Kernel identities for van Diejen's q-difference operators and transformation formulas for multiple basic hypergeometric series

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(Degree)
博士（理学）
(Date of Degree)
2012-03-25
(Date of Publication)
2014-04-01
(Resource Type)
doctoral thesis
(Report Number)
甲5573
(URL)
https://hdl.handle.net/20.500.14094/D1005573

In the theory of Macdonald polynomials of type A, the kernel function of Cauchy type has been used to derive various important properties of Macdonald polynomials. I. G. Macdonald derived the existence of Macdonald polynomials in [Ma1]. K. Mimachi also gives one of the proofs for Macdonald's constant term conjecture [Mi1]. The integral representation of Macdonald polynomials by [MiN] is known as an application of kernel function. Kajihara's Euler transformation formula for multiple basic hypergeometric series can also be regarded as an application of the kernel function of Cauchy type [Ka]. It is known that the kernel function of dual Cauchy type for type A intertwines a q-difference operator $\mathcal{H}_A^x(u;q,t)$ in the x variables with Macdonald difference operator $\mathcal{D}_A^y(u;t,q)$ in the y variables [N2]. The operator $\mathcal{H}_A^x(u;q,t)$ is known as a generating function of "row type" q-difference operators $H_{A,l}^x$. We can obtain this fact as the special case of Kajihara's Euler transformation formula. It is also known that the Macdonald polynomials for type A are the joint eigenfunctions of $\mathcal{H}_A^x(u;q,t)$ and the eigenvalues are given by the special values of the Macdonald polynomials attached to single row partition (l). The commutativity of this family $\{H_{A,l}^x\}_{l=0}^{\infty}$ is proved in [Sa] through the Wronski relations in the elliptic setting.

Recently, Y. Komori, M. Noumi and J. Shiraishi in [KNS] introduced the kernel function $\Phi(x;y|q,t)$ of Cauchy type for type BC in the variables $x=(x_1,\ldots,x_m)$ and $y=(y_1,\ldots,y_n)$ relevant to Koornwinder polynomials. The kernel function $\Phi(x;y|q,t)$ satisfies the following q-difference equation:

$$\langle t\rangle D_1^x\Phi(x;y|q,t)-\langle t\rangle \widetilde{D}_1^y\Phi(x;y|q,t)=\langle t^m\rangle\langle t^{-n}\rangle\langle\alpha^2t^{m-n-1}\rangle\Phi(x;y|q,t), \qquad (1)$$

where $\alpha=\sqrt{abcd/q}$ and $\langle z\rangle$ is the multiplicative notation for trigonometric function

$$\langle z\rangle=z^{\frac{1}{2}}-z^{-\frac{1}{2}}=-z^{-\frac{1}{2}}(1-z).$$

In this identity, D_1^x is the Koornwinder q-difference operator in the x variables and $\widetilde{D}_1^y$ denotes the Koornwinder operator in the y variables with the parameters (a,b,c,d) replaced by $(\sqrt{tq}/a,\sqrt{tq}/b,\sqrt{tq}/c,\sqrt{tq}/d)$. They proved this identity for the elliptic Ruijsenaars-van Diejen second order difference operator. They also derived the explicit formula for the Koornwinder polynomials attached to single row partitions (l) as an application of kernel function. In this paper, we show that the kernel function $\Phi(x;y|q,t)$ intertwines the whole commuting family of van Diejen's q-difference operators, which includes the Koornwinder operator as the first member. From this result, we also derived a transformation formula for certain multiple basic hypergeometric series of type BC. Our paper is organized as follows.

In Section 2, we state our main result Theorem 2.1 and prove it. We first recall the definition of the commuting family of van Diejen's q-difference operators D_r^x in Subsection 2.1. Note that van Diejen's q-difference operators are invariant under the permutations of indices and the inversions of the variables. Van Diejen's q-difference operators have the joint eigenfunctions. Let W_m be the Weyl group of type BC_m acting on the Laurent polynomials in the variables $x=(x_1,\ldots,x_m)$ through the permutations of the indices

and the inversions of the variables. Under the assumption that a,b,c,d,q,t are generic, for each partition $\lambda=(\lambda_1,\ldots,\lambda_m)$ there exists a unique W_m-invariant Laurent polynomial $P_\lambda(x)=P_\lambda(x;a,b,c,d|q,t)$, called the Koornwinder polynomial attached to λ, satisfying the following conditions.

(1) $P_\lambda(x)$ is expanded by the orbit sums $m_\mu(x)=\sum_{\nu\in W.\mu}x^\nu$ as

$$P_\lambda(x)=m_\lambda(x)+\sum_{\mu<\lambda}c_{\lambda\mu}m_\mu(x),$$

where $c_{\lambda\mu}\in\mathbb{C}$ and $<$ means the dominance ordering of the partitions.

(2) $P_\lambda(x)$ is a joint eigenfunction of van Diejen's q-difference operators D_r^x:

$$D_r^xP_\lambda(x)=P_\lambda(x)e_r(\alpha t^{\delta_m}q^\lambda;\alpha|t),$$

where $\delta_m=(m-1,\ldots,1,0)$ and $e_r(x;\alpha|t)$ are the interpolation polynomials of column type defined by

$$e_r(x;\alpha|t)=\sum_{1\le i_1<\cdots<i_r\le m}e(x_{i_1};t^{i_1-1}\alpha)e(x_{i_2};t^{i_2-2}\alpha)\cdots e(x_{i_r};t^{i_r-r}\alpha),$$
$$e(z;w)=\langle zw\rangle\langle zw^{-1}\rangle=z+z^{-1}-w-w^{-1}.$$

Note that $e_r(x;\alpha|t)$ is W_m-invariant and satisfies the following interpolation property: For any partition $\mu\not\supset(1^r)$,

$$e_r(\alpha t^{\delta_m}q^\mu;\alpha|t)=0. \qquad (2)$$

These facts and notation are due to [vD1, Ko, KNS]. In Subsection 2.2, we introduce the kernel function $\Phi(x;y|q,t)$ of Cauchy type for type BC as a solution of linear q-difference equations. We note that the four types of explicit formula for kernel function of Cauchy type are introduced by [KNS]. We also define a generating function $\mathcal{D}(u)$ of van Diejen's q-difference operators. Then the kernel function $\Phi(x;y|q,t)$ of Cauchy type intertwines the q-difference operator $\mathcal{D}^x(u)$ in the x variables with q-difference operator $\widetilde{\mathcal{D}}^y(u)$ in the y variables:

Theorem 2.1: $$\mathcal{D}^x(u)\Phi(x;y|q,t)=e(u;\alpha)_{t,m-n}\widetilde{\mathcal{D}}^y(u)\Phi(x;y|q,t),$$

where $e(z;w)_{q,k}$ is the q-shifted factorial of type BC with base point w. We call this formula a kernel identity of Cauchy type. Since the constant function 1 is a joint eigenfunction of van Diejen's difference operators, this identity for $n=0$ holds. We use this fact as the starting point of our proof. Theorem 2.1 is equivalent to a rational function identity (Theorem 2.2) of x variables and y variables. We prove this rational function identity by induction on the number n of y variables. We regard it as a rational function identity of y_n. Note that this identity is invariant under the inversion and permutation for x and y. By analyzing the residues and computing the limits as $y_n\to\infty$, Theorem 2.1 and Theorem 2.2 are proved. Note that Theorem 2.1 includes the result (1) of [KNS]. In fact, the q-Saalschütz summation formula gives rise to the transformation formula for the base point of q-shifted factorials of type BC. From this transformation formula, we obtain

$$D_r^x\Phi(x;y|q,t)=\sum_{k=0}^{r}\widetilde{D}_k^y\begin{bmatrix}n-k\\r-k\end{bmatrix}_t e(t^{\frac{1}{2}(n-k+1)}/\alpha;t^{\frac{1}{2}(1+n-2m+k)}/\alpha)_{t,r-k}\Phi(x;y|q,t).$$

This formula for $r=1$ recovers the result (1) of [KNS].

We derive two types of transformation formulas for multiple basic hypergeometric series in Section 3. By applying the multiple principal specializations to Theorem 2.2, we have a transformation formula for type BC indexed by four multi-indices (Theorem 3.1). This formula is regarded as a kind of duality transformation formula. In the case of $n=0$, this transformation formula gives rise to a summation formula. Specializing the parameter of Theorem 2.2, we also obtain a rational function identity which include the result of M. Lassale (Theorem 6 in [L]) as a special case. This identity gives rise to a C type transformation formula of multiple basic hypergeometric series by a same method. Note that this formula recovers one of the C type transformation formulas, due to H. Rosengren [R]. Rosengren derived this result from Gunstafson's summation formula of multilateral basic hypergeometric series for type C. Our proof is different from his proof.

In Section 4, we construct an explicit operator $\mathcal{H}^x(u;q,t)$ which satisfies

$$\mathcal{H}^x(u;q,t)\Psi(x;y)=const.\cdot\widehat{\mathcal{D}}^y(u)\Psi(x;y), \tag{3}$$
$$\widehat{\mathcal{D}}^y(u)=\sum_{r=0}^{n}(-1)^r e(u;\widehat{\alpha})_{q,n-r}\widehat{D}_r^y.$$

Here $\Psi(x;y)$ is the kernel function of dual Cauchy type for type BC introduced by Mimachi [Mi2] and $\widehat{\ }$ means the operation of replacing the parameters (q,t) with (t,q). We call this identity a kernel identity of dual Cauchy type. Mimachi proved the following expansion for kernel function of dual Cauchy type:

$$\Psi(x;y)=\sum_{\lambda\subset(n^m)}(-1)^{\lambda^*}P_\lambda(x)\widehat{P}_{\lambda^*}(y),\quad \lambda^*=(m-\lambda'_n,m-\lambda'_{n-1},\ldots,m-\lambda'_1).$$

This formula plays an important role in computing the explicit formula of q-difference operator $\mathcal{H}^x(u;q,t)$.

In Subsection 4.1, we recall Noumi's representations of affine Hecke algebras $\mathcal{H}(W_m^{\mathrm{aff}})$ for type C [N1]. We denote by $\mathbb{C}(x)[T_{q,x}^{\pm1}]$ the ring of q-difference operators with rational coefficients. For any $A^x\in\mathcal{H}(W_m^{\mathrm{aff}})$, A^x is expressed as $\sum_{w\in W_m}A_w^x w$ ($A_w^x\in\mathbb{C}(x)[T_{q,x}^{\pm1}]$). Then we define the q-difference operator L_A^x by $L_A^x=\sum_{w\in W_m}A_w^x$. It is known that the following fact holds. For any W_m-invariant Laurent polynomial $f(\xi)$ in the variables $\xi=(\xi_1,\ldots,\xi_m)$, and for any W_m-invariant Laurent polynomial $\varphi(x)\in\mathbb{C}[x^{\pm1}]^{W_m}$, one has

$$f(Y^x)\varphi(x)=L^x_{f(Y^x)}\varphi(x).$$

Furthermore, the q-difference operator $L_f^x:=L^x_{f(Y^x)}$ satisfies for any partition λ

$$f(Y^x)P_\lambda(x)=L_f^xP_\lambda(x)=P_\lambda(x)f(\alpha t^{\delta_m}q^\lambda). \tag{4}$$

In particular, the q-difference operators L_f^x for the interpolation polynomials $f=e_r(\xi;\alpha|t)$ of column type give rise to van Diejen's operators D_r^x. From this view point, we call D_r^x "column type" q-difference operators. Since $\{e_r(\xi;\alpha|t)\}_{r=1}^m$ is the generator system of the ring $\mathbb{C}[\xi^{\pm1}]^{W_m}$ of W_m-invariant Laurent polynomials in m variables, L_f is an element of $\mathbb{C}[D_1^x,\ldots,D_m^x]$ for any $f(\xi)\in\mathbb{C}[\xi^{\pm1}]^{W_m}$. The operator $\mathcal{H}^x(u;q,t)$ to be constructed in Subsection 4.2 is a generating function of "row type" q-difference operators. This fact (4) guarantees the existence of the q-difference operator $\mathcal{H}^x(u;q,t)$ satisfying the identity (3). We define q-difference operators $H_l^x(l=0,1,2,\ldots)$ by L_f^x for $f=h_l(\xi;\alpha|q,t)$. Here $h_l(\xi;\alpha|q,t)$ is the interpolation polynomials of row type introduced by [KNS]. In this view point, we call H_l^x "row type" q-difference operators. We also introduce the q-difference operator $\mathcal{H}^x(u;q,t)$ as a generating function of row type q-difference operators H_l^x ($l=0,1,2,\ldots$):

$$\mathcal{H}^x(u;q,t)=\sum_{l=0}^{\infty}\frac{H_l^x}{e(u;t^m\sqrt{q/t\alpha})_l}\in\mathbb{C}(x)[T_{q,x}^{\pm1}][[u]].$$

Then the q-difference operator $\mathcal{H}^x(u;q,t)$ and the t-difference operator $\widehat{\mathcal{D}}^y(u)$ satisfy the identity (3). We also obtain the relationship between the q-difference operators H_l^x and van Diejen's difference operators $\widehat{D}_r^y$. However, it is difficult to compute the explicit expressions of difference operators H_l^x by means of the q-Dunkl operators. So, we use the special case $t=q^{-M}$ of Theorem 3.1 for each $M=0,1,2,\ldots$. Then we obtain the kernel identity of dual Cauchy type for $t=q^{-M}$. We compute the explicit formula of q-difference operator H_l^x in the case of $t=q^{-M}$ ($M=l,l+1,\ldots$). Since the form does not depend on M, we obtain the explicit expression for any parameter t. It is known that the Koornwinder polynomials $P_\lambda(x)$ have the duality property [vD2]. Van Diejen derived the Pieri formula of column type by combining the duality with difference operator D_r^x. We present the "Pieri formula of row type" by using the q-difference operators H_l in Subsection 4.4.

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(別紙 1)　　論文審査の結果の要旨

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論文題目	Kernel identities for van Diejen's q-difference operators and transformation formulas for multiple basic hypergeometric series (van Diejen の q 差分作用素に対する核関係式と多重 q 超幾何級数の変換公式)		
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要旨

本論文は, q 差分作用素の可換族に対する核関係式とその多重 q 超幾何級数への応用に関する研究である.

m 変数の $x=(x_1,\ldots,x_m)$ と n 変数の $y=(y_1,\ldots,y_n)$ の有理型函数 $\Phi(x,y)$ について, x 変数に作用する作用素 A_x と y 変数に作用する作用素 B_y が与えられて, 関係式 $A_x\Phi(x,y)=B_y\Phi(x,y)$ が成立するとき, $\Phi(x,y)$ は作用素の組 (A_x,B_y) に対する核函数であるという. この種の核函数は, 線型作用素の固有函数の研究の要であり, 特殊函数論の様々な局面において重要な役割を果たす.

q 差分作用素の核函数の典型例は, A 型の Macdonald q 差分作用素に対する Cauchy 型および双対 Cauchy 型の核函数である. これらは 1980 年代の終り頃からよく知られており, 固有函数である Macdonald 多項式の様々な性質や公式を導出するために用いられてきたものである. 特に本論文と関係するものとして, A 型の q 差分作用素の可換族の Cauchy 型核関係式から, A 型の多重 q 超幾何級数の双対変換公式を導いた梶原康史 (2004) の研究もある.

A 型以外の q 差分作用素の核函数の研究が進展したのは, 比較的最近のことである. 三町 (2001) は, BC_m 型の Koornwinder の q 差分作用素に対する双対 Cauchy 型核函数を発見し, 長方形 (n^m) に対する Koornwinder 多項式の積分表示を構成した. また 小森・白石・野海 (2009) は, Koornwinder q 差分作用素の Cauchy 型核函数を構成し, 分割が 1 列及び 1 行の場合の Koornwinder 多項式の明示公式を与えた. m 変数の Koornwinder の q 差分作用素に対しては, m 個からなる q 差分作用素の可換族で, 代数的に独立なものが van Diejen(1994) によっ

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て構成されており, Koornwinder 多項式はこの van Diejen の可換 q 差分作用素族の同時固有函数となっている. しかし, 三町の核函数についても, 小森・白石・野海の核函数についても, それが van Diejen の可換 q 差分作用素族全体に対してどう振る舞うかは, これまで明らかにされていなかった.

増田恭穂氏の本研究の主定理は, 小森・白石・野海の BC 型核函数が van Diejen の可換 q 差分作用素族全体の核函数となっている事を証明した定理 2.1 である. 増田氏は, この主定理の応用として, 前述の梶原の研究の BC 型への拡張を考察し, BC 型の多重 q 超幾何級数に対する変換公式 (定理 3.1) と C 型の変換公式 (定理 3.3) を導出した. 後者は結果として Rosengren(2003) の変換公式に核函数の観点からの別証明を与えたものとなったが, 前者の BC 型変換公式は全く新しいものであって, 現在までに知られている多重 q 超幾何級数の変換公式の中で最も一般的なものと考えられる. van Diejen の q 差分作用素の可換族は, その固有値が 1 列の補間多項式の系列で表される「列型」の可換族である. これに対して, 増田氏は BC 型の変換公式 (定理 3.1) の応用として, 固有値が 1 行の補間多項式の系列で表されるような 「行型」の q 差分作用素の可換族を構成し, 三町の核函数が, van Diejen の列型の可換族と新しく構成した行型の可換族を交換する役割を担っていることを明らかにした (定理 4.3, 4.4, 4.5).

本研究は, BC 型の van Diejen の q 差分作用素の可換族とその核函数の研究, 及び BC 型の多重 q 超幾何級数の変換公式の研究を飛躍的に, 決定的に深化させたものであり, q 差分作用素と q 超幾何函数の関わる特殊函数論に重要な知見を得た価値ある集積である. よって, 学位申請者 増田 恭穂は, 博士 (理学) の学位を得る資格があると認める.