0$, and hence $\beta_{t+1} = (1 + \delta)\beta_0 \approx \beta_0$.

The above procedure is illustrated by the following diagrams:



(4.24)

4.3 Results

In this section, the numerical results are shown for the extended Henon map system defined in the previous section, which we call the HGP system.

As control experiments, we use a conventional Henon map and a Henon map with the parameter β with added Gaussian noises:

$$\begin{pmatrix} x_{t+1} \\ y_{t+1} \end{pmatrix} = \begin{pmatrix} 1 - \alpha x_t^2 + (\beta_0 + R_G) y_t \\ x_t \end{pmatrix} \tag{4.25}$$

where R_G is a Gaussian noise with a mean $\langle R_G \rangle = 0$ and a standard deviation $\sigma_{R_G} = 10^{-2}$.

Figure 4.3 shows the typical time series of x_t of the HGP. The parameters are set in a conventional Henon map, a fixed point is observed for $\alpha = 0.2$ and $\beta_0 = 0.3$. On the other hand, in HGP, on-off intermittency is observed instead for the same parameter values (Figure 4.3 (a1)-(a2)). Since intermittent motion is not clearly visible on x_t (Figure 4.3 (b1)-(b2)) for other values of α, we observe values of q_t that are defined by

$$q_t = |\psi(\beta_t)(x_t, y_t) - \psi(\beta_0)(x_t, y_t)| = |(\beta_t - \beta_0) y_t| = |\delta \widetilde{\beta}_t y_t|. \tag{4.26}$$

The values of q_t show on-off intermittency (Figure 4.3 (c1)-(c2)).

Figure 4.4 shows the trajectory of the HGP in phase space for $0 \leq t \leq 10^4$. The parameters are set to $\alpha = 0.2$ and $\beta_0 = 0.3$, as in Figure 4.4 (a1)-(a2).

Figure 4.5 gives the bifurcation diagrams on (a) conventional Henon maps, (b) HGP and (c) Henon maps with Gaussian noises. Each diagram shows the values of x_t for $1000 \leq t \leq 4000$ along with the parameter $0.0 \leq \alpha \leq 1.5$. For the left and right figures, the initial parameter β_0 is $\beta_0 = 0.0$ and $\beta_0 = 0.3$, respectively. Generally, intermittency is observed at a critical point between a fixed point (or periodic solution) and a chaotic solution, and thus it is sensitive towards parameter shifts. Contrastingly, in the HGP, the on-off intermittency

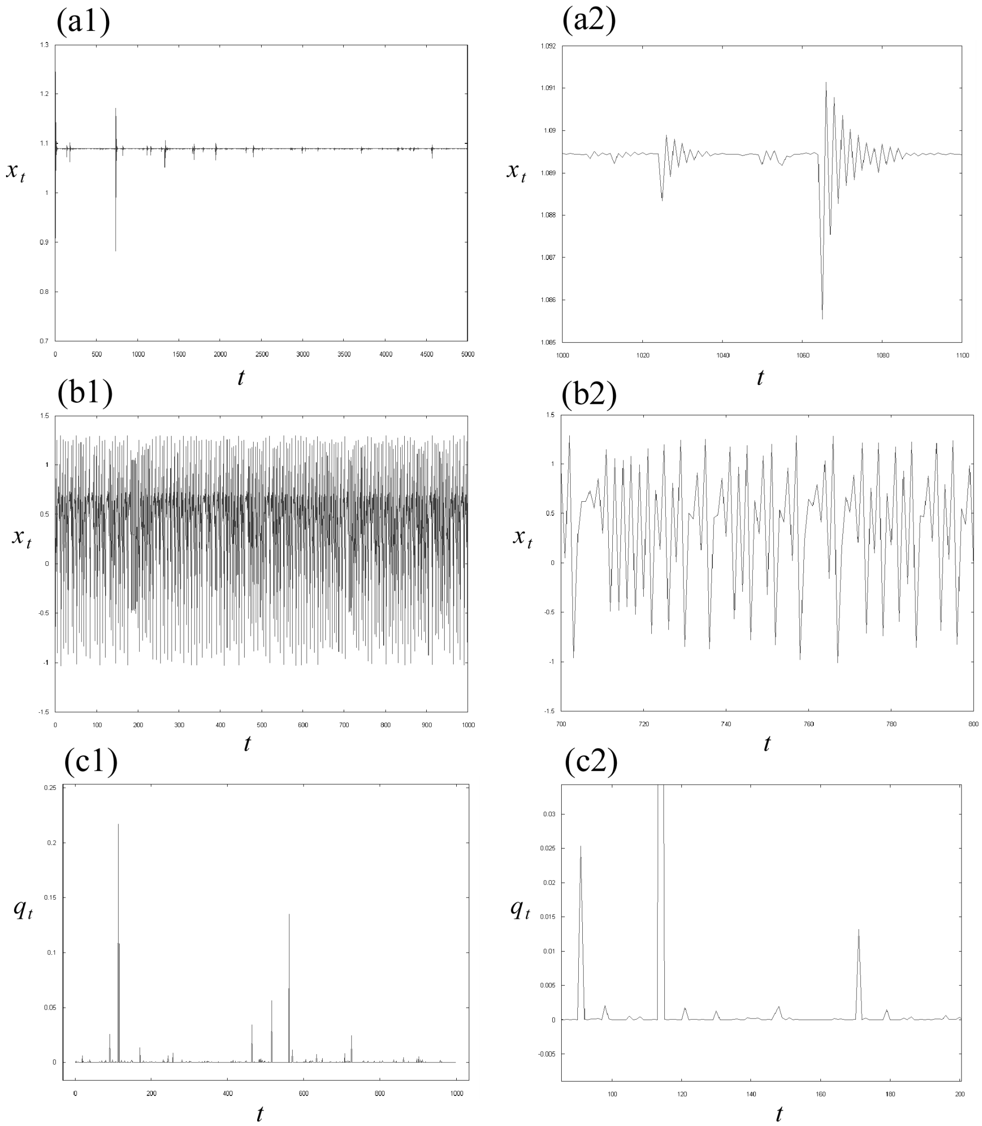


Figure 4.3: (a1)-(b2) are the typical time series of x_t of the HGP: (a1) $\alpha = 0.2$, $\beta_0 = 0.3$ and $0 \leq t \leq 5000$. (a2) $\alpha = 0.2$, $\beta_0 = 0.3$ and $1000 \leq t \leq 1100$. (b1) $\alpha = 1.2$, $\beta_0 = 0.3$ and $0 \leq t \leq 1000$. (b2) $\alpha = 1.2$, $\beta_0 = 0.3$ and $700 \leq t \leq 800$. (c1)-(c2) are the typical time series of q_t derived from that of (b1): (c1) $\alpha = 1.2$, $\beta_0 = 0.3$ and $0 \leq t \leq 1000$. (c2) $\alpha = 1.2$, $\beta_0 = 0.3$ and $85 \leq t \leq 200$.

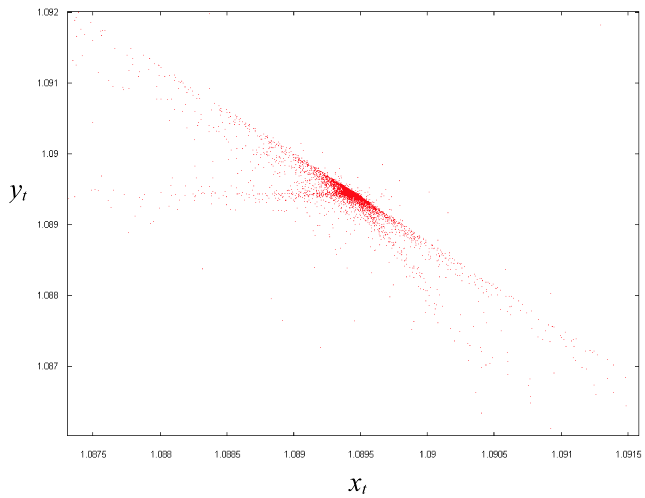


Figure 4.4: The trajectory of the HGP in phase space for $0 \leq t \leq 10^4$. The parameters are set to $\alpha = 0.2$ and $\beta_0 = 0.3$.

occurs within wide regions of α: within $0 < \alpha \lesssim 0.74$ for $\beta_0 = 0.0$ and within $0 < \alpha \lesssim 0.36$ for $\beta_0 = 0.3$ (Figure 4.5 (b)). This points to the coexistence of changeable maps and the occurrence of intermittency with respect to parameter shifts. The results show ubiquitous intermittency featuring robustness instead of stability. The irregular bursts of the intermittency form a bifurcation which is clearly different from the Gaussian noise with constant variance (Figure 4.5 (c)). Moreover, as will hereinafter be described in detail, in the time series of HGP, the value of x_t inevitably diverges after some time. If the divergence occurs sooner than 1000 steps in the time series, a blank region is formed on the bifurcation diagram. Consequently, the bifurcation diagrams of HGP have a window-like structure in the whole parameter region.

Figure 4.6 show the graphs of the magnitude of the fluctuation of β, $|\beta_t - \beta_0| = |\delta\widetilde{\beta}_t|$, versus the probability of that, $\Pr(|\beta_t - \beta_0|) = \Pr(|\delta\widetilde{\beta}_t|)$, on the HGP, where $\beta_0 = 0.3$ and (a) $\alpha = 0.2$ (b) $\alpha = 1.2$. Note that the graph is double logarithmic. These data consist of 100 trials and each trial is observed until $t = 10^4$. The initial state (x_0, y_0) is chosen at random for each cycle. From Figure 4.3 (c1)-(c2), we know the on-off intermittency of HGP is derived from $|\delta\widetilde{\beta}_t| \times |y_t|$ (i.e. the multiplication of the fluctuation of β and the state of HGP). In addition, $\Pr(|\delta\widetilde{\beta}_t|)$ follows a power law. As a result, we can see that $|\delta\widetilde{\beta}_t|$ scales up and down the state in phase space.

In HGP, catastrophe (i.e. divergence of the value of x_t) inevitably occurs at some time. We call the time until the divergence occurs the *lifetime* of the system, T. Figure 4.7 shows the graph of the lifetime of HGP, T, versus the

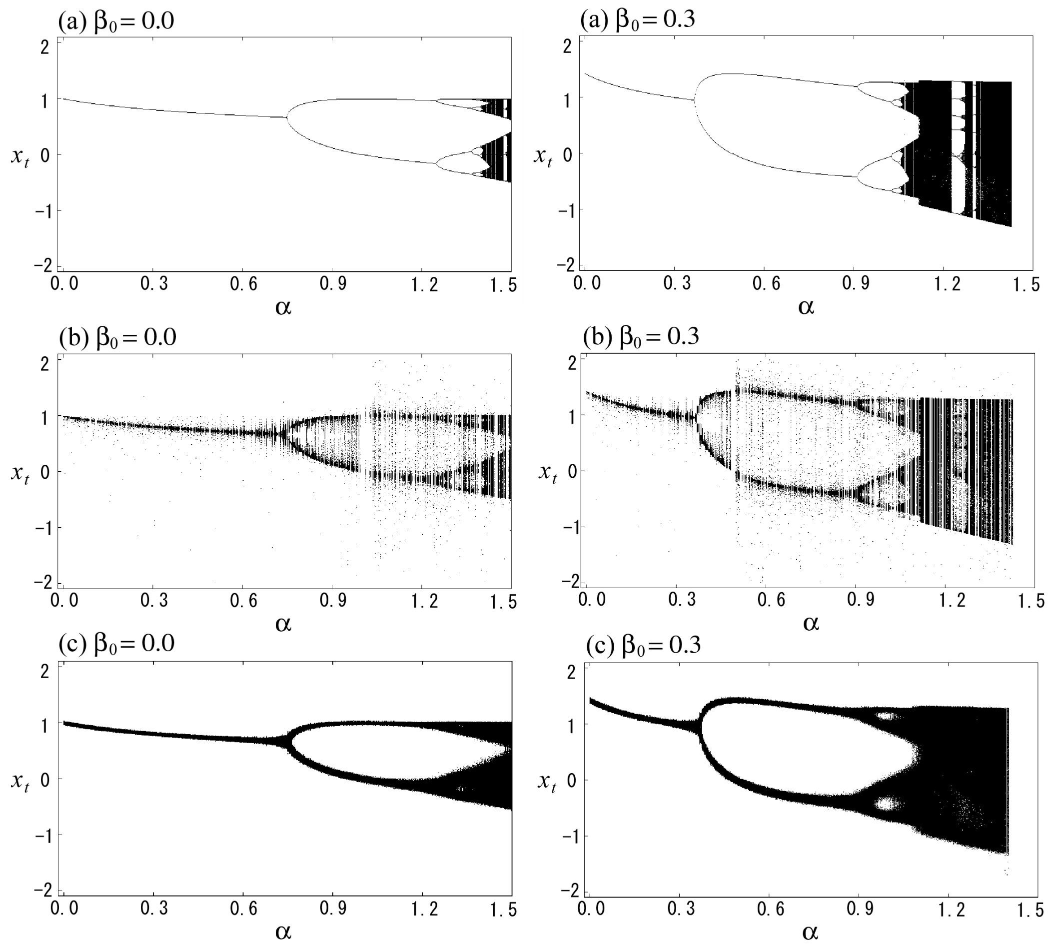


Figure 4.5: The bifurcation diagrams on (a) conventional Henon maps, (b) HGPs and (c) Henon maps with Gaussian noises. Each diagram shows the values of x_t for $1000 \leq t \leq 4000$ along with the parameter $0.0 \leq \alpha \leq 1.5$. For the left and right figures, the initial parameter β_0 is given by $\beta_0 = 0.0$ and $\beta_0 = 0.3$ respectively.

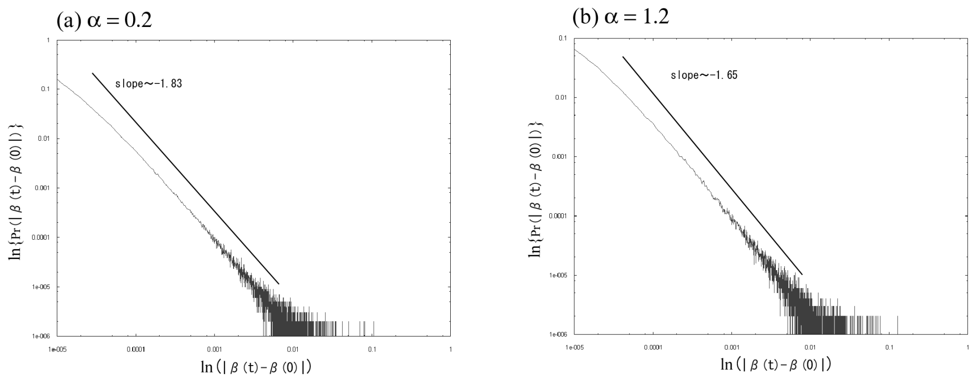


Figure 4.6: The magnitude of fluctuation of β, $|\beta_t - \beta_0|$, versus the probability of that, $\Pr(|\beta_t - \beta_0|)$, on the HGP, where $\beta_0 = 0.3$ and (a) $\alpha = 0.2$ (b) $\alpha = 1.2$. Note that the graph is double logarithmic. These data consist of 100 trials, and each trial is observed until $t = 10^4$. The initial state (x_0, y_0) is chosen randomly for each cycle.

probability of that, $\Pr(T)$. In this case, the parameters of HGP are set to $\alpha = 0.2$ and $\beta_0 = 0.3$, and the divergence point is defined by the threshold $|x_t| > 10^5$. The data consist of 2×10^4 trials and each trial is observed until $t = 4 \times 10^4$. Note that the graph is single logarithmic, i.e. the graph is exponential and the system has a decay time constant $\tau \cong 8023$.

Why does such divergence occur? Figure 4.6 show $\Pr(|\delta\widetilde{\beta_t}|) < 10^{-6}$ such that the fluctuation of β is $|\delta\widetilde{\beta_t}| > 0.1$, and hence big fluctuations occur rarely during the lifetime. If a fluctuation of ± 0.1 for β is attributed to the conventional Henon map, the Henon map has fixed points for $0.2 - 0.1 = 0.1 < \beta < 0.3 = 0.2 + 0.1$ and $\alpha = 0.3$, and the above divergence does not occur. Moreover, $m_t^\sharp$ is a bounded map on $(-1, 1)$, thus it is not the direct cause of the divergence. It is non-trivial that the time series of the HGP diverge by about 8000 steps on the features of Henon maps and $m_t^\sharp$. The divergence of the time series may be derived from the conglomerate of the Henon map and the generative pointer. The statistics and the structure of the system will be analyzed in a separate chapter.

4.4 Conclusion

In this chapter, we introduced the concept of generative pointers, which are extended subobject classifiers with local identity (i.e. an incomplete indication or identification). In a dynamical system, specifying a point or subspace in phase space corresponds to setting the boundary condition which is fulfilled

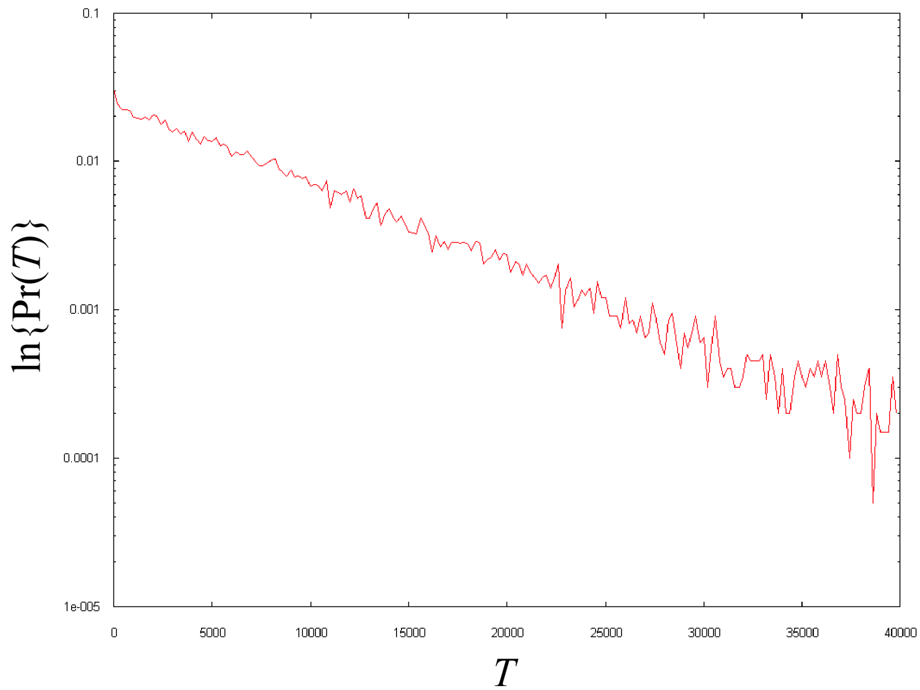


Figure 4.7: The lifetime of HGP, T, versus the probability of that, $\Pr(T)$. The parameters of HGP are set to $\alpha = 0.2$ and $\beta_0 = 0.3$. The divergence point is defined by the threshold $|x_t| > 10^5$. The data consist of 2×10^4 trials, and each trial is observed until $t = 4 \times 10^4$. Note that the graph is single logarithmic, i.e. the graph is exponential and the system has a decay time constant $\tau \cong 8023$.

outside of the dynamical system. The system and its environment are conglomerated through the incomplete identification, and the dynamic changes in the system are derived. Such an extended process of the system is motivated by a generalization of the process of emergence of rational numbers. We regard the concept of emergence as the process of perception of the system from outside.

Generally, intermittency is observed at a critical point between a fixed point (or periodic solution) and a chaotic solution on the bifurcation diagram, and as such it is sensitive with respect to parameter shifts. In contrast, when this line of thought is applied to a Henon map, ubiquitous on-off intermittency is observed for a wide parameter region, which points to robustness with respect to parameter shifts. The concept of structural stability or trajectory stability of dynamical systems is based on the specific invariable map, but robustness, as distinguished from stability, is based on the dynamic change of the map. Thus, in our view, the robustness does not conflict with the emergence but coincides with it.

Moreover, the divergence of the time series of HGP suggests death or extinction in the biological systems. It does not conflict with the claim on the robustness, since the robustness is not the persistence of the bounded motion of the time series but rather the persistence of the intermittency against parameter shifts. In this regard, emergent systems ubiquitously reach complexity and die after some time.

In the sense of conventional dynamical systems, the intermittency occurs at the phase transition points, and the measure of that is zero or sufficiently smaller than that of the usable parameter region. This fact implies the scarcity of life phenomena. Generally, dynamical systems have a closed form on a 1-parameter transformation group, and this fact is based on the vector field as the global structure.

In contrast, the generative pointer corresponds to the connection between the system and its environment, and the local identity corresponds to the incompleteness of that connection. The heterarchical coexistence of the system and the environment induces the dynamic change of the system structure, and derives the ubiquitous complex motion (i.e. the on-off intermittency). Moreover, on-off intermittency can be useful in engineering (Inoue et al. 1991), which implies that our method has the practical advantage of saving the trouble to tune the parameter for the intermittency.

Conclusion

How can our studies be characterized in science ? Mathematics have been motivated by the description of physical phenomena for a long time. In conventional sense, theoretical biology and complex systems science are based on the view of such mathematics or physics: i.e. mathematics based on physics is applied to biological systems. In contrast, we emphasize the probability, such that mathematics motivated by ontological aspects of "bios" is constructed: where "bios" does not always mean a biological system. If we suppose inconsistency / incompleteness / indefiniteness for systems, such as life, economics, computers, proteins, etc, these are regarded as bio-like phenomena, since such inconsistency derives "time": where "time" does not mean a parameter as a master clock, but means emergence as dynamic change with robustness. As two sides of the same coin, if we do not suppose such inconsistency for the systems, these are regarded as machine-like phenomena. Therefore, the difference between "bios" and "machine" is not difference of categories including the objects but that of way for understanding of the objects.

When we accept such inconsistency, we can no longer completely affirm the concept of prediction or that of theoretical explanation. Are theoretical constructions confined to predictions for experimental outcomes or to explanations for that ?

It can be generalized that how to use of theories or models is that of languages. For example, Japanese has multiple usages, such as legal use as judicial order system, explanatory use on encyclopedias and expressive use for poems or for literature, etc. However, Japanese usages are not restricted to their usages, and have been expanded even to accessories such as art of calligraphy on scroll pictures.

Usages of mathematical models are not also restricted a priori. This fact shows our usages of languages (usages of models) are made as customs: i.e. a mathematical model is specific language game on Wittgenstein's language game (Wittgenstein 1953). Although we can indicate such an aspect of mathematical models, effective action is not such indication but actually playing the "games". I am interpreting *Protocomputing and Ontological Measurement* (Gunji 2004) as the above.

Expansion of usage of language is not based on formal guarantee but is based on keeping using such an expansion. The fact that Japanese is used as accessories is not realized by being written in list of usages, but is realized by existence of human that uses it as accessories.

Predictions or explanations are only part of intellectual activity. We emphasize theoretical constructions as a momentum of actuality.

Dialogue and monologue. Although text is written on one's own, monologue is not closed world. It is something to old self, to future self or to others who have not met yet. Monologue is dialogue which destination has not been determined. Therefore, it can make time and can be opened to future.

Appendix A

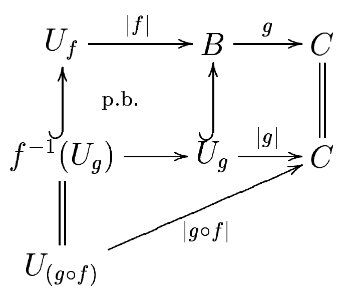
Partial Maps and Pointed Sets

In this appendix, we show an equivalence relation between the category of partial maps and the category of pointed sets. In section 4.1.2, using the framework of the equivalence relation, we generalized the example of the emergence of rational numbers.

Let **Par** be the category of sets and partial maps. An Arrow

$$f : A \to B$$

is a map $|f| : U_f \to B$ for some $U_f \subseteq A$. Identities in **Par** are the same as those in **Sets**, i.e. id_A is the total identitiy function on A. The composite of $f : A \to B$ and $g : B \to C$ is given as follows: Let $U_{(g\circ f)} := f^{-1}(U_g) \subseteq A$, and $|g \circ f| : U_{(g\circ f)} \to C$ is the horizontal composite indicated in the second line from the top in the following diagram, in which the square is a pullback.



(A.1)

It is easy to see that composition is associative and that the identities are units.

The category of pointed sets, $\mathbf{Sets}_*$, has as objects sets A equipped with a distinguished "point" $a \in A$, i.e. pairs: (A, a) with $a \in A$. Arrows are maps that preserve the point, i.e. an arrow $f : (A, a) \to (B, b)$ is a function $f : A \to B$ such that $f(a) = b$.

Lemma A.1 The map $F : \mathbf{Par} \to \mathbf{Sets}_*$ such that it satisfies the following properties 1. and 2. is a functor.

1. For objects, given $A \in \mathbf{Par}$, $F(A) \in \mathbf{Sets}_*$ is defined by:

$$A_* = (A \cup \{*\}, *)$$

   where $*$ is a new point for addition of A.

2. For arrows, given $f : A \to B$, $F(f) : A_* \to B_*$ is defined by:

$$f_*(x) = \begin{cases} f(x) & (\text{if} x \in U_f) \\ * & (\text{if} x \in A_* \backslash U_f) \end{cases}$$

   where $A_* \backslash U_f$ is the difference set of U_f for A_*.

Note that $f_*(*_A) = *_B$ for the definition since $*_A \notin A$ and consequently $*_A \notin U_f$.

Proof We have only to check the preservation of the identity and the composition.

1. (Identity) From the above definitions of F;

$$F(id_A)(x) = (id_A)_*(x) = \begin{cases} id_A(x) & (\text{if} x \in A) \\ *_B & (\text{if} x \in A_* \backslash A) \end{cases}$$

   where $A_* \backslash A = \{*_A\}$, so $(id_A)_*(*_A) = *_B$, and therefore $F(id_A) = id_{A_*}$.

2. (Composition) For arrows in **Par**, $f : A \to B$ and $g : B \to C$, which are defined by the partial maps $|f| : A \supset U_f \to B$ and $|g| : B \supset U_g \to C$, $F(g \circ f)$ is defined by:

$$F(g \circ f)(x) = (g \circ f)_*(x) = \begin{cases} g \circ f(x) & (\text{if} x \in U_{g \circ f}) \\ *_C & (\text{if} x \in A_* \backslash U_{g \circ f}) \end{cases}$$

   On the other hand, for $F(g) \circ F(f) = g_* \circ f_*$,

$$\begin{array}{lll} \text{if} & x \in U_{g \circ f}, & g_* \circ f_*(x) = g_*(f_*(x)) = g_*(f(x)) = g(f(x)) = g \circ f(x) \\ \text{if} & x \in A_* \backslash U_{g \circ f}, & g_* \circ f_*(x) = g_*(f_*(x)) = g_*(*_B) = *_C \end{array}$$

   Therefore, we obtain $F(g \circ f) = F(g) \circ F(f)$.

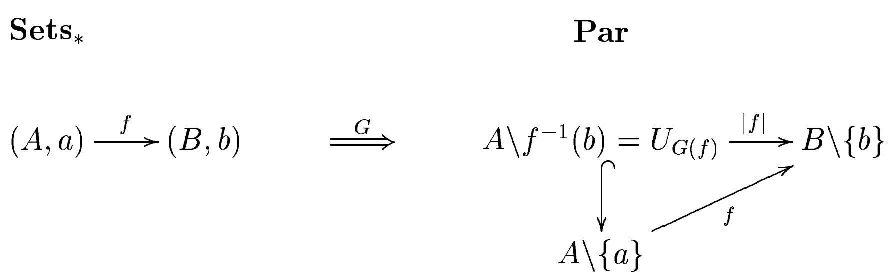
For 1. and 2., the map F is functorial.

■

Lemma A.2 If the map $G : \mathbf{Sets}_* \to \mathbf{Par}$ is defined on an object (A, a) by $G(A, a) = A \backslash \{a\}$, and for an arrow $f : (A, a) \to (B, b)(\iff f : A \to B$ with $(f(a) = b))$,

$$G(f) : A \backslash \{a\} \to B \backslash \{b\}$$

is the map f with domain $U_{G(f)} = A \backslash f^{-1}(b)$, and the map G is a functor.



Proof

1. (Identity) $id_A(a) = a$, thus $id_{(A,a)} = id_A$. Generally, $G(f)$ is defined by f with the domain $U_f = A \backslash f^{-1}(b)$, and thus $G(id_{(A,a)}) = G(id_A)$ is id_A with the domain $U_{id_A} = A \backslash id_A^{-1}(a) = A \backslash \{a\}$. Therefore, $G(id_{(A,a)}) = id_{A \backslash \{a\}}$.

2. (Composition) For arrows in $\mathbf{Sets}_*$, $f : (A, a) \to (B, b)$ and $g : (B, b) \to (C, c)$,

   $$G(g \circ f)(x) = g \circ f(x) \quad (x \in A \backslash (g \circ f)^{-1}(c)).$$

   On the other hand, $G(f)$ is f with the domain $A \backslash f^{-1}(b)$ and $G(g)$ is g with the domain $B \backslash f^{-1}(c)$. By the definition of composition in **Par**,

   $$G(g) \circ G(f)(x) = g \circ f(x) \quad (x \in f^{-1}(U_{G(g)})).$$

   One domain is equal to the other:

   $$f^{-1}(U_{G(g)}) = f^{-1}(B \backslash g^{-1}(c)) = f^{-1}(B) \backslash f^{-1}(g^{-1}(c)) = A \backslash (g \circ f)^{-1}(c)$$

   Note that $f^{-1}(B) = A$ and $f^{-1}(g^{-1}(c)) = (g \circ f)^{-1}(c)$.

   Therefore, we obtain $G(g \circ f) = G(g) \circ G(f)$.

For 1. and 2., the map G is functorial.

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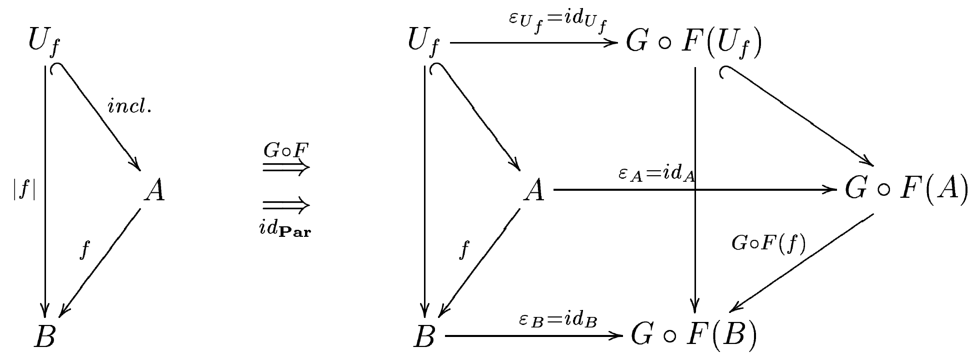
Proposition A.3 (Par $\cong$ Sets$_*$) The categories **Par** and **Sets$_*$** are equivalent. In other words, There are natural isomorphisms $G \circ F \cong id_{\mathbf{Par}}$ and $F \circ G \cong id_{\mathbf{Sets}_*}$ where $F : \mathbf{Par} \to \mathbf{Sets}_*$ and $G : \mathbf{Sets}_* \to \mathbf{Par}$ are the functors indicated in the above lemma 4.1 and 4.2.

Proof

1. The natural transformation $\varepsilon : id_{\mathbf{Par}} \dot{\to} G \circ F$ is the identity on **Par**, since we are merely adding a new point and then removing it through $G \circ F$. This situation is illustrated by the following diagram:



   Note that $G \circ F(A) = G(A_*) = A_* \backslash \{*_A\} = A$, $G \circ F(f) = G(f_*)$ and $G \circ F(U_f) = U_{G(f_*)} = A_* \backslash f_*^{-1}(*_B) = A_* \backslash (A_* \backslash U_f) = U_f$. Thus, $G \circ F(f) = f$, and therefore $G \circ F = id_{\mathbf{Par}}$ is satisfied.

2. For an arbitrary object $(A, a) \in \mathbf{Sets}_*$, we obtain:

$$F \circ G((A, a)) = F(A \backslash \{a\}) = ((A \backslash \{a\}) \cup \{*_A\}, *_A).$$

   Although (A, a) and $F \circ G((A, a))$ are not equal because $a \neq *$, $F \circ G((A, a))$ is the object such that $a \in A$ is replaced by $*_A$.

   For an arbitrary arrow $f : (A, a) \to (B, b)$, we obtain:

$$F \circ G(f)(x) = (G(f))_*(x) = \begin{cases} (G(f))(x) & (\text{if } x \in U_{G(f)} = A \backslash f^{-1}(b)) \\ *_B & (\text{if } x \in ((A \backslash \{a\}) \cup \{*_A\}) \backslash U_{G(f)}) \end{cases}$$

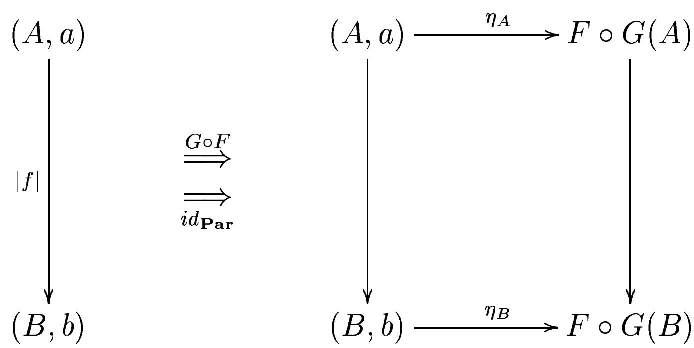
   where $(G(f)) : A \backslash \{a\} \to B \backslash \{b\}$ is an arrow in **Par** with the domain $U_{G(f)} = A \backslash f^{-1}(b)$. Note that:

$$((A \backslash \{a\}) \cup \{*_A\}) \backslash U_{G(f)} = ((A \backslash \{a\}) \cup \{*_A\}) \backslash (A \backslash f^{-1}(b)) = f^{-1}(b) \cup \{*_A\} \backslash \{a\}.$$

   Thus, the natural transformation $\eta : id_{\mathbf{Sets}_*} \dot{\to} F \circ G$ is naturally isomorphic:

$$\eta_A(x) = \begin{cases} x & (\text{if } x \in A \backslash \{a\}) \\ *_A & (\text{if } x \in \{a\}) \end{cases}, \qquad \eta_A^{-1}(x) = \begin{cases} x & (\text{if } x \in A \backslash \{a\}) \\ a & (\text{if } x \in \{*_A\}). \end{cases}$$

Therefore, $F \circ G \cong id_{\mathbf{Sets}_*}$. This situation is illustrated by the following diagram:



For 1. and 2., the equivalence $\mathbf{Par} \cong \mathbf{Sets}_*$ is satisfied.

■

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