The convergence of the exploration process for critical percolation on the k-out graph

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（k-アウトグラフ上のパーコレーションにおける臨界値での探索過程の収束）

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Abstract

We consider the percolation on the k-out graph $G_{\text{out}}(n,k)$. The critical probability of it is $\frac{1}{k+\sqrt{k^2-k}}$. Similarly to the random graph $G(n,p)$, in a scaling window $\frac{1}{k+\sqrt{k^2-k}}\big(1+O(n^{-1/3})\big)$, the sequence of sizes of large components rescaled by $n^{-2/3}$ converges to the excursion lengths of a Brownian motion with some drift. Also, the size of the largest component is $O(\log n)$ in the subcritical phase, and $O(n)$ in the supercritical phase. The proof is based on the analysis of the exploration process.

Contents

Chapter 1

Introduction

The random graph $G(n, p)$ has n vertices $\{1, \dots, n\}$ and each edge $\langle i, j\rangle$ is realized independently with probability p. It is well known that the random graph has "double jump". Namely, for $G(n, c/n)$, the size of the largest component $\mathcal{C}_1$ is of order $\log n$ for $c < 1$, of order $n^{2/3}$ for $c = 1$, and of order n for $c > 1$, see [5]. Furthermore, it became clear that the structure of the random graph rapidly changes in a "scaling window" $\frac{1}{n}(1+O(n^{-1/3}))$, see [6], [7]. In a scaling window, the sequence of sizes of components $(|\mathcal{C}_1|, |\mathcal{C}_2|, \dots)$ aranged in decreasing order, converges when scaled by $n^{-2/3}$, see [8]. Aldous showed that this scaled sequence converges to the sequence of excursions of Brownian motion with some drift. This fact is provided by the analysis of the exprolation process. In this process, we explore one vertex w_t at each time t, i.e., we count vertices connected to w_t by an edge. So, the exploration process reveals the structure of connected components of $G(n, p)$. The convergence to the Brownian excursions appears in another graph model too. Nachmias and Peres investigated it in the random d-regular graph $G_{\text{reg}}(n, d)$. It is known that the d-regular graph is a.s.-connected for $d \geq 2$, see [9], [10]. But when we consider the percolation on $G_{\text{reg}}(n, d)$, i.e., each edge remains with probability p and is removed with probability $1 - p$ independently, we can consider a similar problem for percolation clusters. For this model, in a scaling window $p = \frac{1+O(n^{-1/3})}{d-1}$, the sequence of the sizes of the connected components converges to the sequence of the excursions of a Brownian motion with some drift, both arranged in decreasing order, see [2]

We prove this convergence for the k-out graph, where $k \geq 2$ is a given integer and we fix it throughout this paper. The k-out graph model is explained in [10]. First we construct the graph with directed edges $\vec{G}_{\text{out}}(n, k)$. The vertex set of $\vec{G}_{\text{out}}(n,k)$ is $\mathcal{V} = \{1,\ldots,n\}$. For each $v \in \mathcal{V}$, choose k distinct edges $(v,v_1),\ldots,(v,v_k)$ uniformly from $\{(v,1),(v,2),\ldots,(v,n)\} \setminus \{(v,v)\}$. This is a construction of the $\vec{G}_{\text{out}}(n,k)$. The vertex set of $G_{\text{out}}(n,k)$ is $\mathcal{V}$ too and a non-directed edge $\langle v,w\rangle$ is in the edge set of $G_{\text{out}}(n,k)$ if either (v,w) or (w,v) is an edge of $\vec{G}_{\text{out}}(n,k)$. However, it is convenient to keep the information of the direction of each edge until we construct the exploration process. We also consider percolation on $G_{\text{out}}(n,k)$. Each edge is open with probability p and closed with probability $1-p$ independently. Our exploration process explores open clusters of $G_{\text{out}}(n,k)$.

The k-out graph model is related to the Watts-Strogatz (WS) model, which is commonly used in the study of complex networks. We hope that the analysis of the k-out graph model will lead to knowing the structure of the WS model, hence the structure of real networks. Further, the critical point of percolation on these graphs can be considered as an indicator of robustness of the graph model. Thus this result may be useful to analyze real networks.

Bollobás and Riordan proved the phase transition of the growing k-out graph in [4]. This graph model is similar to the k-out graph, but it has inhomogeneous degree of verteices. The growing k-out graph has more robustness than the k-out graph. In fact, it is interesting to see that the critical point of growing k-out graph is half of the critical point of the k-out graph.

In Chapter 2 we state main results. There are some theorems in each phase of the percolation for the k-out graph. Chapter 3 introduces the construction of the exploration process for the k-out graph. Proofs of Theorem 1, 3, 4 are in Chapter 4. In Chapter 5 the proof of Theorem 2 is provided by the convergence of the exploration process.

For the construction and the calculation of the exploration process of the k-out graph, we follow the argument in [2]. However, we have to overcome some difficulties peculiar to the k-out graph.

Chapter 2

Main results

Let $\mathcal{C}_l$ be the l-th largest open cluster of $G_{\text{out}}(n,k)$.

Theorem 1 (critical phase)**.** *Let $\lambda \in \mathbb{R}$ be fixed and let $p = p(\lambda) = \frac{1+\lambda n^{-1/3}}{k+\sqrt{k^2-k}}$. Then there exist positive constants $c(\lambda,k)$, $C(\lambda,k)$ such that for a large enough $A = O(n^{1/10})$ and a large enough n,*

$$\mathbb{P}\Big(\big|\mathcal{C}_1\big| \geq An^{2/3}\Big) \leq \frac{C(\lambda,k)e^{-c(\lambda,k)A^3}}{A}.$$

Let $\big\{\mathcal{B}(s) : s \in [0,\infty)\big\}$ be a standard Brownian motion. For $\lambda \in \mathbb{R}$, we define the process $\mathcal{B}^\lambda$ by

$$\begin{aligned}\mathcal{B}^\lambda(s) \;&=\; \mathcal{B}\Big(2\big(k-\sqrt{k^2-k}\big)s\Big)\\ &\quad +2\big(k-\sqrt{k^2-k}\big)\lambda s - \big(\sqrt{k^2-k}-k+1\big)s^2 \end{aligned} \tag{2.1}$$

for $s \in [0,\infty)$. Next we define the reflected process of $\mathcal{B}^\lambda$ by

$$W^\lambda(s) = \mathcal{B}^\lambda(s) - \min_{0\leq s'\leq s} \mathcal{B}^\lambda(s'). \tag{2.2}$$

Let $(|\gamma_j|)_{j\geq 1}$ be the sequence of all excursion lengths of W^λ arranged in a decreasing order. Also let $(|\mathcal{C}_j|)_{j\geq 1}$ be the sequence of sizes of open clusters in a decreasing order, concatenated by zeros to form a vector in l^2.

Theorem 2 (scaling window)**.** *Let $\lambda \in \mathbb{R}$ be fixed and let $p = p(\lambda) = \frac{1+\lambda n^{-1/3}}{k+\sqrt{k^2-k}}$. Then*

$$n^{-2/3}\cdot\big(|\mathcal{C}_j|\big)_{j\geq 1} \stackrel{d}{\Longrightarrow} \big(|\gamma_j|\big)_{j\geq 1},$$

where convergence holds with respect to the l^2-norm.

Next, we state the results of the largest open cluster size in subcritical and supercritical phases. Let $p = \frac{c}{k+\sqrt{k^2-k}}$.

Theorem 3 (subcritical phase)**.** *If $c < 1$ and $A > 0$ is sufficiently large, then*

$$\mathbb{P}\big(|\mathcal{C}_1| > A \log n\big) \to 0, \quad \textit{as } n \to \infty.$$

Theorem 4 (supercritical phase)**.** *If $c > 1$ and $\delta > 0$ is sufficiently small, then*

$$\mathbb{P}\big(|\mathcal{C}_1| < \delta n\big) \to 0, \quad \textit{as } n \to \infty.$$

2.1 Notation

Throughout this paper, we write $f = o(g)$ if $\lim_{n\to\infty} f(n)/g(n) = 0$, and $f = O(g)$ if there exists some constant $C > 0$ independent of ω and n, such that $f(n) < Cg(n)$ for a large enough n.

Next, we introduce an abbreviated notation,

$$X \sim NB(n; p) + Bin(m; q) + c,$$

to mean that $X = X_1 + X_2 + c$ for independent random variables X_1 and X_2 and a constant c such that $X_1 \sim NB(n; p)$ and $X_2 \sim Bin(m; q)$.

Chapter 3

Exploration process of the k-out graph

We consider the k-out graph with directed edges $\vec{G}_{\text{out}}(n,k)$. Each vertex has state *neutral* or *active* or *explored*, and each edge has state *checked* or *unchecked*. Let w_t be the vertex that we explore at time t. Let $\mathcal{N}_t$ be the set of neutral vertices at time t, $\mathcal{A}_t$ be the set of active vertices at time t, and $\mathcal{E}_t$ be the set of explored vertices at time t.

Let $\mathcal{N}_t^{(i)}$ be the set of vertices at time t such that, for $v \in \mathcal{N}_t^{(i)}$, (i) v is neutral, and (ii) among k edges starting from v, exactly i edges are checked. Therefore $\mathcal{N}_t$ equals the union of $\mathcal{N}_t^{(0)}, \mathcal{N}_t^{(1)}, \ldots, \mathcal{N}_t^{(k)}$. Simirally, let $\mathcal{A}_t^{(i)}$ be the set of active vertices at time t with exactly i edges being checked. So $\mathcal{A}_t$ equals the union of $\mathcal{A}_t^{(0)}, \mathcal{A}_t^{(1)}, \ldots, \mathcal{A}_t^{(k)}$. Further we define the set $N_t^{(\geq i)}$ as $\cup_{j=i}^{k} \mathcal{N}_t^{(j)}$. Similarly we define $N_t^{(\leq i)}$, $A_t^{(\geq i)}$, $A_t^{(\leq i)}$.

Let N_t and $N_t^{(i)}$ denote the cardinalities of $\mathcal{N}_t$ and $\mathcal{N}_t^{(i)}$, respectivery. Also let A_t and $A_t^{(i)}$ denote the cardinalities of $\mathcal{A}_t$ and $\mathcal{A}_t^{(i)}$, respectivery.

At time 0, all the vertices are in $\mathcal{N}_0^{(0)}$. For $t \geq 1$, given $\mathcal{A}_{t-1}^{(i)}$, $\mathcal{N}_{t-1}^{(i)}$, $0 \leq i \leq k$, we construct $\mathcal{A}_t^{(i)}$, $\mathcal{N}_t^{(i)}$, $0 \leq i \leq k$ in $k+2$ steps in the following way. We write $\mathcal{N}_{t-1,l}^{(i)}$ and $\mathcal{A}_{t-1,l}^{(i)}$ for $\mathcal{N}^{(i)}$ and $\mathcal{A}^{(i)}$ at each step l at time t.

(**Step** 0)[choosing w_t] If $A_{t-1}^{(i)} > 0$ and $A_{t-1}^{(\geq i+1)} = 0$, w_t is chosen from $\mathcal{A}_{t-1}^{(i)}$ uniformly. If $A_{t-1} = 0$, w_t is chosen from $\mathcal{N}_{t-1}$ uniformly. We put $\mathcal{N}_{t-1,0}^{(i)} = \mathcal{N}_{t-1}^{(i)} \setminus \{w_t\}$ and $\mathcal{A}_{t-1,0}^{(i)} = \mathcal{A}_{t-1}^{(i)} \setminus \{w_t\}$ for $0 \leq i \leq k$.

(**Step** l, $1 \leq l \leq k$)[directed edge from w_t] If $w_t \in \mathcal{N}_{t-1}^{(\leq l-1)} \cup \mathcal{A}_{t-1}^{(\leq l-1)}$, we execute arm stretch process (AS process) $\rho_{t-1,l}$. The AS process will be described later. If $w_t \in \mathcal{N}_{t-1}^{(\geq l)} \cup \mathcal{A}_{t-1}^{(\geq l)}$, do nothing and put $\mathcal{N}_{t-1,l}^{(i)} = \mathcal{N}_{t-1,l-1}^{(i)}$ and $\mathcal{A}_{t-1,l}^{(i)} = \mathcal{A}_{t-1,l-1}^{(i)}$ for $0 \leq i \leq k$ at this step.

(**Step** $k+1$)[directed edge to w_t] For $v \in \mathcal{N}_{t-1,k}^{(i)}$, if there is an unchecked edge (v, w_t) in $\vec{G}_{\text{out}}(n,k)$ and this edge is open, then $v \in \mathcal{A}_{t-1,k+1}^{(i+1)}$, if this edge is closed, then $v \in \mathcal{N}_{t-1,k+1}^{(i+1)}$. After this we declare that the edge (v, w_t) is checked. If there is not such an edge, then $v \in \mathcal{N}_{t-1,k+1}^{(i)}$. For $v \in \mathcal{A}_{t-1,k}^{(i)}$, if there is an unchecked edge (v, w_t) in $\vec{G}_{\text{out}}(n,k)$, then $v \in \mathcal{A}_{t-1,k+1}^{(i+1)}$, and we declare (v, w_t) checked. If there is not such an edge, then $v \in \mathcal{A}_{t-1,k+1}^{(i)}$.

When all the above steps are over, we put $\mathcal{E}_t = \mathcal{E}_{t-1} \cup \{w_t\}$, $\mathcal{N}_t^{(i)} = \mathcal{N}_{t-1,k+1}^{(i)}$, $\mathcal{A}_t^{(i)} = \mathcal{A}_{t-1,k+1}^{(i)}$ for $0 \leq i \leq k$, and the exploration of w_t is finished.

Next we explain the AS process $\rho_{t-1,l}$. It is composed of the following algorithm. We start with $T_0 = 0$. First, we check one directed edge from w_t, and write $\eta_{t-1,T_{l-1}+1}$ for the head of this edge. If $\eta_{t-1,T_{l-1}+1}$ is chosen from $\mathcal{N}_{t-1,l-1}^{(0)}$ and the edge $\langle w_t, \eta_{t-1,T_{l-1}+1}\rangle$ is open, we say that $\eta_{t-1,T_{l-1}+1}$ *is good*, and let $\eta_{t-1,T_{l-1}+1}$ change to active. Otherwise we say that $\eta_{t-1,T_{l-1}+1}$ *is bad*. After this we declare that the edge $(w_t, \eta_{t-1T_{l-1}+1})$ is checked. For $x \geq 1$, suppose that we have distinct x good vertices $\eta_{t-1,T_{l-1}+1}, \ldots, \eta_{t-1,T_{l-1}+x}$. Then we check one directed edge from $\eta_{t-1,T_{l-1}+x}$, and write $\eta_{t-1,T_{l-1}+x+1}$ for the head of this edge. If $\eta_{t-1,T_{l-1}+x+1} \in \mathcal{N}_{t-1,l-1}^{(0)} \setminus \{\eta_{t-1,T_{l-1}+1}, \ldots, \eta_{t-1,T_{l-1}+x}\}$ and $\langle \eta_{t-1,T_{l-1}+x}, \eta_{t-1,T_{l-1}+x+1}\rangle$ is open, then we say that $\eta_{t-1,T_{l-1}+x+1}$ is good. Otherwise we say that $\eta_{t-1,T_{l-1}+x+1}$ is bad. After this we declare that the edge $(\eta_{t-1,T_{l-1}+x}, \eta_{t-1,T_{l-1}+x+1})$ is checked. We continue it until we get to a bad vertex, where we stop this process. Let $\{\eta_{t-1,T_{l-1}+1}, \ldots, \eta_{t-1,T_{l-1}+x}\}$ be vertices obtained by the above procedure. Then $\eta_{t-1,T_{l-1}+y}$ is good for $y < x$ and $\eta_{t-1,T_{l-1}+x}$ is bad. We set $T_l = T_{l-1} + x$ and we renew the states of vertices in the following way. First we consider the case where $\eta_{t-1,T_{l-1}+1}$ is bad, so $\eta_{t-1,T_l} = \eta_{t-1,T_{l-1}+1}$.

If η_{t-1,T_l} is chosen from $\mathcal{A}_{t-1,l-1} \setminus \{\eta_{t-1,T_0+1}, \ldots, \eta_{t-1,T_{l-2}+1}\}$,
or η_{t-1,T_l} is chosen from $\mathcal{N}_{t-1,l-1} \setminus \{\eta_{t-1,T_0+1}, \ldots, \eta_{t-1,T_{l-2}+1}\}$ and the edge $\langle w_t, \eta_{t-1,T_l}\rangle$ is closed,
then we put
$\mathcal{N}_{t-1,l}^{(i)} = \mathcal{N}_{t-1,l-1}^{(i)}$ for $0 \leq i \leq k$, $\quad \mathcal{A}_{t-1,l}^{(i)} = \mathcal{A}_{t-1,l-1}^{(i)}$ for $0 \leq i \leq k$.

If η_{t-1,T_l} is chosen from $\mathcal{N}^{(i)}_{t-1,l-1}$ for $1 \le i \le k$ and the edge $\langle w_t, \eta_{t-1,T_l}\rangle$ is open,
then we put
$\mathcal{N}^{(i)}_{t-1,l} = \mathcal{N}^{(i)}_{t-1,l-1} \setminus \{\eta_{t-1,T_l}\}, \quad \mathcal{N}^{(j)}_{t-1,l} = \mathcal{N}^{(j)}_{t-1,l-1}$ for $j \ne i$,
$\mathcal{A}^{(i)}_{t-1,l} = \mathcal{A}^{(i)}_{t-1,l-1} \cup \{\eta_{t-1,T_l}\}, \quad \mathcal{A}^{(j)}_{t-1,l} = \mathcal{A}^{(j)}_{t-1,l-1}$ for $j \ne i$.

Next we consider the case where $\eta_{t-1,T_{l-1}+1}$ is good. Suppose that $\eta_{t-1,T_{l-1}+1}, \dots, \eta_{t-1,T_{l-1}+x-1}$ are good and $\eta_{t-1,T_{l-1}+x}$ is bad for $x \ge 2$. Then $\eta_{t-1,T_l} = \eta_{t-1,T_{l-1}+x}$ and

If η_{t-1,T_l} is chosen from $\mathcal{A}_{t-1,l-1} \cup \{w_t\} \cup \{\eta_{t-1,T_{l-1}+1}, \dots, \eta_{t-1,T_l-1}\}$, or η_{t-1,T_l} is chosen from $\mathcal{N}_{t-1,l-1}$ and the edge $\langle \eta_{t-1,T_l-1}, \eta_{t-1,T_l}\rangle$ is closed,
then we put
$\mathcal{N}^{(0)}_{t-1,l} = \mathcal{N}^{(0)}_{t-1,l-1} \setminus \{\eta_{t-1,T_{l-1}+1}, \dots, \eta_{t-1,T_l-1}\}$,
$\mathcal{N}^{(i)}_{t-1,l} = \mathcal{N}^{(i)}_{t-1,l-1}$ for $1 \le i \le k$,
$\mathcal{A}^{(1)}_{t-1,l} = \mathcal{A}^{(1)}_{t-1,l-1} \cup \{\eta_{t-1,T_{l-1}+1}, \dots, \eta_{t-1,T_l-1}\}$,
$\mathcal{A}^{(i)}_{t-1,l} = \mathcal{A}^{(i)}_{t-1,l-1}$ for $i \ne 1$.

If η_{t-1,T_l} is chosen from $\mathcal{N}^{(1)}_{t-1,l-1}$ and the edge $\langle \eta_{t-1,T_l-1}, \eta_{t-1,T_l}\rangle$ is open,
then we put
$\mathcal{N}^{(0)}_{t-1,l} = \mathcal{N}^{(0)}_{t-1,l-1} \setminus \{\eta_{t-1,T_{l-1}+1}, \dots, \eta_{t-1,T_l-1}\}$,
$\mathcal{N}^{(1)}_{t-1,l} = \mathcal{N}^{(1)}_{t-1,l-1} \setminus \{\eta_{t-1,T_l}\}$,
$\mathcal{N}^{(i)}_{t-1,l} = \mathcal{N}^{(i)}_{t-1,l-1}$ for $2 \le i \le k$,
$\mathcal{A}^{(1)}_{t-1,l} = \mathcal{A}^{(1)}_{t-1,l-1} \cup \{\eta_{t-1,T_{l-1}+1}, \dots, \eta_{t-1,T_l}\}$,
$\mathcal{A}^{(i)}_{t-1,l} = \mathcal{A}^{(i)}_{t-1,l-1}$ for $i \ne 1$.

If η_{t-1,T_l} is chosen from $\mathcal{N}^{(i)}_{t-1,l-1}$ for $2 \le i \le k$ and the edge $\langle \eta_{t-1,T_l-1}, \eta_{t-1,T_l}\rangle$ is open,
then we put
$\mathcal{N}^{(0)}_{t-1,l} = \mathcal{N}^{(0)}_{t-1,l-1} \setminus \{\eta_{t-1,T_{l-1}+1}, \dots, \eta_{t-1,T_l-1}\}$,
$\mathcal{N}^{(i)}_{t-1,l} = \mathcal{N}^{(i)}_{t-1,l-1} \setminus \{\eta_{t-1,T_l}\}$,
$\mathcal{N}^{(j)}_{t-1,l} = \mathcal{N}^{(j)}_{t-1,l-1}$ for $j \ne 0, i$,
$\mathcal{A}^{(1)}_{t-1,l} = \mathcal{A}^{(1)}_{t-1,l-1} \cup \{\eta_{t-1,T_{l-1}+1}, \dots, \eta_{t-1,T_l-1}\}$,
$\mathcal{A}^{(i)}_{t-1,l} = \mathcal{A}^{(i)}_{t-1,l-1} \cup \{\eta_{t-1,T_l}\}$,
$\mathcal{A}^{(j)}_{t-1,l} = \mathcal{A}^{(j)}_{t-1,l-1}$ for $j \ne 1, i$.

When all the above operations are over, the AS process $\rho_{t-1,l}$ is finished.

We difine ξ_t by

$$\xi_t = A_t - A_{t-1,1} - 1. \tag{3.1}$$

Namely, $\xi_t + 1$ is the number of vertices changing their state from neutral to active, from step $(t-1, 2)$ to step $(t-1, k+1)$.

Next we define $N(w_t)$ by $N(w_t) = A_{t-1,1} - A_{t-1,0} + \mathbf{1}_{\{w_t \in \mathcal{N}_{t-1}\}}$. Roughly $N(w_t)$ is the number of vertices changing their state from neutral to active, during step 1 (including w_t). Thus $A_t = A_{t-1} + N(w_t) + \xi_t$.

The process $N(w_t)$ is positive exactly when $A_{t-1} = 0$, and A_t can be written as $A_t = \sum_{s=1}^{t} N(w_s) + \sum_{s=1}^{t} \xi_s$. We will investigate the behavior of

$$X_t = \sum_{s=1}^{t} \xi_s.$$

Remark 1. *By the above construction,* $\mathcal{A}_t^{(0)} = \mathcal{A}_{t,l}^{(0)} = \emptyset$ *for any* t *and* l*.*

Chapter 4

Proof of Theorems 1, 3, and 4

By definition, we have

$$X_t = A_t - Z_t, \tag{4.1}$$

where Z_t is non-decreasing process given by

$$Z_t = \sum_{s=1}^{t} N(w_s).$$

Let $0 = t_0 < t_1 < \dots$ be the times at which A_t vanishes. Then $Z_t = Z_{t_j+1}$ for all $t \in \{t_j + 1, \dots, t_{j+1}\}$. Since $X_{t_j} = -Z_{t_j}$, we have $X_{t_{j+1}} = -Z_{t_{j+1}} = -Z_t < X_t$ for all $t \in \{t_j + 1, \dots, t_{j+1} - 1\}$. Therefore each t_j is one of renewal times of the process $\min_{0 \le s \le t} X_s$.

For vertices u and v, we mean by $u \leftarrow v$ that there is a directed edge (v, u) in $\vec{G}_{\text{out}}(n, k)$, and we mean by $\big\{\eta_{t-1,T_l} \in \mathcal{N}^{(i)}_{t-1,l-1},\ \text{open}\big\}$ that $\eta_{t-1,T_l} \in \mathcal{N}^{(i)}_{t-1,l-1}$ and the edge $\langle \eta_{t-1,T_l-1}, \eta_{t-1,T_l} \rangle$ is open (if $\eta_{t-1,T_l-1} = \eta_{t-1,T_{l-1}}$, then $\langle w_t, \eta_{t-1,T_l} \rangle$ is open). Also, $\big\{w_t \leftarrow v,\ \text{closed}\big\}$ means that $w_t \leftarrow v$ and the edge $\langle w_t, v \rangle$ is closed. $r_{t-1,l}$ denotes the number of good vertices in the AS process $\rho_{t-1,l}$.

Each $N^{(i)}_{t-1,l}$ is described such that

$$\begin{aligned} N^{(0)}_{t-1,1} &= N^{(0)}_{t-1} - \mathbf{1}_{\left\{w_t \in \mathcal{N}^{(0)}_{t-1}\right\}} \left\{1 + r_{t-1,1}\right\}, \\ N^{(0)}_{t-1,k+1} &= N^{(0)}_{t-1,1} - \mathbf{1}_{\left\{w_t \in \mathcal{N}^{(0)}_{t-1}\right\}} \sum_{l=2}^{k} r_{t-1,l} \\ &\quad - \sum_{j=1}^{k} \mathbf{1}_{\left\{w_t \in \mathcal{N}^{(j)}_{t-1} \cup \mathcal{A}^{(j)}_{t-1}\right\}} \sum_{l=j+1}^{k} r_{t-1,l} - \sum_{v \in \mathcal{N}^{(0)}_{t-1,k}} \mathbf{1}_{\left\{w_t \leftarrow v\right\}}, \end{aligned}$$

and for $1 \leq i \leq k$,

$$\begin{aligned} N^{(i)}_{t-1,1} &= N^{(i)}_{t-1} - \mathbf{1}_{\left\{w_t \in \mathcal{N}^{(0)}_{t-1}\right\}} \mathbf{1}_{\left\{\eta_{t-1,T_1} \in \mathcal{N}^{(i)}_{t-1,0},\ \text{open}\right\}} - \mathbf{1}_{\left\{w_t \in \mathcal{N}^{(i)}_{t-1}\right\}}, \\ N^{(i)}_{t-1,k+1} &= N^{(i)}_{t-1,1} - \mathbf{1}_{\left\{w_t \in \mathcal{N}^{(0)}_{t-1}\right\}} \sum_{l=2}^{k} \mathbf{1}_{\left\{\eta_{t-1,T_l} \in \mathcal{N}^{(i)}_{t-1,l-1},\ \text{open}\right\}} \\ &\quad - \sum_{j=1}^{k} \mathbf{1}_{\left\{w_t \in \mathcal{N}^{(j)}_{t-1} \cup \mathcal{A}^{(j)}_{t-1}\right\}} \sum_{l=j+1}^{k} \mathbf{1}_{\left\{\eta_{t-1,T_l} \in \mathcal{N}^{(i)}_{t-1,l-1},\ \text{open}\right\}} \\ &\quad + \sum_{v \in \mathcal{N}^{(i-1)}_{t-1,k}} \mathbf{1}_{\left\{w_t \leftarrow v,\ \text{closed}\right\}} - \sum_{v \in \mathcal{N}^{(i)}_{t-1,k}} \mathbf{1}_{\left\{w_t \leftarrow v\right\}}. \end{aligned}$$

Let

$$r'_{t-1,l} = r_{t-1,l} + \mathbf{1}_{\left\{\eta_{t-1,T_l} \in \mathcal{N}^{(\geq 1)}_{t-1,l-1},\ \text{open}\right\}},$$

i.e., r' is the number of new active vertices in $\rho_{t-1,l}$.

We will use a little different expression of ξ_t. Assume that $w_t \in \mathcal{N}^{(i)}_{t-1} \cup \mathcal{A}^{(i)}_{t-1}$ for some $1 \leq i \leq k-1$. Then we introduce fictitious AS processes $\hat{\rho}_{t-1,l}$ for $l = 1, \ldots, i$, which are independent copies of an AS process with data $\left\{\mathcal{N}^{(j)}_{t-1,l-1} \cup \mathcal{A}^{(j)}_{t-1,l-1}\right\}$, $0 \leq j \leq k$, which always executes, and $\mathcal{N}^{(i)}_{t-1,l}$, $\mathcal{A}^{(i)}_{t-1,l}$ are unchanged from $\mathcal{N}^{(i)}_{t-1,l-1}$, $\mathcal{A}^{(i)}_{t-1,l-1}$. Let $\hat{r}_{t-1,l}$ and $\hat{r}'_{t-1,l}$ be the number of good vertices in $\hat{\rho}_{t-1,l}$ and number of new active vertices in $\hat{\rho}_{t-1,l}$. For $l \geq i+1$, we put $\hat{\rho}_{t-1,l}$ as the l-th AS process as before. Therefore $\hat{r}_{t-1,l} = r_{t-1,l}$ and

$\hat{r}'_{t-1,l} = r'_{t-1,l}$ for $l \geq i+1$. By (3.1), ξ_t is described by

$$\begin{aligned}
\xi_t &= \sum_{l=2}^{k} \mathbf{1}_{\left\{w_t \in \mathcal{N}_{t-1}^{(\leq l-1)} \cup \mathcal{A}_{t-1}^{(\leq l-1)}\right\}} \hat{r}'_{t-1,l} + \sum_{v \in \mathcal{N}_{t-1,k}} \mathbf{1}_{\left\{w_t \leftarrow v,\ \text{open}\right\}} - 1 \\
&= \sum_{l=2}^{k} \hat{r}'_{t-1,l} + \sum_{v \in \mathcal{N}_{t-1,k}} \mathbf{1}_{\left\{w_t \leftarrow v,\ \text{open}\right\}} - 1 \\
&\quad - \sum_{j=2}^{k} \left\{ \mathbf{1}_{\left\{A_{t-1}^{(j)} > 0,\ A_{t-1}^{(\geq j+1)} = 0\right\}} + \mathbf{1}_{\left\{A_{t-1} = 0,\ w_t \in \mathcal{N}_{t-1}^{(j)}\right\}} \right\} \sum_{l=2}^{j} \hat{r}'_{t-1,l}.
\end{aligned} \tag{4.2}$$

Let $\mathcal{F}_t$ be the σ-algebra generated by all the information up to time t, and let $\mathcal{F}_{t-1,l} \supset \mathcal{F}_{t-1}$ be the σ-algebra generated by all the information up to the end of the l-th step of time t.

4.1 Crude estimates

Lemma 1. *Let $0 \leq p < 1$ and $m = m(n) \leq n$ be a sequence such that $m(n) \to \infty$ as $n \to \infty$. Then*

$$\begin{aligned}
&\mathbb{E}\big[\xi_t \big| \mathcal{F}_{t-1}\big] \\
&\quad = (k-1)\frac{p}{1-p} + kp - 1 - \sum_{j=2}^{k} F_j \frac{N_{t-1}^{(j)}}{n-t} - G\frac{A_{t-1}}{n-t} \\
&\quad - \sum_{j=2}^{k} \left\{ \mathbf{1}_{\left\{A_{t-1}^{(j)} > 0,\ A_{t-1}^{(\geq j+1)} = 0\right\}} + \mathbf{1}_{\left\{A_{t-1} = 0,\ w_t \in \mathcal{N}_{t-1}^{(j)}\right\}} \right\} \\
&\quad \times \left\{ (j-1)\frac{p}{1-p} - (j-1)\frac{p^2}{(1-p)^2}\frac{N_{t-1}^{(\geq 1)}}{n-t} - (j-1)\frac{p}{(1-p)^2}\frac{A_{t-1}}{n-t} \right\} \\
&\quad + \frac{\big(N_{t-1,l-1}^{(\geq 1)} + A_{t-1,l-1}\big)^2}{(n-t)^2} O\big(1\big) + O\bigg(\frac{m^{1/3}}{n-t} + m^{1/3} p^{m^{1/3}}\bigg) \\
&\quad + \mathbb{E}\Big[\mathbf{1}_{\left\{N_t^{(0)} < m^{1/3}\right\}} O\big(1\big) \Big| \mathcal{F}_{t-1}\Big],
\end{aligned} \tag{4.3}$$

where $F_j = (k-1)\frac{p^2}{(1-p)^2} + jp$ and $G = (k-1)\frac{p}{(1-p)^2} + kp$. Furthermore, let

$$D_k = 2\big(k - \sqrt{k^2 - k}\big). \tag{4.4}$$

If $p = \frac{1+\epsilon}{k+\sqrt{k^2-k}}$ *for some* $\epsilon = o(1)$*, then*

$$(k-1)\frac{p}{1-p} + kp - 1 = D_k\epsilon + O\bigl(\epsilon^2\bigr). \tag{4.5}$$

To calculate the conditional expectation $\mathbb{E}\bigl[\xi_t \big| \mathcal{F}_{t-1}\bigr]$, we analyze each step. Before the proof, we note that

$$\begin{aligned}
&\mathbf{1}_{\left\{\hat{r}'_{t-1,l}=x\right\}} \\
&\quad= \mathbf{1}_{\left\{\{\eta_{t-1,T_{l-1}+1},\dots,\eta_{t-1,T_{l-1}+x-1}\} \text{ are good}\right\}} \\
&\qquad\times \mathbf{1}_{\left\{\eta_{t-1,T_{l-1}+x} \text{ is bad and changes to active from neutral}\right\}} \\
&\qquad+ \mathbf{1}_{\left\{\{\eta_{t-1,T_{l-1}+1},\dots,\eta_{t-1,T_{l-1}+x}\} \text{ are good}\right\}} \\
&\qquad\times \mathbf{1}_{\left\{\eta_{t-1,T_{l-1}+x+1} \text{ is bad and doesn't change its state}\right\}} \\
&\quad= \mathbf{1}_{\left\{\hat{r}_{t-1,l}=x-1\right\}} \mathbf{1}_{\left\{\eta_{t-1,T_l}\in\mathcal{N}^{(\geq 1)}_{t-1,l-1},\text{ open}\right\}} \\
&\qquad+ \mathbf{1}_{\left\{\hat{r}_{t-1,l}=x\right\}} \left[\mathbf{1}_{\left\{\eta_{t-1,T_l}\in\mathcal{N}_{t-1,l-1},\text{ closed}\right\}} + \mathbf{1}_{\left\{\eta_{t-1,T_l}\in\mathcal{A}_{t-1,l-1}\cup\{w_t\}\right\}}\right].
\end{aligned} \tag{4.6}$$

Remark 2. *Suppose that* w_t *is a good vertex of the AS process* $\rho_{s-1,l}$ *for some* $s < t$ *and some* $1 \leq l \leq k$*. Namely, there exist* $s < t$*,* $1 \leq l \leq k$ *and* $y \in \mathbb{N}$ *such that* $w_t = \eta_{s-1,T_{l-1}+y}$ *and* $T_{l-1} + y < T_l$*. Then we have checked the directed edge* $\bigl(\eta_{s-1,T_{l-1}+y}, \eta_{s-1,T_{l-1}+y+1}\bigr)$*. Hence we have to exclude* $\eta_{s-1,T_{l-1}+y+1}$ *when we check other directed edges from* w_t $(= \eta_{s-1,T_{l-1}+y})$*. Furthermore, when we execute the* l*-th AS process* $\rho_{t-1,l}$*,* $\eta_{t-1,T_{l-1}+1}$ *is chosen from* $\mathcal{N}_{t-1,l-1} \cup \mathcal{A}_{t-1,l-1}$ *outside* $\bigl\{\eta_{t-1,T_0+1}, \eta_{t-1,T_1+1}, \dots, \eta_{t-1,T_{l-2}+1}\bigr\}$*. However, if we take conditional expectation without considering these things, then the error is* $O\bigl(\frac{1}{n-t}\bigr)$*.*

Proof of Lemma 1. First, we consider $\mathbb{E}\bigl[\hat{r}'_{t-1,l} | \mathcal{F}_{t-1,l-1}\bigr]$ for $2 \leq l \leq k$.

By Remark 2,

$$\begin{aligned}
&\mathbb{P}\big(\eta_{t-1,T_{l-1}+1} \text{ is good}\big|\mathcal{F}_{t-1,l-1}\big)\\
&\qquad= \frac{N^{(0)}_{t-1,l-1}}{n-t}p + O\bigg(\frac{1}{n-t}\bigg),\\
&\mathbb{P}\big(\eta_{t-1,T_{l-1}+1} \text{ is bad and change to active from neutral}\big|\mathcal{F}_{t-1,l-1}\big)\\
&\qquad= \frac{N^{(\geq 1)}_{t-1,l-1}}{n-t}p + O\bigg(\frac{1}{n-t}\bigg),\\
&\mathbb{P}\big(\eta_{t-1,T_{l-1}+1} \text{ is bad and doesn't change its state}\big|\mathcal{F}_{t-1,l-1}\big)\\
&\qquad= \frac{N_{t-1,l-1}}{n-t}(1-p) + \frac{A_{t-1,l-1}}{n-t} + O\bigg(\frac{1}{n-t}\bigg).
\end{aligned}$$

In the above equalities and hereafter, $O\big(\frac{1}{n-t}\big)$ can be taken independent of ω. Further, for $y \geq 2$,

$$\begin{aligned}
&\mathbb{P}\Big(\eta_{t-1,T_{l-1}+y} \text{ is good}\Big|\{\eta_{t-1,T_{l-1}+1}, \dots, \eta_{t-1,T_{l-1}+y-1}\} \text{ are good}\Big)\\
&\qquad= \frac{N^{(0)}_{t-1,l-1} - (y-1)}{n-t}p,\\
&\mathbb{P}\Big(\eta_{t-1,T_{l-1}+y} \text{ is bad and change to active from neutral}\Big|\\
&\qquad\qquad\qquad \{\eta_{t-1,T_{l-1}+1}, \dots, \eta_{t-1,T_{l-1}+y-1}\} \text{ are good}\Big)\\
&\qquad= \frac{N^{(\geq 1)}_{t-1,l-1}}{n-t}p,\\
&\mathbb{P}\Big(\eta_{t-1,T_{l-1}+y} \text{ is bad and doesn't change its state}\Big|\\
&\qquad\qquad\qquad \{\eta_{t-1,T_{l-1}+1}, \dots, \eta_{t-1,T_{l-1}+y-1}\} \text{ are good}\Big)\\
&\qquad= \frac{N_{t-1,l-1} - (y-1)}{n-t}(1-p) + \frac{A_{t-1,l-1}+1}{n-t}.
\end{aligned}$$

Therefore, from (4.6),

$$
\begin{aligned}
\mathbb{E}\Big[\hat{r}'_{t-1,l}\Big|\mathcal{F}_{t-1,l-1}\Big] &= \mathbb{E}\Big[\sum_{x=0}^{n} x\mathbf{1}_{\{\hat{r}'_{t-1,l}=x\}}\Big|\mathcal{F}_{t-1,l-1}\Big] \\
&= \sum_{x=1}^{N^{(0)}_{t-1,l-1}} x\cdot\Bigg[\prod_{y=0}^{x-2}\Bigg\{\frac{N^{(0)}_{t-1,l-1}-y}{n-t}p+\mathbf{1}_{\{y=0\}}\cdot O\Big(\frac{1}{n-t}\Big)\Bigg\} \\
&\quad\times\Bigg\{\frac{N^{(\geq 1)}_{t-1,l-1}}{n-t}p+O\Big(\frac{1}{n-t}\Big)\Bigg\} \\
&\quad+\prod_{y=0}^{x-1}\Bigg\{\frac{N^{(0)}_{t-1,l-1}-y}{n-t}p+\mathbf{1}_{\{y=0\}}\cdot O\Big(\frac{1}{n-t}\Big)\Bigg\} \\
&\quad\times\Bigg\{\frac{N_{t-1,l-1}-x}{n-t}(1-p)+\frac{A_{t-1,l-1}+x}{n-t}+O\Big(\frac{1}{n-t}\Big)\Bigg\}\Bigg] \\
&= \frac{N_{t-1,l-1}}{n-t}p\Bigg(1-\frac{N^{(0)}_{t-1,l-1}}{n-t}p\Bigg)\sum_{x=1}^{N^{(0)}_{t-1,l-1}} x\prod_{y=0}^{x-2}\Bigg\{\frac{N^{(0)}_{t-1,l-1}-y}{n-t}p\Bigg\}+O\Big(\frac{1}{n-t}\Big),
\end{aligned}
$$

where we understand that $\prod_{x=0}^{-1}\{\cdot\}=1$.

$$
\begin{aligned}
&\sum_{x=1}^{N^{(0)}_{t-1,l-1}} x\prod_{y=0}^{x-2}\Bigg\{\frac{N^{(0)}_{t-1,l-1}-y}{n-t}p\Bigg\} \\
&= \mathbf{1}_{\{N^{(0)}_{t-1,l-1}\geq m^{1/3}\}}\Bigg[\sum_{x=1}^{\lfloor m^{1/3}\rfloor} x\prod_{y=0}^{x-2}\Bigg\{\frac{N^{(0)}_{t-1,l-1}-y}{n-t}p\Bigg\} \\
&\quad+\sum_{x=\lfloor m^{1/3}\rfloor+1}^{N^{(0)}_{t-1,l-1}} x\prod_{y=0}^{x-2}\Bigg\{\frac{N^{(0)}_{t-1,l-1}-y}{n-t}p\Bigg\}\Bigg] \\
&\quad+\mathbf{1}_{\{N^{(0)}_{t-1,l-1}<m^{1/3}\}}\sum_{x=1}^{N^{(0)}_{t-1,l-1}} x\prod_{y=0}^{x-2}\Bigg\{\frac{N^{(0)}_{t-1,l-1}-y}{n-t}p\Bigg\} \\
&= \frac{1}{\Big(1-\frac{N^{(0)}_{t-1,l-1}}{n-t}p\Big)^2}+O\Big(\frac{m^{1/3}}{n-t}+m^{1/3}p^{m^{1/3}}\Big)+\mathbf{1}_{\{N^{(0)}_{t-1,l-1}<m^{1/3}\}}O(1).
\end{aligned}
$$

Thus

$$\begin{aligned}
&\mathbb{E}\Big[\hat{r}'_{t-1,l}\Big|\mathcal{F}_{t-1,l-1}\Big] \\
&= \frac{N_{t-1,l-1}}{n-t}p\frac{1}{1-\frac{N^{(0)}_{t-1,l-1}}{n-t}p} + O\bigg(\frac{m^{1/3}}{n-t} + m^{1/3}p^{m^{1/3}}\bigg) \\
&\quad +\mathbf{1}_{\left\{N^{(0)}_{t-1,l-1}<m^{1/3}\right\}}O\big(1\big) \\
&= \frac{N_{t-1,l-1}}{n-t}p\bigg[\frac{1}{1-p} - \frac{p}{(1-p)^2}\frac{N^{(\geq 1)}_{t-1,l-1}+A_{t-1,l-1}}{n-t}\bigg] \\
&\quad +\frac{\big(N^{(\geq 1)}_{t-1,l-1}+A_{t-1,l-1}\big)^2}{(n-t)^2}O\big(1\big) + O\bigg(\frac{m^{1/3}}{n-t} + m^{1/3}p^{m^{1/3}}\bigg) \\
&\quad +\mathbf{1}_{\left\{N^{(0)}_{t-1,l-1}<m^{1/3}\right\}}O\big(1\big) \\
&= \frac{p}{1-p} - \frac{p^2}{(1-p)^2}\frac{N^{(\geq 1)}_{t-1,l-1}}{n-t} - \frac{p}{(1-p)^2}\frac{A_{t-1,l-1}}{n-t} \\
&\quad +\frac{\big(N^{(\geq 1)}_{t-1,l-1}+A_{t-1,l-1}\big)^2}{(n-t)^2}O\big(1\big) + O\bigg(\frac{m^{1/3}}{n-t} + m^{1/3}p^{m^{1/3}}\bigg) \\
&\quad +\mathbf{1}_{\left\{N^{(0)}_{t-1,l-1}<m^{1/3}\right\}}O\big(1\big).
\end{aligned}$$

Since

$$\begin{aligned}
\mathbb{E}\bigg[\frac{N^{(\geq 1)}_{t-1,l-1}+A_{t-1,l-1}}{n-t}\bigg|\mathcal{F}_{t-1}\bigg] &= \frac{N^{(\geq 1)}_{t-1}+A_{t-1}}{n-t} + O\bigg(\frac{1}{n-t}\bigg), \\
\mathbb{E}\bigg[\frac{\big(N^{(\geq 1)}_{t-1,l-1}+A_{t-1,l-1}\big)^2}{(n-t)^2}\bigg|\mathcal{F}_{t-1}\bigg] &= \frac{\big(N^{(\geq 1)}_{t-1}+A_{t-1}\big)^2}{(n-t)^2} + O\bigg(\frac{1}{n-t}\bigg),
\end{aligned}$$

we get

$$
\begin{aligned}
&\mathbb{E}\Big[\sum_{l=2}^{k}\hat{r}'_{t-1,l}\Big|\mathcal{F}_{t-1}\Big]\\
&= (k-1)\frac{p}{1-p}-(k-1)\frac{p^2}{(1-p)^2}\frac{N_{t-1}^{(\geq 1)}}{n-t}-(k-1)\frac{p}{(1-p)^2}\frac{A_{t-1}}{n-t}\\
&\quad+\frac{\big(N_{t-1}^{(\geq 1)}+A_{t-1}\big)^2}{(n-t)^2}O\big(1\big)+O\bigg(\frac{m^{1/3}}{n-t}+m^{1/3}p^{m^{1/3}}\bigg)\\
&\quad+\mathbb{E}\Big[\mathbf{1}_{\big\{N_t^{(0)}<m^{1/3}\big\}}O\big(1\big)\Big|\mathcal{F}_{t-1}\Big], \qquad (4.7)
\end{aligned}
$$

where we used that $\mathbf{1}_{\big\{N_{t-1,l-1}^{(0)}<m^{1/3}\big\}}\leq \mathbf{1}_{\big\{N_t^{(0)}<m^{1/3}\big\}}$.

Next, we consider the conditional expectation of step $k+1$. Similarly to Remark 2,

$$
\begin{aligned}
&\mathbb{E}\Big[\sum_{v\in\mathcal{N}_{t-1,k}}\mathbf{1}_{\big\{w_t\leftarrow v,\text{ open}\big\}}\Big|\mathcal{F}_{t-1,k}\Big]\\
&= \sum_{j=0}^{k}N_{t-1,k}^{(j)}\frac{(k-j)p}{n-t}+O\bigg(\frac{1}{n-t}\bigg)\\
&= kp-\sum_{j=1}^{k}jp\frac{N_{t-1,k}^{(j)}}{n-t}-kp\frac{A_{t-1,k}}{n-t}+O\bigg(\frac{1}{n-t}\bigg),
\end{aligned}
$$

hence we have

$$
\begin{aligned}
&\mathbb{E}\Big[\sum_{v\in\mathcal{N}_{t-1,k}}\mathbf{1}_{\big\{w_t\leftarrow v,\text{ open}\big\}}\Big|\mathcal{F}_{t-1}\Big]\\
&= kp-\sum_{j=1}^{k}jp\frac{N_{t-1}^{(j)}}{n-t}-kp\frac{A_{t-1}}{n-t}+O\bigg(\frac{1}{n-t}\bigg). \qquad (4.8)
\end{aligned}
$$

Therefore, combining (4.7) and (4.8), we have

$$
\begin{aligned}
&\mathbb{E}\big[\xi_t\big|\mathcal{F}_{t-1}\big]\\
=\ & (k-1)\frac{p}{1-p}-(k-1)\frac{p^2}{(1-p)^2}\frac{N_{t-1}^{(\geq 1)}}{n-t}-(k-1)\frac{p}{(1-p)^2}\frac{A_{t-1}}{n-t}\\
&+kp-\sum_{j=1}^{k} jp\frac{N_{t-1}^{(j)}}{n-t}-kp\frac{A_{t-1}}{n-t}-1\\
&-\sum_{j=2}^{k}\Big\{\mathbf{1}_{\big\{A_{t-1}^{(j)}>0,\ A_{t-1}^{(\geq j+1)}=0\big\}}+\mathbf{1}_{\big\{A_{t-1}=0,\ w_t\in\mathcal{N}_{t-1}^{(j)}\big\}}\Big\}\\
&\times\Big\{(j-1)\frac{p}{1-p}-(j-1)\frac{p^2}{(1-p)^2}\frac{N_{t-1}^{(\geq 1)}}{n-t}-(j-1)\frac{p}{(1-p)^2}\frac{A_{t-1}}{n-t}\Big\}\\
&+\frac{\big(N_{t-1,l-1}^{(\geq 1)}+A_{t-1,l-1}\big)^2}{(n-t)^2}O\big(1\big)+O\bigg(\frac{m^{1/3}}{n-t}+m^{1/3}p^{m^{1/3}}\bigg)\\
&+\mathbb{E}\Big[\mathbf{1}_{\big\{N_t^{(0)}<m^{1/3}\big\}}O\big(1\big)\Big|\mathcal{F}_{t-1}\Big]
\end{aligned}
$$

Here let $F_j=(k-1)\frac{p^2}{(1-p)^2}+jp$ and $G=(k-1)\frac{p}{(1-p)^2}+kp$, we get (4.3).

(4.5) is trivial. □

Lemma 2. *Let $\{\alpha_{t-1,l}\}$ be independent random vaiables distributed as*

$$\alpha_{t-1,l} \sim Ge(1-p) \quad (1\leq l\leq k), \tag{4.9}$$

$$\alpha_{t-1,k+1} \sim \begin{cases} Bin\Big(n-t;\frac{kp}{n-t}\Big) & (t\leq n-1)\\ Be(kp) & (t=n)\end{cases} \tag{4.10}$$

for each $1\leq t\leq n$ and $1\leq l\leq k+1$. Then we can couple $\{\hat{r}'_{t-1,l}\}$ and $\{\alpha_{t-1,l}\}$ such that almost surely $\hat{r}'_{t-1,l}\leq\alpha_{t-1,l}$ for each $1\leq t\leq n$ and $1\leq l\leq k$. Also we can couple $\big\{\sum_{v\in\mathcal{N}_{t-1,k}}\mathbf{1}_{\{w_t\leftarrow v,\ open\}}\big\}$ and $\{\alpha_{t-1,k+1}\}$ such that almost surely $\sum_{v\in\mathcal{N}_{t-1,k}}\mathbf{1}_{\{w_t\leftarrow v,\ open\}}\leq\alpha_{t-1,k+1}$ for each $1\leq t\leq n$. Further, let $\alpha_t=\sum_{l=2}^{k}\alpha_{t-1,l}+\alpha_{t-1,k+1}-1$. Then α_t is distributed as

$$\alpha_s \sim \begin{cases} NB(k-1;1-p)+Bin\big(n-t;\frac{kp}{n-t}\big)-1 & (t\leq n-1)\\ NB(k-1;1-p)+Be(kp)-1 & (t=n)\end{cases} \tag{4.11}$$

for each $1\leq t\leq n$, and we can couple $\{\xi_t\}$ and $\{\alpha_t\}$ such that almost surely $\xi_t\leq\alpha_t$ for each t.

Proof. We consider a modified AS process for each $(t-1, l)$ for $1 \le l \le k$, which is constructed from the following rule. It continues until it checkes a closed edge even if it hits a bad vertex. Other procedure is just the same as the AS process. Let $\chi_{t-1,l}$ be the number of new active vertices in this process. By definition, almost surely $\hat{r}'_{t-1,l} \le \chi_{t-1,l}$, and we can couple $\chi_{t-1,l}$ and $\alpha_{t-1,l}$ such that almost surely $\chi_{t-1,l} \le \alpha_{t-1,l}$. Therefore we can couple $\hat{r}'_{t-1,l}$ and $\alpha_{t-1,l}$.

Other facts are trivial. □

Lemma 3. *Let $\delta > 0$ be small enough. For any $0 \le t \le \delta n$, there exist positive constants $C_1 = C_1(k,\delta)$ and $C_2 = C_2(k,\delta)$,*

$$\mathbb{E}\big[N_t^{(\ge 1)} + A_t\big] \le C_1 t, \qquad \mathbb{E}\Big[\big(N_t^{(\ge 1)} + A_t\big)^2\Big] \le C_2 t^2.$$

Proof. Let $\zeta_t^{N(\ge 1)+A} = (N_t^{(\ge 1)} + A_t) - (N_{t-1}^{(\ge 1)} + A_{t-1})$. Also, let $\bar{\zeta}_t^{N(\ge 1)+A}$ be a random variable distributed as $NB(k; 1-p) + Bin\big(n; \frac{k}{n-t}\big)$, where $\bar{\zeta}_t^{N(\ge 1)+A}$ is independent of $\mathcal{F}_{t-1}$. We can couple $\zeta_t^{N(\ge 1)+A}$ and $\bar{\zeta}_t^{N(\ge 1)+A}$, such that

$$\zeta_t^{N(\ge 1)+A} \le \bar{\zeta}_t^{N(\ge 1)+A} \quad \text{a.s.}$$

for all $t \le \delta n$. Hence we get some constant $C_1 = C_1(k,\delta)$ depending on k and δ,

$$\begin{aligned} 0 \ &\le\ \mathbb{E}\big[N_t^{(\ge 1)} + A_t\big] = \sum_{s=1}^{t} \mathbb{E}\zeta_s^{N(\ge 1)+A} \\ &\le\ \sum_{s=1}^{t} \mathbb{E}\bar{\zeta}_s^{N(\ge 1)+A} \ \le\ \sum_{s=1}^{t} \left\{ \frac{kp}{1-p} + \frac{kn}{n-s} \right\} \ \le\ C_1 t. \end{aligned}$$

Also there exists $C_2(k,\delta)$ depending on k and δ,

$$\begin{aligned} 0 \le \mathbb{E}\Big[\big(N_t^{(\ge 1)} + A_t\big)^2\Big] \ &=\ \mathbb{E}\Big[\big(N_{t-1}^{(\ge 1)} + A_{t-1} + \zeta_t^{N(\ge 1)+A}\big)^2\Big] \\ &=\ \mathbb{E}\Big[\big(N_{t-1}^{(\ge 1)} + A_{t-1}\big)^2\Big] \\ &\quad +2\mathbb{E}\Big[\big(N_{t-1}^{(\ge 1)} + A_{t-1}\big)\zeta_t^{N(\ge 1)+A}\Big] \\ &\quad +\mathbb{E}\Big[\big(\zeta_t^{N(\ge 1)+A} + 1\big)^2 - 2\zeta_t^{N(\ge 1)+A} - 1\Big] \\ &\le\ \mathbb{E}\Big[\big(N_{t-1}^{(\ge 1)} + A_{t-1}\big)^2\Big] + 2\big(C_1(k,\delta)\big)^2 t + C_1'(k,\delta) \\ &\le\ C_2(k,\delta) t^2, \end{aligned}$$

where $C_1'(k,\delta)$ depends on n, k. □

Lemma 4. *Let δ be in $(0,1)$ and $p<1$.*

If $a > k\big(\frac{p}{1-p}+\frac{1}{1-\delta}\big)$, then there exists some constant $\bar{I}(a)>0$ such that

$$\mathbb{P}\big(N_t^{(0)} < n-t-at\big) \le e^{-\bar{I}(a)t}$$

for any $t\le \delta n$. In particular, if $p<\frac{1}{3}$ and $\delta < \frac{1-2p}{2-3p}$, then there exists some constant $\bar{I}(2k)>0$ such that

$$\mathbb{P}\big(N_t^{(0)} < n-t-2kt\big) \le e^{-\bar{I}(2k)t}$$

for any $t\le\delta n$.

If $b<\frac{k-1-k\delta}{1+k\delta}$ for some $\delta>0$, then there exists some constant $\underline{I}(b)>0$ such that

$$\mathbb{P}\big(N_t^{(0)} > n-t-bt\big) \le e^{-\underline{I}(b)t}$$

for any $t\le\delta n$. In particular, if $\delta\le\frac{2k-3}{k(k+3)}$, then there exists some constant $\underline{I}\big(\frac{k}{3}\big)>0$ such that

$$\mathbb{P}\left(N_t^{(0)} > n-t-\frac{k}{3}t\right)\le e^{-\underline{I}(\frac{k}{3})t}$$

for any $t\le\delta n$.

Proof. Take an $s\le t$. Recall that $\zeta_s^{N^{(\ge1)}+A} = (N_s^{(\ge1)}+A_s)-(N_{s-1}^{(\ge1)}+A_{s-1})$, and let $\{\bar{X}_s\}$ be i.i.d. random variables distributed as $NB(k;1-p)+Bin\big(n;\frac{k}{(1-\delta)n}\big)$ for each $s\le t$, and we couple $\zeta_s^{N^{(\ge1)}+A}$ and $\bar{X}_s$ for each $s\le t$ such that almost surely $\zeta_s^{N^{(\ge1)}+A}\le\bar{X}_s$. Let $\bar{S}_t$ be $\sum_{s=1}^t\bar{X}_s$. Then for any $\theta\in(0,1)$,

$$\mathbb{P}\big(N_t^{(0)}<n-t-at\big)\le\mathbb{P}\Big(\bar{S}_t>at\Big)\le\frac{\mathbb{E}e^{\theta\bar{S}_t}}{e^{\theta at}}=\left(\frac{\mathbb{E}e^{\theta\bar{X}_1}}{e^{\theta a}}\right)^t.$$

Here

$$\mathbb{E}e^{\theta\bar{X}_1}=\left\{\frac{1-p}{1-e^\theta p}\right\}^k\left\{\frac{k}{(1-\delta)n}e^\theta+1-\frac{k}{(1-\delta)n}\right\}^n,$$

for any θ satisfying $e^\theta p < 1$ and $\theta < 1$, thus $\theta < (-\log p) \wedge 1$.

Let $\bar{\phi}(\theta)$ be $\log \mathbb{E} e^{\theta \bar{X}_1}$. Then

$$\begin{aligned} \bar{\phi}'(0) &= \mathbb{E}\bar{X}_1 = k\Big(\frac{p}{1-p} + \frac{1}{1-\delta}\Big), \\ \bar{\phi}''(\theta) &\le \bar{C} \end{aligned}$$

for some constant $\bar{C} > 0$. Therefore

$$\begin{aligned} \mathbb{P}\Big(\bar{S}_t > at\Big) &\le e^{t[\bar{\phi}(\theta) - a\theta]} \\ &\le e^{t\left[\mathbb{E}\bar{X}_1\theta - a\theta + \bar{C}\theta^2\right]}. \end{aligned}$$

We can take this $\bar{C}$ independent of n and t. For some constant a satisfying $a > \mathbb{E}\bar{X}_1 = k\big(\frac{p}{1-p} + \frac{1}{1-\delta}\big)$, take $\bar{C}$ satisfying $\frac{a - \mathbb{E}\bar{X}_1}{2\bar{C}} < (-\log p) \wedge 1$ and let $\bar{I}(a)$ be $\frac{(a-\mathbb{E}\bar{X}_1)^2}{4\bar{C}}$, we get

$$\mathbb{P}\Big(\bar{S}_t > at\Big) \le e^{-\bar{I}(a)t}.$$

In particular, set $a = 2k$ and if $p < \frac{1}{3}$ and $\delta < \frac{1-2p}{2-3p}$, we get

$$\mathbb{P}\big(N_t^{(0)} < n - t - 2kt\big) \le e^{-\bar{I}(2k)t}$$

for any $t \le \delta n$.

Next, let $\underline{X}_s$ be i.i.d. random variables distributed as $B\big(n-(1+b)\delta n; \frac{k}{n}\big) - 1$. We can couple $\zeta_s^{N(\ge 1)+A}$ and $\underline{X}_s$ such that $\zeta_s^{N(\ge 1)+A} \ge \underline{X}_s$ on $\big\{N_{s-1,k}^{(0)} > n - (1+b)\delta n\big\}$ for each $s \le t$, and $\underline{X}_s$ is independent of $\mathcal{F}_{s-1,k}$. We get

$$\mathbb{E} e^{-\theta \underline{X}_s} = \left\{\frac{k}{n} e^{-\theta} + 1 - \frac{k}{n}\right\}^{n-(1+b)\delta n} e^\theta.$$

Let $\underline{\phi}(\theta)$ be $\log \mathbb{E} e^{\theta \underline{X}_1}$. Then

$$\begin{aligned} \underline{\phi}'(0) &= \mathbb{E}\underline{X}_1 = k - 1 - k(1+b)\delta, \\ \underline{\phi}''(\theta) &\le \underline{C} \end{aligned}$$

for some constant $\underline{C} > 0$. Define $\underline{S}_t = \sum_{s=1}^t \underline{X}_s$, we have

$$\begin{aligned} \mathbb{P}\big(\underline{S}_t < bt\big) &\le e^{t\left[\underline{\phi}(-\theta) + b\theta\right]} \\ &\le e^{t\left[-\mathbb{E}\underline{X}_1\theta + b\theta + \underline{C}\theta^2\right]}. \end{aligned}$$

For some $b > 0$ satisfying $b < k - 1 - k(1+b)\delta$, take $\underline{C}$ such that $\frac{\mathbb{E}X_1 - b}{2\underline{C}} < 1$. Let $\underline{I}(b) = \frac{(\mathbb{E}X_1 - b)^2}{4\underline{C}}$. Since $N_{s,l}^{(0)}$ is decreasing for each time and step, we have that

$$\begin{aligned}
\mathbb{P}\big(N_t^{(0)} > n - t - bt\big) &= \mathbb{E}\Big[\mathbf{1}_{\big\{N_t^{(0)} > n-t-bt\big\}}\mathbf{1}_{\big\{N_t^{(0)} > n-t-bt\big\}}\Big] \\
&\leq \mathbb{E}\Big[\mathbf{1}_{\big\{N_t^{(\geq 1)} + A_t < bt\big\}}\mathbf{1}_{\big\{N_t^{(0)} > n-(1+b)\delta n\big\}}\Big] \\
&\leq \mathbb{P}\big(\underline{S}_t < bt\big) \\
&\leq e^{-\underline{I}(b)t}.
\end{aligned}$$

In particular, set $b = \frac{k}{3}$ and if $\delta \leq \frac{2k-3}{k(k+3)}$ then

$$\mathbb{P}\Big(N_t^{(0)} > n - t - \frac{k}{3}t\Big) \leq e^{-\underline{I}(\frac{k}{3})t}$$

for any $t \leq \delta n$. □

4.2 Proof of Theorem 1

By Lemma 4 and the calculation of ξ_t in the proof of Lemma 1, we get the proof of Theorem 1.

Proof of Theorem 1. Let $\{\alpha_s\}$ be random variables given by (4.11), where we extend the definition for $s > n$ as in the following way,

$$\alpha_s \sim \begin{cases} NB(k-1; 1-p) + Bin\big(n-s; \frac{kp}{n-s}\big) - 1 & (s \leq n-1), \\ NB(k-1; 1-p) + Be(kp) - 1 & (s \geq n). \end{cases} \tag{4.12}$$

Let $\{W_t\}$ be the process defined by $W_t = W_0 + \sum_{s=1}^t \alpha_s$, where W_0 is some integer $W_0 > 0$. We fix $W_0 = d$ for some integer $1 \leq d < n^{1/3}$. Let $h = n^{1/3}$ and

$$\gamma_h = \min\big\{t : W_t = 0 \text{ or } W_t \geq h\big\}.$$

When $s \leq n-1$, for any $c > 0$, we get

$$\begin{aligned}
\mathbb{E}e^{-c\alpha_s} &= \left\{\frac{1-p}{1-e^{-c}p}\right\}^{k-1}\left\{\frac{kp}{n-s}e^{-c} + 1 - \frac{kp}{n-s}\right\}^{n-s} e^c \\
&= \exp\Bigg[(k-1)\log\left\{1 - \frac{(1-e^{-c})p}{1-e^{-c}p}\right\} \\
&\quad + (n-s)\log\left\{1 - (1-e^{-c})\frac{kp}{n-s}\right\} + c\Bigg].
\end{aligned}$$

Here, when $c = o(1)$,

$$\begin{aligned}
\frac{(1-e^{-c})p}{1-e^{-c}p} &= \frac{\big(c - \frac{c^2}{2} + O(c^3)\big)p}{1 - \big(1 - c + \frac{c^2}{2} + O(c^3)\big)p} = \frac{\big(c - \frac{c^2}{2}\big)p}{1 - p + cp - \frac{c^2}{2}p} + O(c^3) \\
&= c\frac{p}{1-p} - \frac{c^2}{2}\Big\{\frac{p}{1-p} + 2\Big(\frac{p}{1-p}\Big)^2\Big\} + O(c^3),
\end{aligned}$$

and

$$(1-e^{-c})\frac{kp}{n-s} = c\frac{kp}{n-s} - \frac{c^2}{2}\frac{kp}{n-s} + O\Big(\frac{c^3}{n-s}\Big).$$

Thus,

$$\begin{aligned}
&\mathbb{E}e^{-c\alpha_s} \\
&= \exp\Bigg[(k-1)\log\Bigg\{1 - c\frac{p}{1-p} + \frac{c^2}{2}\Bigg(\frac{p}{1-p} + 2\Big(\frac{p}{1-p}\Big)^2\Bigg) + O(c^3)\Bigg\} \\
&\quad + (n-s)\log\Bigg\{1 - c\frac{kp}{n-s} + \frac{c^2}{2}\frac{kp}{n-s} + O\Big(\frac{c^3}{n-s}\Big)\Bigg\} + c\Bigg] \\
&= \exp\Bigg[c\Bigg\{-(k-1)\frac{p}{1-p} - kp + 1\Bigg\} \\
&\quad + \frac{c^2}{2}\Bigg\{(k-1)\frac{p}{1-p} + (k-1)\Big(\frac{p}{1-p}\Big)^2 + kp - \frac{k^2p^2}{n-s}\Bigg\} + O\big(c^3\big)\Bigg].
\end{aligned}$$

Writing $\epsilon = \lambda n^{-1/3}$ and from (4.5), we have

$$\begin{aligned}
\mathbb{E}e^{-c\alpha_s} &= \exp\Bigg[-D_k c\epsilon + \frac{c^2}{2}\Bigg\{(k-1)\frac{p}{1-p} + (k-1)\Big(\frac{p}{1-p}\Big)^2 \\
&\quad + kp - \frac{k^2p^2}{n-s}\Bigg\} + O\Big(c^3 + c\epsilon^2\Big)\Bigg]. \qquad (4.13)
\end{aligned}$$

When $s \geq n$, for any $c > 0$ we get

$$\mathbb{E}e^{-c\alpha_s} = \Bigg\{\frac{1-p}{1-e^{-c}p}\Bigg\}^{k-1}\Big\{kpe^{-c} + 1 - kp\Big\}e^{c}.$$

So, similarly to the case of $s \leq n-1$, we have

$$\begin{aligned}
&\mathbb{E}e^{-c\alpha_s} \\
&= \exp\Bigg[-D_k c\epsilon + \frac{c^2}{2}\Bigg\{(k-1)\frac{p}{1-p} + (k-1)\Big(\frac{p}{1-p}\Big)^2 + kp - k^2p^2\Bigg\} \\
&\quad + O\big(c^3 + c\epsilon^2\big)\Bigg]. \qquad (4.14)
\end{aligned}$$

It is easy to see that

$$1 < D_k < 2. \tag{4.15}$$

Furthermore,

$$(k-1)\frac{p}{1-p} + (k-1)\Big(\frac{p}{1-p}\Big)^2 + kp = (k-1)\Big(\frac{p}{1-p}\Big)^2 + 1 + O(\epsilon), \tag{4.16}$$

and

$$p = \frac{1}{k+\sqrt{k^2-k}} + O\big(\epsilon\big) \le \frac{1}{2k-1} + O\big(\epsilon\big). \tag{4.17}$$

Using these facts, we will show that

$$\mathbb{P}\big(W_{\gamma_h} > 0\big) = \begin{cases} \frac{4d\lambda}{1-e^{-4\lambda}} n^{-1/3} & \lambda > 0, \\ \frac{-2d\lambda}{e^{-\lambda}-1} n^{-1/3} & \lambda < 0, \\ dn^{-1/3} & \lambda = 0. \end{cases} \tag{4.18}$$

When $\lambda > 0$, let $c = 4\epsilon = 4\lambda n^{-1/3}$. For any $s \ge 1$, by (4.13), (4.14) and (4.16), (4.17), we have

$$\begin{aligned} \mathbb{E}e^{-c\alpha_s} &\ge \exp\Big[8\Big\{-(k-\sqrt{k^2-k}) + (k-1)\frac{p}{1-p} + (k-1)\Big(\frac{p}{1-p}\Big)^2 \\ &\quad +kp - k^2p^2\Big\}\epsilon^2 + O\big(\epsilon^3\big)\Big] \\ &= \exp\Big[8\Big\{-kp + kp^2 - k^2p^2 + 1\Big\}\epsilon^2 + O\big(\epsilon^3\big)\Big] \\ &\ge \exp\Big[8\frac{k^2-2k+1}{(2k-1)^2}\epsilon^2 + O\big(\epsilon^3\big)\Big]. \end{aligned}$$

So, for a small enough $\epsilon > 0$, $\mathbb{E}e^{-c\alpha_s} \ge 1$. Since

$$\frac{e^{-cW_t}}{\prod_{1\le s\le t}\mathbb{E}\big[e^{-c\alpha_s}\big]} \quad \text{is a martingale,}$$

we have

$$\begin{aligned} e^{-cd} &= \mathbb{E}e^{-cW_0} = \mathbb{E}\Big[\frac{e^{-cW_{\gamma_h}}}{\prod_{1\le s\le \gamma_h}\mathbb{E}\big[e^{-c\alpha_s}\big]}\Big] \\ &\le \mathbb{E}e^{-cW_{\gamma_h}} \\ &\le e^{-cn^{1/3}}\mathbb{P}\big(W_{\gamma_h} > 0\big) + \mathbb{P}\big(W_{\gamma_h} = 0\big). \end{aligned}$$

It means that

$$\mathbb{P}\big(W_{\gamma_h} > 0\big) \le \frac{1-e^{-cd}}{1-e^{-cn^{1/3}}} \le \frac{4d\lambda}{1-e^{-4\lambda}} n^{-1/3}.$$

When $\lambda < 0$, let $c = -\epsilon = -\lambda n^{-1/3}$. From (4.17),

$$\frac{p}{1-p} \le \frac{1}{2(k-1)}. \tag{4.19}$$

Using (4.13) - (4.16) and (4.19), for any $s \ge 1$, we have

$$\begin{aligned} \mathbb{E}e^{c\alpha_s} &\le \exp\Big[\epsilon^2\Big\{-1+\frac{1}{2}\Big((k-1)\frac{1}{\big(2(k-1)\big)^2}+1\Big)\Big\}+O\big(\epsilon^3\big)\Big] \\ &\le \exp\Big[-\frac{3}{8}\epsilon^2+O\big(\epsilon^3\big)\Big]. \end{aligned}$$

Thus, for $\epsilon < 0$ with a small enough $|\epsilon|$, $\mathbb{E}e^{c\alpha_s} \le 1$. Since

$$\frac{e^{cW_t}}{\prod_{1\le s\le t}\mathbb{E}e^{c\alpha_s}} \quad \text{is a martingale,}$$

we have

$$\begin{aligned} e^{cd} &= \mathbb{E}e^{cW_0} = \mathbb{E}\Big[\frac{e^{cW_{\gamma_h}}}{\prod_{1\le s\le \gamma_h}\mathbb{E}e^{c\alpha_s}}\Big] \\ &\ge \mathbb{E}e^{cW_{\gamma_h}} \\ &\ge e^{cn^{1/3}}\mathbb{P}\big(W_{\gamma_h}>0\big)+\mathbb{P}\big(W_{\gamma_h}=0\big). \end{aligned}$$

It means that

$$\mathbb{P}\big(W_{\gamma_h} > 0\big) \le \frac{e^{cd}-1}{e^{cn^{1/3}}-1} \le \frac{-2d\lambda}{e^{-\lambda}-1} n^{-1/3},$$

for a large enough n.

When $\lambda = 0$, $\{W_t\}$ is a martingale. By optional stopping theorem, $\mathbb{P}\big(W_{\gamma_h} > 0\big) \le dn^{-1/3}$.

Next, we check the bound of W_t in the following lemma.

Lemma 5. *For a large enough n,*

$$\mathbb{E}W_{\gamma_h} \le \big(3kn^{1/3}+1\big)\mathbb{P}\big(W_{\gamma_h}>0\big), \tag{4.20}$$

$$\mathbb{E}W_{\gamma_h}^2 \le \big(9k^2n^{2/3}+1\big)\mathbb{P}\big(W_{\gamma_h}>0\big). \tag{4.21}$$

Proof. We have

$$\begin{aligned}
\mathbb{E}W_{\gamma_h} &= \mathbb{E}\big[W_{\gamma_h-1}+\alpha_{\gamma_h}\big]\\
&\leq \mathbb{E}\big[n^{1/3}+\alpha_{\gamma_h}\mathbf{1}_{\{\alpha_{\gamma_h}\leq(3k-1)n^{1/3}\}}\big|W_{\gamma_h}>0\big]\mathbb{P}\big(W_{\gamma_h}>0\big)\\
&\quad+\mathbb{E}\big[\alpha_{\gamma_h}\mathbf{1}_{\{\alpha_{\gamma_h}>(3k-1)n^{1/3}\}}\big]\\
&\leq 3kn^{1/3}\mathbb{P}\big(W_{\gamma_h}>0\big)+\mathbb{E}\big[\alpha_{\gamma_h}\mathbf{1}_{\{\alpha_{\gamma_h}>(3k-1)n^{1/3}\}}\big].
\end{aligned}$$

We calculate the second term. First,

$$\begin{aligned}
&\mathbb{E}\big[\alpha_{\gamma_h}\mathbf{1}_{\{\alpha_{\gamma_h}>(3k-1)n^{1/3}\}}\big]\\
&\leq \sum_{x=1}^{n^{1/3}-1}\sum_{s=1}^{\infty}\mathbb{E}\big[\alpha_s\mathbf{1}_{\{\alpha_s>(3k-1)n^{1/3}\}}\big|0<W_1,\ldots,W_{s-2}<n^{1/3},W_{s-1}=x,\\
&\qquad \alpha_s\geq n^{1/3}-x\big]\\
&\qquad \times\mathbb{P}\big(0<W_1,\ldots,W_{s-2}<n^{1/3},W_{s-1}=x,\alpha_s\geq n^{1/3}-x\big)\\
&= \sum_{x=1}^{n^{1/3}-1}\sum_{s=1}^{\infty}\mathbb{E}\big[\alpha_s\mathbf{1}_{\{\alpha_s>(3k-1)n^{1/3}\}}\big|\alpha_s\geq n^{1/3}-x\big]\\
&\qquad \times\mathbb{P}\big(0<W_1,\ldots,W_{s-2}<n^{1/3},W_{s-1}=x,\alpha_s\geq n^{1/3}-x\big)\\
&\leq \sum_{s=1}^{\infty}\frac{\mathbb{E}\big[\alpha_s\mathbf{1}_{\{\alpha_s>(3k-1)n^{1/3}\}}\big]}{\mathbb{P}\big(\alpha_s\geq n^{1/3}\big)}\mathbb{P}\big(\gamma_h=s,\ W_s>0\big)\\
&= \sum_{s=1}^{\infty}\sum_{j=1}^{\infty}\frac{\mathbb{E}\big[\alpha_s\mathbf{1}_{\{(3k-1)jn^{1/3}<\alpha_s\leq(3k-1)(j+1)n^{1/3}\}}\big]}{\mathbb{P}\big(\alpha_s\geq n^{1/3}\big)}\mathbb{P}\big(\gamma_h=s,\ W_s>0\big)\\
&\leq (3k-1)n^{1/3}\sum_{s=1}^{\infty}\sum_{j=1}^{\infty}(j+1)\frac{\mathbb{P}\big(\alpha_s\geq(3k-1)jn^{1/3}\big)}{\mathbb{P}\big(\alpha_s\geq n^{1/3}\big)}\mathbb{P}\big(\gamma_h=s,\ W_s>0\big).
\end{aligned} \tag{4.22}$$

Recall (4.9), (4.10), (4.11). We have

$$\mathbb{P}\big(\alpha_s\geq n^{1/3}\big)\geq\mathbb{P}\big(\alpha_{s-1,2}\geq n^{1/3}\big)=p^{n^{1/3}}.$$

By $k(kp)^k \le k\big(\frac{1}{2-1/k}\big)^k \le \frac{8}{9}$ and

$$
\begin{aligned}
\mathbb{P}\big(\alpha_s \ge (3k-1)jn^{1/3}\big) &\le \sum_{l=2}^{k} \mathbb{P}\big(\alpha_{s-1,l} \ge 2jn^{1/3}\big) \\
&\quad + \mathbb{P}\bigg(\alpha_{s-1,k+1} \ge (k+1)jn^{1/3}\bigg) \\
&\le (k-1)p^{2jn^{1/3}} + C_1(kp)^{(k+1)jn^{1/3}} \qquad (4.23)
\end{aligned}
$$

for some constants $C_1 = C_1(k) > 0$, we get

$$
\begin{aligned}
\frac{\mathbb{P}\big(\alpha_s \ge (3k-1)jn^{1/3}\big)}{\mathbb{P}\big(\alpha_s \ge n^{1/3}\big)} &\le (k-1)p^{(2j-1)n^{1/3}} + C_1\Big\{k(kp)^k\Big\}^{jn^{1/3}} \\
&\le (k-1)p^{jn^{1/3}} + C_1\Big(\frac{8}{9}\Big)^{jn^{1/3}}. \qquad (4.24)
\end{aligned}
$$

From (4.22) and (4.24),

$$
\begin{aligned}
&\mathbb{E}\big[\alpha_{\gamma_h}\mathbf{1}_{\{\alpha_{\gamma_h} > (3k-1)n^{1/3}\}}\big] \\
&\quad \le C_2 n^{1/3}\bigg\{p^{n^{1/3}} + \Big(\frac{8}{9}\Big)^{n^{1/3}}\bigg\}\sum_{s=1}^{\infty}\mathbb{P}\big(\gamma_h = s,\ W_s > 0\big) \\
&\quad \le C_2 n^{1/3}\bigg\{p^{n^{1/3}} + \Big(\frac{8}{9}\Big)^{n^{1/3}}\bigg\}\mathbb{P}\big(W_{\gamma_h} > 0\big) \qquad (4.25)
\end{aligned}
$$

for some constant $C_2 = C_2(k) > 0$, which implies that

$$
\mathbb{E}\big[\alpha_{\gamma_h}\mathbf{1}_{\{\alpha_{\gamma_h} > (3k-1)n^{1/3}\}}\big] \le \mathbb{P}\big(W_{\gamma_h} > 0\big)
$$

for a large enough n. Therefore (4.20) is proved for a large enough n.

Next, we prove (4.21). By (4.25), we have

$$
\begin{aligned}
\mathbb{E}W_{\gamma_h}^2 &= \mathbb{E}\Big[\big(W_{\gamma_h - 1} + \alpha_{\gamma_h}\big)^2\Big] \\
&\le \mathbb{E}\Big[n^{2/3} + \big(2n^{1/3}\alpha_{\gamma_h} + \alpha_{\gamma_h}^2\big)\mathbf{1}_{\big\{\alpha_{\gamma_h} \le (3k-1)n^{1/3}\big\}}\Big| W_{\gamma_h} > 0\Big] \\
&\quad \times \mathbb{P}\big(W_{\gamma_h} > 0\big) + \mathbb{E}\Big[\big(2n^{1/3}\alpha_{\gamma_h} + \alpha_{\gamma_h}^2\big)\mathbf{1}_{\{\alpha_{\gamma_h} > (3k-1)n^{1/3}\}}\Big] \\
&\le 9k^2 n^{2/3}\mathbb{P}\big(W_{\gamma_h} > 0\big) \\
&\quad + 2n^{1/3} \cdot C_2 n^{1/3}\bigg\{p^{n^{1/3}} + \Big(\frac{8}{9}\Big)^{n^{1/3}}\bigg\}\mathbb{P}\big(W_{\gamma_h} > 0\big) \\
&\quad + \mathbb{E}\Big[\alpha_{\gamma_h}^2\mathbf{1}_{\big\{\alpha_{\gamma_h} > (3k-1)n^{1/3}\big\}}\Big].
\end{aligned}
$$

Similary to the proof of (4.20), because of

$$\begin{aligned}
&\frac{\mathbb{E}\big[\alpha_s^2 \mathbf{1}_{\{\alpha_s > (3k-1)n^{1/3}\}}\big]}{\mathbb{P}\big(\alpha_s \geq n^{1/3}\big)} \\
&\quad \leq \sum_{j=1}^{\infty} \frac{\mathbb{E}\big[\alpha_s^2 \mathbf{1}_{\{(3k-1)jn^{1/3} < \alpha_s \leq (3k-1)(j+1)n^{1/3}\}}\big]}{\mathbb{P}\big(\alpha_s \geq n^{1/3}\big)} \\
&\quad \leq (3k-1)^2 n^{2/3} \sum_{j=1}^{\infty} (j+1)^2 \bigg\{ \frac{k-1}{p} p^{jn^{1/3}} + \frac{C_1}{p}\Big(\frac{8}{9}\Big)^{jn^{1/3}} \bigg\} \\
&\quad \leq C_3 n^{2/3} \bigg\{ p^{n^{1/3}} + \Big(\frac{8}{9}\Big)^{n^{1/3}} \bigg\}
\end{aligned}$$

for some constant $C_3 = C_3(k) > 0$, we get

$$\begin{aligned}
&\mathbb{E}\Big[\alpha_{\gamma_h}^2 \mathbf{1}_{\{\alpha_{\gamma_h} > 3(k-1)n^{1/3}\}}\Big] \\
&\quad \leq \sum_{s=1}^{\infty} \frac{\mathbb{E}\big[\alpha_s^2 \mathbf{1}_{\{\alpha_s > (3k-1)n^{1/3}\}}\big]}{\mathbb{P}\big(\alpha_s \geq n^{1/3}\big)} \mathbb{P}\big(\gamma_h = s,\ W_s > 0\big) \\
&\quad \leq C_3 n^{2/3} \bigg\{ p^{n^{1/3}} + \Big(\frac{8}{9}\Big)^{n^{1/3}} \bigg\} \mathbb{P}\big(W_{\gamma_h} > 0\big).
\end{aligned}$$

Therefore (4.21) is proved for a large enough n. □

Using these facts, we have the following bound of $\mathbb{E}\gamma_h$.

Lemma 6. *For a large enough n,*

$$\mathbb{E}\gamma_h \leq 100k^2 d n^{1/3}. \tag{4.26}$$

Proof. Assume first that $\lambda > 1/4$. By $\mathbb{E}\alpha_s = D_k \lambda n^{-1/3} + O(n^{-2/3})$ and $D_k > 1$, $\{W_t - t\lambda n^{-1/3}\}$ is a submartingale for a large enough n. Thus the optional stopping theorem and (4.20) of Lemma 5 provide

$$\begin{aligned}
d = \mathbb{E}W_0 &\leq \mathbb{E}W_{\gamma_h} - \lambda n^{-1/3}\mathbb{E}\gamma_h \\
&\leq (3kn^{1/3}+1)\mathbb{P}\big(W_{\gamma_h} > 0\big) - \lambda n^{-1/3}\mathbb{E}\gamma_h.
\end{aligned}$$

Since $1 - e^{-4\lambda} > 1/2$ for $\lambda > 1/4$, we get from (4.18),

$$\mathbb{E}\gamma_h \leq \frac{3kn^{1/3}}{\lambda n^{-1/3}} \frac{4d\lambda}{1-e^{-4\lambda}} n^{-1/3} \leq 24kdn^{1/3}.$$

When $0 < \lambda \le 1/4$, we consider $\big(W_t - \mathbb{E}W_t\big)^2 - \frac{1}{5}t$. We have

$$\begin{aligned}
&\mathbb{E}\Big[\big(W_t - \mathbb{E}W_t\big)^2 - \frac{1}{5}t\Big|\mathcal{F}_{t-1}\Big] \\
&\quad = \big(W_{t-1} - \mathbb{E}W_{t-1}\big)^2 - \frac{1}{5}(t-1) + \mathbb{E}\alpha_t^2 - \big(\mathbb{E}\alpha_t\big)^2 - \frac{1}{5}.
\end{aligned}$$

It is easy to check that $\mathbb{P}(\alpha_s = 0) < 3/4$ and $\big(\mathbb{E}\alpha_t\big)^2 = O(\epsilon^2)$. Since $\mathbb{E}\alpha_t^2 - \big(\mathbb{E}\alpha_t\big)^2 - \frac{1}{5} \ge 0$, $\big(W_t - \mathbb{E}W_t\big)^2 - \frac{1}{5}t$ is a submartingale. Therefore, by optional stopping theorem,

$$\begin{aligned}
0 = \mathbb{E}\Big[\big(W_0 - \mathbb{E}W_0\big)^2\Big] &\le \mathbb{E}\Big[\big(W_{\gamma_h} - \mathbb{E}W_{\gamma_h}\big)^2 - \frac{1}{5}\gamma_h\Big] \\
&= \mathbb{E}W_{\gamma_h}^2 - \big(\mathbb{E}W_{\gamma_h}\big)^2 - \frac{1}{5}\mathbb{E}\gamma_h,
\end{aligned}$$

so $\mathbb{E}\gamma_h \le 5\mathbb{E}W_{\gamma_h}^2$. Using (4.21) and $\frac{4\lambda}{1-e^{-4\lambda}} \le 2$ for $0 < \lambda \le 1/4$, we get

$$\mathbb{E}\gamma_h \le 5 \cdot 10 \cdot 2k^2 dn^{2/3} \cdot n^{-1/3} \le 100k^2 dn^{1/3}.$$

When $\lambda \le 0$, similarly to the case of $\lambda \in (0, 1/4]$,

$$\mathbb{E}\gamma_h \le 5\mathbb{E}W_{\gamma_h}^2 \le 50k^2 n^{2/3}\mathbb{P}\big(W_{\gamma_h} > 0\big).$$

If $\lambda = 0$,

$$\mathbb{E}\gamma_h \le 50k^2 n^{2/3} dn^{-1/3} \le 50k^2 dn^{1/3}.$$

If $\lambda < 0$, by $e^{-\lambda} - 1 > -\lambda$,

$$\mathbb{E}\gamma_h \le 50k^2 n^{2/3}\frac{-2d\lambda}{e^{-\lambda} - 1}n^{-1/3} \le 100k^2 dn^{1/3}.$$

Therefore, for all $\lambda \in \mathbb{R}$ we have

$$\mathbb{E}\gamma_h \le 100k^2 dn^{1/3}.$$

□

Let $\gamma_h^* = \gamma_h \wedge n^{2/3}$. Then we have

$$\mathbb{P}\big(W_{\gamma_h^*} > 0\big) \le \mathbb{P}\big(W_{\gamma_h} \ge n^{1/3}\big) + \mathbb{P}\big(\gamma_h \ge n^{2/3}\big). \tag{4.27}$$

Using (4.18) for the first term of (4.27), also then, using (4.26) for the second term of (4.27), we have some constant $C_0 = C_0(k, \lambda)$ such that

$$\mathbb{P}\big(W_{\gamma_h^*} > 0\big) \leq C_0 d n^{-1/3}. \tag{4.28}$$

Now we calculate $\mathbb{E}\big[e^{c\xi_t}\big|\mathcal{F}_{t-1}\big]$.

For $2 \leq l \leq k$, we can couple $\hat{r}'_{t-1,l}$ with i.i.d. random variables $\alpha_{t-1,l}$ in (4.9), such that almost surely $\hat{r}'_{t-1,l} \leq \alpha_{t-1,l}$.

We write $\{w_t \overset{(l)}{\leftarrow} v\}$ for the event satisfying the following condition. We choose uniformly l edges from $\Big\{(v, v') : v' \in \mathcal{V} \setminus \big\{\mathcal{E} \cup \{v\}\big\}\Big\}$ and among these l directed edges (v, w_t) is chosen. Note that when $v \in \mathcal{N}_{t-1,k}^{(k-l)}$ this is the same as $w_t \leftarrow v$. But when $v \in \mathcal{N}_{t-1,k}^{(\geq k-l)}$ by the above produce we allow v to choose more edges than it can choose in the original k-out graph. If $t \geq n - l$, we understand that $\mathbf{1}_{\big\{w_t \overset{(l)}{\leftarrow} v\big\}} = 1$. Further, $\big\{w_t \overset{(l)}{\leftarrow} v,\ \text{open}\big\}$ denotes the event that $\big\{w_t \overset{(l)}{\leftarrow} v\big\}$ occurs and the edge $\langle v, w_t\rangle$ is open.

Thus we can couple $\Big\{\{w_t \leftarrow v, \text{open}\} : v \in \mathcal{V} \setminus \big\{\mathcal{E}_{t-1} \cup \{w_t\}\big\}\Big\}$ with

$$\Big\{\{w_t \overset{(l)}{\leftarrow} v, \text{open}\}_{0 \leq l \leq k} : v \in \mathcal{V} \setminus \big\{\mathcal{E}_{t-1} \cup \{w_t\}\big\}\Big\}$$

such that (i) $\{w_t \overset{(l)}{\leftarrow} v, \text{open}\}_{0 \leq l \leq k}$ are independent of

$$\Big\{\{w_t \overset{(l)}{\leftarrow} \tilde{v}, \text{open}\}_{0 \leq l \leq k} : \tilde{v} \in \mathcal{V} \setminus \big\{\mathcal{E}_{t-1} \cup \{w_t, v\}\big\}\Big\}$$

and that

$$\text{(ii) } \mathbf{1}_{\big\{w_t \overset{(m)}{\leftarrow} v,\ \text{open}\big\}} \geq \mathbf{1}_{\big\{w_t \overset{(l)}{\leftarrow} v,\ \text{open}\big\}}$$

almost surely for $m > l$, and that (iii) if $v \in \mathcal{N}_{t-1,k}^{(k-l)}$ then

$$\mathbf{1}_{\big\{w_t \leftarrow v,\ \text{open}\big\}} = \mathbf{1}_{\big\{w_t \overset{(l)}{\leftarrow} v,\ \text{open}\big\}}.$$

Since $N^{(0)}_{t-1} \geq N^{(0)}_{t-1,k}$, we have

$$
\begin{aligned}
&\sum_{v\in\mathcal{N}_{t-1,k}} \mathbf{1}_{\left\{w_t\leftarrow v,\ \text{open}\right\}} \\
&= \sum_{j=0}^{k} \sum_{v\in\mathcal{N}^{(j)}_{t-1,k}} \mathbf{1}_{\left\{w_t\leftarrow v,\ \text{open}\right\}} \\
&\leq \sum_{v\in\mathcal{N}^{(0)}_{t-1,k}} \mathbf{1}_{\left\{w_t\overset{(k)}{\leftarrow} v,\ \text{open}\right\}} + \sum_{v\in\mathcal{N}^{(\geq 1)}_{t-1,k}\cup\mathcal{A}_{t-1,k}} \mathbf{1}_{\left\{w_t\overset{(k-1)}{\leftarrow} v,\ \text{open}\right\}} \\
&\leq \sum_{v\in\mathcal{N}^{(0)}_{t-1}} \mathbf{1}_{\left\{w_t\overset{(k)}{\leftarrow} v,\ \text{open}\right\}} + \sum_{v\in\mathcal{N}^{(\geq 1)}_{t-1}\cup\mathcal{A}_{t-1}} \mathbf{1}_{\left\{w_t\overset{(k-1)}{\leftarrow} v,\ \text{open}\right\}}.
\end{aligned}
$$

Therefore, for $t < n$ and $c > 0$ with $c = o(1)$, from (4.15)

$$
\begin{aligned}
&\mathbb{E}\left[e^{c\xi_t}\middle|\mathcal{F}_{t-1}\right] \\
&\leq \left\{\frac{1-p}{1-e^c p}\right\}^{k-1}\left\{\frac{kp}{n-t}e^c + 1 - \frac{kp}{n-t}\right\}^{N^{(0)}_{t-1}} \\
&\quad\times\left\{\frac{(k-1)p}{n-t}e^c + 1 - \frac{(k-1)p}{n-t}\right\}^{n-t-N^{(0)}_{t-1}} e^{-c} \\
&\leq \exp\Bigg[(k-1)\left\{\frac{p}{1-p}(c+c^2) + \left(\frac{p}{1-p}\right)^2(\sqrt{2}c)^2\right\} \\
&\quad + N^{(0)}_{t-1}\left\{\frac{kp}{n-t}(c+c^2)\right\} + (n-t-N^{(0)}_{t-1})\left\{\frac{(k-1)p}{n-t}(c+c^2)\right\} - c\Bigg] \\
&= \exp\Bigg[\left(D_k\epsilon + O(\epsilon^2)\right)(c+c^2) + c^2 + 2(k-1)\left(\frac{p}{1-p}\right)^2 c^2 \\
&\quad - \frac{n-t-N^{(0)}_{t-1}}{n-t}p(c+c^2)\Bigg] \\
&\leq \exp\Bigg[2(c+c^2)|\epsilon| + \left\{1 + 2(k-1)\left(\frac{p}{1-p}\right)^2\right\}c^2 \\
&\quad - \frac{n-t-N^{(0)}_{t-1}}{n-t}p(c+c^2)\Bigg]
\end{aligned}
\tag{4.29}
$$

for a large enough n.

Now we define the event $\mathcal{D}$ and $\mathcal{D}_t$ such that

$$\begin{aligned}\mathcal{D} &= \Big\{N_t^{(0)} \le n - t - \frac{k}{3}t, \quad \text{for every } t \text{ with } n^{1/3} \le t < \delta n\Big\},\\ \mathcal{D}_t &= \Big\{N_t^{(0)} \le n - t - \frac{k}{3}t\Big\}, \quad n^{1/3} \le t < \delta n,\end{aligned}$$

for some small $\delta > 0$.

Let v be a vertex in the component explored at first, and let $\mathcal{C}(v)$ be the component including v. Let t_1 be the first time the process X_t hits $-N(w_1)$. It means that $|\mathcal{C}(v)| = t_1$. We couple ξ_t and α_t for each t, such that $\xi_t \le \alpha_t$ almost surely and α_t is independent of $\mathcal{F}_{t-1}$. Further, we can couple $\{X_t\}_{t=0}^n$ with $\{W_t\}_{t=0}^n$ such that $X_t \le W_t - W_0$ almost surely for $0 \le t \le n$. We set $W_0 = N(w_1)$. Hence if $|\mathcal{C}(v)| > An^{2/3}$ for $A > 1$, then $W_{\gamma_h^*} > 0$ as well as $X_{\gamma_h^*+(A-1)n^{2/3}} > -N(w_1)$. Thus for a large enough n, we have

$$\begin{aligned}&\mathbb{P}\big(|\mathcal{C}(v)| \ge An^{2/3}\big)\\ &\le \sum_{d=1}^{n^{1/3}-1} \mathbb{P}\big(N(w_1) = d, |\mathcal{C}(v)| \ge An^{2/3}\big) + \mathbb{P}\big(N(w_1) \ge n^{1/3}\big)\\ &\le \sum_{d=1}^{n^{1/3}-1} \mathbb{P}\big(N(w_1) = d, X_{\gamma_h^*+(A-1)n^{2/3}} \ge -d, W_{\gamma_h^*} > 0\big) + \mathbb{P}\big(N(w_1) \ge n^{1/3}\big)\\ &\le \sum_{d=1}^{n^{1/3}-1} \sum_{s=1}^{n^{2/3}} \mathbb{P}\big(N(w_1) = d, \gamma_h^* = s, W_s > 0, X_{s+(A-1)n^{2/3}} \ge -d, \mathcal{D}\big)\\ &\quad + \mathbb{P}\big(\mathcal{D}^C\big) + \mathbb{P}\big(N(w_1) \ge n^{1/3}\big)\\ &\le \sum_{d=1}^{n^{1/3}-1} \sum_{s=1}^{n^{2/3}} \mathbb{P}\Bigg(N(w_1) = d, \gamma_h^* = s, W_s > 0, \sum_{u=s+1}^{s+(A-1)n^{2/3}} \xi_u \ge -W_s - d, \mathcal{D}\Bigg)\\ &\quad + \mathbb{P}\big(\mathcal{D}^C\big) + \mathbb{P}\big(N(w_1) \ge n^{1/3}\big)\\ &\le \sum_{d=1}^{n^{1/3}-1} \sum_{s=1}^{n^{2/3}} \mathbb{E}\Big[e^{c\sum_{u=s+1}^{s+(A-1)n^{2/3}} \xi_u + cW_s + cd} \mathbf{1}_{\{\mathcal{D}\}};\ N(w_1) = d, \gamma_h^* = s, W_s > 0\Big]\\ &\quad + \mathbb{P}\big(\mathcal{D}^C\big) + \mathbb{P}\big(N(w_1) \ge n^{1/3}\big)\end{aligned}$$

for a small constant $c > 0$, where $\mathcal{D}^C$ is complement of $\mathcal{D}$. Since $\epsilon = \lambda n^{-1/3}$, for $n^{1/3} \le t \le \delta n$, (4.29) implies that

$$\begin{aligned}
&\mathbb{E}\Big[e^{c\xi_t}\Big|\mathcal{F}_{t-1}\Big]\mathbf{1}_{\{\mathcal{D}_{t-1}\}} \\
&\quad\le \exp\left[2(c+c^2)|\epsilon| + \left\{1+2(k-1)\left(\frac{p}{1-p}\right)^2\right\}c^2 - \frac{\frac{k}{3}t - \frac{k}{3} - 1}{n-t}p(c+c^2)\right] \\
&\quad\le \exp\left[3c|\epsilon| - \frac{kpct}{3n} + 2c^2\right],
\end{aligned}$$

for a large enough n. Also from (4.29), we have $\mathbb{E}e^{c\xi_t} \le \exp\big[2c|\epsilon| + 2c^2\big]$, and hence

$$\mathbb{E}e^{c\sum_{s=1}^{n^{1/3}}\xi_s} \le \exp\left[2c|\epsilon|n^{1/3} + 2c^2 n^{1/3}\right]$$

for a small enough $c > 0$ and a large enough n. Since $\mathcal{D}_s \supset \mathcal{D}$ for $s \in [n^{1/3}, \delta n]$, we have for $s \ge 0$ and $0 \le s+t \le \delta n$,

$$\begin{aligned}
&\mathbb{E}\Big[e^{c\sum_{u=s+1}^{s+t}\xi_u}\mathbf{1}_{\{\mathcal{D}\}}\Big|\mathcal{F}_s\Big] \\
&\quad\le \exp\Big[\sum_{u=s+1}^{s+t}\left\{3c|\epsilon| - \frac{kpcu}{3n} + 2c^2\right\} + 2c|\epsilon|n^{1/3} + 2c^2n^{1/3}\Big] \\
&\quad\le \exp\left[3c|\epsilon|t - \frac{kpct^2}{6n} + 2c^2t + 2c|\epsilon|n^{1/3} + 2c^2n^{1/3}\right],
\end{aligned}$$

if $c > \frac{kp}{6}n^{-2/3} - \frac{3}{2}|\epsilon|$. Therefore, putting $t = (A-1)n^{2/3}$, we have

$$\begin{aligned}
&\mathbb{P}\big(|\mathcal{C}(v)| \ge An^{2/3}\big) \\
&\quad\le \exp\left[3c|\epsilon|t - \frac{kpct^2}{6n} + 2c^2t + 2c|\epsilon|n^{1/3} + 2c^2n^{1/3}\right] \\
&\qquad\times \sum_{d=1}^{n^{1/3}-1}\sum_{s=1}^{n^{2/3}}\mathbb{E}\left[e^{cW_s};\ N(w_1)=d, \gamma_h^* = s, W_s > 0\right]e^{cd} \\
&\qquad+\mathbb{P}\big(\mathcal{D}^C\big) + \mathbb{P}\big(N(w_1) \ge n^{1/3}\big).
\end{aligned} \tag{4.30}$$

Here,

$$
\begin{aligned}
&\mathbb{E}\Big[e^{cW_s};\ N(w_1) = d, \gamma_h^* = s, W_s > 0\Big] \\
&\quad \le \ e^{cn^{1/3}}\mathbb{E}\Big[e^{c\alpha_s}\mathbf{1}_{\{\alpha_s \le (3k-1)n^{1/3}\}};\ N(w_1) = d, \gamma_h^* = s, W_s > 0\Big] \\
&\qquad + e^{cn^{1/3}}\mathbb{E}\Big[e^{c\alpha_s}\mathbf{1}_{\{\alpha_s > (3k-1)n^{1/3}\}};\ N(w_1) = d, \gamma_h^* = s, W_s > 0\Big] \\
&\quad \le \ e^{3kcn^{1/3}}\mathbb{P}\big(N(w_1) = d, \gamma_h^* = s, W_s > 0\big) + e^{cn^{1/3}}\mathbb{E}\Big[e^{c\alpha_s}\mathbf{1}_{\{\alpha_s > (3k-1)n^{1/3}\}}\Big].
\end{aligned} \tag{4.31}
$$

Using (4.23),

$$
\begin{aligned}
&\mathbb{E}\Big[e^{c\alpha_s}\mathbf{1}_{\{\alpha_s > (3k-1)n^{1/3}\}}\Big] \\
&\quad \le \ \sum_{j=1}^{\infty} \mathbb{E}\Big[e^{c\alpha_s}\mathbf{1}_{\{(3k-1)jn^{1/3} < \alpha_s \le (3k-1)(j+1)n^{1/3}\}}\Big] \\
&\quad \le \ \sum_{j=1}^{\infty} e^{c(3k-1)(j+1)n^{1/3}}\Big\{(k-1)p^{2jn^{1/3}} + C_1(kp)^{(k+1)jn^{1/3}}\Big\}
\end{aligned} \tag{4.32}
$$

for a small enough $c > 0$ and a large enough n. From (4.30) - (4.32), we have

$$
\begin{aligned}
&\mathbb{P}\big(|\mathcal{C}(v)| \ge An^{2/3}\big) \\
&\quad \le \ \exp\left[3c|\epsilon|t - \frac{kpct^2}{6n} + 2c^2t + 2c|\epsilon|n^{1/3} + 2c^2n^{1/3}\right] \\
&\qquad \times \sum_{d=1}^{n^{1/3}-1}\sum_{s=1}^{n^{2/3}}\Big[e^{3kcn^{1/3}}\mathbb{P}\big(N(w_1) = d, \gamma_h^* = s, W_s > 0\big) \\
&\qquad + \sum_{j=1}^{\infty} e^{c(3k-1)(j+1)n^{1/3}}\Big\{(k-1)p^{2jn^{1/3}} + C_1(kp)^{(k+1)jn^{1/3}}\Big\}\Big]e^{cd} \\
&\qquad + \mathbb{P}\big(\mathcal{D}^C\big) + \mathbb{P}\big(N(w_1) \ge n^{1/3}\big)
\end{aligned}
$$

$$
\begin{aligned}
\leq & \exp\Big[3c|\epsilon|t - \frac{kpct^2}{6n} + 2c^2t + 2c|\epsilon|n^{1/3} + 2c^2n^{1/3}\Big] \\
& \times\Big[e^{3kcn^{1/3}}\sum_{d=1}^{n^{1/3}-1} e^{cd}\mathbb{P}\big(N(w_1)=d\big)\mathbb{P}\big(W_{\gamma_h^*}>0\big|N(w_1)=d\big) \\
& + n\sum_{j=1}^{\infty} e^{c\cdot 3k(j+1)n^{1/3}}\Big\{(k-1)p^{2jn^{1/3}} + C_1(kp)^{(k+1)jn^{1/3}}\Big\}\Big] \\
& + \mathbb{P}\big(\mathcal{D}^C\big) + \mathbb{P}\big(N(w_1)\geq n^{1/3}\big).
\end{aligned}
$$

Now recall that $\epsilon = \lambda n^{-1/3}$ and set $t = (A-1)n^{2/3}$,

$$
c = \frac{\frac{kpt^2}{6n} - 3|\epsilon|t - 2|\epsilon|n^{1/3}}{4(t+n^{1/3})}.
$$

If A is large, $c > \frac{kp}{6}n^{-2/3} - \frac{3}{2}|\epsilon|$. From (4.28),

$$
\begin{aligned}
& \mathbb{P}\big(|\mathcal{C}(v)| \geq An^{2/3}\big) \\
\leq & \exp\left[-\frac{\Big\{\frac{kpt^2}{6n} - 3|\epsilon|t - 2|\epsilon|n^{1/3}\Big\}^2}{8\big(t+n^{1/3}\big)}\right] \\
& \times\Big[C_4 n^{-1/3}\sum_{d=1}^{n^{1/3}-1} e^{cd}\mathbb{P}\big(N(w_1)=d\big) \\
& + n\sum_{j=1}^{\infty} e^{c\cdot 3k(j+1)n^{1/3}}\Big\{(k-1)p^{2jn^{1/3}} + C_1(kp)^{(k+1)jn^{1/3}}\Big\}\Big] \\
& + \mathbb{P}\big(\mathcal{D}^C\big) + \mathbb{P}\big(N(w_1)\geq n^{1/3}\big)
\end{aligned}
$$

for some constant $C_4 = C_4(k) > 0$.

By Lemma 4,

$$
\begin{aligned}
\mathbb{P}\big(\mathcal{D}^C\big) &\leq \sum_{s=n^{1/3}}^{\delta n} \mathbb{P}\Big(\Big\{N_s^{(0)} \geq n - s - \frac{k}{3}s\Big\}\Big) \\
&\leq ne^{-\underline{I}(\frac{k}{3})n^{1/3}}
\end{aligned}
$$

for some fixed $\underline{I}(\frac{k}{3}) > 0$. Also note that, $N(w_1)$ is stocasticaly dominated by a random variable distributed as $1 + Ge(1-p)$. Therefore we have

$$
\mathbb{P}\big(|\mathcal{C}(v)| \geq An^{2/3}\big) \leq Cn^{-1/3}e^{-r(A-1)^3} \tag{4.33}
$$

for some constants $C = C(k, \lambda) > 0$ and $r = (k, \lambda) > 0$ and a large enough n. This estimate is correct if $A = O(n^{1/10})$.

Similar argument applies for components explored after the first one.

Denote by S_T the number of vertices contained in components larger than T. Then $|\mathcal{C}_1| > T$ implies that $S_T > T$. So taking $T = An^{2/3}$, we have

$$\mathbb{P}\big(|\mathcal{C}_1| > T\big) \leq \mathbb{P}\big(S_T > T\big) \leq \frac{\mathbb{E}S_T}{T} \leq \frac{n\mathbb{P}\big(|\mathcal{C}(v)| \geq T\big)}{T} \leq \frac{C}{A}e^{-r(A-1)^3},$$

completing the proof. □

4.3 Proofs of Theorems 3, 4

In cases of above or below the critical point, we also use the calcuation of the X_t.

Proof of Theorem 3. Let $\{\alpha_s\}$ be the random variables given by (4.11).

There exists a constant $D > 0$ such that for $s \leq n-1$ and $0 < \theta < D$,

$$\begin{aligned} \mathbb{E}e^{\theta\alpha_s} &\leq \exp\Bigg[(k-1)\bigg\{\frac{p}{1-p}(\theta+\theta^2) + \left(\frac{p}{1-p}\right)^2\big(\sqrt{2}\theta\big)^2\bigg\} \\ &\quad +(n-s)\frac{kp}{n-s}(e^\theta - 1) - \theta\Bigg] \\ &\leq \exp\Bigg[\bigg\{(k-1)\frac{p}{1-p} + kp - 1\bigg\}(\theta+\theta^2) \\ &\quad +\bigg\{2(k-1)\left(\frac{p}{1-p}\right)^2 + 1\bigg\}\theta^2\Bigg]. \end{aligned}$$

The right hand side of the above inequality is not larger than $e^{-r\theta}$ for some $r > 0$ and a small enough $\theta > 0$. We have the same estimate for $s = n$. Let v be a vertex v such that it is included in the component explored at first. $|\mathcal{C}(v)| > A\log n$ implies that $\sum_{s=1}^{A\log n} \alpha_s \geq X_{A\log n} > -N(w_1)$. Also, $N(w_1)$ is stochasticaly dominated by a random variable distributed as $Ge(1-p)+1$.

Therefore we have

$$\begin{aligned}
\mathbb{P}\big(|\mathcal{C}(v)| > A\log n\big) &\leq \sum_{d=1}^{\infty}\mathbb{P}\big(N(w_1)=d\big)\mathbb{P}\big(X_{A\log n} > -d\big) \\
&\leq \sum_{d=1}^{\infty}\mathbb{P}\big(N(w_1)=d\big)\mathbb{P}\big(e^{\theta X_{A\log n}} > e^{-\theta d}\big) \\
&\leq \sum_{d=1}^{\infty}\mathbb{P}\big(N(w_1)=d\big)e^{-r\theta A\log n+\theta d} \\
&\leq Cn^{-r\theta A}
\end{aligned}$$

for some constant $C > 0$. The argument is similar for the components explored after second time, too. Similarly to the proof of Theorem 1,

$$\begin{aligned}
\mathbb{P}\big(|\mathcal{C}_1| > A\log n\big) &\leq \frac{n\mathbb{P}\big(|\mathcal{C}(v)| > A\log n\big)}{A\log n} \\
&\leq \frac{Cn^{1-r\theta A}}{A\log n}.
\end{aligned}$$

If A is large enough, $\mathbb{P}\big(|\mathcal{C}_1| > A\log n\big) \to 0$ as $n\to\infty$. □

Proof of Theorem 4. When $c = k+\sqrt{k^2-k}$ (i.e., $p=1$), since the k-out graph is almost surely connected, $|\mathcal{C}_1| = n$. We consider the case where $1 < c < k+\sqrt{k^2-k}$ (i.e., $p<1$). Let $r = (k-1)\frac{p}{1-p} + kp - 1$. Then r is positive in this case. Let

$$\xi_t' = \sum_{l=2}^{k}\hat{r}'_{t-1,l} + \sum_{v\in\mathcal{N}_{t-1,k}} \mathbf{1}_{\{w_t\leftarrow v,\ \text{open}\}} - 1, \tag{4.34}$$

$$\xi_t'' = \sum_{j=2}^{k}\Big\{\mathbf{1}_{\{A_{t-1}^{(j)}>0,\ A_{t-1}^{(\geq j+1)}=0\}} + \mathbf{1}_{\{A_{t-1}=0,\ w_t\in\mathcal{N}_{t-1}^{(j)}\}}\Big\}\sum_{l=2}^{j}\hat{r}'_{t-1,l}. \tag{4.35}$$

Then $\xi_t = \xi_t' - \xi_t''$. Further, let $X_t' = \sum_{s=1}^{t}\xi_s'$ and $X_t'' = \sum_{s=1}^{t}\xi_s''$. So

$X_t = X'_t - X''_t$. We have for any $r > 0$,

$$\begin{aligned}
\mathbb{P}\big(|\mathcal{C}_1| < \delta n\big) &\le \sum_{t=\delta n+1}^{2\delta n} \mathbb{P}\big(X_t < 0\big) \\
&\le \sum_{t=\delta n+1}^{2\delta n} \Big\{\mathbb{P}\Big(X'_t < \frac{rt}{2}\Big) + \mathbb{P}\Big(X''_t \ge \frac{rt}{2}\Big)\Big\}.
\end{aligned}$$

Take $a > k\big(\frac{p}{1-p} + \frac{1}{1-2\delta}\big)$ with a small $\delta > 0$. For $0 \le s \le 2\delta n$ and $1 \le l \le k$, we define the event $\tilde{\mathcal{D}}_s$ and $\tilde{\mathcal{D}}_{s,l}$ by

$$\begin{aligned}
\tilde{\mathcal{D}}_s &= \Big\{N_s^{(0)} > n - 2(1+a)\delta n\Big\}, \\
\tilde{\mathcal{D}}_{s,l} &= \Big\{N_{s,l}^{(0)} > n - 2(1+a)\delta n - 1 - 2k\delta n\Big\}.
\end{aligned}$$

For $s \le t$ and $2 \le l \le k$, let $\underline{r}_{s-1,l}$ be following independent non-negative random variables;

$$\mathbb{P}\big(\underline{r}_{s-1,l} = x\big) = \prod_{y=0}^{x-1}\Big\{\frac{n - 2(1+a)\delta n - 1 - 2k\delta n - y}{n-s} p\Big\}(1-p),$$

for $x \ge 1$, and it puts on 0 the remaining probability. We can couple $\hat{r}'_{s-1,l}$ and $\underline{r}_{s-1,l}$ such that $\hat{r}'_{s-1,l} \ge \underline{r}_{s-1,l}$ on $\tilde{\mathcal{D}}_{s-1,l-1}$. We also let $b_{s-1,k+1}$ be independent random variables distributed as $Bin\big(n - 2(1+a)\delta n - 1 - 2k\delta n; \frac{kp}{n}\big)$ for each s. We can couple $\sum_{v \in \mathcal{N}_{s-1,k}} \mathbf{1}_{\{w_s \leftarrow v, \text{open}\}}$ and $b_{s-1,k+1}$ such that $\sum_{v \in \mathcal{N}_{s-1,k}} \mathbf{1}_{\{w_s \leftarrow v, \text{open}\}} \ge b_{s-1,k+1}$ on $\tilde{\mathcal{D}}_{s-1,k}$. We get for $\theta > 0$,

$$\begin{aligned}
\mathbb{E}e^{-\theta X'_t} &\le \mathbb{E}\Big[e^{-\theta X'_{t-1}} \prod_{l=2}^{k}\Big\{e^{-\theta \underline{r}_{s-1,l}} \mathbf{1}_{\{\tilde{\mathcal{D}}_{t-1,l-1}\}} + \mathbf{1}_{\{\tilde{\mathcal{D}}^C_{t-1,l-1}\}}\Big\} \\
&\qquad \times \Big\{e^{-\theta b_{s-1,k+1}} \mathbf{1}_{\{\tilde{\mathcal{D}}_{t-1,k}\}} + \mathbf{1}_{\{\tilde{\mathcal{D}}^C_{t-1,k}\}}\Big\} e^{\theta}\Big] \\
&\le \mathbb{E}\prod_{s=1}^{t}\Big[\prod_{l=2}^{k} e^{-\theta \underline{r}_{s-1,l}} e^{-\theta b_{s-1,k+1}} e^{\theta} \\
&\qquad \times \prod_{l=2}^{k}\Big\{\mathbf{1}_{\{\tilde{\mathcal{D}}_{t-1,l-1}\}} + e^{\theta \underline{r}_{s-1,l}} \mathbf{1}_{\{\tilde{\mathcal{D}}^C_{t-1,l-1}\}}\Big\} \\
&\qquad \times \Big\{\mathbf{1}_{\{\tilde{\mathcal{D}}_{t-1,k}\}} + e^{\theta b_{s-1,k+1}} \mathbf{1}_{\{\tilde{\mathcal{D}}^C_{t-1,k}\}}\Big\}\Big]
\end{aligned}$$

$$
\begin{aligned}
&\le \sqrt{\mathbb{E}\Big[\prod_{s=1}^{t}\Big\{\prod_{l=2}^{k} e^{-2\theta \underline{r}_{s-1,l}}\Big\} e^{-2\theta b_{s-1,k+1}} e^{2\theta}\Big]} \\
&\quad \times \sqrt{\mathbb{E}\Big[1+\prod_{s=1}^{t}\Big\{\prod_{l=2}^{k} e^{\theta \underline{r}_{s-1,l}}\Big\} e^{\theta b_{s-1,k+1}} \mathbf{1}_{\{\tilde{\mathcal{D}}^C_{2\delta n,k}\}}\Big]^2}.
\end{aligned}
$$

Here

$$
\begin{aligned}
&\mathbb{E}\Big[1+\prod_{s=1}^{t}\Big\{\prod_{l=2}^{k} e^{\theta \underline{r}_{s-1,l}}\Big\} e^{\theta b_{s-1,k+1}} \mathbf{1}_{\{\tilde{\mathcal{D}}^C_{2\delta n,k}\}}\Big]^2 \\
&\quad \le 1+\mathbb{E}\Big[3\prod_{s=1}^{t}\Big\{\prod_{l=2}^{k} e^{2\theta \underline{r}_{s-1,l}}\Big\} e^{2\theta b_{s-1,k+1}} \mathbf{1}_{\{\tilde{\mathcal{D}}^C_{2\delta n,k}\}}\Big] \\
&\quad \le 1+3\sqrt{\mathbb{E}\Big[\prod_{s=1}^{t}\Big\{\prod_{l=2}^{k} e^{4\theta \underline{r}_{s-1,l}}\Big\} e^{4\theta b_{s-1,k+1}}\Big]}\sqrt{\mathbb{P}\big(\tilde{\mathcal{D}}^C_{2\delta n,k}\big)} \\
&\quad = 1+3\sqrt{\prod_{s=1}^{t}\Big\{\prod_{l=2}^{k} \mathbb{E}e^{4\theta \underline{r}_{s-1,l}}\Big\} \mathbb{E}e^{4\theta b_{s-1,k+1}}}\sqrt{\mathbb{P}\big(\tilde{\mathcal{D}}^C_{2\delta n,k}\big)}.
\end{aligned}
$$

So

$$
\begin{aligned}
\mathbb{E}e^{-\theta X'_t} &\le \sqrt{\mathbb{E}\Big[\prod_{s=1}^{t}\Big\{\prod_{l=2}^{k} e^{-2\theta \underline{r}_{s-1,l}}\Big\} e^{-2\theta b_{s-1,k+1}} e^{2\theta}\Big]} \\
&\quad \times \sqrt{1+3\sqrt{\prod_{s=1}^{t}\Big\{\prod_{l=2}^{k} \mathbb{E}e^{4\theta \underline{r}_{s-1,l}}\Big\} \mathbb{E}e^{4\theta b_{s-1,k+1}}}\sqrt{\mathbb{P}\big(\tilde{\mathcal{D}}^C_{2\delta n,k}\big)}}.
\end{aligned}
$$

Let $\phi_l(\theta) = \log \mathbb{E}e^{\theta \underline{r}_{0,l}}$ for $1 \le l \le k$ and $\phi_k(\theta) = \log \mathbb{E}e^{\theta b_{0,k+1}}$. Then

$$
\begin{aligned}
\phi'_l(0) &= \mathbb{E}\underline{r}_{0,l} = \frac{p}{1-p} + O(\delta), \\
\phi'_k(0) &= \mathbb{E}b_{0,k+1} = kp + O(\delta).
\end{aligned}
$$

Since

$$
\begin{aligned}
\mathbb{P}\big(\tilde{\mathcal{D}}^C_{2\delta n,k}\big) &\le \mathbb{P}\big(\tilde{\mathcal{D}}^C_{2\delta n}\big) + \mathbb{P}\big(N^{(0)}_{2\delta n} - N^{(0)}_{2\delta n,k} \ge 1 + 2k\delta n\big) \\
&\le e^{-\bar{I}(a)\cdot 2\delta n} + kp^{2\delta n}
\end{aligned}
$$

and

$$\begin{aligned}\mathbb{E}e^{-2\theta \underline{r}_{0,l}} &\leq e^{-2\theta\mathbb{E}\underline{r}_{0,l}+\theta^2 C_l}\\ \mathbb{E}e^{-2\theta b_{0,k+1}} &\leq e^{-2\theta\mathbb{E}b_{0,k+1}+\theta^2 C_{k+1}}\end{aligned}$$

for some $C_l > 0$, $2 \leq l \leq k$, $C_{k+1} > 0$ and a small enough $\theta > 0$, we have

$$\begin{aligned}\mathbb{E}e^{-\theta X'_t} &\leq \sqrt{\prod_{s=1}^{t}\Big\{\prod_{l=2}^{k} e^{-2\theta\mathbb{E}\underline{r}_{0,l}+\theta^2 C_l} e^{-2\theta\mathbb{E}b_{0,k+1}+\theta^2 C_{k+1}} e^{2\theta}\Big\}\Big]}\\ &\quad \times\sqrt{1+\sqrt{e^{-\bar{I}(a)\cdot 2\delta n}+p^{2\delta n}}O(1)}\\ &= \exp\Big[-\theta\Big\{(k-1)\frac{p}{1-p}+kp-1+O(\delta)\Big\}t\\ &\quad +\frac{\theta^2}{2}\Big\{\sum_{l=2}^{k} C_l + C_{k+1}\Big\}t\Big]\\ &\quad \times\sqrt{1+\sqrt{e^{-\bar{I}(a)\cdot 2\delta n}+p^{2\delta n}}O(1)}.\end{aligned}$$

Therefore

$$\mathbb{P}\Big(X'_t < \frac{rt}{2}\Big) \leq e^{-r_1 t} \tag{4.36}$$

for some small $r_1 > 0$ and a small enough $\delta > 0$.

To estimate $\mathbb{E}e^{\theta X''_t}$, take a vertex v which appears as $w_s \in \mathcal{N}_{s-1}^{(\geq 2)} \cup \mathcal{A}_{s-1}^{(\geq 2)}$. Then there is some time $u \leq s-1$ such that $v \in \mathcal{N}_{u-1,k}^{(1)} \cup \mathcal{A}_{u-1,k}^{(1)}$ and that $\{w_u \leftarrow v\}$ occurs. Therefore $\sum_{s=1}^{t} \xi''_s \leq \sum_{s=1}^{t} \mathbf{1}_{\{w_s \in \mathcal{N}_{s-1}^{(\geq 2)} \cup \mathcal{A}_{s-1}^{(\geq 2)}\}} \sum_{l=2}^{k} \hat{r}'_{s-1,l}$ and the right hand side is stochastically dominated by

$$\sum_{u=1}^{t} \sum_{v \in \mathcal{N}_{u-1}^{(1)} \cup \mathcal{A}_{u-1}^{(1)}} \mathbf{1}_{\left\{w_u \overset{(k-1)}{\leftarrow} v\right\}} \sum_{l=2}^{k} \alpha_{u-1,l}(v),$$

where $\{\alpha_{u-1,l}(v)\}$ for $1 \leq u \leq t$, $2 \leq l \leq k$, $v \in \mathcal{V}$ are i.i.d. random variables distributed as $Ge(1-p)$. Here we define random variables $f(x)$ and g_s and

h_s by

$$\begin{aligned} f(x) &\sim NB\big((k-1)x; 1-p\big) \\ g_s &\sim Bin\Big(2(1+a)\delta n; \frac{k-1}{(1-2\delta)n}\Big) \\ h_s &\sim Bin\Big(n; \frac{k-1}{(1-2\delta)n}\Big). \end{aligned}$$

Then we have

$$\begin{aligned} \mathbb{E}e^{\theta X_t''} &\leq \mathbb{E}\Big[\prod_{s=1}^{t}\Big\{e^{\theta f(g_s)}\mathbf{1}_{\{\tilde{\mathcal{D}}_{s-1}\}} + e^{\theta f(h_s)}\mathbf{1}_{\{\tilde{\mathcal{D}}_{s-1}^C\}}\Big\}\Big] \\ &\leq \mathbb{E}\Big[\prod_{s=1}^{t} e^{\theta f(g_s)} + \prod_{s=1}^{t} e^{\theta f(h_s)}\mathbf{1}_{\{\tilde{\mathcal{D}}_{2\delta n}^C\}}\Big] \\ &\leq \prod_{s=1}^{t}\mathbb{E}e^{\theta f(g_s)} + \sqrt{\prod_{s=1}^{t}\mathbb{E}e^{2\theta f(h_s)}}\sqrt{\mathbb{P}\big(\tilde{\mathcal{D}}_{2\delta n}^C\big)} \\ &\leq e^{O(\delta)t\theta}. \end{aligned}$$

Therefore

$$\mathbb{P}\Big(X_t'' \geq \frac{rt}{2}\Big) \leq e^{-r_2 t} \tag{4.37}$$

for some $r_2 > 0$.

Combining (4.36) and (4.37), we get the proof of the lemma. □

Chapter 5

Proof of Theorem 2

5.1 Detailed calculation

We start a further analysis of the exploration process. Let $t \le \delta n$ for some small $\delta > 0$. From now on, we assume that $\epsilon = \epsilon(n)$ is a sequence such that $\epsilon(n) \to 0$ as $n \to \infty$ and we write $p = p(n) = \frac{1+\epsilon(n)}{k+\sqrt{k^2-k}}$. Also we assume that s_0 is in $[0, \infty)$ and $m = \lfloor s_0 n^{2/3} \rfloor$.

Lemma 7. *If n is large enough, then for all $t \le s_0 n^{2/3}$,*

$$\mathbb{E}A_t = O\big(\epsilon n^{2/3} + n^{1/3}\big), \tag{5.1}$$

$$\mathbb{E}Z_t = O\big(\epsilon n^{2/3} + n^{1/3}\big). \tag{5.2}$$

Proof. We can couple the ξ_t and α_t, such that almost surely $\xi_t \le \alpha_t$ and α_t is independent of $\mathcal{F}_{t-1}$ for $1 \le t \le \delta n$. Thus, using (4.4) and (4.5), we have

$$\begin{aligned} \mathbb{E}[\xi_t|\mathcal{F}_{t-1}] &\le \mathbb{E}\alpha_t \\ &= (k-1)\frac{p}{1-p} + kp - 1 \\ &= D_k\epsilon + O\big(\epsilon^2\big). \end{aligned} \tag{5.3}$$

Recall that $\{t_0, t_1, \dots\}$ are times such that $A_{t_j} = 0$. By definition of $N(w_t)$, $N(w_t) = 0$ for any $t \ne t_j + 1$, and for $N(w_{t_j+1})$, we can couple it and a random variable κ_{t_j+1} distributed as $Ge(1-p) + 1$ independent of $\mathcal{F}_{t_j}$. So

$$\mathbb{E}\big[N(w_{t_j+1})\big|\mathcal{F}_{t_j}\big] \le \frac{p}{1-p} + 1 \qquad \text{a.s..} \tag{5.4}$$

From (4.1),

$$Z_{t_j+1} = -X_{t_j+1} + N(w_{t_j+1}) + \xi_{t_j+1}. \tag{5.5}$$

So, using (5.3) and (5.4), we get

$$\mathbb{E}\big[Z_{t_j+1}\big|\mathcal{F}_{t_j}\big] \le -\mathbb{E}\big[X_{t_j+1}\big|\mathcal{F}_{t_j}\big] + \frac{p}{1-p} + 1 + D_k\epsilon + O\big(\epsilon^2\big) \qquad \text{a.s..}$$

Thus

$$\begin{aligned} Z_t &= \sum_{j=0}^{\infty} \mathbf{1}_{\left\{t_j<t\le t_{j+1}\right\}} Z_{t_j+1} \\ &= \sum_{j=0}^{\infty} \mathbf{1}_{\left\{t_j<t\le t_{j+1}\right\}} \Big\{-X_{t_j+1} + N(w_{t_j+1}) + \xi_{t_j+1}\Big\} \\ &\le -\min_{s\le t} X_s + \max_{s\le t}\Big\{\kappa_s + \alpha_s\Big\}. \end{aligned} \tag{5.6}$$

Here $X_t - \sum_{s=1}^{t}\mathbb{E}\big[\xi_s\big|\mathcal{F}_{s-1}\big]$ is an $(\mathcal{F}_t)$-martingale. So, using Doob's maximal L^2 inequality and Lemmas 1, 3 and 4, we have

$$\begin{aligned} -\mathbb{E}\Big[\min_{s\le t} X_s\Big] &\le \mathbb{E}\Big[\max_{s\le t}\Big|X_s - \sum_{u=1}^{s}\mathbb{E}\big[\xi_u\big|\mathcal{F}_{u-1}\big]\Big| + \max_{s\le t}\Big|\sum_{u=1}^{s}\mathbb{E}\big[\xi_u\big|\mathcal{F}_{u-1}\big]\Big|\Big] \\ &\le \sqrt{4\mathbb{E}\Big[\Big|X_t - \sum_{s=1}^{t}\mathbb{E}\big[\xi_s\big|\mathcal{F}_{s-1}\big]\Big|^2\Big]} + \mathbb{E}\Big[\sum_{s=1}^{t}\Big|\mathbb{E}\big[\xi_s\big|\mathcal{F}_{s-1}\big]\Big|\Big] \\ &\le \sum_{s=1}^{t}\mathbb{E}\Big[\mathbf{1}_{\left\{A_{s-1}^{(\ge 2)}>0\right\}} + \mathbf{1}_{\left\{A_{s-1}=0,\ w_t\in\mathcal{N}_{s-1}^{(\ge 2)}\right\}}\Big]O\big(1\big) \\ &\quad + O\Big(\sqrt{t} + \epsilon t + \frac{t^2}{n} + \log n\Big) \\ &\le \sum_{s=1}^{t}\mathbb{E}\Big[\mathbf{1}_{\left\{A_{s-1}^{(\ge 2)}>0\right\}} + \mathbf{1}_{\left\{A_{s-1}=0,\ w_t\in\mathcal{N}_{s-1}^{(\ge 2)}\right\}}\Big]O\big(1\big) \\ &\quad + O\big(\epsilon n^{2/3} + n^{1/3}\big). \end{aligned} \tag{5.7}$$

For $1 \le i \le k$, let

$$a_t^{(i)} = A_t^{(i)} - A_{t-1,0}^{(i)}. \tag{5.8}$$

Further we define $a_t^{(\geq i)} = A_t^{(\geq i)} - A_{t-1,0}^{(\geq i)}$. By definition, $a_t^{(i)} \in \mathcal{F}_t$ and

$$\begin{aligned} \mathbf{1}_{\left\{A_{s-1}^{(\geq 2)}>0\right\}} &= \mathbf{1}_{\left\{a_{s-1}^{(\geq 2)}\geq 1\right\}} + \mathbf{1}_{\left\{a_{s-1}^{(\geq 2)}=0\right\}}\mathbf{1}_{\left\{A_{s-2}^{(\geq 2)}\geq 2\right\}} \\ &\leq \mathbf{1}_{\left\{a_{s-1}^{(\geq 2)}\geq 1\right\}} + \mathbf{1}_{\left\{A_{s-2}^{(\geq 2)}\geq 2\right\}}. \end{aligned} \tag{5.9}$$

Now we use the following lemma. We prove this later.

Lemma 8. *For $s \leq t \leq s_0 n^{2/3}$ and $d \in \mathbb{N} \cup \{0\}$,*

$$\mathbb{P}\Big(\big\{a_s^{(\geq 2)} = d\big\} \cap \big\{N_t^{(0)} \geq n - t - 2kt\big\}\Big|\mathcal{F}_{s-1}\Big) \leq 2^{k\mathbf{1}_{\{d\neq 0\}}}\left\{\frac{k(t+2kt)}{n-t}\right\}^d,$$

$$\mathbb{P}\Big(\big\{A_s^{(\geq 2)} \geq 2\big\} \cap \big\{N_t^{(0)} \geq n - t - 2kt\big\}\Big) = O\big(n^{-2/3}\big).$$

Using (5.9) and Lemmas 4, 8, we have

$$\sum_{s=1}^{t} \mathbb{E}\mathbf{1}_{\left\{A_{s-1}^{(\geq 2)}>0\right\}} O\big(1\big) = O\big(n^{1/3}\big). \tag{5.10}$$

Next, we get

$$\mathbb{E}\Big[\mathbf{1}_{\left\{A_{s-1}=0,\ w_t\in\mathcal{N}_{s-1}^{(\geq 2)}\right\}}\Big|\mathcal{F}_{s-1}\Big] \leq \mathbf{1}_{\left\{A_{s-1}=0\right\}}\frac{N_{s-1}^{(\geq 2)}}{N_{s-1}} \leq \frac{N_{s-1}^{(\geq 2)}}{n-s}.$$

By Lemma 3,

$$\sum_{s=1}^{t} \mathbb{E}\mathbf{1}_{\left\{A_{s-1}=0,\ w_t\in\mathcal{N}_{s-1}^{(\geq 2)}\right\}} O\big(1\big) = O\big(n^{1/3}\big). \tag{5.11}$$

Using (5.7), (5.10) and (5.11), we have

$$-\mathbb{E}\Big[\min_{s\leq t} X_s\Big] = O\big(\epsilon n^{2/3} + n^{1/3}\big). \tag{5.12}$$

Therefore, using (5.6) and (5.12), we obtain that

$$\begin{aligned} \mathbb{E}Z_t &\leq -\mathbb{E}\Big[\min_{s\leq t} X_s\Big] + O\big(\log n\big) \\ &= O\big(\epsilon n^{2/3} + n^{1/3}\big). \end{aligned}$$

Also, by Lemmas 1, 3, 4 and (5.10), (5.11), we have

$$\mathbb{E}X_t = O\big(\epsilon n^{2/3} + n^{1/3} + \log n\big) = O\big(\epsilon n^{2/3} + n^{1/3}\big). \tag{5.13}$$

Further, from (4.1) and (5.13),

$$\begin{aligned}\mathbb{E}A_t &= \mathbb{E}X_t + \mathbb{E}Z_t \\ &= O\big(\epsilon n^{2/3} + n^{1/3}\big).\end{aligned}$$

□

Proof of Lemma 8. Let d be a fixed natural number. First, note that

$$\begin{aligned}\mathbf{1}_{\left\{a_s^{(\geq 2)}=d\right\}} &= \sum_{c=0}^{d\wedge k} \mathbf{1}_{\left\{A_{s-1,k}^{(\geq 2)}-A_{s-1,0}^{(\geq 2)}=c\right\}} \mathbf{1}_{\left\{A_{s-1,k+1}^{(\geq 2)}-A_{s-1,k}^{(\geq 2)}=d-c\right\}} \\ &=: \sum_{c=0}^{d\wedge k} \mathbf{1}_{\left\{(\mathrm{I})_c\right\}} \mathbf{1}_{\left\{(\mathrm{II})_{d-c}\right\}}.\end{aligned}$$

For each c with $0 \leq c \leq d \wedge k$, we have

$$\begin{aligned}\mathbf{1}_{\left\{(\mathrm{I})_c\right\}} &\leq \sum_{1\leq l_1<l_2<\cdots<l_c\leq k} \prod_{i=1}^{c} \mathbf{1}_{\left\{A_{s-1,l_i}^{(\geq 2)}-A_{s-1,l_i-1}^{(\geq 2)}=1\right\}} \\ &\qquad \times \mathbf{1}_{\left\{A_{s-1,l'}^{(\geq 2)}-A_{s-1,l'-1}^{(\geq 2)}=0 \text{ for any } l' \neq \{l_1,\ldots,l_c\}\right\}} \\ &\leq \sum_{1\leq l_1<l_2<\cdots<l_c\leq k} \prod_{i=1}^{c} \mathbf{1}_{\left\{A_{s-1,l_i}^{(\geq 2)}-A_{s-1,l_i-1}^{(\geq 2)}=1\right\}} \\ &=: \sum_{1\leq l_1<l_2<\cdots<l_c\leq k} \prod_{i=1}^{c} \mathbf{1}_{\left\{(\mathrm{III})_i\right\}}.\end{aligned}$$

By

$$\begin{aligned}\mathbf{1}_{\left\{(\mathrm{III})_i\right\}} &= \mathbf{1}_{\left\{l_i\text{-th AS process excutes}\right\}} \mathbf{1}_{\left\{\eta_{s-1,T_{l_i}} \in \mathcal{N}_{s-1,l_i-1}^{(\geq 2)}, \text{open}\right\}} \\ &\leq \mathbf{1}_{\left\{\eta_{s-1,T_{l_i}} \in \mathcal{N}_{s-1,l_i-1}^{(\geq 2)}\right\}},\end{aligned}$$

we have

$$\mathbb{E}\Big[\mathbf{1}_{\left\{(\mathrm{III})_i\right\}} \Big| \mathcal{F}_{s-1,l_i-1}\Big] \leq \frac{N_{s-1,l_i-1}^{(\geq 2)}}{n-s}.$$

The inequarity $N^{(0)}_{s-1,l_i-1} \geq n-t-2kt$ means that $N^{(\geq 1)}_{s-1,l_i-1} + A_{s-1,l_i-1} \leq t-s+2kt$, so

$$\begin{aligned}
&\mathbb{E}\Big[\mathbf{1}_{\{(\mathrm{III})_i\}}\mathbf{1}_{\{N^{(0)}_{s-1,l_i-1}\geq n-t-2kt\}}\Big|\mathcal{F}_{s-1,l_i-1}\Big] \\
&\qquad \leq \mathbf{1}_{\{N^{(\geq 1)}_{s-1,l_i-1}+A_{s-1,l_i-1}\leq t-s+2kt\}}\frac{N^{(\geq 2)}_{s-1,l_i-1}}{n-s} \\
&\qquad \leq \frac{t-s+2kt}{n-s} \leq \frac{k(t+2kt)}{n-t}.
\end{aligned}$$

Next, we estimate $(\mathrm{II})_{d-c}$. We have

$$\begin{aligned}
\mathbb{E}\Big[\mathbf{1}_{\{(\mathrm{II})_{d-c}\}}\Big|\mathcal{F}_{s-1,k}\Big] &\leq \binom{N^{(\geq 1)}_{s-1,k}+A_{s-1,k}}{d-c}\left(\frac{k-1}{n-s}\right)^{d-c} \\
&\leq \left\{\left(N^{(\geq 1)}_{s-1,k}+A_{s-1,k}\right)\frac{k}{n-s}\right\}^{d-c},
\end{aligned}$$

so

$$\begin{aligned}
\mathbb{E}\Big[\mathbf{1}_{\{(\mathrm{II})_{d-c}\}}\mathbf{1}_{\{N^{(0)}_{s-1,k}\geq n-t-2kt\}}\Big|\mathcal{F}_{s-1,k}\Big] &\leq \left\{\frac{k(t-s+2kt)}{n-s}\right\}^{d-c} \\
&\leq \left\{\frac{k(t+2kt)}{n-t}\right\}^{d-c}.
\end{aligned}$$

Now, since $N^{(0)}_{t,l}$ is decreasing in both t and l, we have

$$\mathbf{1}_{\{N^{(0)}_{t}\geq n-t-2kt\}} \leq \prod_{s=1}^{t}\prod_{l=0}^{k}\mathbf{1}_{\{N^{(0)}_{s-1,l}\geq n-t-2kt\}}$$

for $s \leq t$. Thus

$$\begin{aligned}
&\mathbb{E}\Big[\mathbf{1}_{\{a^{(\geq 2)}_s=d\}}\mathbf{1}_{\{N^{(0)}_{t}\geq n-t-2kt\}}\Big|\mathcal{F}_{s-1,k}\Big] \\
&\qquad \leq \sum_{c=0}^{d\wedge k}\sum_{1\leq l_1<l_2<\cdots<l_c\leq k}\prod_{i=1}^{c}\mathbf{1}_{\{(\mathrm{III})_i\}}\mathbf{1}_{\{N^{(0)}_{s-1,l_i-1}\geq n-t-2kt\}}\left\{\frac{k(t+2kt)}{n-t}\right\}^{d-c},
\end{aligned}$$

therefore

$$
\begin{aligned}
&\mathbb{E}\Big[\mathbf{1}_{\left\{a_s^{(\geq 2)}=d\right\}}\mathbf{1}_{\left\{N_{s-1,k}^{(0)}\geq n-t-2kt\right\}}\Big|\mathcal{F}_{s-1}\Big] \\
&\qquad \leq \sum_{c=0}^{d\wedge k}\binom{k}{c}\left\{\frac{k(t+2kt)}{n-t}\right\}^c\left\{\frac{k(t+2kt)}{n-t}\right\}^{d-c} \\
&\qquad \leq 2^k\left\{\frac{k(t+2kt)}{n-t}\right\}^d.
\end{aligned}
$$

Furthermore, for $d \in \mathbb{N} \cup \{0\}$,

$$
\mathbb{E}\Big[\mathbf{1}_{\left\{a_s^{(\geq 2)}=d\right\}}\mathbf{1}_{\left\{N_t^{(0)}\geq n-t-2kt\right\}}\Big|\mathcal{F}_{s-1}\Big] \leq 2^{k\mathbf{1}_{\{d\neq 0\}}}\left\{\frac{k(t+2kt)}{n-t}\right\}^d.
$$

To prove the second part of this lemma, let $\left\{d(q)\right\}_{q=r}^{s}$ be non-negative integers for $0 \leq r \leq s < t$. Using notations $\mathbf{1}_{\{(\mathrm{I})\}}$, $\mathbf{1}_{\{(\mathrm{II})\}}$ and $\mathbf{1}_{\{(\mathrm{III})\}}$ again, we get

$$
\begin{aligned}
&\prod_{q=r}^{s}\mathbf{1}_{\left\{a_q^{(\geq 2)}=d(q)\right\}}\mathbf{1}_{\left\{N_t^{(0)}\geq n-t-2kt\right\}} \\
&\qquad \leq \prod_{q=r}^{s}\sum_{c(q)=0}^{d(q)\wedge k}\sum_{1\leq l_1<l_2<\cdots<l_{c(q)}\leq k}\prod_{i=1}^{c(q)}\mathbf{1}_{\left\{(\mathrm{III})_{q,i}\right\}}\mathbf{1}_{\left\{N_{q-1,l_i-1}^{(0)}\geq n-t-2kt\right\}} \\
&\qquad\qquad \times\mathbf{1}_{\left\{(\mathrm{II})_{q,d(q)-c(q)}\right\}}\mathbf{1}_{\left\{N_{q-1,k}^{(0)}\geq n-t-2kt\right\}}.
\end{aligned}
$$

Thus, by the first part of this lemma,

$$
\begin{aligned}
&\mathbb{E}\Big[\prod_{q=r}^{s}\mathbf{1}_{\left\{a_q^{(\geq 2)}=d(q)\right\}}\mathbf{1}_{\left\{N_t^{(0)}\geq n-t-2kt\right\}}\Big|\mathcal{F}_{r-1}\Big] \\
&\qquad \leq \prod_{q=r}^{s}2^{k\mathbf{1}_{\{d(q)\neq 0\}}}\left\{\frac{k(t+2kt)}{n-t}\right\}^{d(q)} = 2^{k\sum_{q=r}^{s}\mathbf{1}_{\{d(q)\neq 0\}}}\left\{\frac{k(t+2kt)}{n-t}\right\}^{\sum_{q=r}^{s}d(q)} \\
&\qquad \leq \left\{\frac{2^k k(t+2kt)}{n-t}\right\}^{\sum_{q=r}^{s}d(q)}. \qquad (5.14)
\end{aligned}
$$

Now

$$
\begin{aligned}
&\mathbf{1}_{\left\{A_s^{(\geq 2)}\geq 2\right\}}\\
&\quad\leq\ \mathbf{1}_{\left\{A_s^{(\geq 2)}\geq 2\right\}}\Bigg[\mathbf{1}_{\left\{\forall u\leq s,\ a_u^{(\geq 2)}\leq 2\right\}}+\mathbf{1}_{\left\{\exists u\leq s,\ a_u^{(\geq 2)}=3,\ \forall u'\neq u,\ a_u^{(\geq 2)}\leq 2\right\}}\\
&\qquad+\mathbf{1}_{\left\{\exists u,u'\leq s,\ a_u^{(\geq 2)}=a_{u'}^{(\geq 2)}=3\right\}}+\mathbf{1}_{\left\{\exists u\leq s,\ a_u^{(\geq 2)}\geq 4\right\}}\Bigg].
\end{aligned}
$$

When $A_{u-1}^{(\geq 2)}=0$, if $a_u^{(\geq 2)}=x$ then $A_u^{(\geq 2)}=x$. When $A_{u-1}^{(\geq 2)}\geq 1$, since w_u is chosen from $\mathcal{A}_{u-1}^{(\geq 2)}$, if $a_u^{(\geq 2)}=x$ then $A_u^{(\geq 2)}=A_{u-1}^{(\geq 2)}+x-1$. On the event $\left\{A_s^{\geq 2}\geq 2,\ \forall u\leq s,\ a_u^{(\geq 2)}\leq 2\right\}$, if

$$
\sharp\left\{u\in[s-r,s];\ a_u^{(\geq 2)}=2\right\}-\sharp\left\{u\in[s-r,s];\ a_u^{(\geq 2)}=0\right\}\leq 0
$$

for all $r\leq s$, then $A_0^{(\geq 2)}>0$, which is a contradiction. So there exists a time $r\in[0,s]$ such that

$$
\sharp\left\{u\in[s-r,s];\ a_u^{(\geq 2)}=2\right\}-\sharp\left\{u\in[s-r,s];\ a_u^{(\geq 2)}=0\right\}=1.
$$

Therefore

$$
\begin{aligned}
&\mathbf{1}_{\left\{A_s^{(\geq 2)}\geq 2\right\}}\mathbf{1}_{\left\{\forall u\leq s,\ a_u^{(\geq 2)}\leq 2\right\}}\\
&\quad\leq\ \sum_{r=0}^{s}\mathbf{1}_{\left\{\sharp\left\{u\in[s-r,s];\ a_u^{(\geq 2)}=2\right\}-\sharp\left\{u\in[s-r,s];\ a_u^{(\geq 2)}=0\right\}=1\right\}}\mathbf{1}_{\left\{\forall u\leq s,\ a_u^{(\geq 2)}\leq 2\right\}}\\
&\quad=\ \sum_{r=0}^{s}\sum_{x=0}^{\lfloor r/2\rfloor}\mathbf{1}_{\left\{\sharp\left\{u\in[s-r,s];\ a_u^{(\geq 2)}=0\right\}=x\right\}}\\
&\qquad\times\mathbf{1}_{\left\{\sharp\left\{u\in[s-r,s];\ a_u^{(\geq 2)}=2\right\}=x+1\right\}}\mathbf{1}_{\left\{\sharp\left\{u\in[s-r,s];\ a_u^{(\geq 2)}=1\right\}=r-2x\right\}}.
\end{aligned}
$$

Thus, from (5.14),

$$\begin{aligned}
&\mathbb{E}\Big[\mathbf{1}_{\{A_s^{(\geq 2)}\geq 2\}}\mathbf{1}_{\{\forall u\leq s,\ a_u^{(\geq 2)}\leq 2\}}\mathbf{1}_{\{N_t^{(0)}\geq n-t-2kt\}}\Big]\\
&\quad\leq \sum_{r=0}^{s}\sum_{x=0}^{\lfloor r/2\rfloor}\frac{(r+1)!}{x!(x+1)!(r-2x)!}\left\{\frac{2^k k(t+2kt)}{n-t}\right\}^{2(x+1)+r-2x}\\
&\quad\leq \sum_{r=0}^{s}\left\{\frac{2^k k(t+2kt)}{n-t}\right\}^{r+2}3^{r+1}\\
&\quad= O\left(\frac{t^2}{n^2}\right).
\end{aligned}$$

Next, we have

$$\begin{aligned}
&\mathbf{1}_{\{A_s^{(\geq 2)}\geq 2\}}\mathbf{1}_{\{\exists u\leq s,\ a_u^{(\geq 2)}=3,\ \forall u'\neq u,\ a_u^{(\geq 2)}\leq 2\}}\\
&\quad\leq\ \mathbf{1}_{\{a_s^{(\geq 2)}\geq 2\}}+\mathbf{1}_{\{a_{s-1}^{(\geq 2)}\geq 2\}}\\
&\qquad+\left\{\mathbf{1}_{\{a_s^{(\geq 2)}=1\}}+\mathbf{1}_{\{a_{s-1}^{(\geq 2)}=1\}}\right\}\mathbf{1}_{\{\exists u\leq s-2,\ a_u^{(\geq 2)}=3\}}\\
&\qquad+\mathbf{1}_{\{\exists u\leq s-2,\ a_u^{(\geq 2)}=3,\ \forall u'\neq u,\ a_u^{(\geq 2)}\leq 2\}}\mathbf{1}_{\{a_s^{(\geq 2)}=a_{s-1}^{(\geq 2)}=0\}}\mathbf{1}_{\{A_{s-2}^{(\geq 2)}\geq 4\}}.
\end{aligned}\tag{5.15}$$

From the first part of this lemma, we know that

$$\begin{aligned}
&\mathbb{E}\Big[\Big\{\text{the first and the second term of (5.15)}\Big\}\mathbf{1}_{\{N_t^{(0)}\geq n-t-2kt\}}\Big]\\
&\quad=\ O\left(\frac{t^2}{n^2}\right),
\end{aligned}$$

and from the first part of this lemma,

$$\mathbb{E}\Big[\Big\{\text{the third term of (5.15)}\Big\}\mathbf{1}_{\{N_t^{(0)}\geq n-t-2kt\}}\Big]=O\left(\frac{t}{n}t\frac{t^3}{n^3}\right)=O\left(\frac{t^2}{n^2}\right).$$

We consider the fourth term of (5.15). Similarly to the above inequality, we have

$$
\begin{aligned}
&\mathbf{1}_{\left\{\exists u\le s,\ a_u^{(\ge 2)}=3,\ \forall u'\ne u,\ a_{u'}^{(\ge 2)}\le 2\right\}}\mathbf{1}_{\left\{a_s^{(\ge 2)}=a_{s-1}^{(\ge 2)}=0\right\}}\mathbf{1}_{\left\{A_{s-2}^{(\ge 2)}\ge 4\right\}}\\
&\le \sum_{u=1}^{s-3}\mathbf{1}_{\left\{a_u^{(\ge 2)}=3,\ \forall u'\ne u,\ a_{u'}^{(\ge 2)}\le 2\right\}}\\
&\quad\times\sum_{r=0}^{s-3-u}\mathbf{1}_{\left\{\sharp\left\{u'\in[s-4-r,s-4];\ a_{u'}^{(\ge 2)}=2\right\}-\sharp\left\{u'\in[s-4-r,s-4];\ a_{u'}^{(\ge 2)}=0\right\}=1\right\}}\\
&\quad+\sum_{u=2}^{s-2}\mathbf{1}_{\left\{a_u^{(\ge 2)}=3,\ \forall u'\ne u,\ a_{u'}^{(\ge 2)}\le 2\right\}}\\
&\quad\times\sum_{r=0}^{u-2}\mathbf{1}_{\left\{\sharp\left\{u'\in[u-1-r,u-1];\ a_{u'}^{(\ge 2)}=2\right\}-\sharp\left\{u'\in[u-1-r,u-1];\ a_{u'}^{(\ge 2)}=0\right\}=1\right\}}\\
&= \sum_{u=1}^{s-3}\mathbf{1}_{\left\{a_u^{(\ge 2)}=3\right\}}\sum_{r=0}^{s-3-u}\sum_{x=0}^{\lfloor r/2\rfloor}\mathbf{1}_{\left\{\sharp\left\{u'\in[s-4-r,s-4];\ a_{u'}^{(\ge 2)}=0\right\}=x\right\}}\\
&\quad\times\mathbf{1}_{\left\{\sharp\left\{u'\in[s-4-r,s-4];\ a_{u'}^{(\ge 2)}=2\right\}=x+1\right\}}\mathbf{1}_{\left\{\sharp\left\{u'\in[s-4-r,s-4];\ a_{u'}^{(\ge 2)}=1\right\}=r-2x\right\}}\\
&\quad+\sum_{u=2}^{s-2}\mathbf{1}_{\left\{a_u^{(\ge 2)}=3\right\}}\sum_{r=0}^{u-2}\sum_{x=0}^{\lfloor r/2\rfloor}\mathbf{1}_{\left\{\sharp\left\{u'\in[u-1-r,u-1];\ a_{u'}^{(\ge 2)}=0\right\}=x\right\}}\\
&\quad\times\mathbf{1}_{\left\{\sharp\left\{u'\in[u-1-r,u-1];\ a_{u'}^{(\ge 2)}=2\right\}=x+1\right\}}\mathbf{1}_{\left\{\sharp\left\{u'\in[u-1-r,u-1];\ a_{u'}^{(\ge 2)}=1\right\}=r-2x\right\}}.
\end{aligned}
$$

By similar calculation as above, we have

$$
\begin{aligned}
&\mathbb{E}\left[\mathbf{1}_{\left\{\exists u\le s,\ a_u^{(\ge 2)}=3,\ \forall u'\ne u,\ a_u^{(\ge 2)}\le 2\right\}}\mathbf{1}_{\left\{a_s^{(\ge 2)}=a_{s-1}^{(\ge 2)}=0\right\}}\mathbf{1}_{\left\{A_{s-2}^{(\ge 2)}\ge 4\right\}}\mathbf{1}_{\left\{N_t^{(0)}\ge n-t-2kt\right\}}\right]\\
&\le \sum_{u=1}^{s-3}\sum_{r=0}^{s-3-u}\sum_{x=0}^{\lfloor r/2\rfloor}\frac{(r+1)!}{x!(x+1)!(r-2x)!}\left\{\frac{2^k k(t+2kt)}{n-t}\right\}^{3+2(x+1)+r-2x}\\
&\quad+\sum_{u=2}^{s-2}\sum_{r=0}^{u-2}\sum_{x=0}^{\lfloor r/2\rfloor}\frac{(r+1)!}{x!(x+1)!(r-2x)!}\left\{\frac{2^k k(t+2kt)}{n-t}\right\}^{3+2(x+1)+r-2x}\\
&= O\left(\frac{t^6}{n^5}\right) = O\left(\frac{1}{n}\right).
\end{aligned}
$$

Therefore

$$\mathbb{E}\Big[\mathbf{1}_{\big\{A_s^{(\geq 2)}\geq 2\big\}}\mathbf{1}_{\big\{\exists u\leq s,\ a_u^{(\geq 2)}=3,\ \forall u'\neq u,\ a_u^{(\geq 2)}\leq 2\big\}}\mathbf{1}_{\big\{N_t^{(0)}\geq n-t-2kt\big\}}\Big]=O\bigg(\frac{t^2}{n^2}\bigg).$$

Finally, by a straightforward calculation

$$\begin{aligned}
&\mathbb{E}\Big[\mathbf{1}_{\big\{\exists u,u'\leq s-2,\ a_u^{(\geq 2)}=a_{u'}^{(\geq 2)}=3\big\}}\mathbf{1}_{\big\{N_t^{(0)}\geq n-t-2kt\big\}}\Big]\\
&\qquad=O\bigg(t^2\frac{t^3}{n^3}\frac{t^3}{n^3}\bigg)=O\bigg(\frac{t^2}{n^2}\bigg),\\
&\mathbb{E}\Big[\mathbf{1}_{\big\{\exists u\leq s-2,\ a_u^{(\geq 2)}\geq 4\big\}}\mathbf{1}_{\big\{N_t^{(0)}\geq n-t-2kt\big\}}\Big]\\
&\qquad=O\bigg(t\frac{t^4}{n^4}\bigg)=O\bigg(\frac{t^2}{n^2}\bigg).
\end{aligned}$$

Thus we have $\mathbb{E}\Big[\mathbf{1}_{\big\{A_{s-2}^{(\geq 2)}\geq 2\big\}}\mathbf{1}_{\big\{N_t^{(0)}\geq n-t-2kt\big\}}\Big]=O\big(\frac{t^2}{n^2}\big)$, it is the second part of this lemma. □

Lemma 9. *If n is large enough, then for all $t\leq s_0 n^{2/3}$,*

$$\begin{aligned}
\mathbb{E}N_t^{(0)} &= n-\bigg\{(k-1)\frac{p}{1-p}+k\bigg\}t+O\big(\epsilon n^{2/3}+n^{1/3}\big),\\
\mathbb{E}N_t^{(1)} &= k(1-p)t+O\big(\epsilon n^{2/3}+n^{1/3}\big),\\
\mathbb{E}N_t^{(i)} &= O\big(\epsilon n^{2/3}+n^{1/3}\big), \qquad \textit{for } 2\leq i\leq k.
\end{aligned}$$

Proof. We have

$$\begin{aligned}
N_t^{(0)} &= N_{t-1}^{(0)}-\mathbf{1}_{\big\{w_t\in\mathcal{N}_{t-1}^{(0)}\big\}}-\sum_{l=1}^{k}\mathbf{1}_{\big\{w_t\in\mathcal{A}_{t-1}^{(\leq l)}\cup\mathcal{N}_{t-1}^{(\leq l)}\big\}}\hat{r}_{t-1,l}\\
&\quad-\sum_{v\in\mathcal{N}_{t-1,k}^{(0)}}\mathbf{1}_{\big\{w_t\leftarrow v\big\}}\\
&= N_{t-1}^{(0)}\\
&\quad-\sum_{l=2}^{k}\hat{r}_{t-1,l}-\sum_{v\in\mathcal{N}_{t-1,k}^{(0)}}\mathbf{1}_{\big\{w_t\leftarrow v\big\}}-\mathbf{1}_{\big\{A_{t-1}=0,w_t\in\mathcal{N}_{t-1}^{(0)}\big\}}\Big\{1+\hat{r}_{t-1,1}\Big\}\\
&\quad+\sum_{j=2}^{k}\bigg\{\mathbf{1}_{\big\{A_{t-1}^{(j)}>0,\ A_{t-1}^{(\geq j+1)}=0\big\}}+\mathbf{1}_{\big\{A_{t-1}=0,\ w_t\in\mathcal{N}_{t-1}^{(j)}\big\}}\bigg\}\sum_{l=2}^{j}\hat{r}_{t-1,l}.
\end{aligned}$$

So, take conditional expectation of both sides, we get

$$\begin{aligned}\mathbb{E}\Big[N_t^{(0)}\Big|\mathcal{F}_{t-1}\Big] &= N_{t-1}^{(0)} - \sum_{l=2}^{k}\mathbb{E}\Big[\hat{r}_{t-1,l}\Big|\mathcal{F}_{t-1}\Big] - N_{t-1}^{(0)}\frac{k}{n-t} \\ &\quad + \mathbf{1}_{\left\{A_{t-1}=0\right\}}O(1) + \mathbf{1}_{\left\{A_{t-1}^{(\geq 2)}>0\right\}}O(1) + O\bigg(\frac{1}{n}\bigg).\end{aligned}$$

Here recall that we have put $m = \lfloor s_0 n^{2/3}\rfloor$ for some $s_0 \in [0,\infty)$. Then

$$\begin{aligned}\mathbb{E}\big[\hat{r}_{t-1,l}\big|\mathcal{F}_{t-1,l-1}\big] &= \mathbb{E}\Big[\sum_{x=0}^{n} x\mathbf{1}_{\left\{\hat{r}_{t-1,l}=x\right\}}\Big|\mathcal{F}_{t-1,l-1}\Big] \\ &= \sum_{x=1}^{N_{t-1,l-1}^{(0)}} x\prod_{y=0}^{x-1}\bigg\{\frac{N_{t-1,l-1}^{(0)}-y}{n-t}p + \mathbf{1}_{\left\{y=0\right\}}\cdot O\bigg(\frac{1}{n}\bigg)\bigg\} \\ &\quad \times\bigg\{\frac{N_{t-1,l-1}^{(0)}-x}{n-t}(1-p) + \frac{N_{t-1,l-1}^{(\geq 1)}+A_{t-1,l-1}+x}{n-t} + O\bigg(\frac{1}{n}\bigg)\bigg\} \\ &= \sum_{x=1}^{N_{t-1,l-1}^{(0)}} x\prod_{y=0}^{x-1}\bigg\{\frac{N_{t-1,l-1}^{(0)}-y}{n-t}p\bigg\}\bigg\{1-\frac{N_{t-1,l-1}^{(0)}}{n-t}p\bigg\} + O\bigg(\frac{1}{n}\bigg) \\ &= \frac{N_{t-1,l-1}^{(0)}}{n-t}p\bigg\{1-\frac{N_{t-1,l-1}^{(0)}}{n-t}p\bigg\}\sum_{x=1}^{N_{t-1,l-1}^{(0)}} x\prod_{y=1}^{x-1}\bigg\{\frac{N_{t-1,l-1}^{(0)}-y}{n-t}p\bigg\} + O\bigg(\frac{1}{n}\bigg) \\ &= \frac{N_{t-1,l-1}^{(0)}}{n-t}p\frac{1}{\Big(1-\frac{N_{t-1,l-1}^{(0)}}{n-t}p\Big)^2}\bigg\{1-\frac{N_{t-1,l-1}^{(0)}}{n-t}p\bigg\} \\ &\quad + O\bigg(\frac{m^{1/3}}{n} + m^{1/3}p^{m^{1/3}}\bigg) + \mathbf{1}_{\left\{N_{t-1,l-1}^{(0)}<m^{1/3}\right\}}\cdot O(1) \\ &= \frac{p}{1-p} + \frac{N_{t-1,l-1}^{(\geq 1)}+A_{t-1,l-1}}{n}O\big(1\big) + O\big(n^{-2/3}\big) + \mathbf{1}_{\left\{N_{t-1,l-1}^{(0)}<m^{1/3}\right\}}O\big(1\big).\end{aligned}$$

Therefore

$$\begin{aligned}\mathbb{E}\big[N_t^{(0)}\big|\mathcal{F}_{t-1}\big] &= N_{t-1}^{(0)} - (k-1)\frac{p}{1-p} - k + \mathbf{1}_{\left\{A_{t-1}^{(\geq 2)}>0\right\}}O(1) \\ &\quad + \mathbf{1}_{\left\{A_{t-1}=0\right\}}O(1) + \frac{N_{t-1,l-1}^{(\geq 1)} + A_{t-1,l-1}}{n}O\big(1\big) \\ &\quad + O\big(n^{-2/3}\big) + \mathbf{1}_{\left\{N_{t-1,l-1}^{(0)}<m^{1/3}\right\}}O\big(1\big).\end{aligned}$$

Using Lemma 7, we have also

$$\mathbb{E}\Big[\sum_{s=1}^{t}\mathbf{1}_{\left\{A_{s-1}=0\right\}}\Big] \leq \mathbb{E}Z_t = O\big(\epsilon n^{2/3} + n^{1/3}\big),$$

and by Lemma 4 and Lemma 8,

$$\mathbb{E}\Big[\sum_{s=1}^{t}\mathbf{1}_{\left\{A_{s-1}^{(\geq 2)}>0\right\}}\Big] = O\Big(\log n + \frac{t^2}{n}\Big) \;=\; O\big(n^{1/3}\big).$$

Furthermore by Lemma 3,

$$\mathbb{E}N_t^{(0)} = n - \Big\{(k-1)\frac{p}{1-p} + k\Big\}t + O\big(\epsilon n^{2/3} + n^{1/3}\big).$$

Next, for $i \geq 1$

$$\begin{aligned}N_t^{(i)} &= N_{t-1}^{(i)} - \sum_{l=2}^{k}\mathbf{1}_{\left\{\eta_{t,T_l}\in\mathcal{N}_{t-1,l-1}^{(i)},\ \text{open}\right\}} + \sum_{v\in\mathcal{N}_{t-1,k}^{(i-1)}}\mathbf{1}_{\left\{w_t\leftarrow v,\ \text{closed}\right\}} \\ &\quad - \sum_{v\in\mathcal{N}_{t-1,k}^{(i)}}\mathbf{1}_{\left\{w_t\leftarrow v\right\}} \\ &\quad + \sum_{j=2}^{k}\Big\{\mathbf{1}_{\left\{A_{t-1}^{(j)}>0,\ A_{t-1}^{(\geq j+1)}=0\right\}} + \mathbf{1}_{\left\{A_{t-1}=0,\ w_t\in\mathcal{N}_{t-1}^{(j)}\right\}}\Big\} \\ &\quad \times\sum_{l=2}^{j}\mathbf{1}_{\left\{\eta_{t,T_l}\in\mathcal{N}_{t-1,l-1}^{(i)},\ \text{open}\right\}} \\ &\quad - \mathbf{1}_{\left\{w_t\in\mathcal{N}_{t-1}^{(i)}\right\}} - \mathbf{1}_{\left\{w_t\in\mathcal{N}_{t-1}^{(0)}\right\}}\mathbf{1}_{\left\{\eta_{t,T_1}\in\mathcal{N}_{t-1,0}^{(i)},\ \text{open}\right\}}.\end{aligned}$$

So, we have

$$\begin{aligned}\mathbb{E}\big[N_t^{(i)}\big|\mathcal{F}_{t-1}\big] &= N_{t-1}^{(i)} + N_{t-1}^{(i-1)}\frac{(k-i-1)(1-p)}{n-t} + \mathbf{1}_{\left\{A_{t-1}=0\right\}}O(1) \\ &\quad + \mathbf{1}_{\left\{A_{t-1}^{(\geq 2)}>0\right\}}O(1) + \frac{N_{t-1,l-1}^{(\geq 1)} + A_{t-1,l-1}}{n}O\big(1\big) \\ &\quad + O\big(n^{-2/3}\big) + \mathbf{1}_{\left\{N_{t-1,l-1}^{(0)}<m^{1/3}\right\}}O\big(1\big).\end{aligned}$$

When $i=1$,

$$\mathbb{E}N_t^{(1)} = k(1-p)t + O\big(\epsilon n^{2/3} + n^{1/3}\big),$$

and when $i \geq 2$,

$$\mathbb{E}N_t^{(i)} = O\big(\epsilon n^{2/3} + n^{1/3}\big).$$

□

Lemma 10. *If n is large enough, then for all $t \leq s_0 n^{2/3}$ and $0 \leq i \leq k$,*

$$\mathbb{E}\Big|N_t^{(i)} - \mathbb{E}N_t^{(i)}\Big| = O\big(\epsilon n^{2/3} + n^{1/3}\big).$$

Proof. By Lemma 9, For $i \geq 2$, $\mathbb{E}|N_t^{(i)} - \mathbb{E}N_t^{(i)}| \leq 2\mathbb{E}N_t^{(i)} = O\big(\epsilon n^{2/3} + n^{1/3}\big)$, also by Lemma 7, $\mathbb{E}|A_t - \mathbb{E}A_t| \leq O\big(\epsilon n^{2/3} + n^{1/3}\big)$. So we consider for $i=1$. By

$$\begin{aligned}N_t^{(1)} &= \sum_{s=1}^{t}\sum_{v\in\mathcal{N}_{s-1,k}^{(0)}} \mathbf{1}_{\left\{w_s\leftarrow v,\ \text{closed}\right\}} \\ &\quad + \sum_{s=1}^{t}\Bigg[-\sum_{l=2}^{k}\mathbf{1}_{\left\{\eta_{s-1,T_l}\in\mathcal{N}_{s-1,l-1}^{(1)},\ \text{open}\right\}} - \sum_{v\in\mathcal{N}_{s-1,k}^{(1)}}\mathbf{1}_{\left\{w_s\leftarrow v\right\}}\Bigg] \\ &\quad + \sum_{s=1}^{t}\Bigg[\sum_{j=2}^{k}\Big\{\mathbf{1}_{\left\{A_{s-1}^{(j)}>0,\ A_{s-1}^{(\geq j+1)}=0\right\}} + \mathbf{1}_{\left\{A_{s-1}=0,\ w_t\in\mathcal{N}_{s-1}^{(j)}\right\}}\Big\} \\ &\quad \times\sum_{l=2}^{j}\mathbf{1}_{\left\{\eta_{s-1,T_l}\in\mathcal{N}_{s-1,l-1}^{(1)},\ \text{open}\right\}}\Bigg] \\ &\quad - \sum_{s=1}^{t}\Bigg[\mathbf{1}_{\left\{w_s\in\mathcal{N}_{s-1}^{(0)}\right\}}\mathbf{1}_{\left\{\eta_{s-1,T_1}\in\mathcal{N}_{s-1,0}^{(1)},\ \text{open}\right\}} + \mathbf{1}_{\left\{w_s\in\mathcal{N}_{s-1}^{(1)}\right\}}\Bigg],\end{aligned}$$

we have

$$
\begin{aligned}
&\mathbb{E}|N_t^{(1)} - \mathbb{E}N_t^{(1)}| \\
&\qquad \le \mathbb{E}\Big|\sum_{s=1}^{t} \sum_{v\in\mathcal{N}_{s-1,k}^{(0)}} \mathbf{1}_{\{w_s\leftarrow v,\ \text{closed}\}} - \mathbb{E}\sum_{s=1}^{t} \sum_{v\in\mathcal{N}_{s-1,k}^{(0)}} \mathbf{1}_{\{w_s\leftarrow v,\ \text{closed}\}}\Big| \\
&\qquad +O\big(\epsilon n^{2/3} + n^{1/3}\big).
\end{aligned}
$$

Here recall the definition of $\{w_s \overset{(l)}{\leftarrow} v\}$ in the proof of Theorem 1,

$$
\begin{aligned}
&\mathbb{E}\Big|\sum_{s=1}^{t} \sum_{v\in\mathcal{N}_{s-1,k}^{(0)}} \mathbf{1}_{\{w_s\leftarrow v,\ \text{closed}\}} - \mathbb{E}\sum_{s=1}^{t} \sum_{v\in\mathcal{N}_{s-1,k}^{(0)}} \mathbf{1}_{\{w_s\leftarrow v,\ \text{closed}\}}\Big| \\
&\qquad = \mathbb{E}\Big|\sum_{s=1}^{t} \sum_{v\notin\mathcal{E}_s} \mathbf{1}_{\{w_s\overset{(k)}{\leftarrow} v,\ \text{closed}\}} - \mathbb{E}\sum_{s=1}^{t} \sum_{v\notin\mathcal{E}_s} \mathbf{1}_{\{w_s\overset{(k)}{\leftarrow} v,\ \text{closed}\}} \\
&\qquad\quad + \sum_{s=1}^{t} \sum_{v\in\mathcal{N}_{s-1,k}^{(0)}} \mathbf{1}_{\{w_s\leftarrow v,\ \text{closed}\}} - \mathbb{E}\sum_{s=1}^{t} \sum_{v\in\mathcal{N}_{s-1,k}^{(0)}} \mathbf{1}_{\{w_s\leftarrow v,\ \text{closed}\}} \\
&\qquad\quad - \sum_{s=1}^{t} \sum_{v\notin\mathcal{E}_s} \mathbf{1}_{\{w_s\overset{(k)}{\leftarrow} v,\ \text{closed}\}} + \mathbb{E}\sum_{s=1}^{t} \sum_{v\notin\mathcal{E}_s} \mathbf{1}_{\{w_s\overset{(k)}{\leftarrow} v,\ \text{closed}\}}\Big|.
\end{aligned}
$$

By Schwarz's inequality,

$$
\begin{aligned}
&\mathbb{E}\Big|\sum_{s=1}^{t} \sum_{v\notin\mathcal{E}_s} \mathbf{1}_{\{w_s\overset{(k)}{\leftarrow} v,\ \text{closed}\}} - \mathbb{E}\sum_{s=1}^{t} \sum_{v\notin\mathcal{E}_s} \mathbf{1}_{\{w_s\overset{(k)}{\leftarrow} v,\ \text{closed}\}}\Big| \\
&\qquad \le \sqrt{Var\Big(\sum_{s=1}^{t} \sum_{v\notin\mathcal{E}_s} \mathbf{1}_{\{w_s\overset{(k)}{\leftarrow} v,\ \text{closed}\}}\Big)} \\
&\qquad = \sqrt{\sum_{s=1}^{t} (n-s)\frac{k(1-p)}{n-s}\left(1-\frac{k(1-p)}{n-s}\right)} \\
&\qquad = O\big(\sqrt{t}\big) \;=\; O\big(n^{1/3}\big).
\end{aligned}
$$

Also,

$$\mathbb{E}\Big|\sum_{s=1}^{t}\sum_{v\in\mathcal{N}^{(0)}_{s-1,k}}\mathbf{1}_{\{w_s\leftarrow v,\text{ closed}\}}-\mathbb{E}\sum_{s=1}^{t}\sum_{v\in\mathcal{N}^{(0)}_{s-1,k}}\mathbf{1}_{\{w_s\leftarrow v,\text{ closed}\}}$$
$$-\sum_{s=1}^{t}\sum_{v\notin\mathcal{E}_s}\mathbf{1}_{\{w_s\overset{(k)}{\leftarrow}v,\text{ closed}\}}+\mathbb{E}\sum_{s=1}^{t}\sum_{v\notin\mathcal{E}_s}\mathbf{1}_{\{w_s\overset{(k)}{\leftarrow}v,\text{ closed}\}}\Big|$$
$$\leq\ 2\sum_{s=1}^{t}\mathbb{E}\Big|\sum_{v\in\mathcal{N}^{(\geq 1)}_{s-1,k}\cup\mathcal{A}_{s-1,k}}\mathbf{1}_{\{w_s\overset{(k)}{\leftarrow}v,\text{ closed}\}}\Big|$$
$$+O\big(\epsilon n^{2/3}+n^{1/3}\big),$$

therefore

$$\mathbb{E}\Big|N_t^{(1)}-\mathbb{E}N_t^{(1)}\Big|=O\big(\epsilon n^{2/3}+n^{1/3}\big).$$

Finally by $N_t^{(0)}=n-\sum_{i=1}^{k}N_t^{(i)}-A_t-t$,

$$\mathbb{E}\Big|N_t^{(0)}-\mathbb{E}N_t^{(0)}\Big|=O\big(\epsilon n^{2/3}+n^{1/3}\big).$$

□

5.2 Convergence of the exploration process

Recall times $t_0<t_1<\dots$ such that $A_{t_j}=0$. We define the process $\hat{X}_t$ by $\hat{X}_0=X_0=0$ and for any $t\in[t_j,t_{j+1})$,

$$\hat{X}_t=\begin{cases}X_t & \text{if } X_t\geq X_{t_j},\\ X_{t_j} & \text{othewise},\end{cases}\tag{5.16}$$

and $\hat{X}_t=X_n$ for any $t\geq n$. By this definition, the times when the minima of $\hat{X}_t$ updates are only $\{t_j\}$.

Theorem 5. *Let λ be in $\mathbb{R}$ and $\epsilon=\lambda n^{-1/3}$. Then*

$$n^{-1/3}\hat{X}_{n^{2/3}\cdot}\overset{d}{\Longrightarrow}\mathcal{B}^{\lambda}(\cdot),\qquad \textit{as } n\to\infty,$$

where this convergence is on finite intervals.

Remark 3. *Recording minima for the process $\hat{X}_t$ occurs only when $A_t = 0$. Since we explore one vertex at each time, the size of the j-th explored open cluster is $t_{j+1} - t_j$. Therefore the analysis of the process $\hat{X}_t$ characterizes open clusters of the k-out graph.*

Recall that $s_0 \in [0, \infty)$ and $m = \lfloor s_0 n^{2/3} \rfloor$. Let t be in $[0, 1]$. We define $\xi_s^{(m)}$ by

$$\begin{aligned}\xi_s^{(m)} = &\sum_{l=2}^{k} \hat{r}'_{s-1,l} \mathbf{1}_{\left\{\hat{r}'_{s-1,l} \le m^{1/3}\right\}} \\ &+ \sum_{j=0}^{k} \sum_{v \in \mathcal{N}_{s-1,k}^{(j)}} \mathbf{1}_{\left\{w_s \leftarrow v,\ \text{open}\right\}} \mathbf{1}_{\left\{\sum_{v \in \mathcal{N}_{s-1,k}^{(j)}} \mathbf{1}_{\left\{w_s \leftarrow v,\ \text{open}\right\}} \le m^{1/3}\right\}} - 1,\end{aligned} \tag{5.17}$$

and let $X_s^{(m)} = \sum_{u=1}^{s} \xi_u^{(m)}$.

To prove the convergence of the process X, we use a central limit theorem for martingales (cf. [11] Chapter7, Theorem 7.2.). Namely,

1. $\left| m^{-1/2} \left(\xi_s^{(m)} - \mathbb{E}\left[\xi_s^{(m)} \middle| \mathcal{F}_{s-1} \right] \right) \right| \le \epsilon_m$ for all $s \le m$ with $\epsilon_m \to 0$, and
2. $m^{-1} \sum_{s=1}^{\lfloor mt \rfloor} \mathbb{E}\left[\left(\xi_s^{(m)} - \mathbb{E}\left[\xi_s^{(m)} \middle| \mathcal{F}_{s-1} \right] \right)^2 \middle| \mathcal{F}_{s-1} \right] \to Ct$ i.p. for any $t \in [0, 1]$ and some $C > 0$,

then $m^{-1/2} \sum_{s=1}^{\lfloor mt \rfloor} \left(\xi_s^{(m)} - \mathbb{E}\left[\xi_s^{(m)} \middle| \mathcal{F}_{s-1} \right] \right) \overset{\text{d}}{\Longrightarrow} \mathcal{B}(Ct)$, where $\mathcal{B}(t)$ is a standard Brownian motion.

We start the calculation of $\mathbb{E}\left[\xi_s^{(m)} \middle| \mathcal{F}_{s-1} \right]$. Similarly to the calculation of

$\mathbb{E}\big[\hat{r}'_{s-1,l}\big|\mathcal{F}_{s-1,l-1}\big]$ in the proof of Lemma 1,

$$
\begin{aligned}
&\mathbb{E}\big[\hat{r}'_{s-1,l}\mathbf{1}_{\left\{\hat{r}'_{s,l}\le m^{1/3}\right\}}\big|\mathcal{F}_{s-1,l-1}\big]\\
&\quad=\frac{N_{s-1,l-1}}{n-s}p\bigg(1-\frac{N^{(0)}_{s-1,l-1}}{n-s}p\bigg)\sum_{x=1}^{\lfloor m^{1/3}\rfloor\wedge N^{(0)}_{s-1,l-1}}x\prod_{y=0}^{x-2}\bigg\{\frac{N^{(0)}_{s-1,l-1}-y}{n-s}p\bigg\}\\
&\quad+O\bigg(\frac{1}{n}\bigg)\\
&\quad=\frac{N_{s-1,l-1}}{n-s}p\frac{1}{1-\frac{N^{(0)}_{s-1,l-1}}{n-s}p}+O\bigg(\frac{m^{1/3}}{n}+m^{1/3}p^{m^{1/3}}\bigg)\\
&\quad+\mathbf{1}_{\left\{N^{(0)}_{s-1,l-1}<m^{1/3}\right\}}O\big(1\big)\\
&\quad=\frac{p}{1-p}-\frac{p^2}{(1-p)^2}\frac{N^{(1)}_{s-1,l-1}}{n}+O\bigg(\frac{m^{1/3}}{n}+m^{1/3}p^{m^{1/3}}\bigg)\\
&\quad+\frac{N^{(\ge 2)}_{s-1,l-1}+A_{s-1,l-1}}{n}O\big(1\big)+\frac{\big(N^{(\ge 1)}_{s-1,l-1}+A_{s-1,l-1}\big)^2}{n^2}O\big(1\big)\\
&\quad+\mathbf{1}_{\left\{N^{(0)}_{s-1,l-1}<m^{1/3}\right\}}O\big(1\big).
\end{aligned}
$$

Also, similarly to the calculation of $\mathbb{E}\big[\sum_{v\in\mathcal{N}_{s-1,k}}\mathbf{1}_{\left\{w_s\leftarrow v,\text{ open}\right\}}\big|\mathcal{F}_{s-1,k}\big]$ in the proof of Lemma 1, we have

$$\mathbb{E}\Big[\sum_{j=0}^{k}\sum_{v\in\mathcal{N}_{s-1,k}^{(j)}}\mathbf{1}_{\{w_t\leftarrow v,\ \text{open}\}}\mathbf{1}_{\{\sum_{v\in\mathcal{N}_{s-1,k}^{(j)}}\mathbf{1}_{\{w_s\leftarrow v,\ \text{open}\}}\leq m^{1/3}\}}\Big|\mathcal{F}_{s-1,k}\Big]$$

$$= \sum_{j=0}^{k}\sum_{x=0}^{\lfloor m^{1/3}\rfloor\wedge N_{s-1,k}^{(j)}} x\binom{N_{s-1,k}^{(j)}}{x}\left(\frac{(k-j)p}{n-s}\right)^{x}\left(1-\frac{(k-j)p}{n-s}\right)^{N_{s-1,k}^{(j)}-x}$$
$$+\mathbf{1}_{\{N_{s-1,k}^{(0)}<m^{1/3}\}}O(1)$$

$$= \sum_{j=0}^{k}\Bigg\{N_{s-1,k}^{(j)}\frac{(k-j)p}{n-s}$$
$$-\sum_{x=\lfloor m^{1/3}\rfloor+1}^{N_{s-1,k}^{(j)}} x\binom{N_{s-1,k}^{(j)}}{x}\left(\frac{(k-j)p}{n-s}\right)^{x}\left(1-\frac{(k-j)p}{n-s}\right)^{N_{s-1,k}^{(j)}-x}\Bigg\}$$
$$+\mathbf{1}_{\{N_{s-1,k}^{(0)}<m^{1/3}\}}O(1)$$

$$= \sum_{j=0}^{k}N_{s-1,k}^{(j)}\frac{(k-j)p}{n-s}+O\big(m^{1/3}(kp)^{m^{1/3}}\big)+\mathbf{1}_{\{N_{s-1,k}^{(0)}<m^{1/3}\}}O(1)$$

$$= kp-\frac{N_{s-1,k}^{(1)}}{n-s}p+O\big(m^{1/3}(kp)^{m^{1/3}}\big)$$
$$+\frac{N_{s-1,k}^{(\geq 2)}+A_{s-1,k}}{n}O(1)+\mathbf{1}_{\{N_{s-1,k}^{(0)}<m^{1/3}\}}O(1),$$

where we understand that $\sum_{x=1}^{b} \cdot = 0$ if $a > b$. Therefore

$$
\begin{aligned}
&\mathbb{E}\big[\xi_s^{(m)}\big|\mathcal{F}_{s-1}\big] \\
= \;& (k-1)\frac{p}{1-p} - (k-1)\frac{p^2}{(1-p)^2}\frac{N_{s-1}^{(1)}}{n-s} + kp - \sum_{j=1}^{k} jp\frac{N_{s-1}^{(j)}}{n-s} - 1 \\
&+O\bigg(\frac{m^{1/3}}{n} + m^{1/3}(kp)^{m^{1/3}}\bigg) + \frac{N_{s-1}^{(\geq 2)} + A_{s-1}}{n}O\big(1\big) \\
&+\frac{\big(N_{s-1}^{(\geq 1)} + A_{s-1}\big)^2}{n^2}O\big(1\big) + \sum_{l=1}^{k}\mathbb{E}\Big[\mathbf{1}_{\big\{N_{s-1,l}^{(0)} < m^{1/3}\big\}}\Big|\mathcal{F}_{s-1}\Big]O\big(1\big) \\
= \;& D_k\epsilon - p\bigg\{(k-1)\frac{p}{(1-p)^2} + 1\bigg\}\frac{N_{s-1}^{(1)}}{n-s} + O\bigg(\frac{m^{1/3}}{n} + m^{1/3}(kp)^{m^{1/3}} + \epsilon^2\bigg) \\
&+\frac{N_{s-1}^{(\geq 2)} + A_{s-1}}{n}O\big(1\big) + \frac{\big(N_{s-1}^{(\geq 1)} + A_{s-1}\big)^2}{n^2}O\big(1\big) \\
&+\mathbb{E}\Big[\mathbf{1}_{\big\{N_{s}^{(0)} < m^{1/3}\big\}}\Big|\mathcal{F}_{s-1}\Big]O\big(1\big). \qquad (5.18)
\end{aligned}
$$

In the last line we used the fact that $N_{s-1,l}^{(0)} > N_s^{(0)}$. Furthermore using Lemmas 4, 7, 9, we get for a large enough n,

$$
\begin{aligned}
\mathbb{E}\xi_s^{(m)} \;=\;& D_k\epsilon - kp(1-p)\bigg\{(k-1)\frac{p}{(1-p)^2} + 1\bigg\}\frac{s}{n-s} \\
&+O\bigg(\frac{m^{1/3}}{n} + m^{1/3}(kp)^{m^{1/3}} + \frac{\epsilon n^{2/3} + n^{1/3}}{n} + \epsilon^2\bigg) \\
=\;& D_k\epsilon - kp(1-p)\bigg\{(k-1)\frac{p}{(1-p)^2} + 1\bigg\}\frac{s}{n-s} \\
&+O\big(\epsilon^2 + \epsilon n^{-1/3} + n^{-2/3}\big). \qquad (5.19)
\end{aligned}
$$

Lemma 11. *If n is large enough, then for all $s \leq s_0 n^{2/3}$*

(i) $\mathbb{E}\xi_s^{(m)} - D_k\epsilon + kp(1-p)\Big\{(k-1)\frac{p}{(1-p)^2} + 1\Big\}\frac{s}{n-s} = O\big(\epsilon^2 + \epsilon n^{-1/3} + n^{-2/3}\big)$.

(ii) $\mathbb{E}\Big|\mathbb{E}\big[\xi_s^{(m)}\big|\mathcal{F}_{s-1}\big] - \mathbb{E}\xi_s^{(m)}\Big| = O\big(\epsilon^2 + \epsilon n^{-1/3} + n^{-2/3}\big)$.

(iii) $\mathbb{E}\big[\big(\xi_s^{(m)}\big)^2\big|\mathcal{F}_{s-1}\big] = 2(k - \sqrt{k^2-k})$
$+ O\big(\epsilon + n^{-2/3}\big) + \big(N_{s-1}^{(\geq 1)} + A_{s-1}\big)O\big(\frac{1}{n}\big) + \mathbb{E}\Big[\mathbf{1}_{\big\{N_s^{(0)} < m^{1/3}\big\}}\Big|\mathcal{F}_{s-1}\Big]O\big(1\big)$.

Proof. (i) has already been confirmed. Also, by (5.18) and Lemmas 3, 7, 9, 10,

$$\begin{aligned}
&\mathbb{E}\Big|\mathbb{E}\big[\xi_s^{(m)}\big|\mathcal{F}_{s-1}\big] - \mathbb{E}\xi_s^{(m)}\Big| \\
&\quad = p\bigg\{(k-1)\frac{p}{(1-p)^2}+1\bigg\}\frac{1}{n-s}\mathbb{E}\Big|N_{s-1}^{(1)} - \mathbb{E}N_{s-1}^{(1)}\Big| \\
&\qquad + O\bigg(\frac{m^{1/3}}{n} + m^{1/3}(kp)^{m^{1/3}} + \frac{\epsilon n^{2/3}+n^{1/3}}{n} + \epsilon^2\bigg) \\
&\quad = O\big(\epsilon^2 + \epsilon n^{-1/3} + n^{-2/3}\big),
\end{aligned}$$

which proves (ii).

To prove the (iii), we consider $\big(\xi_s^{(m)}+1\big)^2$. We get

$$\begin{aligned}
&\mathbb{E}\Big[\big(\xi_s^{(m)}+1\big)^2\Big|\mathcal{F}_{s-1}\Big] \\
&= 2\sum_{2\le l<l'\le k}\mathbb{E}\Big[\hat{r}'_{s,l}\mathbf{1}_{\{\hat{r}'_{s,l}\le m^{1/3}\}}\hat{r}'_{s,l'}\mathbf{1}_{\{\hat{r}'_{s,l'}\le m^{1/3}\}}\Big|\mathcal{F}_{s-1}\Big] \\
&+\sum_{l=2}^{k}\mathbb{E}\Big[\big(\hat{r}'_{s,l}\big)^2\mathbf{1}_{\{\hat{r}'_{s,l}\le m^{1/3}\}}\Big|\mathcal{F}_{s-1}\Big] \\
&+\mathbb{E}\Bigg[\bigg\{\sum_{v\in\mathcal{N}_{s-1,k}^{(0)}}\mathbf{1}_{\{w_s\leftarrow v,\text{ open}\}}\mathbf{1}_{\{\sum_{v\in\mathcal{N}_{s-1,k}^{(0)}}\mathbf{1}_{\{w_s\leftarrow v,\text{ open}\}}\le m^{1/3}\}}\bigg\}^2\Bigg|\mathcal{F}_{s-1}\Bigg] \\
&+2\mathbb{E}\Bigg[\sum_{v\in\mathcal{N}_{s-1,k}^{(0)}}\mathbf{1}_{\{w_s\leftarrow v,\text{ open}\}}\mathbf{1}_{\{\sum_{v\in\mathcal{N}_{s-1,k}^{(0)}}\mathbf{1}_{\{w_s\leftarrow v,\text{ open}\}}\le m^{1/3}\}} \\
&\times\sum_{j=1}^{k}\sum_{v\in\mathcal{N}_{s-1,k}^{(j)}}\mathbf{1}_{\{w_s\leftarrow v,\text{ open}\}}\mathbf{1}_{\{\sum_{v\in\mathcal{N}_{s-1,k}^{(j)}}\mathbf{1}_{\{w_s\leftarrow v,\text{ open}\}}\le m^{1/3}\}}\Bigg|\mathcal{F}_{s-1}\Bigg] \\
&+\mathbb{E}\Bigg[\bigg\{\sum_{j=1}^{k}\sum_{v\in\mathcal{N}_{s-1,k}^{(j)}}\mathbf{1}_{\{w_s\leftarrow v,\text{ open}\}}\mathbf{1}_{\{\sum_{v\in\mathcal{N}_{s-1,k}^{(j)}}\mathbf{1}_{\{w_s\leftarrow v,\text{ open}\}}\le m^{1/3}\}}\bigg\}^2\Bigg| \\
&\quad\mathcal{F}_{s-1}\Bigg]
\end{aligned}$$

$$+2\sum_{l=2}^{k}\mathbb{E}\Big[\hat{r}'_{s,l}\mathbf{1}_{\{\hat{r}'_{s,l}\le m^{1/3}\}}$$
$$\times\sum_{j=0}^{k}\sum_{v\in\mathcal{N}^{(j)}_{s-1,k}}\mathbf{1}_{\{w_s\leftarrow v,\ \text{open}\}}\mathbf{1}_{\{\sum_{v\in\mathcal{N}^{(j)}_{s-1,k}}\mathbf{1}_{\{w_s\leftarrow v,\ \text{open}\}}\le m^{1/3}\}}\Big|\mathcal{F}_{s-1}\Big].$$

We analyze each of them, similarly to the calculation of $\mathbb{E}\big[\xi_s\big|\mathcal{F}_{s-1}\big]$ in the proof of Lemma 1. For $l<l'$, we have

$$\begin{aligned}
&\mathbb{E}\Big[\hat{r}'_{s,l}\mathbf{1}_{\{\hat{r}'_{s,l}\le m^{1/3}\}}\hat{r}'_{s,l'}\mathbf{1}_{\{\hat{r}'_{s,l'}\le m^{1/3}\}}\Big|\mathcal{F}_{s-1,l'-1}\Big]\\
=\ &\hat{r}'_{s,l}\mathbf{1}_{\{\hat{r}'_{s,l}\le m^{1/3}\}}\bigg\{\frac{p}{1-p}+O\big(n^{-2/3}\big)+\frac{N^{(\ge1)}_{s-1,l'-1}+A_{s-1,l'-1}}{n}O\big(1\big)\\
&+\mathbf{1}_{\{N^{(0)}_{s-1,l'-1}<m^{1/3}\}}O\big(1\big)\bigg\},
\end{aligned}$$

thus

$$\begin{aligned}
&\mathbb{E}\Big[\hat{r}'_{s,l}\mathbf{1}_{\{\hat{r}'_{s,l}\le m^{1/3}\}}\hat{r}'_{s,l'}\mathbf{1}_{\{\hat{r}'_{s,l'}\le m^{1/3}\}}\Big|\mathcal{F}_{s-1}\Big]\\
=\ &\frac{p^2}{\big(1-p\big)^2}+O\big(n^{-2/3}\big)+\frac{N^{(\ge1)}_{s-1}+A_{s-1}}{n}O\big(1\big)\\
&+\mathbb{E}\Big[\mathbf{1}_{\{N^{(0)}_{s-1,l-1}<m^{1/3}\}}\Big|\mathcal{F}_{s-1}\Big]O\big(1\big)+\mathbb{E}\Big[\mathbf{1}_{\{N^{(0)}_{s-1,l'-1}<m^{1/3}\}}\Big|\mathcal{F}_{s-1}\Big]O\big(1\big).
\end{aligned}$$

Next,

$$\begin{aligned}
&\mathbb{E}\Big[\big(\hat{r}'_{s,l}\big)^2\mathbf{1}_{\{\hat{r}'_{s,l}\le m^{1/3}\}}\Big|\mathcal{F}_{s-1}\Big]=\mathbb{E}\Bigg[\sum_{x=0}^{N^{(0)}_{s-1,l-1}\wedge m^{1/3}}x^2\mathbf{1}_{\{\hat{r}'_{s,l}=x\}}\Bigg|\mathcal{F}_{s-1}\Bigg]\\
=\ &\frac{p(1+p)}{(1-p)^2}+O\big(n^{-2/3}\big)+\frac{N^{(\ge1)}_{s-1}+A_{s-1}}{n}O\big(1\big)\\
&+\mathbb{E}\Big[\mathbf{1}_{\{N^{(0)}_{s-1,l-1}<m^{1/3}\}}\Big|\mathcal{F}_{s-1}\Big]O\big(1\big).
\end{aligned}$$

On the other hand, for a large enough n,

$$\begin{aligned}
&\mathbb{E}\Bigg[\bigg\{\sum_{v\in\mathcal{N}^{(0)}_{s-1,k}}\mathbf{1}_{\left\{w_s\leftarrow v,\text{ open}\right\}}\mathbf{1}_{\left\{\sum_{v\in\mathcal{N}^{(0)}_{s-1,k}}\mathbf{1}_{\left\{w_s\leftarrow v,\text{ open}\right\}}\leq m^{1/3}\right\}}\bigg\}^2\bigg|\mathcal{F}_{s-1,k}\Bigg]\\
&= \mathbb{E}\Bigg[\sum_{v,v'\in\mathcal{N}^{(0)}_{s-1,k},v\neq v'}\mathbf{1}_{\left\{w_s\leftarrow v,\text{ open}\right\}}\mathbf{1}_{\left\{w_s\leftarrow v',\text{ open}\right\}}\\
&\quad\times\mathbf{1}_{\left\{\sum_{v\in\mathcal{N}^{(0)}_{s-1,k}}\mathbf{1}_{\left\{w_s\leftarrow v,\text{ open}\right\}}\leq m^{1/3}\right\}}\bigg|\mathcal{F}_{s-1,k}\Bigg]\\
&\quad+\mathbb{E}\Bigg[\sum_{v\in\mathcal{N}^{(0)}_{s-1,k}}\mathbf{1}_{\left\{w_s\leftarrow v,\text{ open}\right\}}\mathbf{1}_{\left\{\sum_{v\in\mathcal{N}^{(0)}_{s-1,k}}\mathbf{1}_{\left\{w_s\leftarrow v,\text{ open}\right\}}\leq m^{1/3}\right\}}\bigg|\mathcal{F}_{s-1,k}\Bigg]\\
&= N^{(0)}_{s-1,k}\big(N^{(0)}_{s-1,k}-1\big)\frac{(kp)^2}{(n-s)^2}+N^{(0)}_{s-1,k}\frac{kp}{n-s}+O\big(n^{-2/3}\big)\\
&\quad+\mathbf{1}_{\left\{N^{(0)}_{s-1,k}<m^{1/3}\right\}}O\big(1\big),
\end{aligned}$$

thus

$$\begin{aligned}
&\mathbb{E}\Bigg[\bigg\{\sum_{v\in\mathcal{N}^{(0)}_{s-1,k}}\mathbf{1}_{\left\{w_s\leftarrow v,\text{ open}\right\}}\mathbf{1}_{\left\{\sum_{v\in\mathcal{N}^{(0)}_{s-1,k}}\mathbf{1}_{\left\{w_s\leftarrow v,\text{ open}\right\}}\leq m^{1/3}\right\}}\bigg\}^2\bigg|\mathcal{F}_{s-1}\Bigg]\\
&= k^2p^2+kp+O\big(n^{-2/3}\big)+\frac{N^{(\geq 1)}_{s-1}+A_{s-1}}{n}O\big(1\big)\\
&\quad+\mathbb{E}\Big[\mathbf{1}_{\left\{N^{(0)}_{s-1,k}<m^{1/3}\right\}}\Big|\mathcal{F}_{s-1}\Big]O\big(1\big)
\end{aligned}$$

Similarly we have

$$
\begin{aligned}
&2\mathbb{E}\Bigg[\sum_{v\in\mathcal{N}_{s-1,k}^{(0)}}\mathbf{1}_{\left\{w_s\leftarrow v,\ \text{open}\right\}}\mathbf{1}_{\left\{\sum_{j=0}^{k}\sum_{v\in\mathcal{N}_{s-1,k}^{(0)}}\mathbf{1}_{\left\{w_s\leftarrow v,\ \text{open}\right\}}\leq m^{1/3}\right\}}\\
&\times\sum_{j=1}^{k}\sum_{v\in\mathcal{N}_{s-1,k}^{(j)}}\mathbf{1}_{\left\{w_s\leftarrow v,\ \text{open}\right\}}\mathbf{1}_{\left\{\sum_{j=0}^{k}\sum_{v\in\mathcal{N}_{s-1,k}^{(j)}}\mathbf{1}_{\left\{w_s\leftarrow v,\ \text{open}\right\}}\leq m^{1/3}\right\}}\Bigg|\mathcal{F}_{s-1}\Bigg]\\
&+\mathbb{E}\Bigg[\Bigg\{\sum_{j=1}^{k}\sum_{v\in\mathcal{N}_{s-1,k}^{(j)}}\mathbf{1}_{\left\{w_s\leftarrow v,\ \text{open}\right\}}\mathbf{1}_{\left\{\sum_{j=0}^{k}\sum_{v\in\mathcal{N}_{s-1,k}^{(j)}}\mathbf{1}_{\left\{w_s\leftarrow v,\ \text{open}\right\}}\leq m^{1/3}\right\}}\Bigg\}^2\Bigg|\\
&\quad\mathcal{F}_{s-1}\Bigg]\\
&\qquad=\frac{N_{s-1}^{(\geq 1)}}{n}O(1).
\end{aligned}
$$

Finally,

$$
\begin{aligned}
&\mathbb{E}\Bigg[\hat{r}'_{s,l}\mathbf{1}_{\left\{\hat{r}'_{s,l}\leq m^{1/3}\right\}}\\
&\times\sum_{j=0}^{k}\sum_{v\in\mathcal{N}_{s-1,k}^{(j)}}\mathbf{1}_{\left\{w_s\leftarrow v,\ \text{open}\right\}}\mathbf{1}_{\left\{\sum_{j=0}^{k}\sum_{v\in\mathcal{N}_{s-1,k}^{(j)}}\mathbf{1}_{\left\{w_s\leftarrow v,\ \text{open}\right\}}\leq m^{1/3}\right\}}\Bigg|\mathcal{F}_{s-1}\Bigg]\\
&=\quad\frac{p}{1-p}kp+O\big(n^{-2/3}\big)+\frac{N_{s-1}^{(\geq 1)}+A_{s-1}}{n}O(1)\\
&\qquad+\mathbb{E}\Big[\mathbf{1}_{\left\{N_{s-1,l-1}^{(0)}<m^{1/3}\right\}}\Big|\mathcal{F}_{s-1}\Big]O(1)+\mathbb{E}\Big[\mathbf{1}_{\left\{N_{s-1,k}^{(0)}<m^{1/3}\right\}}\Big|\mathcal{F}_{s-1}\Big]O(1).
\end{aligned}
$$

Therefore we have

$$\begin{aligned}
&\mathbb{E}\big[\big(\xi_s^{(m)}+1\big)^2\big|\mathcal{F}_{s-1}\big] \\
&= (k-1)(k-2)\frac{p^2}{(1-p)^2}+(k-1)\frac{p(1+p)}{(1-p)^2}+k^2p^2+kp \\
&\quad +2(k-1)\frac{p}{1-p}kp \\
&\quad +O\big(n^{-2/3}\big)+\frac{N_{s-1}^{(\geq 1)}+A_{s-1}}{n}O(1)+\sum_{l=1}^{k}\mathbb{E}\Big[\mathbf{1}_{\{N_{s,l}^{(0)}<m^{1/3}\}}\Big|\mathcal{F}_{s-1}\Big]O(1) \\
&= 2(k-\sqrt{k^2-k})+1+O\big(\epsilon+n^{-2/3}\big)+\frac{N_{s-1}^{(\geq 1)}+A_{s-1}}{n}O(1) \\
&\quad +\mathbb{E}\Big[\mathbf{1}_{\{N_s^{(0)}<m^{1/3}\}}\Big|\mathcal{F}_{s-1}\Big]O(1),
\end{aligned}$$

where we used $N_{s-1,l}^{(0)} > N_s^{(0)}$ again. Also

$$\mathbb{E}\big[\big(\xi_s^{(m)}+1\big)^2\big|\mathcal{F}_{s-1}\big]=\mathbb{E}\big[\big(\xi_s^{(m)}\big)^2\big|\mathcal{F}_{s-1}\big]+2\mathbb{E}\big[\xi_s^{(m)}\big|\mathcal{F}_{s-1}\big]+1,$$

so we obtain (iii). □

Lemma 12. *Let λ be in $\mathbb{R}$ and $\epsilon=\lambda n^{-1/3}$. Then for $t\in[0,1]$,*

$$\begin{aligned}
m^{-1/2}X_{\lfloor mt\rfloor}^{(m)} \quad &\overset{d}{\Longrightarrow} \quad \mathcal{B}\Big(2(k-\sqrt{k^2-k})t\Big) \\
&\quad +2\big(k-\sqrt{k^2-k}\big)\lambda\sqrt{s_0}t-\Big(1-\big(k-\sqrt{k^2-k}\big)^2\Big)\frac{s_0^{3/2}t^2}{2}
\end{aligned}$$

in $D[0,1]$.

Proof. By definition, the condition 1 of ([11] Chapter7, Theorem 7.2.) is satisfied. Also by Lemmas 3, 4 and the part (iii) of Lemma 11,

$$\mathbb{E}\big[\big(\xi_s^{(m)}\big)^2\big|\mathcal{F}_{s-1}\big]\to 2(k-\sqrt{k^2-k}) \qquad \text{i.p.}$$

for $1\leq s\leq\lfloor mt\rfloor$, thus

$$m^{-1}\sum_{s=1}^{\lfloor mt\rfloor}\mathbb{E}\Big[\Big(\xi_s^{(m)}-\mathbb{E}\big[\xi_s^{(m)}\big|\mathcal{F}_{s-1}\big]\Big)^2\Big|\mathcal{F}_{s-1}\Big]\to 2(k-\sqrt{k^2-k})t \qquad \text{i.p.},$$

it means that the condition 2 is satisfied. Therefore

$$m^{-1/2}\sum_{s=1}^{\lfloor mt\rfloor}\Big\{\xi_s^{(m)}-\mathbb{E}\big[\xi_s^{(m)}\big|\mathcal{F}_{s-1}\big]\Big\}\overset{\mathrm{d}}{\Longrightarrow}\mathcal{B}\Big(2(k-\sqrt{k^2-k})t\Big).$$

On the other hand,

$$\begin{aligned}m^{-1/2}\sum_{s=1}^{\lfloor mt\rfloor}\mathbb{E}\big[\xi_s^{(m)}\big|\mathcal{F}_{t-1}\big] &= m^{-1/2}\sum_{s=1}^{\lfloor mt\rfloor}\mathbb{E}\big[\xi_s^{(m)}\big]\\ &\quad+m^{-1/2}\sum_{s=1}^{\lfloor mt\rfloor}\Big\{\mathbb{E}\big[\xi_s^{(m)}\big|\mathcal{F}_{s-1}\big]-\mathbb{E}\big[\xi_s^{(m)}\big]\Big\}.\end{aligned}$$

Here, by the part (i) of Lemma 11,

$$\begin{aligned}&m^{-1/2}\sum_{s=1}^{\lfloor mt\rfloor}\mathbb{E}\xi_s^{(m)}\\ &= m^{-1/2}\sum_{s=1}^{\lfloor mt\rfloor}\bigg[D_k\epsilon-kp(1-p)\Big\{(k-1)\frac{p}{(1-p)^2}+1\Big\}\frac{s}{n}\\ &\quad+O\big(\epsilon^2+\epsilon n^{-1/3}+n^{-2/3}\big)\bigg]\\ &= m^{-1/2}\bigg[D_k\epsilon mt-kp(1-p)\Big\{(k-1)\frac{p}{(1-p)^2}+1\Big\}\frac{m^2t^2}{2n}\bigg]+O\big(n^{-1/3}\big)\\ &= \frac{1}{\sqrt{s_0}}\bigg[D_k\lambda s_0 t-kp\Big\{(k-1)\frac{p}{1-p}+1-p\Big\}\frac{s_0^2t^2}{2}\bigg]+O\big(n^{-1/3}\big)\\ &= 2\big(k-\sqrt{k^2-k}\big)\lambda\sqrt{s_0}t-\Big(1-\big(k-\sqrt{k^2-k}\big)^2\Big)\frac{s_0^{3/2}t^2}{2}+O\big(n^{-1/3}\big)\\ &\to 2\big(k-\sqrt{k^2-k}\big)\lambda\sqrt{s_0}t-\Big(1-\big(k-\sqrt{k^2-k}\big)^2\Big)\frac{s_0^{3/2}t^2}{2}.\end{aligned}$$

Also, by the part (ii) of Lemma 11,

$$\begin{aligned}&\mathbb{E}\bigg|m^{-1/2}\sum_{s=1}^{\lfloor mt\rfloor}\Big\{\mathbb{E}\big[\xi_s^{(m)}\big|\mathcal{F}_{s-1}\big]-\mathbb{E}\xi_s^{(m)}\Big\}\bigg|\\ &\qquad\le m^{-1/2}\sum_{s=1}^{\lfloor mt\rfloor}O\big(\epsilon^2+\epsilon n^{-1/3}+n^{-2/3}\big)\\ &\qquad\to 0.\end{aligned}$$

Therefore,

$$m^{-1/2}X^{(m)}_{\lfloor mt\rfloor} \quad \overset{\mathrm{d}}{\Longrightarrow} \quad \mathcal{B}\Big(\big(k-\sqrt{k^2-k}+1\big)t\Big) \\ +2\big(k-\sqrt{k^2-k}\big)\lambda\sqrt{s_0}t-\Big(1-\big(k-\sqrt{k^2-k}\big)^2\Big)\frac{s_0^{3/2}t^2}{2}.$$

□

Next, we show the convergence of $m^{-1/2}\sum_{s=1}^{\lfloor mt\rfloor}\mathbf{1}_{\left\{A^{(2)}_{s-1}>0,A^{(\geq 3)}_{s-1}=0\right\}}\hat{r}'_{s,2}$.

Lemma 13. *Let λ be in $\mathbb{R}$ and $\epsilon=\lambda n^{-1/3}$. Then for $t\in[0,1]$,*

$$m^{-1/2}\sum_{s=1}^{\lfloor mt\rfloor}\mathbf{1}_{\left\{A^{(2)}_{s-1}>0,A^{(\geq 3)}_{s-1}=0\right\}}\hat{r}'_{s,2}\to\frac{k-1}{k}\big(k-\sqrt{k^2-k}\big)^2\frac{s_0^{3/2}t^2}{2},\qquad i.p.$$

as $n\to\infty$.

Proof. Let $h_s=\mathbf{1}_{\left\{A^{(2)}_{s-1}>0,A^{(\geq 3)}_{s-1}=0\right\}}\hat{r}'_{s,2}$ for $1\leq s\leq\lfloor mt\rfloor$. We have

$$\begin{aligned}\mathbb{E}\Bigg|m^{-1/2}\sum_{s=1}^{\lfloor mt\rfloor}\mathbf{1}_{\left\{A^{(2)}_{s-1}>0,A^{(\geq 3)}_{s-1}=0\right\}}\hat{r}'_{s,2}-m^{-1/2}\sum_{s=1}^{\lfloor mt\rfloor}\mathbb{E}\Big[\mathbf{1}_{\left\{A^{(2)}_{s-1}>0,A^{(\geq 3)}_{s-1}=0\right\}}\hat{r}'_{s,2}\Big]\Bigg| \\ \leq\ \mathbb{E}\Bigg|m^{-1/2}\sum_{s=1}^{\lfloor mt\rfloor}h_s-m^{-1/2}\sum_{s=1}^{\lfloor mt\rfloor}\mathbb{E}\big[h_s\big|\mathcal{F}_{s-2}\big]\Bigg| \\ +\mathbb{E}\Bigg|m^{-1/2}\sum_{s=1}^{\lfloor mt\rfloor}\mathbb{E}\big[h_s\big|\mathcal{F}_{s-2}\big]-m^{-1/2}\sum_{s=1}^{\lfloor mt\rfloor}\mathbb{E}h_s\Bigg|.\end{aligned}$$

First, by Lemmas 4, 8 and (5.9) imply that

$$\mathbb{E}h_s^2=O\big(n^{-1/3}\big)$$

for $\frac{1}{I(2k)}\log n\leq s\leq\lfloor mt\rfloor$. Also,

$$\sum_{s=1}^{\frac{1}{I(2k)}\log n}\mathbb{E}h_s^2=O\big(\log n\big). \tag{5.20}$$

Therefore

$$
\begin{aligned}
&\mathbb{E}\Big|m^{-1/2}\sum_{s=1}^{\lfloor mt\rfloor} h_s - m^{-1/2}\sum_{s=1}^{\lfloor mt\rfloor}\mathbb{E}\big[h_s\big|\mathcal{F}_{s-2}\big]\Big|^2 \\
&\le\ 2m^{-1}\sum_{s=3}^{\lfloor mt\rfloor}\mathbb{E}\Big[\Big\{h_{s-1}-\mathbb{E}\big[h_{s-1}\big|\mathcal{F}_{s-3}\big]\Big\}\Big\{h_s-\mathbb{E}\big[h_s\big|\mathcal{F}_{s-2}\big]\Big\}\Big] \\
&\quad +m^{-1}\sum_{s=2}^{\lfloor mt\rfloor}\mathbb{E}\Big[\Big\{h_s-\mathbb{E}\big[h_s\big|\mathcal{F}_{s-2}\big]\Big\}^2\Big] \\
&=\ O\big(n^{-1/3}+n^{-2/3}\log n\big)\ =\ O\big(n^{-1/3}\big). \qquad (5.21)
\end{aligned}
$$

Next, from (4.7) in the proof of Lemma 1,

$$
\begin{aligned}
\mathbb{E}\Big[h_s\Big|\mathcal{F}_{s-1}\Big] \ =\ & \mathbf{1}_{\left\{A_{s-1}^{(2)}>0,A_{s-1}^{(\ge3)}=0\right\}}\Bigg\{\frac{p}{1-p}+\frac{N_{s-1}^{(\ge1)}+A_{s-1}}{n}O(1) \\
&+\frac{\big(N_{s-1}^{(\ge1)}+A_{s-1}\big)^2}{n^2}O(1)+O\Big(\frac{m^{1/3}}{n}+m^{1/3}p^{m^{1/3}}\Big) \\
&+\mathbb{E}\Big[\mathbf{1}_{\left\{N_s^{(0)}<m^{1/3}\right\}}O(1)\Big|\mathcal{F}_{s-1}\Big]\Bigg\}.
\end{aligned}
$$

So we have

$$
\begin{aligned}
\mathbb{E}\Big[h_s\Big|\mathcal{F}_{s-2}\Big] \ =\ & \mathbb{E}\Big[\mathbf{1}_{\left\{A_{s-1}^{(2)}>0,A_{s-1}^{(\ge3)}=0\right\}}\Big|\mathcal{F}_{s-2}\Big] \\
&\times\Bigg\{\frac{p}{1-p}+\frac{N_{s-2}^{(\ge1)}+A_{s-2}}{n}O(1)+\frac{\big(N_{s-2}^{(\ge1)}+A_{s-2}\big)^2}{n^2}O(1) \\
&+O\Big(\frac{m^{1/3}}{n}+m^{1/3}p^{m^{1/3}}\Big)\Bigg\} \\
&+\mathbb{E}\Big[\mathbf{1}_{\left\{N_s^{(0)}<m^{1/3}\right\}}\Big|\mathcal{F}_{s-2}\Big]O(1).
\end{aligned}
$$

Here, recall $a_s^{(i)}$ from (5.8), and let s be $\frac{1}{I(2k)}\log n\le s\le\lfloor mt\rfloor$.

$$
\begin{aligned}
\mathbf{1}_{\left\{A_{s-1}^{(2)}>0,A_{s-1}^{(\ge3)}=0\right\}} \ =\ & \mathbf{1}_{\left\{A_{s-1}^{(2)}>0,A_{s-1}^{(\ge3)}=0,a_{s-1}^{(2)}\ge1\right\}}+\mathbf{1}_{\left\{A_{s-1}^{(2)}>0,A_{s-1}^{(\ge3)}=0,a_{s-1}^{(2)}\le0\right\}} \\
=\ & \mathbf{1}_{\left\{a_{s-1}^{(2)}\ge1\right\}}-\mathbf{1}_{\left\{A_{s-1}^{(\ge3)}>0,a_{s-1}^{(2)}\ge1\right\}} \\
&+\mathbf{1}_{\left\{A_{s-1}^{(2)}>0,A_{s-1}^{(\ge3)}=0,a_{s-1}^{(2)}\le0\right\}}
\end{aligned}
$$

Using Lemmas 4 and 7 - 9,

$$\begin{aligned}
\mathbb{E}\mathbf{1}_{\left\{A_{s-1}^{(\geq 3)}>0,a_{s-1}^{(2)}\geq 1\right\}} &\leq \mathbb{E}\mathbf{1}_{\left\{A_{s-1}^{(\geq 3)}>0\right\}} = \mathbb{E}\mathbf{1}_{\left\{a_{s-1}^{(\geq 3)}\geq 1\right\}} + O\big(n^{-2/3}\big), \\
\mathbb{E}\mathbf{1}_{\left\{A_{s-1}^{(2)}>0,A_{s-1}^{(\geq 3)}=0,a_{s-1}^{(2)}\leq 0\right\}} &\leq \mathbb{E}\mathbf{1}_{\left\{A_{s-2}^{(\geq 2)}\geq 2\right\}} = O\big(n^{-2/3}\big).
\end{aligned}$$

Furthermore, $a_{s-1}^{(\geq 3)} \geq 1$ implies that some bad vertex of the AS process is chosen from $\mathcal{N}_{s-2}^{(\geq 3)}$ or there is a directed edge from $\mathcal{N}_{s-2,k}^{(\geq 2)} \cup \mathcal{A}_{s-2,k}^{(2)}$ to w_{s-1}. So, by Lemmas 7, 9,

$$\mathbb{E}\mathbf{1}_{\left\{a_{s-1}^{(\geq 3)}\geq 1\right\}} = O\big(n^{-2/3}\big). \tag{5.22}$$

We again use Lemmas 3, 4, 8 and above facts to obtain

$$\begin{aligned}
\mathbb{E}h_s &= \mathbb{E}\left[\left\{\frac{p}{1-p} + \frac{N_{s-1}^{(1)} + A_{s-1}}{n} O\big(1\big)\right\}\mathbf{1}_{\left\{a_{s-1}^{(2)}\geq 1\right\}}\right] + O\big(n^{-2/3}\big) \\
&= \frac{p}{1-p}\mathbb{E}\mathbf{1}_{\left\{a_{s-1}^{(2)}\geq 1\right\}} + O\big(n^{-2/3}\big). \qquad (5.23)
\end{aligned}$$

Now

$$\begin{aligned}
\mathbf{1}_{\left\{a_{s-1}^{(2)}\geq 1\right\}} &= \mathbf{1}_{\left\{a_{s-1}^{(2)}=1\right\}} + \mathbf{1}_{\left\{a_{s-1}^{(2)}\geq 2\right\}} \\
&= \mathbf{1}_{\left\{A_{s-2,k}^{(2)}-A_{s-2,0}^{(2)}=0,A_{s-2,k+1}^{(2)}-A_{s-2,k}^{(2)}=1\right\}} \\
&\quad +\mathbf{1}_{\left\{A_{s-2,k}^{(2)}-A_{s-2,0}^{(2)}\geq 1,a_{s-1}^{(2)}=1\right\}} + \mathbf{1}_{\left\{a_{s-1}^{(2)}\geq 2\right\}} \\
&= \mathbf{1}_{\left\{A_{s-2,k+1}^{(2)}-A_{s-2,k}^{(2)}=1\right\}} - \mathbf{1}_{\left\{A_{s-2,k}^{(2)}-A_{s-2,0}^{(2)}\geq 1,A_{s-2,k+1}^{(2)}-A_{s-2,k}^{(2)}=1\right\}} \\
&\quad +\mathbf{1}_{\left\{A_{s-2,k}^{(2)}-A_{s-2,0}^{(2)}\geq 1,a_{s-1}^{(2)}=1\right\}} + \mathbf{1}_{\left\{a_{s-1}^{(2)}\geq 2\right\}}.
\end{aligned}$$

$A_{s-2,k}^{(2)} - A_{s-2,0}^{(2)} \geq 1$ implies that some bad vertex of the AS process is chosen from $\mathcal{N}_{s-2}^{(2)}$. So, by Lemmas 8, 9,

$$\mathbb{E}\mathbf{1}_{\left\{a_{s-1}^{(2)}\geq 1\right\}} = \mathbb{E}\mathbf{1}_{\left\{A_{s-2,k+1}^{(2)}-A_{s-2,k}^{(2)}=1\right\}} + O\big(n^{-2/3}\big).$$

Furthermore,

$$
\begin{aligned}
&\mathbf{1}_{\left\{A^{(2)}_{s-2,k+1}-A^{(2)}_{s-2,k}=1\right\}}\\
&\quad= \mathbf{1}_{\left\{\forall v\in\mathcal{A}^{(2)}_{s-2,k},w_{s-1}\not\leftarrow v,A^{(2)}_{s-2,k+1}-A^{(2)}_{s-2,k}=1\right\}}\\
&\qquad+\mathbf{1}_{\left\{\exists v\in\mathcal{A}^{(2)}_{s-2,k},w_{s-1}\leftarrow v,A^{(2)}_{s-2,k+1}-A^{(2)}_{s-2,k}=1\right\}}\\
&\quad= \mathbf{1}_{\left\{\forall v\in\mathcal{A}^{(2)}_{s-2,k},w_{s-1}\not\leftarrow v,A^{(2)}_{s-2,k+1}-A^{(2)}_{s-2,k}=1\right\}}\mathbf{1}_{\left\{N^{(0)}_{s-2,k}\ge n-t-2kt\right\}}\\
&\qquad+\mathbf{1}_{\left\{\forall v\in\mathcal{A}^{(2)}_{s-2,k},w_{s-1}\not\leftarrow v,A^{(2)}_{s-2,k+1}-A^{(2)}_{s-2,k}=1\right\}}\mathbf{1}_{\left\{N^{(0)}_{s-2,k}< n-t-2kt\right\}}\\
&\qquad+\mathbf{1}_{\left\{\exists v\in\mathcal{A}^{(2)}_{s-2,k},w_{s-1}\leftarrow v,A^{(2)}_{s-2,k+1}-A^{(2)}_{s-2,k}=1\right\}}.
\end{aligned}
$$

So,

$$
\begin{aligned}
&\mathbb{E}\Big[\mathbf{1}_{\left\{A^{(2)}_{s-2,k+1}-A^{(2)}_{s-2,k}=1\right\}}\Big|\mathcal{F}_{s-2,k}\Big]\\
&\quad= N^{(1)}_{s-2,k}\frac{(k-1)p}{n-s+1}\Big(1+O\big(n^{-1/3}\big)\Big)\mathbf{1}_{\left\{N^{(0)}_{s-2,k}\ge n-t-2kt\right\}}\\
&\qquad+\mathbf{1}_{\left\{N^{(0)}_{s-2,k}<n-t-2kt\right\}}O\big(1\big)+\frac{A^{(2)}_{s-2,k}}{n}O\big(1\big).
\end{aligned}
\tag{5.24}
$$

By Lemmas 3, 4, and (5.23), (5.24), we have

$$
\mathbb{E}h_s=\mathbb{E}\left[\frac{p}{1-p}\frac{(k-1)p}{n-s+1}N^{(1)}_{s-2}\right]+O\big(n^{-2/3}\big). \tag{5.25}
$$

Therefore, by Lemma 10 and (5.20),

$$
\begin{aligned}
&\mathbb{E}\left|m^{-1/2}\sum_{s=1}^{\lfloor mt\rfloor}\mathbb{E}\big[h_s\big|\mathcal{F}_{s-2}\big]-m^{-1/2}\sum_{s=1}^{\lfloor mt\rfloor}\mathbb{E}h_s\right|\\
&\quad\le m^{-1/2}\sum_{s=\frac{1}{I(2k)}\log n}^{\lfloor mt\rfloor}\frac{p}{1-p}\frac{(k-1)p}{n-s+1}\mathbb{E}\left|N^{(1)}_{s-2}-\mathbb{E}N^{(1)}_{s-2}\right|+O\big(n^{-1/3}\log n\big)\\
&\quad= O\big(n^{-1/3}\log n\big).
\end{aligned}
\tag{5.26}
$$

Using (5.21) and (5.26), we get

$$
\begin{aligned}
&\mathbb{E}\Bigg|m^{-1/2}\sum_{s=1}^{\lfloor mt\rfloor}\mathbf{1}_{\left\{A^{(2)}_{s-1}>0,A^{(\ge 3)}_{s-1}=0\right\}}\hat{r}'_{s,2}\\
&\qquad-m^{-1/2}\sum_{s=1}^{\lfloor mt\rfloor}\mathbb{E}\Big[\mathbf{1}_{\left\{A^{(2)}_{s-1}>0,A^{(\ge 3)}_{s-1}=0\right\}}\hat{r}'_{s,2}\Big]\Bigg|\to 0,
\end{aligned}
$$

as $n \to \infty$. So, if we get the convergence of $m^{-1/2} \sum_{s=1}^{\lfloor mt \rfloor} \mathbb{E} h_s$, then the proof of this lemma is complete. However, by (5.25) and Lemma 9,

$$\begin{aligned}
\mathbb{E} h_s &= \frac{p}{1-p} \frac{(k-1)p}{n-s+1} k(1-p)s + O\big(n^{-2/3}\big) \\
&= k(k-1)p^2 \frac{s}{n} + O\big(n^{-2/3}\big),
\end{aligned}$$

for $\frac{2}{I(2k)} \log n \le s \le \lfloor mt \rfloor$. Therefore, by this and (5.20),

$$\begin{aligned}
m^{-1/2} \sum_{s=1}^{\lfloor mt \rfloor} \mathbb{E} h_s &= k(k-1)p^2 \frac{s_0^{3/2} t^2}{2} + O\big(n^{-1/3} \log n\big) \\
&\to \frac{k-1}{k} \big(k - \sqrt{k^2 - k}\big)^2 \frac{s_0^{3/2} t^2}{2}.
\end{aligned}$$

□

Proof of Theorem 5. Recall the definition (5.16) of $\hat{X}_t$, which means that

$$\big| \hat{X}_t - X_t \big| \le N(w_{t_j+1})$$

for each $t \in [t_j, t_{j+1})$. We can couple $N(w_t)$ and a random variable κ_t distributed as $1 + Ge(1+p)$, hence we have

$$\begin{aligned}
m^{-1/2} \mathbb{E} \Big| \hat{X}_t - X_t \Big| &\le m^{-1/2} n^{1/4} \mathbb{P} \Big[\max_{1 \le s \le \lfloor m \rfloor} \kappa_s \le n^{1/4} \Big] \\
&\quad + m^{-1/2} \mathbb{E} \Big[\max_{1 \le s \le \lfloor m \rfloor} \kappa_s : \max_{1 \le s \le \lfloor m \rfloor} \kappa_s > n^{1/4} \Big] \\
&\le m^{-1/2} n^{1/4} + m^{-1/2} \sum_{x > n^{1/4}} x \mathbb{P} \Big[\max_{1 \le s \le \lfloor m \rfloor} \kappa_s \ge x \Big] \\
&\to 0
\end{aligned}$$

as $n \to \infty$ for $t \le \lfloor m \rfloor$. Therefore it suffices to prove the convergence of X_t.

Using $X_s = X_s^{(m)} + X_s - X_s^{(m)}$,

$$
\begin{aligned}
& m^{-1/2} X_{\lfloor mt \rfloor} \\
&= m^{-1/2} X_{\lfloor mt \rfloor}^{(m)} \\
&\quad + m^{-1/2} \sum_{s=1}^{\lfloor mt \rfloor} \Bigg[\sum_{l=2}^{k} \hat{r}'_{s,l} \mathbf{1}_{\left\{\hat{r}'_{s,l} > m^{1/3}\right\}} \\
&\quad + \sum_{j=0}^{k} \sum_{v \in \mathcal{N}_{s-1,k}^{(j)}} \mathbf{1}_{\left\{w_s \leftarrow v,\ \text{open}\right\}} \mathbf{1}_{\left\{\sum_{v \in \mathcal{N}_{s-1,k}^{(j)}} \mathbf{1}_{\left\{w_s \leftarrow v,\ \text{open}\right\}} > m^{1/3}\right\}} \\
&\quad - \sum_{j=2}^{k} \left\{ \mathbf{1}_{\left\{A_{s-1}^{(j)} > 0,\ A_{s-1}^{(\geq j+1)} = 0\right\}} + \mathbf{1}_{\left\{A_{s-1} = 0,\ w_t \in \mathcal{N}_{s-1}^{(j)}\right\}} \right\} \sum_{l=2}^{j} \hat{r}'_{s,l} \Bigg] \\
&= m^{-1/2} X_{\lfloor mt \rfloor}^{(m)} - m^{-1/2} \sum_{s=1}^{\lfloor mt \rfloor} \mathbf{1}_{\left\{A_{s-1}^{(2)} > 0, A_{s-1}^{(\geq 3)} = 0\right\}} \hat{r}'_{s,2} \\
&\quad + m^{-1/2} \sum_{s=1}^{\lfloor mt \rfloor} \Bigg[\sum_{l=3}^{k} \hat{r}'_{s,l} \mathbf{1}_{\left\{\hat{r}'_{s,l} > m^{1/3}\right\}} \\
&\quad + \sum_{j=0}^{k} \sum_{v \in \mathcal{N}_{s-1,k}^{(j)}} \mathbf{1}_{\left\{w_s \leftarrow v,\ \text{open}\right\}} \mathbf{1}_{\left\{\sum_{v \in \mathcal{N}_{s-1,k}^{(j)}} \mathbf{1}_{\left\{w_s \leftarrow v,\ \text{open}\right\}} > m^{1/3}\right\}} \\
&\quad - \sum_{j=3}^{k} \mathbf{1}_{\left\{A_{s-1}^{(j)} > 0,\ A_{s-1}^{(\geq j+1)} = 0\right\}} \sum_{l=2}^{j} \hat{r}'_{s,l} - \sum_{j=2}^{k} \mathbf{1}_{\left\{A_{s-1} = 0,\ w_s \in \mathcal{N}_{s-1}^{(j)}\right\}} \sum_{l=2}^{j} \hat{r}'_{s,l} \Bigg].
\end{aligned}
$$

$\hat{r}'_{s,l}$ is stochasticaly dominated by a random variable distributed as $Ge(1-p)$. Also $\sum_{v \in \mathcal{N}_{s-1,k}} \mathbf{1}_{\left\{w_s \leftarrow v,\ \text{open}\right\}}$ is stochasticaly dominated by a random variable distributed as $Bin(n-s; \frac{kp}{n-s})$. By Lemmas 4, 8 and (5.22), we have

$$
\begin{aligned}
\mathbb{E}\left[\sum_{j=3}^{k} \mathbf{1}_{\left\{A_{s-1}^{(j)} > 0,\ A_{s-1}^{(\geq j+1)} = 0\right\}} \right] &\leq \mathbb{E}\mathbf{1}_{\left\{A_{s-1}^{(\geq 3)} > 0\right\}} \\
&\leq \mathbb{E}\mathbf{1}_{\left\{a_{s-1}^{(\geq 3)} \geq 1\right\}} + \mathbb{E}\mathbf{1}_{\left\{A_{s-2}^{(\geq 3)} \geq 2\right\}} = O\left(n^{-2/3}\right),
\end{aligned}
$$

for $\frac{1}{I(2k)}\log n \le s \le \lfloor mt \rfloor$. Lemma 9 implies that

$$\mathbb{E}\Big[\sum_{j=2}^{k} \mathbf{1}_{\{A_{s-1}=0,\ w_s\in\mathcal{N}_{s-1}^{(j)}\}}\Big] \le \mathbb{E}\mathbf{1}_{\{w_s\in\mathcal{N}_{s-1}^{(\ge 2)}\}} = O\big(n^{-2/3}\big).$$

Therefore,

$$\begin{aligned}
& m^{-1/2}\sum_{s=1}^{\lfloor mt \rfloor}\Bigg[\sum_{l=3}^{k}\hat{r}'_{s,l}\mathbf{1}_{\{\hat{r}'_{s,l}>m^{1/3}\}} \\
&\quad +\sum_{j=0}^{k}\sum_{v\in\mathcal{N}_{s-1,k}^{(j)}}\mathbf{1}_{\{w_s\leftarrow v,\ \text{open}\}}\mathbf{1}_{\{\sum_{v\in\mathcal{N}_{s-1,k}^{(j)}}\mathbf{1}_{\{w_s\leftarrow v,\ \text{open}\}}>m^{1/3}\}} \\
&\quad -\sum_{j=3}^{k}\mathbf{1}_{\{A_{s-1}^{(j)}>0,\ A_{s-1}^{(\ge j+1)}=0\}}\sum_{l=2}^{j}\hat{r}'_{t,l}-\sum_{j=2}^{k}\mathbf{1}_{\{A_{s-1}=0,\ w_t\in\mathcal{N}_{s-1}^{(j)}\}}\sum_{l=2}^{j}\hat{r}'_{s,l}\Bigg] \\
& \to\ 0, \qquad \text{i.p..}
\end{aligned}$$

Therefore, combining Lemma 12 and Lemma 13, we get

$$\begin{aligned}
m^{-1/2}X_{\lfloor mt \rfloor} \quad & \overset{\mathrm{d}}{\Longrightarrow} \quad \mathcal{B}\Big(2\big(k-\sqrt{k^2-k}\big)t\Big) \\
& \qquad +2\big(k-\sqrt{k^2-k}\big)\lambda\sqrt{s_0}t \\
& \qquad -\bigg(1-\big(k-\sqrt{k^2-k}\big)^2+\frac{k-1}{k}\big(k-\sqrt{k^2-k}\big)^2\bigg)\frac{s_0^{3/2}t^2}{2} \\
& = \quad \mathcal{B}\Big(2\big(k-\sqrt{k^2-k}\big)t\Big) \\
& \qquad +2\big(k-\sqrt{k^2-k}\big)\lambda\sqrt{s_0}t-\big(\sqrt{k^2-k}-k+1\big)s_0^{3/2}t^2.
\end{aligned}$$

It means that

$$\begin{aligned}
n^{-1/3}\hat{X}_{(n^{2/3}s_0t)} \quad & \overset{\mathrm{d}}{\Longrightarrow} \quad \mathcal{B}\Big(2\big(k-\sqrt{k^2-k}\big)s_0t\Big) \\
& \qquad +2\big(k-\sqrt{k^2-k}\big)\lambda s_0t-\big(\sqrt{k^2-k}-k+1\big)s_0^2t^2.
\end{aligned}$$

□

We have proved the convergence of the exploration process, but to state the convergence of the series of large component sizes, we need a bit more work.

Lemma 14. *Let λ be in $\mathbb{R}$ and $\epsilon = \lambda n^{-1/3}$. We define $\mathcal{C}_1^{(s_0 n^{2/3})}$ by the lagest component that we started exploring after time $s_0 n^{2/3}$. Then for any $\alpha > 0$ we have*

$$\lim_{s_0\to\infty} \limsup_{n\to\infty} \mathbb{P}\Big(\big|\mathcal{C}_1^{(s_0 n^{2/3})}\big| \geq \alpha n^{2/3}\Big) = 0.$$

Proof. Recall that $D_k = 2(k - \sqrt{k^2 - k}) < 2$. Similarly to (4.29),

$$\begin{aligned}
\mathbb{E}\big[\xi_t \big| \mathcal{F}_{t-1}\big] &\leq (k-1)\frac{p}{1-p} + N_{t-1}^{(0)}\frac{kp}{n-t} + \big(n - t - N_{t-1}^{(0)}\big)\frac{(k-1)p}{n-t} - 1 \\
&= D_k\epsilon - \frac{n - t - N_{t-1}^{(0)}}{n-t}p + O\big(\epsilon^2\big) \\
&\leq 2\epsilon - \frac{n - t - N_{t-1}^{(0)}}{n-t}p
\end{aligned}$$

for a large enough n. Let some $\delta > 0$ be fixed. We define

$$\begin{aligned}
\mathcal{D} &= \Big\{N_t^{(0)} \leq n - t - \frac{k}{3}t, \quad \text{for every } t \text{ with } s_0 n^{2/3} \leq t \leq \delta n\Big\}, \\
\mathcal{D}_t &= \Big\{N_t^{(0)} \leq n - t - \frac{k}{3}t\Big\}, \quad \text{for each } t \text{ with } s_0 n^{2/3} \leq t \leq \delta n.
\end{aligned}$$

Then we have

$$\mathbb{E}\Big[\xi_t \mathbf{1}_{\{\mathcal{D}_{t-1}\}} \Big| \mathcal{F}_{t-1}\Big] \leq 2\lambda n^{-1/3} - \frac{kt}{3(n-t)}p$$

for all $t \in (s_0 n^{2/3}, \delta n]$ and a large enough n. Now if $s_0 = s_0(\delta, \lambda) > 0$ is large enough, then for all $t \in (s_0 n^{2/3}, \delta n]$,

$$\mathbb{E}\Big[\xi_t \mathbf{1}_{\{\mathcal{D}_{t-1}\}} \Big| \mathcal{F}_{t-1}\Big] \leq -\delta^{-1} n^{-1/3}. \tag{5.27}$$

Let $\hat{t}_0 > s_0 n^{2/3}$ be the first time after $s_0 n^{2/3}$ such that $A_t = 0$. We define the stopping time γ such that

$$\gamma = \min\big\{t > 0 : X_{\hat{t}_0 + t} = X_{\hat{t}_0} - N(w_{\hat{t}_0+1})\big\}.$$

We can couple $N(w_{\hat{t}_0+1})$ and a random variable κ distributed as $Ge(1-p)+1$. Let $N(w_{\hat{t}_0+1}) = d$ be fixed. We define an $(\mathcal{F}_t)$-supermartingale Q_t by $Q_t =$

$\sum_{s=1}^{t} \xi_{\hat{t}_0+s} \mathbf{1}_{\{\mathcal{D}_{\hat{t}_0+s-1}\}} + \delta^{-1} n^{-1/3} t$. By (5.27) and optional stopping theorem,

$$\begin{aligned} 0 &= \mathbb{E}\big[Q_0 \big| d\big] \\ &\geq \mathbb{E}\big[Q_{\gamma \wedge \delta n} \big| d\big] \\ &\geq \mathbb{E}\Big[\sum_{s=1}^{\gamma\wedge\delta n} \xi_{\hat{t}_0+s} - \sum_{s=1}^{\gamma\wedge\delta n} \xi_{\hat{t}_0+s} \mathbf{1}_{\{\mathcal{D}^C_{\hat{t}_0+s-1}\}} + \delta^{-1} n^{-1/3} \{\gamma \wedge \delta n\} \Big| d\Big] \\ &\geq -d + \mathbb{P}\big(\mathcal{D}^C_{\hat{t}_0}\big) O\big(n\big) + \delta^{-1} n^{-1/3} \mathbb{E}\big[\gamma \wedge \delta n \big| d\big]. \end{aligned}$$

By Lemma 4, $\mathbb{P}\big(\mathcal{D}^C_{\hat{t}_0}\big) \leq n^{-2}$. Thus we have $\mathbb{E}[\gamma \wedge \delta n | d] \leq 2\delta d n^{1/3}$ and it provides

$$\begin{aligned} \mathbb{E}\big[\gamma \wedge \delta n\big] &\leq \sum_{d=1}^{\infty} \mathbb{P}\big(N(w_{\hat{t}_0+1}) = d\big) \mathbb{E}\big[\gamma \wedge \delta n \big| d\big] \\ &\leq \sum_{d=1}^{\infty} \mathbb{P}\big(N(w_{\hat{t}_0+1}) = d\big) \cdot 2\delta d n^{1/3} \\ &\leq \frac{2}{1-p} \delta n^{1/3}. \end{aligned}$$

Also by $\gamma = |\mathcal{C}(w_{\hat{t}_0+1})|$ and Theorem 1, $\mathbb{P}(\gamma > \delta n) \leq n^{-1}$ for a large enough n. Thus

$$\begin{aligned} \mathbb{E}\gamma &\leq n\mathbb{P}(\gamma > \delta n) + \mathbb{E}\big[\gamma \mathbf{1}_{\{\gamma \leq \delta n\}}\big] \leq 1 + \mathbb{E}\big[\gamma \wedge \delta n\big] \\ &\leq \frac{3}{1-p} \delta n^{1/3} \end{aligned}$$

for a large enough $s_0 > 0$. Similar argument applies to components explored after second time. Hence $\mathbb{E}|C(v)| = O(\delta) n^{1/3}$ for any $v \in \mathcal{V} \setminus \mathcal{E}_{s_0 n^{2/3}}$. Therefore for any fixed $\alpha > 0$, we get

$$\mathbb{P}\big(|C(v)| > \alpha n^{2/3}\big) = O(\delta) n^{-1/3}.$$

Let S be the number of vertices $v \in \mathcal{V} \setminus \mathcal{E}_{s_0 n^{2/3}}$ such that $|C(v)| > \alpha n^{2/3}$. We have checked that $\mathbb{E}S \leq n\mathbb{P}(|C(v)| > \alpha n^{2/3}) = O(\delta) n^{2/3}$. Also $\big|\mathcal{C}_1^{(s_0 n^{2/3})}\big| > \alpha n^{2/3}$ implies that $S > \alpha n^{2/3}$. Hence

$$\mathbb{P}\Big(\big|\mathcal{C}_1^{(s_0 n^{2/3})}\big| > \alpha n^{2/3}\Big) \leq \mathbb{P}\big(S > \alpha n^{2/3}\big) = \frac{O(\delta) n^{2/3}}{\alpha n^{2/3}} = O(\delta).$$

Since $\delta > 0$ was arbitrary and s_0 was large enough depending only on δ and λ, this completes the proof. □

Finally, we prove Theorem 2. This can be done parallel to the proof of Theorem 5 in [2].

Proof of Theorem 2. We define for $f \in C[0,s]$,

$$\begin{aligned} \Xi &= \Big\{(r,l) \subset [0,s] : f(r) = f(l) = \min_{u \le l} f(u), \\ &\qquad \text{and } f(x) > f(r) \text{ for every } x \text{ with } r < x < l \Big\} \end{aligned}$$

and

$$\big(\mathcal{L}_1, \mathcal{L}_2, \dots\big) = \big(l_1 - r_1, l_2 - r_2, \cdots : \ l_i - r_i \ge l_{i+1} - \hat{r}_{i+1} \text{ for all } i\big),$$

i.e., $\big(\mathcal{L}_1, \mathcal{L}_2, \dots\big)$ is the decreasing sequence of the length of the excursion of $f(r) - \min_{q \le r} f(q)$. We call l an *ending point* if $(r,l) \in \Xi$ for some $0 \le r < l$. If for almost every $x \in [0,s]$, there exists $(r,l) \in \Xi$ such that $r < x < l$, we say *the function* $f \in C[0,s]$ *is good*. For $m \in \mathbb{N}$, we define the function $\phi_m : C[0,s] \to \mathbb{R}^m$ by

$$\phi_m(f) = \big(\mathcal{L}_1, \mathcal{L}_2, \dots, \mathcal{L}_m\big).$$

Now we use the following proposition in [2].

Proposition 1 (Nachmias and Peres [2])**.** *If $f \in C[0,s]$ is good, then $\phi_m(f)$ is continuous at f with respect to the $\| \cdot \|_\infty$ norm.*

A sample path of a Brownian motion is good with probability 1. By Cameron Martin Theorem, the process $\mathcal{B}^\lambda(\cdot)$ is good with probability 1 too, see [3]. Thus $\phi_m(f)$is continuous on almost every sample point of $\mathcal{B}^\lambda$. Furthermore, using Theorem 5 and Theorem 2.3 in Durret [11], Chapter 2, we have

$$n^{-2/3}\phi_m\big(\hat{X}\big) \overset{\mathrm{d}}{\Longrightarrow} \phi_m\big(\mathcal{B}^\lambda\big).$$

Thus we rescale by $n^{-2/3}$ and order the sequence of the sizes of components explored before time $sn^{2/3}$. Then this sequence converges in distribution to the ordered sequence of excursion sizes of $W^\lambda[0,s]$. By Lemma 14, we get the proof of Theorem 2. □

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