Parametrized diamond principles and their applications to set theory of reals

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Parametrized diamond principles and their applications to set theory of reals

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Contents

Chapter 1

Introduction

By $\wp(\omega)$ we denote the power set of the set of ω of natural numbers. In set theory the infinitary combinatorics of $(\wp(\omega)/fin, \leq_{fin})$ has been studied, where $\wp(\omega)/fin$ is the power set of natural numbers ω modulo the finite sets ordered by $\leq_{fin}$ where $[A] \leq_{fin} [B]$ if $A \setminus B$ is finite. Here we denote by $[A]$ the equivalence class of a set $A \subset \omega$. To investigate combinatorial structures of $(\wp(\omega)/fin, \leq_{fin})$ cardinal invariants of the continuum are introduced and analyzed. For example the reaping number $\mathfrak{r}$ is the least size of a family $\mathcal{R}$ of infinite subset of the natural number such that for every 2-coloring of ω there is a monochromatic set in $\mathcal{R}$.

One can easily show that the reaping number is strictly larger than ω and $\mathfrak{r}$ is at most the cardinality $\mathfrak{c}$ of the continuum. Then it is natural to ask how large $\mathfrak{r}$ is. The answer of this question is that it depends on the underlying model. Assuming Zermero-Fraenkel set theory with Axiom of Choice ZFC and the Continuum Hypothesis CH, the answer is trivial, that is, $\omega_1 = \mathfrak{r} = \mathfrak{c}$. Also assuming ZFC with Martin's Axiom MA, $\mathfrak{r}$ is equal to $\mathfrak{c}$ and strictly larger than ω_1. With the forcing method we can show that $\mathfrak{r} < \mathfrak{c}$ is consistent with ZFC.

So it doesn't seem reasonable to ask how large cardinal invariant is in ZFC. But there are relations which is provable in ZFC. For example, let $\mathfrak{b}$ is the least size of a family of ω^ω which cannot eventually dominated by a function in ω^ω, then it is provable in ZFC that $\mathfrak{b}$ is smaller than $\mathfrak{r}$. Therefore it is reasonable to ask whether relationships between cardinal invariants is provable or unprovable in ZFC. The relationship between cardinal invariants related to the infinitary combinatorics of $(\wp(\omega)/fin, \leq_{fin})$ has been investigated and is displayed in van Dowen's diagram.

1.1 van Douwen's Diagram

Throughout this thesis, we will assume ZFC.

By $[\omega]^\omega$ we denote the set of all infinite subsets of ω. We denote the set of all finite subsets of ω by $[\omega]^{<\omega}$. ω^ω and $\omega^{<\omega}$ stand for all function from ω to ω

and all finite sequence of ω respectively.

We introduce several cardinal invariants and display the interaction between them, called **van Douwen's diagram**.

For $X, Y \in [\omega]^\omega$ X is **almost included** by Y, we write $X \subset^* Y$ if $|X \setminus Y| < \aleph_0$. For $\mathcal{F} \subset [\omega]^\omega$ A is a **pseudointersection** of $\mathcal{F}$ if $A \subset^* F$ for $F \in \mathcal{F}$.

$\mathcal{T} = \langle t_\alpha : \alpha < \kappa \rangle$ is a **tower** if

1. t_α is an infinite subset of ω for $\alpha < \kappa$,
2. $t_\beta \subset^* t_\alpha$ for $\alpha < \beta < \kappa$ and
3. there is no pseudointersection of $\mathcal{T}$.

The tower number $\mathfrak{t}$ is the least length of a tower.

$\mathcal{P}$ has the strong finite intersection property if every non-empty finite subfamily has infinite intersection. The pseudointersection number $\mathfrak{p}$ is the least size of a $\mathcal{P} \subset [\omega]^\omega$ which has the strong finite intersection property with no infinite pseudointersection.

$\mathcal{D} \subset [\omega]^\omega$ is open if $X \in \mathcal{D}$, then $Y \in \mathcal{D}$ for $Y \subset^* X$. $\mathcal{D} \subset [\omega]^\omega$ is dense if for $X \in [\omega]^\omega$ there exists $Y \subset X$ such that $Y \in \mathcal{C}$. The distributivity number $\langle$ is the least size of a open dense families with empty intersection. $\mathfrak{h}$

For $X, Y \in [\omega]^\omega$ X and Y are almost disjoint if $|X \cap Y| < \omega$. $\mathcal{A} \subset [\omega]^\omega$ is almost disjoint family if $|\mathcal{A}| \geq \omega$ and pairwise almost disjoint. The maximal almost disjoint number $\mathfrak{a}$ is the least size of a maximal almost disjoint family.

For $X, Y \in [\omega]^\omega$ X splits Y if $|X \cap Y| = \aleph_0$ and $|Y \setminus X| = \aleph_0$. $\mathcal{S} \subset [\omega]^\omega$ is a splitting family if for $Y \in [\omega]^\omega$ there exists $X \in \mathcal{S}$ such that X splits Y. The splitting number $\mathfrak{s}$ is the least size of a splitting family.

$\mathcal{R} \subset [\omega]^\omega$ is a reaping family if for $X \in [\omega]^\omega$ there exists $Y \in \mathcal{R}$ such that X cannot split Y i.e., $|X \cap Y| < \omega$ or $Y \subset^* X$. The reaping number $\mathfrak{r}$ is the least number of a reaping family.

For $f, g \in \omega^\omega$ f **eventually dominates** g, denotes $f \leq^* g$ if for all but finitely many $n \in \omega$ $f(n) \leq g(n)$. $\mathcal{F} \subset \omega^\omega$ is a **dominating family** if for each $g \in \omega^\omega$ there exists $f \in \mathcal{F}$ such that $g \leq^* f$. The dominating number $\mathfrak{d}$ is the least size of a dominating family.

$\mathcal{G} \subset \omega^\omega$ is an **unbounded family** if for each $f \in \omega^\omega$ there exists $g \in \mathcal{G}$ such that $g \not\leq^* f$ i.e., there exists infinitely many $n \in \omega$ such that $g(n) > f(n)$. The unbounded number $\mathfrak{b}$ is the least size of an unbounded family.

$\mathcal{I} \subset [\omega]^\omega$ is a independence family if for any finite subsets $\mathcal{A}, \mathcal{B} \subset \mathcal{I}$ with $\mathcal{A} \cap \mathcal{B} = \emptyset$ $\bigcap \mathcal{A} \cap \bigcap \{\omega \setminus B : B \in \mathcal{B}\}$ is infinite. The independence number $\mathfrak{i}$ is the least size of a maximal independence family.

$\mathcal{F}$ is a filter on ω if

1. if $X, Y \in \mathcal{F}$, then $X \cap Y \in \mathcal{F}$.
2. if $X \in \mathcal{F}$, then $Y \in \mathcal{F}$ for $X \subset Y$.
3. $\emptyset \notin \mathcal{F}$.

We will consider only the filters which contains all cofinite subset of ω. $\mathcal{U}$ is a ultrafilter on ω if $\mathcal{U}$ is a filter and $X \in \mathcal{U}$ or $\omega \setminus X \in \mathcal{U}$ for $X \in \wp(\omega)$. $\mathcal{B}$ is a base for a filter $\mathcal{F}$ if $\mathcal{B} \subset \mathcal{F}$ and for each $X \in \mathcal{F}$ there exists $Y \in \mathcal{B}$ such that $Y \subset X$. The ultrafilter number $\mathfrak{u}$ is the least size of a base for a ultrafilter.

$$
\begin{tikzcd}
 & & \mathfrak{c} \arrow[dl] \arrow[dr] \arrow[ddrr] & & \\
 & \mathfrak{i} \arrow[d] \arrow[dr] & & \mathfrak{u} \arrow[dl] & \\
 & \mathfrak{d} \arrow[dl] \arrow[dr] & \mathfrak{r} \arrow[d] & & \mathfrak{a} \arrow[dll] \\
\mathfrak{s} \arrow[dr] & & \mathfrak{b} \arrow[dl] & & \\
 & \mathfrak{h} \arrow[d] & & & \\
 & \mathfrak{t} \arrow[d] & & & \\
 & \mathfrak{p} \arrow[d] & & & \\
 & \omega_1 & & &
\end{tikzcd}
$$

van Douwen's diagram.
($\rightarrow$ means that the inequality $\leq$ is provable in ZFC).

1.1.1 Cichoń's diagram

We will introduce cardinal invariants related to an ideal on $\mathbb{R}$. Let $\mathcal{I}$ be a ideal on $\mathbb{R}$.

$$
\begin{aligned}
\operatorname{add}(\mathcal{I}) &= \min\{|\mathcal{A}| : \mathcal{A} \subset \mathcal{I} \wedge \forall X \in \mathcal{I} \exists Y \in \mathcal{A}(X \not\supset Y)\} \\
\operatorname{cov}(\mathcal{I}) &= \min\{|\mathcal{A}| : \mathcal{A} \subset \mathcal{I} \wedge \mathbb{R} = \bigcup \mathcal{A}\} \\
\operatorname{non}(\mathcal{I}) &= \min\{|X|\, X \subset \mathbb{R} \wedge \bigcup X \notin \mathcal{I}\} \\
\operatorname{cof}(\mathcal{I}) &= \min\{|\mathcal{A}| : \mathcal{A} \subset \mathcal{I} \wedge \forall X \in \mathcal{I} \exists Y \in \mathcal{A}(X \subset Y)\}.
\end{aligned}
$$

Let $\mathcal{N}$ be a null ideal. Let $\mathcal{M}$ be a meager ideal. Then we have the following relations.

$$\begin{tikzcd}
\mathrm{cov}(\mathcal{M}) \arrow[dd] & \mathrm{non}(\mathcal{M}) \arrow[l] \arrow[d] & \mathrm{cof}(\mathcal{M}) \arrow[l] \arrow[d] & \mathrm{cof}(\mathcal{N}) \arrow[l] \arrow[dd] \\
 & \mathfrak{b} \arrow[d] & \mathfrak{d} \arrow[l] \arrow[d] & \\
\mathrm{add}(\mathcal{N}) & \mathrm{add}(\mathcal{M}) \arrow[l] & \mathrm{cov}(\mathcal{M}) \arrow[l] & \mathrm{non}(\mathcal{M}) \arrow[l]
\end{tikzcd} .$$

Cichoń's diagram.
($\rightarrow$ means that the inequality $\leq$ is provable in ZFC).

1.2 Motivation

The aim of this thesis is to deal with two kinds of problems concerning to cardinal invariants.

The one is the relationship between combinatorial principles called "parametrized $\diamondsuit$ principles" related to cardinal invariants in Cichoń's diagram and the other is the properties of cardinal invariants related to the structure $((\omega)^\omega, \leq^*)$.

In [20], Jensen showed V=L implies Suslin's hypothesis doesn't hold. To prove this result he introduced the $\diamondsuit$-principle:

$\diamondsuit$ There exists a sequence $\langle A_\alpha \subset \alpha : \alpha < \omega_1 \rangle$ such that for all $X \subset \omega_1$ the set $\{\alpha < \omega_1 : X \cap \alpha = A_\alpha\}$ is stationary.

It is known that many statements which are independent of ZFC follow from $\diamondsuit$ (see [15], [17]). In [19] Hrušák introduced the $\diamondsuit$-like principle $\diamondsuit_{\mathfrak{d}}$:

$\diamondsuit_{\mathfrak{d}}$ There exists a sequence $\langle g_\alpha : \omega \leq \alpha < \omega_1 \rangle$ such that g_α is a function from α to ω and for every $f : \omega_1 \to \omega$ there is an $\alpha \geq \omega$ with $f \restriction \alpha \leq^* g_\alpha$.

The purpose of this principle was to give a partial solution to a question of J. Roitman who asked whether $\mathfrak{d} = \omega_1$ implies $\mathfrak{a} = \omega_1$ and to answer a question of Brendle who asked whether $\mathfrak{a} = \omega_1$ in any model obtained by adding a single Laver real. In [31] Moore, Hrušák, and Džamonja provided a broad framework of "parametrized $\diamondsuit$-principles" and techniques to force them and to force the negation of them.

Here we introduce other techniques to force parametrized $\diamondsuit$ principles and to force the negation of parametrized $\diamondsuit$ principles. We prove several consistency of the propositions on parametrized $\diamondsuit$ principles related to cardinal invariants in Cichoń's diagram.

Next we investigate cardinal invariants on the structure $((\omega)^\omega, \leq^*)$ of infinite partitions of ω ordered by $\leq^*$ where $A \leq^* B$ if all but finitely many blocks of A is union of a subset of B. In recent decade cardinal invariants related to the similar structures to $(\wp(\omega)/fin, \leq_{fin})$ are defined and investigated to understand the similarities and differences between their properties. For example the interesting works on $(Dense(\mathbb{Q}), \subset_{nwd})$ are done in [2] and [11].

On $((\omega)^\omega, \leq^*)$ a dualized version of Ramsey's theorem proved by Simpson and Carlson in [13] inspire a number of papers. Cardinal invariants of $((\omega)^\omega, \leq^*)$ was initiated by Matet in [25], and investigated comprehensively by Cichoń, Krawczyk, Majcher-Iwanow and Węglorz in [14], Halbeisen in [18], Spinas in [33] and Brendle in [10]. However the relation between cardinal invariants related to $((\omega)^\omega, \leq^*)$ and cardinal invariants in Cichoń's diagram has not investigated so much except dual-splitting number $\mathrm{cov}(\mathcal{M}) \leq \mathfrak{s}_d$ in [14]. We investigate the relationship dual cardinals and cardinals in Cichoń's diagram.

In Chapter 2 we will investigate parametrized $\diamondsuit$ principle and study the forcing method to force them and the negation of them.

In Chapter 3 we shall survey the relations between cardinal invariants related to $((\omega), \leq^*)$ and other cardinal invariants.

In Chapter 4 we will study interaction between the cardinal invariants related to $((\omega)^\omega, \leq^*)$ and forcings.

Chapter 2

Parametrized diamond principles and c.c.c forcing

The purpose of this chapter is to present some techniques to force parametrized $\diamondsuit$ principles.

2.1 Definition of parametrized diamonds and their applications

In this section, some properties of parametrized $\diamondsuit$ principles are introduced. Firstly we define parametrized $\diamondsuit$ principles and state their properties.

2.1.1 Borel invariants and parametrized diamonds, and their properties

In [36] Vojtáš introduced a framework to describe many cardinal invariants.

Definition 1. [36][31] The triple (A, B, E) is an *invariant* if

(1) $|A|, |B| \leq |\mathbb{R}|$,

(2) $E \subset A \times B$,

(3) For each $a \in A$ there exists $b \in B$ such that $(a, b) \in E$ and

(4) For each $b \in B$ there exists $a \in A$ such that $(a, b) \notin E$.

We will write aEb instead of $(a, b) \in E$. If A and B are Borel subsets of some Polish spaces and E is a Borel subset of their product, we call the triple (A, B, E) "*Borel invariant*".

Borel invariants were introduced in [6]. In the present paper we are interested only in Borel invariants.

Definition 2. Suppose (A, B, E) is an invariant. Then its *evaluation* is defined by

$$\langle A, B, E\rangle = \min\{|X| : X \subset B \text{ and } \forall a \in A \exists b \in X\ (aEb)\}.$$

If $A = B$, we will write (A, E) and $\langle A, E\rangle$ instead of (A, B, E) and $\langle A, B, E\rangle$.

Example 1. The following Borel invariants $(\mathcal{N}, \not\supset)$, $(\mathcal{N}, \subset)$, $(\mathbb{R}, \mathcal{M}, \in)$, $(\mathcal{M}, \mathbb{R}, \not\ni)$, $(\omega^\omega, <^*)$, $(\omega^\omega, \not>^*)$ and $([\omega]^\omega, \text{is split by})$ have the evaluations $\mathrm{add}(\mathcal{N})$, $\mathrm{cof}(\mathcal{N})$, $\mathrm{cov}(\mathcal{M})$, $\mathrm{non}(\mathcal{M})$, $\mathfrak{d}$, $\mathfrak{b}$ and $\mathfrak{s}$ respectively.

Definition 3. Suppose A is a Borel subset in some Polish space. Then $F : 2^{<\omega_1} \to A$ is *Borel* if for every $\alpha < \omega_1$ $F \restriction 2^\alpha$ is a Borel function.

In [15, 32] the principle "weak diamond principle" was introduced by Devlin and Shelah. This was the starting point for the parametrized diamond principles introduced by Moore, Hrušák and Džamonja [31].

Definition 4. [31](Parametrized diamond principle)
Suppose (A, B, E) is a Borel invariant. Then $\diamondsuit(A, B, E)$ is the following statement:

$\diamondsuit(A, B, E)$ For all Borel $F : 2^{<\omega_1} \to A$ there exists $g : \omega_1 \to B$ such that for every $f : \omega_1 \to 2$ the set $\{\alpha \in \omega_1 : F(f \restriction \alpha) E g(\alpha)\}$ is stationary.

The witness g for a given F in this statement will be called $\diamondsuit(A, B, E)$-sequence for F.

$\diamondsuit(A, B, E)$ and $\diamondsuit$ have the following relation:

Proposition 2.1.1. [31] Let (A, B, E) be a Borel invariant. $\diamondsuit$ implies $\diamondsuit(A, B, E)$.

$\diamondsuit(A, B, E)$ and $\langle A, B, E\rangle$ have the following relation:

Proposition 2.1.2. [31] Suppose (A, B, E) is a Borel invariant and $\diamondsuit(A, B, E)$ holds. Then $\langle A, B, E\rangle \leq \omega_1$ holds.

If two Borel invariants (A_1, B_1, E_1),(A_2, B_2, E_2) are comparable in the Borel Tukey order, then $\diamondsuit(A_1, B_1, E_1)$ and $\diamondsuit(A_2, B_2, E_2)$ have some relation:

Definition 5. (Borel Tukey ordering [6]) Given a pair of Borel invariants (A_1, B_1, E_1) and (A_2, B_2, E_2), we say that $(A_1, B_1, E_1) \leq_T^B (A_2, B_2, E_2)$ if there exist Borel maps $\phi : A_1 \to A_2$ and $\psi : B_2 \to B_1$ such that $(\phi(a), b) \in E_2$ implies $(a, \psi(b)) \in E_1$.

Proposition 2.1.3. [31] Let (A_1, B_1, E_1) and (A_2, B_2, E_2) be Borel invariants. Suppose $(A_1, B_1, E_1) \leq_T^B (A_2, B_2, E_2)$ and $\diamondsuit(A_2, B_2, E_2)$ holds. Then $\diamondsuit(A_1, B_1, E_1)$ holds.

By Proposition 2.1.3 if $(A_1, B_1, E_1) \leq_T^B (A_2, B_2, E_2)$ and $\diamondsuit(A_2, B_2, E_2)$, then $\diamondsuit(A_1, B_1, E_1)$ holds. But if we can separate $\langle A_1, B_1, E_1\rangle$, $\langle A_2, B_2, E_2\rangle$, then can we separate $\diamondsuit(A_1, B_1, E_1)$ from $\diamondsuit(A_2, B_2, E_2)$?

Question 1. *[31] If (A_1, B_1, E_1) and (A_2, B_2, E_2) are two Borel invariants such that the inequality $\langle A_1, B_1, E_1\rangle < \langle A_2, B_2, E_2\rangle$ is consistent, is it consistent that $\diamondsuit(A_1, B_1, E_1)$ holds and $\diamondsuit(A_2, B_2, E_2)$ fails in the presence of CH?*

Concerning $\leq^B_T$, we know the following diagram holds.

(Cichoń's diagram)

$$\begin{tikzcd}
(\mathbb{R}, \mathcal{N}, \in) \arrow[dd] & (\mathcal{M}, \mathbb{R}, \not\ni) \arrow[l] \arrow[d] & (\mathcal{M}, \subset) \arrow[l] \arrow[d] & (\mathcal{N}, \subset) \arrow[l] \arrow[dd] \\
 & (\omega^\omega, \not\geq^*) \arrow[d] & (\omega^\omega, \leq^*) \arrow[l] \arrow[d] & \\
(\mathcal{N}, \not\supset) & (\mathcal{M}, \not\supset) \arrow[l] & (\mathbb{R}, \mathcal{M}, \in) \arrow[l] & (\mathcal{N}, \mathbb{R}, \not\ni) \arrow[l]
\end{tikzcd}$$

(The direction of the arrow is from larger to smaller in the Borel Tukey order).
Hence the following holds:

$$\begin{tikzcd}
 & & & & \diamondsuit \; . \arrow[dl] \\
\diamondsuit(\mathrm{cov}(\mathcal{N})) \arrow[dd] & \diamondsuit(\mathrm{non}(\mathcal{M})) \arrow[l] \arrow[d] & \diamondsuit(\mathrm{cof}(\mathcal{M})) \arrow[l] \arrow[d] & \diamondsuit(\mathrm{cof}(\mathcal{N})) \arrow[l] \arrow[dd] & \\
 & \diamondsuit(\mathfrak{b}) \arrow[d] & \diamondsuit(\mathfrak{d}) \arrow[l] \arrow[d] & & \\
\diamondsuit(\mathrm{add}(\mathcal{N})) & \diamondsuit(\mathrm{add}(\mathcal{M})) \arrow[l] & \diamondsuit(\mathrm{cov}(\mathcal{M})) \arrow[l] & \diamondsuit(\mathrm{non}(\mathcal{N})) \arrow[l] &
\end{tikzcd}$$

(The direction of the arrow is the direction of the implication.)

We call this diagram "Cichoń's diagram for parametrized diamonds".

Note When we deal with Borel invariants in Cichoń's diagram, we will use the standard notation for their evaluations to denote the Borel invariants themselves (e.g., we will use $\diamondsuit(\mathrm{add}(\mathcal{N}))$ to denote $\diamondsuit(\mathcal{N}, \not\supset)$).

So this suggests the following interesting question:

Question 2. If we can construct a model M such that some cardinals in Cichoń's diagram are ω_1 and others are ω_2, then under CH can we construct a model such that for the invariants which are ω_1 in M, the corresponding parametrized diamond principle holds but for the others it doesn't hold?

In this question the hypothesis "CH holds" is important since an ω_2-stage countable support iteration of definable forcings which forces $\langle A_1, B_1, E_1\rangle = \omega_1$ and $\langle A_2, B_2, E_2\rangle = \omega_2$ also forces $\diamondsuit(A_1, B_1, E_1)$ (see [31]). In this paper we will give some general technique dealing with this problem and give some results.

2.1.2 Parametrized $\diamondsuit$ principles and cardinal invariants

We shall show some cardinal invariants are influenced by the parametrized *diamondsuit* principles.

Theorem 2.1.4. *[31] $\diamondsuit(\mathbb{R}, \neq)$ implies $\mathfrak{t} = \omega_1$.*

Theorem 2.1.5. *[31] $\diamondsuit(\mathfrak{b})$ implies $\mathfrak{a} = \omega_1$.*

Theorem 2.1.6. *[31] $\diamondsuit(\mathfrak{r})$ implies $\mathfrak{u} = \omega_1$.*

Theorem 2.1.7. *[31] $\diamondsuit(\mathfrak{r}_{\mathbb{Q}})$ implies $\mathfrak{i} = \omega_1$.*

2.2 ω_1-stage finite support iteration and parametrized $\diamondsuit$ principles

In this section, some techniques for dealing with parametrized $\diamondsuit$ principles are introduced. Firstly we shall construct parametrized $\diamondsuit$ principles by using ω_1-stage finite support iteration.

2.2.1 Construction of diamonds

We present a technique to construct $\diamondsuit(A, B, E)$. In [31] some methods to construct models of $\diamondsuit(A, B, E)$ are given.

Theorem 2.2.1. [31] Let $\mathbb{C}_{\omega_1}$ and $\mathbb{B}_{\omega_1}$ be the Cohen and random forcing corresponding to the product space 2^{ω_1}. Then $V^{\mathbb{C}_{\omega_1}} \models$ "$\diamondsuit(\text{non}(\mathcal{M}))$" and $V^{\mathbb{B}_{\omega_1}} \models$ "$\diamondsuit(\text{non}(\mathcal{N}))$".

Similarly we can prove the following theorem.

Theorem 2.2.2. Let $\mathbb{P}_{\omega_1}$ be an ω_1-stage finite support iteration of c.c.c forcings such that for any $\alpha \in \omega_1$ there exists $b \in B \cap V^{\mathbb{P}_{\omega_1}}$ such that aEb for any $a \in A \cap V^{\mathbb{P}_\alpha}$. Then $V^{\mathbb{P}_{\omega_1}} \models \diamondsuit^*(A, B, E)$ where $\diamondsuit^*(A, B, E)$ is the statement obtained by replacing "stationary" by "club " in $\diamondsuit(A, B, E)$.

Remark 1. If A is Borel set and $\mathbb{P}$ is a forcing notion, then we will write $A \cap V^{\mathbb{P}}$ for the interpretation of a Borel code for A in $V^{\mathbb{P}}$.

Proof of Theorem. Let $F \in V^{\mathbb{P}_{\omega_1}}$ be such that $F : 2^{<\omega_1} \to A$ is a Borel function. For each $\delta \in \omega_1$, let $r_\delta \in V^{\mathbb{P}_{\omega_1}}$ be a real coding $F \restriction 2^\delta$. Then define $f : \omega_1 \to \omega_1$ strictly increasing such that $r_\delta \in V^{\mathbb{P}_{f(\delta)}}$.

Then define $g : \omega_1 \to B$ so that

$$g(\alpha) = b \text{ where } b \text{ satisfies (2) for } f(\alpha).$$

Claim 1. g is $\diamondsuit^*(A, B, E)$-sequence for F

Let $h : \omega_1 \to 2$. Then define $C_h = \{\alpha \in \omega_1 : h \restriction \alpha \in V^{\mathbb{P}_\alpha}\}$. Since $\mathbb{P}_{\omega_1}$ is c.c.c, C_h is club. Then by construction if $\alpha \in C_h$, then $F(h \restriction \alpha) \in A \cap V^{\mathbb{P}_{f(\alpha)}}$. So $F(h \restriction \alpha) E g(\alpha)$. Hence g is a $\diamondsuit^*(A, B, E)$-sequence for F.

Claim$\blacksquare$ Theorem $\square$

2.2.2 Preservation of non-diamond

We present a technique to preserve $\neg\diamondsuit(A, B, E)$.

Theorem 2.2.3. (General preservation of $\neg\diamondsuit(A, B, E)$)
Let (A, B, E) be a Borel invariant and let $\mathbb{P}$ be a forcing notion which doesn't collapse ω_1.

(i) Suppose $V^{\mathbb{P}} \models \diamondsuit(A, B, E)$. If for each Borel function $F : 2^{<\omega_1} \to A$ in V and for a $\diamondsuit(A, B, E)$-sequence $\dot{g} : \omega_1 \to B$ for F in $V^{\mathbb{P}}$ there exists $g^* : \omega_1 \to B$ in V such that

$$\forall a \in A \cap V \left[\Big(\exists p \in \mathbb{P}(\ p \Vdash \check{a} E \dot{g}(\alpha)\ ) \Big) \text{ implies } aEg^*(\alpha) \right], \tag{2.1}$$

then $V \models \diamondsuit(A, B, E)$.

(ii) If $\mathbb{P}$ is a forcing notion such that for any $\mathbb{P}$-name $\dot{b}$ with $\Vdash \dot{b} \in B$ there exists $b' \in B \cap V$ such that

$$\forall a \in A \cap V \left[\Big(\exists p \in \mathbb{P}\ (p \Vdash \check{a} E \dot{b}) \Big) \text{ implies } aEb'\ \right], \tag{2.2}$$

then $V \models \neg\diamondsuit(A, B, E) \Rightarrow V^{\mathbb{P}} \models \neg\diamondsuit(A, B, E)$.

Proof. (ii) follows from (i). So we shall show only (i).

Suppose $\diamondsuit(A, B, E)$ holds in $V^{\mathbb{P}}$. Let $F : 2^{<\omega_1} \to A$ be a Borel function in V and $\dot{g}$ be a $\mathbb{P}$-name for a $\diamondsuit(A, B, E)$-sequence for F in $V^{\mathbb{P}}$. Then by (1) there exists $g^* : \omega_1 \to B$ in V such that

$$\forall \alpha \in \omega_1 \forall a \in A \cap V \left[\Big(\exists p \in \mathbb{P}\ (\ p \Vdash \check{a} E \dot{g}(\alpha)\ ) \Big) \text{ implies } aEg^*(\alpha) \right].$$

Let $f : \omega_1 \to 2$ be in V. Then $\{\alpha \in \omega_1 :$ there exists $p \in \mathbb{P}$ such that $p \Vdash$ "$F(f \restriction \alpha) E \dot{g}(\alpha)$"$\}$ is stationary. Since $\{\alpha \in \omega_1 : F(f \restriction \alpha) E g^*(\alpha)\}$ contains this set, it is also stationary. Hence $V \models \diamondsuit(A, B, E)$.

□

In the Kitami set theory seminar, Yasuo Yoshinobu pointed out the following fact.

Proposition 2.2.4. Let (A, B, E) be an Borel invariant. Then (ii) in Theorem 2.2.3 implies that $\mathbb{P}$ has $\langle A, B, E\rangle$-c.c.

Proof. Suppose there is an antichain $\mathcal{A} \subset \mathbb{P}$ with cardinality $\langle A, B, E\rangle$ and $\mathcal{D} \subset B$ witnesses $\langle A, B, E\rangle$. Then we have a $\mathbb{P}$-name $\dot{b}$ such that for all $a \in A \cap V$ there exists $p \in \mathbb{P}$ such that $p \Vdash$ "$aE\dot{b}$". If (ii) in Theorem 2.2.3 holds, then there exists $b \in B \cap V$ such that for all $a \in A \cap V$ aEb holds. But this is a contradiction to (4) in Definition 1.

□

So if CH holds in V, then $\mathbb{P}$ should have c.c.c in V.

2.3 Cichoń's diagram and Parametrized diamond under CH

We would like to show that under CH we can separate parametrized diamond principles for Borel invariants in Cichoń's diagram. In [19] Hrušák showed the following:

Theorem 2.3.1. [19] Con(CH+$\diamondsuit_{\mathfrak{d}}$ + $\neg\diamondsuit$).

In the proof Hrušák shows that if $V \models$ "CH+$\neg\diamondsuit$", then $V^{\mathbb{D}_{\omega_1}} \models$ "CH+$\diamondsuit_{\mathfrak{d}}$+ $\neg\diamondsuit$". Similarly we will start with a model in which the "weak" parametrized diamond principle fails. By [31], CH doesn't imply the "weak" parametrized diamond principle:

Proposition 2.3.2. [31] $(\mathbb{R}^\omega, \not\supseteq) \leq^B_T (\mathcal{N}, \not\supset)$ where $\not\supseteq$ is a relation on $\mathbb{R}^\omega$ such that for $x, y \in \mathbb{R}^\omega$ $x \not\supseteq y$ if $\mathrm{rng}(x) \not\supset \mathrm{rng}(y)$.

Theorem 2.3.3. [31] It is relatively consistent that CH + $\neg\diamondsuit(\mathbb{R}^\omega, \not\supseteq)$. Hence it is relatively consistent that CH + $\neg\diamondsuit(\mathrm{add}(\mathcal{N}))$ by Proposition 2.3.2.

But ω_1-stage countable support iteration of non-trivial proper forcing is not suitable to solve Question 2.

Theorem 2.3.4. Let $\mathbb{P} = \langle P_\alpha, \dot{Q}_\alpha : \alpha < \omega_1 \rangle$ be ω_1-stage countable support iteration of non-trivial proper forcing. Then $\Vdash_{\mathbb{P}} \diamondsuit$.

This is well-known and can be proved like Theorem 8.3 of Ch.VII, §8 in [22]. So if we want to use countable support iteration, we cannot use ω_1-stage iteration. Hence in this paper we use finite support iteration. But finite support iteration has some limitation.

Theorem 2.3.5. Finite support iterations of non-trivial forcing notions add Cohen reals in limit stages of cofinality ω. Hence ω_1-stage iterations of nontrivial c.c.c forcing result in models of $\diamondsuit(\mathrm{non}(\mathcal{M}))$. More precisely $\diamondsuit^*(\mathrm{non}(\mathcal{M}))$ holds.

But by using finite support iteration of c.c.c forcing we have the following results:

Theorem 2.3.6. (**Main theorem**) Each of the following are relatively consistent with ZFC:

(1) CH + $\diamondsuit(\mathrm{non}(\mathcal{M}))$ + $\neg\diamondsuit(\mathrm{cov}(\mathcal{M}))$ (see Diagram 1),

(2) CH + $\diamondsuit(\mathrm{non}(\mathcal{N}))$ + $\neg\diamondsuit(\mathfrak{b})$ + $\neg\diamondsuit(\mathrm{cov}(\mathcal{N}))$ (see Diagram 2),

(3) CH + $\diamondsuit(\mathrm{non}(\mathcal{M}))$ + $\diamondsuit(\mathrm{non}(\mathcal{N}))$ + $\neg\diamondsuit(\mathfrak{d})$ (see Diagram 3),

(4) CH + $\diamondsuit(\mathrm{cov}(\mathcal{M}))$ + $\diamondsuit(\mathrm{non}(\mathcal{M}))$ + $\neg\diamondsuit(\mathfrak{d})$ + $\neg\diamondsuit(\mathrm{non}(\mathcal{N}))$ (see Diagram 4),

(5) CH + $\diamondsuit(\mathrm{cof}(\mathcal{M}))$ + $\neg\diamondsuit(\mathrm{non}(\mathcal{N}))$ (see Diagram 5),

(6) CH + $\diamondsuit(\mathrm{cof}(\mathcal{M}))$ + $\diamondsuit(\mathrm{non}(\mathcal{N}))$ + $\neg\diamondsuit(\mathrm{cof}(\mathcal{N}))$ (see Diagram 6),

(7) CH + $\diamondsuit(\mathrm{cof}(\mathcal{N}))$ + $\neg\diamondsuit$ (see Diagram 7).

2.3.1 Cohen forcing and random forcing

Firstly we use Cohen forcing, random forcing and ω_1-stage finite support iteration of random forcing. In this paper we write $(\mathbb{B})_{\omega_1}$ for ω_1-stage finite support iteration of random forcing.

Proposition 2.3.7. (1) If $V \models \neg\diamondsuit(\mathrm{cov}(\mathcal{N}))$, then $V^{\mathbb{B}_{\omega_1}} \models \neg\diamondsuit(\mathrm{cov}(\mathcal{N}))$.

(2) If $V \models \neg\diamondsuit(\mathrm{cov}(\mathcal{M}))$, then $V^{\mathbb{C}_{\omega_1}} \models \neg\diamondsuit(\mathrm{cov}(\mathcal{M}))$.

To show this, we use the following theorem:

Theorem 2.3.8. [3, p.145 Lemma 3.3.17 for (1), p.125 Lemma 3.2.39 for (2)]

(1) Let $A \in \mathcal{M} \cap V^{\mathbb{C}_\kappa}$. Then there exists $B \in \mathcal{M} \cap V$ such that

$$\forall x \in \mathbb{R} \cap V \left[\Big(\exists p \in \mathbb{C}_\kappa \ (p \Vdash x \in A) \Big) \text{ implies } x \in B \right].$$

(2) Let $A \in \mathcal{N} \cap V^{\mathbb{B}_\kappa}$. Then there exists $B \in \mathcal{N} \cap V$ such that

$$\forall x \in \mathbb{R} \cap V \ \left[\Big(\exists p \in \mathbb{B}_\kappa \ (p \Vdash x \in A) \Big) \text{ implies } x \in B \right].$$

Proof of Proposition. We only show (2). Let $F : 2^{<\omega_1} \to \mathbb{R}$ in V and let $g : \omega_1 \to \mathcal{M}$ in $V^{\mathbb{C}_\kappa}$ be a $\diamondsuit(\mathrm{cov}(\mathcal{M}))$-sequence for F. Then by Theorem 2.3.8 and 2.2.3, we can find a $\diamondsuit(\mathrm{cov}(\mathcal{M}))$-sequence for F in V.

□

Proposition 2.3.9. Let $\mathbb{B}$ be a measure algebra. If $V \models \neg\diamondsuit(\mathfrak{d})$, then $V^{\mathbb{B}} \models \neg\diamondsuit(\mathfrak{d})$. Similarly if $V \models \neg\diamondsuit(\mathfrak{b})$, then $V^{\mathbb{B}} \models \neg\diamondsuit(\mathfrak{b})$.

Proof. Assume on the contrary that for each $F : 2^{<\omega_1} \to \omega^\omega$ Borel, there is a $\diamondsuit(\mathfrak{d})$-sequence $g : \omega_1 \to \omega^\omega$ in $V[G]$. Let F be a Borel function in V. By ω^ω-bounding and c.c.c, there is g^* such that $\Vdash g(\alpha) \leq^* g^*(\alpha)$ for all α. Let $f : \omega_1 \to 2$ in V. Then $\{\alpha \in \omega_1 : F(f \restriction \alpha) \leq^* g^*(\alpha)\}$ is stationary.

□

More generally we have the following result:

Proposition 2.3.10. If a c.c.c forcing notion $\mathbb{P}$ doesn't add dominating reals, then $V \models \neg\diamondsuit(\mathfrak{b}) \Rightarrow V^{\mathbb{P}} \models \neg\diamondsuit(\mathfrak{d})$.

Proof. Let $F : 2^{<\omega_1} \to \omega^\omega$ in V be a Borel function. Suppose $\diamondsuit(\mathfrak{d})$ holds and let $g : \omega_1 \to \omega^\omega$ be a $\diamondsuit(\mathfrak{d})$-sequence for F in $V^{\mathbb{P}}$. Since $\mathbb{P}$ doesn't add dominating reals and has the c.c.c, for each $\alpha < \omega_1$ there exists $f_\alpha \in \omega^\omega$ such that $\Vdash g(\alpha) \not>^* f_\alpha$. Define $g^* : \omega_1 \to \omega^\omega$ by $g^*(\alpha) = f_\alpha$. Then $\exists p \in \mathbb{P}$ ($p \Vdash f <^* g(\alpha)$) implies $f \not>^* g^*(\alpha)$. So g^* is a $\diamondsuit(\mathfrak{b})$-sequence for F.

□

Theorem 2.3.11. (1) If $V \models \mathrm{CH}+\neg\diamondsuit(\mathrm{cov}(\mathcal{M}))$, then $V^{\mathbb{C}_{\omega_1}} \models \mathrm{CH}+\diamondsuit(\mathrm{non}(\mathcal{M}))+\neg\diamondsuit(\mathrm{cov}(\mathcal{M}))$ (see Diagram 1).

(2) If $V \models \mathrm{CH}+\neg\diamondsuit(\mathfrak{b})+\neg\diamondsuit(\mathrm{cov}(\mathcal{N}))$, then $V^{\mathbb{B}_{\omega_1}} \models \mathrm{CH}+\neg\diamondsuit(\mathfrak{b})+\neg\diamondsuit(\mathrm{cov}(\mathcal{N}))+\diamondsuit(\mathrm{non}(\mathcal{N}))$ (see Diagram 2).

(3) If $V \models \mathrm{CH}+\neg\diamondsuit(\mathfrak{b})$, then $V^{(\mathbb{B})_{\omega_1}} \models \mathrm{CH}+\diamondsuit(\mathrm{non}(\mathcal{M}))+\diamondsuit(\mathrm{non}(\mathcal{N}))+\neg\diamondsuit(\mathfrak{d})$ (see Diagram 3).

Proof. (1): From Proposition 2.3.7 (2), and Theorem 2.2.1, this statement holds.
(2): From Proposition 2.3.7 (1), 2.3.9 and Theorem 2.2.1, this statement holds.
(3) To show this we use the following theorem:

Theorem 2.3.12. [3, p.100 Lemma 3.1.2, p.313 Lemma 6.5.1 and Theorem 6.5.4] [8] Finite support iteration of random forcing doesn't add dominating reals.

From the above theorem and 2.3.10, $V^{(\mathbb{B})_{\omega_1}} \models \neg\diamondsuit(\mathfrak{d})$. By 2.2.2 and 2.3.5, $V^{(\mathbb{B})_{\omega_1}} \models \diamondsuit(\mathrm{non}(\mathcal{M})) + \diamondsuit(\mathrm{non}(\mathcal{N}))$.

$\square$

By Theorem 2.3.11 it is relatively consistent with ZFC and CH that Cichoń's diagram for parametrized diamond looks as follows where a black square means the corresponding parametrized diamond fails while the others hold:

$$\begin{tikzcd}
\diamondsuit(\mathrm{cov}(\mathcal{N})) \arrow[r, dash] \arrow[dd, dash] & \diamondsuit(\mathrm{non}(\mathcal{M})) \arrow[r, dash] \arrow[d, dash] & \blacksquare \arrow[r, dash] \arrow[d, dash] & \blacksquare \arrow[dd, dash] \\
 & \diamondsuit(\mathfrak{b}) \arrow[r, dash] \arrow[d, dash] & \blacksquare \arrow[d, dash] & \\
\diamondsuit(\mathrm{add}(\mathcal{N})) \arrow[r, dash] & \diamondsuit(\mathrm{add}(\mathcal{M})) \arrow[r, dash] & \blacksquare \arrow[r, dash] & \blacksquare
\end{tikzcd}$$

Diagram 1

$$\begin{tikzcd}
\blacksquare \arrow[r, dash] \arrow[dd, dash] & \blacksquare \arrow[r, dash] \arrow[d, dash] & \blacksquare \arrow[r, dash] \arrow[d, dash] & \blacksquare \arrow[dd, dash] \\
 & \blacksquare \arrow[r, dash] \arrow[d, dash] & \blacksquare \arrow[d, dash] & \\
\diamondsuit(\mathrm{add}(\mathcal{N})) \arrow[r, dash] & \diamondsuit(\mathrm{add}(\mathcal{M})) \arrow[r, dash] & \diamondsuit(\mathrm{cov}(\mathcal{M})) \arrow[r, dash] & \diamondsuit(\mathrm{non}(\mathcal{N}))
\end{tikzcd}$$

Diagram 2

$$\begin{tikzcd}
\Diamond(\mathrm{cov}(\mathcal{N})) \arrow[r, dash] \arrow[dd, dash] & \Diamond(\mathrm{non}(\mathcal{M})) \arrow[r, dash] \arrow[d, dash] & \blacksquare \arrow[r, dash] \arrow[d, dash] & \blacksquare \arrow[dd, dash] \\
 & \Diamond(\mathfrak{b}) \arrow[r, dash] \arrow[d, dash] & \blacksquare \arrow[d, dash] & \\
\Diamond(\mathrm{add}(\mathcal{N})) \arrow[r, dash] & \Diamond(\mathrm{add}(\mathcal{M})) \arrow[r, dash] & \Diamond(\mathrm{cov}(\mathcal{M})) \arrow[r, dash] & \Diamond(\mathrm{non}(\mathcal{N}))
\end{tikzcd}$$

Diagram 3

2.3.2 σ-centered forcing

Secondly we will deal with σ-centered forcings.

Definition 6. The Hechler forcing notion is defined as follows:

$$\langle s, f\rangle \in \mathbb{D} \text{ if } s \in \omega^{<\omega},\ f \in \omega^{\omega} \text{ and } s \subset f.$$

It is ordered by

$$\langle s, f\rangle \leq \langle t, g\rangle \text{ if } s \supset t \text{ and } g \leq f.$$

It is clear that the following statement holds.

Proposition 2.3.13. Hechler forcing $\mathbb{D}$ adds a dominating real:

$$\text{There exists } f \in \omega^{\omega} \cap V^{\mathbb{D}} \text{ such that } \Vdash \text{“}g <^* f\text{” for all } g \in \omega^{\omega} \cap V.$$

Definition 7. The eventually different forcing notion is defined as follows:

$$\langle s, H\rangle \in \mathbb{E} \text{ if } s \in \omega^{<\omega} \text{ and } H \in [\omega^{\omega}]^{<\omega}$$

It is ordered by $\langle s, H\rangle \leq \langle s', H'\rangle$ if $s \supset s', H \supset H'$ and

$$\text{for all } f \in H' \text{ for all } j \in [|s'|, |s|)\ s(j) \neq f(j).$$

Proposition 2.3.14. Eventually different forcing adds an eventually different real:

$$\text{There exists } f \in \omega^{\omega} \cap V^{\mathbb{E}} \text{ such that } \Vdash \text{“}\forall^{\infty} n\ f(n) \neq g(n)\text{” for all } g \in \omega^{\omega} \cap V.$$

So there is $M \in \mathcal{M} \cap V^{\mathbb{E}}$ such that $2^{\omega} \cap V \subset M$.

Now we use these two forcing notions. They have the following property:

Definition 8. Let $\mathbb{P}$ be a forcing notion.

(1) Let $\mathcal{A} \subset \mathbb{P}$. Then $\mathcal{A}$ is centered if every finite subset of $\mathcal{A}$ has a lower bound.

(2) $\mathbb{P}$ is σ-centered if $\mathbb{P} = \bigcup_{n\in\omega} P_n$ where each P_n is centered.

σ-centered forcing has the following property:

Theorem 2.3.15. [3, p.321 Lemma 6.5.26, p.322 Theorem 6.5.29]
σ-centered forcing doesn't add random reals. More precisely, if a $\mathbb{P}$-name $\dot{x}$ for an element of 2^ω is given, then there is a null set $N \in V$ such that $\Vdash \dot{x} \in N$.

Proposition 2.3.16. (1) If a forcing notion $\mathbb{P}$ doesn't add Cohen reals and has c.c.c, then $V \models \neg\diamondsuit(\mathrm{add}(\mathcal{M})) \Rightarrow V^{\mathbb{P}} \models \neg\diamondsuit(\mathrm{non}(\mathcal{M}))$.

(2) If a forcing notion $\mathbb{P}$ doesn't add random reals and has c.c.c, then $V \models \neg\diamondsuit(\mathrm{add}(\mathcal{N})) \Rightarrow V^{\mathbb{P}} \models \neg\diamondsuit(\mathrm{non}(\mathcal{N}))$.

Proof. We show only the random case. Let $F : 2^{<\omega_1} \to \mathcal{N}$ be a Borel function in V. Suppose $\diamondsuit(\mathrm{non}(\mathcal{N}))$ holds in $V^{\mathbb{P}}$. Let $g : \omega_1 \to \mathbb{R}$ be a $\diamondsuit(\mathrm{non}(\mathcal{N}))$-sequence for F. Since $\mathbb{P}$ doesn't add random reals and has c.c.c, for each $\alpha < \omega_1$ there exists $N_\alpha \in \mathcal{N} \cap V$ such that $\Vdash g(\alpha) \in N_\alpha$. Define $g^* : \omega_1 \to \mathcal{N}$ by $g^*(\alpha) = N_\alpha$. Let $f : \omega_1 \to 2$ be given. Then $\Big(\exists p \in \mathbb{P}\ (p \Vdash F(f \restriction \alpha) \not\ni g(\alpha)\ )\Big)$ implies $F(f \restriction \alpha) \not\supset g^*(\alpha)$. So g^* is a $\diamondsuit(\mathrm{add}(\mathcal{N}))$-sequence for F.

□

Proposition 2.3.17. Suppose $\mathbb{P}$ is a σ-centered forcing notion. $V \models \neg\diamondsuit(\mathrm{add}(\mathcal{N})) \Rightarrow V^{\mathbb{P}} \models \neg\diamondsuit(\mathrm{non}(\mathcal{N}))$.

Proof. Follows from Theorem 2.3.15 and Proposition 2.3.16

□

To treat ω_1-stage iteration of $\mathbb{D}$ or $\mathbb{E}$, we use the following result:

Proposition 2.3.18. [1] An ω_1-stage finite support iteration of σ-centered forcing notions is σ-centered.

Theorem 2.3.19. If $V \models \mathrm{CH} + \neg\diamondsuit(\mathrm{add}(\mathcal{N}))$, then $V^{\mathbb{E}_{\omega_1}} \models \mathrm{CH} + \diamondsuit(\mathrm{non}(\mathcal{M})) + \diamondsuit(\mathrm{cov}(\mathcal{M})) + \neg\diamondsuit(\mathfrak{d}) + \neg\diamondsuit(\mathrm{non}(\mathcal{N}))$ (see Diagram 4).

By Theorem 2.2.2, Proposition 2.3.14 and Proposition 2.3.17, it is clear that $V \models \mathrm{CH} + \neg\diamondsuit(\mathrm{add}(\mathcal{M}))$ implies $V^{\mathbb{E}_{\omega_1}} \models \mathrm{CH} + \neg\diamondsuit(\mathrm{non}(\mathcal{N})) + \diamondsuit(\mathrm{non}(\mathcal{M})) + \diamondsuit(\mathrm{cov}(\mathcal{M}))$. To show $V^{\mathbb{E}_{\omega_1}} \models \neg\diamondsuit(\mathfrak{d})$, we use following Theorem:

Theorem 2.3.20. [3, p.367,Theorem 7.4.9] Neither $\mathbb{E}$ nor $\mathbb{E}_{\omega_1}$ add dominating reals.

Using this Theorem and Proposition 2.3.10, we have $V^{\mathbb{E}_{\omega_1}} \models \neg\diamondsuit(\mathfrak{d})$.

□

$$\begin{tikzcd}
\diamondsuit(\mathrm{cov}(\mathcal{N})) \arrow[r, dash] \arrow[dd, dash] & \diamondsuit(\mathrm{non}(\mathcal{M})) \arrow[r, dash] \arrow[d, dash] & \blacksquare \arrow[r, dash] \arrow[d, dash] & \blacksquare \arrow[dd, dash] \\
 & \diamondsuit(\mathfrak{b}) \arrow[r, dash] \arrow[d, dash] & \blacksquare \arrow[d, dash] & \\
\diamondsuit(\mathrm{add}(\mathcal{N})) \arrow[r, dash] & \diamondsuit(\mathrm{add}(\mathcal{M})) \arrow[r, dash] & \diamondsuit(\mathrm{cov}(\mathcal{M})) \arrow[r, dash] & \blacksquare
\end{tikzcd}$$

Diagram 4

Theorem 2.3.21. If $V \models \mathrm{CH}+\neg\diamondsuit(\mathrm{add}(\mathcal{N}))$, then $V^{\mathbb{D}_{\omega_1}} \models \mathrm{CH}+\neg\diamondsuit(\mathrm{non}(\mathcal{N}))+\diamondsuit(\mathrm{cof}(\mathcal{M}))$ (see Diagram 5).

By Theorem 2.2.2 and Proposition 2.3.17, $V^{\mathbb{D}_{\omega_1}} \models \diamondsuit(\mathfrak{d}) + \diamondsuit(\mathrm{non}(\mathcal{M})) + \neg\diamondsuit(\mathrm{non}(\mathcal{N}))$. To show $V^{\mathbb{D}_{\omega_1}} \models \diamondsuit(\mathrm{cof}(\mathcal{M}))$, we use the following Theorem which is analogous to $\mathrm{cof}(\mathcal{M}) = \max\{\mathfrak{d}, \mathrm{non}(\mathcal{M})\}$.

Theorem 2.3.22. If $\diamondsuit^*(\mathrm{non}(\mathcal{M}))$ and $\diamondsuit(\mathfrak{d})$, then $\diamondsuit(\mathrm{cof}(\mathcal{M}))$ holds. Similarly $\diamondsuit(\mathrm{non}(\mathcal{M}))$ and $\diamondsuit^*(\mathfrak{d})$, then $\diamondsuit(\mathrm{cof}(\mathcal{M}))$ holds.

Proof. We use the following statement:

Claim 2. [5] There are functions $\Phi : 2^\omega \times \omega^\omega \to \mathcal{M}$ and $\Psi : 2^\omega \times \mathcal{M} \to \omega^\omega$ such that for each $f \in 2^\omega, A \in \mathcal{M}, \Phi(f, \cdot)$ and $\Psi(\cdot, A)$ are Borel functions, and if $f \in \omega^\omega, A \in \mathcal{M}, x \in 2^\omega, x \notin A + 2^{<\omega}$, and $f \geq^* \Psi(x, A)$ then $A \subset \Phi(x, f)$. Here for $s \in 2^{<\omega}$, $f \in 2^\omega$, $f + s$ is in 2^ω such that

$$(f+s)(n) := \begin{cases} f(n) + s(n) \pmod 2 & \text{if } n \in |s|, \\ f(n) & \text{o.w.} \end{cases}$$

And $A + 2^{<\omega} := \{f + s : f \in A, s \in 2^{<\omega}\}$

We will show that $\diamondsuit^*(\mathrm{non}(\mathcal{M}))$ and $\diamondsuit(\mathfrak{d})$ implies $\diamondsuit(\mathrm{cof}(\mathcal{M}))$.

Let $F : 2^{<\omega_1} \to \mathcal{M}$ be a Borel function . Let $\{\sigma_n : n \in \omega\}$ be an enumeration of $2^{<\omega}$.

By assumption, there is a $\diamondsuit^*(\mathrm{non}(\mathcal{M}))$-sequence g for F^* where F^* is $F + 2^{<\omega}$, that is, $F^*(h) = \{f \in 2^\omega : \text{there exists } s \in 2^{<\omega} \text{ such that } f + s \in F(h)\}$. Then $\{\alpha \in \omega_1 : F^*(f \restriction \alpha) \not\ni g(\alpha)\}$ is club for any $f : \omega_1 \to 2$. Note that g is also a $\diamondsuit^*(\mathrm{non}(\mathcal{M}))$-sequence for F. Define a Borel function $G : 2^{<\omega_1} \to \omega^\omega$ by $G(f \restriction \alpha) := \Psi(g(\alpha), F(f \restriction \alpha))$. Let h be a $\diamondsuit(\mathfrak{d})$-sequence for G. Then $\{\alpha : g(\alpha) \notin F(f \restriction \alpha) \text{ and } G(f \restriction \alpha) \leq^* h(\alpha)\}$ is stationary for any $f : \omega_1 \to 2$. By definition of G and the Claim, $\{\alpha : F(f \restriction \alpha) \subset \Phi(g(\alpha), h(\alpha))\}$ is stationary. Hence $\Phi(g(\alpha), h(\alpha))$ witnesses a $\diamondsuit(\mathrm{cof}(\mathcal{M}))$-sequence for F.

Theorem 2.3.22 $\square$

So by Theorem 2.3.22, $V^{\mathbb{D}_{\omega_1}} \models \diamondsuit(\mathrm{cof}(\mathcal{M}))$.

Theorem 2.3.21 $\square$

$$\begin{tikzcd}
\diamondsuit(\mathrm{cov}(\mathcal{N})) \arrow[r, dash] \arrow[dd, dash] & \diamondsuit(\mathrm{non}(\mathcal{M})) \arrow[r, dash] \arrow[d, dash] & \diamondsuit(\mathrm{cof}(\mathcal{M})) \arrow[r, dash] \arrow[d, dash] & \blacksquare \arrow[dd, dash] \\
& \diamondsuit(\mathfrak{b}) \arrow[r, dash] \arrow[d, dash] & \diamondsuit(\mathfrak{d}) \arrow[d, dash] & \\
\diamondsuit(\mathrm{add}(\mathcal{N})) \arrow[r, dash] & \diamondsuit(\mathrm{add}(\mathcal{M})) \arrow[r, dash] & \diamondsuit(\mathrm{cov}(\mathcal{M})) \arrow[r, dash] & \blacksquare
\end{tikzcd}$$

Diagram 5

Question 3. (1) Does the conjunction of $\diamondsuit(\mathrm{non}(\mathcal{M}))$ and $\diamondsuit(\mathfrak{d})$ imply $\diamondsuit(\mathrm{cof}(\mathcal{M}))$?

(2) Does $\diamondsuit(\mathrm{add}(\mathcal{M}))$ imply the disjunction of $\diamondsuit(\mathrm{cov}(\mathcal{M}))$ and $\diamondsuit(\mathfrak{b})$?

(3) Are there models under CH such that

$$\begin{tikzcd}
\diamondsuit(\mathrm{cov}(\mathcal{N})) \arrow[r, dash] \arrow[dd, dash] & \diamondsuit(\mathrm{non}(\mathcal{M})) \arrow[r, dash] \arrow[d, dash] & \blacksquare \arrow[r, dash] \arrow[d, dash] & \blacksquare \arrow[dd, dash] \\
& \diamondsuit(\mathfrak{b}) \arrow[r, dash] \arrow[d, dash] & \diamondsuit(\mathfrak{d}) \arrow[d, dash] & \\
\diamondsuit(\mathrm{add}(\mathcal{N})) \arrow[r, dash] & \diamondsuit(\mathrm{add}(\mathcal{M})) \arrow[r, dash] & \diamondsuit(\mathrm{cov}(\mathcal{M})) \arrow[r, dash] & \blacksquare
\end{tikzcd}$$

holds?

By Theorem 2.3.22, we should add $\diamondsuit(\mathfrak{d})$ and $\diamondsuit(\mathrm{non}(\mathcal{M}))$ without $\diamondsuit^*(\mathfrak{d})$ nor $\diamondsuit^*(\mathrm{non}(\mathcal{M}))$. But ω_1-stage finite support iteration adds $\diamondsuit^*(\mathrm{non}(\mathcal{M}))$. Since ω_1-stage countable support iteration adds $\diamondsuit$, ω_1-stage countable support iteration is not suitable. The candidate is "mixed support iteration" or totally proper forcing or some other forcings.

2.3.3 The forcing $(\mathbb{B} * \dot{\mathbb{D}})_{\omega_1}$

Thirdly we will deal with c.c.c forcing notion which preserves $\neg\diamondsuit(\mathrm{cof}(\mathcal{N}))$.

Theorem 2.3.23. If $V \models \mathrm{CH}+\neg\diamondsuit(\mathrm{add}(\mathcal{N}))$, then $V^{(\mathbb{B}*\dot{\mathbb{D}})_{\omega_1}} \models \mathrm{CH}+\diamondsuit(\mathrm{cof}(\mathcal{M}))+\diamondsuit(\mathrm{non}(\mathcal{N}))+\neg\diamondsuit(\mathrm{cof}(\mathcal{N}))$ (see Diagram 6).

By Theorem 2.2.2, $V^{(\mathbb{B}*\dot{\mathbb{D}})_{\omega_1}} \models \diamondsuit(\mathrm{cof}(\mathcal{M}))+\diamondsuit(\mathrm{non}(\mathcal{N}))$.

Proposition 2.3.24. If $V \models \neg\diamondsuit(\mathrm{add}(\mathcal{N}))$, then $V^{(\mathbb{B}*\dot{\mathbb{D}})_{\omega_1}} \models \neg\diamondsuit(\mathrm{cof}(\mathcal{N}))$ where $(\mathbb{B}*\dot{\mathbb{D}})_{\omega_1}$ is finite support iteration .

To show this theorem we use the following lemma.

Lemma 2.3.25. [3, p.317 Theorem 6.5.14 - Lemma 6.5.18]
$\Vdash_{(\mathbb{B}*\dot{\mathbb{D}})_{\omega_1}} \bigcup \mathcal{N}\cap V \notin \mathcal{N}$.

<u>Lemma 2.3.25⇒ Proposition 2.3.24</u>
Suppose $V^{(\mathbb{B}*\dot{\mathbb{D}})_{\omega_1}} \models \diamondsuit(\mathrm{cof}(\mathcal{N}))$. Let $F : 2^{<\omega_1} \to \mathcal{N}$ be a Borel function in V . Let $g : \omega_1 \to \mathcal{N}$ be a $\diamondsuit(\mathrm{cof}(\mathcal{N}))$-sequence in $V^{(\mathbb{B}*\dot{\mathbb{D}})_{\omega_1}}$. By Lemma 2.3.25, $\Vdash_{(\mathbb{B}*\dot{\mathbb{D}})_{\omega_1}} \bigcup \mathcal{N} \cap V \notin \mathcal{N}$. So for each $\alpha \in \omega_1$ there exists $N_\alpha \in \mathcal{N} \cap V$ such that $\Vdash N_\alpha \not\subset g(\alpha)$. Then define $g^* : \omega_1 \to \mathcal{N}$ by $g^*(\alpha) = N_\alpha$. It is clear that $g^* \in V$.

Claim 3. g^* is a $\diamondsuit(\mathrm{add}(\mathcal{N}))$-sequence for F.

Let $N \in \mathcal{N} \cap V$. If $\Vdash$ "$N \subset g(\alpha)$", then $N \not\supset g^*(\alpha)$ by $\Vdash g^*(\alpha) \not\subset g(\alpha)$. So for each $f : \omega_1 \to 2$ in V, $\{\alpha \in \omega_1 :$ there exists $p \in \mathbb{P}$ such that $p \Vdash F(f \restriction \alpha) \subset g(\alpha)\} \subset \{\alpha \in \omega_1 : F(f \restriction \alpha) \not\supset g^*(\alpha)\}$. Hence g^* is a $\diamondsuit(\mathrm{add}(\mathcal{N}))$-sequence for F.

Claim ■

Hence $V \models \diamondsuit(\mathrm{add}(\mathcal{N}))$.

Lemma 2.3.25⇒ Proposition 2.3.24 □ Theorem 2.3.23□

$$\begin{tikzcd}
\diamondsuit(\mathrm{cov}(\mathcal{N})) \arrow[r, dash] \arrow[dd, dash] & \diamondsuit(\mathrm{non}(\mathcal{M})) \arrow[r, dash] \arrow[d, dash] & \diamondsuit(\mathrm{cof}(\mathcal{M})) \arrow[r, dash] \arrow[d, dash] & \blacksquare \arrow[dd, dash] \\
 & \diamondsuit(\mathfrak{b}) \arrow[r, dash] \arrow[d, dash] & \diamondsuit(\mathfrak{d}) \arrow[d, dash] & \\
\diamondsuit(\mathrm{add}(\mathcal{N})) \arrow[r, dash] & \diamondsuit(\mathrm{add}(\mathcal{M})) \arrow[r, dash] & \diamondsuit(\mathrm{cov}(\mathcal{M})) \arrow[r, dash] & \diamondsuit(\mathrm{non}(\mathcal{N}))
\end{tikzcd}$$

Diagram 6

2.3.4 Amoeba forcing

Finally we will deal with Amoeba forcing.

Definition 9. (Amoeba forcing) [34] The Amoeba forcing notion $\mathbb{A}$ is defined as follows:

$$(U, \varepsilon) \in \mathbb{A} \text{ if } U \subset 2^\omega, \text{ open and } 0 < \varepsilon \leq 1 \ \mu(U) < \epsilon.$$

For (U, ε), $(V, \delta) \in \mathbb{A}$ they are ordered by

$$(U, \varepsilon) \leq (V, \delta) \text{ if } U \supset V \text{ and } \varepsilon \leq \delta.$$

Lemma 2.3.26. [3, p.106 Lemma 3.1.12] $\mathbb{A}$ is σ-linked, that is, $\mathbb{A} = \bigcup_{n \in \omega} A_n$ where A_n consists of pairwise compatible elements (we will say A_n is linked).

$\mathbb{A}$ has the following property:

Theorem 2.3.27. [35] $V^{\mathbb{A}} \models$ "$\mu(Ra(V)) = 1$" where $Ra(V)$ is the set of random reals over V. So $\Vdash \bigcup \mathcal{N} \cap V \in \mathcal{N}$.

Since σ-linked forcing notion has c.c.c, $\mathbb{A}_{\omega_1}$ preserves $\neg\diamondsuit$.

Proposition 2.3.28. [22, Exercise (H.29) p.248]
Let $\mathbb{P}$ be a forcing notion with c.c.c, then $V \models \neg\diamondsuit$ implies $V^{\mathbb{P}} \models \neg\diamondsuit$.

Theorem 2.3.29. If $V \models \neg\diamondsuit$, then $V^{\mathbb{A}_{\omega_1}} \models \diamondsuit(\mathrm{cof}(\mathcal{N})) + \neg\diamondsuit$ (see Diagram 7).

Proof. By Theorem 2.2.2 and Proposition 2.3.28 this statement holds.

□

By Theorem 2.3.29 it is relatively consistent with ZFC and CH that Diagram 7 holds where the black square means $\neg\diamondsuit$.

$$\begin{tikzcd}
& & & & \blacksquare . \\
\diamondsuit(\mathrm{cov}(\mathcal{N})) \arrow[r, dash] \arrow[dd, dash] & \diamondsuit(\mathrm{non}(\mathcal{M})) \arrow[r, dash] \arrow[d, dash] & \diamondsuit(\mathrm{cof}(\mathcal{M})) \arrow[r, dash] \arrow[d, dash] & \diamondsuit(\mathrm{cof}(\mathcal{N})) \arrow[ur, dash] \arrow[dd, dash] & \\
& \diamondsuit(\mathfrak{b}) \arrow[r, dash] \arrow[d, dash] & \diamondsuit(\mathfrak{d}) \arrow[d, dash] & & \\
\diamondsuit(\mathrm{add}(\mathcal{N})) \arrow[r, dash] & \diamondsuit(\mathrm{add}(\mathcal{M})) \arrow[r, dash] & \diamondsuit(\mathrm{cov}(\mathcal{M})) \arrow[r, dash] & \diamondsuit(\mathrm{non}(\mathcal{N})) &
\end{tikzcd}$$

Diagram 7

So we proved the **Main Theorem**.

□

2.4 ω_2-stage finite support iteration and parametrized $\diamondsuit$ principles

In [27] by using ω_1-stage finite support iteration several models which satisfy CH and some $\diamondsuit(A, B, E)$ while others fail are constructed. For countable support iteration, there is a general theorem to construct $\diamondsuit(A, B, E)$.

Theorem 2.4.1. *[31] Suppose that $\langle \mathcal{Q}_\alpha : \alpha < \omega_2 \rangle$ is a sequence of Borel partial orders such that for each $\alpha < \omega_2$ $\mathcal{Q}_\alpha$ is equivalent to $\wp(2)^+ \times \mathcal{Q}_\alpha$ as a forcing notion and let $\mathcal{P}_{\omega_2}$ be the countable support iteration of this sequence. If $\mathcal{P}_{\omega_2}$ is proper and (A, B, E) is a Borel invariant then $\mathcal{P}_{\omega_2}$ forces $\langle A, B, E \rangle \leq \omega_1$ iff $\mathcal{P}_{\omega_2}$ forces $\diamondsuit(A, B, E)$.*

This result is best possible because the following proposition holds.

Proposition 2.4.2. *Let (A, B, E) be a Borel invariant. If $\diamondsuit(A, B, E)$ holds, then $\langle A, B, E \rangle \leq \omega_1$.*

In this paper we shall prove the consistency of $\diamondsuit(\mathfrak{x}) + \mathfrak{y} = \omega_2$ for several pairs $(\mathfrak{x}, \mathfrak{y})$ of cardinal invariants of the continuum. As mentioned above (Theorem2.4.1) this has been achieved before by Moore, Hrušák and Džamonja in [31]. They used countable support iteration to show $\diamondsuit(\mathfrak{x})+\mathfrak{y} = \omega_2$. But our approach is completely different from the methods of Moore, Hrušák and Džamonja. We shall use finite support iteration of Suslin c.c.c forcing notions to prove the consistency of $\diamondsuit(\mathfrak{x}) + \mathfrak{y} = \omega_2$.

And our results are more general. We can obtain the consistency of $\diamondsuit(\mathfrak{x})+\mathfrak{y} = \kappa$ not just $\diamondsuit(\mathfrak{x}) + \mathfrak{y} = \omega_2$.

Along the way, new preservation results for finite support iteration are established. These are interesting in their own right.

The present paper is organized as follows. Section 2 shows some properties of Suslin forcing. Section 3 presents several models satisfying parametrized diamond principles by using ω_2-stage finite support iteration of Suslin forcing notions.

2.4.1 Suslin c.c.c forcing and complete embedding

In this section we will study some properties of a family of c.c.c forcing notions which have a nice definition.

Definition 10. *[3, p.168] A forcing notion $\mathbb{P} = \langle \mathbb{P}, \leq_{\mathbb{P}} \rangle$ has a Suslin definition if $\mathbb{P} \subset \omega^\omega$, $\leq_{\mathbb{P}} \subset \omega^\omega \times \omega^\omega$ and $\perp_{\mathbb{P}} \subset \omega^\omega \times \omega^\omega$ are Σ^1_1.*

$\mathbb{P}$ is Suslin if $\mathbb{P}$ is c.c.c and has a Suslin definition.

Definition 11. *[3, p.168] Let $M \models ZFC^*$. A Suslin forcing $\mathbb{P}$ is in M if all the parameters used in the definitions of $\mathbb{P}$, $\leq_{\mathbb{P}}$ and $\perp_{\mathbb{P}}$ are in M.*

We will interpret Suslin forcing notion in forcing extensions using its code rather than taking the ground model forcing notion.

Definition 12. *Let $\mathbb{A}$ and $\mathbb{B}$ be forcing notions. Then $i : \mathbb{A} \to \mathbb{B}$ is a complete embedding if*

(1) for all $a, a' \in \mathbb{A}$ if $a \leq a'$, then $i(a) \leq i(a')$,

(2) for all a_1, $a_2 \in \mathbb{A}$ $a_1 \perp a_2$ if and only if $i(a_1) \perp i(a_2)$ and

(3) for all $\mathcal{A} \subset \mathbb{P}$ if $\mathcal{A}$ is a maximal antichain in $\mathbb{A}$ then $i[\mathcal{A}]$ is a maximal antichain in $\mathbb{B}$.

If there is a complete embedding from $\mathbb{A}$ to $\mathbb{B}$ then we write $\mathbb{A} \lessdot \mathbb{B}$.

Lemma 2.4.3. *Assume $\mathbb{A} \lessdot \mathbb{B}$ and $\mathcal{P}$ is a Suslin forcing notion. Then $\mathbb{A} * \dot{\mathcal{P}} \lessdot \mathbb{B} * \dot{\mathcal{P}}$ where $\dot{\mathcal{P}}$ are names for interpretation of the code for the Suslin forcing notion in each model.*

Proof of Lemma. Let $i : \mathbb{A} \to \mathbb{B}$ be a complete embedding. Define $\hat{i} : \mathbb{A} * \dot{\mathcal{P}} \to \mathbb{B} * \dot{\mathcal{P}}$ by $\hat{i}(\langle a, \dot{x} \rangle) = \langle i(a), i_*(\dot{x}) \rangle$ where i_* is the class map from $\mathbb{A}$-names to $\mathbb{B}$-names induced by i (see [22, p.222]). We will show if $\mathcal{A} \subset \mathbb{A} * \dot{\mathcal{P}}$ is a maximal antichain, then $\hat{i}[\mathcal{A}]$ is also a maximal antichain. It is clear $\hat{i}[\mathcal{A}]$ is an antichain. Let $\mathcal{A}$ be a maximal antichain of $\mathbb{A} * \dot{\mathcal{P}}$ and put $\mathcal{A} = \{\langle a_\alpha, \dot{p}_\alpha \rangle : \alpha < \kappa\}$. Assume there exists $\langle b, \dot{p} \rangle \in \mathbb{B} * \dot{\mathcal{P}}$ such that $\langle b, \dot{p} \rangle$ and $\hat{i}(\langle a_\alpha, \dot{p}_\alpha \rangle)$ are incompatible for all $\alpha < \kappa$. Let G be a $(\mathbb{B}, V)$-generic such that $b \in G$ and let $H = i^{-1}[G]$. Let $\mathcal{A}' = \{\dot{p}_\alpha[H] : i(a_\alpha) \in G\} \in V[H]$.

Subclaim 1. $V[H] \models$ "$\mathcal{A}'$ *is a maximal antichain*".
Proof of subclaim. Firstly we shall show $\mathcal{A}'$ is an antichain. Suppose $\dot{p}_\alpha[H]$, $\dot{p}_\beta[H] \in \mathcal{A}'$. Since a_α, $a_\beta \in H$, a_α and a_β are compatible. Since $\langle a_\alpha, \dot{p}_\alpha \rangle$ and $\langle a_\beta, \dot{p}_\beta \rangle$ are incompatible, for all $r \leq a_\alpha, a_\beta$ there exists $s \leq r$ such that $s \Vdash$ "$\dot{p}_\alpha$ and $\dot{p}_\beta$are incompatible". So $\dot{p}_\alpha[H]$ and $\dot{p}_\beta[H]$ are incompatible. Hence $\mathcal{A}'$ is an antichain.

From now on we shall show maximality of $\mathcal{A}'$. Assume to the contrary, there exists $p \in \mathcal{P}$ such that p and $\dot{p}_\alpha[H]$ are incompatible for any $\dot{p}_\alpha[H] \in \mathcal{A}'$. So there exists $a \in H$ and an $\mathbb{A}$-name $\dot{\mathcal{P}}$ for p such that $a \Vdash \forall \alpha < \kappa (a_\alpha \in \dot{H} \rightarrow$ $\dot{p}$ and $\dot{p}_\alpha$ are compatible). Hence $\langle a, \dot{p} \rangle$ and $\langle a_\alpha, \dot{p}_\alpha \rangle$ are incompatible. But it contradicts the maximality of $\mathcal{A}$.

subclaim ■

$V[H] \models$ "$\mathcal{A}'$ is a maximal antichain in $\mathcal{P}$" and "$\mathcal{A}'$ is a maximal antichain in $\mathcal{P}$" is a Π^1_1 statement with parameter $\mathcal{A}'$, $\mathcal{P}$, $\leq_\mathcal{P}$ and $\perp_\mathcal{P}$. Hence by Π^1_1-absoluteness $V[G] \models \mathcal{A}'$ is a maximal antichain in $\mathcal{P}$. But this is a contradiction to the fact $V[G] \models$ "$\dot{p}[G] \perp i_*(\dot{p}_\alpha)[G]$" for $i(a_\alpha) \in G$.

□

Theorem 2.4.4. *Let* $\langle Q_\alpha : \alpha < \kappa \rangle$ *be a sequence of Suslin forcing notions. Let* $\mathbb{P}_\kappa$ *be the limit of the finite support iteration of* $\langle \mathbb{P}_\alpha, \dot{Q}_\alpha : \alpha < \kappa \rangle$. *Then* $\mathbb{A} \lessdot \mathbb{B}$ *implies* $\mathbb{A} * \dot{\mathbb{P}}_\kappa \lessdot \mathbb{B} * \dot{\mathbb{P}}_\kappa$
Proof. By induction on κ. Limit stage is clear. Successor stage follows from above Lemma.

□

Corollary 2.4.5. *Let* $\langle Q_\alpha : \alpha < \kappa \rangle$ *be a sequence of Suslin forcing notions. Let* $I \subset \kappa$. *Then* $\mathbb{P}_I \lessdot \mathbb{P}_\kappa$ *where* $\mathbb{P}_I$ *is the limit of the iteration of* $\langle \mathbb{P}^\alpha_I, \dot{R}_\alpha : \alpha < \kappa \rangle$ *where* $\Vdash_{\mathbb{P}^\alpha_I} \dot{R}_\alpha = \begin{cases} \dot{Q}_\alpha & \alpha \in I \\ \{1\} & \textit{otherwise.} \end{cases}$

□

2.4.2 Construction of Parametrized $\diamondsuit$ principles

We shall construct several models by finite support iteration of Suslin forcing notions.

Definition 13. *(1) The Hechler forcing notion is defined as follows:*

$$\langle s, f \rangle \in \mathbb{D} \text{ if } s \in \omega^{<\omega}, f \in \omega^\omega \text{ and } s \subset f.$$

It is ordered by

$$\langle s, f\rangle \leq \langle t, g\rangle \text{ if } s \supset t \text{ and } g \leq f.$$

(2) *The eventually different forcing notion is defined as follows:*

$$\langle s, H\rangle \in \mathbb{E} \text{ if } s \in \omega^{<\omega} \text{ and } H \in [\omega^\omega]^{<\omega}.$$

It is ordered by $\langle s, H\rangle \leq \langle t, G\rangle$ *if* $s \supset t$, $H \supset G$ *and*

$$\text{for all } g \in G \text{ for all } j \in [|t|, |s|)\, s(j) \neq g(j).$$

(3) *Let* $\textbf{Borel}(2^\omega)$ *be the smallest* σ*-algebra containing all open subsets of* 2^ω. *Let* μ *be the standard product measure on* 2^ω *and let* $\mathcal{N} = \{A \in \textbf{Borel}(2^\omega) : \mu(A) = 0\}$. *For* $A, B \in \textbf{Borel}(2^\omega)$ *let* $A \cong_{\mathcal{N}} B$ *if* $A \triangle B \in \mathcal{N}$. *Let* $[A]_{\mathcal{N}}$ *be the equivalence class of the set* A *with respect to the equivalence relation* $\cong_{\mathcal{N}}$.

Define

$$\mathbb{B} = \{[A]_{\mathcal{N}} : A \in \textbf{Borel}(2^\omega)\}.$$

It is ordered by $[A]_{\mathcal{N}} \leq [B]_{\mathcal{N}}$ *if* $A \setminus B \in \mathcal{N}$.

Theorem 2.4.6. *Let* κ *be an ordinal with* $cf(\kappa) > \omega_1$. *Let* $\mathbb{D}_\kappa$, $\mathbb{E}_\kappa$, $\mathbb{B}_\kappa$ *and* $(\mathbb{B} * \dot{\mathbb{D}})_\kappa$ *be the* κ*-stage finite support iteration of* $\mathbb{D}$, $\mathbb{E}$, $\mathbb{B}$ *and* $\mathbb{B} * \dot{\mathbb{D}}$ *respectively. Then the following statements hold:*

(1) $V^{\mathbb{D}_\kappa} \models \diamondsuit(cov(\mathcal{N}))$.

(2) $V^{\mathbb{E}_\kappa} \models \diamondsuit(cov(\mathcal{N}))$ *and* $\diamondsuit(\mathfrak{b})$.

(3) $V^{\mathbb{B}_\kappa} \models \diamondsuit(\mathfrak{b})$.

(4) $V^{(\mathbb{B} * \dot{\mathbb{D}})_\kappa} \models \diamondsuit(add(\mathcal{N}))$.

Proof. (1) Let Π be a partition of ω into finite intervals I_n with $|I_n| = n+1$ for $n \in \omega$. Define a relation $=^\infty_\Pi$ so that $x =^\infty_\Pi y$ if there exist infinitely many $n \in \omega$ such that $x \restriction I_n = y \restriction I_n$. We will show $V^{\mathbb{D}_\kappa} \models \diamondsuit(2^\omega, =^\infty_\Pi)$. Let $\dot{F}$ be a $\mathbb{D}_\kappa$-name such that $\Vdash_{\mathbb{D}_\kappa} \dot{F} : 2^{<\omega_1} \to 2^\omega$. Since $\mathbb{D}_\kappa$ has the c.c.c, a real $\dot{r}_\alpha$ coding the Borel function $\dot{F} \restriction 2^\alpha$ appears at an intermediate stage. By $cf(\kappa) > \omega_1$ we can assume $\dot{F}$ is a $\mathbb{D}_\beta$-name for some $\beta < \kappa$. Furthermore we can assume $\dot{F}$ is a Borel function in ground model. Let F be a Borel function in ground model. Let $\dot{f}$ be a $\mathbb{D}_\kappa$-name such that $\Vdash_{\mathbb{D}_\kappa} \dot{f} : \omega_1 \to 2$. Then the following claim holds:

Claim 4. *Define* $C_{\dot{f}} \subset \omega_1$ *by*

$$C_{\dot{f}} = \{\alpha < \omega_1 : \dot{f} \restriction \alpha \text{ is } \mathbb{D}_{\alpha \cup [\omega_1, \kappa)}\text{-name}\}.$$

Then $C_{\dot{f}}$ *contains a club.*

Remark 2. *More precisely we should write*

$$C_{\dot{f}} = \{\alpha < \omega_1 : \textit{there exists } \mathbb{D}_{\alpha\cup[\omega_1,\kappa)}\textit{-name } \dot{x}_\alpha \textit{ such that } \Vdash_{\mathbb{D}_\kappa} \dot{f}\restriction\alpha = i_*(\dot{x}_\alpha)\}$$

where i_* is a class function from $\mathbb{D}_{\alpha\cup[\omega_1,\kappa)}$-names to $\mathbb{D}_\kappa$-names induced by the complete embedding $i : \mathbb{D}_{\alpha\cup[\omega_1,\kappa)} \lessdot \mathbb{D}_\kappa$. But for convenience we will think of a $\mathbb{D}_\kappa$-name $\dot{x}$ as $\mathbb{D}_I$-name if there exists a $\mathbb{D}_I$-name $\dot{y}$ such that $\Vdash_{\mathbb{D}_\kappa} \dot{x} = i_{I*}(\dot{y})$ where i_I is a complete embedding from $\mathbb{D}_I$ to $\mathbb{D}_\kappa$.

For $\alpha \in C_{\dot{f}}$ let $\dot{x}_\alpha$ be a $\mathbb{D}_{\alpha\cup[\omega_1,\kappa)}$-name such that $\Vdash_{\mathbb{D}_{\alpha\cup[\omega_1,\kappa)}} F(\dot{f}\restriction\alpha) = \dot{x}_\alpha$. Let $\dot{c}_\alpha$ be a $\mathbb{D}_{\omega_1}$-name such that for all $\dot{x} \in 2^\omega \cap V^{\mathbb{D}_\alpha} \Vdash_{\mathbb{D}_{\omega_1}} \exists^\infty n\,(\dot{c}_\alpha\restriction I_n = \dot{x}\restriction I_n)$. We can obtain such $\dot{c}_\alpha$. For example put $\dot{c}_\alpha$ a $\mathbb{D}_{\omega_1}$-name for a Cohen real over $V^{\mathbb{D}_\alpha}$.

We shall show $\Vdash_{\mathbb{D}_\kappa} \exists^\infty n\,(\dot{c}_\alpha\restriction I_n = \dot{x}_\alpha\restriction I_n)$. To prove this we will work in $V^{\mathbb{D}_\alpha}$ and show the following lemma.

Lemma 2.4.7. *Suppose γ is an ordinal and $\mathbb{P}$ is a forcing notion which has a $\mathbb{P}$-name $\dot{c}$ such that for all $x \in 2^\omega \cap V \Vdash_{\mathbb{P}} \exists^\infty n\,(x\restriction I_n = \dot{c}\restriction I_n)$. Let $\dot{x}$ be a $\mathbb{D}_\gamma$-name such that $\Vdash \dot{x} \in 2^\omega$. Then $\Vdash_{\mathbb{P}*\dot{\mathbb{D}}_\gamma} \exists^\infty n\,(\dot{c}\restriction I_n = \dot{x}\restriction I_n)$. Here precisely we should write $\Vdash_{\mathbb{P}*\dot{\mathbb{D}}_\gamma} \exists^\infty n\,(\dot{c}\restriction I_n = i_*(\dot{x})\restriction I_n)$ where i_* is a canonical map from $\mathbb{D}_\gamma$-names to $\mathbb{P}*\dot{\mathbb{D}}_\gamma$-names induced by the complete embedding $i : \mathbb{D}_\gamma \to \mathbb{P}*\dot{\mathbb{D}}_\gamma$.*
Proof. We proceed by induction on γ.
First step
Let $\dot{x}$ be a $\mathbb{D}$-name such that $\Vdash_{\mathbb{D}} \dot{x} \in 2^\omega$. Let $\dot{c}$ be a $\mathbb{P}$-name such that $\Vdash_{\mathbb{P}}$ "$\exists^\infty n \in \omega\,(\dot{c}\restriction I_n = x\restriction I_n)$" for all $x \in V \cap 2^\omega$. Let $(p_0, \dot{q}_0) \in \mathbb{P}*\dot{\mathbb{D}}$ and $m \in \omega$.

It suffices to show there exist $(p_1, \dot{q}_1) \leq_{\mathbb{P}*\dot{\mathbb{D}}} (p_0, \dot{q}_0)$ and $n \geq m$ such that $(p_1, \dot{q}_1) \Vdash_{\mathbb{P}*\dot{\mathbb{D}}} \dot{x}\restriction I_n = \dot{c}\restriction I_n$.

Without loss of generality we can assume $p_0 \Vdash \dot{q}_0 = \langle s, \dot{g}\rangle$ for some $s \in \omega^{<\omega}$. Let $x_s \in V \cap 2^\omega$ such that

$$\forall j \in \omega \forall g' \in \omega^\omega\,(g' \supset s \to \neg\langle s, g'\rangle \Vdash_{\mathbb{D}} \dot{x}\restriction I_j \neq x_s\restriction I_j)\,.$$

Let $r \leq p_0$ such that $r \Vdash_{\mathbb{P}} x_s\restriction I_n = \dot{c}\restriction I_n$ for some $n \geq m$. Then define $\langle r_k : k \in \omega\rangle$ a decreasing sequence of $\mathbb{P}$ and $g^* \in 2^\omega \cap V$ such that $r_0 \leq_{\mathbb{P}} r$ and $r_k \Vdash_{\mathbb{P}} \dot{g}\restriction(|s|+k) = g^*\restriction(|s|+k)$.

By definition of x_s there is $\langle t, h\rangle \leq_{\mathbb{D}} \langle s, g^*\rangle$ such that $\langle t, h\rangle \Vdash_{\mathbb{D}} x_s\restriction I_n = \dot{x}\restriction I_n$. Since $\langle t, h\rangle \leq_{\mathbb{D}} \langle s, g^*\rangle$, for all $l \in [|s|, |t|)$ $t(l) \geq g^*(l)$. Since $r_{|t|} \Vdash_{\mathbb{P}} \forall i \in |t|\,(\dot{g}(i) = g^*(i) \leq t(i))$, $r_{|t|} \Vdash_{\mathbb{P}} \langle t, h\rangle$ and $\langle s, \dot{g}\rangle$ are compatible. Put $p_1 = r_{|t|}$ and put a $\mathbb{P}$-name $\dot{q}_1$ so that $p_1 \Vdash_{\mathbb{P}} \dot{q}_1 \leq_{\mathbb{D}} \langle s, \dot{g}\rangle, \langle t, h\rangle$. Then $(p_1, \dot{q}_1) \leq_{\mathbb{P}*\dot{\mathbb{D}}} (p_0, \dot{q}_0)$ and $p_1 \Vdash_{\mathbb{P}} x_s\restriction I_n = \dot{c}\restriction I_n$ by $p_1 \leq_{\mathbb{P}} r$ and $p_1 \Vdash_{\mathbb{P}}$ "$\dot{q}_1 \Vdash_{\mathbb{D}} x_s\restriction I_n = \dot{x}\restriction I_n$" by $p_1 \Vdash_{\mathbb{P}} \dot{q}_1 \leq_{\mathbb{D}} \langle t, h\rangle$. Therefore $(p_1, \dot{q}_1) \Vdash_{\mathbb{P}*\dot{\mathbb{D}}} \dot{x}\restriction I_n = x_s\restriction I_n = \dot{c}\restriction I_n$.
Successor step:
Suppose the lemma holds for γ. Let $\dot{x}$ be a $\mathbb{D}_{\gamma+1}$-name such that $\Vdash_{\mathbb{D}_{\gamma+1}} \dot{x} \in 2^\omega$. Let $(p_0, \dot{q}_0) \in \mathbb{P}*\dot{\mathbb{D}}_{\gamma+1}$ and $m \in \omega$. Without loss of generality we can assume $(p_0, \dot{q}_0\restriction\gamma) \Vdash_{\mathbb{P}*\dot{\mathbb{D}}_\gamma} \dot{q}_0(\gamma) = \langle \check{s}, \dot{g}\rangle$ for some $s \in \omega^{<\omega}$.

Let $\dot{x}_s$ be a $\mathbb{D}_\gamma$-name such that

$$\Vdash_{\mathbb{D}_\gamma} \forall j \in \omega \forall \dot{g}' \in \dot{\omega^\omega}\,(\dot{g}' \supset \check{s} \to \neg\langle \check{s}, \dot{g}'\rangle \Vdash_{\mathbb{D}} \dot{x}_s\restriction I_j \neq \dot{x}\restriction I_j)\,.$$

By induction hypothesis there is $(p', \dot{q}') \in \mathbb{P} * \dot{\mathbb{D}}_\gamma$ and $n \geq m$ such that $(p', \dot{q}') \leq_{\mathbb{P}*\dot{\mathbb{D}}_\gamma}$ $(p_0, \dot{q}_0 \restriction \gamma)$ and $(p', \dot{q}') \Vdash_{\mathbb{P}*\dot{\mathbb{D}_\gamma}} \dot{x}_s \restriction I_n = \dot{c} \restriction I_n$.

Since $\mathbb{D}_\gamma \lessdot \mathbb{P} * \dot{\mathbb{D}}_\gamma$, there is a $\mathbb{D}_\gamma$-name $\dot{Q}$ for a partial order such that $\mathbb{P} * \dot{\mathbb{D}}_\gamma \cong \mathbb{D}_\gamma * \dot{Q}$. Let q^* be the projection of $(p', \dot{q}')$ to $\mathbb{D}_\gamma$.

Define $\mathbb{D}_\gamma$-names $\dot{g}^*$ and $\langle \dot{r}_k : k \in \omega \rangle$ such that

(i) $\Vdash_{\mathbb{D}_\gamma} \dot{g}^* \in \omega^\omega$ and $\dot{r}_k \in \dot{Q}$ for $k \in \omega$,

(ii) $(q^*, \dot{r}_0) \leq (p', \dot{q}')$,

(iii) $\Vdash_{\mathbb{D}_\gamma} \dot{r}_{k+1} \leq_{\dot{Q}} \dot{r}_k$ for $k \in \omega$ and

(vi) $\Vdash_{\mathbb{D}_\gamma}$ "$\dot{r}_k \Vdash_{\dot{Q}} \dot{g}(k) = \dot{g}^*(k)$".

Let $q_1^* \leq_{\mathbb{D}_\gamma} q^*$ such that there exists $t \in \omega^{<\omega}$ and $\mathbb{D}_\gamma$-name $\dot{h}$ for a function from ω to ω such that $q_1^* \Vdash_{\mathbb{D}_\gamma}$ "$\langle \check{t}, \dot{h} \rangle \leq_{\mathbb{D}} \langle s, \dot{g}^* \rangle$ and $\langle \check{t}, \dot{h} \rangle \Vdash_{\mathbb{D}} \dot{x} \restriction I_n = \dot{x}_s \restriction I_n$". Since $(q_1^*, \dot{r}_{|t|}) \Vdash \forall i \in |t| \left(\dot{g}(i) = \dot{g}^*(i) \leq \dot{h}(i) \right)$, $(q_1^*, \dot{r}_{|t|}) \Vdash \langle t, \dot{h} \rangle$ and $\langle s, \dot{g} \rangle$ are compatible.

Put $(p_1, \dot{q}_1) \in \mathbb{P} * \dot{\mathbb{D}}_{\gamma+1}$ so that $(p_1, \dot{q}_1 \restriction \gamma) = (q_1^*, \dot{r}_{|t|})$ and $(p_1, \dot{q}_1 \restriction \gamma) = (q^*, \dot{r}_{|t|}) \Vdash_{\mathbb{D}_\gamma * \dot{Q}} \dot{q}_1(\gamma) \leq_{\mathbb{D}} \langle t, \dot{h} \rangle, \langle s, \dot{g} \rangle$. Then $(p_1, \dot{q}_1 \restriction \gamma) \Vdash_{\mathbb{P}*\dot{\mathbb{D}_\gamma}}$ "$\dot{c} \restriction I_n = \dot{x}_s \restriction I_n$ and $\dot{q}_1(\gamma) \Vdash_{\dot{\mathbb{D}}} \dot{x}_s \restriction I_n = \dot{x} \restriction I_n$". Therefore $(p_1, \dot{q}_1) \Vdash_{\mathbb{P}*\dot{\mathbb{D}}_{\gamma+1}} \dot{c} \restriction I_n = \dot{x} \restriction I_n$.

Limit step:

Suppose γ is a limit ordinal and for $\beta < \gamma$ the lemma holds. Without loss of generality we can assume the cofinality of γ is ω. Let $\langle \gamma_i : i \in \omega \rangle$ be a strictly increasing sequence converging to γ. Let $(p_0, \dot{q}_0) \in \mathbb{P} * \dot{\mathbb{D}}_\gamma$, $m \in \omega$ and $\dot{x}$ be a $\mathbb{D}_\gamma$-name such that $\Vdash_{\mathbb{D}_\gamma} \dot{x} \in 2^\omega$. Suppose $(p_0, \dot{q}_0) \in \mathbb{P} * \dot{\mathbb{D}}_{\gamma_j}$.

In $V^{\mathbb{D}_{\gamma_j}}$ let $\langle r_k : k \in \omega \rangle$ be a decreasing sequence of $\mathbb{D}_{[\gamma_j, \gamma)}$ such that $r_k \Vdash_{\mathbb{D}_{[\gamma_j, \gamma)}}$ "$\dot{x} \restriction I_k = x_j \restriction I_k$" where $x_j \in 2^\omega \cap V^{\mathbb{D}_{\gamma_j}}$.

Back into V let $\dot{r}_k$ and $\dot{x}_j$ be $\mathbb{D}_{\gamma_j}$-names such that $\Vdash_{\mathbb{D}_{\gamma_j}}$ "$\langle \dot{r}_k : k \in \omega \rangle$ and $\dot{x}_j$ satisfies the above".

By induction hypothesis there exists $\langle p', \dot{q}' \rangle \leq_{\mathbb{P}*\dot{\mathbb{D}}_{\gamma_j}} \langle p_0, \dot{q}_0 \rangle$ and $n \geq m$ such that $(p', \dot{q}') \Vdash_{\mathbb{P}*\dot{\mathbb{D}}_{\gamma_j}} \dot{c} \restriction I_n = \dot{x}_j \restriction I_n$. Put $p_1 = p'$ and $\Vdash_{\mathbb{P}} \dot{q}_1 = \dot{q}' {}^\frown \dot{r}_n$

Then $\langle p_1, \dot{q}_1 \rangle \Vdash_{\mathbb{P}*\dot{\mathbb{D}}_\gamma} \dot{c} \restriction I_n = \dot{x}_j \restriction I_n = \dot{x} \restriction I_n$.

Lemma $\square$

Let $\dot{c}_\alpha$ be a $\mathbb{D}_{\omega_1}$-name such that $\Vdash_{\mathbb{D}_{\omega_1}} \exists^\infty n \left(\dot{c}_\alpha \restriction I_n = \dot{x} \restriction I_n \text{ for } \dot{x} \in 2^\omega \cap V^{\mathbb{D}_\alpha} \right)$. By the above lemma if $\alpha \in C_{\dot{f}}$, then $\Vdash_{\mathbb{D}_\kappa} \exists^\infty n \left(\dot{x}_\alpha \restriction I_n = F(\dot{f} \restriction \alpha) \restriction I_n = \dot{c}_\alpha \restriction I_n \right)$. Hence $\Vdash_{\mathbb{D}_\kappa} \langle \dot{c}_\alpha : \alpha \in \omega_1 \rangle$ is a $\diamondsuit(2^\omega, =^\infty_\Pi)$-sequence for F.

Let $\phi : 2^\omega \to \mathcal{N}$ be the function such that

$$\phi(x) = \{ y \in 2^\omega : \exists^\infty n \, (x \restriction I_n = y \restriction I_n) \}.$$

Then $\phi : 2^\omega \to \mathcal{N}$ and the identity function $id : 2^\omega \to 2^\omega$ witness $(2^\omega, \mathcal{N}, \in) \leq^B_T (2^\omega, =^\infty_\Pi)$ (see [5, Theorem 5.11]). So $V^{\mathbb{D}_\kappa} \models \diamondsuit(2^\omega, \mathcal{N}, \in)$.

(1) □

(2) $\Vdash_{\mathbb{E}_\kappa} \diamondsuit(\mathrm{cov}(\mathcal{N}))$ is similar to (1). We shall only show $\Vdash_{\mathbb{E}_\kappa} \diamondsuit(\omega^\omega, \not\geq^*)$. To prove this it suffices to show the following lemma:

Lemma 2.4.8. *Suppose γ is an ordinal and $\mathbb{P}$ is a forcing notion which has a $\mathbb{P}$-name $\dot{c}$ such that for all $x \in \omega^\omega \cap V \Vdash_{\mathbb{P}} \exists^\infty n\,(x(n) < \dot{c}(n))$. Let $\dot{x}$ be a $\mathbb{E}_\gamma$-name such that $\Vdash_{\mathbb{E}_\gamma} \dot{x} \in \omega^\omega$. Then $\Vdash_{\mathbb{P}*\dot{\mathbb{E}}_\gamma} \exists^\infty n\,(\dot{x}(n) < \dot{c}(n))$.*

Proof. We proceed by induction on γ. We shall only prove the successor step. The rest of the proof is similar to the proof of Lemma2.4.7.

Successor step:

Suppose the lemma holds for γ. Let $\dot{x}$ be a $\mathbb{E}_{\gamma+1}$-name such that $\Vdash_{\mathbb{E}_{\gamma+1}} \dot{x} \in \omega^\omega$. Let $(p_0, \dot{q}_0) \in \mathbb{P} * \dot{\mathbb{E}}_{\gamma+1}$ and $m \in \omega$. Without loss of generality we can assume $(p_0, \dot{q}_0 \restriction \gamma) \Vdash_{\mathbb{P}*\dot{\mathbb{E}}_\gamma}$ "$\dot{q}_0(\gamma) = \langle s, \dot{F}\rangle$ and $\dot{F} = \{\dot{f}_j : j < l\}$" for some $l \in \omega$ and $s \in \omega^{<\omega}$. Let $\dot{x}_{s,l}$ be a $\mathbb{E}_\gamma$-name such that

$$\Vdash_{\mathbb{E}_\gamma} \dot{x}_{s,l}(i) = \min\{j : \forall \dot{H} \subset \omega^\omega \text{ with } |\dot{H}| = l \left(\neg\langle s, \dot{H}\rangle \Vdash \dot{x}(i) > j\right)\}.$$

By induction hypothesis there is $(p', \dot{q}') \in \mathbb{P}*\dot{\mathbb{E}}_\gamma$ and $n \geq m$ such that $(p', \dot{q}') \leq_{\mathbb{P}*\dot{\mathbb{E}}_\gamma} (p_0, \dot{q}_0 \restriction \gamma)$ and $(p', \dot{q}') \Vdash_{\mathbb{P}*\dot{\mathbb{E}}_\gamma} \dot{c}(n) > \dot{x}_{s,l}(n)$. Since $\mathbb{E}_\gamma \lessdot \mathbb{P} * \dot{\mathbb{E}}_\gamma$, there is a $\mathbb{E}_\gamma$-name $\dot{Q}$ for a partial order such that $\mathbb{P} * \dot{\mathbb{E}}_\gamma \cong \mathbb{E}_\gamma * \dot{Q}$. Let q^* be a projection of $(p', \dot{q}')$ to $\mathbb{E}_\gamma$. Find $\mathbb{E}_\gamma$-names $\langle \dot{r}_k : k \in \omega\rangle$ and $\dot{F}^*$ such that

(i) $\Vdash_{\mathbb{E}_\gamma} \dot{F}^* = \{\dot{f}^*_j : j < l\} \subset \omega^\omega$ and $\dot{r}_k \in \dot{Q}$ for $k \in \omega$,

(ii) $(q^*, \dot{r}_0) \leq (p', \dot{q}')$,

(iii) $\Vdash_{\mathbb{E}_\gamma} \dot{r}_{k+1} \leq_{\dot{Q}} \dot{r}_k$ for $k \in \omega$ and,

(iv) $(q^*, \dot{r}_k) \Vdash_{\mathbb{E}_\gamma * \dot{Q}} \forall j < l \left(\dot{f}^*_j(k) = \dot{f}_j(k)\right)$ for $k \in \omega$.

Then there are $q_1^* \leq_{\mathbb{E}_\gamma} q^*$, $t \in \omega^{<\omega}$ and $\mathbb{E}_\gamma$-name $\dot{G}$ such that $q_1^* \Vdash_{\mathbb{E}_\gamma}$ "$\langle t, \dot{G}\rangle \leq_{\mathbb{E}} \langle s, \dot{F}^*\rangle$ and $\langle t, \dot{G}\rangle \Vdash_{\dot{\mathbb{E}}} \dot{x}(n) \leq \dot{x}_{s,l}(n)$".

Since $(q^*, \dot{r}_{|t|}) \Vdash_{\mathbb{E}_\gamma * \dot{Q}}$ "$\forall j < l\ \forall k < |t| \left(\dot{f}_j(k) = \dot{f}^*_j(k)\right)$" and $q_1^* \Vdash_{\mathbb{E}_\gamma} \forall j < n \forall k \in [|s|, |t|) \left(\dot{f}^*_j(k) \neq t(k)\right)$, $(q_1^*, \dot{r}_{|t|}) \Vdash_{\mathbb{E}_\gamma * \dot{Q}} \langle t, \dot{G}\rangle$ and $\langle s, \dot{F}\rangle$ are compatible.

Put $(p_1, \dot{q}_1) \in \mathbb{P} * \dot{\mathbb{E}}_{\gamma+1}$ so that $(p_1, \dot{q}_1 \restriction \gamma) = (q_1^*, \dot{r}_{|t|})$ and $(p_1, \dot{q}_1 \restriction \gamma) \Vdash_{\mathbb{P}*\dot{\mathbb{E}}_\gamma} \dot{q}_1(\gamma) \leq_{\mathbb{E}} \langle s, \dot{F}\rangle, \langle t, \dot{G}\rangle$. Then $(p_1, \dot{q}_1 \restriction \gamma) \Vdash_{\mathbb{P}*\dot{\mathbb{E}}_\gamma}$ "$\dot{x}_{s,l}(n) < \dot{c}(n)$ and $\dot{q}_1(\gamma) \Vdash_{\dot{\mathbb{E}}} \dot{x}(n) \leq \dot{x}_{s,l}(n)$". Therefore $(p_1, \dot{q}_1) \Vdash_{\mathbb{P}*\dot{\mathbb{E}}_{\gamma+1}} \dot{x}(n) < \dot{c}(n)$.

□

(3) To prove (3) it suffices to show the following lemma:

Lemma 2.4.9. *Suppose γ is an ordinal and $\mathbb{P}$ is a forcing notion which has a $\mathbb{P}$-name $\dot{c}$ such that for all $x \in \omega^\omega \cap V \Vdash_{\mathbb{P}} \exists^\infty n\,(x(n) < \dot{c}(n))$. Let $\dot{x}$ be a $\mathbb{B}_\gamma$-name such that $\Vdash \dot{x} \in \omega^\omega$. Then $\Vdash_{\mathbb{P}*\dot{\mathbb{B}}_\gamma} \exists^\infty n\,(\dot{x}(n) < \dot{c}(n))$.*

Proof of lemma. We proceed by induction on γ. We shall prove only the successor step.

Successor step:

Suppose for γ the lemma holds. Let μ be a measure on $\mathbb{B}$. Let $\dot{x}$ be a $\mathbb{B}_{\gamma+1}$-name such that $\Vdash_{\mathbb{B}_{\gamma+1}} \dot{x} \in \omega^\omega$. Let $\dot{x}^*$ be a $\mathbb{B}_\gamma$-name such that

$$\Vdash_{\mathbb{B}_\gamma} \mu([\![\dot{x}(k) \le \dot{x}^*(k)]\!]_{\dot{\mathbb{B}}}) \ge 1 - \frac{1}{2^k}.$$

Let $(p_0, \dot{q}_0) \in \mathbb{P} * \dot{\mathbb{B}}_{\gamma+1}$ and $m \in \omega$. Without loss of generality we can assume $(p_0, \dot{q}_0 \restriction \gamma) \Vdash_{\mathbb{P}*\dot{\mathbb{B}}_\gamma} \mu(\dot{q}_0(\gamma)) \ge \frac{1}{2^l}$. By induction hypothesis there is $(p', \dot{q}') \in \mathbb{P} * \dot{\mathbb{B}}_\gamma$ such that $(p', \dot{q}') \le_{\mathbb{P}*\dot{\mathbb{B}}_\gamma} (p, \dot{q} \restriction \gamma)$ and $n \ge m, l$ such that $(p', \dot{q}') \Vdash_{\mathbb{P}*\dot{\mathbb{B}}_\gamma} \dot{x}^*(n) < \dot{c}(n)$. Put $(p_1, \dot{q}_1) \in \mathbb{P} * \dot{\mathbb{B}}_{\gamma+1}$ so that $(p_1, \dot{q}_1 \restriction \gamma) = (p', \dot{q}')$ and $(p_1, \dot{q}_1 \restriction \gamma) \Vdash_{\mathbb{P}*\dot{\mathbb{B}}_\gamma} \dot{q}_1(\gamma) \le_{\mathbb{B}} \dot{q}_0(\gamma), [\![\dot{x}(n) \le \dot{x}^*(n)]\!]_{\mathbb{B}}$. Then $(p_1, \dot{q}_1 \restriction \gamma) \Vdash_{\mathbb{P}*\dot{\mathbb{B}}_\gamma}$ "$\dot{x}^*(n) < \dot{c}(n)$ and $\dot{q}_1(\gamma) \Vdash_{\mathbb{B}} \dot{x}(n) \le \dot{x}^*(n)$". Therefore $(p_1, \dot{q}_1) \Vdash_{\mathbb{P}*\dot{\mathbb{B}}_{\gamma+1}} \dot{x}(n) \le \dot{x}^*(n) < \dot{c}(n)$.

Lemma $\square$

(4) To prove (4) we shall show $V^{(\mathbb{B}*\dot{\mathbb{D}})_\kappa} \models \diamondsuit(\mathbb{LOC}, \omega^\omega, \not\sqsupset)$ where $\mathbb{LOC} = \{\phi : \phi \text{ is a function from } \omega \text{ to } \omega^{<\omega} \text{ such that } \exists k \in \omega\, |\phi(n)| \le n^k \text{ for } n \in \omega\}$ and $\phi \sqsupset x$ if $\forall^\infty n\,(\phi(n) \ni x(n))$ for $\phi \in \mathbb{LOC}$ and $x \in \omega^\omega$. Without loss of generality we can assume $\mathbb{B} * \dot{\mathbb{D}}$ is a complete Boolean algebra with strictly positive finitely additive measure μ [3, p319 Lemma 6.5.18]. So it suffices to show the following lemma:

Lemma 2.4.10. *Suppose γ is an ordinal and $\mathbb{P}$ is a forcing notion which has a $\mathbb{P}$-name $\dot{c}$ such that for all $\phi \in \mathbb{LOC} \cap V \Vdash_{\mathbb{P}} \exists^\infty n\,(\phi(n) \not\ni \dot{c}(n))$. Let $\mathcal{B}_\gamma$ be a γ-stage finite support iteration of complete Boolean algebras with strictly additive measure μ for each γ. Let $\dot{\phi}$ be a $\mathcal{B}_\gamma$-name such that $\Vdash_{\mathcal{B}_\gamma} \dot{\phi} \in \mathbb{LOC}$. Then $\Vdash_{\mathbb{P}*\dot{\mathcal{B}}_\gamma} \dot{\phi} \not\sqsupset \dot{c}$.*

Proof. We proceed by induction on γ. We shall prove only the successor step.

Successor step:

Suppose for γ the lemma holds. Let $\dot{\phi}$ be a $\mathcal{B}_{\gamma+1}$-name such that $\Vdash_{\mathcal{B}_{\gamma+1}} \dot{\phi} \in \mathbb{LOC}$. Let $\dot{\psi}_i$ $(i < \omega)$, $\dot{p}_i$ $(i < \omega)$ and $\dot{k}_i$ $(i < \omega)$ be $\mathcal{B}_\gamma$-names such that

- $\Vdash_{\mathcal{B}_\gamma} \dot{\psi}_i \in \mathbb{LOC}$, $\dot{p}_i \in \dot{\mathcal{B}}$ and $\dot{k}_i \in \omega$ for $i < \omega$,
- $\Vdash_{\mathcal{B}_\gamma}$ "$\dot{p}_i \Vdash_{\dot{\mathcal{B}}} \forall n \in \omega \left(\dot{\phi}_i(n) \le n^{\dot{k}_i}\right)$" and
- $\Vdash_{\mathcal{B}_\gamma} \dot{\psi}_i(n) = \{j : \mu\left([\![j \in \dot{\phi}(n)]\!]_{\dot{\mathcal{B}}} \wedge \dot{p}_i\right) \ge \frac{1}{n}\}$.

Claim 5. $\Vdash_{\mathcal{B}_\gamma} \left|\dot\psi_i(n)\right| \leq n^{\dot k_i+1}$.

Let $m \in \omega$ and $(p_0, \dot q_0) \in \mathbb{P} * \dot{\mathcal{B}}_{\gamma+1}$. Without loss of generality we can find $i \in \omega$ and $n_i \in \omega$ such that $(p, \dot q \restriction \gamma) \Vdash_{\mathbb{P}*\dot{\mathcal{B}}_\gamma}$ "$\mu\left(\dot q(\gamma) \wedge \dot p_i\right) \geq \dfrac{1}{n_i}$". By induction hypothesis there exist $(p', \dot q') \leq_{\mathbb{P}*\dot{\mathcal{B}}_\gamma} (p, \dot q \restriction \gamma)$ and $n \geq n_i, m$ such that $(p', \dot q') \Vdash_{\mathbb{P}*\dot{\mathcal{B}}_\gamma}$ $\dot c(n) \notin \dot\psi_i(n)$. Without loss of generality we can assume p' decides $\dot c(n)$ and $p' \Vdash_{\mathcal{B}}$ "$\dot c(n) = l$" for some $l \in \omega$. Since $(p', \dot q') \Vdash_{\mathbb{P}*\dot{\mathcal{B}}_\gamma} l \notin \dot\psi_i(n)$, $(p', \dot q') \Vdash_{\mathbb{P}*\dot{\mathcal{B}}_\gamma}$ $\mu\left(\llbracket l \in \dot\phi(n)\rrbracket_{\dot{\mathcal{B}}} \wedge \dot p_i\right) < \dfrac{1}{n}$. So $(p', \dot q') \Vdash_{\mathbb{P}*\dot{\mathcal{B}}_\gamma} \mu\left(\llbracket l \notin \dot\phi(n)\rrbracket_{\dot{\mathcal{B}}} \wedge \dot p_i \wedge \dot q(\gamma)\right) > 0$. Put $(p_1, \dot q_1) \in \mathbb{P} * \dot{\mathcal{B}}_\gamma$ so that $(p_1, \dot q_1 \restriction \gamma) = (p', \dot q')$ and $(p_1, \dot q_1 \restriction \gamma) \Vdash_{\mathbb{P}*\dot{\mathcal{B}}_\gamma} \dot q_1(\gamma) = \llbracket l \notin \dot\phi(n)\rrbracket_{\dot{\mathcal{B}}} \wedge \dot p_i \wedge \dot q(\gamma)$. Then $(p_1, \dot q_1) \Vdash_{\mathbb{P}*\dot{\mathcal{B}}_{\gamma+1}}$ "$\dot c(n) = l \notin \dot\phi(n)$".

Lemma $\square$

So We have $V^{(\mathbb{B}*\dot{\mathbb{D}})_\kappa} \models \diamondsuit(\mathbb{LOC}, \omega^\omega, \not\ni)$.

Let $\{C_{i,j}\}$ be a family of independent open sets with $\mu(C_{i,j}) = \frac{1}{(i+1)^2}$ for all i, j. Let $\Phi : \omega^\omega \to \mathcal{N}$ be the function such that

$$\Phi(f) = \bigcup_n \bigcap_{i \geq n} C_{i, f(i)}.$$

For each $B \in \mathcal{N}$ fix a compact set $K_B \subset \omega^\omega \setminus B$ with $\mu(K_B \cap U) > 0$ for any open set U with $K_B \cap U \neq \emptyset$. Let $\{\sigma_n^B : n \in \omega\}$ list all $\sigma \in \omega^{<\omega}$ with $K_B \cap [\sigma] \neq \emptyset$. Put

$$g(B, n, i) = \{j : K_B \cap [\sigma_n^B] \cap C_{i,j} = \emptyset\}$$

for $i, n \in \omega$. Fix $k(B, n)$ such that

$$|g(B, n, i)| \leq \frac{(i+1)^2}{2^{n+1}}$$

for $i \geq k(B, n)$. Define $\Psi : \mathcal{N} \to \mathbb{LOC}$ by

$$\Psi(B)(i) = \bigcup_{k(B,n) \leq i} g(B, n, i).$$

Then Ψ and Φ witness $(\mathcal{N}, \mathcal{N}, \not\supseteq) \leq_B^T (\mathbb{LOC}, \omega^\omega, \not\ni)$ (see [3, Theorem 2.3.9]). So $V^{(\mathbb{B}*\dot{\mathbb{D}})_\kappa} \models \diamondsuit(\mathcal{N}, \mathcal{N}, \not\supseteq)$.

Theorem $\square$

Corollary 2.4.11. *Each of the following are relatively consistent with ZFC:*

(i) $\mathfrak{c} = add(\mathcal{M}) = \omega_2 + \diamondsuit(cov(\mathcal{N}))$ *(see Diagram 1).*

(ii) $\mathfrak{c} = cov(\mathcal{N}) = cov(\mathcal{M}) = \omega_2 + \diamondsuit(\mathfrak{b})$ *(see Diagram 2).*

(iii) $\mathfrak{c} = non(\mathcal{M}) = cov(\mathcal{M}) = \omega_2 + \diamondsuit(\mathfrak{b}) + \diamondsuit(cov(\mathcal{N}))$ *(see Diagram 3).*

(iv) $\mathfrak{c} = cov(\mathcal{N}) = add(\mathcal{M}) = \omega_2 + \diamondsuit(add(\mathcal{N}))$ *(see Diagram 4).*

Proof. (i) Suppose $V \models$ CH. By Theorem 2.4.6 (1) $V^{\mathbb{D}_{\omega_2}} \models \diamondsuit(\mathrm{cov}(\mathcal{N}))$. Since $\mathbb{D}_{\omega_2}$ adds ω_2-many dominating reals and Cohen reals, $V^{\mathbb{D}_{\omega_2}} \models \mathfrak{c} = \mathfrak{b} = \mathrm{cov}(\mathcal{M}) = \omega_2$. Since $\mathrm{add}(\mathcal{M}) = \min\{\mathfrak{b}, \mathrm{cov}(\mathcal{M})\}$ (see [3], [26]),

$$V^{\mathbb{D}_{\omega_2}} \models \diamondsuit(\mathrm{cov}(\mathcal{N})) + \mathfrak{c} = \mathrm{add}(\mathcal{M}) = \omega_2.$$

Cichoń's diagram for parametrized diamond looks as follows where an ω_2 means the corresponding evaluation of the Borel invariant is ω_2 while the parametrized diamond principle for the others hold.

$$\begin{tikzcd}
\diamondsuit(\mathrm{cov}(\mathcal{N})) \arrow[r, dash] \arrow[dd, dash] & \omega_2 \arrow[r, dash] \arrow[d, dash] & \omega_2 \arrow[r, dash] \arrow[d, dash] & \omega_2 \arrow[dd, dash] \\
 & \omega_2 \arrow[r, dash] \arrow[d, dash] & \omega_2 \arrow[d, dash] & \\
\diamondsuit(\mathrm{add}(\mathcal{N})) \arrow[r, dash] & \omega_2 \arrow[r, dash] & \omega_2 \arrow[r, dash] & \omega_2
\end{tikzcd}$$

Diagram 1.

(ii) Suppose $V \models$ CH. By Theorem 2.4.6 (2) $V^{\mathbb{B}_{\omega_2}} \models \diamondsuit(\mathfrak{b})$. Since $\mathbb{B}_{\omega_2}$ adds ω_2 many Cohen and random reals, $V^{\mathbb{B}_{\omega_2}} \models \mathfrak{c} = \mathrm{cov}(\mathcal{N}) = \mathrm{cov}(\mathcal{M}) = \omega_2$. Hence

$$V^{\mathbb{B}_{\omega_2}} \models \diamondsuit(\mathfrak{b}) + \mathfrak{c} = \mathrm{cov}(\mathcal{N}) = \mathrm{cov}(\mathcal{M}) = \omega_2.$$

$$\begin{tikzcd}
\omega_2 \arrow[r, dash] \arrow[dd, dash] & \omega_2 \arrow[r, dash] \arrow[d, dash] & \omega_2 \arrow[r, dash] \arrow[d, dash] & \omega_2 \arrow[dd, dash] \\
 & \diamondsuit(\mathfrak{b}) \arrow[r, dash] \arrow[d, dash] & \omega_2 \arrow[d, dash] & \\
\diamondsuit(\mathrm{add}(\mathcal{N})) \arrow[r, dash] & \diamondsuit(\mathrm{add}(\mathcal{M})) \arrow[r, dash] & \omega_2 \arrow[r, dash] & \omega_2
\end{tikzcd}$$

Diagram 2.

(iii) Suppose $V \models$ CH. By Theorem 2.4.6 (3) $V^{\mathbb{E}_{\omega_2}} \models \diamondsuit(\mathrm{cov}(\mathcal{N})) + \diamondsuit(\mathfrak{b})$. Since $\mathbb{E}_{\omega_2}$ adds ω_2 many Cohen and eventually different reals, $\mathfrak{c} = \mathrm{non}(\mathcal{M}) = \mathrm{cov}(\mathcal{M}) = \omega_2$. Hence

$$V^{\mathbb{E}_{\omega_2}} \models \diamondsuit(\mathrm{cov}(\mathcal{N})) + \diamondsuit(\mathfrak{b}) + \mathfrak{c} = \mathrm{non}(\mathcal{M}) = \mathrm{cov}(\mathcal{M}).$$

$$\begin{tikzcd}
\diamondsuit(\mathrm{cov}(\mathcal{N})) \arrow[r, dash] \arrow[dd, dash] & \omega_2 \arrow[r, dash] \arrow[d, dash] & \omega_2 \arrow[r, dash] \arrow[d, dash] & \omega_2 \arrow[dd, dash] \\
 & \diamondsuit(\mathfrak{b}) \arrow[r, dash] \arrow[d, dash] & \omega_2 \arrow[d, dash] & \\
\diamondsuit(\mathrm{add}(\mathcal{N})) \arrow[r, dash] & \diamondsuit(\mathrm{add}(\mathcal{M})) \arrow[r, dash] & \omega_2 \arrow[r, dash] & \omega_2
\end{tikzcd}$$

Diagram 3.

(iv) Suppose $V \models \mathrm{CH}$. By Theorem 2.4.6 (4) $V^{(\mathbb{B}*\dot{\mathbb{D}})_{\omega_2}} \models \diamondsuit(\mathrm{add}(\mathcal{N}))$. Since $(\mathbb{B}*\dot{\mathbb{D}})_{\omega_2}$ adds ω_2 many random, Cohen and dominating reals, $\mathfrak{c} = \mathrm{cov}(\mathcal{N}) = \mathrm{add}(\mathcal{M}) = \min\{\mathfrak{b}, \mathrm{cov}(\mathcal{M})\} = \omega_2$. Hence

$$V^{(\mathbb{B}*\dot{\mathbb{D}})_{\omega_2}} \models \diamondsuit(\mathrm{add}(\mathcal{N})) + \mathfrak{c} = \mathrm{cov}(\mathcal{N}) = \mathrm{add}(\mathcal{M}) = \omega_2.$$

$$\begin{tikzcd}
\omega_2 \arrow[r, dash] \arrow[dd, dash] & \omega_2 \arrow[r, dash] \arrow[d, dash] & \omega_2 \arrow[r, dash] \arrow[d, dash] & \omega_2 \arrow[dd, dash] \\
 & \omega_2 \arrow[r, dash] \arrow[d, dash] & \omega_2 \arrow[d, dash] & \\
\diamondsuit(\mathrm{add}(\mathcal{N})) \arrow[r, dash] & \omega_2 \arrow[r, dash] & \omega_2 \arrow[r, dash] & \omega_2
\end{tikzcd}$$

Diagram 4

$\square$

Hrušák asked the following question after a talk I gave at the 33rd Winter School on Abstract Analysis -Section of Topology held in the Czech Republic (2005 January).

Question 4 (Hrušák)**.** *Let $\mathbb{A}$ be a amoeba forcing. Then $V^{\mathbb{A}_{\omega_2}} \models \diamondsuit(\mathfrak{s})$?*

Chapter 3

partitions of ω

The structure $([\omega]^\omega, \subset^*)$ of the set of all infinite subsets of ω ordered by "almost inclusion" is well studied in set theory. To describe much of the combinatorial structure of $([\omega]^\omega, \subset^*)$ cardinal invariants of the continuum are introduced like, for example, the reaping number $\mathfrak{r}$ or the independence number $\mathfrak{i}$.

In recent years partial orders similar to $([\omega]^\omega, \subset^*)$ have been focused on and analogous cardinal invariants have been defined and investigated. For example $((\omega)^\omega, \leq^*)$, the set of all infinite partitions of ω ordered by "almost coarser", and the cardinal invariants $\mathfrak{p}_d$, $\mathfrak{t}_d$, $\mathfrak{s}_d$, $\mathfrak{r}_d$, $\mathfrak{a}_d$ and $\mathfrak{h}_d$ have been defined and investigated in [10], [14] and [18].

3.1 Cardinal invariants related to partitions of ω

We say that X is a partition of ω if X is a subset of $\wp(\omega)$, pairwise disjoint and $\bigcup X = \omega$. (ω) denotes the set of all partitions of ω. We say a partition is finite if it has finitely many pieces. By $(\omega)^{<\omega}$ we denote the set of all finite partitions of ω. Also by $(\omega)^\omega$ we denote the set of all infinite partitions of ω.

For $X, Y \in (\omega)$ X is coarser than Y, we write $X \leq Y$ if each element of X is a union of elements of Y. Note that $((\omega), \leq)$ is lattice. By $X \wedge Y$ we denote the infinimum of X and Y for $X, Y \in (\omega)$.

For $X, Y \in (\omega)^\omega$ X is almost coarser than Y, We write $X \leq^* Y$ if all but finite element of X is a union of elements of Y. X is almost orthogonal Y, we write $X \perp Y$ if $X \wedge Y \in (\omega)^{<\omega}$. We say X and Y are compatible, we write $X \| Y$ if X is not orthogonal Y, i.e., $X \wedge Y \notin (\omega)^{<\omega}$.

For $X, Y \in (\omega)^\omega$ X dual-splits Y if $X \| Y$ and $Y \not\leq^* X$. We call $\mathcal{S} \subset (\omega)^\omega$ is dual-splitting family if for each $Y \in (\omega)^\omega$ there exists $X \in \mathcal{S}$ such that X dual-splits Y. We call $\mathcal{R} \subset (\omega)^\omega$ is dual-reaping family if for each $Y \in (\omega)^\omega$ X

cannot be dual-split by Y i.e., there exists $X \in \mathcal{R}$ such that $X \perp Y$ or $X \leq^* Y$.

$$\begin{aligned} \mathfrak{r}_d &= \min\{|\mathcal{R}| : \mathcal{R} \subset (\omega)^\omega \wedge \mathcal{R} \text{ is dual-reaping family}\} \\ \mathfrak{s}_d &= \min\{|\mathcal{S}| : \mathcal{S} \subset (\omega)^\omega \wedge \mathcal{S} \text{ is dual-splitting family}\} \end{aligned}$$

$\mathcal{T} \subset (\omega)^\omega$ is a tower if $\mathcal{T}$ is a decreasing sequence ordered by $\leq^*$ and no lower bound.

$$\mathfrak{t}_d = \min\{|\mathcal{T}| : \mathcal{T} \subset (\omega)^\omega \wedge \mathcal{T} \text{ is a tower}\}.$$

$\mathcal{P} \subset (\omega)^\omega$ is $\leq^*$-centered family if for each finite $\mathcal{P}_0 \subset \mathcal{P}$ there is some $X \in (\omega)^\omega$ such that $X \leq^* Y$ for all $Y \in \mathcal{P}_0$.

$$\mathfrak{p}_d = \min\{|\mathcal{P}| : \mathcal{P} \subset (\omega)^\omega \wedge \mathcal{P} \text{ is a } \leq^* \text{-centered family with no lower bound}\}.$$

By $(\omega)^c$ we denote the set of partitions of ω which is not almost finer than $\{\{n\} : n \in \omega\}$. $\mathcal{A} \subset (\omega)^c$ is a maximal almost orthogonal family (mao family) if $\mathcal{A}$ is a maximal family of pairwise orthogonal partitions.

$$\mathfrak{a}_d = \min\{|\mathcal{A}| : \mathcal{A} \subset (\omega)^\omega \wedge \text{ is maximal almost orthogonal family}\}.$$

A family $\mathbf{F}$ of mao families of partitions shatters a partition $A \in (\omega)^\omega$ if there are $\mathcal{F} \in \mathbf{F}$, and two distinct partitions $X, Y \in \mathcal{F}$ such that A is compatible with both X and Y.

$$\mathfrak{h}_d = \min\{|\mathbf{F}| : \mathbf{F} \text{ is mao families and} \forall X \in (\omega)^\omega (\mathbf{F} \text{ shatters } X)\}.$$

3.2 dual van Douwen diagram

The relationship between cardinal invariants of $(\wp(\omega)/fin, \subset^*)$ is displayed in van Douwen diagram. We also display the relationship between cardinal invariants of $((\omega)^\omega, \leq^*)$ in dual van Douwen diagram.

$$\begin{tikzcd}
 & & \mathfrak{c} \arrow[dl] \arrow[dr] \arrow[ddrr] & & \\
 & \mathfrak{i} \arrow[d] \arrow[dr] & & \mathfrak{u} \arrow[dl] & \\
 & \mathfrak{d} \arrow[dl] \arrow[dr] & \mathfrak{r} \arrow[d] & & \mathfrak{a} \arrow[dll] \\
\mathfrak{s} \arrow[dr] & & \mathfrak{b} \arrow[dl] & & \\
 & \mathfrak{h} \arrow[d] & & & \\
 & \mathfrak{t} \arrow[d] & & & \\
 & \mathfrak{p} \arrow[d] & & & \\
 & \omega_1 & & &
\end{tikzcd}$$

van Douwen's diagram.

By the following property, $\mathfrak{r}_d$ is not countable.

Lemma 3.2.1. *[14] If $\{X_n : n \in \omega\}$ be a countable subset of $(\omega)^c$, then there exists Y such that Y dual-splits X_n for $n \in \omega$. Therefore $\omega_1 \leq \mathfrak{r}_d$.*

As $\mathfrak{h} \leq \mathfrak{s}$ and $\mathfrak{t} \leq \mathfrak{h}$, we can prove the followings:

Theorem 3.2.2. *[14] $\mathfrak{h}_d \leq \mathfrak{s}_d$.*

Theorem 3.2.3. *[14] $\mathfrak{t}_d \leq \mathfrak{h}_d$.*

Some cardinal invariants is just ω_1 or $\mathfrak{c}$.

Theorem 3.2.4. *[14] $\mathfrak{a}_d = \mathfrak{c}$.*

Theorem 3.2.5. *[25] $\mathfrak{p}_d = \mathfrak{t}_d = \omega_1$.*

$$\begin{tikzcd}
 & \mathfrak{h}_d \arrow[dl] & \mathfrak{s}_d \arrow[l] & \mathfrak{c} \arrow[l] \arrow[dl] \\
\omega_1 & & \mathfrak{r}_d \arrow[ll] &
\end{tikzcd}$$

dual van Douwen diagram.
(The direction of the arrow is from larger to smaller cardinal).

This diagram doesn't collapse. Halbeisen proved the following result by using dual-Mathias forcing.

Theorem 3.2.6. *[18] It is consistent that $\omega_1 < \mathfrak{h}_d$.*

Also he prove the following consistency by using Mathias forcing:

Theorem 3.2.7. *[18] It is consistent that $\mathfrak{h}_d < \mathfrak{h}$. Therefore it is consistent $\mathfrak{h}_d < \mathfrak{s}_d$.*

In [14] by using finite support iteration of c.c.c forcing, it is proved the following statement:

Theorem 3.2.8. *[14] It is consistent that $\mathfrak{r}_d, \mathfrak{s}_d < \mathfrak{c}$.*

3.3 Relationship with other cardinal invariants

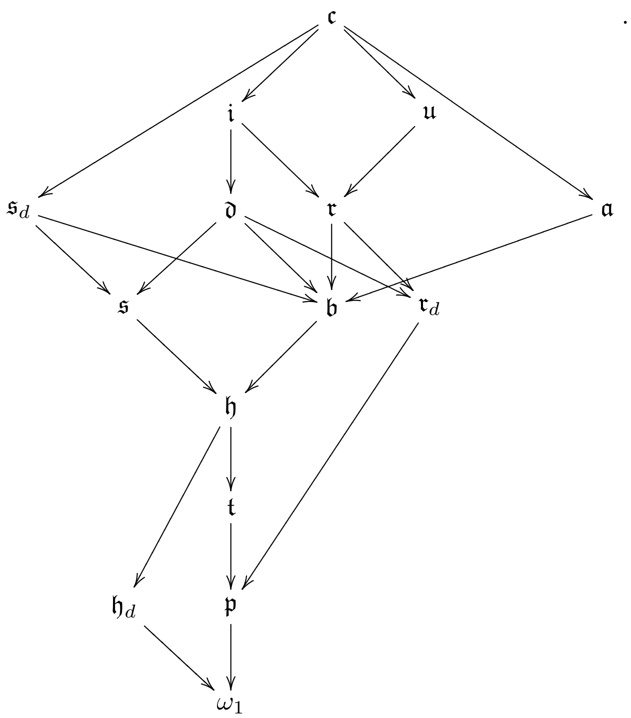
In this section we shall investigate the relationship between cardinal invariants related to $(\omega)^\omega$ and cardinal invariants in Cichoń's diagram and van Douwen's diagram.

Theorem 3.3.1. *[18] $\mathfrak{h}_d \leq \mathfrak{h}$.*

Theorem 3.3.2. *[14][21] $\mathfrak{s}_d \geq \mathfrak{s}$ and $\mathfrak{r}_d \leq \mathfrak{r}$.*

Theorem 3.3.3. *(Kamo) $\mathfrak{r}_d \leq \mathfrak{d}$ and $\mathfrak{s}_d \geq \mathfrak{b}$.*

Proposition 3.3.4. *There exists a σ-centered forcing which add a new partitions of ω which dual-splits every partitions of ω in ground model. Therefore $\mathfrak{p} \leq \mathfrak{r}_d$.*



This diagram doesn't collapse. By using Cohen forcing, we can prove the following:

Proposition 3.3.5. *It is consistent that* $\mathfrak{s}_d > \mathfrak{s}, \mathfrak{b}$. *It is consistent that* $\mathfrak{r}_d < \mathfrak{r}, \mathfrak{d}$.

In chapter 4 we shall prove it is consistent that $\mathfrak{b} < \mathfrak{r}_d$. As results we can say the followings.

Proposition 3.3.6. *It is consistent that* $\mathfrak{p} < \mathfrak{r}_d$.

We shall state relationship with Cichoń's diagram.

$$
\begin{tikzcd}
\mathrm{cov}(\mathcal{N}) \arrow[dd] & \mathrm{non}(\mathcal{M}) \arrow[l] \arrow[d] & \mathrm{cof}(\mathcal{M}) \arrow[l] \arrow[d] & \mathrm{cof}(\mathcal{N}) \arrow[l] \arrow[dd] \; . \\
 & \mathfrak{b} \arrow[d] & \mathfrak{d} \arrow[l] \arrow[d] & \\
\mathrm{add}(\mathcal{N})) & \mathrm{add}(\mathcal{M}) \arrow[l] & \mathrm{cov}(\mathcal{M}) \arrow[l] & \mathrm{non}(\mathcal{N}) \arrow[l]
\end{tikzcd}
$$

Cichoń's diagram.

Theorem 3.3.7. *[14]* $\mathfrak{r}_d \leq cov(\mathcal{M})$.

Theorem 3.3.8. *(Brendle)* $\mathfrak{r}_d \leq cov(\mathcal{N})$ *and* $\mathfrak{s}_d \geq non(\mathcal{N}), \mathfrak{s}_d \geq non(\mathcal{M})$.

Therefore we have the following diagram.

$$
\begin{tikzcd}
 & & \mathfrak{s}_d \arrow[dll] \arrow[ddl] \arrow[dddr] & & \; . \\
\mathrm{cov}(\mathcal{N}) \arrow[dd] & \mathrm{non}(\mathcal{M}) \arrow[l] \arrow[d] \arrow[dddr] & & \mathrm{cof}(\mathcal{M}) \arrow[lll] \arrow[d] & \mathrm{cof}(\mathcal{N}) \arrow[l] \arrow[dd] \\
 & \mathfrak{b} \arrow[d] & & \mathfrak{d} \arrow[ll] \arrow[d] \arrow[ddl] & \\
\mathrm{add}(\mathcal{N})) & \mathrm{add}(\mathcal{M}) \arrow[l] & & \mathrm{cov}(\mathcal{M}) \arrow[ll] & \mathrm{non}(\mathcal{N}) \arrow[l] \arrow[dll] \\
 & & \mathfrak{r}_d & &
\end{tikzcd}
$$

This diagram doesn't collapse.

Proposition 3.3.9. *It is consistent that* $\mathfrak{r}_d < non(\mathcal{M}), non(\mathcal{N}), \mathfrak{d}$. *Also it is consistent that* $\mathfrak{s}_d > cov(\mathcal{M}), cov(\mathcal{N}), \mathfrak{b}$.

In chapter 4 we shall prove more strong statement which say that it is consistent $\mathfrak{r}_d < \mathrm{add}(\mathcal{M})$ and it is consistent that $\mathfrak{s}_d > \mathrm{cof}(\mathcal{M})$.

Chapter 4

forcing and cardinal invariants for partitions of ω

4.1 dual-ultrafilter number for partitions of ω

Let $(\mathbb{P}, \leq)$ be a partial order. Then $\mathcal{F} \subset \mathbb{P}$ is a filter if

(1) if $X \in \mathcal{F}$, then $Y \in \mathcal{F}$ for $Y \geq X$ and

(2) if $X, Y \in \mathcal{F}$, then there exists $Z \in \mathcal{F}$ such that $Z \leq X, Y$.

For a filter $\mathcal{F}$ on $\mathbb{P}$, $\mathcal{B} \subset \mathcal{F}$ is a base for $\mathcal{F}$ if for any $X \in \mathcal{F}$ there exists $Y \in \mathcal{B}$ such that $Y \leq X$. For a filter $\mathcal{F}$ on $\mathbb{P}$, $\mathcal{F}$ is a maximal filter if for each $X \in \mathbb{P}$ $X \in \mathcal{F}$ or $X \notin \mathcal{F}$. For a filter U on $\wp(\omega)$ U is a ultrafilter if for any $X \in \wp(\omega)$ $X \in U$ or $\omega \setminus X \in U$. Notice that on $\wp(\omega)$, U is a ultrafilter if and only if U is a maximal filter.

For $\mathcal{F} \subset \wp(\omega)$ $\mathcal{F}$ is a non-trivial filter if $\mathcal{F}$ contains $\{X \in \wp(\omega) : \omega \subset^* X\}$. For $\mathcal{F} \subset (\omega)^\omega$ $\mathcal{F}$ is a non-trivial filter if $\mathcal{F}$ contains $\{X \in (\omega)^\omega : \{\{n\} : n \in \omega\} \leq^* X\}$. Then define ultrafilter number $\mathfrak{u}$ and dual-ultrafilter number $\mathfrak{u}_d$ by

$$\begin{aligned} \mathfrak{u} &= \min\{|\mathcal{B}| : \mathcal{B} \text{ is a base for a non-trivial maximal filter on } \wp(\omega)\}. \\ \mathfrak{u}_d &= \min\{|\mathcal{B}| : \mathcal{B} \text{ is a base for a non-trivial maximal filter on } (\omega)^\omega\}. \end{aligned}$$

Then we have the following relationship.

Theorem 4.1.1. *(Minami)* $\mathfrak{u}_d \leq \mathfrak{u}$.

Proof. Let H be a non-principle filter on ω. Put $\mathcal{F}_H$ be a filter on $(\omega)^\omega$ which generated by $\{X_A : A \in H\}$ where $X_A = \{\{n\} : n \in A\} \cup \{\omega \setminus A\}$. Then following statement holds.

Claim 6. *[25] $\mathcal{F}_H$ is maximal iff H is an ultrafilter.*

Proof of Claim. Suppose $\mathcal{F}_H$ is a maximal filter on $(\omega)^\omega$. Let $A \in [\omega]^\omega$ such that $A \notin H$. Choose $Y \in \mathcal{F}_H$ with $Y \wedge X_A = \mathbf{0}$. Now let $B \in H$ such that $X_B \leq Y$. Then $B \cap A = \emptyset$. Hence $B \subset \omega \setminus A$. Therefore H is an ultrafilter.

Conversely suppose H is an ultrafilter on ω. Let $Y \in (\omega)^\omega \setminus \mathcal{F}_H$. If $Min(Y) = \{\min(y) : y \in Y\} \notin H$, then $Y \wedge X_{\omega \setminus Min(Y)} = \mathbf{0}$.
Assume $Min(Y) \in H$. If $Min^*(Y) = \{\min(y) : y \in Y \text{ and } |y| \geq 2\} \notin H$, then $Y \wedge X_{\omega \setminus Min^*(Y)} = \mathbf{0}$.
If $Min^*(Y) \in H$, then $X_{Min(Y)} \leq Y$. It is contradict to $Y \notin \mathcal{F}_H$. □

As $\mathfrak{r} \leq \mathfrak{u}$, we have the following result.

Theorem 4.1.2. *(Brendle)* $\mathfrak{r}_d \leq \mathfrak{u}_d$.

Proof. Let $\mathcal{B} \subset (\omega)^\omega$ be a base for a maximal filter with $|\mathcal{B}| = \mathfrak{u}_d$. We shall prove $\mathcal{B}$ is a dual-reaping family. Let $X \in (\omega)^\omega$ and let $\mathcal{F}$ be a maximal filter generated by $\mathcal{B}$. If X is compatible with all element of $\mathcal{B}$, then $\{X\} \cup \mathcal{B}$ generate a filter. Since $\mathcal{B}$ is a base for a maximal filter, $X \in \mathcal{F}$. Since $\mathcal{B}$ is a base for $\mathcal{F}$, there exists $Y \in \mathcal{B}$ such that $Y \leq X$. □

$$\begin{tikzcd}
& \mathfrak{r} \arrow[dl] & \mathfrak{u} \arrow[l] \arrow[dl] \\
\mathfrak{r}_d & \mathfrak{u}_d \arrow[l] &
\end{tikzcd}$$

It is natural to ask this diagram collapse.

Question 5. *Is it consistent that* $\mathfrak{r}_d < \mathfrak{u}_d$*?*

But it is difficult to show it is consistent that $\mathfrak{r}_d \leq \mathfrak{u}_d$ because of influence from $\diamondsuit$. For $\mathfrak{r}$ and $\mathfrak{u}$ there is the following influence from $\diamond$.

Theorem 4.1.3. *[31]* $\diamondsuit(\mathfrak{r})$ *implies that there exists a P-point of character* ω_1. *In Particular* $\diamondsuit(\mathfrak{r})$ *implies* $\mathfrak{u} = \omega_1$.

For filters on $(\omega)^\omega$ we introduce the notion corresponding to P-point.

Definition 14. *[25] A filter* $\mathcal{F} \subset (\omega)^\omega$ *has the property P if, for every descending sequence* $X_0 \geq X_1 \geq \ldots \geq X_n \geq \ldots$ *of members of* $\mathcal{F}$ *, there exists* $X \in \mathcal{F}$ *such that* $X \leq^* X_n$ *for all* $n \in \omega$.

As influence of $\diamondsuit(\mathfrak{r})$, we have the following theorem.

Theorem 4.1.4. *(Minami)* $\diamondsuit(\mathfrak{r}_d)$ *implies there exists a maximal filter on* $(\omega)^\omega$ *with property P of character* ω_1. *In particular* $\diamondsuit(\mathfrak{r}_d)$ *implies* $\mathfrak{u}_d = \omega_1$.

Proof. For each $\delta < \omega_1$ fix a bijection $e_\delta : \delta \to \omega$. The domain of the function F we will consider will consist of pairs $(\vec{U}, C)$ such that $\vec{U} = \langle U_\xi : \xi \leq \delta \rangle$ is a countable $\leq^*$-decreasing sequence of infinite partition of ω and C is a infinite partition of ω. Given $\vec{U}$ as above, let $B(U)$ be the set $\{x_i : i \in \omega\}$ where x_i is a subset of ω such that

(1) $\forall j < i\, (x_i \cap x_j = \emptyset)$,

(2) $\forall j < i+1 \left(x_i \text{ is a union of blocks of } U_{e_\delta^{-1}(j)} \right)$ and

(3) $0 < \min(x_i) < \min(x_j)$ for $i < j$.

(4) $x_0 = \omega \setminus \bigcup_{i>0} x_i$

Note that $B(\vec{U})$ is infinite partition of ω and almost contained in U_ξ for every $xi < \delta$. Let

$$F(\vec{U}, C) = \begin{cases} \{\{i \in \omega : x_i \subset y\} : y \in B(\vec{U}) \wedge C\} & \text{if } B(\vec{U}) \| C \\ \mathbf{1} & \textit{otherwise.} \end{cases}$$

Now suppose that $g : \omega_1 \to (\omega)^\omega$ is a $\diamondsuit(\mathfrak{r}_d)$-sequence for F. Construct a $\leq^*$-decreasing sequence $\langle U_\xi : \xi < \omega_1 \rangle$ of infinite partition of ω by recursion. Let $U_n = \{n\} \cup \{\{k\} : k \geq n\}$. Having defined $\vec{U} = \langle U_\xi : \xi < \delta \rangle$ let $U_\delta = \{\bigcup i \in a : a \in g(\delta)\}$ where $B(\vec{U}) = \{x_i : i \in \omega\}$. The family $\langle U_\xi : \xi < \omega_1 \rangle$ generates a filter with property P. To see that it is a maximal filter, note $C \in (\omega)^\omega$ is given and g guesses $\vec{U}, C$ at δ.

Case 1. $F(\vec{U} \restriction \delta, C)^* \geq g(\delta)$.

(1) $B(\vec{U} \restriction \delta) \| C$.

Since $F(\vec{U} \restriction \delta, C)^* \geq g(\delta)$, $B(\vec{U} \restriction \delta) \wedge C^* \geq U_\delta$. So $C^* \geq U_\delta$.

(2) $B(\vec{U} \restriction \delta) \perp C$. Then $U_\delta = B(\vec{U} \restriction \delta)$. So $U_\delta \perp C$.

Case 2. $F(\vec{U} \restriction \delta, C) \perp g(\delta)$.

(1) $B(\vec{U} \restriction \delta) \| C$.

Since $F(\vec{U} \restriction \delta, C) \perp g(\delta)$, $B(\vec{U} \restriction \delta) \wedge C \perp U_\delta$. Since $U_\delta \leq^* B(\vec{U} \restriction \delta)$, $U_\delta \perp C$.

(2) $B(\vec{U} \restriction \delta) \perp C$.

Then $\mathbf{1} \perp g(\delta)$. It is impossible.

Therefore $U_\delta \perp C$ or $C^* \geq U_\delta$. □

Corollary 4.1.5. *(Minami) It is consistent that* $\mathfrak{u}_d < \mathfrak{r}$.

Proof. By product lemma $\mathbb{C}(\omega_2) = \mathbb{C}(\omega_2) * \mathbb{C}(\omega_1)$. Since Cohen forcing adds a partitions of ω which is almost orthogonal to every non-trivial partitions of ω. So $V^{\mathbb{C}(\omega_2)} \models \diamondsuit(\mathfrak{r}_d)$. But Cohen forcing enlarge $\mathfrak{r}$. Therefore $V^{\mathbb{C}_{\omega_2}} \models \mathfrak{u}_d < \mathfrak{r}$. □

4.2 independence number for partitions of ω

In this section we will define the dual-independence number $\mathfrak{i}_d$ analogous to the independence number $\mathfrak{i}$ and get a consistency result.

Once we define dual-independence number $\mathfrak{i}_d$, we can prove the following proposition similar to the proof of $\mathfrak{r} \leq \mathfrak{i}$.

Proposition 4.2.1. *[Brendle] $\mathfrak{r}_d \leq \mathfrak{i}_d$.*

And $\mathfrak{r}_d$ has the following property.

Theorem 4.2.2. *[14] MA implies $\mathfrak{r}_d = \mathfrak{c}$.*

So it is consistent that $\mathfrak{i}_d = \mathfrak{c}$. And it is natural to ask the following question.

Question 6. *Is it consistent that $\mathfrak{i}_d < \mathfrak{c}$?*

4.2.1 $(\omega)^\omega$ and dual-independent family

We will define the dual-independence number and study its properties.

As $([\omega]^\omega, \subset^*)$, $((\omega)^\omega, \leq^*)$ has the following properties:

Lemma 4.2.3. *[14] Suppose that $X_0 \geq X_1 \geq X_2 \geq \ldots$ is a decreasing sequence of $(\omega)^\omega$. Then there exists $Y \in (\omega)^\omega$ such that $Y \leq^* X_n$ for $n \in \omega$.*

Lemma 4.2.4. *[14] For $X, Y \in (\omega)^\omega$ if $\neg(X \leq^* Y)$, then there exists $Z \in (\omega)^\omega$ such that $Z \leq^* X$ and $Z \perp Y$.*

So $((\omega)^\omega, \leq^*)$ is similar to $([\omega]^\omega, \subset^*)$. On the other hand there is a serious difference: $([\omega]^\omega, \subset^*)$ is a Boolean algebra but $((\omega)^\omega, \leq^*)$ is just a lattice and not a Boolean algebra.

In general when we define independence, we use complementation. But $((\omega)^\omega, \leq^*)$ doesn't have any natural complementation. So we will define independence for $((\omega)^\omega, \leq^*)$ without mentioning complementation.

Definition 15. *Let $\mathcal{I}$ be a subset of $(\omega)^\omega$. $\mathcal{I}$ is dual-independent if for all $\mathcal{A}$ and $\mathcal{B}$ finite subsets of $\mathcal{I}$ with $\mathcal{A} \cap \mathcal{B} = \emptyset$ there exists $C \in (\omega)^\omega$ such that*

(i) $C \leq^* A$ for $A \in \mathcal{A}$ and

(ii) $C \perp B$ for $B \in \mathcal{B}$.

Then define dual-independence number $\mathfrak{i}_d$ by

$$\mathfrak{i}_d = \min\{|\mathcal{I}| : \mathcal{I} \textit{ is a maximal dual-independent family}\}.$$

Since there is no natural complementation for an element of $((\omega)^\omega, \leq^*)$, it becomes more difficult to handle dual-independent families than to handle independent families for a Boolean algebra. But the following lemmata helps to handle dual-independent families.

Lemma 4.2.5. *[14] If $X, Y \in (\omega)^\omega$ and $\neg(X \leq^* Y)$, then there exists an infinite sequence $\{a_n\}_{n\in\omega}$ of different elements of X such that*

$$\forall n \in \omega \exists y \in Y \, (y \cap a_{2n} \neq \emptyset \wedge y \cap a_{2n+1} \neq \emptyset)$$

or there exists a finite subset A of X such that the set

$$\{x \in X \setminus A : \exists y \in Y (x \cap y \neq \emptyset \wedge \bigcup A \cap y \neq \emptyset)\}$$

is infinite.

Proof. Suppose that we have defined a sequence $\{a_n\}_{n<2k}$ but for any two $a, b \in X \setminus \{a_0, \ldots, a_{2k-1}\}$ and $y \in Y$ we have $a \cap y = \emptyset$ or $b \cap y = \emptyset$. Let A denote the finite family $\{a_0, \ldots, a_{2k-1}\}$ and let

$$\mathcal{F} = \{x \in X \setminus A : \exists y \in Y \left(x \cap y \neq \emptyset \wedge \bigcup A \cap y \neq \emptyset\right)\}.$$

If $\mathcal{F}$ is finite, then the partition

$$X_* = \{\bigcup A \cup \bigcup \mathcal{F}\} \cup (X \setminus A \cup \mathcal{F})$$

is a finite modification of X which is coarser than Y. It is a contradiction to $\neg(X \leq^* Y)$. $\square$

By this lemma we can prove the following useful lemma.

Lemma 4.2.6. *If $X \in (\omega)^\omega$ and $\mathcal{B}$ is a finite subset of $(\omega)^\omega$ such that $\neg(X \leq^* B)$ for $B \in \mathcal{B}$, then there exists $Z \leq X$ such that $Z \perp B$ for $B \in \mathcal{B}$.*

Proof. Let $\mathcal{B} = \{B_i : i < n\}$. By the above lemma for each $i < n$ there exists an infinite sequence $\{a^i_k\}_{k\in\omega}$ of different elements of X such that

$$\forall k \in \omega \exists b \in B_i (b \cap a^i_{2k} \neq \emptyset \wedge b \cap a^i_{2k+1} \neq \emptyset)$$

or there exists a finite subset A_i of X and an infinite sequence $\{a^i_k\}_{k\in\omega}$ of different elements of $X \setminus A_i$ such that

$$\forall k \in \omega \exists b \in B_i (b \cap a^i_k \neq \emptyset \wedge \bigcup A_i \cap b \neq \emptyset).$$

In the first case we define $A_i = \emptyset$.

Recursively we shall construct a subsequence $\{b^i_k\}_{k\in\omega}$ of $\{a^i_k\}_{k\in\omega}$ for $i < n$. Given $\{b^i_l\}_{l<2k}$ for $i < n$ and b^i_{2k}, b^i_{2k+1} for $i < j$ for some $j < n$.
$\underline{A_j = \emptyset}$ Choose $k_0 \in \omega$ such that

$$\{a^j_{2k_0}, a^j_{2k_0+1}\} \cap \left(\bigcup_{i<n} A_i \cup \{b^i_l : i < n \wedge l < 2k\} \cup \{b^i_{2k}, b^i_{2k+1} : i < j\}\right) = \emptyset.$$

Put $b^j_{2k} = a^j_{2k_0}$ and $b^j_{2k+1} = a^j_{2k_0+1}$.

$\underline{A_j \neq \emptyset}$ Choose $k_0 < k_1 \in \omega$ such that

$$\{a^j_{k_0}, a^j_{k_1}\} \cap \left(\bigcup_{i<n} A_i \cup \{b^i_l : i < n \wedge l < 2k\} \cup \{b^i_{2k}, b^i_{2k+1} : i < j\} \right) = \emptyset.$$

Put $b^j_{2k} = a^j_{k_0}$ and $b^j_{2k+1} = a^j_{k_1}$.

Define $Z = \{\bigcup_{i<n} b^i_{2k} : k \in \omega\} \cup \{\omega \setminus \bigcup_{k\in\omega} \bigcup_{i<n} b^i_{2k}\}$. Then $Z \leq X$ and for each $z \in Z$ and $i < n$ there exists $b \in B_i$ such that

$$b \cap z \neq \emptyset \wedge (\omega \setminus \bigcup_{k\in\omega} \bigcup_{i<n} b^i_{2k}) \cap b \neq \emptyset.$$

Hence $Z \perp B_i$ for $i < n$. $\square$

So it becomes easier to check dual-independence.

Corollary 4.2.7. *$\mathcal{I}$ is dual-independent if and only if for each finite subset $\mathcal{A}$ of $\mathcal{I}$ and $B \in \mathcal{I} \setminus \mathcal{A}$*

$$\bigwedge \mathcal{A} \not\leq^* B.$$

By using corollary we can prove Proposition 4.2.1.

Proof. (Proposition 4.2.1) Let $\mathcal{I}$ be a maximal dual-independence family. For $\mathcal{A} \in [\mathcal{I}]^{<\omega}$ and $B \in \mathcal{I} \setminus \mathcal{A}$ fix $C_{\mathcal{A}}, B \in (\omega)^\omega$ such that

(i) $C_{\mathcal{A},B} \leq^* A$ for $A \in \mathcal{A}$ and

(ii) $C_{\mathcal{A},B} \perp B$.

Let $\mathcal{R} = \{C_{\mathcal{A},B} : \mathcal{A} \in [\mathcal{I}]^{<\omega} \wedge B \in (\mathcal{I} \setminus \mathcal{A}) \cup \{\emptyset\}\}$. We shall show $\mathcal{R}$ is a dual-reaping family.

Assume to the contrary, there exists $X \in (\omega)^\omega$ such that X dual-splits Y for $Y \in \mathcal{R}$. Then $X \| Y$ and $Y \not\leq^* X$ for $Y \in \mathcal{R}$. So for each $\mathcal{A} \in [\mathcal{I}]^{<\omega}$ $C_{\mathcal{A},B} \leq^* \bigwedge \mathcal{A} \not\leq^* X$. And for each $\mathcal{A} \in [\mathcal{I}]^{<\omega}$ and $B \in \mathcal{I} \setminus \mathcal{A}$, $\bigwedge \mathcal{A} \wedge X \not\leq^* B$ because $C_{\mathcal{A},B} \| X$. Therefore $\{X\} \cup \mathcal{I}$ is dual-independent. It is contradiction. $\square$

4.2.2 Cohen forcing and dual-independence number

By using Cohen forcing we will prove it is consistent that $\mathfrak{i}_d < \mathfrak{c}$.

Theorem 4.2.8. *Suppose $V \models CH$. Then $V^{\mathbb{C}(\omega_2)} \models \mathfrak{i}_d = \omega_1$.*

To prove Theorem 4.2.8 we use the following lemma.

Lemma 4.2.9. *Assume $p \in \mathbb{C}$, $\mathcal{I}$ is a countable dual-independent family and $\dot{X}$ is a $\mathbb{C}$-name such that $p \Vdash$ "$\dot{X}$ is a non-trivial infinite partition of ω and $\{\dot{X}\} \cup \mathcal{I}$ is dual-independent". Then there exists $X^* \in (\omega)^\omega \cap V$ such that $\{X^*\} \cup \mathcal{I}$ is dual-independent and $p \Vdash \dot{X} \perp X^*$.*

Proof of 4.2.8 from 4.2.9 Within the ground model we shall define a maximal dual-independent family $\mathcal{I}$ of size ω_1. It suffices to verify maximality of $\mathcal{I}$ in the extension via $\mathbb{C}$ (see [22] pp256).

By CH, let $\langle p_\xi, \tau_\xi \rangle$ $\xi < \omega_1$ enumerate all pairs $\langle p, \tau \rangle$ such that $p \in \mathbb{C}$ and τ is a nice name for an infinite partition of ω. By recursion, pick an infinite partition of ω as follows. Given $\{X_\eta : \eta < \xi\}$ for some $\xi < \omega_1$. Choose X_ξ so that

(1) $\{X_\xi\} \cup \{X_\eta : \eta < \xi\}$ is dual-independent.

(2) If $p_\xi \Vdash$ "$\{\tau_\xi\} \cup \{X_\eta : \eta < \xi\}$ is dual-independent", then $p_\xi \Vdash X_\xi \perp \tau_\xi$.

(2) is possible by Lemma 4.2.9. Let $\mathcal{I} = \{X_\eta : \eta < \omega_1\}$. We shall prove $\mathcal{I}$ is maximal. If $\mathcal{I}$ is not maximal in $V[G]$ for some $\mathbb{C}$-generic G, then there exists $p_\xi \in G$ and τ_ξ such that $p_\xi \Vdash \{\tau_\xi\} \cup \mathcal{I}$ is dual-independent. By construction there exists $X_\xi \in \mathcal{I}$ and $p_\xi \Vdash \tau_\xi \perp X_\xi$. It is a contradiction.

□

Proof of 4.2.9. Let $\mathbb{P}(\mathcal{I})$ be a partial order such that $\langle \sigma, \mathcal{H} \rangle \in \mathbb{P}(\mathcal{I})$ if σ is a partition of a finite subset of ω and $\mathcal{H}$ is a finite subset of $\mathcal{I}$. It is ordered by $\langle \sigma, \mathcal{H} \rangle \leq \langle \tau, \mathcal{G} \rangle$ if

(i) $\forall x \in \tau \exists x' \in \sigma (x \subset x')$,

(ii) $\mathcal{H} \supset \mathcal{G}$,

(iii) $\forall x_0 \neq x_1 \in \tau \forall x'_0 \in \sigma \, (x_0 \subset x'_0 \to x_1 \cap x'_0 = \emptyset)$,

(iv) $\forall Y \in \mathcal{G} \forall y_0, y_1 \in (Y \wedge \tau) \forall y'_0, y'_1 \in (Y \wedge \sigma)$

$$\Big(y_0 \cap y_1 = \emptyset \wedge \bigcup \tau \cap y_0 \neq \emptyset \wedge \bigcup \tau \cap y_1 \neq \emptyset \wedge y_0 \subset y'_0 \wedge y_1 \subset y'_1 \to y'_0 \cap y'_1 = \emptyset\Big).$$

Claim 7. *The following sets are dense.*

(i) $D_n = \{\langle \sigma, \mathcal{H} \rangle : n \in \bigcup \sigma\}$ *for* $n \in \omega$.

(ii) $D^l_{\mathcal{A}} = \{\langle \sigma, \mathcal{H} \rangle : \mathcal{A} \subset \mathcal{H} \wedge |\{h \in (\bigwedge \mathcal{H} \wedge \sigma) : h \cap \bigcup \sigma \neq \emptyset\}| \geq l\}$ *for finite subsets* $\mathcal{A}$ *of* $\mathcal{I}$ *and* $l \in \omega$.

(iii) $D_{\mathcal{A},l} = \{\langle \sigma, \mathcal{H} \rangle : \mathcal{A} \subset \mathcal{H} \wedge \exists x \in \sigma \, (|\{h \in \bigwedge \mathcal{H} : x \cap h \neq \emptyset\}| \geq l)\}$ *for finite subsets* $\mathcal{A}$ *of* $\mathcal{I}$ *and* $l \in \omega$.

(iv) Let $\mathcal{A}$ *be a finite subset of* $\mathcal{I}$, $B \in \mathcal{I} \setminus \mathcal{A}$ *and* $A = \bigwedge \mathcal{A}$. *Since* $\neg(A \leq^* B)$ *and by Lemma 4.2.5, there exists* $\{a_n\}_{n \in \omega}$ *such that*

$$\forall n \in \omega \exists b \in B \, (a_{2n} \cap b \neq \emptyset \wedge a_{2n+1} \cap b \neq \emptyset) \tag{4.1}$$

or there exists a finite subset A_0 *of* A *such that the set*

$$\mathcal{F}_{A_0} = \{a \in A \setminus A_0 : \exists y \in Y \left(y \cap a \neq \emptyset \wedge y \cap \bigcup A_0 \neq \emptyset\right)\} \tag{4.2}$$

is infinite. If (4.1) holds, fix $\{a_n\}_{n \in \omega}$. *If (4.2) holds, fix* A_0 *and* $\mathcal{F}_{A_0}$

(4.1) Let $D_{\mathcal{A},B,l} = \{\langle\sigma,\mathcal{H}\rangle : \exists\{a^i : i < 2l\} \subset (A\wedge\sigma)\ \big(\forall i < 2l(\bigcup\sigma\cap a^i \neq \emptyset)\wedge$ $\wedge\{a^i : i < 2l\}$ *is pairwise disjoint* $\wedge\,\forall i < l\exists b \in B(a^{2i}\cap b \neq \emptyset \wedge a^{2i+1}\cap b \neq \emptyset)\big)\}$.

(4.2) Let $D_{\mathcal{A},B,l} = \{\langle\sigma,\mathcal{H}\rangle : \exists\{a^i : i < l\} \subset (A\wedge\sigma)\ \big(\forall i < l(\bigcup\sigma\cap a^i \neq \emptyset)\wedge$ $\{a^i : i < l\}$ *is pairwise disjoint* $\wedge\,\forall i < l(\bigcup A_0\cap a^i = \emptyset)\wedge$ $\forall a \in A_0(a\cap\bigcup\sigma \neq \emptyset)\wedge\forall i < l\exists b \in B(b\cap a^i \neq \emptyset\wedge b\cap\bigcup A_0 \neq \emptyset)\big)\}$.

(v) Let $\{\dot{x}_i : i \in \omega\}$ *be* $\mathbb{C}$*-names such that* $\Vdash \dot{X} = \{\dot{x}_i : i \in \omega\}$ *and* $\min\dot{x}_i < \min\dot{x}_{i+1}$. *Put* $D_{\dot{X},l,q} = \{\langle\sigma,\mathcal{H}\rangle : \exists r \leq q\left(r \Vdash \exists x \in (\dot{X}\wedge\sigma)(\bigcup_{i<l}\dot{x}_i \subset x)\right)\}$ *for* $q \leq p$ *and* $l \in \omega$.

Proof of Claim.
(i) Clear.
(ii) Let $\langle\tau,\mathcal{H}\rangle \in \mathbb{P}(\mathcal{I})$. Without loss of generality, we can assume $\mathcal{A} \subset \mathcal{H}$. Let $H = \wedge\mathcal{H}$. Choose $h_i \in H$ for $i < l$ such that $h_i\cap\bigcup\tau = \emptyset$. Choose $n_i \in h_i$. Put $\sigma = \tau\cup\{\{n_i\} : i < l\}$. Then $\{h_i : i < l\} \subset \{h \in (H\wedge\sigma) : h\cap\bigcup\sigma \neq \emptyset\}$. So $\langle\sigma,\mathcal{H}\rangle \in D^l_{\mathcal{A}}$.

We shall prove $\langle\sigma,\mathcal{H}\rangle \leq \langle\tau,\mathcal{H}\rangle$. Let $Y \in \mathcal{H}$. Since $h_i\cap\bigcup\tau = \emptyset$ and $n_i \in h_i$ for $i < l$, $\{y \in (Y\wedge\sigma) : y\cap\bigcup\sigma \neq \emptyset\} = \{y \in (Y\wedge\tau) : y\cap\bigcup\sigma \neq \emptyset\}\dot{\cup}\{y \in Y : \exists i < l(n_i \in y)\}$. Hence $\langle\sigma,\mathcal{H}\rangle \leq \langle\tau,\mathcal{H}\rangle$. (iii) Let $\langle\tau,\mathcal{H}\rangle \in \mathbb{P}(\mathcal{I})$. Without loss of generality, we can assume $\mathcal{A} \subset \mathcal{H}$. Let $H = \bigwedge\mathcal{H}$. Choose $\{h_i : i < l\}$ distinct elements of H such that $h_i\cap\bigcup\tau = \emptyset$ for $i < l$. Choose $n_i \in h_i$ for $i < l$. Put $\sigma = \tau\cup\{\{n_i : i < l\}\}$. Then $\{h \in H : \{n_i : i < l\}\cap h \neq \emptyset\} = \{h_i : i < l\}$. So $\langle\sigma,\mathcal{H}\rangle \in D_{\mathcal{A},l}$.

We shall prove $\langle\sigma,\mathcal{H}\rangle \leq \langle\tau,\mathcal{H}\rangle$.

Since $h_i\cap\bigcup\tau = \emptyset$ and $n_i \in h_i$ for $i < l$, $\{y \in (Y\wedge\sigma) : y\cap\bigcup\sigma \neq \emptyset\} = \{y \in (Y\wedge\tau) : y\cap\bigcup\tau \neq \emptyset\}\dot{\cup}\{\bigcup\{y \in Y : \exists i < l(n_i \in y)\}\}$. Hence $\langle\sigma,\mathcal{H}\rangle \leq \langle\tau,\mathcal{H}\rangle$. (iv) (1) Let $\langle\tau,\mathcal{H}\rangle \in \mathbb{P}(\mathcal{I})$. Choose distinct $i_j \in \omega$ for $j \leq l$ so that $\bigcup\tau\cap a_{2i_j} = \emptyset$ and $\bigcup\tau\cap a_{2i_j+1} = \emptyset$ for $j < l$. Let $k_n = \min a_n$ for $n \in \omega$. Put $\sigma = \tau\cup\{\{k_{2i_j}\},\{k_{2i_j+1}\} : j < l\}$. Since $\bigcup\tau\cap a_{2i_j} = \bigcup\tau\cap a_{2i_j+1} = \emptyset$ and $k_n \in a_n$, $\{a_{2i_j}, a_{2i_j+1} : j < l\} \subset (A\wedge\sigma)$, $\{a_{2i_j}, a_{2i_j+1} : j < l\}$ is pairwise distinct and for $i < l$ there exists $b \in B$ such that $b\cap a_{2i_j} \neq \emptyset$ and $b\cap a_{2i_j+1} \neq \emptyset$. So $\langle\sigma,\mathcal{H}\rangle \in D_{\mathcal{A},B,l}$.

We shall prove $\langle\sigma,\mathcal{H}\rangle \leq \langle\tau,\mathcal{H}\rangle$ Let $Y \in \mathcal{H}$. Since $\bigcup\tau\cap a_{2i_j} = \bigcup\tau\cap a_{2i_j+1} = \emptyset$, $\{y \in (Y\wedge\sigma) : y\cap\bigcup\sigma \neq \emptyset\} = \{y \in (Y\wedge\tau) : y\cap\bigcup\sigma \neq \emptyset\}\cup\{y \in (Y\wedge\tau) : \exists j < l(k_{2i_j} \in y\vee k_{2i_j+1} \in y)\}$. Hence $\langle\sigma,\mathcal{H}\rangle \leq \langle\tau,\mathcal{H}\rangle$.
(2) Let $\langle\tau,\mathcal{H}\rangle \in \mathbb{P}(\mathcal{I})$. Without loss of generality we can assume $\bigcup\tau\cap a \neq \emptyset$ for $a \in A_0$. Choose distinct a^i for $i < l$ so that $a^i\cap\bigcup\tau = \emptyset$ and $a^i \in \mathcal{F}_{A_0}$. Let $k_i = \min a^i$ and $\sigma = \tau\cup\{\{k_i\} : i < l\}$. Since $\bigcup\tau\cap a^i = \emptyset$, $a^i \in \mathcal{F}_{A_0}$ and $k_i \in a^i$, $\{a^i : j < l\} \subset (A\wedge\sigma)$, $\{a^i : i < l\}$ is pairwise distinct, $\bigcup A_0\cap a^i = \emptyset$ and for each $i < l$ there exists $b \in B$ such that $b\cap a^i \neq \emptyset$ and $b\cap\bigcup A_0 \neq \emptyset$. So $\langle\sigma,\mathcal{H}\rangle \in D_{\mathcal{A},B,l}$.

We shall prove $\langle\sigma,\mathcal{H}\rangle \leq \langle\tau,\mathcal{H}\rangle$. Let $Y \in \mathcal{H}$. Then $\{y \in (Y\wedge\sigma) : y\cap\bigcup\sigma \neq \emptyset\} = \{y \in (Y\wedge\tau) : y\cap\bigcup\tau \neq \emptyset\}\cup\{y \in (Y\wedge\tau) : \exists i < l(k_i \in y)\}$. Hence $\langle\sigma,\mathcal{H}\rangle \leq \langle\tau,\mathcal{H}\rangle$. (v) Let $\langle\tau,\mathcal{H}\rangle \in \mathbb{P}(\mathcal{I})$ and $q \in \mathbb{C}$. Let $H = \bigwedge\mathcal{H}$. Let $q' \leq q$ and $n_i \in \omega$ such that $q' \Vdash n_i \in \dot{x}_i$ for $i < l$. Without loss of generality we can assume $n_i \in \bigcup \tau$. Since $p \Vdash \{\dot{X}\} \cup \mathcal{I}$ is dual-independent, $p \Vdash \neg(H \leq^* \dot{X})$. So $p \Vdash$ "$\exists \langle h_n : n \in \omega \rangle \subset H \Big(\forall n \in \omega \exists x \in \dot{X} (h_{2n} \cap x \neq \emptyset \wedge h_{2n+1} \cap x \neq \emptyset) \Big)$ or $\exists H_0 \subset H$ finite $\Big(\Big| \{h \in H \setminus H_0 : \exists x \in \dot{X} (x \cap h \neq \emptyset \wedge x \cap \bigcup H_0 \neq \emptyset)\} \Big| = \omega \Big)$". Without loss of generality we can assume

$$q' \Vdash \text{“}\exists \langle h_n : n \in \omega \rangle \subset H \Big(\forall n \in \omega \exists x \in \dot{X} (h_{2n} \cap x \neq \emptyset \wedge h_{2n+1} \cap x \neq \emptyset) \Big)\text{”} \tag{4.3}$$

or

$$q' \Vdash \text{“}\exists \text{finite } H_0 \subset H \Big(\Big| \{h \in H \setminus H_0 : \exists x \in \dot{X} (x \cap h \neq \emptyset \wedge x \cap \bigcup H_0 \neq \emptyset)\} \Big| = \omega \Big)\text{”}. \tag{4.4}$$

<u>case(4.3)</u> Let $r \leq q'$, $\langle h_i : i < 2l \rangle \subset H$ and $\langle k_i : i < 2l \rangle$ such that $\bigcup \sigma \cap h_i = \emptyset$, h_i are pairwise disjoint and

$$r \Vdash \forall i < l \exists x \in \dot{X} \left(k_{2i} \in x \cap h_{2i} \wedge k_{2i+1} \in x \cap h_{2i+1} \right).$$

Put $k_{-1} = k_0$. Then put $\sigma = \{s' : s' = s \cup \{k_{2i}, k_{2i-1} : n_i \in s\} \text{ for } s \in \tau\}$.

We shall prove $\langle \sigma, \mathcal{H} \rangle \in D_{\dot{X},l,q}$. Let $\dot{x}$ be a $\mathbb{C}$-name such that $r \Vdash$ "$\dot{x} \in (\dot{X} \wedge \sigma) \wedge \dot{x}_i \subset \dot{x}$" for some $i < l$. Since $r \Vdash n_i \in \dot{x}_i$, $r \Vdash n_i \in \dot{x}$. Since there exists $s' \in \sigma$ such that $\{n_i, k_{2i}, k_{2i-1}\} \subset s'$, $r \Vdash k_{2i} \in \dot{x}$. Since $r \Vdash$ "$\exists x \in \dot{X}(\{k_{2i}, k_{2i+1}\} \subset x)$" and there exists $s' \in \sigma$ such that $\{k_{2i+1}, k_{2i+2}, n_{i+1}\} \subset s'$, $r \Vdash n_{i+1} \in \dot{x}$. So $r \Vdash \bigcup_{i<l} \dot{x}_i \subset \dot{x}$. Hence $\langle \sigma, \mathcal{H} \rangle \in D_{\dot{X},l,q}$.

Finally we shall prove $\langle \sigma, \mathcal{H} \rangle \leq \langle \tau, \mathcal{H} \rangle$. Let $Y \in \mathcal{H}$ and $y_i \in Y$ such that $k_i \in y_i$ for $i < 2l$. Then $\{y \in (Y \wedge \sigma) : y \cap \bigcup \sigma \neq \emptyset\} = \{y \cup \bigcup \{y_{2i}, y_{2i-1} : \exists i < l (n_i \in y)\} : y \in (Y \wedge \tau) \wedge y \cap \bigcup \tau \neq \emptyset\}$. Since $H \leq Y$, $\{h_i : i < 2l\}$ is pairwise disjoint and $\bigcup \tau \cap h_i = \emptyset$ for $i < 2l$, $\{y_i : i < 2l\}$ is pairwise disjoint and $\bigcup \tau \cap y_i = \emptyset$ for $i < l$. So if $y \neq y' \in (Y \wedge \tau)$ with $y \cap \bigcup \tau \neq \emptyset \wedge y' \cap \bigcup \tau \neq \emptyset$, then $(y \cup \bigcup \{y_{2i}, y_{2i-1} : n_i \in y\}) \cap (y' \cup \bigcup \{y_{2i}, y_{2i-1} : n_i \in y'\}) = \emptyset$. Hence $\langle \sigma, \mathcal{H} \rangle \leq \langle \tau, \mathcal{H} \rangle$.

<u>case(4.4)</u> Let G be $\mathbb{C}$-generic over V with $q' \in G$. We will work in $V[G]$. Let H_0 be a finite subset of H such that the set

$$\{h \in H \setminus H_0 : \exists x \in \dot{X}[G] : h \cap x \neq \emptyset \wedge x \cap \bigcup H_0 \neq \emptyset\}$$

is infinite where $\dot{X}[G]$ is the interpretation of $\dot{X}$ in $V[G]$. Since H_0 is finite, there exists $h' \in H_0$ such that the set

$$\{h \in H \setminus \{h'\} : \exists x \in \dot{X}[G] \left(h \cap x \neq \emptyset \wedge x \cap h' \neq \emptyset \right)\}$$

is infinite.

Let $\langle h_j : j \in \omega \rangle$ be an enumeration of the set

$$\{h \in H \setminus \{h'\} : \exists x \in \dot{X}[G] \Big(h \cap x \neq \emptyset \wedge x \cap h' \neq \emptyset \wedge h \cap \bigcup \tau = \emptyset \Big)\}$$

and $\langle k_j : j \in \omega \rangle$ be natural numbers such that

$$\exists x \in \dot{X}[G] (k_{2j} \in x \cap h_j \wedge k_{2j+1} \in x \cap h').$$

Let $\{Y_i : i < m\}$ be an enumeration of $\mathcal{H}$. By induction we shall construct decreasing sequence $\{A_j : j < m\}$ of infinite sets of natural numbers. Put $A_{-1} = \{k_{2i+1} : i \in \omega\} \setminus \bigcup \tau$.

Suppose we already have A_j. Let $A_j \restriction Y_{j+1} = \{A_j \cap y : y \in Y_{j+1}\} \setminus \{\emptyset\}$. If $A_j \restriction Y_{j+1}$ is infinite, put

$$A_{j+1} = \bigcup \{A_j \cap y : y \cap \bigcup \tau = \emptyset \wedge y \in Y_{j+1}\}.$$

If $A_j \restriction Y_{j+1}$ is finite, then choose $y \in Y_{j+1}$ so that $A_j \cap y$ is infinite and put

$$A_{j+1} = y \cap A_j.$$

In both cases A_{j+1} is infinite. Choose j_i for $i < l$ so that $k_{2j_i+1} \in A_{m-1}$ for $i < l$. Then define $\sigma = \{s' : s' = s \cup \{k_{2j_i} : n_i \in s\}$ for $s \in \tau\} \cup \{\{k_{2j_i+1} : i < l\}\}$.

From now on we will work in V and prove $\langle \sigma, \mathcal{H} \rangle \in D_{\dot{X},q,l}$. Let $r \leq q'$ such that

$$r \Vdash \forall i < l \exists x \in \dot{X} \left(k_{2j_i} \in x \cap h_{j_i} \wedge k_{2j_i+1} \in x \cap h'\right).$$

Suppose $r \Vdash$ "$\dot{x} \in (X \wedge \sigma) \wedge \dot{x}_i \subset \dot{x}$" for some $i < l$ and a $\mathbb{C}$-name $\dot{x}$. Since $r \Vdash \dot{x}_i \subset \dot{x}$, $r \Vdash n_i \in \dot{x}$. Since there exists $s' \in \sigma$ such that $\{k_{2j_i}, n_i\} \subset s'$, $r \Vdash \{k_{2j_i}, n_i\} \subset \dot{x}$. Since $r \Vdash \exists x \in \dot{X}(k_{2j_i} \in x \cap h_{j_i} \wedge k_{2j_i+1} \in x \cap h')$, $r \Vdash \{k_{2j_i}, k_{2j_i+1}\} \subset \dot{x}$. Since $\{k_{2j_i+1} : i < l\} \in \sigma$, $r \Vdash k_{2j_{i+1}+1} \in \dot{x}$. By similar argument, we have $r \Vdash \dot{x}_{i+1} \subset \dot{x}$. Therefore $r \Vdash \bigcup_{i<l} \dot{x}_i \subset \dot{x}$. Hence $\langle \sigma, \mathcal{H} \rangle \in D_{\dot{X},q,l}$.

Finally we shall prove $\langle \sigma, \mathcal{H} \rangle \leq \langle \tau, \mathcal{H} \rangle$. Let $Y \in \mathcal{H}$. By construction of $\{A_j : j < m\}$, there is $y \in Y$ such that $\{k_{2j_i+1} : i < l\} \subset y$ or for $i < l$ and $y \in Y$ if $k_{2j_i+1} \in y$, then $y \cap \bigcup \tau = \emptyset$.

<u>case 1. There is $y \in Y$ such that $\{k_{2j_i+1} : i < l\} \subset y$.</u>

For each $y \in Y$ let $y_\tau \in (Y \wedge \tau)$ such that $y \subset y_\tau$. Let $y' \in Y$ such that $\{k_{2j_i+1} : i < l\} \subset y'$. Then $\{y \in (Y \wedge \sigma) : y \cap \bigcup \sigma \neq \emptyset\} = \{y'_\tau\} \cup \{y_\tau \cup \bigcup \{y^* \in Y : \exists i < l\, (k_{2j_i} \in y^* \wedge n_i \in y_\tau)\} : y \cap \bigcup \tau \neq \emptyset \wedge y \in Y\}$.

Suppose $y'_\tau \neq y_\tau$ for some $y \in Y$ with $y \cap \bigcup \tau \neq \emptyset$. Since $H \leq Y$, $\{h_{j_i} : i < l\} \cup \{h'\}$ is pairwise disjoint, $y' \subset h'$, $k_{2j_i} \in h_{j_i}$ and $\bigcup \sigma \cap h_i = \emptyset$, $y'_\sigma \cap y_\sigma = y'_\tau \cap (y_\tau \cup \{y^* \in Y : \exists i < l\, (k_{2j_i} \in y^* \wedge n_i \in y_\tau)\}) = \emptyset$.

Let $y^0_\tau \neq y^1_\tau$ such that $y^0_\tau \neq y'_\tau$, $y^1_\tau \neq y'_\tau$, $y^0 \cap \bigcup \tau \neq \emptyset$ and $y^1 \cap \bigcup \tau \neq \emptyset$. Since $H \leq Y$, $\{h_{j_i} : i < l\}$ is pairwise disjoint, $y' \subset h'$, $k_{2j_i} \in h_{j_i}$ and $\bigcup \sigma \cap h_i = \emptyset$, $y^0_\sigma \cap y^1_\sigma = (y^0_\tau \cup \bigcup \{y^* \in Y : \exists i < l\, \left(k_{2j_i} \in y^* \wedge n_i \in y^0_\tau\right)\}) \cap (y^1_\tau \cup \bigcup \{y^* \in Y : \exists i < l\, \left(k_{2j_i} \in y^* \wedge n_i \in y^1_\tau\right)\} = \emptyset$. Hence $\forall y^0, y^1 \in Y$

$$\left(y^0_\tau \cap y^1_\tau = \emptyset \wedge \bigcup \tau \cap y^0 \neq \emptyset \wedge \bigcup \tau \cap y^1 \neq \emptyset \rightarrow y^0_\sigma \cap y^1_\sigma = \emptyset\right).$$

<u>case 2. for $i < l$ and $y \in Y$ if $k_{2j_i+1} \in y$.</u>

If $\forall i < l \forall y \in Y\, (k_{2j_i} \in y \rightarrow y \cap \bigcup \tau = \emptyset)$, $\{y \in (Y \wedge \sigma) : y \cap \bigcup \sigma \neq \emptyset\} = \{\bigcup \{y \in Y : \exists i < l(k_{2j_i+1} \in y)\}\} \cup \{y_\tau \cup \bigcup \{y^* \in Y : \exists i < l\, (k_{2j_i} \in y^* \wedge n_i \in y_\tau)\} : y \cap \bigcup \tau \neq \emptyset \wedge y \in Y\}$. Since $k_{2j_i+1} \in y$ implies $y \cap \bigcup \tau = \emptyset$, $\bigcup \{y \in Y : \exists i < l(k_{2j_i+1} \in y)\} \cap \bigcup \tau = \emptyset$.

Let $y^0_\tau \neq y^1_\tau$ with $y^0 \cap \bigcup \tau \neq \emptyset$ and $y^1 \cap \bigcup \tau \neq \emptyset$. Since $H \leq Y$ and $\{h_{j_i} : i < l\}$ is pairwise disjoint, $(y^0_\tau \cup \bigcup\{y^* \in Y : \exists i < l(k_{2j_i} \in y^* \wedge n_i \in y^0_\tau)\}) \cap (y^1_\tau \cup \bigcup\{y^* \in Y : \exists i < l(k_{2j_i} \in y^* \wedge n_i \in y^1_\tau)\}) = \emptyset$. Hence $\forall y^0, y^1 \in Y$

$$\left(y^0_\tau \cap y^1_\tau = \emptyset \wedge \bigcup \tau \cap y^0 \neq \emptyset \wedge \bigcup \tau \cap y^1 \neq \emptyset \rightarrow y^0_\sigma \cap y^1_\sigma = \emptyset\right).$$

Therefore $\langle \sigma, \mathcal{H}\rangle \leq \langle \tau, \mathcal{H}\rangle$.

Claim ■

Let $\mathcal{D} = \{D_n : n \in \omega\} \cup \{D^l_{\mathcal{A}} : \mathcal{A}$ is a finite subset of $\mathcal{I} \wedge l \in \omega\} \cup \{D_{\mathcal{A},l} : \mathcal{A}$ is a finite subset of $\mathcal{I} \wedge l \in \omega\} \cup \{D_{\mathcal{A},B,l} : \mathcal{A}$ is a finite subset of $\mathcal{I} \wedge B \in \mathcal{I} \setminus \mathcal{A} \wedge l \in \omega\} \cup \{D_{\dot{X},l,q} : q \leq p \wedge l \in \omega\}$ and G is $\mathcal{D}$-generic for $\mathbb{P}(\mathcal{I})$.

Let X_G be a partition generated by $\equiv_G$ where $\equiv_G$ is defined by

$$n \equiv_G m \text{ if } \exists \langle \sigma, \mathcal{H}\rangle \exists x \in \sigma\,(\{n, m\} \subset x).$$

Then by (i) and (ii) $X_G \in (\omega)^\omega$. By (ii) $X_G \wedge \bigwedge \mathcal{A} \in (\omega)^\omega$ for finite $\mathcal{A} \subset \mathcal{I}$. By (iii) $\neg(\bigwedge \mathcal{A} \leq^* X_G)$ for finite $\mathcal{A} \subset \mathcal{I}$. By (iv) $\neg(X_G \wedge \bigwedge \mathcal{A} \leq^* Y)$ for finite $\mathcal{A} \subset \mathcal{I}$ and $Y \in \mathcal{I} \setminus \mathcal{A}$. Therefore $\{X_G\} \cup \mathcal{I}$ is dual-independent by Corollary 4.2.7. By (v) $p \Vdash \dot{X} \perp X_G$. Hence X_G is a required partition.

□

4.3 reaping number and splitting number for partitions of ω

In this section we shall investigate the relationship between $\mathfrak{r}_d$, $\mathfrak{s}_d$, $\mathfrak{b}$ and $\mathfrak{d}$.

Definition 16. *[14] Let $\mathbb{DS}$ be a forcing notion such that $\langle \sigma, A\rangle \in \mathbb{DS}$ such that*

(1) σ is a partition of finite subset of ω,

(2) $A \in (\omega)^{<\omega}$,

(3) for $s \in \sigma$ there exists $a \in A$ such that $s \subset a$ and

(4) for $a \in A$ the set $\{s \in \sigma : s \subset a\}$ has cardinality at most one.

$\mathbb{DS}$ is ordered by $\langle \sigma, A\rangle \leq \langle \tau, B\rangle$ if

(i) $\forall t \in \tau \exists s \in \sigma\,(t \subset s)$,

(ii) $A \geq B$.

Theorem 4.3.1 (Brendle)**.** *The followings are consistent;*

(i) $\mathfrak{s}_d < \mathfrak{d}$,

(ii) $\mathfrak{r}_d > \mathfrak{b}$

Proof. It suffices to show following claim.

Claim 8. *Let $\dot{f}$ be a $\mathbb{DS}$-name such that $\Vdash_{\mathbb{DS}} \dot{f} \in \omega^\omega$. There exists $\langle f_n : n \in \omega\rangle \in V$ such that $f_n \in \omega^\omega$ and for any $g \in \omega^\omega \cap V$ if $g \not\leq^* f_n$, then $\Vdash_{\mathbb{DS}} g \not\leq^* \dot{f}$*

Proof of Claim. Let $\dot{f}$ be a $\mathbb{DS}$-name for a function from ω to ω. Let $DS_{\sigma,k}$ be a subset of $\mathbb{DS}$ such that $\langle \tau, A\rangle \in DS_{\sigma,k}$ if $\langle \tau, A\rangle \in \mathbb{DS}$, $\tau = \sigma$ and $|A| \leq k$. Define $f_{\sigma,k} \in \omega^\omega \cap V$ so that $f_{\sigma,k}(n) = \min\{m : \forall \langle \sigma, A\rangle \in DS_{\sigma,k} \neg\langle \sigma, A\rangle \Vdash \dot{f}(n) \geq m\}$.
Subclaim. $f_{\sigma,k}$ is well-defined.
Proof of subclaim. Suppose not. Then there exists $n \in \omega$ such that for each $j \in \omega$ there exists $A_j \in (\omega)^{\leq k}$ such that $\langle \sigma, A_j\rangle \Vdash \dot{f}(n) \geq j$. For $A \in (\omega)^{\leq k}$ with $\{a_i : i < k\}$ such that $\min a_i < \min a_j$ for $i < j$ define $h_A \in k^\omega$ such that $h_A(l) = i$ if $l \in a_i$. By compactness of k^ω there exists $A \in (\omega)^{\leq k}$ and $\langle j_l : l \in \omega\rangle$ such that $\lim_{l\to\infty} h_{A_{j_l}} = h_A$. Then $\langle \sigma, A\rangle \in \mathbb{DS}$ since for large enough $i \in \omega$ $h_{A_{j_i}} \upharpoonright \cup\sigma = h_A \upharpoonright \cup\sigma$ and $\langle \sigma, A_{j_i}\rangle \in \mathbb{DS}$.

Let $\langle \tau, B\rangle \leq \langle \sigma, A\rangle$ and $m \in \omega$ such that $\langle \tau, B\rangle \Vdash \dot{f}(n) = m$. Since $h_{j_i} \to h_A$, there exists i_0 such that $i \geq i_0$ implies $j_i > m$ and $h_{A_{j_i}} \upharpoonright \cup\tau = h_A \upharpoonright \cup\tau$, so is $\langle \tau, B\rangle$ and $\langle \sigma, A_{j_i}\rangle$ compatible. But it is contradiction to $\langle \sigma, A_{j_i}\rangle \Vdash \dot{f}(n) \geq j_i > m$.

subclaim ■

Let $g \in \omega^\omega \cap V$ such that $g \not\leq^* f_{\sigma,k}$ for a partition σ of a finite subset of ω and $k \in \omega$. Let $n \in \omega$ and $\langle \sigma, A\rangle \in \mathbb{DS}$ with $|A| \leq k$. Then there exists $m \geq n$ such that $g(m) > f_{\sigma,k}(m)$. By definition of $f_{\sigma,k}$, there exists $\langle \tau, B\rangle \leq \langle \sigma, A\rangle$ such that $\langle \tau, B\rangle \Vdash \dot{f}(m) \leq f_{\sigma,k}(m) < g(m)$. So $\Vdash_{\mathbb{DS}} g \not\leq^* \dot{f}$.

Claim ■ Theorem □

By this theorem it looks that there is no relation between $\mathfrak{r}_d$ and $\mathfrak{d}$, $\mathfrak{s}_d$ and $\mathfrak{b}$. But Kamo at Osaka prefecture university prove the following Theorem.

Theorem 4.3.2. *(Kamo) $\mathfrak{b} \leq \mathfrak{s}_\mathfrak{d}$. $\mathfrak{d} \geq \mathfrak{r}_d$.*

To prove this theorem we use the following lemma.

Lemma 4.3.3. *Suppose $M \models ZFC^-$. Let $d \in \omega^\omega$ such that $f \leq^* d$ for $f \in \omega^\omega \cap M$. Let $a = rng(d)$. Then $x \setminus a$ is infinite for $x \in M \cap \omega^\omega$.*

□

[$\mathfrak{b} \leq \mathfrak{s}_d$]
Let $M_0 \subset M_1 \subset M_2 \subset \ldots$ be a sequence of ZFC^- model. Let $\{d_{n+1} : n \in \omega\}$ be a sequence such that

- $d_{n+1} \in M_{n+1} \cap \omega^{\uparrow\omega}$,
- $f \leq^* d_{n+1}$ for $f \in M_n \cap \omega^\omega$ for $n \in \omega$,

- $rng(d_{n+1}) \supset rng(d_n)$,
- $d_{n+1}(0) > n$ and
- for $f \in M_0 \cap \omega^\omega$ for all but finite $n \in \omega$ $|[f(n), f(n+1)) \cap rng(d_1)| \leq 1$.

For each $n \in \omega$ put $a_0 = \omega$ and $a_n = rng(d_1) \cap rng(d_2) \cap \ldots \cap rng(d_n)$. Then $\omega = a_0 \subset a_1 \subset \ldots$ and $\bigcap_{n<\omega} a_n = \emptyset$. Put $b_n = a_n \setminus a_{n+1}$ for $n \in \omega$. By Lemma 4.3.3 $b_n \in [\omega]^\omega$. Put $B = \{b_n : n \in \omega\}$. Then $B \in (\omega)^\omega$ and $B \subset [\omega]^\omega$.

Lemma 4.3.4. *For $x \in M_0 \cap [\omega]^\omega$ and $n \in \omega$ $x \cap a_n$ is infinite if and only if $x \cap b_n$ is infinite.*

Proof. ($\Leftarrow$) It is clear.

($\Rightarrow$) By Lemma 4.3.3 and $x \cap a_n \in [\omega]^\omega \cap M_n$, $x \cap a_n \setminus a_{n+1} = x \cap (a_n \setminus a_{n+1}) = x \cap b_n$ is infinite.

□

Lemma 4.3.5. *For $X \in M_0 \cap (\omega)^\omega$ if $X \| B$, then $B \leq^* B$.*

$\mathfrak{b} \leq \mathfrak{s}_d$ follows directly from Main lemma. Because by Lemma **??** $|M_0 \cap \omega^\omega| < \mathfrak{b}$ implies $M_0 \cap (\omega)^\omega$ cannot dual-split B.

Lemma 4.3.6. *(1) For $x \in X \cap [\omega]^\omega$ there exists $n < \omega$ such that $x \cap a_n = \emptyset$.*

(2) There exists $n \in \omega$ such that $x \cap a_n = \emptyset$. Therefore $x \cap a_n = \emptyset$ for all but finite $n \in \omega$

Proof. (1) To get a contradiction, assume that $x \in X \cap [\omega]^\omega$ and $x \cap a_n \neq \emptyset$ for all $n < \omega$. Then it holds $x \cap a_n$ infinite. So by Lemma 4.3.4 $x \cap b_n$ is infinite for all $n < \omega$. So x glue all elements of B. Hence $X \perp B$. It is contradict to $X \| B$.
(2) By (1) for each $x \in X \cap [\omega]^\omega$, put $k_x = \max\{n < \omega : x \cap a_n$ is infinite$\}$. Note that for each $x \in X \cap [\omega]^\omega$ $x \cap b_j$ is infinite for $j \leq k_x$. Since $X \| B$, we have that $k = \sup\{k_x : x \in X \cap [\omega]^\omega\}$. Put $y = \bigcup (X \cap [\omega]^\omega) \setminus \bigcup_{n \leq k} b_n$. Since $X \| B$, there is an $m < \omega$ with $m > k$ such that $y \cap b_m = \emptyset$. Then we have that $y \cap a_m$ is finite. So there exists an $n \geq m$ such that $y \cap a_n = \emptyset$. □

Set $Y = \{x \in X : 2 \leq |x| < \omega\}$.

Lemma 4.3.7. *For all but finite $x \in Y$ $|x \cap a_1| \leq 1$.*

Proof. Since $Y \subset [\omega]^{<\omega}$ is pairwise disjoint, we can take $f, g \in M_0 \cap \omega^{\uparrow\omega}$ such that for all $x \in Y$ there exists $n < \omega$ such that $x \subset [f(n), f(n+1))$ or $x \subset [g(n), g(n+1))$. Take $m < \omega$ such that for $n \geq m$ $|[f(n), f(n+1)) \cap a_1| \leq 1$ and $|[g(n), g(n+1)) \cap a_1| \leq 1$. Then it holds that for $x \in Y$ if $\min x \geq \max(f(n), g(n))$, then $|x \cap a_1| \leq 1$. □

Lemma 4.3.8. $(\bigcup Y) \cap a_n = \emptyset$ *for some $n \in \omega$. Therefore $(\bigcup Y) \cap a_n = \emptyset$ for all but finite n by definition of a_n.*

Proof. Suppose not. Since $\bigcup Y \in M_0 \cap [\omega]^\omega$, we have that $(\bigcup Y) \cap b_n$ is infinite for all $n < \omega$. By Lemma 4.3.7 for all but finite $x \in Y$ $|x \cap a_1| \leq 1$. So for $x \in Y$ $x \cap a_1 = \emptyset$ $(x \subset b_0)$ or $|x \cap a_1| = (x \cap b_0 \neq \emptyset)$. Since $\bigcup Y \cap b_n \neq \emptyset$, there exists $x \in Y$ such that $x \cap b_n \neq \emptyset$. But this $x \in Y$ satisfies $x \cap b_0 \neq \emptyset$. Hence $X \perp B$. It is contradict to $X \| B$. So Lemma holds. □

Proof. (Lemma 4.3.5) By Lemma 4.3.6 and 4.3.8 take $n < \omega$ such that $x \cap a_n = \emptyset$ for $x \in X \cap [\omega]^\omega$ and $x \cap a_n = \emptyset$ for $x \in Y$. Then it holds that for $m \geq n$ for $j \in b_m$ $j \in X$. So we have that $B \leq^* X$. □

$[\mathfrak{r}_d \leq \mathfrak{d}]$ We shall prove the following theorem.

Theorem 4.3.9. *Suppose $M \models ZFC^-$ and $M \cap \omega^\omega$ is dominating family. Then $M \cap (\omega)^\omega$ is a dual-reaping family.*

Proof. Let $X \in (\omega)^\omega$.
<u>case 1</u> If there exists $x \in X$ such that x is infinite:

Take $d \in M \cap \omega^\omega$ so that $d(0) = 0$ and $x \cap [d(n), d(n+1)) \neq \emptyset$ for $n < \omega$. And put $A = \{[d(n), d(n+1)) : n \in \omega\}$. Then $A \in M \cap (\omega)^\omega$ and $A \perp X$.
<u>case 2</u> If $X \subset [\omega]^{<\omega}$:

Take $d \in M \cap \omega^{\uparrow\omega}$ so that $d(0) = 0$ and for $x \in X$ there exists $n < \omega$ such that $x \subset (d(n), d(n+2))$.

Put $a = rng(d)$, $b_1 = \{j \in a : \{j\} \in X\}$ and $b_2 = a \setminus b_1$.
<u>case 2.1</u> If b_1 is finite:

Put $A = \{\omega \setminus a\} \cup \{\{j\} : j \in a\}$. Then $A \in M$ and if $|x| \geq 2$, then $x \cap (\omega \setminus a) \neq \emptyset$ since for $x \in X$ there exists $n < \omega$ such that $x \subset (d(n), d(n+2))$. Therefore $A \perp X$.
<u>case 2.2</u> If b_2 is finite:

Put $A = \{\omega \setminus a\} \cup \{\{j\} : j \in a\}$. Then $A \in M$ and $A \leq^* X$.
<u>case 2.3</u> If both b_1 and b_1 are infinite:

Pick $e \in M \cap \omega^{\uparrow\omega}$ so that $e(0) = 0$, $b_1 \cap [e(n), e(n+1)) \neq \emptyset$ and $b_2 \cap [e(n), e(n+1)) \neq \emptyset$ for $n \in \omega$. Put $A = \{\omega \setminus a\} \cup \{[e(n), e(n+1)) \cap a : n \in \omega\}$.

Then $A \in M$ and $A \perp X$. Because if $\{j\} \in X$ and $j \in a$, then there is $n < \omega$ such that $j \in [e(n), e(n+1)) \cap a$. Pick $x \in X$ with $|x| \geq 2$ and $[e(n), e(n+1)) \neq \emptyset$. Then since there is $m \in \omega$ such that $x \subset (d(m), d(m+2))$, $x \setminus a \neq \emptyset$. So $x \cap (\omega \setminus a) \neq \emptyset$. Therefore x joints $[e(n), e(n+1)) \cap a$ and $\{\omega \setminus a\}$. Therefore $A \perp X$. □

So the following diagram holds.

$$\begin{tikzcd}
 & & & \mathfrak{s}_d \arrow[dlll] \arrow[ddl] \arrow[dddr] & & & \\
\mathrm{cov}(\mathcal{N}) \arrow[dd] & & \mathrm{non}(\mathcal{M}) \arrow[ll] \arrow[d] \arrow[dddr] & & \mathrm{cof}(\mathcal{M}) \arrow[ll] \arrow[d] & & \mathrm{cof}(\mathcal{N}) \arrow[ll] \arrow[dd] \\
 & & \mathfrak{b} \arrow[d] & & \mathfrak{d} \arrow[ll] \arrow[d] \arrow[ddl] & & \\
\mathrm{add}(\mathcal{N})) & & \mathrm{add}(\mathcal{M}) \arrow[ll] & & \mathrm{cov}(\mathcal{M}) \arrow[ll] & & \mathrm{non}(\mathcal{N}) \arrow[ll] \arrow[dlll] \\
 & & & \mathfrak{r}_d & & &
\end{tikzcd}$$

Between $\mathfrak{r}$, $\mathfrak{s}$ and cardinal invariants in Cichoń's diagram there is the following relationship.

$$\begin{tikzcd}
 & & & \mathfrak{r} \arrow[dlll] \arrow[ddl] \arrow[dddr] & & & \\
\mathrm{cov}(\mathcal{N}) \arrow[dd] & & \mathrm{non}(\mathcal{M}) \arrow[ll] \arrow[d] \arrow[dddr] & & \mathrm{cof}(\mathcal{M}) \arrow[ll] \arrow[d] & & \mathrm{cof}(\mathcal{N}) \arrow[ll] \arrow[dd] \\
 & & \mathfrak{b} \arrow[d] & & \mathfrak{d} \arrow[ll] \arrow[d] \arrow[ddl] & & \\
\mathrm{add}(\mathcal{N})) & & \mathrm{add}(\mathcal{M}) \arrow[ll] & & \mathrm{cov}(\mathcal{M}) \arrow[ll] & & \mathrm{non}(\mathcal{N}) \arrow[ll] \arrow[dlll] \\
 & & & \mathfrak{s} & & &
\end{tikzcd}$$

Also we know the following relationship.

Theorem 4.3.10. *[14][21]* $\mathfrak{s}_d \geq \mathfrak{s}$. $\mathfrak{r}_d \leq \mathfrak{r}$.

For $\mathfrak{s}$ and $\mathfrak{r}$ we have the following consistency result.

Theorem 4.3.11. *[7] It is consistent that* $\mathfrak{u} < \mathfrak{s}$. *Therefore it is consistent that* $\mathfrak{r} < \mathfrak{s}$.

By this Theorem we can say it is consistent that $\mathfrak{s}_d > \mathfrak{r}$ and $\mathfrak{r}_d < \mathfrak{s}$. So there is the following question:

Question 7. *Is it consistent that* $\mathfrak{s}_d < \mathfrak{r}$? $\mathfrak{r}_d > \mathfrak{s}$?

In this context it is natural to ask the following question.

Question 8. *Does* $\mathbb{DS}$ *preserve* $\mathfrak{r}$?

Theorem 4.3.12. *If $V \models \mathfrak{b} = \mathfrak{c}$, then $V^{\mathbb{DS}_{\omega_1}} \models \mathfrak{r} = \mathfrak{c}$.*

Lemma 4.3.13. *Let $\dot{\Pi}$ be a $\mathbb{DS}_\delta$-name for an interval partition. Then there is $\Pi_n \in IP \cap V$ such that if $\Pi \in IP \cap V = \langle I_k : k \in \omega\rangle$ dominates Π_n for $n \in \omega$, then for any $p \in \mathbb{DS}_\delta$ there exists $k_0 \in \omega$ and $q \in \mathbb{DS}_\delta$ such that for each $k \geq k_0$ there exists $r \leq q$ such that*

$$r \Vdash \exists I \in \dot{\Pi}(J \subset I_k).$$

Proof. We shall prove by induction on δ.
<u>$\delta = 1$</u> Let σ be a partition of a finite subset of ω and $k \geq |\sigma|$. Then define $g_{\sigma,k} : \omega \to \omega$ so that

$$\forall \langle \sigma, A\rangle \in DS_{\sigma,k} \neg\langle \sigma, A\rangle \Vdash \exists I \in \dot{\Pi}\left(I \subset [n, g_{\sigma,k}(n))\right).$$

Then by compact trick, $g_{\sigma,k}$ is well-defined. Define $\Pi_{\sigma,k} = \{\left[g^l_{\sigma,k}(0), g^{l+1}_{\sigma,k}(0)\right) : l \in \omega\}$ where $g^0_{\sigma,k}(0) = 0$.

Claim 9. $\Pi_{\sigma,k}(\sigma, k \geq |\sigma|)$ *satisfy required condition.*

Proof. Suppose $\Pi \in IP \cap V$ dominates all $\Pi_{\sigma,k}$. Let $p = \langle \sigma_p, A_p\rangle \in \mathbb{DS}$ and $k_p = |A_p|$. Let $k_0 \in \omega$ such that for $k \geq k_0$ there exists $J \in \Pi_{\sigma_p,k_p}$ such that $J \subset I_k$. By construction of Π_{σ_p,k_p} there exists $r \leq p$ such that

$$r \Vdash \exists I \in \dot{\Pi}(I \subset J).$$

Therefore $r \Vdash \exists I \in \dot{\Pi}(I \subset I_k)$. $\square$

<u>$\delta = \alpha + 1$</u> Suppose for α the induction hypothesis holds. Let $p \in \mathbb{DS}_{\alpha+1}$ and $\dot{\Pi}$ be a $\mathbb{DS}_{\alpha+1}$-name for an interval partition of ω. Then for each partition σ of a finite subset of ω and $l \geq |\sigma|$ let $\dot{\Pi}_{\sigma,l} = \langle \dot{I}^{\sigma,l}_m : m \in \omega\rangle$ be a $\mathbb{DS}_\alpha$-name such that

$$\Vdash_{\mathbb{DS}_\alpha} \forall \langle \sigma, A\rangle \in DS_{\sigma,l} \forall m \in \omega \neg\langle \sigma, A\rangle \Vdash_{\mathbb{DS}} \neg\left[\exists I \in \dot{\Pi}(I \subset I^{\sigma,l}_m)\right].$$

Then by induction hypothesis for each partition σ of a finite subset of ω and $l \geq |\sigma|$ there exists $\Pi^j_{\sigma,l}(j \in \omega) \in IP \cap V$ which satisfies the induction hypothesis on α.

Suppose $\Pi \in IP \cap V = \langle I_n : n \in \omega\rangle$ dominates all π. Extend p to p_0 so that there exists a partition σ of a finite subset of ω and $i \in \omega$ such that $p_0 \restriction \alpha \Vdash_{\mathbb{DS}_\alpha} p_0(\alpha) = \langle \sigma, A\rangle$ and $|A| = i$. By induction hypothesis there exists $q' \leq_{\mathbb{DS}} p_0 \restriction \alpha$ and $k_0 \in \omega$ such that for $k \geq k_0$ there exists $r' \leq_{\mathbb{DS}_\alpha} q'$ such that

$$r \Vdash \exists I \in \dot{\Pi}_{\sigma,i}(I \subset I_k).$$

By definition of $\Pi_{\sigma,i}$ $q = q'^\frown\langle \sigma, A\rangle$ and k_0 satisfies desired condition.
<u>δ is limit ordinal.</u> It is enough to show the case $cf(\delta) = \omega$. Let δ_n be a increasing sequence converging to δ as $n \to \omega$. Let $\dot{\Pi} = \langle \dot{I}_m : m \in \omega\rangle$ be a $\mathbb{DS}$-name for an interval partition of ω. For $n \in \omega$ let $\dot{\Pi}_n = \langle \dot{I}^n_m : m \in \omega\rangle$ and $\langle \dot{p}^n_m : m \in \omega\rangle$ be a

$\mathbb{DS}_{\delta_n}$-name such that $\Vdash_{\mathbb{DS}_{\delta_n}}$ "$\langle \dot{p}^n_m : m \in \omega\rangle$ is a decreasing sequence of $\mathbb{DS}_{[\delta_n,\delta)}$ and for each $m \in \omega$ $\dot{p}^n_m \Vdash_{\mathbb{DS}_{[\delta_n,\delta)}} \dot{I}_m = \dot{I}^n_m$".

For each $n \in \omega$ there exists $\Pi^j_n (j \in \omega) \in IP \cap V$ which witness induction hypothesis for $\dot{\Pi}_n$.

Suppose $p \in \mathbb{DS}_{\delta_n}$ and $\Pi \in IP \cap V = \langle I_k : k \in \omega\rangle$ dominating all Π^j_n for $j, n \in \omega$. Then there exists $q \leq_{\mathbb{DS}_{\delta_n}} p$ and k_0 which witness induction hypothesis for $\dot{\Pi}_n$ on δ_n. So for each $k \geq k_0$ there exists $r \leq_{\mathbb{DS}_{\delta_n}} q$ such that $r \Vdash \exists I \in \dot{\Pi}_n (I \subset I_k)$. Extend r to r' so that there exists $m \in \omega$ such that $r' \Vdash \dot{I}^n_m \subset I_k$. Then put $r^* = r' {}^\frown \dot{p}^n_m$. Then $r_0 \Vdash \dot{I}_m = \dot{I}^n_m \subset I_k$. Therefore q and k_0 satisfies desired property. □

Proof of Theorem from Lemma. Let $\dot{X}$ be a $\mathbb{DS}_\delta$-name for an infinite subset of ω. Let $\dot{\Pi}$ be a $\mathbb{DS}_\delta$-name for an interval partition of ω such that $\Vdash \forall I \in \dot{\Pi}(I \cap \dot{X} \neq \emptyset)$. Let $\Pi = \langle I_n : n \in \omega\rangle$ be an interval partition witnessing Lemma for $\dot{\Pi}$. Let X be an infinite and coinfinite subset of ω in V. By lemma for each p and $l \in \omega$ we can find $q_0, q_1 \in \mathbb{DS}_\delta$ and $l_0, l_1 \geq l$ such that $l_0 \in X$, $l_1 \notin X$, $q_0 \Vdash \exists I \in \dot{\Pi}(I \subset I_{l_0})$ and $q_1 \Vdash \exists I \in \dot{\Pi}(I \subset I_{l_1})$. Therefore $\Vdash \bigcup_{n \in X} I_n$ split $\dot{X}$.

□

Corollary 4.3.14. *It is consistent that $\mathfrak{s}_d < \mathfrak{r}$. Also it is consistent that $\mathfrak{r}_d > \mathfrak{s}$.*

4.4 additivity of $\mathcal{M}$, cofinality of $\mathcal{M}$, $\mathfrak{r}_d$ and $\mathfrak{s}_d$

To investigate $\mathfrak{r}_d$ and $\mathfrak{s}_d$ we introduce new cardinal invariants pair-splitting number $\mathfrak{s}_{pair}$ and pair-reaping number $\mathfrak{r}_{pair}$.

For $X \in [\omega]^\omega, A \subset [\omega]^2$ infinite, X pair-splits A if there exists infinitely many $a \in A$ such that $a \cap x \neq \emptyset$ and $a \setminus x \neq \emptyset$. We call $\mathcal{S} \subset (\omega)^\omega$ is a pair splitting family if for $A \subset [\omega]^2$ there exists $X \in \mathcal{S}$ such that if $|A| = \aleph_0$, then X pair-splits A.

$$\mathfrak{s}_{pair} = \min \{|\mathcal{S}| : \mathcal{S} \subset (\omega)^\omega \wedge \mathcal{S} \text{ is pair-splitting family }\}.$$

We call $\mathcal{R} \subset \wp([\omega]^2)$ is a pair-reaping family if $|A| = \aleph_0$ for $A \in \mathcal{R}$ and for each $X \in [\omega]^\omega$ there exists $A \in \mathcal{R}$ such that X cannot pair-split A i.e., for all but finite $a \in A$ $a \subset X$ or $a \cap X = \emptyset$.

$$\mathfrak{r}_{pair} = \min \left\{|\mathcal{R}| : \mathcal{R} \subset \wp([\omega]^2) \wedge \mathcal{R} \text{ is a pair-reaping family}\right\}.$$

$\mathfrak{r}_{pair}$ and $\mathfrak{s}_{pair}$ have the following properties.

Proposition 4.4.1. *(1)* $\mathfrak{r}_{pair} \leq \mathfrak{r}$.

(2) $\mathfrak{s} \leq \mathfrak{s}_{pair}$

(3) $\mathfrak{r}_{pair} \leq \mathfrak{s}_d$.

Proof. (1) Let $\mathcal{R} \subset [\omega]^\omega$ be a reaping family. Then for $R \in \mathcal{R}$ pick A_R so that $A_R \subset [R]^2$ and pairwise disjoint. Then $\{A_R : R \in \mathcal{R}\}$ witness pair-reaping family.
(2) Let $\mathcal{S}$ be a pair splitting family. Then for $Y \in [\omega]^\omega$ define $A_Y = [Y]^2$. If $X \in \mathcal{S}$ pair-splits A_Y, then X splits Y. Hence $\mathcal{S}$ is a splitting family.
(3) Let $\kappa < \mathfrak{r}_{pair}$ and $\mathcal{S} \subset (\omega)^\omega$ with $|\mathcal{S}| = \kappa$. For each $S \in \mathcal{S}$ fix $A_S \subset [\omega]^2$ such that A_S is infinite, pairwise disjoint and for $a \in A_S$ there exists $x \in S$ such that $a \subset x$. Put $\mathcal{A} = \{A_S : S \in \mathcal{S}\}$. Then $\mathcal{A} \subset \wp([\omega]^2)$ and $|\mathcal{A}| = \kappa$. Since $\kappa < \mathfrak{r}_{pair}$, there exists $y_0 \in [\omega]^\omega$ such that x_0 pair-splits A for $A \in \mathcal{A}$.

Define $\mathcal{S}_0 = \{y_A : A \in \mathcal{A} \wedge y_A = \bigcup \{a \setminus y_0 : a \cap y_0 \neq \emptyset \wedge a \in A\}\}$. Then $\mathcal{S}_0 \subset [\omega]^\omega$ and $|\mathcal{S}_0| = \kappa < \mathfrak{r}_{pair} \leq \mathfrak{r}$. So there exists $y_1 \in [\omega \setminus y_0]^\omega$ such that y_1 splits y for $y \in \mathcal{S}_0$. Recursively define $y_{i+1} \in [\omega \setminus \bigcup_{j<i+1} y_j]^\omega$ and $\mathcal{S}_{i+1} \subset [\omega \setminus \bigcup_{j \leq i+1} y_j]^\omega$ so that y_{i+1} splits all y for $y \in \mathcal{S}_i$ and $\mathcal{S}_{i+1} = \{y \setminus y_{i+1} : y \in \mathcal{S}_i\}$. Without loss of generality we can assume $\bigcup \{y_i : i \in \omega\} = \omega$. Let $Y = \{y_i : i \in \omega\} \in (\omega)^\omega$. Then by construction $Y \perp S$ for $S \in \mathcal{S}$. Hence $\mathcal{S}$ is not dual-splitting family. $\square$

Proposition 4.4.2. $\mathfrak{s}_{pair} \leq$ *non*$(\mathcal{M})$, *non*$(\mathcal{N})$. $\mathfrak{r}_{pair} \geq$ *cov*$(\mathcal{M})$, *cov*$(\mathcal{N})$.

Proof. For a countable pairwise disjoint subset $A \subset [\omega]^2$, $D_A = \{X \in [\omega]^\omega : X$ pair-splits $A\}$ is comeager and measure 1 subset of 2^ω. Therefore if $\kappa < \mathrm{cov}(\mathcal{M})$ and $\langle A_\alpha : \alpha < \kappa \rangle$ is a family of countable pairwise disjoint subsets of $[\omega]^2$, $\bigcap_{\alpha<\kappa} D_{A_\alpha} \neq \emptyset$. Let $X \in \bigcap_{\alpha<\kappa} D_{A_\alpha}$. Then X pair-split all A_α. Hence $\mathfrak{r}_{pair} \geq \mathrm{cov}(\mathcal{M})$. The rest of proof is similar. $\square$

Theorem 4.4.3. *It is consistent that* $\mathfrak{r}_d <$ *add*$(\mathcal{M})$. *Also it is consistent that* $\mathfrak{s}_d >$ *cof*$(\mathcal{M})$.

Proof. Let $\dot{X}$ be a $\mathbb{D}_\alpha$-name for a non-trivial infinite partition of ω. If we can find $Y \in (\omega)^\omega \cap V$ such that $\Vdash_{\mathbb{D}_\alpha} \dot{X} \perp Y$, then we can prove the desired statement. To show this, we shall prove the following lemma.

Lemma 4.4.4. *Let* $\dot{\mathcal{A}} = \langle \dot{A}_n : n \in \omega \rangle$ *and* $\dot{\mathcal{C}} = \langle \dot{C}_n : n \in \omega \rangle$ *be* $\mathbb{D}_\alpha$*-names such that*

$$\Vdash_{\mathbb{D}_\alpha} \forall n \in \omega (\dot{A}_n \subset [\omega]^2, \forall m \in \omega \exists a \in \dot{A}_n (a \cap m = \emptyset) \text{ and } \dot{C}_n \in [\omega]^\omega).$$

Then there exists $\mathcal{A} = \langle A_n : n \in \omega \rangle \in ([\omega]^2)^\omega \cap V$ *and* $\mathcal{C} = \langle C_n : n \in \omega \rangle \in ([\omega]^\omega)^\omega \cap V$ *such that if there exists* $y \in [\omega]^\omega \cap V$ *such that* y *pair-splits* A_n *and* y *splits* C_n *for* $n \in \omega$*, then*

$$\Vdash_{\mathbb{D}_\alpha} \forall A \in \dot{\mathcal{A}} (y \text{ pair-splits } A) \text{ and } \forall C \in \dot{\mathcal{C}} (y \text{ splits } C).$$

Proof. Induction on α.
$\underline{\alpha = 1}$ Given $\mathbb{D}$-names $\dot{\mathcal{A}} = \langle \dot{A}_n : n \in \omega \rangle$ and $\dot{\mathcal{C}} = \langle \dot{C}_n : n \in \omega \rangle$. Since Hechler forcing preserves $\mathfrak{s}$, there are $\mathcal{C}^* = \langle C_n^i : n \in \omega \wedge i \in \omega \rangle$ such that if $y \in [\omega]^\omega$ splits C_n^i for $i < \omega$, then $\Vdash_{\mathbb{D}} y$ splits C_n^i. So it is enough to think about $\dot{\mathcal{A}}$.

We will use rank argument as [4]. Let $t \in \omega^{<\uparrow\omega}$, $E \subset \omega^{<\uparrow\omega}$. Define by recursion on the ordinal when $\mathrm{rk}(t, E) = \alpha$.

1. $\mathrm{rk}(t, E) = 0$ if $t \in E$.
2. $\mathrm{rk}(t, E) = \alpha$ if $\neg(\beta < \alpha \wedge \mathrm{rk}(t, E) = \beta)$ and $\exists m \in \omega \exists t_k \in \omega^{<\uparrow\omega}(k \in \omega)$ such that $\forall k \in \omega(t \subset t_k)$, $|t_k| = m$ and $t_k(|t|) \geq k$.

Recall the following theorem:

Theorem 4.4.5. *[4] If $I \subset \mathbb{D}$ is dense, $E = \left\{t \in \omega^{<\uparrow\omega} : \exists f \in \omega^{\uparrow\omega} : \langle t, f\rangle \in I\right\}$, then $rk(t, E) < \omega_1$ for any $t \in \omega^{<\uparrow\omega}$.*

For each $m \in \omega$ let $D_m = \{\langle s, f\rangle \in \mathbb{D} : \exists k_0, k_1 \geq m\, (\langle s, f\rangle \Vdash \{k_0, k_1\} \in A_n\}$. Then D_m is dense open subset of $\mathbb{D}$. Let $E_m = \left\{s \in \omega^{<\uparrow\omega} : \exists f \in \omega^{\uparrow\omega}\, \langle s, f\rangle \in D_m\right\}$. By above Theorem, $\mathrm{rk}(t, E_m)$ is always defined. By induction of rk, define

- when t is $\begin{cases} t \text{ is bad} \\ t \text{ is so-so} \\ t \text{ is good} \\ t \text{ is neither} \end{cases}$ for m.
- For bad t, define $k^0_{t,m} < k^1_{t,m} \in \omega \setminus (m + 1)$.
- For so-so t, define $k^0_{t,m} \in \omega$ and $C_{t,m} \in [\omega]^\omega$.
- For good t, define $A_{t,m}$ countable subset of $[\omega]^2$.

Basic step

$\mathrm{rk}(t, E_m) = 0$. Then t is bad for m.

Since $\mathrm{rk}(t, E_m) = 0$, $t \in A_m$. So $\exists f$ such that $\langle t, f\rangle \in D_m$. Hence there exists $k^0_{t,m}$ and $k^1_{t,m}$ such that $m \leq k^0_{t,m} < k^1_{t,m}$.

Recursion step

$\mathrm{rk}(t, E_m) > 0$. Choose t_i , $i \in \omega$ such that $t \subset t_i$, $|t_i| = |t_j|$, $t_i(|t|) \geq i$, $\mathrm{rk}(t_i, E_m) < \mathrm{rk}(t, E_m)$.

Case 1 Almost all t_i bad.

Subcase (a) $\exists k^0_{t,m}, k^1_{t,m}$ such that $\exists^\infty i \in \omega(k^0_{t_i,m} = k^0_{t,m} \wedge k^1_{t_i,m} = k^1_{t,m})$.

Then t bad.

Subcase (b)$\neg$(Subcase (a)) and $\exists k^0_{t,m}$ such that $\exists^\infty i \in \omega(k^0_{t_i,m} = k^0_{t,m})$.

Then t is so-so and $C_{t,m} = \left\{k^1_{t_i,m} : \exists i \in \omega(k^0_{t_i,m} = k^0_{t,m})\right\} \in [\omega \setminus (k^0_{t_i,m} + 1)]^\omega$.

Subcase (c)$\neg$(Subcase$(a) \vee$ Subcase(b)).

Then t is good and $A_{t,m} = \left\{\left\{k^0_{t_i,m}, k^1_{t_i,m}\right\} : i \in \omega\right\}$ infinite subset of $[\omega]^2$.

Case 2 Infinitely many t_i are not good.

Then t is neither.

Now we shall construct $\mathcal{A}$ and $\mathcal{C}$:

If t is bad with respect to almost all m, then put $A_t = \left\{\left\{k^0_{t,m}, k^1_{t,m}\right\} : m \in \omega\right\}$. Then put $\mathcal{A} = \{A_t : t \text{ is bad for almost all } m\} \cup \{A_{t,m} : t \text{ is good for } m\}$ and $\mathcal{C} = \{C_{t,m} : t \text{ is so-so for } m\} \cup \mathcal{C}^*$.

We shall show if $y \in [\omega]^\omega \cap V$ such that y pair-splits A for $A \in \mathcal{A}$ and y splits C for $C \in \mathcal{C}$, then

$$\Vdash_{\mathbb{D}} \forall A \in \dot{\mathcal{A}}(y \text{ pair-splits } A) \text{ and } \forall C \in \dot{\mathcal{C}}(y \text{ splits } C).$$

Suppose $y \in [\omega]^\omega \cap V$ such that y pair-splits A for $A \in \mathcal{A}$ and y splits C for $C \in \mathcal{C}$. Since y splits C for $C \in \mathcal{C}^* \subset \mathcal{C}$, $\Vdash_{\mathbb{D}} y$ splits C for $C \in \dot{\mathcal{C}}$.

So we shall prove $\Vdash_{\mathbb{D}} \forall A \in \dot{\mathcal{A}}(y$ pair-splits $A)$.

Fix $\langle s, f\rangle \in \mathbb{D}$, $m \in \omega$ and $\mathbb{D}$-name $\dot{A}$ such that $\Vdash_{\mathbb{D}} \dot{A} \in \dot{\mathcal{A}}$. We need to find $\langle t, g\rangle \leq \langle s, f\rangle$ and k^0, $k^1 \geq m$ such that

$$\langle t, g\rangle \Vdash_{\mathbb{D}} \left\{k^0, k^1\right\} \in \dot{A} \wedge \left\{k^0, k^1\right\} \cap y \neq \emptyset \wedge \left\{k^0, k^1\right\} \setminus y \neq \emptyset.$$

Case 1 $\forall m^* \geq m$ s bad for m^*.

Since y pair-splits A_s, there exists m^* and $k^0_{s,m^*}, k^1_{s,m^*} \geq m^*$ such that $y \cap \left\{k^0_{t,m^*}, k^1_{t,m^*}\right\} \neq \emptyset$ and $\left\{k^0_{t,m^*}, k^1_{t,m^*}\right\} \setminus y \neq \emptyset$. By construction of A_s there exists s_i such that $s \subset s_i$, $f(j) \leq s_i(j)$ for $j \in |s_i|$, $\mathrm{rk}(s_i, E_{m^*}) < \mathrm{rk}(s, E_{m^*})$, $k^0_{s_i,m^*} = k^0_{s,m^*}$, $k^1_{s_i,m^*} = k^1_{s,m^*}$ and s_i bad for m^*.

By induction on rank, we see there exists t such that $s_i \subset t$, $t(j) \geq f(j)$ for $j \in |t|$, $\mathrm{rk}(t, E_{m^*}) = 0$, $k^0_{t,m^*} = k^0_{s_i,m^*} = k^0_{s,m^*}$ and $k^1_{t,m^*} = k^1_{s_i,m^*} = k^1_{s,m^*}$. By definition $t \in E_{m^*}$, so there exists $g \in \omega^{\uparrow\omega}$ such that $\langle t, g\rangle \in D_{m^*}$ with $\langle t, g\rangle \Vdash \left\{k^0_{t,m^*}, k^1_{t,m^*}\right\} \in \dot{A}$. Without loss, $g \geq f$. Therefore $\langle t, g\rangle \leq \langle s, f\rangle$.

Case 2 $\exists m^* \geq m$ s is not bad for m^*.

So s is neither, so-so or good. By induction on rank we can see there exists s^* such that $s \subset s^*$, $s^*(i) \geq f(i)$ for $i \in |s^*|$ and s^* is good or so-so.

Subcase (i) s^* is so-so for m^*.

So we have $k^0_{s^*,m^*}$ and C_{s^*,m^*}. Assume $k^0_{s^*,m^*} \in y$. Since y splits C_{s^*,m^*}, there exists s^*_i such that $s^* \subset s^*_i$, $s^*_i(j) \geq f(j)$ for $j \in |s^*_i|$, $\mathrm{rk}(s^*_i, E_{m^*}) < \mathrm{rk}(s^*, E_{m^*})$, s^*_i is bad, $k^0_{s^*_i,m^*} = k^0_{s*,m^*}$ and $k^1_{s^*_i,m^*} \in C_{s^*,m^*} \setminus y$. By induction on rank, we see there exists t such that $s^*_i \subset t$, $t(j) \geq f(j)$ for $j \in |t|$, $\mathrm{rk}(t, E_{m^*}) = 0$, $k^0_{t,m^*} = k^0_{s^*_i,m^*} = k^0_{s^*,m^*}$ and $k^1_{t,m^*} = k^1_{s^*_i,m^*}$. By definition, $t \in E_{m^*}$, so there exists $g \in \omega^{\uparrow\omega}$ such that $\langle t, g\rangle \Vdash_{\mathbb{D}} \left\{k^0_{s^*,m^*}, k^1_{s^*,m^*}\right\} \in \dot{A}$. Without loss of generality, $g \geq f$. Therefore $\langle t, g\rangle \leq \langle s, f\rangle$ and

$\langle t, g\rangle \Vdash_{\mathbb{D}} \left\{k^0_{s^*,m^*}, k^1_{s^*,m^*}\right\} \in \dot{A}$, $k^0_{s^*,m^*} \in y$ and $k^1_{s^*,m^*} \notin y$.

Subcase (ii) s^* is good for m^*.

So we have A_{s^*,m^*} countable subset of $[\omega]^2$. Since y pair-splits A_{s^*}, there exists s^*_i such that $s^* \subset s^*_i$, $\mathrm{rk}(s^*_i, E_{m^*}) < \mathrm{rk}(s^*, E_{m^*})$, $s^*_i(j) \geq f(j)$ for $j \in |s^*_i|$, s^*_i is bad, $\left\{k^0_{s^*_i,m}, k^1_{s^*_i,m}\right\} \in A_{s^*,m}$, $\left\{k^0_{s^*_i,m}, k^1_{s^*_i,m}\right\} \cap y \neq \emptyset$ and $\left\{k^0_{s^*_i,m}, k^1_{s^*_i,m}\right\} \setminus y \neq \emptyset$.

Assume $k^0_{s^*_i,m} \in y$ and $k^1_{s^*_i,m} \notin y$. By induction on rank, we see there exists t such that $s^*_i \subset t$, $t(j) \geq f(j)$ for $j \in |t|$, $\mathrm{rk}(t, E_{m^*})=0$, $k^0_{t,m^*} = k^0_{s^*_i,m^*}$ and $k^1_{t,m^*} = k^1_{s^*_i,m^*}$. By definition, $t \in E_{m^*}$, so there exists $g \in \omega^{\uparrow\omega}$ such that $\langle t, g\rangle \Vdash_{\mathbb{D}} \left\{k^0_{s^*_i,m^*}, k^1_{s^*_i,m^*}\right\} \in \dot{A}$. Without loss of generality, $g \geq f$. Therefore $\langle t, g\rangle \leq \langle s, f\rangle$ and

$\langle t, g\rangle \Vdash_{\mathbb{D}} \left\{k^0_{s^*_i,m^*}, k^1_{s^*_i,m^*}\right\} \in \dot{A}$, $k^0_{s^*_i,m^*} \in y$ and $k^1_{s^*_i,m^*} \notin y$.

α is a successor ordinal or limit ordinal

We use following theorem.

Theorem 4.4.6. *Let $\langle \sqsubset_n : n \in \omega\rangle$ be a increasing sequence of two-place relation on ω^ω or similar space. Put $\sqsubset = \bigcup_{n\in\omega} \sqsubset_n$. Assume $\forall f \in \omega^\omega, \{g : f \sqsubset_n g\}$ is closed. Take $\mathcal{F} \subset \omega^\omega$ in V such that for a countable $X \subset \omega^\omega$ there exists $f \in \mathcal{F}$ such that $f \not\sqsubset g$ for $g \in X$.*

Let $\left\langle \mathbb{P}_\alpha, \dot{\mathbb{Q}}_\alpha : \alpha < \delta \right\rangle$ be an finite support iteration of c.c.c p.o's. Assume for $\alpha < \delta$

$\Vdash_\alpha$ "$\Vdash_{\dot{\mathbb{Q}}_\alpha}$ "for all countable $X \subset \omega^\omega$ there exists a countable $Y \subset \omega^\omega \cap V[\dot{G}_\alpha]$ in $V[\dot{G}_\alpha]$ such that $\forall f \in \check{\mathcal{F}}(\forall g \in Y(f \not\sqsubset g) \to \forall g \in X(f \not\sqsubset g))$"". Then

$\Vdash_\delta$ "for all countable $X \subset \omega^\omega$ there exists a countable $Y \subset \omega^\omega \cap V$ in V such that $(\forall f \in \check{\mathcal{F}}(\forall g \in Y(f \not\sqsubset g) \to \forall g \in X(f \not\sqsubset g)))$".

So it suffices to find relations $\sqsubset$ and $\langle \sqsubset_n : n \in \omega\rangle$ such that for $y \in [\omega]^\omega$ and $\langle X, A\rangle \in [\omega]^\omega \times \{B \subset [\omega]^2 : |B| = \omega \wedge \forall n \in \omega(B \cap [\omega \setminus n]^2 \neq \emptyset)\} = \mathcal{G}$ $y \not\sqsubset \langle X, A\rangle$ if y splits X, y pair-splits A and for $y \in [\omega]^\omega$ for $n \in \omega$ $\{\langle X, A\rangle \in \mathcal{G} : y \sqsubset_n \langle X, A\rangle\}$ is closed. Define $y \sqsubset_n \langle X, A\rangle$ if $y \cap X \subset n$ or $X \setminus y \subset n$ and $|rng(f \restriction a))| = 1$ for $a \in A \cap [\omega \setminus n]^2$. Then $\sqsubset_n$is required.

□

(Lemma 4.4.4⇒Theorem 4.4.3)

Let $\dot{X}$ be a $\mathbb{D}_\alpha$-name for a non-trivial partition of ω. Then there is a $\mathbb{D}_\alpha$-name $\dot{A}$ for a countable subset of $[\omega]^2$ such that

$$\Vdash_{\mathbb{D}_\alpha} \forall a \in \dot{A} \exists x \in \dot{X}(a \subset x) \wedge \forall m \in \omega \exists a \in \dot{A}(a \cap n = \emptyset).$$

Then by Lemma 4.4.4 there exists $\mathcal{A} = \langle A_n : n \in \omega\rangle \in ([\omega]^2)^\omega \cap V$ and $\mathcal{C} = \langle C_n : n \in \omega\rangle \in ([\omega]^\omega)^\omega \cap V$ such that if $y \in [\omega]^\omega$ satisfies that y pair-splits A_n for all $n \in \omega$ and y splits C_n for $n \in \omega$, then $\Vdash_{\mathbb{D}_\alpha} y$ pair-splits $\dot{A}$. Fix such $y \in [\omega]^\omega$. Recursively we will construct $\langle y_n : n \in \omega\rangle = Y \in (\omega)^\omega$:

1. $y_0 = y$.
2. Suppose given y_i for $i < n$. Then pick $y_n \in [\omega \setminus \bigcup_{i<n} y_i]^\omega$ so that $\Vdash_{\mathbb{D}_\alpha} y_i$ splits $\bigcup_{a\in\dot{A}} \{a \setminus y : a \cap y \neq \emptyset\} \setminus \bigcup_{i<n} y_i$.

Second condition is possible since the finite support iteration of Hechler forcing preserve $\mathfrak{s}$. By construction $\Vdash$ "$\exists a \in \dot{A}(a \cap y_i \neq \emptyset \wedge a \cap y_0 \neq \emptyset)$" for each $i \geq 1$. So $\Vdash \exists x \in \dot{X}(x \cap y_0 \neq \emptyset \wedge x \cap y_i \neq \emptyset)$. Therefore $\Vdash Y \perp \dot{X}$.

□

Question 9. $\mathfrak{r}_d \leq \mathfrak{s}_{pair}$*?*

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