Kernel identities for van Diejen's q-difference operators and transformation formulas for multiple basic hypergeometric series

Masuda, Yasuho

(Degree)
博士（理学）
(Date of Degree)
2012-03-25
(Date of Publication)
2014-04-01
(Resource Type)
doctoral thesis
(Report Number)
甲5573
(URL)
https://hdl.handle.net/20.500.14094/D1005573

博　士　論　文

Kernel identities for van Diejen's q-difference operators
and transformation formulas for multiple basic
hypergeometric series
(van Diejen の q 差分作用素に対する核関係式と
多重 q 超幾何級数の変換公式)

平成 24 年 1 月

神戸大学大学院理学研究科

増田　恭穂

Kernel identities for van Diejen's q-difference operators and transformation formulas for multiple basic hypergeometric series

Yasuho Masuda
Department of Mathematics, Graduate School of Science, Kobe University

Abstract

In this paper, we show that the kernel function of Cauchy type for type BC intertwines the commuting family of van Diejen's q-difference operators. This result gives rise to a transformation formula for certain multiple basic hypergeometric series of type BC. We also construct a new infinite family of commuting q-difference operators for which the Koornwinder polynomials are joint eigenfunctions.

Contents

1 Introduction

In the theory of Macdonald polynomials of type A, the kernel function of Cauchy type has been used to derive various important properties of Macdonald polynomials ([Ma1, Mi1, KiN]). Kajihara's Euler transformation formula for multiple basic hypergeometric series can also be regarded as an application of the kernel function of Cauchy type [Ka].

Recently, Y. Komori, M. Noumi and J. Shiraishi in [KNS] introduced the kernel function $\Phi(x;y|q,t)$ of type BC in the variables $x=(x_1,\ldots,x_m)$ and $y=(y_1,\ldots,y_n)$ relevant to Koornwinder polynomials. The kernel function $\Phi(x;y|q,t)$ satisfies the following q-difference equation:

$$\langle t\rangle D_1^x\Phi(x;y|q,t)-\langle t\rangle\widetilde{D}_1^y\Phi(x;y|q,t)=\langle t^m\rangle\langle t^{-n}\rangle\langle\alpha^2t^{m-n-1}\rangle\Phi(x;y|q,t), \tag{1.1}$$

where $\alpha=\sqrt{abcd/q}$ and $\langle z\rangle$ is the multiplicative notation for trigonometric function

$$\langle z\rangle=z^{\frac{1}{2}}-z^{-\frac{1}{2}}=-z^{-\frac{1}{2}}(1-z). \tag{1.2}$$

In this identity, D_1^x is the Koornwinder q-difference operator in the x variables

$$D_1^x=\sum_{i=1}^m A_i(x)(T_{q,x_i}-1)+\sum_{i=1}^m A_i(x^{-1})(T_{q,x_i}^{-1}-1), \tag{1.3}$$

$$A_i(x)=\frac{\langle ax_i\rangle\langle bx_i\rangle\langle cx_i\rangle\langle dx_i\rangle}{\langle x_i^2\rangle\langle qx_i^2\rangle}\prod_{\substack{1\le j\le m\\ j\ne i}}\frac{\langle tx_ix_j\rangle\langle tx_ix_j^{-1}\rangle}{\langle x_ix_j\rangle\langle x_ix_j^{-1}\rangle}, \tag{1.4}$$

$$T_{q,x_i}f(x_1,\ldots,x_i,\ldots,x_m)=f(x_1,\ldots,qx_i,\ldots,x_m), \tag{1.5}$$

and $\widetilde{D}_1^y$ denotes the Koornwinder operator in the y variables with the parameters (a,b,c,d) replaced by $(\sqrt{tq}/a,\sqrt{tq}/b,\sqrt{tq}/c,\sqrt{tq}/d)$. In this paper, we show that $\Phi(x;y|q,t)$ intertwines the whole commuting family of van Diejen's q-difference operators, which includes the Koornwinder operator as the first member.

In Subsection 2.1, we recall some basic facts on van Diejen's q-difference operators. We also state our main result in Subsection 2.2 and prove it in Subsection 2.3. In the proof of the main result, we show a rational function identity of x variables and y variables. By using a method of principal specialization, from this identity we derive two types of transformation formulas for multiple q-series (Theorem 3.1 and Theorem 3.3). Theorem 3.3 recovers one of the C type transformation formulas, due to H. Rosengren [R]. From the special case of Theorem 3.1, we construct a family of explicit q-difference operators of "row type" for which the Koornwinder polynomials are the eigenfunctions.

Throughout the present paper, we assume that the base q is a complex number such that $0<|q|<1$. We also assume that a,b,c,d,q,t are generic complex numbers.

2 Kernel identity of Cauchy type

2.1 Van Diejen's q-difference operators

In this subsection, we recall some basic properties of the family of van Diejen's q-difference operators. For further details, we refer the reader to [vD1, Ko, KNS].

The family of van Diejen's q-difference operators $\{D_r^x(a,b,c,d|q,t)\}_{r=0}^m$ in the variables $x=(x_1,\ldots,x_m)$ is defined as follows:

$$D_r^x:=D_r^x(a,b,c,d|q,t)=\sum_{\substack{I\subset\{1,\ldots,m\}\\ 0\le|I|\le r\\ \epsilon_i=\pm1(i\in I)}}V_{\epsilon I,I^c}(x)U_{I^c,r-|I|}(x)T_{q,x}^{(I,\epsilon)}, \tag{2.1}$$

$$V_{\epsilon I,J}(x)=\prod_{i\in I}\frac{\langle ax_i^{\epsilon_i},bx_i^{\epsilon_i},cx_i^{\epsilon_i},dx_i^{\epsilon_i}\rangle}{\langle x_i^{2\epsilon_i},qx_i^{2\epsilon_i}\rangle}\prod_{\substack{i,j\in I\\ i<j}}\frac{\langle tx_i^{\epsilon_i}x_j^{\epsilon_j},tqx_i^{\epsilon_i}x_j^{\epsilon_j}\rangle}{\langle x_i^{\epsilon_i}x_j^{\epsilon_j},qx_i^{\epsilon_i}x_j^{\epsilon_j}\rangle}\prod_{\substack{i\in I\\ j\in J}}\frac{\langle tx_i^{\epsilon_i}x_j^{\pm1}\rangle}{\langle x_i^{\epsilon_i}x_j^{\pm1}\rangle},\tag{2.2}$$

$$U_{J,r}(x)=\sum_{\substack{I\subset J\\ |I|=r\\ \delta_i=\pm1(i\in I)}}(-1)^r\prod_{i\in I}\frac{\langle ax_i^{\delta_i},bx_i^{\delta_i},cx_i^{\delta_i},dx_i^{\delta_i}\rangle}{\langle x_i^{2\delta_i},qx_i^{2\delta_i}\rangle}\prod_{\substack{i,j\in I\\ i<j}}\frac{\langle tx_i^{\delta_i}x_j^{\delta_j},qx_i^{\delta_i}x_j^{\delta_j}/t\rangle}{\langle x_i^{\delta_i}x_j^{\delta_j},qx_i^{\delta_i}x_j^{\delta_j}\rangle}\prod_{\substack{i\in I\\ j\in J\backslash I}}\frac{\langle tx_i^{\delta_i}x_j^{\pm1}\rangle}{\langle x_i^{\delta_i}x_j^{\pm1}\rangle},\tag{2.3}$$

where $T_{q,x}^{(I,\epsilon)}=\prod_{i\in I}T_{q,x_i}^{\epsilon_i}$ and we used the shorthand notation

$$\langle z_1,\ldots,z_k\rangle=\langle z_1\rangle\cdots\langle z_k\rangle,\quad \langle zw^{\pm1}\rangle=\langle zw,zw^{-1}\rangle.\tag{2.4}$$

We will use the following notation of q-shifted factorial in this paper:

$$\langle z\rangle_{q,l}=\prod_{i=1}^{l}\langle q^{i-1}z\rangle=(-1)^l q^{-\frac{1}{2}\binom{l}{2}}z^{-\frac{l}{2}}(z;q)_l\quad (l=0,1,2,\ldots),\tag{2.5}$$

where $(z;q)_l=\prod_{i=1}^{l}(1-q^{i-1}z)$. For these two types of q-shifted factorials, we use the shorthand notation as

$$\langle z_1,\ldots,z_k\rangle_{q,l}=\prod_{1\le i\le k}\langle z_i\rangle_{q,l},\quad \langle zw^{\pm1}\rangle_{q,l}=\langle zw\rangle_{q,l}\langle zw^{-1}\rangle_{q,l},\tag{2.6}$$

$$(z_1,\ldots,z_k;q)_l=\prod_{1\le i\le k}(z_i;q)_l,\quad (zw^{\pm1};q)_l=(zw;q)_l(zw^{-1};q)_l.\tag{2.7}$$

Let $w(z)$ and $v(z)$ denote the following rational functions, respectively:

$$w(z)=\frac{\langle az,bz,cz,dz\rangle}{\langle z^2,qz^2\rangle},\quad v(z)=\frac{\langle tz\rangle}{\langle z\rangle}.\tag{2.8}$$

Then $V_{\epsilon I,J}(x),U_{J,r}(x)$ are also expressed as

$$V_{\epsilon I,J}(x)=\prod_{i\in I}w(x_i^{\epsilon_i})\prod_{\substack{i,j\in I\\ i<j}}v(x_i^{\epsilon_i}x_j^{\epsilon_j})v(qx_i^{\epsilon_i}x_j^{\epsilon_j})\prod_{\substack{i\in I\\ j\in J}}v(x_i^{\epsilon_i}x_j^{\pm1}),\tag{2.9}$$

$$U_{J,r}(x)=\sum_{\substack{I\subset J\\ |I|=r\\ \delta_i=\pm1(i\in I)}}(-1)^r\prod_{i\in I}w(x_i^{\delta_i})\prod_{\substack{i,j\in I\\ i<j}}v(x_i^{\delta_i}x_j^{\delta_j})v(q^{-1}x_i^{-\delta_i}x_j^{-\delta_j})\prod_{\substack{i\in I\\ j\in J\backslash I}}v(x_i^{\delta_i}x_j^{\pm1}),\tag{2.10}$$

where $v(x_ix_j^{\pm1})$ means $v(x_ix_j)\cdot v(x_ix_j^{-1})$.

Let W_m be the Weyl group of type BC_m acting on the Laurent polynomials in the variables $x=(x_1,\ldots,x_m)$ through the permutations of the indices and the inversions of the variables. Under the assumption that a,b,c,d,q,t are generic, for each partition $\lambda=(\lambda_1,\ldots,\lambda_m)$ there exists a unique W_m-invariant Laurent polynomial $P_\lambda(x)=P_\lambda(x;a,b,c,d|q,t)$, called the Koornwinder polynomial attached to λ, satisfying the following conditions.

(1) $P_\lambda(x)$ is expanded by the orbit sums $m_\mu(x)=\sum_{\nu\in W.\mu}x^\nu$ as

$$P_\lambda(x)=m_\lambda(x)+\sum_{\mu<\lambda}c_{\lambda\mu}m_\mu(x),\tag{2.11}$$

where $c_{\lambda\mu} \in \mathbb{C}$ and $<$ means the dominance ordering of the partitions.

(2) $P_\lambda(x)$ is a joint eigenfunction of van Diejen's q-difference operators D_r^x:

$$D_r^x P_\lambda(x) = P_\lambda(x) e_r(\alpha t^{\delta_m} q^\lambda; \alpha|t), \tag{2.12}$$

where $\delta_m = (m-1, \dots, 1, 0)$ and $e_r(x;\alpha|t)$ are the interpolation polynomials of column type defined by

$$\begin{aligned} e_r(x;\alpha|t) &= \sum_{1\le i_1<\cdots<i_r\le m} e(x_{i_1}; t^{i_1-1}\alpha) e(x_{i_2}; t^{i_2-2}\alpha) \cdots e(x_{i_r}; t^{i_r-r}\alpha) & (2.13)\\ &= \sum_{1\le i_1<\cdots<i_r\le m} e(x_{i_1}; t^{m-i_1-r+1}\alpha) e(x_{i_2}; t^{m-i_2-r+2}\alpha) \cdots e(x_{i_r}; t^{m-i_r}\alpha), & (2.14)\\ e(z;w) &= \langle zw\rangle\langle zw^{-1}\rangle = z + z^{-1} - w - w^{-1}. & (2.15) \end{aligned}$$

Note that $e_r(x;\alpha|t)$ is W_m-invariant and satisfies the following interpolation property (See [KNS]): For any partition $\mu \not\supset (1^r)$,

$$e_r(\alpha t^{\delta_m} q^\mu; \alpha|t) = 0. \tag{2.16}$$

2.2 Main result

We recall the definition of the kernel function $\Phi(x;y|q,t)$ of Cauchy type associated with the root systems of type BC in the variables $x = (x_1, \dots, x_m)$ and $y = (y_1, \dots, y_n)$. The kernel function $\Phi(x;y|q,t)$ is defined as a solution of the following linear q-difference equations:

$$T_{q,x_i}\Phi(x;y|q,t) = \Phi(x;y|q,t) \prod_{1\le l\le n} \frac{e(\sqrt{q/t}x_i; y_l)}{e(\sqrt{tq}x_i; y_l)} \quad (1 \le i \le m), \tag{2.17}$$

$$T_{q,y_k}\Phi(x;y|q,t) = \Phi(x;y|q,t) \prod_{1\le j\le m} \frac{e(\sqrt{q/t}y_k; x_j)}{e(\sqrt{tq}y_k; x_j)} \quad (1 \le k \le n). \tag{2.18}$$

Such a $\Phi(x;y|q,t)$ is a multiple of the function

$$\Phi_0(x;y|q,t) = (x_1\cdots x_m)^{n\gamma} \prod_{\substack{1\le i\le m\\ 1\le k\le n}} \frac{(\sqrt{tq}x_i y_k^{\pm1}; q)_\infty}{(\sqrt{q/t}x_i y_k^{\pm1}; q)_\infty}, \tag{2.19}$$

by a q-periodic function with respect to all the variables x and y. Here $(z;q)_\infty = \prod_{i=0}^\infty (1 - q^i z)$ and γ is a complex number such that $q^\gamma = t$. We note that four types of explicit formulas for kernel function of Cauchy type including $\Phi_0(x;y|q,t)$ are introduced by [KNS].

For any integer l, let $e(z;w)_{q,l}$ be the q-shifted factorial of type BC with base point w defined by

$$e(z;w)_{q,l} = \begin{cases} e(z;w)e(z;qw)\cdots e(z;q^{l-1}w) & (l \ge 0),\\ \dfrac{1}{e(z;q^l w)e(z;q^{l+1}w)\cdots e(z;q^{-1}w)} & (l < 0). \end{cases} \tag{2.20}$$

We also define a generating function of D_r^x and that of $\widetilde{D}_r^y$ by

$$\mathcal{D}^x(u) := \mathcal{D}^x(u;a,b,c,d|q,t) = \sum_{r=0}^m (-1)^r D_r^x e(u;\alpha)_{t,m-r}, \tag{2.21}$$

$$\widetilde{\mathcal{D}}^y(u) := \mathcal{D}^y(u;\widetilde{a},\widetilde{b},\widetilde{c},\widetilde{d}|q,t) = \sum_{r=0}^{n}(-1)^r \widetilde{D}_r^y e(u;\widetilde{\alpha})_{t,n-r}, \tag{2.22}$$

where $(\widetilde{a},\widetilde{b},\widetilde{c},\widetilde{d}) = (\sqrt{tq}/a, \sqrt{tq}/b, \sqrt{tq}/c, \sqrt{tq}/d)$, so $\widetilde{\alpha} = t/\alpha$. We also denoted $D_r^y(\widetilde{a},\widetilde{b},\widetilde{c},\widetilde{d}|q,t)$ by $\widetilde{D}_r^y$. For any function $f(z) = f(z;a,b,c,d)$ depending on the parameters (a,b,c,d), we write $\widetilde{f}(z) = f(z;\widetilde{a},\widetilde{b},\widetilde{c},\widetilde{d})$. Then we have the following theorem.

Theorem. 2.1. *The kernel function $\Phi(x;y|q,t)$ intertwines the q-difference operator $\mathcal{D}^x(u)$ in the x variables with the q-difference operator $\widetilde{\mathcal{D}}^y(u)$ in the y variables:*

$$\mathcal{D}^x(u)\Phi(x;y|q,t) = e(u;\alpha)_{t,m-n}\widetilde{\mathcal{D}}^y(u)\Phi(x;y|q,t). \tag{2.23}$$

We call this equation a kernel identity of Cauchy type.

A proof of this theorem will be given in the next subsection. We now give some remarks related to Theorem 2.1. Firstly, it is known that Theorem 2.1 in the case of $n=0$ holds. Namely, the constant function 1 is the eigenfunction of van Diejen's q-difference operators [vD1]:

$$\mathcal{D}^x(u)\cdot 1 = e(u;\alpha)_{t,m}. \tag{2.24}$$

We will use this fact as the starting point of our proof. It is also known by [KNS] that

$$\sum_{r=0}^{m}(-1)^r e_r(x;\alpha|t)e(u;\alpha)_{t,m-r} = \prod_{i=1}^{m} e(u;x_i). \tag{2.25}$$

In general, for any partition λ we have

$$D^x(u)P_\lambda(x) = P_\lambda(x)\prod_{i=1}^{m} e(u;\alpha t^{m-i}q^{\lambda_i}). \tag{2.26}$$

Secondly, comparing the coefficient of $e(u;\alpha)_{t,m-1}$ in the left-hand side of (2.23) with that in the right-hand side, we obtain (1.1). In fact, the q-Saalschütz sum gives the transformation formula for the base points of the q-shifted factorials of type BC:

$$e(w;b)_{t,l} = \sum_{0\le r\le l}(-1)^r \begin{bmatrix} l \\ r \end{bmatrix}_t e(t^{\frac{1}{2}(l-1)}b; t^{\frac{1}{2}(1-l)}/a)_{t,r}e(w;a)_{t,l-r}, \tag{2.27}$$

$$\begin{bmatrix} l \\ r \end{bmatrix}_t = (-1)^r\frac{\langle t^{-l}\rangle_{t,r}}{\langle t\rangle_{t,r}}. \tag{2.28}$$

It follows from this formula that

$$\begin{aligned} &e(u;\alpha)_{t,m-n}e(u;t/\alpha)_{t,n-k} \\ &= \sum_{0\le l\le n-k}(-1)^l \begin{bmatrix} n-k \\ l \end{bmatrix}_t e(t^{\frac{1}{2}(n-k+1)}/\alpha; t^{\frac{1}{2}(1+n-2m+k)}/\alpha)_{t,l}e(u;\alpha)_{t,m-k-l}. \end{aligned} \tag{2.29}$$

Comparing the coefficients of $e(u;\alpha)_{t,m-r}$ in the both sides of (2.23), we have

$$D_r^x\Phi(x;y|q,t) = \sum_{k=0}^{r}\widetilde{D}_k^y \begin{bmatrix} n-k \\ r-k \end{bmatrix}_t e(t^{\frac{1}{2}(n-k+1)}/\alpha; t^{\frac{1}{2}(1+n-2m+k)}/\alpha)_{t,r-k}\Phi(x;y|q,t). \tag{2.30}$$

The formula (2.30) for $r=1$ recovers the result (1.1) of [KNS].

2.3 Proof of the main result

It is enough to show the case where $m \geq n \geq 0$. The identity (2.23) is equivalent to

$$\Phi(x;y|q,t)^{-1}\mathcal{D}^x(u)\Phi(x;y|q,t) = \Phi(x;y|q,t)^{-1}\widetilde{\mathcal{D}}^y(u)\Phi(x;y|q,t)e(u;\alpha)_{t,m-n}. \tag{2.31}$$

Regarding this as a rational function identity of the variable y_n, we prove it by computing the residues and the limits as $y_n \to \infty$.

The generating function $\mathcal{D}^x(u)$ is expanded as

$$\mathcal{D}^x(u) = \sum_{\substack{I\subset\{1,\dots,m\}\\ \epsilon_i=\pm1(i\in I)}} (-1)^{|I|}V_{\epsilon I,I^c}(x)U_{I^c}(u;x)T_{q,x}^{(I,\epsilon)}, \tag{2.32}$$

$$U_J(u;x) = \sum_{\substack{I\subset J\\ \delta_i=\pm1(i\in I)}} e(u;\alpha)_{t,|J|-|I|}\prod_{i\in I} w(x_i^{\delta_i}) \prod_{\substack{i,j\in I\\ i<j}} v(x_i^{\delta_i}x_j^{\delta_j})v(q^{-1}x_i^{-\delta_i}x_j^{-\delta_j}) \prod_{\substack{i\in I\\ j\in J\setminus I}} v(x_i^{\delta_i}x_j^{\pm1}). \tag{2.33}$$

Similarly, we expand $\widetilde{\mathcal{D}}_y(u)$. We also define the rational function $F(z;w)$ in the variables $z = (z_1,\dots,z_r)$ and $w = (w_1,\dots,w_s)$ by

$$F(z;w) = \prod_{\substack{1\leq i\leq r\\ 1\leq k\leq s}} \frac{e(\sqrt{q/t}z_i;w_k)}{e(\sqrt{tq}z_i;w_k)}. \tag{2.34}$$

For any subset $I = \{i_1,\dots,i_r\} \subset \{1,\dots,m\}, |I| = r$ and signs $\epsilon_i = \pm1 (i \in I)$, we write $x_I^\epsilon = (x_{i_1}^{\epsilon_{i_1}},\dots,x_{i_r}^{\epsilon_{i_r}})$. Then, (2.31) is expressed as

$$\begin{aligned}&\sum_{\substack{I\subset\{1,\dots,m\}\\ \epsilon_i=\pm1(i\in I)}} (-1)^{|I|}V_{\epsilon I,I^c}(x)U_{I^c}(u;x)F(x_I^\epsilon;y)\\ &= e(u;\alpha)_{t,m-n}\sum_{\substack{K\subset\{1,\dots,n\}\\ \epsilon_k=\pm1(k\in K)}} (-1)^{|K|}\widetilde{V}_{\epsilon K,K^c}(y)\widetilde{U}_{K^c}(u;y)F(y_K^\epsilon;x).\end{aligned} \tag{2.35}$$

We prove this identity by induction on n, starting with (2.24) of the case $n = 0$. We assume that our identity holds when the number of y variables is less than n.

Firstly, we consider the residues of the both sides. The left-hand side of (2.35) may have the poles at

$$y_n = \sqrt{tq}x_i^{\pm1}, \quad \frac{1}{\sqrt{tq}}x_i^{\pm1}\ (1 \leq i \leq m). \tag{2.36}$$

On the other hand, there may be the poles at

$$\begin{aligned} y_n =&\sqrt{tq}x_i^{\pm1}, \quad \frac{1}{\sqrt{tq}}x_i^{\pm1}\ (1 \leq i \leq m), \quad \pm1, \quad \pm q^{1/2},\\ &\pm q^{-1/2}, \quad y_k^{\pm1}, \quad qy_k^{\pm1}, \quad q^{-1}y_k^{\pm1}\ (1 \leq k \leq n)\end{aligned} \tag{2.37}$$

in the right-hand side. However, we can check by direct calculation that the points other than $y_n = \sqrt{tq}x_i^{\pm1}, \frac{1}{\sqrt{tq}}x_i^{\pm1}$ $(i = 1,\dots,m)$ are apparent singular points. Since (2.35) is invariant under the inversions and permutations for x and y, we have only to analyze the residue at the point $y_n = \sqrt{tq}x_m$.

In the left-hand side, the term indexed by (I, ϵ) has a pole at $y_n = \sqrt{tq}x_m$ if and only if $m \in I$ and $\epsilon_m = 1$. Note that

$$F(x_{I'}^{\epsilon}; \sqrt{tq}x_m) = \prod_{i \in I'} \frac{\langle qx_m x_i^{\epsilon_i}, tx_m/x_i^{\epsilon_i}\rangle}{\langle tqx_m x_i^{\epsilon_i}, x_m/x_i^{\epsilon_i}\rangle} = \prod_{i \in I'} \frac{v(x_m x_i^{-\epsilon_i})}{v(qx_m x_i^{\epsilon_i})}, \tag{2.38}$$

where $I' = I \setminus \{m\}$. Thus it follows that the residue is equal to

$$-\frac{\sqrt{t}\langle ax_m, bx_m, cx_m, dx_m\rangle(\sqrt{tq} - \sqrt{q/t})x_m}{\langle x_m^2, tqx_m^2\rangle} \prod_{1 \le j \le m-1} v(x_j^{\pm 1} x_m)$$
$$\cdot F(x_m; y') \times (\text{l.h.s. for the case of } (x'; y')), \tag{2.39}$$

where $x' = (x_1, \ldots, x_{m-1})$ and $y' = (y_1, \ldots, y_{n-1})$.

In the right-hand side, the term indexed by (K, ϵ) cannot have a pole at $y_n = \sqrt{tq}x_m$ unless $n \in K$ and $\epsilon_n = -1$. The corresponding residue is equal to

$$-\frac{\sqrt{t}\langle ax_m, bx_m, cx_m, dx_m\rangle(\sqrt{tq} - \sqrt{q/t})x_m}{\langle x_m^2, tqx_m^2\rangle} \prod_{1 \le j \le m-1} v(x_j^{\pm 1} x_m)$$
$$\cdot F(x_m; y') \times (\text{r.h.s. for the case of } (x'; y')). \tag{2.40}$$

Therefore it follows from the induction hypothesis that the residues of the both sides at the point $y_n = \sqrt{tq}x_m$ are equal.

Next, we calculate the limits of the both sides as $y_n \to \infty$. It is easy to check

$$\lim_{y_n \to \infty} (\text{l.h.s.}) = (\text{l.h.s. for the case of } (x; y')). \tag{2.41}$$

We consider the limit of the individual terms of the right-hand side in the following three cases:

(i) $n \in K$ and $\epsilon_n = 1$, (ii) $n \in K$ and $\epsilon_n = -1$, (iii) $n \notin K$.

By direct calculation, we can check in the case (i) and (ii) respectively as follows:

$$\lim_{y_n \to \infty} \left(\sum_{\substack{n \in K \\ \epsilon_n = 1}} (-1)^{|K|} \widetilde{V}_{\epsilon K, K^c}(y) \widetilde{U}_{K^c}(u; y) F(y_K^{\epsilon}; x) \right)$$
$$= -\widetilde{\alpha} t^{n-m-1} \sum_{\substack{K' \subset \{1, \ldots, n-1\} \\ \epsilon_k = \pm 1 (k \in K')}} (-1)^{|K'|} \widetilde{V}_{\epsilon K', K'^c}(y') \widetilde{U}_{K'^c}(u; y') F(y_{K'}^{\epsilon}; x), \tag{2.42}$$

$$\lim_{y_n \to \infty} \left(\sum_{\substack{n \in K \\ \epsilon_n = -1}} (-1)^{|K|} \widetilde{V}_{\epsilon K, K^c}(y) \widetilde{U}_{K^c}(u; y) F(y_K^{\epsilon}; x) \right)$$
$$= -\widetilde{\alpha}^{-1} t^{m-n+1} \sum_{\substack{K' \subset \{1, \ldots, n-1\} \\ \epsilon_k = \pm 1 (k \in K')}} (-1)^{|K'|} \widetilde{V}_{\epsilon K', K'^c}(y') \widetilde{U}_{K'^c}(u; y') F(y_{K'}^{\epsilon}; x). \tag{2.43}$$

In the case (iii), we divide the sum

$$\widetilde{U}_{K^c}(u;y) = \sum_{\substack{L\subset K^c\\ \delta_k=\pm1(k\in L)}} e(u;\widetilde{\alpha})_{t,|K^c|-|L|}\prod_{k\in L}\widetilde{w}(y_k^{\delta_k})\prod_{\substack{k,l\in L\\ k<l}} v(y_k^{\delta_k}y_l^{\delta_l})v(q^{-1}y_k^{-\delta_k}y_l^{-\delta_l})\prod_{\substack{k\in L\\ l\in K^c\setminus L}} v(y_k^{\delta_k}y_l^{\pm1}) \tag{2.44}$$

into the three groups of terms as

(a) $n\in L$ and $\delta_n=1$, (b) $n\in L$ and $\delta_n=-1$, (c) $n\notin L$.

Combining the limits of these three cases, we obtain

$$\begin{aligned}&\lim_{y_n\to\infty}\left(\sum_{n\notin K}(-1)^{|K|}\widetilde{V}_{\epsilon K,K^c}(y)\widetilde{U}_{K^c}(u;y)F(y_K^\epsilon;x)\right)\\ &=(u+u^{-1})\sum_{\substack{K'\subset\{1,\ldots,n-1\}\\ \epsilon_k=\pm1(k\in K')}}(-1)^{|K'|}\widetilde{V}_{\epsilon K',K'^c}(y')\widetilde{U}_{K'^c}(u;y')F(y_{K'}^\epsilon;x).\end{aligned} \tag{2.45}$$

From (2.42), (2.43) and (2.45), it follows that

$$\lim_{y_n\to\infty}(\text{r.h.s.}) = (\text{r.h.s. for the case of } (x;y')), \tag{2.46}$$

and hence we complete the proof of Theorem 2.1.

Replacing (q,t) by (t,q), the formula (2.35) can be rewritten explicitly as follows.

Theorem. 2.2. *Given two sets of variables $x=(x_1,\ldots,x_m)$ and $y=(y_1,\ldots,y_n)$, the following identity holds:*

$$\begin{aligned}&\sum_{\substack{I\subset\{1,\ldots,m\}\\ \epsilon_i=\pm1(i\in I)}}\left((-1)^{|I|}\prod_{i\in I}\frac{\langle ax_i^{\epsilon_i},bx_i^{\epsilon_i},cx_i^{\epsilon_i},dx_i^{\epsilon_i}\rangle}{\langle x_i^{2\epsilon_i},tx_i^{2\epsilon_i}\rangle}\prod_{\substack{i,j\in I\\ i<j}}\frac{\langle qx_i^{\epsilon_i}x_j^{\epsilon_j},tqx_i^{\epsilon_i}x_j^{\epsilon_j}\rangle}{\langle x_i^{\epsilon_i}x_j^{\epsilon_j},tx_i^{\epsilon_i}x_j^{\epsilon_j}\rangle}\prod_{\substack{i\in I\\ j\in I^c}}\frac{\langle qx_i^{\epsilon_i}x_j^{\pm1}\rangle}{\langle x_i^{\epsilon_i}x_j^{\pm1}\rangle}\right.\\
&\cdot\sum_{\substack{J\subset I^c\\ \delta_i=\pm1(i\in J)}}\left(e(u;\sqrt{q/t}\alpha)_{q,|I^c|-|J|}\prod_{i\in J}\frac{\langle ax_i^{\delta_i},bx_i^{\delta_i},cx_i^{\delta_i},dx_i^{\delta_i}\rangle}{\langle x_i^{2\delta_i},tx_i^{2\delta_i}\rangle}\prod_{\substack{i,j\in J\\ i<j}}\frac{\langle qx_i^{\delta_i}x_j^{\delta_j},tx_i^{\delta_i}x_j^{\delta_j}/q\rangle}{\langle x_i^{\delta_i}x_j^{\delta_j},tx_i^{\delta_i}x_j^{\delta_j}\rangle}\right.\\
&\left.\left.\cdot\prod_{\substack{i\in J\\ j\in I^c\setminus J}}\frac{\langle qx_i^{\delta_i}x_j^{\pm1}\rangle}{\langle x_i^{\delta_i}x_j^{\pm1}\rangle}\right)\prod_{\substack{i\in I\\ 1\le k\le n}}\frac{e(\sqrt{t/q}x_i^{\epsilon_i};y_k)}{e(\sqrt{tq}x_i^{\epsilon_i};y_k)}\right)\\
&=e(u;\sqrt{q/t}\alpha)_{q,m-n}\sum_{\substack{K\subset\{1,\ldots,n\}\\ \epsilon_k=\pm1(k\in K)}}\left((-1)^{|K|}\prod_{k\in K}\frac{\langle\sqrt{tq}y_k^{\epsilon_k}/a,\sqrt{tq}y_k^{\epsilon_k}/b,\sqrt{tq}y_k^{\epsilon_k}/c,\sqrt{tq}y_k^{\epsilon_k}/d\rangle}{\langle y_k^{2\epsilon_k},ty_k^{2\epsilon_k}\rangle}\right.\\
&\cdot\prod_{\substack{k,l\in K\\ k<l}}\frac{\langle qy_k^{\epsilon_k}y_l^{\epsilon_l},tqy_k^{\epsilon_k}y_l^{\epsilon_l}\rangle}{\langle y_k^{\epsilon_k}y_l^{\epsilon_l},ty_k^{\epsilon_k}y_l^{\epsilon_l}\rangle}\prod_{\substack{k\in K\\ l\in K^c}}\frac{\langle qy_k^{\epsilon_k}y_l^{\pm1}\rangle}{\langle y_k^{\epsilon_k}y_l^{\pm1}\rangle}\sum_{\substack{L\subset K^c\\ \delta_k=\pm1(k\in L)}}\left(e(u;\sqrt{tq}/\alpha)_{q,|K^c|-|L|}\right.\\
&\cdot\prod_{k\in L}\frac{\langle\sqrt{tq}y_k^{\delta_k}/a,\sqrt{tq}y_k^{\delta_k}/b,\sqrt{tq}y_k^{\delta_k}/c,\sqrt{tq}y_k^{\delta_k}/d\rangle}{\langle y_k^{2\delta_k},ty_k^{2\delta_k}\rangle}\prod_{\substack{k,l\in L\\ k<l}}\frac{\langle qy_k^{\delta_k}y_l^{\delta_l},ty_k^{\delta_k}y_l^{\delta_l}/q\rangle}{\langle y_k^{\delta_k}y_l^{\delta_l},ty_k^{\delta_k}y_l^{\delta_l}\rangle}\end{aligned}$$

$$\cdot \prod_{\substack{k\in L\\ l\in K^c\setminus L}} \frac{\langle q y_k^{\delta_k} y_l^{\pm 1}\rangle}{\langle y_k^{\delta_k} y_l^{\pm 1}\rangle}\Bigg) \prod_{\substack{k\in K\\ 1\le i\le m}} \frac{e(\sqrt{t/q}\, y_k^{\epsilon_k}; x_i)}{e(\sqrt{tq}\, y_k^{\epsilon_k}; x_i)}\Bigg). \tag{2.47}$$

3 Transformation formulas for multiple basic hypergeometric series

In the case of type A, the kernel function of Cauchy type intertwines the Macdonald q-difference operators [MiN]. This property gives the rational function identity which is similar to (2.47). Applying certain specializations to this identity, Kajihara derived the Euler transformation formula for multiple basic hypergeometric series. In the same way, we propose two types of transformation formulas for multiple basic hypergeometric series.

3.1 Type BC case

In this subsection, we derive a transformation formula of type BC from Theorem 2.2. Let $\alpha = (\alpha_1, \dots, \alpha_M) \in \mathbb{N}^M$ and $\beta = (\beta_1, \dots, \beta_N) \in \mathbb{N}^N$ be the multi-indices such that $|\alpha| := \sum_{i=1}^M \alpha_i = m, |\beta| = \sum_{k=1}^N \beta_k = n$. Here $\mathbb{N}$ is the set of non-negative integers. Then we consider the following specializations:

$$x = p_\alpha(z;q) := (z_1, qz_1, \dots, q^{\alpha_1-1}z_1; z_2, qz_2, \dots, q^{\alpha_2-1}z_2; \dots; z_M, qz_M, \dots, q^{\alpha_M-1}z_M), \tag{3.1}$$
$$y = p_\beta(w;q) := (w_1, qw_1, \dots, q^{\beta_1-1}w_1; w_2, qw_2, \dots, q^{\beta_2-1}w_2; \dots; w_N, qw_N, \dots, q^{\beta_N-1}w_N). \tag{3.2}$$

These specializations are called multiple principal specializations. We apply these to (2.47).

For each pair (I, ϵ), we divide the subset I as $I^+ \sqcup I^-$ by setting

$$I^+ = \{i \in I | \epsilon_i = 1\}, \quad I^- = \{i \in I | \epsilon_i = -1\}. \tag{3.3}$$

Similarly, we divide the subset J by $J^+ = \{i \in J | \delta_i = 1\}$ and $J^- = \{i \in J | \delta_i = -1\}$ for each pair (J, δ). We also denote the complement $\{1, \dots, m\}\setminus(I \cup J)$ by C. Using these symbols, we rewrite the left-hand side of (2.47) as

$$\sum_{\substack{I^+\sqcup I^-\sqcup J^+\sqcup J^-\sqcup C\\ =\{1,\dots,m\}}} \Bigg((-1)^{|I^+|+|I^-|} e(u;\sqrt{q/t}\,\alpha)_{q,|C|} \prod_{i\in I^+\sqcup J^+} \frac{\langle ax_i, bx_i, cx_i, dx_i\rangle}{\langle x_i^2, tx_i^2\rangle}$$
$$\cdot \prod_{i\in I^-\sqcup J^-} \frac{\langle a/x_i, b/x_i, c/x_i, d/x_i\rangle}{\langle x_i^{-2}, tx_i^{-2}\rangle} \prod_{\substack{i,j\in I^+\\ i<j}} \frac{\langle qx_ix_j\rangle_{t,2}}{\langle x_ix_j\rangle_{t,2}} \prod_{\substack{i,j\in I^-\\ i<j}} \frac{\langle qx_i^{-1}x_j^{-1}\rangle_{t,2}}{\langle x_i^{-1}x_j^{-1}\rangle_{t,2}} \prod_{\substack{i\in I^+\\ j\in I^-}} \frac{\langle qx_ix_j^{-1}\rangle_{t,2}}{\langle x_ix_j^{-1}\rangle_{t,2}}$$
$$\cdot \prod_{\substack{i\in I^+\\ j\in J^-\sqcup C\sqcup J^+}} \frac{\langle qx_ix_j^{\pm 1}\rangle}{\langle x_ix_j^{\pm 1}\rangle} \prod_{\substack{i\in I^-\\ j\in J^-\sqcup C\sqcup J^+}} \frac{\langle qx_i^{-1}x_j^{\pm 1}\rangle}{\langle x_i^{-1}x_j^{\pm 1}\rangle} \prod_{\substack{i,j\in J^+\\ i<j}} \frac{\langle qx_ix_j, tx_ix_j/q\rangle}{\langle x_ix_j\rangle_{t,2}} \prod_{\substack{i,j\in J^-\\ i<j}} \frac{\langle qx_i^{-1}x_j^{-1}, tx_i^{-1}x_j^{-1}/q\rangle}{\langle x_i^{-1}x_j^{-1}\rangle_{t,2}}$$
$$\cdot \prod_{\substack{i\in J^+\\ j\in J^-}} \frac{\langle qx_ix_j^{-1}, tx_ix_j^{-1}/q\rangle}{\langle x_ix_j^{-1}\rangle_{t,2}} \prod_{\substack{i\in J^+\\ j\in C}} \frac{\langle qx_ix_j^{\pm 1}\rangle}{\langle x_ix_j^{\pm 1}\rangle} \prod_{\substack{i\in J^-\\ j\in C}} \frac{\langle qx_i^{-1}x_j^{\pm 1}\rangle}{\langle x_i^{-1}x_j^{\pm 1}\rangle}$$

$$\cdot \prod_{\substack{i\in I^+\\ 1\le k\le n}} \frac{\langle \sqrt{t/q}x_iy_k^{\pm1}\rangle}{\langle \sqrt{tq}x_iy_k^{\pm1}\rangle} \prod_{\substack{i\in I^-\\ 1\le k\le n}} \frac{\langle \sqrt{t/q}x_i^{-1}y_k^{\pm1}\rangle}{\langle \sqrt{tq}x_i^{-1}y_k^{\pm1}\rangle}\Bigg). \tag{3.4}$$

We first consider the principal specialization $x_i = q^{i-1}z$ $(1 \le i \le m)$ of a single block. Noting that $\langle qx_i/x_{i+1}\rangle = 0$ $(i = 1, \dots, m-1)$, we find that non-zero terms arise from the divisions of the following form:

$$\begin{aligned} &I^- = \{1, 2, \dots, i_1\}, \quad J^- = \{i_1+1, i_1+2, \dots, i_2\}, \quad C = \{i_2+1, i_2+2, \dots, i_3\}, \\ &J^+ = \{i_3+1, i_3+2, \dots, i_4\}, \quad I^+ = \{i_4+1, i_4+2, \dots, m\}, \quad 0 \le i_1 \le i_2 \le i_3 \le i_4 \le m. \end{aligned} \tag{3.5}$$

Next, we consider the multiple principal specializations $x = p_\alpha(z;q), y = p_\beta(w;q)$. We replace the index set $\{1, \dots, m\}$ by

$$\{1, \dots, m\} = \{(i,a) | 1 \le i \le M,\ 0 \le a < \alpha_i\} \tag{3.6}$$

and write $x_{(i,a)} = q^a z_i$. For any two multi-indices $\mu, \nu \in \mathbb{N}^M$, if $\mu_i \le \nu_i$ $(i = 1, \dots, M)$ then we write $\mu \le \nu$. From the same argument as above applied to each block, I^-, J^-, C, J^+, I^+ are parametrized by the four multi-indices $\mu^-, \nu^-, \nu^+, \mu^+ \in \mathbb{N}^M$ such that $0 \le \mu^- \le \nu^- \le \nu^+ \le \mu^+ \le \alpha$ as follows:

$$\begin{aligned} I^- &= \{(i,a) | 1 \le i \le M,\ 0 \le a < \mu_i^-\}, \\ J^- &= \{(i,a) | 1 \le i \le M, \mu_i^- \le a < \nu_i^-\}, \\ C &= \{(i,a) | 1 \le i \le M,\ \nu_i^- \le a < \nu_i^+\}, \\ J^+ &= \{(i,a) | 1 \le i \le M,\ \nu_i^+ \le a < \mu_i^+\}, \\ I^+ &= \{(i,a) | 1 \le i \le M,\ \mu_i^+ \le a < \alpha_i\}. \end{aligned} \tag{3.7}$$

In the following, we omit the base q in the q-shifted factorials $\langle z\rangle_{q,k}$ and $e(z;w)_{q,k}$.

With this parametrization (3.6) of indices, the formula (3.4) specialized by $x = p_\alpha(z;q), y = p_\beta(w;q)$ gives rise to

$$\prod_{1\le i\le M} \frac{\langle az_i, bz_i, cz_i, dz_i\rangle_{\alpha_i}}{\langle z_i^2, tz_i^2\rangle_{\alpha_i}} \prod_{1\le i<j\le M} \frac{\langle q^{\alpha_j}z_iz_j, tq^{\alpha_j}z_iz_j\rangle_{\alpha_i}}{\langle z_iz_j, tz_iz_j\rangle_{\alpha_i}} \prod_{\substack{1\le i\le M\\ 1\le k\le N}} \frac{\langle \sqrt{t/q}z_iw_k, \sqrt{tq}z_i/q^{\beta_k}w_k\rangle_{\alpha_i}}{\langle q^{\beta_k}\sqrt{t/q}z_iw_k, \sqrt{tq}z_i/w_k\rangle_{\alpha_i}}$$

$$\cdot \sum_{0\le\mu^-\le\nu^-\le\nu^+\le\mu^+\le\alpha} \Bigg((-1)^{|\alpha|+|\nu^+|+|\nu^-|} e(u;\sqrt{q/t}\alpha)_{|\nu^+|-|\nu^-|} \prod_{1\le i\le M} \frac{\langle z_i/a, z_i/b, z_i/c, z_i/d\rangle_{\nu_i^-}}{\langle az_i, bz_i, cz_i, dz_i\rangle_{\nu_i^+}}$$

$$\cdot \prod_{1\le i\le j\le M} \frac{\langle q^{\mu_i^-+\mu_j^-}z_iz_j/tq, tq^{\mu_i^++\mu_j^+}z_iz_j/q\rangle}{\langle z_iz_j/tq, tz_iz_j/q\rangle} \prod_{1\le i<j\le M} \frac{\langle q^{\mu_i^--\mu_j^-}z_i/z_j, q^{\mu_i^+-\mu_j^+}z_i/z_j\rangle}{\langle z_i/z_j, z_i/z_j\rangle}$$

$$\cdot \prod_{1\le i<j\le M} \frac{\langle q^{\nu_i^--\nu_j^-}z_i/z_j, q^{\nu_i^+-\nu_j^+}z_i/z_j\rangle \langle z_iz_j/t, z_iz_j/q\rangle_{\nu_i^-+\nu_j^-}}{\langle z_i/z_j, z_i/z_j\rangle \langle z_iz_j, tz_iz_j/q\rangle_{\nu_i^++\nu_j^+}} \prod_{1\le i,j\le M} \frac{\langle q^{\mu_i^-+\mu_j^+}z_iz_j/q, q^{\nu_i^-+\nu_j^+}z_iz_j/q\rangle}{\langle z_iz_j/q, z_iz_j/q\rangle}$$

$$\cdot \prod_{1\le i,j\le M} \frac{\langle q^{\mu_i^--\mu_j^+}z_i/tz_j\rangle \langle z_iz_j/q\rangle_{\mu_i^-+\nu_j^+} \langle tz_iz_j/q\rangle_{\mu_i^++\nu_j^+} \langle z_iz_j/tq\rangle_{\mu_i^-} \langle z_iz_j/q\rangle_{\mu_i^+} \langle z_i/q^{\nu_j^-}z_j, z_i/tq^{\alpha_j}z_j\rangle_{\mu_i^-}}{\langle z_i/tq^{\mu_j^+}z_j\rangle \langle z_iz_j/t\rangle_{\mu_i^-+\nu_j^-} \langle z_iz_j\rangle_{\mu_i^++\nu_j^-} \langle q^{\alpha_j}z_iz_j\rangle_{\mu_i^-} \langle tq^{\alpha_j}z_iz_j\rangle_{\mu_i^+} \langle qz_i/tq^{\nu_j^+}z_j, qz_i/z_j\rangle_{\mu_i^-}}$$

$$\cdot \prod_{1\le i,j\le M} \frac{\langle z_i/q^{\alpha_j} z_j\rangle_{\mu_i^+} \langle qz_i/tq^{\nu_j^+} z_j, z_i/q^{\nu_j^+} z_j\rangle_{\nu_i^-} \langle z_i/q^{\mu_j^+} z_j\rangle_{\nu_i^+}}{\langle qz_i/z_j\rangle_{\mu_i^+} \langle qz_i/tq^{\mu_j^+} z_j, qz_i/z_j\rangle_{\nu_i^-} \langle qz_i/z_j\rangle_{\nu_i^+}}$$

$$\cdot \prod_{\substack{1\le i\le M\\ 1\le k\le N}} \frac{\langle q^{\beta_k} z_i w_k/\sqrt{tq}, \sqrt{q/t} z_i/w_k\rangle_{\mu_i^-} \langle q^{\beta_k}\sqrt{t/q} z_i w_k, \sqrt{tq} z_i/w_k\rangle_{\mu_i^+}}{\langle z_i w_k/\sqrt{tq}, \sqrt{q/t} z_i/q^{\beta_k} w_k\rangle_{\mu_i^-} \langle \sqrt{t/q} z_i w_k, \sqrt{tq} z_i/q^{\beta_k} w_k\rangle_{\mu_i^+}} \Bigg). \tag{3.8}$$

We used $\langle a\rangle_k$ version of some formulas of Appendix I in [GR]. We also used the formula

$$\prod_{1\le i\ne j\le m} \frac{\langle qx_i/x_j\rangle_{\lambda_i}}{\langle x_i/q^{\lambda_j} x_j\rangle_{\lambda_i}} = \prod_{1\le i<j\le m} \frac{\langle q^{\lambda_i-\lambda_j} x_i/x_j\rangle}{\langle x_i/x_j\rangle}, \tag{3.9}$$

due to Milne.

We apply the same specializations to the right-hand side of (2.47). If we denote (3.8) by $F^{(M,N)}_{\alpha,\beta}(z;w;a,b,c,d)$, Theorem 2.2 implies the following duality transformation formula:

$$F^{(M,N)}_{\alpha,\beta}(z;w;a,b,c,d) = e(u;\sqrt{q/t}\alpha)_{M-N} F^{(N,M)}_{\beta,\alpha}(w;z;\sqrt{tq}/a,\sqrt{tq}/b,\sqrt{tq}/c,\sqrt{tq}/d). \tag{3.10}$$

Relabeling M and N by m and n, and replacing the variables and parameters by

$$\begin{gathered} z_i \to \sqrt{tq}x_i \ (i=1,\dots,m), \quad w_k \to y_k \ (k=1,\dots,n), \\ (a,b,c,d) \to (\sqrt{tq}/a_1, \sqrt{tq}/a_2, \sqrt{tq}/a_3, \sqrt{tq}/a_4), \end{gathered} \tag{3.11}$$

we obtain the following theorem.

Theorem. 3.1. *Let* $a_0 = \sqrt{a_1a_2a_3a_4/q}$. *Take two sets of variables* $x = (x_1,\dots,x_m), y = (y_1,\dots,y_n)$ *and two multi-indices* $\alpha = (\alpha_1,\dots,\alpha_m) \in \mathbb{N}^m, \beta = (\beta_1,\dots,\beta_n) \in \mathbb{N}^n$. *Then the following identity holds:*

$$\prod_{1\le i\le m} \frac{\langle tqx_i/a_1, tqx_i/a_2, tqx_i/a_3, tqx_i/a_4\rangle_{\alpha_i}}{\langle tqx_i^2, t^2qx_i^2\rangle_{\alpha_i}} \prod_{1\le i<j\le m} \frac{\langle tq^{\alpha_j+1}x_ix_j, t^2q^{\alpha_j+1}x_ix_j\rangle_{\alpha_i}}{\langle tqx_ix_j, t^2qx_ix_j\rangle_{\alpha_i}}$$

$$\cdot \prod_{\substack{1\le i\le m\\ 1\le k\le n}} \frac{\langle tx_iy_k, tq^{1-\beta_k}x_i/y_k\rangle_{\alpha_i}}{\langle tq^{\beta_k}x_iy_k, tqx_i/y_k\rangle_{\alpha_i}} \sum_{0\le\mu^-\le\nu^-\le\nu^+\le\mu^+\le\alpha} \Bigg((-1)^{|\alpha|+|\nu^+|+|\nu^-|} e(u;\sqrt{tq}/a_0)_{|\nu^+|-|\nu^-|}$$

$$\cdot \prod_{1\le i\le m} \frac{\langle a_1x_i, a_2x_i, a_3x_i, a_4x_i\rangle_{\nu_i^-}}{\langle tqx_i/a_1, tqx_i/a_2, tqx_i/a_3, tqx_i/a_4\rangle_{\nu_i^+}} \prod_{1\le i\le j\le m} \frac{\langle q^{\mu_i^-+\mu_j^-}x_ix_j, t^2q^{\mu_i^++\mu_j^+}x_ix_j\rangle}{\langle x_ix_j, t^2x_ix_j\rangle}$$

$$\cdot \prod_{1\le i<j\le m} \frac{\langle q^{\mu_i^--\mu_j^-}x_i/x_j, q^{\mu_i^+-\mu_j^+}x_i/x_j, q^{\nu_i^--\nu_j^-}x_i/x_j, q^{\nu_i^+-\nu_j^+}x_i/x_j\rangle \langle tx_ix_j, qx_ix_j\rangle_{\nu_i^-+\nu_j^-}}{\langle x_i/x_j, x_i/x_j, x_i/x_j, x_i/x_j\rangle \langle t^2x_ix_j, tqx_ix_j\rangle_{\nu_i^++\nu_j^+}}$$

$$\cdot \prod_{1\le i,j\le m} \frac{\langle tq^{\mu_i^-+\mu_j^+}x_ix_j, tq^{\nu_i^-+\nu_j^+}x_ix_j, q^{\mu_i^--\mu_j^+}x_i/tx_j\rangle \langle tx_ix_j\rangle_{\mu_i^-+\nu_j^+} \langle t^2x_ix_j\rangle_{\mu_i^++\nu_j^+}}{\langle tx_ix_j, tx_ix_j, x_i/tq^{\mu_j^+}x_j\rangle \langle qx_ix_j\rangle_{\mu_i^-+\nu_j^-} \langle tqx_ix_j\rangle_{\mu_i^++\nu_j^-}}$$

$$\cdot \prod_{1\le i,j\le m} \frac{\langle x_ix_j, x_i/q^{\nu_j^-}x_j, x_i/tq^{\alpha_j}x_j\rangle_{\mu_i^-} \langle tx_ix_j, x_i/q^{\alpha_j}x_j\rangle_{\mu_i^+} \langle qx_i/tq^{\nu_j^+}x_j, x_i/q^{\nu_j^+}x_j\rangle_{\nu_i^-}}{\langle tq^{\alpha_j+1}x_ix_j, qx_i/tq^{\nu_j^+}x_j, qx_i/x_j\rangle_{\mu_i^-} \langle t^2q^{\alpha_j+1}x_ix_j, qx_i/x_j\rangle_{\mu_i^+} \langle qx_i/tq^{\mu_j^+}x_j, qx_i/x_j\rangle_{\nu_i^-}}$$

$$\cdot \prod_{1\le i,j\le m} \frac{\langle x_i/q^{\mu_j^+} x_j\rangle_{\nu_i^+}}{\langle qx_i/x_j\rangle_{\nu_i^+}} \prod_{\substack{1\le i\le m\\ 1\le k\le n}} \frac{\langle q^{\beta_k} x_i y_k, qx_i/y_k\rangle_{\mu_i^-} \langle tq^{\beta_k} x_i y_k, tqx_i/y_k\rangle_{\mu_i^+}}{\langle x_i y_k, q^{1-\beta_k} x_i/y_k\rangle_{\mu_i^-} \langle tx_i y_k, tq^{1-\beta_k} x_i/y_k\rangle_{\mu_i^+}} \Bigg)$$

$$= e(u;\sqrt{tq}/a_0)_{|\alpha|-|\beta|} \prod_{1\le k\le n} \frac{\langle a_1y_k, a_2y_k, a_3y_k, a_4y_k\rangle_{\beta_k}}{\langle y_k^2, ty_k^2\rangle_{\beta_k}} \prod_{1\le k<l\le n} \frac{\langle q^{\beta_l} y_k y_l, tq^{\beta_l} y_k y_l\rangle_{\beta_k}}{\langle y_k y_l, ty_k y_l\rangle_{\beta_k}}$$

$$\cdot \prod_{\substack{1\le k\le n\\ 1\le i\le m}} \frac{\langle ty_k x_i, q^{-\alpha_i} y_k/x_i\rangle_{\beta_k}}{\langle tq^{\alpha_i} y_k x_i, y_k/x_i\rangle_{\beta_k}} \sum_{0\le\lambda^-\le\kappa^-\le\kappa^+\le\lambda^+\le\beta} \Bigg((-1)^{|\beta|+|\kappa^+|+|\kappa^-|} e(u;\sqrt{q/t}a_0)_{|\kappa^+|-|\kappa^-|}$$

$$\cdot \prod_{1\le k\le n} \frac{\langle y_k/a_1, y_k/a_2, y_k/a_3, y_k/a_4\rangle_{\kappa_k^-}}{\langle a_1y_k, a_2y_k, a_3y_k, a_4y_k\rangle_{\kappa_k^+}} \prod_{1\le k\le l\le n} \frac{\langle q^{\lambda_k^- +\lambda_l^- -1} y_k y_l/t, tq^{\lambda_k^+ +\lambda_l^+ -1} y_k y_l\rangle}{\langle y_k y_l/tq, ty_k y_l/q\rangle}$$

$$\cdot \prod_{1\le k<l\le n} \frac{\langle q^{\lambda_k^- -\lambda_l^-} y_k/y_l, q^{\lambda_k^+ -\lambda_l^+} y_k/y_l, q^{\kappa_k^- -\kappa_l^-} y_k/y_l, q^{\kappa_k^+ -\kappa_l^+} y_k/y_l\rangle \langle y_k y_l/q, y_k y_l/t\rangle_{\kappa_k^- +\kappa_l^-}}{\langle y_k/y_l, y_k/y_l, y_k/y_l, y_k/y_l\rangle \langle ty_k y_l/q, y_k y_l\rangle_{\kappa_k^+ +\kappa_l^+}}$$

$$\cdot \prod_{1\le k,l\le n} \frac{\langle q^{\lambda_k^- +\lambda_l^+ -1} y_k y_l, q^{\kappa_k^- +\kappa_l^+ -1} y_k y_l, q^{\lambda_k^- -\lambda_l^+} y_k/ty_l\rangle \langle y_k y_l/q\rangle_{\lambda_k^- +\kappa_l^+} \langle ty_k y_l/q\rangle_{\lambda_k^+ +\kappa_l^+}}{\langle y_k y_l/q, y_k y_l/q, y_k/tq^{\lambda_l^+} y_l\rangle \langle y_k y_l/t\rangle_{\lambda_k^- +\kappa_l^-} \langle y_k y_l\rangle_{\lambda_k^+ +\kappa_l^-}}$$

$$\cdot \prod_{1\le k,l\le n} \frac{\langle y_k y_l/tq, y_k/q^{\kappa_l^-} y_l, y_k/tq^{\beta_l} y_l\rangle_{\lambda_k^-} \langle y_k y_l/q, y_k/q^{\beta_l} y_l\rangle_{\lambda_k^+} \langle qy_k/tq^{\kappa_l^+} y_l, y_k/q^{\kappa_l^+} y_l\rangle_{\kappa_k^-} \langle y_k/q^{\lambda_l^+} y_l\rangle_{\kappa_k^+}}{\langle q^{\beta_l} y_k y_l, qy_k/tq^{\kappa_l^+} y_l, qy_k/y_l\rangle_{\lambda_k^-} \langle tq^{\beta_l} y_k y_l, qy_k/y_l\rangle_{\lambda_k^+} \langle qy_k/tq^{\lambda_l^+} y_l, qy_k/y_l\rangle_{\kappa_k^-} \langle qy_k/y_l\rangle_{\kappa_k^+}}$$

$$\cdot \prod_{\substack{1\le k\le n\\ 1\le i\le m}} \frac{\langle q^{\alpha_i} y_k x_i, y_k/tx_i\rangle_{\lambda_k^-} \langle tq^{\alpha_i} y_k x_i, y_k/x_i\rangle_{\lambda_k^+}}{\langle y_k x_i, y_k/tq^{\alpha_i} x_i\rangle_{\lambda_k^-} \langle ty_k x_i, q^{-\alpha_i} y_k/x_i\rangle_{\lambda_k^+}} \Bigg). \tag{3.12}$$

We give some remarks of Theorem 3.1. As the special case $n = 0$, we obtain the following summation formula:

$$\sum_{0\le\mu^-\le\nu^-\le\nu^+\le\mu^+\le\alpha} \Bigg((-1)^{|\alpha|+|\nu^+|+|\nu^-|} e(u;\sqrt{tq}/a_0)_{|\nu^+|-|\nu^-|}$$

$$\cdot \prod_{1\le i\le m} \frac{\langle a_1x_i, a_2x_i, a_3x_i, a_4x_i\rangle_{\nu_i^-}}{\langle tqx_i/a_1, tqx_i/a_2, tqx_i/a_3, tqx_i/a_4\rangle_{\nu_i^+}} \prod_{1\le i\le j\le m} \frac{\langle q^{\mu_i^- +\mu_j^-} x_i x_j, t^2 q^{\mu_i^+ +\mu_j^+} x_i x_j\rangle}{\langle x_i x_j, t^2 x_i x_j\rangle}$$

$$\cdot \prod_{1\le i<j\le m} \frac{\langle q^{\mu_i^- -\mu_j^-} x_i/x_j, q^{\mu_i^+ -\mu_j^+} x_i/x_j, q^{\nu_i^- -\nu_j^-} x_i/x_j, q^{\nu_i^+ -\nu_j^+} x_i/x_j\rangle \langle tx_i x_j, qx_i x_j\rangle_{\nu_i^- +\nu_j^-}}{\langle x_i/x_j, x_i/x_j, x_i/x_j, x_i/x_j\rangle \langle t^2 x_i x_j, tqx_i x_j\rangle_{\nu_i^+ +\nu_j^+}}$$

$$\cdot \prod_{1\le i,j\le m} \frac{\langle tq^{\mu_i^- +\mu_j^+} x_i x_j, tq^{\nu_i^- +\nu_j^+} x_i x_j, q^{\mu_i^- -\mu_j^+} x_i/tx_j\rangle \langle tx_i x_j\rangle_{\mu_i^- +\nu_j^+} \langle t^2 x_i x_j\rangle_{\mu_i^+ +\nu_j^+} \langle x_i/q^{\mu_j^+} x_j\rangle_{\nu_i^+}}{\langle tx_i x_j, tx_i x_j, x_i/tq^{\mu_j^+} x_j\rangle \langle qx_i x_j\rangle_{\mu_i^- +\nu_j^-} \langle tqx_i x_j\rangle_{\mu_i^+ +\nu_j^-} \langle qx_i/x_j\rangle_{\nu_i^+}}$$

$$\cdot \prod_{1\le i,j\le m} \frac{\langle x_i x_j, x_i/q^{\nu_j^-} x_j, x_i/tq^{\alpha_j} x_j\rangle_{\mu_i^-} \langle tx_i x_j, x_i/q^{\alpha_j} x_j\rangle_{\mu_i^+} \langle qx_i/tq^{\nu_j^+} x_j, x_i/q^{\nu_j^+} x_j\rangle_{\nu_i^-}}{\langle tq^{\alpha_j+1} x_i x_j, qx_i/tq^{\nu_j^+} x_j, qx_i/x_j\rangle_{\mu_i^-} \langle t^2 q^{\alpha_j+1} x_i x_j, qx_i/x_j\rangle_{\mu_i^+} \langle qx_i/tq^{\mu_j^+} x_j, qx_i/x_j\rangle_{\nu_i^-}} \Bigg)$$

$$= e(u;\sqrt{tq}/a_0)_{|\alpha|}\prod_{1\le i\le m}\frac{\langle tqx_i^2,t^2qx_i^2\rangle_{\alpha_i}}{\langle tqx_i/a_1,tqx_i/a_2,tqx_i/a_3,tqx_i/a_4\rangle_{\alpha_i}}\prod_{1\le i<j\le m}\frac{\langle tqx_ix_j,t^2qx_ix_j\rangle_{\alpha_i}}{\langle tq^{\alpha_j+1}x_ix_j,t^2q^{\alpha_j+1}x_ix_j\rangle_{\alpha_i}}. \tag{3.13}$$

Note that the q-Saalschütz sum (2.27) implies

$$\begin{aligned}
&e(u;\sqrt{tq}/a_0)_{|\alpha|-|\beta|}e(u;\sqrt{q/t}a_0)_{|\kappa^+|-|\kappa^-|}\\
&=\sum_{r=0}^{|\kappa^+|-|\kappa^-|}\Bigg(\frac{\langle q^{|\kappa^-|-|\kappa^+|}\rangle_r}{\langle q\rangle_r}e(t^{-\frac12}q^{\frac12(|\kappa^+|-|\kappa^-|)}a_0;t^{-\frac12}q^{\frac12(|\kappa^-|-|\kappa^+|-2|\alpha|+2|\beta|)}a_0)_r\\
&\quad\cdot e(u;\sqrt{q/t}/a_0)_{|\alpha|-|\beta|+|\kappa^+|-|\kappa^-|-r}\Bigg).
\end{aligned}\tag{3.14}$$

Hence, the right-hand side of (3.12) is also expressed as

$$\begin{aligned}
&\prod_{1\le k\le n}\frac{\langle a_1y_k,a_2y_k,a_3y_k,a_4y_k\rangle_{\beta_k}}{\langle y_k^2,ty_k^2\rangle_{\beta_k}}\prod_{1\le k<l\le n}\frac{\langle q^{\beta_l}y_ky_l,tq^{\beta_l}y_ky_l\rangle_{\beta_k}}{\langle y_ky_l,ty_ky_l\rangle_{\beta_k}}\prod_{\substack{1\le k\le n\\1\le i\le m}}\frac{\langle ty_kx_i,q^{-\alpha_i}y_k/x_i\rangle_{\beta_k}}{\langle tq^{\alpha_i}y_kx_i,y_k/x_i\rangle_{\beta_k}}\\
&\cdot\sum_{0\le\lambda^-\le\kappa^-\le\kappa^+\le\lambda^+\le\beta}\Bigg(\sum_{r=0}^{|\kappa^+|-|\kappa^-|}\Bigg((-1)^{|\beta|+|\kappa^+|+|\kappa^-|}\frac{\langle q^{|\kappa^-|-|\kappa^+|}\rangle_r}{\langle q\rangle_r}\\
&\cdot e(t^{-\frac12}q^{\frac12(|\kappa^+|-|\kappa^-|)}a_0;t^{-\frac12}q^{\frac12(|\kappa^-|-|\kappa^+|-2|\alpha|+2|\beta|)}a_0)_r e(u;\sqrt{tq}/a_0)_{|\alpha|-|\beta|+|\kappa^+|-|\kappa^-|-r}\\
&\cdot\prod_{1\le k\le n}\frac{\langle y_k/a_1,y_k/a_2,y_k/a_3,y_k/a_4\rangle_{\kappa_k^-}}{\langle a_1y_k,a_2y_k,a_3y_k,a_4y_k\rangle_{\kappa_k^+}}\prod_{1\le k\le l\le n}\frac{\langle q^{\lambda_k^-+\lambda_l^--1}y_ky_l/t,tq^{\lambda_k^++\lambda_l^+-1}y_ky_l\rangle}{\langle y_ky_l/tq,ty_ky_l/q\rangle}\\
&\cdot\prod_{1\le k<l\le n}\frac{\langle q^{\lambda_k^--\lambda_l^-}y_k/y_l,q^{\lambda_k^+-\lambda_l^+}y_k/y_l,q^{\kappa_k^--\kappa_l^-}y_k/y_l,q^{\kappa_k^+-\kappa_l^+}y_k/y_l\rangle\langle y_ky_l/q,y_ky_l/t\rangle_{\kappa_k^-+\kappa_l^-}}{\langle y_k/y_l,y_k/y_l,y_k/y_l,y_k/y_l\rangle\langle ty_ky_l/q,y_ky_l\rangle_{\kappa_k^++\kappa_l^+}}\\
&\cdot\prod_{1\le k,l\le n}\frac{\langle q^{\lambda_k^-+\lambda_l^+-1}y_ky_l,q^{\kappa_k^-+\kappa_l^+-1}y_ky_l,q^{\lambda_k^--\lambda_l^+}y_k/ty_l\rangle\langle y_ky_l/q\rangle_{\lambda_k^-+\kappa_l^+}\langle ty_ky_l/q\rangle_{\lambda_k^++\kappa_l^+}}{\langle y_ky_l/q,y_ky_l/q,y_k/tq^{\lambda_l^+}y_l\rangle\langle y_ky_l/t\rangle_{\lambda_k^-+\kappa_l^-}\langle y_ky_l\rangle_{\lambda_k^++\kappa_l^-}}\\
&\cdot\prod_{1\le k,l\le n}\frac{\langle y_ky_l/tq,y_k/q^{\kappa_l^-}y_l,y_k/tq^{\beta_l}y_l\rangle_{\lambda_k^-}\langle y_ky_l/q,y_k/q^{\beta_l}y_l\rangle_{\lambda_k^+}\langle qy_k/tq^{\kappa_l^+}y_l,y_k/q^{\kappa_l^+}y_l\rangle_{\kappa_k^-}\langle y_k/q^{\lambda_l^+}y_l\rangle_{\kappa_k^+}}{\langle q^{\beta_l}y_ky_l,qy_k/tq^{\kappa_l^+}y_l,qy_k/y_l\rangle_{\lambda_k^-}\langle tq^{\beta_l}y_ky_l,qy_k/y_l\rangle_{\lambda_k^+}\langle qy_k/tq^{\lambda_l^+}y_l,qy_k/y_l\rangle_{\kappa_k^-}\langle qy_k/y_l\rangle_{\kappa_k^+}}\\
&\cdot\prod_{\substack{1\le k\le n\\1\le i\le m}}\frac{\langle q^{\alpha_i}y_kx_i,y_k/tx_i\rangle_{\lambda_k^-}\langle tq^{\alpha_i}y_kx_i,y_k/x_i\rangle_{\lambda_k^+}}{\langle y_kx_i,y_k/tq^{\alpha_i}x_i\rangle_{\lambda_k^-}\langle ty_kx_i,q^{-\alpha_i}y_k/x_i\rangle_{\lambda_k^+}}\Bigg)\Bigg).
\end{aligned}\tag{3.15}$$

3.2 Type C case

In the previous subsection, we derived the transformation formula of type BC by a method of principal specializations. In this subsection, we apply the same method to derive another type of transformation formula. We specialize in advance the parameters of (2.47) so that $(c,d,t)=$

$(q^{1/2}, -q^{1/2}, q)$:

$$\sum_{\substack{I\subset\{1,\dots,m\}\\ \epsilon_i=\pm1(i\in I)}} \left((-1)^{|I|} \prod_{i\in I} \frac{\sqrt{-1}\langle ax_i^{\epsilon_i}, bx_i^{\epsilon_i}\rangle}{\langle x_i^{2\epsilon_i}\rangle} \prod_{\substack{i,j\in I\\ i<j}} \frac{\langle q^2 x_i^{\epsilon_i} x_j^{\epsilon_j}\rangle}{\langle x_i^{\epsilon_i} x_j^{\epsilon_j}\rangle} \prod_{\substack{i\in I\\ j\in I^c}} \frac{\langle q x_i^{\epsilon_i} x_j^{\pm1}\rangle}{\langle x_i^{\epsilon_i} x_j^{\pm1}\rangle} \prod_{\substack{i\in I\\ 1\le k\le n}} \frac{\langle x_i^{\epsilon_i} y_k^{\pm1}\rangle}{\langle q x_i^{\epsilon_i} y_k^{\pm1}\rangle} \right.$$

$$\left. \cdot \sum_{\substack{J\subset I^c\\ \delta_i=\pm1(i\in J)}} e(u;\alpha)_{|I^c|-|J|} \prod_{i\in J} \frac{\sqrt{-1}\langle ax_i^{\delta_i}, bx_i^{\delta_i}\rangle}{\langle x_i^{2\delta_i}\rangle} \prod_{\substack{i\in J\\ j\in I^c\setminus J}} \frac{\langle q x_i^{\delta_i} x_j^{\pm1}\rangle}{\langle x_i^{\delta_i} x_j^{\pm1}\rangle} \right)$$

$$= e(u;\alpha)_{m-n} \sum_{\substack{K\subset\{1,\dots,n\}\\ \epsilon_k=\pm1(k\in K)}} \left((-1)^{|K|} \prod_{k\in K} \frac{-\sqrt{-1}\langle q y_k^{\epsilon_k}/a, q y_k^{\epsilon_k}/b\rangle}{\langle y_k^{2\epsilon_k}\rangle} \prod_{\substack{k,l\in K\\ k<l}} \frac{\langle q^2 y_k^{\epsilon_k} y_l^{\epsilon_l}\rangle}{\langle y_k^{\epsilon_k} y_l^{\epsilon_l}\rangle} \prod_{\substack{k\in K\\ l\in K^c}} \frac{\langle q y_k^{\epsilon_k} y_l^{\pm1}\rangle}{\langle y_k^{\epsilon_k} y_l^{\pm1}\rangle} \right.$$

$$\left. \cdot \prod_{\substack{k\in K\\ 1\le i\le m}} \frac{\langle y_k^{\epsilon_k} x_i^{\pm1}\rangle}{\langle q y_k^{\epsilon_k} x_i^{\pm1}\rangle} \sum_{\substack{L\subset K^c\\ \delta_k=\pm1(k\in L)}} e(u;q/\alpha)_{|K^c|-|L|} \prod_{k\in L} \frac{-\sqrt{-1}\langle q y_k^{\delta_k}/a, q y_k^{\delta_k}/b\rangle}{\langle y_k^{2\delta_k}\rangle} \prod_{\substack{k\in L\\ l\in K^c\setminus L}} \frac{\langle q y_k^{\delta_k} y_l^{\pm1}\rangle}{\langle y_k^{\delta_k} y_l^{\pm1}\rangle} \right). \tag{3.16}$$

Note that $\alpha = \sqrt{-ab}$. In this setting, the internal sum of each side simplifies drastically.

Lemma. 3.2.

$$\sum_{\substack{J\subset I^c\\ \delta_i=\pm1(i\in J)}} e(u;\alpha)_{|I^c|-|J|} \prod_{i\in J} \frac{\sqrt{-1}\langle ax_i^{\delta_i}, bx_i^{\delta_i}\rangle}{\langle x_i^{2\delta_i}\rangle} \prod_{\substack{i\in J\\ j\in I^c\setminus J}} \frac{\langle q x_i^{\delta_i} x_j^{\pm1}\rangle}{\langle x_i^{\delta_i} x_j^{\pm1}\rangle} = \left(u+\frac{1}{u}\right)^{|I^c|}, \tag{3.17}$$

$$\sum_{\substack{L\subset K^c\\ \delta_k=\pm1(k\in L)}} e(u;q/\alpha)_{|K^c|-|L|} \prod_{k\in L} \frac{-\sqrt{-1}\langle q y_k^{\delta_k}/a, q y_k^{\delta_k}/b\rangle}{\langle y_k^{2\delta_k}\rangle} \prod_{\substack{k\in L\\ l\in K^c\setminus L}} \frac{\langle q y_k^{\delta_k} y_l^{\pm1}\rangle}{\langle y_k^{\delta_k} y_l^{\pm1}\rangle} = \left(u+\frac{1}{u}\right)^{|K^c|}. \tag{3.18}$$

We can prove this lemma in the same way as in Theorem 2.1 by analyzing the residues. From this lemma, if $u = \sqrt{-1}$, (3.16) reduces to

$$\sum_{\substack{\epsilon_i=\pm1\\ i\in\{1,\dots,m\}}} (-1)^m \prod_{1\le i\le m} \frac{\sqrt{-1}\langle ax_i^{\epsilon_i}, bx_i^{\epsilon_i}\rangle}{\langle x_i^{2\epsilon_i}\rangle} \prod_{1\le i<j\le m} \frac{\langle q^2 x_i^{\epsilon_i} x_j^{\epsilon_j}\rangle}{\langle x_i^{\epsilon_i} x_j^{\epsilon_j}\rangle} \prod_{\substack{1\le i\le m\\ 1\le k\le n}} \frac{\langle x_i^{\epsilon_i} y_k^{\pm1}\rangle}{\langle q x_i^{\epsilon_i} y_k^{\pm1}\rangle}$$

$$= e(\sqrt{-1};\alpha)_{m-n} \sum_{\substack{\epsilon_k=\pm1\\ k\in\{1,\dots,n\}}} (-1)^n \prod_{1\le k\le n} \frac{-\sqrt{-1}\langle q y_k^{\epsilon_k}/a, q y_k^{\epsilon_k}/b\rangle}{\langle y_k^{2\epsilon_k}\rangle} \prod_{1\le k<l\le n} \frac{\langle q^2 y_k^{\epsilon_k} y_l^{\epsilon_l}\rangle}{\langle y_k^{\epsilon_k} y_l^{\epsilon_l}\rangle} \prod_{\substack{1\le k\le n\\ 1\le i\le m}} \frac{\langle y_k^{\epsilon_k} x_i^{\pm1}\rangle}{\langle q y_k^{\epsilon_k} x_i^{\pm1}\rangle}. \tag{3.19}$$

Since $e(\sqrt{-1};\alpha)_{m-n} = (\sqrt{-1})^{m-n}(-1)^m \frac{\langle ab\rangle_{q^2,m}}{\langle q^{2-2m}/ab\rangle_{q^2,n}}$, (3.19) is equal to

$$\sum_{\substack{\epsilon_i=\pm1\\ i\in\{1,\dots,m\}}} \prod_{1\le i\le m} \frac{\langle ax_i^{\epsilon_i}, bx_i^{\epsilon_i}\rangle}{\langle x_i^{2\epsilon_i}\rangle} \prod_{1\le i<j\le m} \frac{\langle q^2 x_i^{\epsilon_i} x_j^{\epsilon_j}\rangle}{\langle x_i^{\epsilon_i} x_j^{\epsilon_j}\rangle} \prod_{\substack{1\le i\le m\\ 1\le k\le n}} \frac{\langle x_i^{\epsilon_i} y_k^{\pm1}\rangle}{\langle q x_i^{\epsilon_i} y_k^{\pm1}\rangle}$$

$$= \frac{\langle ab\rangle_{q^2,m}}{\langle q^{2-2m}/ab\rangle_{q^2,n}} \sum_{\substack{\epsilon_k=\pm 1\\ k\in\{1,\dots,n\}}} \prod_{1\le k\le n} \frac{\langle qy_k^{\epsilon_k}/a, qy_k^{\epsilon_k}/b\rangle}{\langle y_k^{2\epsilon_k}\rangle} \prod_{1\le k<l\le n} \frac{\langle q^2 y_k^{\epsilon_k} y_l^{\epsilon_l}\rangle}{\langle y_k^{\epsilon_k} y_l^{\epsilon_l}\rangle} \prod_{\substack{1\le k\le n\\ 1\le i\le m}} \frac{\langle y_k^{\epsilon_k} x_i^{\pm 1}\rangle}{\langle q y_k^{\epsilon_k} x_i^{\pm 1}\rangle}. \tag{3.20}$$

Let I^-, I^+, K^-, K^+ be

$$\begin{aligned} I^- = \{i | 1\le i\le m, \epsilon_i = -1\}, \quad I^+ = \{i | 1\le i\le m, \epsilon_i = 1\}, \\ K^- = \{k | 1\le k\le n, \epsilon_k = -1\}, \quad K^+ = \{k | 1\le k\le n, \epsilon_k = 1\}. \end{aligned} \tag{3.21}$$

Using this notation and replacing the parameter q with $q^{1/2}$, we obtain

$$\sum_{\substack{I^-\sqcup I^+\\ =\{1,\dots,m\}}} \Bigg(\prod_{i\in I^+} \frac{\langle ax_i, bx_i\rangle}{\langle x_i^2\rangle} \prod_{i\in I^-} \frac{-\langle x_i/a, x_i/b\rangle}{\langle x_i^2\rangle} \prod_{\substack{i,j\in I^+\\ i<j}} \frac{\langle qx_ix_j\rangle}{\langle x_ix_j\rangle} \prod_{\substack{i,j\in I^-\\ i<j}} \frac{\langle x_ix_j/q\rangle}{\langle x_ix_j\rangle}$$

$$\cdot \prod_{\substack{i\in I^+\\ j\in I^-}} \frac{\langle qx_i/x_j\rangle}{\langle x_i/x_j\rangle} \prod_{\substack{i\in I^-\\ 1\le k\le n}} \frac{\langle \sqrt{q}x_iy_k^{\pm 1}\rangle}{\langle x_iy_k^{\pm 1}/\sqrt{q}\rangle} \Bigg)$$

$$= \frac{\langle ab\rangle_m}{\langle q^{1-m}/ab\rangle_n} \prod_{\substack{1\le i\le m\\ 1\le k\le n}} \frac{\langle y_k/\sqrt{q}x_i\rangle}{\langle \sqrt{q}y_k/x_i\rangle} \sum_{\substack{K^-\sqcup K^+\\ =\{1,\dots,n\}}} \Bigg(\prod_{k\in K^+} \frac{\langle \sqrt{q}y_k/a, \sqrt{q}y_k/b\rangle}{\langle y_k^2\rangle} \prod_{k\in K^-} \frac{-\langle ay_k/\sqrt{q}, by_k/\sqrt{q}\rangle}{\langle y_k^2\rangle}$$

$$\cdot \prod_{\substack{k,l\in K^+\\ k<l}} \frac{\langle qy_ky_l\rangle}{\langle y_ky_l\rangle} \prod_{\substack{k,l\in K^-\\ k<l}} \frac{\langle y_ky_l/q\rangle}{\langle y_ky_l\rangle} \prod_{\substack{k\in K^+\\ l\in K^-}} \frac{\langle qy_k/y_l\rangle}{\langle y_k/y_l\rangle} \prod_{\substack{k\in K^-\\ 1\le i\le m}} \frac{\langle \sqrt{q}y_kx_i^{\pm 1}\rangle}{\langle y_kx_i^{\pm 1}/\sqrt{q}\rangle} \Bigg). \tag{3.22}$$

Specializing this formula as $x = p_\alpha(z;q), y = p_\beta(w;q)$ ($\alpha \in \mathbb{N}^M, \beta \in \mathbb{N}^N, |\alpha| = m, |\beta| = n$), we obtain

$$\sum_{0\le\mu\le\alpha} \Bigg(\prod_{1\le i\le M} \frac{\langle z_i/a, z_i/b\rangle_{\mu_i}}{\langle az_i, bz_i\rangle_{\mu_i}} \prod_{1\le i<j\le M} \frac{\langle q^{\mu_i-\mu_j} z_i/z_j\rangle}{\langle z_i/z_j\rangle} \prod_{1\le i\le j\le M} \frac{\langle q^{\mu_i+\mu_j} z_iz_j/q\rangle}{\langle z_iz_j/q\rangle}$$

$$\cdot \prod_{1\le i,j\le M} \frac{\langle z_iz_j/q, z_i/q^{\alpha_j}z_j\rangle_{\mu_i}}{\langle q^{\alpha_j}z_iz_j, qz_i/z_j\rangle_{\mu_i}} \prod_{\substack{1\le i\le M\\ 1\le k\le N}} \frac{\langle q^{\beta_k}z_iw_k/\sqrt{q}, \sqrt{q}z_i/w_k\rangle_{\mu_i}}{\langle z_iw_k/\sqrt{q}, \sqrt{q}z_i/q^{\beta_k}w_k\rangle_{\mu_i}} \Bigg)$$

$$= \frac{\langle ab\rangle_{|\alpha|}}{\langle q^{1-|\alpha|}/ab\rangle_{|\beta|}} \frac{\prod_{1\le k\le N}\langle \sqrt{q}w_k/a, \sqrt{q}w_k/b\rangle_{\beta_k}}{\prod_{1\le i\le M}\langle az_i, bz_i\rangle_{\alpha_i}} \frac{\prod_{1\le i,j\le M}\langle z_iz_j\rangle_{\alpha_i}}{\prod_{1\le i<j\le M}\langle z_iz_j\rangle_{\alpha_i+\alpha_j}} \frac{\prod_{1\le k<l\le N}\langle w_kw_l\rangle_{\beta_k+\beta_l}}{\prod_{1\le k,l\le N}\langle w_kw_l\rangle_{\beta_k}}$$

$$\cdot \prod_{\substack{1\le k\le N\\ 1\le i\le M}} \frac{\langle \sqrt{q}w_k/q^{\alpha_i}z_i\rangle_{\beta_k}}{\langle \sqrt{q}w_k/z_i\rangle_{\beta_k}} \sum_{0\le\nu\le\beta} \Bigg(\prod_{1\le k\le N} \frac{\langle aw_k/\sqrt{q}, bw_k/\sqrt{q}\rangle_{\nu_k}}{\langle \sqrt{q}w_k/a, \sqrt{q}w_k/b\rangle_{\nu_k}} \prod_{1\le k<l\le N} \frac{\langle q^{\nu_k-\nu_l}w_k/w_l\rangle}{\langle w_k/w_l\rangle}$$

$$\cdot \prod_{1\le k\le l\le N} \frac{\langle q^{\nu_k+\nu_l}w_kw_l/q\rangle}{\langle w_kw_l/q\rangle} \prod_{1\le k,l\le N} \frac{\langle w_kw_l/q, w_k/q^{\beta_l}w_l\rangle_{\nu_k}}{\langle q^{\beta_l}w_kw_l, qw_k/w_l\rangle_{\nu_k}} \prod_{\substack{1\le k\le N\\ 1\le i\le M}} \frac{\langle q^{\alpha_i}w_kz_i/\sqrt{q}, \sqrt{q}w_k/z_i\rangle_{\nu_k}}{\langle w_kz_i/\sqrt{q}, \sqrt{q}w_k/q^{\alpha_i}z_i\rangle_{\nu_k}} \Bigg). \tag{3.23}$$

Relabeling M and N by m and n, and replacing the variables and parameters by

$$z_i \to \sqrt{q}x_i \ (i = 1,\dots,m), \quad w_k \to y_k \ (k = 1,\dots,n), \quad (a,b) \to (\sqrt{q}/a_1, \sqrt{q}/a_2), \tag{3.24}$$

we obtain a transformation formulas of type C, due to Rosengren [R] (Corollary 4.4).

Theorem. 3.3. *For $\alpha \in \mathbb{N}^m$ and $\beta \in \mathbb{N}^n$, the following identity holds:*

$$\sum_{0\le\mu\le\alpha}\Bigg(\prod_{1\le i\le m}\frac{\langle a_1x_i,a_2x_i\rangle_{\mu_i}}{\langle qx_i/a_1,qx_i/a_2\rangle_{\mu_i}}\prod_{1\le i<j\le m}\frac{\langle q^{\mu_i-\mu_j}x_i/x_j\rangle}{\langle x_i/x_j\rangle}\prod_{1\le i\le j\le m}\frac{\langle q^{\mu_i+\mu_j}x_ix_j\rangle}{\langle x_ix_j\rangle}$$
$$\cdot\prod_{1\le i,j\le m}\frac{\langle x_ix_j,x_i/q^{\alpha_j}x_j\rangle_{\mu_i}}{\langle q^{\alpha_j+1}x_ix_j,qx_i/x_j\rangle_{\mu_i}}\prod_{\substack{1\le i\le m\\1\le k\le n}}\frac{\langle q^{\beta_k}x_iy_k,qx_i/y_k\rangle_{\mu_i}}{\langle x_iy_k,q^{1-\beta_k}x_i/y_k\rangle_{\mu_i}}\Bigg)$$
$$=\frac{\langle q/a_1a_2\rangle_{|\alpha|}}{\langle a_1a_2/q^{|\alpha|}\rangle_{|\beta|}}\frac{\prod_{1\le k\le n}\langle a_1y_k,a_2y_k\rangle_{\beta_k}}{\prod_{1\le i\le m}\langle qx_i/a_1,qx_i/a_2\rangle_{\alpha_i}}\frac{\prod_{1\le i,j\le m}\langle qx_ix_j\rangle_{\alpha_i}}{\prod_{1\le i<j\le m}\langle qx_ix_j\rangle_{\alpha_i+\alpha_j}}\frac{\prod_{1\le k<l\le n}\langle y_ky_l\rangle_{\beta_k+\beta_l}}{\prod_{1\le k,l\le n}\langle y_ky_l\rangle_{\beta_k}}$$
$$\cdot\prod_{\substack{1\le k\le n\\1\le i\le m}}\frac{\langle y_k/q^{\alpha_i}x_i\rangle_{\beta_k}}{\langle y_k/x_i\rangle_{\beta_k}}\sum_{0\le\nu\le\beta}\Bigg(\prod_{1\le k\le n}\frac{\langle y_k/a_1,y_k/a_2\rangle_{\nu_k}}{\langle a_1y_k,a_2y_k\rangle_{\nu_k}}\prod_{1\le k<l\le n}\frac{\langle q^{\nu_k-\nu_l}y_k/y_l\rangle}{\langle y_k/y_l\rangle}$$
$$\cdot\prod_{1\le k\le l\le n}\frac{\langle q^{\nu_k+\nu_l}y_ky_l/q\rangle}{\langle y_ky_l/q\rangle}\prod_{1\le k,l\le n}\frac{\langle y_ky_l/q,y_k/q^{\beta_l}y_l\rangle_{\nu_k}}{\langle q^{\beta_l}y_ky_l,qy_k/y_l\rangle_{\nu_k}}\prod_{\substack{1\le k\le n\\1\le i\le m}}\frac{\langle q^{\alpha_i}y_kx_i,y_k/x_i\rangle_{\nu_k}}{\langle y_kx_i,y_k/q^{\alpha_i}x_i\rangle_{\nu_k}}\Bigg).\tag{3.25}$$

Rosengren derived this result from Gustafson's summation formula of multilateral basic hypergeometric series for type C. We remark that Lassalle has derived a special case of Theorem 3.3 from a rational function identity by the method of principal specialization ([L], Theorem 11). His rational function identity (Theorem 6) corresponds to (3.20) with $a=q,b=-q$.

4 New family of q-difference operators

In the A type case, it is known that there exists an explicit operator $\mathcal{H}_A^x(u;q,t)$ satisfying the following equation [N2]:

$$(u;q)_\infty\mathcal{H}_A^x(u;q,t)\Psi_A(x;y)=(t^mq^nu;q)_\infty\mathcal{D}_A^y(u;t,q)\Psi_A(x;y),\tag{4.1}$$

where $\Psi_A(x;y)=\prod_{1\le i\le m}\prod_{1\le k\le n}(x_i-y_k)$ is the kernel function of dual Cauchy type and $\mathcal{D}_A^y(u;q,t)$ is the Macdonald q-difference operator:

$$\mathcal{D}_A^y(u;q,t)=\sum_{r=0}^n(-u)^rD_{A,r}^y(q,t),\tag{4.2}$$

$$D_{A,r}^y(q,t)=t^{\binom{r}{2}}\sum_{\substack{K\subset\{1,\dots,n\}\\|K|=r}}\prod_{\substack{k\in K\\l\notin K}}\frac{ty_k-y_l}{y_k-y_l}\prod_{k\in K}T_{q,y_k}.\tag{4.3}$$

The operator $\mathcal{H}_A^x(u;q,t)$ is defined by

$$\mathcal{H}_A^x(u;q,t)=\sum_{l=0}^\infty u^lH_{A,l}^x,\tag{4.4}$$

$$H_{A,l}^x=\sum_{\substack{\mu\in\mathbb{N}^m\\|\mu|=l}}\prod_{1\le i<j\le m}\frac{q^{\mu_i}x_i-q^{\mu_j}x_j}{x_i-x_j}\prod_{1\le i,j\le m}\frac{(tx_i/x_j;q)_{\mu_i}}{(qx_i/x_j;q)_{\mu_i}}\prod_{1\le i\le m}T_{q,x_i}^{\mu_i}.\tag{4.5}$$

We can obtain this fact as the special case of Kajihara's Euler transformation formula. It is also known that the Macdonald polynomials $P_{A,\lambda}(x|q,t)$ for type A are the joint eigenfunctions of $\mathcal{H}_A^x(u;q,t)$:

$$\mathcal{H}_A^x(u;q,t)P_{A,\lambda}(x|q,t) = P_{A,\lambda}(x|q,t)\prod_{1\le i\le m}\frac{(ut^{m-i+1}q^{\lambda_i};q)_\infty}{(ut^{m-i}q^{\lambda_i};q)_\infty}. \tag{4.6}$$

The commutativity of this family $\{H_{A,l}^x\}_{l=0}^\infty$ is proved in [Sa] through the Wronski relations in the elliptic setting.

In this section, we give the BC type analogue of (4.1). Namely, we construct an explicit operator $\mathcal{H}^x(u;q,t)$ which satisfies

$$\mathcal{H}^x(u;q,t)\Psi(x;y) = const.\cdot\widehat{\mathcal{D}}^y(u)\Psi(x;y), \tag{4.7}$$

$$\widehat{\mathcal{D}}^y(u) = \sum_{r=0}^{n}(-1)^r e(u;\widehat{\alpha})_{q,n-r}\widehat{D}_r^y. \tag{4.8}$$

Here $\Psi(x;y) := \prod_{\substack{1\le i\le m\\1\le k\le n}} e(x_i;y_k)$ is the kernel function of dual Cauchy type for type BC introduced by Mimachi [Mi2] and $\widehat{\ }$ means the operation of replacing the parameters (q,t) with (t,q). Therefore, $\widehat{\alpha} = \sqrt{q/t}\alpha$ and $\widehat{D}_r^y$ are the t-difference operators $D_r^y(a,b,c,d|t,q)$. Note that $\Psi(x;y)$ is expanded by the Koornwinder polynomials $P_\lambda(x)$ as follows [Mi2]:

$$\Psi(x;y) = \sum_{\lambda\subset(n^m)}(-1)^{\lambda^*}P_\lambda(x)\widehat{P}_{\lambda^*}(y), \tag{4.9}$$

$$\lambda^* = (m-\lambda'_n, m-\lambda'_{n-1},\dots,m-\lambda'_1). \tag{4.10}$$

4.1 Affine Hecke algebras

In order to guarantee the existence of $\mathcal{H}^x(u;q,t)$ satisfying (4.7), we use the framework of affine Hecke algebras, due to Cherednik [C] and Macdonald [Ma2]. In this subsection, we recall Noumi's representations of affine Hecke algebras of type C and the fundamental facts of q-Dunkl operators. Our notation is due to [N1].

We use the parameters t_0, t_m, u_0, u_m are defined such that

$$(a,b,c,d) = (t_m^{\frac12}u_m^{\frac12}, -t_m^{\frac12}u_m^{-\frac12}, q^{\frac12}t_0^{\frac12}u_0^{\frac12}, -q^{\frac12}t_0^{\frac12}u_0^{-\frac12}). \tag{4.11}$$

We define the Lustig operators $T_0^x, T_1^x,\dots,T_m^x$ as

$$T_0^x = t_0^{\frac12} + t_0^{-\frac12}\frac{(1-t_0^{\frac12}u_0^{\frac12}q^{\frac12}x_1^{-1})(1+t_0^{\frac12}u_0^{-\frac12}q^{\frac12}x_1^{-1})}{1-qx_1^{-2}}(s_0^x-1), \tag{4.12}$$

$$T_i^x = t^{\frac12} + t^{-\frac12}\frac{1-tx_i/x_{i+1}}{1-x_i/x_{i+1}}(s_i^x-1) \quad (i=1,\dots,m-1), \tag{4.13}$$

$$T_m^x = t_m^{\frac12} + t_m^{-\frac12}\frac{(1-t_m^{\frac12}u_m^{\frac12}x_m)(1+t_m^{\frac12}u_m^{-\frac12}x_m)}{1-x_m^2}(s_m^x-1). \tag{4.14}$$

Here $s_0^x, s_1^x,\dots,s_m^x$ are the simple reflections

$$(s_0^x f)(x) = f(qx_1^{-1},x_2,\dots,x_m), \tag{4.15}$$

$$(s_i^x f)(x) = f(x_1, \ldots, x_{i+1}, x_i, \ldots, x_m) \quad (i = 1, \ldots, m-1), \tag{4.16}$$
$$(s_m^x f)(x) = f(x_1, x_2, \ldots, x_m^{-1}). \tag{4.17}$$

Note that these operators $s_1^x, \ldots, s_m^x$ generate the Weyl group W_m of type BC_m. The algebra $\mathcal{H}(W_m^{\mathrm{aff}})$ generated by $T_0^x, T_1^x, \ldots, T_m^x$ is isomorphic to the affine Hecke algebra of type C_m:

$$(T_i^x - t_i^{\frac{1}{2}})(T_i^x + t_i^{-\frac{1}{2}}) = 0 \quad (i = 0, 1, \ldots, m), \tag{4.18}$$
$$T_i^x T_{i+1}^x T_i^x = T_{i+1}^x T_i^x T_{i+1}^x \quad (i = 1, \ldots, m-2), \tag{4.19}$$
$$T_i^x T_{i+1}^x T_i^x T_{i+1}^x = T_{i+1}^x T_i^x T_{i+1}^x T_i^x \quad (i = 0, m-1), \tag{4.20}$$
$$T_i^x T_j^x = T_j^x T_i^x \quad (|i-j| \geq 2). \tag{4.21}$$

Here we wrote $t_1 = \cdots = t_{m-1} = t$. The q-Dunkl operators $Y_1^x, \ldots, Y_m^x$ are defined by

$$Y_i^x = (T_i^x \cdots T_{m-1}^x)(T_m^x \cdots T_0^x)((T_1^x)^{-1} \cdots (T_{i-1}^x)^{-1}) \ (i = 1, \ldots, m). \tag{4.22}$$

We denote by $\mathbb{C}(x)[T_{q,x}^{\pm 1}]$ the ring of q-difference operators with rational coefficients. For any $A^x \in \mathcal{H}(W_m^{\mathrm{aff}})$, A^x is expressed as $\sum_{w \in W_m} A_w^x w$ $(A_w^x \in \mathbb{C}(x)[T_{q,x}^{\pm 1}])$. Then we define the q-difference operator L_A^x by $L_A^x = \sum_{w \in W_m} A_w^x$. It is known that the following fact holds.

For any W_m-invariant Laurent polynomial $f(\xi)$ in the variables $\xi = (\xi_1, \ldots, \xi_m)$, and for any W_m-invariant Laurent polynomial $\varphi(x) \in \mathbb{C}[x^{\pm 1}]^{W_m}$, one has

$$f(Y^x)\varphi(x) = L_{f(Y^x)}^x \varphi(x). \tag{4.23}$$

Furthermore, the q-difference operator $L_f^x := L_{f(Y^x)}^x$ satisfies for any partition λ

$$f(Y^x)P_\lambda(x) = L_f^x P_\lambda(x) = P_\lambda(x) f(\alpha t^{\delta_m} q^\lambda). \tag{4.24}$$

In particular, the q-difference operators L_f^x for the interpolation polynomials $f = e_r(\xi; \alpha|t)$ of column type give rise to van Diejen's operators D_r^x. From this view point, we call D_r^x "column type" q-difference operators. Since $\{e_r(\xi; \alpha|t)\}_{r=1}^m$ is the generator system of the ring $\mathbb{C}[\xi^{\pm 1}]^{W_m}$ of W_m-invariant Laurent polynomials in m variables, L_f^x is an element of $\mathbb{C}[D_1^x, \ldots, D_m^x]$ for any $f(\xi) \in \mathbb{C}[\xi^{\pm 1}]^{W_m}$. The operator $\mathcal{H}^x(u; q, t)$ to be constructed in the next subsection is a generating function of "row type" q-difference operators.

4.2 Construction of row type q-difference operators

The results of this subsection are based on a discussion with M. Noumi.

Let $\xi = (\xi_1, \ldots, \xi_m)$ and $\eta = (\eta_1, \ldots, \eta_n)$. We define

$$H(u; \xi) = \Phi_0(u; x|q, t) = u^{m\gamma} \prod_{i=1}^m \frac{(\sqrt{tq}u\xi_i^{\pm 1}; q)_\infty}{(\sqrt{q/t}u\xi_i^{\pm 1}; q)_\infty}, \quad E(u; \eta) = \prod_{k=1}^n e(u; \eta), \tag{4.25}$$

where γ is a complex number such that $q^\gamma = t$. Note that $E(u; \eta)$ is a generating function of $e_r(\eta; \alpha|t)$:

$$E(u; \eta) = \sum_{r=0}^n (-1)^r e_r(\eta; \alpha|t) e(u; \alpha)_{t, n-r}. \tag{4.26}$$

This implies that

$$E(u;Y^y)P_\lambda(y)=P_\lambda(y)\prod_{i=1}^n e(u;\alpha t^{n-i}q^{\lambda_i}). \tag{4.27}$$

Namely, the operator L_f^y for $f=E(u;\eta)$ is the generating function $D^y(u)$ of column type operator D_r^y. In the following, we regard $H(u;\xi)$ as an element of $u^{-m\gamma}\mathbb{C}[\xi^{\pm1}]^{W_m}[[u]]$. Namely, $u^{m\gamma}H(u;\xi)$ is a formal power series in u with coefficients in the ring of W_m-invariant Laurent polynomials in ξ.

Lemma. 4.1. *The operators $H(u;Y^x)$ and $E(u;Y^y)$ satisfy the following identity as formal power series in u:*

$$\frac{H(u;Y^x)}{H(u;\alpha t^{\delta_m})}\Psi(x;y)=\frac{e(u;\widehat{\alpha})_n}{e(u;t^m\widehat{\alpha})_n}\frac{E(u;\widehat{Y}^y)}{E(u;\widehat{\alpha}q^{\delta_n})}\Psi(x;y)=\frac{E(u;\widehat{Y}^y)}{e(u;t^m\widehat{\alpha})_n}\Psi(x;y). \tag{4.28}$$

Proof. Note first that $E(u;\widehat{\alpha}q^{\delta_n})=e(u;\widehat{\alpha})_n$. From (4.9) and (4.24), the formula (4.28) is equivalent to the identity on the eigenvalue:

$$\frac{H(u;\alpha t^{\delta_m}q^{\lambda})}{H(u;\alpha t^{\delta_m})}=\frac{e(u;\widehat{\alpha})_n}{e(u;t^m\widehat{\alpha})_n}\frac{E(u;\widehat{\alpha}q^{\delta_n}t^{\lambda^*})}{E(u;\widehat{\alpha}q^{\delta_n})}\quad(\lambda\subset(n^m)). \tag{4.29}$$

The left-hand side is equal to

$$\prod_{1\le i\le m}\frac{(\sqrt{q/t}u\alpha t^{m-i};q)_{\lambda_i}(\sqrt{tq}u\alpha^{-1}t^{-m+i}q^{-\lambda_i};q)_{\lambda_i}}{(\sqrt{tq}u\alpha t^{m-i};q)_{\lambda_i}(\sqrt{q/t}u\alpha^{-1}t^{-m+i}q^{-\lambda_i};q)_{\lambda_i}}. \tag{4.30}$$

Using the notation $c_{ij}=\alpha t^{m-i}q^{j-1}$ $(1\le i\le m, 1\le j\le n)$, we obtain

$$(4.30)=\prod_{(i,j)\in\lambda}\frac{t(1-\sqrt{q/t}c_{ij}u)(1-\sqrt{q/t}c_{ij}u^{-1})}{(1-\sqrt{tq}c_{ij}u)(1-\sqrt{tq}c_{ij}u^{-1})}=\prod_{(i,j)\in\mu}\frac{\langle u^{\pm1}\sqrt{q/t}c_{ij}\rangle}{\langle u^{\pm1}\sqrt{tq}c_{ij}\rangle}. \tag{4.31}$$

On the other hand, since we compute

$$\begin{aligned}\frac{E(u;\widehat{\alpha}q^{\delta_n}t^{\lambda^*})}{E(u;\widehat{\alpha}q^{\delta_n})}&=\prod_{k=1}^n\frac{\langle\sqrt{q/t}u\alpha q^{n-k}t^{\lambda_k^*}\rangle\langle\sqrt{t/q}u\alpha^{-1}q^{k-n}t^{-\lambda_k^*}\rangle}{\langle\sqrt{q/t}u\alpha q^{n-k}\rangle\langle\sqrt{t/q}u\alpha^{-1}q^{k-n}\rangle}\\&=\prod_{k=1}^n\frac{\langle\sqrt{tq}u\alpha q^{n-k}t^{\lambda_k^*-1}\rangle\langle\sqrt{tq}u^{-1}\alpha q^{n-k}t^{\lambda_k^*-1}\rangle}{\langle\sqrt{q/t}u\alpha q^{n-k}\rangle\langle\sqrt{q/t}u^{-1}\alpha q^{n-k}\rangle}\\&=\prod_{(i,j)\in(n^m)\backslash\lambda}\frac{\langle u^{\pm1}\sqrt{tq}c_{ij}\rangle}{\langle u^{\pm1}\sqrt{q/t}c_{ij}\rangle},\end{aligned} \tag{4.32}$$

and hence the right-hand side of (4.29) is equal to

$$\begin{aligned}\frac{\langle\sqrt{q/t}\alpha u^{\pm1}\rangle_{q,n}}{\langle\sqrt{tq}\alpha t^{m-1}u^{\pm1}\rangle_{q,n}}\prod_{(i,j)\in(n^m)\backslash\lambda}\frac{\langle u^{\pm1}\sqrt{tq}c_{ij}\rangle}{\langle u^{\pm1}\sqrt{q/t}c_{ij}\rangle}&=\prod_{(i,j)\in(n^m)}\frac{\langle u^{\pm1}\sqrt{q/t}c_{ij}\rangle}{\langle u^{\pm1}\sqrt{tq}c_{ij}\rangle}\prod_{(i,j)\in(n^m)\backslash\lambda}\frac{\langle u^{\pm1}\sqrt{tq}c_{ij}\rangle}{\langle u^{\pm1}\sqrt{q/t}c_{ij}\rangle}\\&=\prod_{(i,j)\in\lambda}\frac{\langle u^{\pm1}\sqrt{q/t}c_{ij}\rangle}{\langle u^{\pm1}\sqrt{tq}c_{ij}\rangle}.\end{aligned} \tag{4.33}$$

□

Next we show that $\frac{H(u;\xi)}{H(u;\alpha t^{\delta_m})}$ is a generating function of the row type interpolation polynomials $h_l(\xi;\alpha|q,t)$ introduced by [KNS]:

$$h_l(\xi;\alpha|q,t) = \sum_{\substack{\nu\in\mathbb{N}^m \\ |\nu|=l}} \frac{\langle t\rangle_{\nu_1}\cdots\langle t\rangle_{\nu_m}}{\langle q\rangle_{\nu_1}\cdots\langle q\rangle_{\nu_m}} e(\xi_1;\alpha)_{\nu_1} e(\xi_2;tq^{\nu_1}\alpha)_{\nu_2}\cdots e(\xi_m;t^{m-1}q^{\nu_1+\cdots\nu_{m-1}}\alpha)_{\nu_m}. \tag{4.34}$$

Note that the Laurent polynomial $h_l(\xi;\alpha|q,t)$ is W_m-invariant and satisfies the following interpolation property: For any partition $\mu \not\supset (l)$,

$$h_l(\alpha t^{\delta_m} q^{\mu};\alpha|q,t) = 0. \tag{4.35}$$

Lemma. 4.2. *The following identity holds as formal power series in u:*

$$\frac{H(u;\xi)}{H(u;\alpha t^{\delta_m})} = \sum_{l=0}^{\infty} \frac{h_l(\xi;\alpha|q,t)}{e(u;t^m\sqrt{q/t}\alpha)_l}. \tag{4.36}$$

Proof. If $t = q^{-k}$ $(k = 0,1,2,\ldots)$, from Lemma 5.4 in [KNS] one has

$$H(u;\xi) = \prod_{1\le i\le m} e(u;q^{\frac{1}{2}(1-k)}\xi_i)_k = \sum_{l=0}^{km} h_l(\xi;\alpha|q,t)e(u;\sqrt{tq}/\alpha)_{km-l}. \tag{4.37}$$

Since $H(u;\alpha t^{\delta_m})$ with $t = q^{-k}$ equals $e(u;\sqrt{tq}/\alpha)_{km}$, by dividing the both sides of (4.37) by $e(u;\sqrt{tq}/\alpha)_{km}$, we obtain this lemma in the case of $t = q^{-k}$. In the formal power series of u in each side of (4.36), all the coefficients are the rational function in $t^{\frac{1}{2}}$. Hence the identity (4.36) follows from its validity at infinitely many values of $t = q^{-k}$ $(k = 0,1,2,\ldots)$. □

From this lemma, it follows that

$$\frac{H(u;Y^x)}{H(u;\alpha t^{\delta_m})} = \sum_{l=0}^{\infty} \frac{h_l(Y^x;\alpha|q,t)}{e(u;t^m\sqrt{q/t}\alpha)_l}. \tag{4.38}$$

We now define the q-difference operators $H_l^x := H_l^x(a,b,c,d|q,t)$ $(l = 0,1,2,\ldots)$ to be L_f^x for $f = h_l(\xi;\alpha|q,t)$, so that

$$H_l^x P_\lambda(x) = P_\lambda(x) h_l(\alpha t^{\delta_m} q^{\lambda};\alpha|q,t). \tag{4.39}$$

We call these operators H_l^x $(l = 0,1,2,\ldots)$ "row type" q-difference operators. We also introduce a generating function $\mathcal{H}^x(u) := \mathcal{H}^x(u;q,t)$ of H_l^x by

$$\mathcal{H}^x(u;q,t) = \sum_{l=0}^{\infty} \frac{H_l^x}{e(u;t^m\sqrt{q/t}\alpha)_l} \in \mathbb{C}(x)[T_{q,x}^{\pm 1}][[u]]. \tag{4.40}$$

From Lemma 4.1 , we obtain a "kernel identity of dual Cauchy type".

Theorem. 4.3. *The kernel function of dual Cauchy type intertwines the q-difference operator $\mathcal{H}^x(u)$ with the t-difference operator $\widehat{\mathcal{D}}^y(u)$:*

$$\mathcal{H}^x(u)\Psi(x;y) = \frac{\widehat{\mathcal{D}}^y(u)}{e(u;t^m\widehat{\alpha})_n}\Psi(x;y). \tag{4.41}$$

Theorem 4.3 gives the relationship between H_l and van Diejen's operators D_r.

Theorem. 4.4. *For any integer* $l = 0, 1, 2, \ldots, n$, *the following equation holds:*

$$(-1)^l H_l^x \Psi(x;y) = \sum_{0\le s\le l} \frac{\langle q^{n-l+1}\rangle_s}{\langle q\rangle_s} \langle q^{1-l}t^{-m}, t^{m-1}q^n\alpha^2\rangle_s \widehat{D}_{l-s}^y \Psi(x;y). \tag{4.42}$$

Proof. Since $H_l^x P_\mu(x) = 0$ if $\mu \subset (n^m)$ and $l > n$,

$$\mathcal{H}^x(u)\Psi(x;y) = \left(1 + \frac{1}{e(u;t^m\sqrt{q/t\alpha})}H_1^x + \cdots + \frac{1}{e(u;t^m\sqrt{q/t\alpha})_n}H_n^x\right)\Psi(x;y). \tag{4.43}$$

By using the q-Saalschütz sum (2.27), the right-hand side of (4.41) is expressed by

$$\frac{\widehat{D}_y(u)}{e(u;t^m\widehat{\alpha})_n}\Psi(x;y) = \sum_{l=0}^n \frac{1}{e(u;t^m\sqrt{q/t\alpha})_l}\sum_{0\le r\le l}(-1)^l \begin{bmatrix} n-r \\ l-r\end{bmatrix}_q \langle t^{-m}q^{1-l}, t^{m-1}q^n\alpha^2\rangle_{l-r}\widehat{D}_r^y\Psi(x;y). \tag{4.44}$$

Comparing the coefficient of $\frac{1}{e(u;t^m\sqrt{q/t\alpha})_l}$ in (4.43) with that in (4.44) for each l, we obtain (4.42).

□

4.3 Explicit formulas of H_l

In the previous subsection, we defined the row type q-difference operators H_l^x by q-Dunkl operators and showed the relationship between H_l^x and $\widehat{D}_r^y$. However it is difficult to compute the explicit expressions of operators H_l^x by means of the q-Dunkl operators. In this subsection, by using the special case of Theorem 3.1, we give the explicit formulas of H_l^x.

We consider Theorem 3.1 in the case of $\alpha = (M, \ldots, M) \in \mathbb{N}^m, \beta = (1, \ldots, 1) \in \mathbb{N}^n$:

$$\prod_{1\le i\le m}\frac{\langle tqx_i/a_1, tqx_i/a_2, tqx_i/a_3, tqx_i/a_4\rangle_M}{\langle tqx_i^2, t^2qx_i^2\rangle_M}\prod_{1\le i<j\le m}\frac{\langle tq^{M+1}x_ix_j, t^2q^{M+1}x_ix_j\rangle_M}{\langle tqx_ix_j, t^2qx_ix_j\rangle_M}$$

$$\cdot \sum_{0\le\mu^-\le\nu^-\le\nu^+\le\mu^+\le(M^m)}\Bigg((-1)^{mM+|\nu^+|+|\nu^-|}e(u;\sqrt{tq}/a_0)_{|\nu^+|-|\nu^-|}$$

$$\cdot\prod_{1\le i\le m}\frac{\langle a_1x_i, a_2x_i, a_3x_i, a_4x_i\rangle_{\nu_i^-}}{\langle tqx_i/a_1, tqx_i/a_2, tqx_i/a_3, tqx_i/a_4\rangle_{\nu_i^+}}\prod_{1\le i\le j\le m}\frac{\langle q^{\mu_i^-+\mu_j^-}x_ix_j, t^2q^{\mu_i^++\mu_j^+}x_ix_j\rangle}{\langle x_ix_j, t^2x_ix_j\rangle}$$

$$\cdot\prod_{1\le i<j\le m}\frac{\langle q^{\mu_i^--\mu_j^-}x_i/x_j, q^{\mu_i^+-\mu_j^+}x_i/x_j, q^{\nu_i^--\nu_j^-}x_i/x_j, q^{\nu_i^+-\nu_j^+}x_i/x_j\rangle\langle tx_ix_j, qx_ix_j\rangle_{\nu_i^-+\nu_j^-}}{\langle x_i/x_j, x_i/x_j, x_i/x_j, x_i/x_j\rangle\langle tqx_ix_j, t^2x_ix_j\rangle_{\nu_i^++\nu_j^+}}$$

$$\cdot\prod_{1\le i,j\le m}\frac{\langle tq^{\mu_i^-+\mu_j^+}x_ix_j, tq^{\nu_i^-+\nu_j^+}x_ix_j, q^{\mu_i^--\mu_j^+}x_i/tx_j\rangle\langle tx_ix_j\rangle_{\mu_i^-+\nu_j^+}\langle t^2x_ix_j\rangle_{\mu_i^++\nu_j^+}}{\langle tx_ix_j, tx_ix_j, x_i/tq^{\mu_j^+}x_j\rangle\langle qx_ix_j\rangle_{\mu_i^-+\nu_j^-}\langle tqx_ix_j\rangle_{\mu_i^++\nu_j^-}}$$

$$\cdot\prod_{1\le i,j\le m}\frac{\langle x_ix_j, x_i/q^{\nu_j^-}x_j, x_i/tq^Mx_j\rangle_{\mu_i^-}\langle tx_ix_j, x_i/q^Mx_j\rangle_{\mu_i^+}\langle qx_i/tq^{\nu_j^+}x_j, x_i/q^{\nu_j^+}x_j\rangle_{\nu_i^-}}{\langle tq^{M+1}x_ix_j, qx_i/tq^{\nu_j^+}x_j, qx_i/x_j\rangle_{\mu_i^-}\langle t^2q^{M+1}x_ix_j, qx_i/x_j\rangle_{\mu_i^+}\langle qx_i/tq^{\mu_j^+}x_j, qx_i/x_j\rangle_{\nu_i^-}}$$

$$\cdot \prod_{1\le i,j\le m} \frac{\langle x_i/q^{\mu_j^+}x_j\rangle_{\nu_i^+}}{\langle qx_i/x_j\rangle_{\nu_i^+}} \prod_{\substack{1\le i\le m\\1\le k\le n}} \frac{\langle qx_iy_k, qx_i/y_k\rangle_{\mu_i^-}\langle tq^{\mu_i^+}x_iy_k, tq^{\mu_i^+}x_i/y_k\rangle_{M-\mu_i^+}}{\langle x_iy_k, x_i/y_k\rangle_{\mu_i^-}\langle tq^{\mu_i^++1}x_iy_k, tq^{\mu_i^++1}x_i/y_k\rangle_{M-\mu_i^+}}\Bigg)$$

$$= e(u;\sqrt{tq}/a_0)_{mM-n} \prod_{1\le k\le n} \frac{\langle a_1y_k, a_2y_k, a_3y_k, a_4y_k\rangle}{\langle y_k^2, ty_k^2\rangle} \prod_{1\le k<l\le n} \frac{\langle qy_ky_l, tqy_ky_l\rangle}{\langle y_ky_l, ty_ky_l\rangle}$$

$$\cdot \sum_{0\le\lambda^-\le\kappa^-\le\kappa^+\le\lambda^+\le(1^n)} (-1)^{n+|\kappa^+|+|\kappa^-|}\Bigg(e(u;\sqrt{q/t}a_0)_{|\kappa^+|-|\kappa^-|}$$

$$\cdot \prod_{1\le k\le n} \frac{\langle y_k/a_1, y_k/a_2, y_k/a_3, y_k/a_4\rangle_{\kappa_k^-}}{\langle a_1y_k, a_2y_k, a_3y_k, a_4y_k\rangle_{\kappa_k^+}} \prod_{1\le k\le l\le n} \frac{\langle q^{\lambda_k^-+\lambda_l^--1}y_ky_l/t, tq^{\lambda_k^++\lambda_l^+-1}y_ky_l\rangle}{\langle y_ky_l/tq, ty_ky_l/q\rangle}$$

$$\cdot \prod_{1\le k<l\le n} \frac{\langle q^{\lambda_k^--\lambda_l^-}y_k/y_l, q^{\lambda_k^+-\lambda_l^+}y_k/y_l, q^{\kappa_k^--\kappa_l^-}y_k/y_l, q^{\kappa_k^+-\kappa_l^+}y_k/y_l\rangle\langle y_ky_l/q, y_ky_l/t\rangle_{\kappa_k^-+\kappa_l^-}}{\langle y_k/y_l, y_k/y_l, y_k/y_l, y_k/y_l\rangle\langle ty_ky_l/q, y_ky_l\rangle_{\kappa_k^++\kappa_l^+}}$$

$$\cdot \prod_{1\le k,l\le n} \frac{\langle q^{\lambda_k^-+\lambda_l^+-1}y_ky_l, q^{\kappa_k^-+\kappa_l^+-1}y_ky_l, q^{\lambda_k^--\lambda_l^+}y_k/ty_l\rangle\langle y_ky_l/q\rangle_{\lambda_k^-+\kappa_l^+}\langle ty_ky_l/q\rangle_{\lambda_k^++\kappa_l^+}}{\langle y_ky_l/q, y_ky_l/q, y_k/tq^{\lambda_l^+}y_l\rangle\langle y_ky_l/t\rangle_{\lambda_k^-+\kappa_l^-}\langle y_ky_l\rangle_{\lambda_k^++\kappa_l^-}}$$

$$\cdot \frac{\langle y_ky_l/tq, y_k/q^{\kappa_l^-}y_l, y_k/tqy_l\rangle_{\lambda_k^-}\langle y_ky_l/q, y_k/qy_l\rangle_{\lambda_k^+}\langle qy_k/tq^{\kappa_l^+}y_l, y_k/q^{\kappa_l^+}y_l\rangle_{\kappa_k^-}\langle y_k/q^{\lambda_l^+}y_l\rangle_{\kappa_k^+}}{\langle qy_ky_l, qy_k/tq^{\kappa_l^+}y_l, qy_k/y_l\rangle_{\lambda_k^-}\langle tqy_ky_l, qy_k/y_l\rangle_{\lambda_k^+}\langle qy_k/tq^{\lambda_l^+}y_l, qy_k/y_l\rangle_{\kappa_k^-}\langle qy_k/y_l\rangle_{\kappa_k^+}}$$

$$\cdot \prod_{\substack{1\le k\le n\\1\le i\le m}} \frac{\langle q^My_kx_i, y_k/tx_i\rangle_{\lambda_k^-}\langle tq^{\lambda_k^+}y_kx_i, q^{\lambda_k^+-M}y_k/x_i\rangle_{1-\lambda_k^+}}{\langle y_kx_i, y_k/tq^Mx_i\rangle_{\lambda_k^-}\langle tq^{M+\lambda_k^+}y_kx_i, q^{\lambda_k^+}y_k/x_i\rangle_{1-\lambda_k^+}}\Bigg). \tag{4.45}$$

Then the both sides are W_m-invariant for the variables x and W_n-invariant for y. Since

$$\prod_{\substack{1\le i\le m\\1\le k\le n}} \frac{\langle qx_iy_k, qx_i/y_k\rangle_{\mu_i^-}\langle tq^{\mu_i^+}x_iy_k, tq^{\mu_i^+}x_i/y_k\rangle_{M-\mu_i^+}}{\langle x_iy_k, x_i/y_k\rangle_{\mu_i^-}\langle tq^{1+\mu_i^+}x_iy_k, tq^{\mu_i^++1}x_i/y_k\rangle_{M-\mu_i^+}}$$

$$= \prod_{\substack{1\le i\le m\\1\le k\le n}} \frac{T_{q,x_i}^{\mu_i^-}\langle x_iy_k, x_i/y_k\rangle}{\langle x_iy_k, x_i/y_k\rangle} \prod_{\substack{1\le i\le m\\1\le k\le n}} \frac{T_{q,x_i}^{-(M-\mu_i^+)}\langle tq^Mx_iy_k, tq^Mx_i/y_k\rangle}{\langle tq^Mx_iy_k, tq^Mx_i/y_k\rangle}, \tag{4.46}$$

the left-hand side of (4.45) is expressed as the bilinear form:

$$\sum_{0\le\mu^-\le\mu^+\le M} A_{\mu^-,\mu^+}(u;x) \prod_{\substack{1\le i\le m\\1\le k\le n}} \frac{T_{q,x_i}^{\mu_i^-}\langle x_iy_k, x_i/y_k\rangle}{\langle x_iy_k, x_i/y_k\rangle} \prod_{\substack{1\le i\le m\\1\le k\le n}} \frac{T_{q,x_i}^{-(M-\mu_i^+)}\langle tq^Mx_iy_k, tq^Mx_i/y_k\rangle}{\langle tq^Mx_iy_k, tq^Mx_i/y_k\rangle}. \tag{4.47}$$

In terms of this form, Theorem 3.1 is regarded as a kind of bilinear transformation formula.

We consider the special case $t = q^{-M}$ in (4.45). Then the factors involving both variables x

and y are expressed by

$$\prod_{\substack{1\le i\le m\\1\le k\le n}} \frac{\langle qx_iy_k, qx_i/y_k\rangle_{\mu_i^-}\langle tq^{\mu_i^+}x_iy_k, tq^{\mu_i^+}x_i/y_k\rangle_{M-\mu_i^+}}{\langle x_iy_k, x_i/y_k\rangle_{\mu_i^-}\langle tq^{1+\mu_i^+}x_iy_k, tq^{\mu_i^++1}x_i/y_k\rangle_{M-\mu_i^+}} = \prod_{1\le i\le m}\left[\frac{T_{q,x_i}^{\mu_i^-}\Psi(x_i;y)}{\Psi(x_i;y)}\frac{T_{q,x_i}^{-(M-\mu_i^+)}\Psi(x_i;y)}{\Psi(x_i;y)}\right], \tag{4.48}$$

$$\prod_{\substack{1\le k\le n\\1\le i\le m}} \frac{\langle q^My_kx_i, y_k/tx_i\rangle_{\lambda_k^-}\langle tq^{\lambda_k^+}y_kx_i, q^{\lambda_k^+-M}y_k/x_i\rangle_{1-\lambda_k^+}}{\langle y_kx_i, y_k/tq^Mx_i\rangle_{\lambda_k^-}\langle tq^{M+\lambda_k^+}y_kx_i, q^{\lambda_k^+}y_k/x_i\rangle_{1-\lambda_k^+}} = \prod_{1\le k\le n}\left[\frac{T_{t,y_k}^{1-\lambda_k^+}\Psi(x;y_k)}{\Psi(x;y_k)}\frac{T_{t,y_k}^{-\lambda_k^-}\Psi(x;y_k)}{\Psi(x;y_k)}\right]. \tag{4.49}$$

We check that each side of (4.45) does not have a pole at the point $t = q^{-M}$. We have only to examine the following factor in the left-hand side:

$$\prod_{1\le i\le m} \frac{\langle q^{\mu_i^- - \mu_i^+}/t\rangle\langle q^{-M}/t\rangle_{\mu_i^-}\langle q^{1-\nu_i^+}/t\rangle_{\nu_i^-}}{\langle q^{-\mu_i^+}/t\rangle\langle q^{1-\nu_i^+}/t\rangle_{\mu_i^-}\langle q^{1-\mu_i^+}/t\rangle_{\nu_i^-}}. \tag{4.50}$$

If $\mu_i^+ = M$ for some $i = 1, \ldots, m$, the denominator has a zero at $t = q^{-M}$. But when $\mu_i^- = 0$, since $\langle q^{\mu_i^- - \mu_i^+}/t\rangle = \langle q^{-\mu_i^+}/t\rangle$,

$$(4.50) = \prod_{1\le i\le m} \frac{\langle q^{1-\nu_i^+}/t\rangle_{\nu_i^-}}{\langle q/t\rangle_{\nu_i^-}}. \tag{4.51}$$

Also, when $\mu_i^- > 0$, since $\langle q^{-\mu_i^+}/t\rangle = \langle q^{-M}/t\rangle$,

$$(4.50) = \prod_{1\le i\le m} \frac{\langle q^{\mu_i^-}\rangle\langle q^{1-M}/t\rangle_{\mu_i^- -1}\langle q^{1-\nu_i^+}/t\rangle_{\nu_i^-}}{\langle q^{1-\nu_i^+}/t\rangle_{\mu_i^-}\langle q\rangle_{\nu_i^-}}. \tag{4.52}$$

Therefore the point $t = q^{-M}$ in the left-hand side is an apparent singularity. Note that unless $\mu_i^- = 0$ or $\mu_i^+ = M$ for each $1 \le i \le m$, the corresponding term in the left-hand side of (4.45) is zero. We can also check that the point $t = q^{-M}$ in the right-hand side is an apparent singularity.

From the argument above, specializing Theorem 3.1 as $\alpha = (M^m), \beta = (1^n)$ and $t = q^{-M}$ ($M = 0, 1, 2, \ldots$), we find that the factor involving both variables x and y in each side of (4.45) simplifies to the form

$$\prod_{i:\mu_i^->0} \frac{T_{q,x_i}^{\mu_i^-}\Psi(x_i;y)}{\Psi(x_i;y)} \prod_{i:\mu_i^+<M} \frac{T_{q,x_i}^{-(M-\mu_i^+)}\Psi(x_i;y)}{\Psi(x_i;y)} = \frac{T_{q,x}^{\nu}\Psi(x;y)}{\Psi(x;y)}, \tag{4.53}$$

$$\prod_{k:\lambda_k^+=0} \frac{T_{t,y_k}^{1-\lambda_k^+}\Psi(x;y_k)}{\Psi(x;y_k)} \prod_{k:\lambda_k^-=1} \frac{T_{t,y_k}^{-\lambda_k^-}\Psi(x;y_k)}{\Psi(x;y_k)} = \frac{T_{t,y}^{\kappa}\Psi(x;y)}{\Psi(x;y)}, \tag{4.54}$$

respectively. Here we set ν and κ as follows:

$$\nu_i = \begin{cases} \mu_i^- & (\mu_i^- > 0, \mu_i^+ = M), \\ -(M-\mu_i^+) & (\mu_i^+ < M, \mu_i^- = 0), \\ 0 & (\text{otherwise}), \end{cases} \qquad \kappa_k = \begin{cases} 1 & (\lambda_k^+ = 0, \lambda_k^- = 0), \\ -1 & (\lambda_k^+ = 1, \lambda_k^- = 1), \\ 0 & (\text{otherwise}). \end{cases} \tag{4.55}$$

In this way, the left-hand side can be interpreted as the action of a q-difference operator on the kernel function of dual Cauchy type. The right-hand side is also expressed by the action of a t-difference operator, and in fact is equal to

$$\Psi(x;y)^{-1}e(u;\sqrt{tq}/a_0)_{mM-n}\mathcal{D}^y(u;a_1,a_2,a_3,a_4|t,q)\Psi(x;y). \tag{4.56}$$

We replace the parameters (a_1,a_2,a_3,a_4) with (a,b,c,d). Then the left-hand side of (4.45) can be expressed as

$$\Psi(x;y)^{-1}\left(\sum_{l=0}^{mM}e(u;\sqrt{tq}/\alpha)_{mM-l}K_l^x\right)\Psi(x;y) \tag{4.57}$$

for some q-difference operators K_l^x for which we will determine the explicit formulas later. Hence we have

$$\Psi(x;y)^{-1}\left(\sum_{l=0}^{mM}e(u;\sqrt{tq}/\alpha)_{mM-l}K_l^x\right)\Psi(x;y)=\Psi(x;y)^{-1}e(u;\sqrt{tq}/\alpha)_{mM-n}\widehat{\mathcal{D}}^y(u)\Psi(x;y). \tag{4.58}$$

Comparing (4.41) with (4.58), we obtain that

$$\begin{aligned}
&\Psi(x;y)^{-1}e(u;t^m\widehat{\alpha})_n e(u;\sqrt{tq}/\alpha)_{mM-n}\mathcal{H}^x(u)\Psi(x;y)\\
&=\Psi(x;y)^{-1}\left(\sum_{l=0}^{mM}e(u;\sqrt{tq}/\alpha)_{mM-l}K_l^x\right)\Psi(x;y)\\
&=\prod_{1\le i\le m}\frac{\langle tqx_i/a,tqx_i/b,tqx_i/c,tqx_i/d\rangle_M}{\langle tqx_i^2,t^2qx_i^2\rangle_M}\prod_{1\le i<j\le m}\frac{\langle tq^{M+1}x_ix_j,t^2q^{M+1}x_ix_j\rangle_M}{\langle tqx_ix_j,t^2qx_ix_j\rangle_M}\\
&\cdot\sum_{0\le\mu^-\le\nu^-\le\nu^+\le\mu^+\le(M^m)}\Bigg((-1)^{mM+|\nu^+|+|\nu^-|}e(u;\sqrt{tq}/\alpha)_{|\nu^+|-|\nu^-|}\\
&\cdot\prod_{1\le i\le m}\frac{\langle ax_i,bx_i,cx_i,dx_i\rangle_{\nu_i^-}}{\langle tqx_i/a,tqx_i/b,tqx_i/c,tqx_i/d\rangle_{\nu_i^+}}\prod_{1\le i\le j\le m}\frac{\langle q^{\mu_i^-+\mu_j^-}x_ix_j,t^2q^{\mu_i^++\mu_j^+}x_ix_j\rangle}{\langle x_ix_j,t^2x_ix_j\rangle}\\
&\cdot\prod_{1\le i<j\le m}\frac{\langle q^{\mu_i^--\mu_j^-}x_i/x_j,q^{\mu_i^+-\mu_j^+}x_i/x_j,q^{\nu_i^--\nu_j^-}x_i/x_j,q^{\nu_i^+-\nu_j^+}x_i/x_j\rangle\langle tx_ix_j,qx_ix_j\rangle_{\nu_i^-+\nu_j^-}}{\langle x_i/x_j,x_i/x_j,x_i/x_j,x_i/x_j\rangle\langle tqx_ix_j,t^2x_ix_j\rangle_{\nu_i^++\nu_j^+}}\\
&\cdot\prod_{1\le i,j\le m}\frac{\langle tq^{\mu_i^-+\mu_j^+}x_ix_j,tq^{\nu_i^-+\nu_j^+}x_ix_j,q^{\mu_i^--\mu_j^+}x_i/tx_j\rangle\langle tx_ix_j\rangle_{\mu_i^-+\nu_j^+}\langle t^2x_ix_j\rangle_{\mu_i^++\nu_j^+}}{\langle tx_ix_j,tx_ix_j,x_i/tq^{\mu_j^+}x_j\rangle\langle qx_ix_j\rangle_{\mu_i^-+\nu_j^-}\langle tqx_ix_j\rangle_{\mu_i^++\nu_j^-}}\\
&\cdot\prod_{1\le i,j\le m}\frac{\langle x_ix_j,x_i/q^{\nu_j^-}x_j,x_i/tq^Mx_j\rangle_{\mu_i^-}\langle tx_ix_j,x_i/q^Mx_j\rangle_{\mu_i^+}\langle qx_i/tq^{\nu_j^+}x_j,x_i/q^{\nu_j^+}x_j\rangle_{\nu_i^-}}{\langle tq^{M+1}x_ix_j,qx_i/tq^{\nu_j^+}x_j,qx_i/x_j\rangle_{\mu_i^-}\langle t^2q^{M+1}x_ix_j,qx_i/x_j\rangle_{\mu_i^+}\langle qx_i/tq^{\mu_j^+}x_j,qx_i/x_j\rangle_{\nu_i^-}}\\
&\cdot\prod_{1\le i,j\le m}\frac{\langle x_i/q^{\mu_j^+}x_j\rangle_{\nu_i^+}}{\langle qx_i/x_j\rangle_{\nu_i^+}}\frac{T_{q,x}^{\nu}\Psi(x;y)}{\Psi(x;y)}\Bigg).
\end{aligned} \tag{4.59}$$

Since

$$e(u;t^m\widehat{\alpha})_n e(u;\sqrt{tq}/\alpha)_{mM-n} = e(u;\sqrt{tq}/\alpha)_{mM} = H(u;\alpha t^{\delta_m}), \tag{4.60}$$

the left-hand side of (4.59) equals

$$\Psi(x;y)^{-1}\left(\sum_{l=0}^{mM} e(u;\sqrt{tq}/\alpha)_{mM-l}H_l^x\right)\Psi(x;y). \tag{4.61}$$

Hence we have $H_l^x\Psi(x;y) = K_l^x\Psi(x;y)$ for each $l = 0,1,2,\ldots,mM$. From the formula (4.9) proved by Mimachi, we obtain $H_l^xP_\lambda(x) = K_l^xP_\lambda(x)$ for any partition $\lambda \subset (n^m)$. Since n is the arbitrary non-negative integer, H_l^x equals K_l^x as a q-difference operator. We see that the row type q-difference operators $H_l^x(l \le mM)$ correspond to the terms of the right-hand side of (4.59) such that $|\nu^+| - |\nu^-| = mM - l$.

We compute the explicit formula of $H_l^x = K_l^x$ for $l \le M$. As we will see below their coefficients are expressed as rational functions in $t^{\frac{1}{2}}$. Note also that the operator H_l^x does not depend on the non-negative integer n. Since H_l^x are the W_m-invariant operators, it is enough to calculate the coefficients $H_\nu^{(l)}(x) := H_\nu^{(l)}(x;a,b,c,d)$ of $\prod_{1\le i\le m} T_{q,x_i}^{\nu_i}$ $(\nu \in \mathbb{N}^m, 0 \le |\nu| \le l)$. The coefficients $H_\nu^{(l)}(x)$ have the following form:

$$H_\nu^{(l)}(x) = \sum_{\substack{\nu\le\nu^- \\ |\nu|\le|\nu^-|\le l}} \sum_{\substack{\nu^-\le\nu^+\le(M^m) \\ |\nu^+|=mM-l+|\nu^-|}} A^{(l)}_{\nu,\nu^-,\nu^+}(x). \tag{4.62}$$

Relabeling the indices of summations, we obtain

$$\begin{aligned}
H_\nu^{(l)}(x) = \sum_{\substack{\nu\le\nu^+ \\ |\nu|\le|\nu^+|\le l}} \Bigg(\sum_{\substack{0\le\nu^-\le(l^m)-\nu^+ \\ |\nu^-|=l-|\nu^+|}} \Bigg((-1)^l \prod_{1\le i\le m} \frac{\langle ax_i,bx_i,cx_i,dx_i\rangle_{\nu_i^+}\langle a/x_i,b/x_i,c/x_i,d/x_i\rangle_{\nu_i^-}}{\langle x_i^2\rangle_{\nu_i+\nu_i^+}\langle x_i^{-2}\rangle_{\nu_i+\nu_i^-}} \\
\cdot \prod_{1\le i<j\le m} \frac{\langle q^{\nu_i-\nu_j}x_i/x_j, q^{\nu_i^+-\nu_j^+}x_i/x_j, q^{\nu_j^--\nu_i^-}x_i/x_j\rangle\langle tx_ix_j,qx_ix_j\rangle_{\nu_i^++\nu_j^+}\langle tx_i^{-1}x_j^{-1},qx_i^{-1}x_j^{-1}\rangle_{\nu_i^-+\nu_j^-}}{\langle x_i/x_j,x_i/x_j,x_i/x_j\rangle\langle x_ix_j\rangle_{\nu_i+\nu_j^+}\langle qx_ix_j\rangle_{\nu_i^++\nu_j}\langle x_i^{-1}x_j^{-1}\rangle_{\nu_i+\nu_j^-}\langle qx_i^{-1}x_j^{-1}\rangle_{\nu_i^-+\nu_j}} \\
\cdot \prod_{1\le i\le j\le m} \frac{\langle q^{\nu_i+\nu_j}x_ix_j, x_i^{-1}x_j^{-1}\rangle}{\langle q^{\nu_i+\nu_j^+}x_ix_j, q^{\nu_i+\nu_j^-}x_i^{-1}x_j^{-1}\rangle} \prod_{1\le i,j\le m} \frac{\langle q^{\nu_i^+-\nu_j^-}x_ix_j\rangle\langle q^{-\nu_j^+}x_i/x_j\rangle_{\nu_i}}{\langle x_ix_j\rangle\langle q^{\nu_j^-+1}x_i/x_j\rangle_{\nu_i}} \\
\cdot \prod_{1\le i,j\le m} \frac{\langle q^{\nu_j^-+1}x_i/x_j, tx_i/x_j\rangle_{\nu_i^+}\langle tq^{\nu_j^+}x_j/x_i\rangle_{\nu_i^-}\langle x_ix_j, q^{\nu_j^-+1}x_i^{-1}x_j^{-1}\rangle_{\nu_i}}{\langle qx_i/x_j,qx_i/x_j\rangle_{\nu_i^+}\langle qx_j/x_i\rangle_{\nu_i^-}\langle qx_ix_j\rangle_{\nu_i^+}\langle q^{1-\nu_j}x_i^{-1}x_j^{-1}\rangle_{\nu_i^-}}\Bigg)\Bigg).
\end{aligned} \tag{4.63}$$

In particular, for $|\nu| = l, \nu \in \mathbb{N}^m$ we have

$$H_\nu^{(l)}(x) = \prod_{1\le i\le m} \frac{\langle ax_i,bx_i,cx_i,dx_i\rangle_{\nu_i}}{\langle x_i^2\rangle_{2\nu_i}} \prod_{1\le i<j\le m} \frac{\langle tx_ix_j\rangle_{\nu_i+\nu_j}\langle q^{\nu_i-\nu_j}x_i/x_j\rangle}{\langle x_ix_j\rangle_{\nu_i+\nu_j}\langle x_i/x_j\rangle} \prod_{1\le i,j\le m} \frac{\langle tx_i/x_j\rangle_{\nu_i}}{\langle qx_i/x_j\rangle_{\nu_i}}. \tag{4.64}$$

Although we computed the explicit formula of q-difference operator H_l^x in the case of $t = q^{-M}$, for a fixed l this expression with $t = q^{-M}$ is valid for any $M = l, l+1, \ldots$. Thus the explicit formula (4.63) is valid for any parameter t.

Theorem. 4.5. *For any $\nu=(\epsilon_1\nu'_1,\ldots,\epsilon_m\nu'_m)\in\mathbb{Z}^m$ $(\epsilon_i=\pm1,\nu'_i\in\mathbb{N})$ such that $\sum_{i=1}^m\nu'_i\le l$, we write $|\nu|=\sum_{i=1}^m\nu'_i$ and set $H^{(l)}_\nu(x;a,b,c,d)=H^{(l)}_{\nu'}(x_1^{\epsilon_1},\ldots,x_m^{\epsilon_m};a,b,c,d)$. Then the row type q-difference operators H^x_l $(l=0,1,2,\ldots)$ are expressed explicitly as*

$$H^x_l=\sum_{\substack{\nu\in\mathbb{Z}^m\\0\le|\nu|\le l}}H^{(l)}_\nu(x;a,b,c,d)\prod_{1\le i\le m}T^{\nu_i}_{q,x_i}. \tag{4.65}$$

To summarize: The Koornwinder polynomials $P_\lambda(x)$ are the joint eigenfunctions of H^x_l $(l=0,1,2,\ldots)$:

$$H^x_lP_\lambda(x)=P_\lambda(x)h_l(\alpha t^{\delta_m}q^\lambda;\alpha|q,t). \tag{4.66}$$

Thus the kernel function of dual Cauchy type intertwines the q-difference operators H^x_l with t-difference operators $\widehat{D}^y_r$:

$$\mathcal{H}^x(u)\Psi(x;y)=\frac{\widehat{\mathcal{D}}^y(u)}{e(u;t^m\widehat{\alpha})_{q,n}}\Psi(x;y). \tag{4.67}$$

4.4 Pieri formulas

It is known that the Koornwinder polynomials have the duality property [vD2, S]:

$$\frac{P_\lambda(aq^\mu t^{\delta_m};a,b,c,d|q,t)}{P_\lambda(at^{\delta_m};a,b,c,d|q,t)}=\frac{P_\mu(\alpha q^\lambda t^{\delta_m};\alpha,\beta,\gamma,\delta|q,t)}{P_\mu(\alpha t^{\delta_m};\alpha,\beta,\gamma,\delta|q,t)}, \tag{4.68}$$

where the parameters $\alpha,\beta,\gamma,\delta$ are defined by

$$\alpha=\sqrt{abcd/q},\beta=ab/\alpha,\gamma=ac/\alpha,\delta=ad/\alpha. \tag{4.69}$$

Van Diejen derived the Pieri formula of column type from the duality of the Koornwinder polynomials and D_r. In this subsection, we present the "Pieri formulas of row type" by using the q-difference operators H_l. By direct calculation, we obtain the following lemma.

Lemma. 4.6. *Let μ be a partition. For $\nu\in\mathbb{Z}^m$, if $\mu+\nu=(\mu_1+\nu_1,\ldots,\mu_m+\nu_m)$ is not a partition, $H^{(l)}_\nu(aq^\mu t^{\delta_m};a,b,c,d)=0$.*

For any partition μ , by substituting $x=aq^\mu t^\delta$ in (4.66), we obtain

$$\sum_{\substack{\nu\in\mathbb{Z}^m\\0\le|\nu|\le l}}H^{(l)}_\nu(aq^\mu t^{\delta_m};a,b,c,d)\frac{P_\lambda(aq^{\mu+\nu}t^{\delta_m})}{P_\lambda(at^{\delta_m})}=h_l(\alpha q^\lambda t^{\delta_m};\alpha|q,t)\frac{P_\lambda(aq^\mu t^{\delta_m})}{P_\lambda(at^{\delta_m})}. \tag{4.70}$$

From Lemma 4.6, we can apply the duality of Koornwinder polynomials to (4.70) to obtain

$$\sum_{\substack{\nu\in\mathbb{Z}^m\\0\le|\nu|\le l}}H^{(l)}_\nu(aq^\mu t^{\delta_m};a,b,c,d)\frac{P_{\mu+\epsilon\nu}(aq^\lambda t^{\delta_m};\alpha,\beta,\gamma,\delta|q,t)}{P_{\mu+\epsilon\nu}(at^{\delta_m};\alpha,\beta,\gamma,\delta|q,t)}$$

$$=h_l(\alpha q^\lambda t^{\delta_m};\alpha|q,t)\frac{P_\mu(aq^\lambda t^{\delta_m};\alpha,\beta,\gamma,\delta|q,t)}{P_\mu(at^{\delta_m};\alpha,\beta,\gamma,\delta|q,t)}. \tag{4.71}$$

Replacing $aq^\lambda t^\delta$ and the parameters $(\alpha,\beta,\gamma,\delta)$ with x and (a,b,c,d), respectively, we obtain the following Pieri formula of row type.

Theorem. 4.7. *For any $l = 0, 1, 2, \ldots$, we have the Pieri formula of row type:*

$$h_l(x; a|q,t)\frac{P_\lambda(x)}{P_\lambda(at^{\delta_m})} = \sum_{\substack{\nu \in \mathbb{Z}^m \\ 0 \le |\nu| \le l \\ \mu := \lambda + \nu \in P^+}} H_\nu^{(l)}(\alpha q^\mu t^{\delta_m}; \alpha, \beta, \gamma, \delta)\frac{P_\mu(x)}{P_\mu(at^{\delta_m})}. \tag{4.72}$$

Here P^+ is a set of the partitions λ with $l(\lambda) \le m$:

$$P^+ = \{\lambda \in \mathbb{N}^m | \lambda_1 \ge \cdots \ge \lambda_m \ge 0\}. \tag{4.73}$$

Acknowledgments

The author would like to express his thanks to Professors Masatoshi Noumi and Yasushi Komori for various advices.

References

[C] I. Cherednik, *Double affine Hecke algebras and Macdonald's conjectures*, Ann. of Math. **141** (1995), 191–216.

[vD1] J. F. van Diejen, *Integrability of difference Calogero-Moser systems*, J. Math. Phys. **35** (1994), 2983–3004.

[vD2] J. F. van Diejen, *Self-dual Koornwinder-Macdonald polynomials*, Invent. Math. **126** (1996), 319–339.

[GR] G. Gasper and M. Rahman, *Basic hypergeometric series*, second edition, Encyclopedia of Mathematics and its Applications, **96**, Cambridge University Press, Cambridge, 2004.

[Ka] Y. Kajihara, *Euler transformation formula for multiple basic hypergeometric series of type A and some applications*, Adv. Math. **187** (2004), 53–97.

[KiN] A. N. Kirillov and M. Noumi, *Affine Hecke algebras and raising operators for Macdonald polynomials*, Duke Math. J. **93** (1998), 1–39.

[KNS] Y. Komori, M.Noumi and J. Shiraishi, *Kernel Functions for Difference Operators of Ruijsenaars Type and Their Applications*, SIGMA Symmetry Integrability Geom. Methods Appl. **5** (2009), Paper 054, 40pp, arXiv:0812.0279.

[Ko] T. Koornwinder, *Askey-Wilson polynomials for root systems of type BC*, in Hypergeometric functions on domains of positivity, Jack polynomials, and applications, Contemp. Math., **138**, Amer. Math. Soc. Providence, RI, 1992, 189–204.

[L] M. Lassalle, *An (inverse) Pieri formula for Macdonald polynomials of type C*, Transform. Groups **15** (2010), 154–183, arXiv:0905.0412.

[Ma1] I. G. Macdonald, *Symmetric Functions and Hall Polynomials*, second edition, Oxford Mathematical Monographs, Oxford Science Publications, The Clarendon Press, Oxford University Press, New York, 1995.

[Ma2] I. G. Macdonald, *Affine Hecke algebras and orthogonal polynomials*, Cambridge Tracts in Mathematics, **157**, Cambridge University Press, Cambridge, 2003.

[Mi1] K. Mimachi, *A new derivation of the inner product formula for the Macdonald symmetric polynomials*, Composit. Math. **113** (1998), 117–122, arXiv:q-alg/9610018v1.

[Mi2] K. Mimachi, *A duality of MacDonald-Koornwinder polynomials and its application to integral representations*, Duke Math. J. **107** (2001), 265–281.

[MiN] K. Mimachi and M. Noumi, *An integral representation of eigenfunctions for Macdonald's q-difference operators*, Tohoku Math. J. **49** (1997), 517–525.

[N1] M. Noumi, *Macdonald-Koornwinder polynomials and affine Hecke rings*, RIMS Kōkyūroku **919** (1995), 44–55 (in Japanese).

[N2] M. Noumi, *private communication.*

[R] H. Rosengren, *Karlsson-Minton type hypergeometric functions on the root system C_n*, J. Math. Anal. Appl. **281** (2003), 332–345, arXiv:math/0205108.

[S] S. Sahi, *Nonsymmetric Koornwinder polynomials and Duality*, Ann. Math, **150** (1999), 267–282.

[Sa] A. Sano, *A commuting family of difference operators involving elliptic functions as coefficients* , master thesis (Kobe Univ. 2008, in Japanese).