



Instanton counting and Virasoro algebra

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博 士 論 文

Instanton counting and Virasoro algebra
(インスタントンの数え上げとビラソロ代数)

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Abstract

This thesis deals with several topics on the conjecture proposed by Alday, Gaiotto and Tachikawa, which is now called AGT conjecture.

The conjecture comes from the string theory in physics. Its main proposal is the correspondence between an $N = 2$ four-dimensional supersymmetric Yang-Mills theory and a two-dimensional conformal field theory.

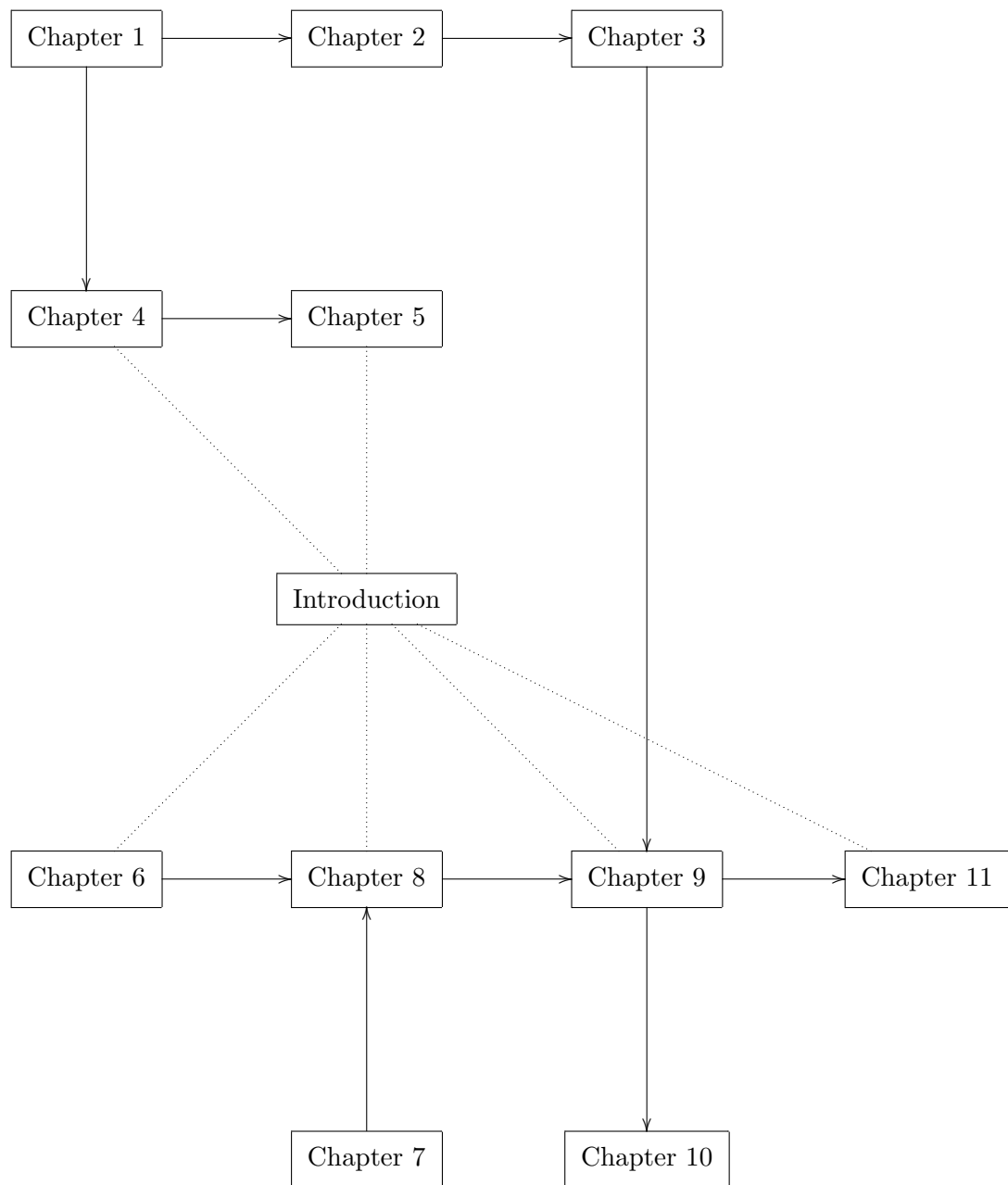
In mathematical terminology, it predicts the existence of certain action of Virasoro algebra on the direct sum of equivariant cohomology groups of framed moduli spaces of $SU(2)$ -instantons on $\mathbb{R}^4$ with various instanton numbers. Hence natural expectations have risen on both algebraic studies and geometric ones.

Our viewpoint is mainly algebraic one. This thesis presents the following three consequences:

- The first is a proof of the formula on norm of logarithmic primary of Virasoro algebra, which was proposed by Al. Zamolodchikov. This result is the final piece of the proof of AGT relation for pure $SU(2)$ case. (Chapter 10)
- The second is an explicit formula of Whittaker vector for Virasoro algebra in terms of Jack symmetric functions. This is a differential version of the conjecture proposed by Awata and Yamada. (Chapter 3)
- The third concerns with K -theoretic pure $SU(2)$ AGT conjecture proposed by Awata and Yamada. We propose a q -deformed version of Zamolodchikov recursive formula for the norm of deformed Gaiotto state, and show that it is satisfied by the K -theoretic pure $SU(2)$ Nekrasov partition function. Thus the K -theoretic conjecture is reduced to our conjectural formula. (Chapter 11)

The logical dependence of the chapters are shown in the next page.

Logical dependence of the chapters



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Introduction: What is the AGT conjecture?

In the striking paper [2], a remarkable proposal, now called AGT conjecture, was given on the equivalence between $N = 2$ four dimensional supersymmetric Yang-Mills theory and two dimensional conformal field theory.

Although the original proposal is based on physical arguments, it implies non-trivial mathematical claims on geometry of moduli spaces and infinite-dimensional algebras. In particular, the conjecture predicts an existence of an action of Virasoro algebra on the direct sum of equivariant cohomology groups of framed moduli spaces of $SU(2)$ -instantons on $\mathbb{R}^4$ with various instanton numbers.

At the present time, there are several generalizations of AGT conjecture. Here we quickly review those proposals from mathematical viewpoint.

Geometric settings

In order to state those proposals, let us start with the following situation, based on the description of [18] and [20]. Denote by $\mathfrak{g}$ a simple Lie algebra over $\mathbb{C}$, and by G its associated (simply connected) Lie group. Let Bun_G^n be the moduli of framed principal G -bundle over $\mathbb{CP}^2$. Set-theoretically it consists of the following data:

$$\text{Bun}_G^n = \left\{ (E, \varphi) \left| \begin{array}{l} E : \text{principal } G\text{-bundle over } \mathbb{CP}^2, c_2(E) = n \\ \varphi : \text{trivialization on } \ell_\infty \end{array} \right. \right\} / \sim.$$

Here $\ell_\infty = \{[0 : z_1 : z_2] \in \mathbb{CP}^2\} \subset \mathbb{CP}^2$ is the line at infinity on $\mathbb{CP}^2$, and $\sim$ means the natural isomorphism.

Let $\mathcal{U}_G^n$ be the Uhlenbeck (partial) compactification. Set-theoretically it has a decomposition

$$\mathcal{U}_G^n = \coprod_{0 \leq k \leq n} \text{Bun}_G^k \times \text{Sym}^{n-k}(\mathbb{C}^2).$$

Here $\text{Sym}^k(\mathbb{C}^2)$ denotes the k -th symmetric product of $\mathbb{C}^2$, and the affine plane $\mathbb{C}^2$ is considered as the subspace of $\mathbb{CP}^2$ by $\mathbb{C}^2 = \{[1 : z_1 : z_2] \in \mathbb{CP}^2\} \subset \mathbb{CP}^2$. Especially $\mathcal{U}_G^n$ is singular in general.

Both spaces admit an action of $G \times \text{GL}(\mathbb{C}^2)$, where G acts by changing the framing at ℓ_∞ and $\text{GL}(\mathbb{C}^2)$ acts naturally on $\mathbb{C}^2$. Let T be the maximal torus of G , and set $\tilde{T} := T \times (\mathbb{C}^*)^2$. Now let $\text{IH}_{\tilde{T}}^*(\mathcal{U}_G^n)$ be the $\tilde{T}$ -equivariant intersection cohomology group. By the general theory of equivariant cohomology, $\text{IH}_{\tilde{T}}^*(\mathcal{U}_G^n)$ is an $S^*(\text{Lie}(\tilde{T})^*)$ -module. Let us set $S := S^*(\text{Lie}(\tilde{T})^*)$ and denote by $Q(S)$ its quotient field. Set $\text{IH}_{\tilde{T}}^*(\mathcal{U}_G^n)_{\text{loc}} := \text{IH}_{\tilde{T}}^*(\mathcal{U}_G^n) \otimes_S Q(S)$. Then the Poincaré pairing induces a symmetric bilinear form

$$\langle \cdot, \cdot \rangle : \text{IH}_{\tilde{T}}^*(\mathcal{U}_G^n)_{\text{loc}} \otimes \text{IH}_{\tilde{T}}^*(\mathcal{U}_G^n)_{\text{loc}} \rightarrow Q(S).$$

Now let us define

$$\int_{\mathcal{U}_G^n} \omega := \langle 1_{\mathcal{U}_G^n}, \omega \rangle.$$

The generating function (considered as formal series in x)

$$Z := \sum_{n=0}^{\infty} x^n \int_{\mathcal{U}_G^n} 1_{\mathcal{U}_G^n} \tag{0.0.1}$$

is called *the Nekrasov partition function*, following the proposal by Nekrasov [74].

Actually, Nekrasov proposed a version of this partition function (0.0.1) for the case $G = \mathrm{SL}(r, \mathbb{C})$, where the concerning moduli space is replaced by a smooth one. Then he derived an explicit formula of the generating function (0.0.1), using the localization theorem for equivariant cohomology. He also conjectured that a certain limit of (0.0.1) should coincide with the Seiberg-Witten prepotential. This conjecture was solved by [72] and [75] in the case $G = \mathrm{SL}(r, \mathbb{C})$. For general G , a proof is given by [18] and [19].

Statement of the conjecture

Now we can state the proposal of AGT [2] and its degenerated version proposed by Gaiotto [42]:

CONJECTURE (AGT conjecture for $G = \mathrm{SL}(2, \mathbb{C})$). For the case $G = \mathrm{SL}(2, \mathbb{C})$, there exists an action of Virasoro algebra Vir on $V := \oplus_{n=0}^{\infty} \mathrm{IH}_{\tilde{T}}^*(\mathcal{U}_G^n)_{\mathrm{loc}}$ such that

- (1) V is isomorphic to Verma module of Vir .
- (2) The generators L_m and L_{-m} are adjoint with respect to the pairing $\langle \cdot, \cdot \rangle$.
- (3) For any $n \in \mathbb{Z}_{\geq 0}$, $L_m 1_{\mathcal{U}_G^n} = 0$ for $m \in \mathbb{Z}_{\geq 2}$ and $L_1 1_{\mathcal{U}_G^n} = 1_{\mathcal{U}_G^{n-1}}$.

In particular, it predicts

$$Z \stackrel{?}{=} \text{norm of Whittaker vector of Virasoro algebra.} \quad (0.0.2)$$

See Chapter 1 for the detail of Virasoro algebra, and Chapter 3 for its Whittaker vector. We will also explain in detail this simplest version of the conjecture in Chapter 9. There we call the conjecture by $\mathrm{SU}(2)$ AGT conjecture, since the partition function relates to the $\mathrm{SU}(2)$ gauge theory.

In Chapter 10, we will present a proof of the formula (0.0.2), based on the Zamolodchikov-type formula. Let us remark that this strategy was proposed by [78] and that some parts of the proof are shown by [30].

Let us remark that for the case $G = \mathrm{SL}(r, \mathbb{C})$, one can replace the singular space $\mathcal{U}_G^n$ by a smooth one $M(r, n)$. This space is the framed moduli space of torsion free sheaves. Since this space is smooth, the localization theorem of equivariant homology gives an explicit expression of the partition function. We will review these aspects in Chapters 6–8.

Now let us comment on the generalized conjectures. Let ${}^L\mathfrak{g}$ be the Langlands dual of $\mathfrak{g}$ and denote by $W({}^L\mathfrak{g})$ the associated $\mathcal{W}$ -algebra with respect to the principal nilpotent element. (See Chapters 4 and 5 for details on $\mathcal{W}$ -algebras.)

CONJECTURE (AGT conjecture for G). For general G , there exists an action of $W({}^L\mathfrak{g})$ on V such that V is isomorphic to Verma module of $W({}^L\mathfrak{g})$.

In [84], a direct check of the formula of type (0.0.2) is given for the case $G = \mathrm{SL}(3)$.

One may see that, roughly speaking, the word ‘AGT conjecture’ suggests a realization of ‘ $\mathcal{W}$ -algebra’ as the hidden symmetry of the ‘instanton moduli space’.

By the quantum Drinfeld-Sokolov reduction invented by Feigin and Frenkel (see Chapter 4), a generalized $\mathcal{W}$ -algebra is associated to the pair $(\mathfrak{g}, e)$ of a simple Lie algebra $\mathfrak{g}$ and its nilpotent element e . There is a proposal [20] for the corresponding generalized AGT conjecture, involving the arbitrary $\mathcal{W}$ -algebra as the symmetry of moduli spaces. To state those in full generality, one should introduce moduli spaces $\tilde{\mathcal{U}}_{G,P}^n$ of parabolic bundles with framing, which we do not give the detail. Here $P \subset G$ denotes a parabolic subspace of G . Let us mention, however, that the case $P = B$, the concerning $\mathcal{W}$ -algebra is ${}^L\mathfrak{g}_{\mathrm{aff}} = {}^L\mathfrak{g}[t^{\pm 1}] \oplus \mathbb{C}K \oplus \mathbb{C}D$, the affine Lie algebra of the Langlands dual ${}^L\mathfrak{g}$, which is shown in [18].

For physics literatures on these generalizations, let us mention to [1], [89], [90] and [61]. Recent studies focus on the super-symmetric analogue of the conjecture. See [11] and [10] for this direction.

There is another direction of the analogue: q -deformed version. This is initiated by [7], and will be explained in Chapter 11. See also [8].

Finally, let us mention that there is a finite analogue, concerning the moduli ${}^b\text{QMap}(\mathbb{CP}, G/P)$ of quasi-map and finite $\mathcal{W}$ -algebra. For the type A case, this conjecture is solved by [20].

Now we summarize the conjectures in the table below.

moduli space	gauge group	algebra
$\mathcal{U}_G^n$	$G = \text{SL}(2, \mathbb{C})$	Vir
	$G = \text{SL}(r, \mathbb{C})$	$W(\mathfrak{sl}(r, \mathbb{C}))$
	G : general	$W({}^L\mathfrak{g})$
$\tilde{\mathcal{U}}_{G,P}^n$	G : general, $P = B$	${}^L\mathfrak{g}_{\text{aff}}$
	G, P : general	$W({}^L\mathfrak{g}, e)$
${}^b\text{QMap}(\mathbb{CP}, G/P)$	$G = \text{GL}(r)$, P : general	finite $W(\mathfrak{gl}(r, \mathbb{C}), e)$ -algebra
	G, P : general	finite $W({}^L\mathfrak{g}, e)$ -algebra

TABLE 0.0.1. AGT conjectures

Part 1

Algebra

CHAPTER 1

Virasoro algebra

In this chapter we recall several notations and well-known facts on Virasoro algebra. Such topics are often treated in the textbooks of conformal field theory, such as [58], [53], [24], [81], [91] and [52].

1.1. Definition of Virasoro Lie algebra

Virasoro algebra is the Lie algebra over $\mathbb{C}$ generated by L_n ($n \in \mathbb{Z}$) and C (central) with the relation

$$[L_m, L_n] = (m - n)L_{m+n} + \frac{C}{12}m(m^2 - 1)\delta_{m+n,0}, \quad [L_n, C] = 0.$$

We denote Virasoro algebra by Vir .

Vir has a triangular decomposition

$$\text{Vir} = \text{Vir}_+ \oplus \text{Vir}_0 \oplus \text{Vir}_-$$

with

$$\text{Vir}_+ := \bigoplus_{n \in \mathbb{Z}_{>0}} \mathbb{C}L_n, \quad \text{Vir}_0 := \mathbb{C}C \oplus \mathbb{C}L_0, \quad \text{Vir}_- := \bigoplus_{n \in \mathbb{Z}_{>0}} \mathbb{C}L_{-n}.$$

Let us also denote

$$\text{Vir}_{\geq 0} := \text{Vir}_0 \oplus \text{Vir}_+.$$

The universal enveloping algebra of Vir will be denoted by $U(\text{Vir})$. By the Poincaré-Birkhoff-Witt theorem, $U(\text{Vir})$ has a basis consisting of the elements

$$L_{n_1}L_{n_2} \cdots L_{n_k}C \quad (k \geq 0, n_1 \geq n_2 \geq \cdots \geq n_k).$$

1.2. Verma Module

First we recall the notion of highest weight vector/module.

Let M be a Vir -module. A non-zero element v of M is called a highest weight vector of highest weight $h \in \mathbb{C}$ and central charge $c \in \mathbb{C}$ if it satisfies the following conditions:

- (1) $Cv = cv$, $L_0v = hv$.
- (2) $L_nv = 0$ for all $n \in \mathbb{Z}_{>0}$.

A Vir -module M is called a highest weight module if it satisfies

- (1) M has a highest weight vector v .
- (2)

$$M = \sum_{k \geq 0} \sum_{n_1 \geq n_2 \geq \cdots \geq n_k \geq 0} \mathbb{C}L_{-n_k}L_{-n_{k-1}} \cdots L_{-n_1}.$$

Now we introduce Verma module. Let c and h be complex numbers. Let $\mathbb{C}_{c,h}$ be the one-dimensional representation of the subalgebra $\text{Vir}_{\geq 0}$, where Vir_+ acts trivially, L_0 acts as multiplication by h , and C acts as multiplication by c . Then Verma module $M(c, h)$ of Vir is defined by

$$M(c, h) := \text{Ind}_{\text{Vir}_{\geq 0}}^{\text{Vir}} \mathbb{C}_{c,h}.$$

Obeying the notation in physics literature, we denote by $|c, h\rangle$ a fixed basis of $\mathbb{C}_{c,h}$. Then one has $\mathbb{C}_{c,h} = \mathbb{C}|c, h\rangle$ and $M(c, h) = U(\text{Vir})|c, h\rangle$.

$M(c, h)$ has an L_0 -weight decomposition:

$$M(c, h) = \bigoplus_{n \in \mathbb{Z}_{\geq 0}} M(c, h)_n$$

with

$$M(c, h)_n := \{v \in M(c, h) \mid L_0 v = (h + n)v\}.$$

A basis of $M(c, h)_n$ can be described by partitions. Our notations for partitions are explained in §A.1. For a partition $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_k)$ of n we use the symbol

$$L_{-\lambda} := L_{-\lambda_k} L_{-\lambda_{k-1}} \cdots L_{-\lambda_1} \quad (1.2.1)$$

for the product of elements in the enveloping algebra $U(\text{Vir}_-)$ of the subalgebra Vir_- . Then the set

$$\{L_{-\lambda} | c, h \rangle \mid \lambda \vdash n\}$$

forms a basis of $M(c, h)_n$.

Thus $M(c, h)$ is a highest weight module, and $|c, h\rangle \in M(c, h)$ is a highest weight vector.

We also need the (restricted) dual Verma module $M^*(c, h)$. This is a right Vir -module generated by $\langle c, h|$ with

$$\langle c, h| \text{Vir}_+ = 0, \quad \langle c, h| L_0 = h \langle c, h|, \quad \langle c, h| C = c \langle c, h|.$$

It has an L_0 -weight decomposition

$$\bigoplus_{n \in \mathbb{Z}_{\geq 0}} M^*(c, h)_n$$

with

$$M^*(c, h)_n := \{v \in M^*(c, h) \mid v L_0 = (h - n)v\}.$$

$M^*(c, h)_n$ has a basis

$$\{\langle c, h| L_\lambda \mid \lambda \vdash n\}$$

with

$$L_\lambda := L_{\lambda_1} L_{\lambda_2} \cdots L_{\lambda_k}. \quad (1.2.2)$$

for a partition $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_k)$.

1.3. Contravariant form

Next we introduce the contravariant form (also called the Shapovalov form). It is a bilinear map on the modules

$$\cdot : M^*(c, h) \times M(c, h) \rightarrow \mathbb{C} \quad (1.3.1)$$

determined by

$$\langle h| \cdot |h\rangle = 1, \quad \langle h| u_1 u_2 \cdot |h\rangle = \langle h| u_1 \cdot u_2 |h\rangle = \langle h| \cdot u_1 u_2 |h\rangle \quad (u_1, u_2 \in \text{Vir}).$$

We usually omit the symbol $\cdot$ and write $\langle h| u |h\rangle := \langle h| u \cdot |h\rangle$ as in the physics literature. By counting L_0 -weights one can easily see that

$$M^*(c, h)_m \cdot M(c, h)_n = 0 \text{ if } m \neq n. \quad (1.3.2)$$

This bilinear form is contravariant in the following sense:

$$\langle h| L_\lambda L_{-\mu} |h\rangle = \langle h| L_\mu L_{-\lambda} |h\rangle \quad \text{for any } \lambda, \mu. \quad (1.3.3)$$

Recalling the basis (1.2.1) and the dual basis (1.2.2), we define

$$K_{\lambda, \mu}(c, h) := \langle c, h| L_\mu L_{-\lambda} |c, h\rangle.$$

Then the properties of the contravariant form are encoded in the (infinite size) matrix $(K_{\lambda, \mu})$, where λ and μ run over the set of all partitions.

However, because of (1.3.2), we only need to consider the $p(n) \times p(n)$ matrix

$$K_n := (K_{\lambda, \mu})_{\lambda, \mu \vdash n} \quad (1.3.4)$$

for each $n \in \mathbb{Z}_{\geq 0}$. By (1.3.3), it is a symmetric matrix.

Let us write down some examples:

$$\begin{aligned}
K_1 &= (K_{(1),(1)}) = (2h), \\
K_2 &= \begin{pmatrix} K_{(1^2),(1^2)} & K_{(1^2),(2)} \\ K_{(2),(1^2)} & K_{(2),(2)} \end{pmatrix} = \begin{pmatrix} 4h(1+2h) & 6h \\ 6h & 4h+c/2 \end{pmatrix}, \\
K_3 &= \begin{pmatrix} K_{(1^3),(1^3)} & K_{(1^3),(2,1)} & K_{(1^3),(3)} \\ K_{(2,1),(1^3)} & K_{(2,1),(2,1)} & K_{(2,1),(3)} \\ K_{(3),(1^3)} & K_{(3),(2,1)} & K_{(3),(3)} \end{pmatrix} \\
&= \begin{pmatrix} 24h(1+h)(1+2h) & 12h(1+3h) & 24h \\ 12h(1+3h) & 8h^2+8h+ch & 10h \\ 24h & 10h & 6h+2c \end{pmatrix}, \\
K_4 &= \begin{pmatrix} K_{(1^4),(1^4)} & K_{(1^4),(2,1^2)} & K_{(1^4),(2,2)} & K_{(1^4),(3,1)} & K_{(1^4),(4)} \\ K_{(2,1^2),(1^4)} & K_{(2,1^2),(2,1^2)} & K_{(2,1^2),(2,2)} & K_{(2,1^2),(3,1)} & K_{(2,1^2),(4)} \\ K_{(2,2),(1^4)} & K_{(2,2),(2,1^2)} & K_{(2,2),(2,2)} & K_{(2,2),(3,1)} & K_{(2,2),(4)} \\ K_{(3,1),(1^4)} & K_{(3,1),(2,1^2)} & K_{(3,1),(2,2)} & K_{(3,1),(3,1)} & K_{(3,1),(4)} \\ K_{(4),(1^4)} & K_{(4),(2,1^2)} & K_{(4),(2,2)} & K_{(4),(3,1)} & K_{(4),(4)} \end{pmatrix} \quad (1.3.5) \\
&= \begin{pmatrix} 0 & 48h(1+2h)(2+3h) & 72h(2+3h) & 48h(1+4h) & 120h \\ * & 0 & 6h(8+8h+c) & 20h(1+2h) & 36h \\ * & * & 0 & 30h & 3(8h+c) \\ * & * & * & 0 & 14h \\ * & * & * & * & 0 \end{pmatrix} \\
&\quad + \text{diag}(96h(1+h)(1+2h)(3+2h), 2h(16+c+58h+2ch+16h^2), \\
&\quad \frac{1}{2}(8h+c)(8+8h+c), 4h(3+3h+c), 8h+5c).
\end{aligned}$$

In the expression of K_4 , we omit the lower-triangular part. The determinant $\det K_n$ is called the Kac determinant. It has the following factored expression:

$$\begin{aligned}
\det K_1(c, h) &= 2h, \\
\det K_2(c, h) &= 2h(16h^2 - (10 - 2c)h + c), \\
\det K_3(c, h) &= 48h^2(16h^2 - (10 - 2c)h + c)(3h^2 - (7 - c)h + 2 + c), \\
\det K_3(c, h) &= 384h^3(16h^2 - (10 - 2c)h + c)(3h^2 - (7 - c)h + 2 + c) \\
&\quad \times (16h^2 - (82 - 10c)h + c)(8h - 1 + c).
\end{aligned} \quad (1.3.6)$$

As conjectured in [55] and shown by [35] and [36], it has the factored form

$$\det K_n(c, h) = \prod_{\lambda \vdash n} 2^{\ell(\lambda)} z_\lambda \times \prod_{\substack{r, s \in \mathbb{Z}_{\geq 1} \\ rs \leq n}} (h - h_{r,s})^{p(n-rs)}. \quad (1.3.7)$$

Here $\ell(\lambda)$ is the length of the partition λ , z_λ is given by

$$z_\lambda := \prod_{i \in \mathbb{Z}_{\geq 1}} i^{m_i(\lambda)} m_i(\lambda)! \quad \text{with} \quad m_i(\lambda) := \#\{1 \leq i \leq \ell(\lambda) \mid \lambda_j = i\}, \quad (1.3.8)$$

and $p(n) := \#\{\lambda \mid \lambda \vdash n\}$ is the partition number of n .

To describe the zeros $h_{r,s}$, let us introduce a parametrization of c :

$$c = c(t) := 13 - 6(t + t^{-1}). \quad (1.3.9)$$

Then the rewritten form $h_{r,s}(t) := h_{r,s}|_{c=c(t)}$ is given by

$$h_{r,s}(t) := \frac{(r-st)^2 - (t-1)^2}{4t}. \quad (1.3.10)$$

Note that by the symmetry

$$c(t) = c(t^{-1}), \quad (1.3.11)$$

we have

$$M(c(t), h) = M(c(t^{-1}), h).$$

1.4. Singular vector

DEFINITION 1.4.1. Let M be a Vir-module. An element v of M is called a singular vector if

$$L_k v = 0 \quad \text{for any } k \in \mathbb{Z}_{>0}. \quad (1.4.1)$$

Thus a highest weight vector is a singular vector.

An element v of $M(c, h)_n$ is called a singular vector of level n if it is a singular vector of $M(c, h)$.

The definition (1.4.1) of the singular vector v indicates that in the expansion

$$v = \sum_{\lambda \vdash n} c_\lambda L_{-\lambda} |c, h\rangle$$

with respect to the basis (1.2.1), the set of coefficients $(c_\lambda)_{\lambda \vdash n}$ forms an eigenvector of the matrix (1.3.4) with eigenvalue 0. Thus if a singular vector exists in $M(c, h)$, then the Kac determinant (1.3.7) vanishes, i.e., the pair (c, h) is expressed as $(c(t), h_{r,s}(t))$ for some $r, s \in \mathbb{Z}_{\geq 1}$ with $rs \leq n$.

Several studies explored explicit forms of singular vectors. First let us mention to

FACT 1.4.2 ([40]). One can write the singular vector $|\chi_{r,s}\rangle$ of $M(c(t), h_{r,s}(t))$ as

$$|\chi_{r,s}\rangle = P_{r,s}(t) |c(t), h_{r,s}(t)\rangle,$$

with

$$P_{r,s}(t) = L_{-1}^{rs} + \cdots \in U(\text{Vir}_-) \otimes \mathbb{C}[t, t^{-1}]. \quad (1.4.2)$$

The point is that the coefficients in (1.4.2) are Laurent polynomials of t .

By direct calculations using the matrices (1.3.5), one can obtain examples for small r and s :

$$\begin{aligned} P_{1,2}(t) &= P_{2,1}(t^{-1}) = L_{-1}^2 - tL_{-2}, \\ P_{1,3}(t) &= P_{3,1}(t^{-1}) = L_{-1}^3 - 4tL_{-2}L_{-1} + 2t(2t-1)L_{-3}, \\ P_{1,4}(t) &= P_{4,1}(t^{-1}) = L_{-1}^4 - 10tL_{-2}L_{-1}^2 + 9t^2L_{-2}^2 \\ &\quad + 2t(12t-5)L_{-3}L_{-1} - 6t(6t^2-4t+1)L_{-4}, \\ P_{2,2}(t) &= P_{2,2}(t^{-1}) = L_{-1}^4 - 2(t+t^{-1})L_{-2}L_{-1}^2 + (t^2-2+t^{-2})L_{-2}^2 \\ &\quad - 2(t-3+t^{-1})L_{-3}L_{-1} - 3(t-2+t^{-1})L_{-4}. \end{aligned} \quad (1.4.3)$$

In these examples the condition (1.4.2) is clearly satisfied.

As another simple remark, we have the equality

$$P_{r,s}(t) = P_{s,r}(t^{-1}), \quad (1.4.4)$$

which is an easy consequence of the symmetries (1.3.11) and

$$h_{r,s}(t) = h_{s,r}(t^{-1}).$$

In the late 1980s and the early 1990s, a series of works tried to write down $P_{r,s}(t)$ explicitly. [14] gave an explicit formula for $P_{1,s}(t)$. [9] gave an algorithm for constructing general $P_{r,s}(t)$ from $P_{1,s}(t)$. See also [24, §8.A] for this algorithm. [63] gave a formula of $P_{r,s}(t)$ using ‘analytic continuation’ of $P_{1,s}(t)$. See also [40] for a mathematically rigorous treatment of this ‘analytic continuation’. Although these formulas for $P_{r,s}(t)$ were enough for some studies on representation theory (see [64] for example), an ‘explicit’ formula for $P_{r,s}(t)$ could not be obtained.

The paper [68] shed a new light on this problem. It discovered that the following two objects coincide up to normalization: the integral expression of the Jack symmetric function $J_{(s^r)}^{(t)}$ [66, Chap.VI §10], and the expression of $|\chi_{r,s}\rangle$ in terms of the screening operators for the Feigin-Fuchs bosonization of Virasoro algebra [35]. See Fact 2.4.2 in Chapter 2 §2.4 for the precise statement.

CHAPTER 2

Bosonization of Virasoro algebra

Here we explain the Feigin-Fuchs bosonization of Virasoro algebra.

2.1. Heisenberg algebra and Fock module

Consider the Heisenberg algebra $\mathcal{H}$ generated by a_n ($n \in \mathbb{Z}$) with the relation

$$[a_m, a_n] = m\delta_{m+n,0}.$$

$\mathcal{H}$ has the triangular decomposition $\mathcal{H} = \mathcal{H}_+ \oplus \mathcal{H}_0 \oplus \mathcal{H}_-$ with $\mathcal{H}_\pm := \bigoplus_{\pm n \in \mathbb{Z}_{\geq 1}} \mathbb{C}a_n$ and $\mathcal{H}_0 := \mathbb{C}a_0$. Let $\mathbb{C}_\alpha = \mathbb{C}|\alpha\rangle_{\mathcal{F}}$ be the one-dimensional representation of $\mathcal{H}_0 \oplus \mathcal{H}_+$ with the action $a_0|\alpha\rangle_{\mathcal{F}} = \alpha|\alpha\rangle_{\mathcal{F}}$ and $a_n|\alpha\rangle_{\mathcal{F}} = 0$ ($n \in \mathbb{Z}_{\geq 1}$). Then the Fock space $\mathcal{F}_\alpha$ is defined to be

$$\mathcal{F}_\alpha := \text{Ind}_{\mathcal{H}_0 \oplus \mathcal{H}_-}^{\mathcal{H}} \mathbb{C}_\alpha.$$

It has a weight decomposition

$$\mathcal{F}_\alpha = \bigoplus_{n \in \mathbb{Z}_{\geq 0}} \mathcal{F}_{\alpha,n}, \quad (2.1.1)$$

where each weight space $\mathcal{F}_{\alpha,n}$ has a basis

$$\{a_{-\lambda}|\alpha\rangle_{\mathcal{F}} \mid \lambda \vdash n\} \quad (2.1.2)$$

with $a_{-\lambda} := a_{-\lambda_k} a_{-\lambda_{k-1}} \cdots a_{-\lambda_1}$ for a partition $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_k)$. Then the action of $\widehat{U}(\mathcal{H})$ on $\mathcal{F}_\alpha$ is well-defined.

Similarly the dual Fock space $\mathcal{F}_\alpha^*$ is defined to be $\text{Ind}_{\mathcal{H}_0 \oplus \mathcal{H}_-}^{\mathcal{H}} \mathbb{C}_\alpha^*$, where $\mathbb{C}_\alpha^* = \mathbb{C} \cdot {}_{\mathcal{F}}\langle \alpha |$ is the one-dimensional representation of $\mathcal{H}_0 \oplus \mathcal{H}_-$ with the action ${}_{\mathcal{F}}\langle \alpha | a_0 = \alpha \cdot {}_{\mathcal{F}}\langle \alpha |$ and ${}_{\mathcal{F}}\langle \alpha | a_{-n} = 0$ ($n \in \mathbb{Z}_{\geq 1}$).

We also have the bilinear form

$$\cdot : \mathcal{F}_\alpha^* \times \mathcal{F}_\alpha \rightarrow \mathbb{C}$$

defined by

$$\begin{aligned} {}_{\mathcal{F}}\langle \alpha | \cdot |\alpha\rangle_{\mathcal{F}} &= 1, \\ {}_{\mathcal{F}}\langle \alpha | uu' \cdot |\alpha\rangle_{\mathcal{F}} &= {}_{\mathcal{F}}\langle \alpha | u \cdot u' |\alpha\rangle_{\mathcal{F}} = {}_{\mathcal{F}}\langle \alpha | \cdot uu' |\alpha\rangle_{\mathcal{F}} \quad (u, u' \in \mathcal{H}). \end{aligned}$$

As in the physics literature, we often omit the symbol $\cdot$ and simply write ${}_{\mathcal{F}}\langle \alpha | \alpha \rangle_{\mathcal{F}}$, ${}_{\mathcal{F}}\langle \alpha | u |\alpha\rangle_{\mathcal{F}}$ and so on.

2.2. Feigin-Fuchs bosonization

For a fixed $\rho \in \mathbb{C}$, consider the correspondence

$$L_n \mapsto \mathcal{L}_n := \frac{1}{2} \sum_{m \in \mathbb{Z}} \circ a_m a_{n-m} \circ - (n+1)\rho a_n, \quad C \mapsto 1 - 12\rho^2, \quad (2.2.1)$$

where the symbol $\circ \circ$ means the (usual) normal ordering of $\mathcal{H}$:

$$\circ a_m a_n \circ := \begin{cases} a_m a_n & n \geq 0, \\ a_n a_m & n < 0. \end{cases}$$

This correspondence (2.2.1) determines a well-defined morphism

$$\varphi : U(\text{Vir}) \rightarrow \widehat{U}(\mathcal{H}).$$

Here $\widehat{U}(\mathcal{H})$ is the completion of the universal enveloping algebra $U(\mathcal{H})$ in the following sense (see also [34], [39] and [38]). For $n \in \mathbb{Z}_{\geq 0}$, let I_n be the left ideal of $U(\mathcal{H})$ generated by all polynomials in a_m ($m \in \mathbb{Z}_{\geq 1}$) of degrees greater than or equal to n (where we defined the degree by $\deg a_m := m$). Then we define

$$\widehat{U}(\mathcal{H}) := \varprojlim_n \widehat{U}(\mathcal{H})/I_n.$$

Next we recall the functorial correspondence of the representations. First, let us define the Fock representation $\mathcal{F}_\alpha$ of $\mathcal{H}$. (2.2.1) is compatible with the map

$$\psi : M(c, h) \rightarrow \mathcal{F}_\alpha, \quad L_{-\lambda} |c, h\rangle \mapsto \mathcal{L}_{-\lambda} |\alpha\rangle_{\mathcal{F}} \quad (2.2.2)$$

with $\mathcal{L}_{-\lambda} := \mathcal{L}_{-\lambda_1} \mathcal{L}_{-\lambda_2} \cdots \mathcal{L}_{-\lambda_k}$ for $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_k)$ and

$$c = 1 - 12\rho^2, \quad h = \frac{1}{2}\alpha(\alpha - 2\rho). \quad (2.2.3)$$

In other words, we have

$$\psi(xv) = \varphi(x)\psi(v) \quad (x \in \text{Vir}, v \in M(c, h))$$

under the parametrization (2.2.3) of highest weights.

Note that from the parametrizations (1.3.9), (1.3.10) and the correspondence (2.2.3), a singular vector occurs at

$$c = 1 - 12\rho(t)^2, \quad h = h_{r,s} = \frac{1}{2}\alpha_{r,s}(t)(\alpha_{r,s} - 2\rho(t))$$

with

$$\rho(t) := \frac{1}{\sqrt{2}}(t^{-1/2} - t^{1/2}), \quad \alpha_{r,s}(t) := \frac{1}{\sqrt{2}}((r+1)t^{-1/2} - (s+1)t^{1/2}). \quad (2.2.4)$$

2.3. Bosonization and singular vector

The Fock space $\mathcal{F}_\alpha$ is naturally identified with the space Λ of symmetric functions. The notations for symmetric functions are explained in the appendix Chapter A, § A.3.

Following [6], we define the isomorphism between $\mathcal{F}_\alpha$ and $\Lambda_{\mathbb{C}(t^{1/2})}$:

$$\iota : \mathcal{F}_\alpha \otimes \mathbb{C}(t^{1/2}) \rightarrow \Lambda_{\mathbb{C}(t^{1/2})}, \quad v \mapsto {}_{\mathcal{F}}\langle \alpha | \exp\left(\frac{1}{\sqrt{2}t} \sum_{n=1}^{\infty} \frac{1}{n} p_n a_n\right) v. \quad (2.3.1)$$

Under this morphism, an element $a_{-\lambda} |\alpha\rangle_{\mathcal{F}}$ of the base (2.1.2) is mapped to

$$\iota(a_{-\lambda} |\alpha\rangle_{\mathcal{F}}) = \frac{p_\lambda(x)}{(\sqrt{2}t)^{\ell(\lambda)}}.$$

Since $\{p_\lambda\}$ is a basis of $\Lambda_{\mathbb{Q}}$, ι is actually an isomorphism.

Using the examples (1.4.3) and the Feigin-Fuchs bosonization (2.2.1), one can calculate some examples of the images of singular vectors:

$$\iota\left(\varphi(P_{r,s}(t)) |\alpha_{r,s}(t)\rangle\right) = \iota \circ \psi(P_{r,s}(t) |c(t), h_{r,s}(t)\rangle) = \iota \circ \psi(|\chi_{r,s}\rangle).$$

The result is

$$\begin{aligned}
\iota \circ \psi(|\chi_{1,1}\rangle) &= J_{(1)}^{(t)} \cdot (t^{-1} - 1), \\
\iota \circ \psi(|\chi_{2,1}\rangle) &= J_{(1^2)}^{(t)} \cdot (t^{-1} - 1)(2t^{-1} - 1), \\
\iota \circ \psi(|\chi_{3,1}\rangle) &= J_{(1^3)}^{(t)} \cdot (t^{-1} - 1)(2t^{-1} - 1)(3t^{-1} - 1), \\
\iota \circ \psi(|\chi_{4,1}\rangle) &= J_{(1^4)}^{(t)} \cdot (t^{-1} - 1)(2t^{-1} - 1)(3t^{-1} - 1)(4t^{-1} - 1), \\
\iota \circ \psi(|\chi_{2,2}\rangle) &= J_{(2^2)}^{(t)} \cdot (t^{-1} - 1)(t^{-1} - 2)(2t^{-1} - 1)(2t^{-1} - 2).
\end{aligned} \tag{2.3.2}$$

Here $J_{\lambda}^{(t)}$ is the integral Jack symmetric function, whose definition will be recalled in the next subsection. Thus if one expresses $|\chi_{r,s}\rangle$ in terms of the Heisenberg generators a_n 's and identifies a_{-n} with the power sum symmetric function p_n , then the expression of $|\chi_{r,s}\rangle$ is proportional to $J_{(s^r)}^{(t)}$.

2.4. Jack symmetric function

Now we recall the definition and some properties of Jack symmetric function (see [66, Chap.VI §10] and [83]). Let t be an indeterminate and define an inner product on $\Lambda_{\mathbb{Q}(t)}$ by

$$\langle p_{\lambda}, p_{\mu} \rangle_t := \delta_{\lambda, \mu} z_{\lambda} t^{\ell(\lambda)}. \tag{2.4.1}$$

Here z_{λ} is given in (1.3.8).

REMARK 2.4.1. Our parameter t is usually denoted by α in the literature, for example, in [66]. We avoid using α since it is already defined to be the highest weight of the Heisenberg Fock space $\mathcal{F}_{\alpha}$.

Then the monic Jack symmetric function $P_{\lambda}^{(t)}$ is determined uniquely by the following two conditions:

(i): It has an expansion via monomial symmetric function m_{ν} in the form

$$P_{\lambda}^{(t)} = m_{\lambda} + \sum_{\mu < \lambda} c_{\lambda, \mu}(t) m_{\mu}.$$

Here $c_{\lambda, \mu}(t) \in \mathbb{Q}(t)$ and the ordering $<$ among the partitions is the dominance semi-ordering.

(ii): The family of Jack symmetric functions is an orthogonal basis with respect to $\langle \cdot, \cdot \rangle_t$:

$$\langle P_{\lambda}^{(t)}, P_{\mu}^{(t)} \rangle_t = 0 \quad \text{if } \lambda \neq \mu.$$

In order to define the integral Jack symmetric function $J_{\lambda}^{(t)}$, it is necessary to express the norm of $P_{\lambda}^{(t)}$. One can simply write it down using Young diagrams.

Using these combinatorial notations, one can write down the norm of monic Jack symmetric function as

$$\langle P_{\lambda}^{(t)}, P_{\lambda}^{(t)} \rangle_t = \prod_{\square \in \lambda} \frac{ta_{\lambda}(\square) + \ell_{\lambda}(\square) + t}{ta_{\lambda}(\square) + \ell_{\lambda}(\square) + 1}, \tag{2.4.2}$$

where $\square \in \lambda$ means that the box $\square$ runs over the boxes in the Young diagram associated to λ .

Then the integral Jack symmetric function $J_{\lambda}^{(t)}$ is defined to be

$$J_{\lambda}^{(t)} := P_{\lambda}^{(t)} \cdot \prod_{\square \in \lambda} (ta_{\lambda}(\square) + \ell_{\lambda}(\square) + 1) \tag{2.4.3}$$

It is known that for a partition λ of n ,

$$J_{\lambda}^{(t)} = \sum_{\mu \vdash n} u_{\lambda, \mu}(t) p_{\mu}, \quad u_{\lambda, (1^n)}(t) = 1, \quad u_{\lambda, \mu}(t) \in \mathbb{Z}[t].$$

This is the origin of the word ‘integral’ Jack symmetric function.

Finally we can state the fact obtained in [68].

FACT 2.4.2. (1) [68] The bosonization ψ and the isomorphism ι map the singular vector $|\chi_{r,s}\rangle$ to the integral Jack symmetric function:

$$\iota \circ \psi(|\chi_{r,s}\rangle) \propto J_{(s^r)}^{(t)}.$$

(2) [79] The proportional factor in the above equation is equal to

$$B_{r,s}(t) := \prod_{k=1}^r \prod_{l=1}^s (kt^{-1} - l). \quad (2.4.4)$$

CHAPTER 3

Whittaker vectors via Jack symmetric function

We give an explicit formula of Whittaker vector for Virasoro algebra in terms of Jack symmetric functions. Our fundamental tools are Feigin-Fuchs bosonization and the split expression of Calogero-Sutherland model given by Awata, Matsuo, Odake and Shiraishi.

This chapter is based on the author's paper [93].

3.1. Main theorem

The proposal of AGT conjecture implies that the inner product of a certain element in Verma module of Virasoro algebra coincides with Nekrasov partition function for four dimensional $N = 2$ pure gauge theory. The element considered is a kind of Whittaker vector in Verma module of Virasoro algebra.

Whittaker vectors and Whittaker modules are important gadgets in the representation theory since its emergence in the study of finite dimensional Lie algebras [65]. Although numerous analogues and generalizations have been proposed for other algebras, such as affine algebras and quantum groups, not so many investigations have been given for the Whittaker vectors of Virasoro algebra. A general theory on the properties of Whittaker modules for Virasoro algebra was recently given in [76].

In this chapter, we give an explicit expression of the Whittaker vector for Verma module of Virasoro algebra in terms of Jack symmetric functions [66, Chap.VI §10]. We use Feigin-Fuchs bosonization [35] to identify Verma module and the space of symmetric functions, and then utilize the split expression of Calogero-Sutherland Hamiltonian [82] to derive a recursion relation on the coefficients of Whittaker vector in the expansion with respect to Jack symmetric functions.

Our result is related to a conjecture given by Awata and Yamada in [7]. They proposed the five-dimensional AGT conjecture for pure $SU(2)$ gauge theory using the deformed Virasoro algebra, and as a related topic, they also proposed a conjectural formula on the explicit form of the deformed Gaiotto state in terms of Macdonald symmetric functions [7, (3.18)]. Our formula is the non-deformed Virasoro, or four-dimensional, counterpart of their conjectural formula.

The motivation of our study also comes from the work [68], where singular vectors of Virasoro algebra are expressed by Jack symmetric polynomials.

Now we can state the main theorem in this chapter.

Consider the Verma module $M(c, h)$ of Virasoro algebra Vir with generic complex numbers c and h . Let a be an arbitrary complex number, and let v_G be an element of $M(c, h)$ such that

$$L_1 v_G = a v_G, \quad L_n v_G = 0 \quad (n \geq 2).$$

Then v_G exists uniquely up to scalar multiplication (see Fact 3.3.2).

Introduce the complex numbers ρ , α and β by the relations

$$c = 1 - 12\rho^2, \quad h = \frac{1}{12}\alpha(\alpha - 2\rho), \quad \rho = \frac{\beta^{1/2} - \beta^{-1/2}}{\sqrt{2}}.$$

Then by the Feigin-Fuchs bosonization $\psi : M(c, h) \rightarrow \mathcal{F}_\alpha$ (2.2.2) and the isomorphism $\iota_\beta := \iota|_{t=\beta^{-1}}$: (2.3.1)

$$\begin{array}{ccc} \iota_\beta : \mathcal{F}_\alpha & \xrightarrow{\quad} & \Lambda_{\mathbb{C}}, \\ \quad \quad \quad \downarrow \Psi & & \downarrow \Psi \\ v & \mapsto & {}_{\mathcal{F}}\langle \alpha | \exp \left(\sqrt{\frac{\beta}{2}} \sum_{n=1}^{\infty} \frac{1}{n} p_n a_n \right) v, \end{array}$$

one has an element $\iota_\beta \circ \psi(v_G) \in \Lambda_{\mathbb{C}}$.

THEOREM. We have

$$\iota_\beta \circ \psi(v_G) = \sum_{\lambda \in \mathcal{P}} a^{|\lambda|} c_\lambda(\alpha, \beta) P_\lambda^{(\beta^{-1})}, \quad (3.1.1)$$

where λ runs over all the partitions and the coefficient $c_\lambda(\alpha, \beta)$ is given by

$$\begin{aligned} & c_\lambda(\alpha, \beta) \\ &= \prod_{(i,j) \in \lambda} \frac{1}{\lambda_i - j + 1 + \beta(\lambda_j^\vee - i)} \prod_{\substack{(i,j) \in \lambda \\ (i,j) \neq (1,1)}} \frac{\beta}{(j+1) + \sqrt{2}\beta^{1/2}\alpha - (i+1)\beta}. \end{aligned} \quad (3.1.2)$$

(See Chapter A §A.2 for the symbol “ $(i, j) \in \lambda$ ”.)

The proof of this theorem will be given in §3.3.1.

In the main theorem above, the element v_G is the Whittaker vector associated to the degenerate (Lie algebra) homomorphism $\eta : \text{Vir}_+ \rightarrow \mathbb{C}$, that is, $\eta(L_2) = 0$. We will call this element by “Gaiotto state”, following [7]. A general theory of Whittaker vectors usually assumes the non-degeneracy of the homomorphism η , i.e., $\eta(L_1) \neq 0$ and $\eta(L_2) \neq 0$. This non-degenerate case will be treated in Proposition 3.3.11, although there seem no factored expressions for the coefficients as (3.1.2).

The content of this chapter is as follows. In §3.2, we recall the split expression of the Calogero-Sutherland Hamiltonian, which is a key point in our proof. In §3.3, we investigate the Whittaker vectors in terms of symmetric functions. The main theorem will be proved §3.3.1, using some combinatorial identities shown in §3.4. The Whittaker vector with respect to the non-degenerate homomorphism will be treated in §3.3.3.

3.2. Calogero-Sutherland Hamiltonian

This section is a preliminary for the proof of the main theorem. We need the following Definition 3.2.1 and Proposition 3.2.2:

DEFINITION 3.2.1. (1) Let λ be a partition and b, β be generic complex numbers. Define $f_\lambda^{(b, \beta)} \in \mathcal{F}_\alpha$ to be the element such that

$$\iota_\beta(f_\lambda^{(b, \beta)}) = P_\lambda^{(b)}, \quad (3.2.1)$$

where ι_β is the isomorphism given in (2.3.1).

(2) For a complex number β , define an element of $\widehat{U}(\mathcal{H})$ by

$$\widehat{E}_\beta = \sqrt{2\beta} \sum_{n>0} a_{-n} \mathcal{L}_n + \sum_{n>0} a_{-n} a_n \left(\beta - 1 - \sqrt{2\beta} a_0 \right). \quad (3.2.2)$$

Here $\mathcal{L}_n \in \widehat{U}(\mathcal{H})$ is the bosonized Virasoro generator, and we have put the assumption

$$\rho = (\beta^{1/2} - \beta^{-1/2})/\sqrt{2}. \quad (3.2.3)$$

PROPOSITION 3.2.2. For a generic complex number β , we have

$$\widehat{E}_\beta f_\lambda^{(\beta^{-1}, \beta)} = \epsilon_\lambda(\beta) f_\lambda^{(\beta^{-1}, \beta)}, \quad (3.2.4)$$

$$\epsilon_\lambda(\beta) := \sum_i (\lambda_i^2 + \beta(1 - 2i)\lambda_i) \quad (3.2.5)$$

for any partition λ .

The proof of this proposition is rather complicated, since we should utilize Jack symmetric polynomials with finite variables.

3.2.1. Jack symmetric polynomials. As in Chapter A §A.3, we denoted by Λ_N the space of symmetric polynomials of N variables, and by Λ_N^d its degree d homogeneous subspace. In order to denote N -variable symmetric polynomials, we put the superscript “ (N) ” on the symbols for the infinite-variable symmetric functions. For example, we denote by $p_\lambda^{(N)}(x) := p_{\lambda_1}^{(N)}(x)p_{\lambda_2}^{(N)}(x) \cdots$ the product of the power sum polynomials $p_k^{(N)}(x) := \sum_{i=1}^N x_i^k$, and by $m_\lambda^{(N)}(x)$ the monomial symmetric polynomial.

Let us fix $N \in \mathbb{Z}_{\geq 1}$ and an indeterminate t . For a partition λ with $\ell(\lambda) \leq N$, the N -variable Jack symmetric polynomial $P_\lambda^{(N)}(x; t)$ is uniquely specified by the following two properties.

(i):

$$P_\lambda^{(N)}(x; t) = m_\lambda^{(N)}(x) + \sum_{\mu < \lambda} \tilde{c}_{\lambda, \mu}(t) m_\mu^{(N)}(x), \quad \tilde{c}_{\lambda, \mu}(t) \in \mathbb{Q}(t).$$

(ii):

$$H_t^{(N)} P_\lambda^{(N)}(x; t) = \epsilon_\lambda^{(N)}(t) P_\lambda^{(N)}(x; t), \quad (3.2.6)$$

$$H_t^{(N)} := \sum_{i=1}^N \left(x_i \frac{\partial}{\partial x_i} \right)^2 + t \sum_{1 \leq i < j \leq N} \frac{x_i + x_j}{x_i - x_j} \left(x_i \frac{\partial}{\partial x_i} - x_j \frac{\partial}{\partial x_j} \right), \quad (3.2.7)$$

$$\epsilon_\lambda^{(N)}(t) := \sum_i (\lambda_i^2 + t(N + 1 - 2i)\lambda_i). \quad (3.2.8)$$

The differential operator (3.2.7) is equivalent to the Calogero-Sutherland Hamiltonian (see [6, §2] for the detailed explanation.) In (i), we used the dominance partial semi-ordering on the partitions. If $N \geq d$, then $\{P_\lambda^{(N)}(x; t)\}_{\lambda \vdash d}$ is a basis of $\Lambda_{N, \mathbb{Q}(t)}^d$.

REMARK 3.2.3. In the literature, the indeterminate t is usually denoted by $\beta = \alpha^{-1}$, and we will also identify it with our β given in (3.2.3) later. But at this moment we don't use it to avoid confusion.

DEFINITION 3.2.4. For $M \geq N$, we denote the restriction map from Λ_M to Λ_N by

$$\begin{aligned} \rho_{M, N} : \quad \Lambda_M &\rightarrow \Lambda_N \\ \Psi &\Psi \\ f(x_1, \dots, x_M) &\mapsto f(x_1, \dots, x_N, 0, \dots, 0), \end{aligned}$$

and the induced restriction map from Λ to Λ_N by

$$\begin{aligned} \rho_N : \quad \Lambda &\rightarrow \Lambda_N. \\ \Psi &\Psi \\ f(x_1, x_2, \dots) &\mapsto f(x_1, \dots, x_N, 0, \dots). \end{aligned}$$

We denote the maps on the tensored spaces $\Lambda_{M, \mathbb{C}} \rightarrow \Lambda_{N, \mathbb{C}}$ and $\Lambda_{\mathbb{C}} \rightarrow \Lambda_{N, \mathbb{C}}$ by the same symbols $\rho_{M, N}$ and ρ_N .

FACT 3.2.5. For any $\lambda \in \mathcal{P}$, every $N \in \mathbb{Z}_{\geq 1}$ with $N \geq \ell(\lambda)$, and any generic $t \in \mathbb{C}$ we have

$$\rho_N(P_\lambda^{(t^{-1})}) = P_\lambda^{(N)}(x; t).$$

3.2.2. Split form of the Calogero-Sutherland Hamiltonian. We recall the collective field method in the Calogero-Sutherland model following [6, §3]. Recall that the Calogero-Sutherland Hamiltonian $H_t^{(N)}$ (3.2.7) acts on the space of symmetric polynomials $\Lambda_{N, \mathbb{Q}(t)}$.

FACT 3.2.6. (1) Let t, t' be non-zero complex numbers. Define an element of $\widehat{U}(\mathcal{H})$ by

$$\widehat{H}_{t, t'}^{(N)} := \sum_{n, m > 0} \left(t' a_{-m-n} a_m a_n + \frac{t}{t'} a_{-m} a_{-n} a_{m+n} \right)$$

$$+ \sum_{n>0} (n(1-t) + Nt) a_{-n} a_n.$$

Then for any $v \in \mathcal{F}_\alpha$ and every $N \in \mathbb{Z}_{\geq 1}$ we have

$$\rho_N \circ \iota_t(\widehat{H}_{t,t'}^{(N)} v) = H_t^{(N)}(\rho_N \circ \iota_t(v)). \quad (3.2.9)$$

(2) Under the relation

$$\rho = (t^{1/2} - t^{-1/2})/\sqrt{2}$$

we have

$$\widehat{H}_{t,\sqrt{t/2}}^{(N)} = \sqrt{2t} \sum_{n>0} a_{-n} \mathcal{L}_n + \sum_{n>0} a_{-n} a_n (Nt + t - 1 - \sqrt{2ta_0}). \quad (3.2.10)$$

Here $\mathcal{L}_n \in \widehat{U}(\mathcal{H})$ is the bosonized Virasoro generator.

PROOF. These are well-known results (for example, see [81, Prop. 4.47], [6] and the references therein). We only show the sketch of the proof.

As for (1), note that $\{a_{-\lambda} |\alpha\rangle_{\mathcal{F}} \mid \lambda \in \mathcal{P}\}$ is a basis of $\mathcal{F}$. So it is enough to show (3.2.9) for each λ . One can calculate the left hand side using the commutation relation of $\mathcal{H}$ only. On the right hand side, one may use $\iota_t(a_{-\lambda} |\alpha\rangle_{\mathcal{F}}) \propto p_\lambda$, and calculate it using the expression (3.2.7).

(2) is proved by direct calculation. $\square$

REMARK 3.2.7. The form (3.2.10) is called the split expression in [82, §1].

3.2.3. Proof of Proposition 3.2.2. By Fact 3.2.6, we have the left commuting diagram in (3.2.11). Note that we set the parameters t and t' in $\widehat{H}_{t,t'}^{(N)}$ to be β and $\sqrt{\beta/2}$, so that we may use Fact 3.2.6 (2). In the right diagram of (3.2.11) we show how the element $f_\lambda^{(b,\beta)} \in \mathcal{F}_\alpha$ given in (3.2.1) behaves under the maps indicated in the left diagram. Here we set the parameter b to be β^{-1} so that $\iota_\beta f_\lambda^{(\beta^{-1},\beta)} = P_\lambda^{(\beta^{-1})} \in \Lambda_{\mathbb{C}}$ and $\rho_N \circ \iota_\beta f_\lambda^{(\beta^{-1},\beta)} = P_\lambda^{(N)}(x; \beta) \in \Lambda_{N,\mathbb{C}}$. At the bottom line we used the eigen-equation of Jack symmetric polynomial (3.2.6).

$$\begin{array}{ccc} \mathcal{F}_\alpha & \xrightarrow{\widehat{H}_{\beta,\sqrt{\beta/2}}^{(N)}} & \mathcal{F}_\alpha \\ \downarrow \iota_\beta & & \downarrow \iota_\beta \\ \Lambda_{\mathbb{C}} & \circ & \Lambda_{\mathbb{C}} \\ \downarrow \rho_N & & \downarrow \rho_N \\ \Lambda_{N,\mathbb{C}} & \xrightarrow{H_\beta^{(N)}} & \Lambda_{N,\mathbb{C}} \end{array} \quad \begin{array}{ccc} f_\lambda^{(\beta^{-1},\beta)} & \mapsto & \widehat{H}_{\beta,\sqrt{\beta/2}}^{(N)}(f_\lambda^{(\beta^{-1},\beta)}) \\ \downarrow & & \downarrow \\ P_\lambda^{(\beta^{-1})} & & \iota_\beta \circ \widehat{H}_{\beta,\sqrt{\beta/2}}^{(N)}(f_\lambda^{(\beta^{-1},\beta)}) \\ \downarrow & & \downarrow \\ P_\lambda^{(N)}(x; \beta) & \mapsto & P_\lambda^{(N)}(x; \beta) \cdot \epsilon_\lambda^{(N)}(\beta) \end{array} \quad (3.2.11)$$

Since this diagram holds for every N with $N \geq \ell(\lambda)$, we have

$$\widehat{H}_{\beta,\sqrt{\beta/2}}^{(N)} f_\lambda^{(\beta^{-1},\beta)} = \epsilon_\lambda^{(N)}(\beta) f_\lambda^{(\beta^{-1},\beta)}.$$

Therefore we have

$$\begin{aligned} & \left[\sqrt{2t} \sum_{n>0} a_{-n} \mathcal{L}_n + \sum_{n>0} a_{-n} a_n (Nt + t - 1 - \sqrt{2ta_0}) \right] f_\lambda^{(\beta^{-1},\beta)} \\ &= f_\lambda^{(\beta^{-1},\beta)} \cdot \sum_i (\lambda_i^2 + t(N+1-2i)\lambda_i). \end{aligned}$$

We can subtract N -dependent terms from both sides. The result is nothing but the desired statement of Proposition 3.2.2.

3.3. Whittaker vectors

Recall the notion of the Whittaker vector for a finite dimensional Lie algebra $\mathfrak{g}$ given in [65]. Let $\mathfrak{n}$ be a maximal nilpotent Lie subalgebra of $\mathfrak{g}$ and $\eta : \mathfrak{n} \rightarrow \mathbb{C}$ be a homomorphism. Let V be any $U(\mathfrak{g})$ -module. Then a vector $w \in V$ is called a Whittaker vector with respect to η if $xw = \eta(x)w$ for all $x \in \mathfrak{n}$.

We will discuss an analogue of this Whittaker vector in the Virasoro algebra Vir with the triangular decomposition $\text{Vir} = \text{Vir}_+ \oplus \text{Vir}_0 \oplus \text{Vir}_-$. Note that the elements $L_1, L_2 \in \text{Vir}_+$ generate Vir_+ . Thus if we take Vir_+ as the η in the above definition, what we should consider is a homomorphism $\eta : \text{Vir}_+ \rightarrow \mathbb{C}$, which is determined by $\eta(L_1)$ and $\eta(L_2)$.

In [76], a characterization of Whittaker vectors in general $U(\text{Vir})$ -modules are given under the assumption that η is non-degenerate, i.e. $\eta(L_1) \neq 0$ and $\eta(L_2) \neq 0$.

In this section we will express Whittaker vectors in the Verma module $M(c, h)$ using Jack symmetric functions. Before starting the general treatment, we first investigate a degenerate version of the Whittaker vector, i.e. we assume $\eta(L_2) = 0$. We will call this vector by Gaiotto state of Virasoro algebra, although the paper [42] treated both degenerate and non-degenerate Whittaker vectors.

3.3.1. Gaiotto state via Jack symmetric functions.

DEFINITION 3.3.1. Fix a non-zero complex number a . Let v_G be a non-zero element of the completed Verma module $\widehat{M}(c, h)$ satisfying

$$L_1 v_G = a v_G, \quad L_n v_G = 0 \quad (n \in \mathbb{Z}_{\geq 2}).$$

We call such an element v_G a Gaiotto state of $M(c, h)$.

Here the completion is taken with respect to the L_0 -weight decomposition

$$M(c, h) = \bigoplus_{n \in \mathbb{Z}_{\geq 0}} M(c, h)_n.$$

FACT 3.3.2. Assume that c and h are generic. Then v_G exists uniquely up to constant multiplication.

PROOF. This statement is shown in [76]. □

LEMMA 3.3.3. Decompose a Gaiotto state v_G as

$$v_G = \sum_{n \in \mathbb{Z}_{\geq 0}} a^n v_{G,n}, \quad v_{G,n} \in M(c, h)_n.$$

Then we have

$$v_{G,n} = L_1 v_{G,n+1} \quad (n \in \mathbb{Z}_{\geq 0}). \tag{3.3.1}$$

PROOF. This follows from the commutation relation $[L_0, L_1] = -L_1$. □

Now consider the bosonized Gaiotto state

$$w_{G,n} := \psi(v_{G,n}) \in \mathcal{F}_{\alpha,n},$$

where $\psi : M(c, h) \rightarrow \mathcal{F}_{\alpha}$ is the Feigin-Fuchs bosonization (2.2.2) and $\mathcal{F}_{\alpha,n}$ is the a_0 -weight space (2.1.1). At this moment the Heisenberg parameters ρ, α are related to the Virasoro parameters c, h by the relations

$$c = 1 - 12\rho^2, \quad h = \frac{1}{12}\alpha(\alpha - 2\rho).$$

From the condition (3.3.1) we have

$$\mathcal{L}_1 w_{G,n+1} \in \mathcal{F}_{h,n}, \quad w_{G,n} = \mathcal{L}_1 w_{G,n+1}. \tag{3.3.2}$$

Next we map this bosonized state $w_{G,n}$ into a symmetric function by the isomorphism $\iota_\beta : \mathcal{F}_\alpha \rightarrow \Lambda_\mathbb{C}$ (2.3.1):

$$\iota_\beta(w_{G,n}) = \iota_\beta \circ \psi(v_{G,n}) \in \Lambda_\mathbb{C}^n.$$

Here $\Lambda_\mathbb{C}^n$ is the space of degree n symmetric functions. We take the parameter β so that the Heisenberg parameter ρ is expressed by

$$\rho = (\beta^{1/2} - \beta^{-1/2})/\sqrt{2}.$$

Recall also that the family of Jack symmetric functions $\{P_\lambda^{(\beta^{-1})} \mid \lambda \vdash n\}$ is a basis of $\Lambda_\mathbb{C}^n$ for a generic $\beta \in \mathbb{C}$. Thus we can expand $\iota_\beta(w_{G,n}) \in \Lambda_\mathbb{C}^n$ by $P_\lambda^{(\beta^{-1})}$'s. Let us express this expansion as:

$$\iota_\beta(w_{G,n}) = \iota_\beta \circ \psi(v_{G,n}) = \sum_{\lambda \vdash n} c_\lambda(\alpha, \beta) P_\lambda^{(\beta^{-1})}, \quad c_\lambda(\alpha, \beta) \in \mathbb{C}. \quad (3.3.3)$$

Note that this expansion is equivalent to

$$w_{G,n} = \sum_{\lambda \vdash n} c_\lambda(\alpha, \beta) f_\lambda^{(\beta^{-1}, \beta)} \in \mathcal{F}_\alpha \quad (3.3.4)$$

by (3.2.1). Now the correspondence of the parameters becomes:

$$c = 1 - 12\rho^2, \quad h = \frac{1}{12}\alpha(\alpha - 2\rho_0), \quad \rho = \frac{\beta^{1/2} - \beta^{-1/2}}{\sqrt{2}}. \quad (3.3.5)$$

The main result of this chapter is

THEOREM 3.3.4. Assume that c and h are generic. (Then v_G exists uniquely up to constant multiplication by Fact 3.3.2.) If $c_\emptyset(\alpha, \beta)$ is set to be 1 in the expansion (3.3.3), then the other coefficients are given by

$$\begin{aligned} c_\lambda(\alpha, \beta) &= \prod_{(i,j) \in \lambda} \frac{1}{\lambda_i - j + 1 + \beta(\lambda'_j - i)} \\ &\quad \times \prod_{(i,j) \in \lambda \setminus \{(1,1)\}} \frac{1}{(j+1)\beta + \sqrt{2}\beta^{1/2}\alpha - (i+1)}. \end{aligned} \quad (3.3.6)$$

3.3.2. Proof of Theorem 3.3.4. Before starting the proof, we need to prepare the following Proposition 3.3.7. Recall the Pieri formula of Jack symmetric function. We only need the case of “adding one box”, that is, the case of multiplying the degree one power sum function $p_1 = x_1 + x_2 + \cdots$.

DEFINITION 3.3.5. For partitions μ and λ , we denote $\mu <_k \lambda$ if $|\mu| = |\lambda| - k$ and $\mu \subset \lambda$.

FACT 3.3.6 ([66, Chap.VI (6.24), Chap.VI (10.10)]). We have

$$p_1 P_\mu^{(\beta^{-1})} = \sum_{\lambda >_1 \mu} \psi'_{\lambda/\mu}(\beta) P_\lambda^{(\beta^{-1})}, \quad (3.3.7)$$

$$\psi'_{\lambda/\mu}(\beta) := \prod_{i=1}^{I-1} \frac{\lambda_i - \lambda_I + \beta(I - i + 1)}{\lambda_i - \lambda_I + 1 + \beta(I - i)} \frac{\lambda_i - \lambda_I + 1 + \beta(I - i - 1)}{\lambda_i - \lambda_I + \beta(I - i)}. \quad (3.3.8)$$

In the expression in (3.3.8) the partitions λ and μ are related by $\lambda_I = \mu_I + 1$ and $\lambda_i = \mu_i$ for $i \neq I$.

PROPOSITION 3.3.7. $c_\lambda(\alpha, \beta)$ satisfies the next recursion relation:

$$\left(\epsilon_\lambda(\beta) + |\lambda|(1 + \sqrt{2\beta}\alpha - \beta) \right) c_\lambda(\alpha, \beta) = \beta \sum_{\mu <_1 \lambda} \psi'_{\lambda/\mu}(\beta) c_\mu(\alpha, \beta). \quad (3.3.9)$$

Here the function $\epsilon_\lambda(\beta)$ is given in (3.2.5).

PROOF. We will calculate $\widehat{E}_\beta w_{G,n} \in \mathcal{F}_\alpha$ in two ways. By comparing both expression we obtain the recursion relation.

First, by the definition of $\widehat{E}_\beta$ given in (3.2.2) and by the condition (3.3.2) of $v_{G,n}$ we have

$$\begin{aligned}\widehat{E}_\beta w_{G,n} &= \left[\sqrt{2\beta} \sum_{m \geq 1} a_{-m} \mathcal{L}_m + \sum_{m \geq 1} a_{-m} a_m (\beta - 1 - \sqrt{2\beta} a_0) \right] w_{G,n} \\ &= \left[\sqrt{2\beta} a_{-1} \mathcal{L}_1 + \sum_{m \geq 1} a_{-m} a_m (\beta - 1 - \sqrt{2\beta} a_0) \right] w_{G,n} \\ &= \sqrt{2\beta} a_{-1} w_{G,n-1} + n(\beta - 1 - \sqrt{2\beta} \alpha) w_{G,n} \in \mathcal{F}_\alpha.\end{aligned}$$

Now applying the isomorphism $\iota_\beta : \mathcal{F}_\alpha \rightarrow \Lambda_{\mathbb{C}}$ on both sides and substituting $w_{G,n}$ and $w_{G,n-1}$ by their expansions (3.3.3), we have

$$\begin{aligned}\iota_\beta(\widehat{E}_\beta w_{G,n}) &= \beta p_1 \sum_{\mu \vdash n-1} c_\mu(\alpha, \beta) P_\mu^{(\beta-1)} \\ &\quad + n(\beta - 1 - \sqrt{2\beta} \alpha) \sum_{\lambda \vdash n} c_\lambda(\alpha, \beta) P_\lambda^{(\beta-1)} \in \Lambda_{\mathbb{C}}.\end{aligned}$$

Using the Pieri formula (3.3.7) in the first term, we have

$$\begin{aligned}\iota_\beta(\widehat{E}_\beta w_{G,n}) &= \beta \sum_{\mu \vdash n-1} c_\mu(\alpha, \beta) \sum_{\lambda >_1 \mu} \psi'_{\lambda/\mu}(\beta) P_\lambda^{(\beta-1)} \\ &\quad + n(\beta - 1 - \sqrt{2\beta} \alpha) \sum_{\lambda \vdash n} c_\lambda(\alpha, \beta) P_\lambda^{(\beta-1)}.\end{aligned}\tag{3.3.10}$$

Next, by (3.3.4) and by (3.2.4) we have

$$\widehat{E}_\beta w_{G,n} = \widehat{E}_\beta \sum_{\lambda \vdash n} c_\lambda(\alpha, \beta) f_\lambda^{(\beta-1, \beta)} = \sum_{\lambda \vdash n} c_\lambda(\alpha, \beta) \epsilon_\lambda(\beta) f_\lambda^{(\beta-1, \beta)} \in \mathcal{F}_\alpha.$$

Therefore we have

$$\iota_\beta(\widehat{E}_\beta w_{G,n}) = \sum_{\lambda \vdash n} c_\lambda(\alpha, \beta) \epsilon_\lambda(\beta) P_\lambda^{(\beta-1)} \in \Lambda_{\mathbb{C}}.\tag{3.3.11}$$

Then comparing (3.3.10) and (3.3.11) we have

$$\begin{aligned}&\sum_{\lambda \vdash n} (\epsilon_\lambda(\beta) + n(1 + \sqrt{2\beta} \alpha - \beta)) c_\lambda(\alpha, \beta) P_\lambda^{(\beta-1)} \\ &= \beta \sum_{\mu \vdash n-1} c_\mu(\alpha, \beta) \sum_{\lambda >_1 \mu} \psi'_{\lambda/\mu}(\beta) P_\lambda^{(\beta-1)} \in \Lambda_{\mathbb{C}}^n.\end{aligned}$$

Since $\{P_\lambda^{(\beta)} \mid \lambda \vdash n\}$ is a basis of $\Lambda_{\mathbb{C}}^n$, we have

$$(\epsilon_\lambda(\beta) + n(1 + \sqrt{2\beta} \alpha - \beta)) c_\lambda(\alpha, \beta) = \beta \sum_{\mu <_1 \lambda} c_\mu(\alpha, \beta) \psi'_{\lambda/\mu}(\beta).$$

□

PROOF OF THEOREM 3.3.4. The recursion relation (3.3.9) of Propositions 3.3.7 determines $c_\lambda(\alpha, \beta)$ uniquely if we set the value of $c_\emptyset(\alpha, \beta)$. Since the existence and uniqueness of v_G is known by Fact 3.3.2, we only have to show that the ansatz (3.3.6) satisfies (3.3.9).

For partitions λ and μ which are related by $\lambda_I = \mu_I + 1$ and $\lambda_i = \mu_i$ for $i \neq I$, we have the following two formulas:

$$\begin{aligned}&\left[\prod_{(i,k) \in \mu} \frac{1}{\lambda_i - k + 1 + \beta(\lambda'_k - i)} \right] \Big/ \left[\prod_{(i,k) \in \lambda} \frac{1}{\lambda_i - k + 1 + \beta(\lambda'_k - i)} \right] \\ &= \prod_{i=1}^{I-1} \frac{\lambda_i - \lambda_I + 1 + \beta(I - i)}{\lambda_i - \lambda_I + 1 + \beta(I - 1 - i)} \times \prod_{i=1}^{\lambda_I - 1} \frac{\lambda_I - i + 1 + \beta(\lambda'_i - I)}{\lambda_I - i + \beta(\lambda'_i - I)},\end{aligned}$$

$$\begin{aligned}
& \left[\prod_{\substack{(i,k) \in \mu \\ \mu \neq (1,1)}} \frac{\beta}{(k+1) + \sqrt{2\beta}\alpha - (i+1)\beta} \right] / \left[\prod_{\substack{(i,k) \in \lambda \\ \mu \neq (1,1)}} \frac{\beta}{(k+1) + \sqrt{2\beta}\alpha - (i+1)\beta} \right] \\
&= \frac{(\lambda_I + 1) + \sqrt{2\beta}\alpha - (I+1)\beta}{\beta}.
\end{aligned}$$

Substituting the $c_\mu(\alpha, \beta)$ in the right hand side of (3.3.9) by the ansatz (3.3.6) and using the above two equations, we have

$$\begin{aligned}
& \text{RHS of (3.3.9)} \\
&= \sum_{(I, \lambda_I) \in C(\lambda)} \prod_{i=1}^{I-1} \frac{\lambda_i - \lambda_I + \beta(I-i+1)}{\lambda_i - \lambda_I + \beta(I-i)} \times \prod_{i=1}^{\lambda_I-1} \frac{\lambda_I - i + 1 + \beta(\lambda'_i - I)}{\lambda_I - i + \beta(\lambda'_i - I)} \\
& \quad \times \left((\lambda_I + 1) + \sqrt{2\beta}\alpha - (I+1)\beta \right) c_\lambda(\alpha, \beta),
\end{aligned} \tag{3.3.12}$$

where $C(\lambda)$ is the set of boxes $\square$ in the Young diagram of λ such that $\lambda \setminus \{\square\}$ is also associated to a partition. In particular, if $\square = (I, \lambda_I) \in C(\lambda)$, then $\mu := \lambda \setminus \{\square\}$ is the partition satisfying $\mu_I = \lambda_I - 1$ and $\mu_i = \lambda_i$ for $i \neq I$, recovering the previous description.

As for the left hand side of (3.3.9), we have by (3.2.5):

$$\epsilon_\lambda(\beta) + |\lambda|(1 + \sqrt{2\beta}\alpha - \beta) = |\lambda|(1 + \sqrt{2\beta}\alpha) + \sum_i (\lambda_i^2 - 2i\lambda_i\beta). \tag{3.3.13}$$

Thus by (3.3.12) and (3.3.12), the equation (3.3.9) under the substitution (3.3.6) is equivalent to the next one:

$$\begin{aligned}
& |\lambda|(1 + \sqrt{2\beta}\alpha) + \sum_i (\lambda_i^2 - 2i\lambda_i\beta) \\
&= \sum_{(I, \lambda_I) \in C(\lambda)} \prod_{i=1}^{I-1} \frac{\lambda_i - \lambda_I + \beta(I-i+1)}{\lambda_i - \lambda_I + \beta(I-i)} \times \prod_{i=1}^{\lambda_I-1} \frac{\lambda_I - i + 1 + \beta(\lambda'_i - I)}{\lambda_I - i + \beta(\lambda'_i - I)} \\
& \quad \times \left(1 + \sqrt{2\beta}\alpha + \lambda_I - (I+1)\beta \right).
\end{aligned}$$

This is verified by Propositions 3.4.1 and 3.4.2, which will be shown in the next §3.4. $\square$

3.3.3. Non-degenerate Whittaker vector via Jack symmetric functions.

DEFINITION 3.3.8. Fix non-zero complex numbers a and b . Let v_W be an element of the completed Verma module $\widehat{M}(c, h)$ satisfying

$$L_1 v_W = a v_W, \quad L_2 v_W = b v_W, \quad L_n v_W = 0 \quad (n \in \mathbb{Z}_{\geq 3}).$$

We call such an element v_W by (non-degenerate) Whittaker vector of $M(c, h)$.

FACT 3.3.9. Assume that c and h are generic complex numbers. Then v_W exists uniquely up to scalar multiplication.

PROOF. This is shown in [76]. $\square$

LEMMA 3.3.10. Let us decompose v_W as

$$v_W = \sum_{n \in \mathbb{Z}_{\geq 0}} a^n v_{W,n}, \quad v_{W,n} \in M(c, h)_n.$$

Then we have

$$L_1 v_{W,n+1} = v_{W,n}, \quad L_2 v_{W,n+2} = a^{-2} b \cdot v_{W,n}.$$

PROOF. This follows from the commutation relations $[L_0, L_1] = -L_1$ and $[L_0, L_2] = -2L_2$. $\square$

Now we expand the bosonized Whittaker vector

$$w_{W,n} := \psi(v_{W,n}) \in \mathcal{F}_{\alpha,n}$$

by $f_{\lambda}^{(\beta^{-1}, \beta)}$'s (3.2.1) and express it as

$$w_{W,n} = \sum_{\lambda \vdash n} d_{\lambda}(\alpha, \beta) f_{\lambda}^{(\beta^{-1}, \beta)}, \quad d_{\lambda}(\alpha, \beta) \in \mathbb{C}.$$

PROPOSITION 3.3.11. Using the notation $\lambda >_k \mu$ given in Definition 3.3.5, we have the next recursion relation for $d_{\lambda}(\alpha, \beta)$:

$$\begin{aligned} & (\epsilon_{\lambda}(\beta) + |\lambda|(1 + \sqrt{2\beta}\alpha - \beta))c_{\lambda}(\alpha, \beta) \\ &= \beta \sum_{\nu <_2 \lambda} d_{\nu}(\alpha, \beta) \psi'_{\lambda/\nu}^{(2)}(\beta) + \beta \sum_{\mu <_1 \lambda} d_{\mu}(\alpha, \beta) \psi'_{\lambda/\mu}(\beta). \end{aligned} \quad (3.3.14)$$

$\psi'_{\lambda/\nu}^{(2)}(\beta)$ is the coefficient in the next Pieri formula:

$$p_2 P_{\nu}^{(\beta^{-1})} = \sum_{\lambda} \psi'_{\lambda/\nu}^{(2)}(\beta) P_{\lambda}^{(\beta^{-1})}.$$

PROOF. Similar as the proof of Proposition 3.3.7 □

REMARK 3.3.12. The author doesn't know whether d_{λ} has a good explicit formula, although c_{λ} has the factored formula (3.3.6).

3.4. Combinatorial identities of rational functions

PROPOSITION 3.4.1. For a partition λ , let $C(\lambda)$ be the set of boxes $\square$ of λ such that $\lambda \setminus \{\square\}$ is also a partition. Then

$$\sum_{(I, \lambda_I) \in C(\lambda)} \prod_{i=1}^{I-1} \frac{\lambda_i - \lambda_I + \beta(I - i + 1)}{\lambda_i - \lambda_I + \beta(I - i)} \times \prod_{i=1}^{\lambda_I-1} \frac{\lambda_I - i + 1 + \beta(\lambda'_i - I)}{\lambda_I - i + \beta(\lambda'_i - I)} = |\lambda|. \quad (3.4.1)$$

PROOF. Let λ be the partition such that

$$\lambda = (\overbrace{n_1, \dots, n_1}^{j_1}, \overbrace{n_2, \dots, n_2}^{j_2}, \dots, \overbrace{n_l, \dots, n_l}^{j_l}). \quad (3.4.2)$$

Then we have

$$C(\lambda) = \{(m_1, n_1), (m_2, n_2), \dots, (m_l, n_l)\}$$

with $m_k := j_1 + j_2 + \dots + j_k$ ($k = 1, 2, \dots, l$), where we used the coordinate (i, j) of Young diagram associated to λ as explained in §A.2.

Let us choose an element $\square = (m_k, n_k)$ of $C(\lambda)$, and calculate the corresponding factor in (3.4.1). The first product reads

$$\begin{aligned} & \prod_{1 \leq i \leq m_1} \frac{(n_1 - n_k) + \beta(m_k - i + 1)}{(n_1 - n_k) + \beta(m_k - i)} \prod_{m_1+1 \leq j \leq m_2} \frac{(n_2 - n_k) + \beta(m_k - i + 1)}{(n_2 - n_k) + \beta(m_k - i)} \\ & \times \dots \times \prod_{m_{k-1}+1 \leq i \leq m_k-1} \frac{m_k - i + 1}{m_k - i} \\ &= \prod_{i=1}^{k-1} \frac{(n_i - n_k) + \beta(m_k - m_{i-1})}{(n_i - n_k) + \beta(m_k - m_i)} \times (m_k - m_{k-1}). \end{aligned}$$

Here we used the notation $m_0 := 0$. The second product reads

$$\begin{aligned} & \prod_{1 \leq j \leq n_l} \frac{(n_k - j + 1) + \beta(m_l - n_k)}{(n_k - j) + \beta(m_l - n_k)} \prod_{n_l+1 \leq j \leq n_{l-1}} \frac{(n_k - j + 1) + \beta(m_{l-1} - n_k)}{(n_k - j) + \beta(m_{l-1} - n_k)} \\ & \times \dots \times \prod_{n_{k+1}+1 \leq j \leq n_k-1} \frac{(n_k - j + 1)}{(n_k - j)} \end{aligned}$$

$$= (n_k - n_{k+1}) \times \prod_{j=k+1}^l \frac{(n_k - n_{j+1}) + \beta(m_j - m_k)}{(n_k - n_j) + \beta(m_j - m_k)}.$$

Here we used the notation $n_{l+1} := 0$.

Now let us define

$$\begin{aligned} F_1(\{m_k\}, \{n_k\}, \beta) &:= \sum_{k=1}^l F_{1,k}(\{m_k\}, \{n_k\}, \beta), \\ F_{1,k}(\{m_k\}, \{n_k\}, \beta) &:= (m_k - m_{k-1})(n_k - n_{k+1}) \\ &\times \prod_{i=1}^{k-1} \frac{(n_i - n_k) + \beta(m_k - m_{i-1})}{(n_i - n_k) + \beta(m_k - m_i)} \prod_{j=k+1}^l \frac{(n_k - n_{j+1}) + \beta(m_j - m_k)}{(n_k - n_j) + \beta(m_j - m_k)}. \end{aligned}$$

Then, for the proof of (3.4.1), it is enough to show that F_1 is equal to $|\lambda|$ if $\{m_k\}$ and $\{n_k\}$ correspond to λ as in (3.4.2) and (3.4.3).

Hereafter we consider F_1 as a rational function of the valuables $\{m_k\}$, $\{n_k\}$ and β . As a rational function of β , F_1 has the apparent poles at

$$\beta_{j,k} := -\frac{n_j - n_k}{m_k - m_j}$$

for $j = 1, 2, \dots, k-1, k+1, \dots, l$. We may assume that these apparent poles are mutually different so that all the poles are at most single. Then the residue at $\beta = \beta_{j,k}$ comes from the factors $F_{1,j}$ and $F_{1,k}$. Now we may assume $j < k$. Then the direct computation yields

$$\text{Res}_{\beta=\beta_{j,k}} F_{1,j} = \frac{(m_j - m_{j-1})(n_j - n_{j+1})(n_k - n_{k+1})}{(m_j - m_k)} \quad (3.4.3)$$

$$\times \prod_{i=1}^{j-1} \frac{(n_i - n_j)(m_k - m_j) - (n_j - n_k)(m_j - m_{i-1})}{(n_i - n_j)(m_k - m_j) - (n_j - n_k)(m_j - m_i)} \quad (3.4.4)$$

$$\times \prod_{i=j+1}^{k-1} \frac{(n_j - n_{i+1})(m_k - m_j) - (n_j - n_k)(m_i - m_j)}{(n_j - n_i)(m_k - m_j) - (n_j - n_k)(m_i - m_j)} \quad (3.4.5)$$

$$\times \prod_{i=k+1}^l \frac{(n_j - n_{i+1})(m_k - m_j) - (n_j - n_k)(m_i - m_j)}{(n_j - n_i)(m_k - m_j) - (n_j - n_k)(m_i - m_j)}, \quad (3.4.6)$$

and

$$\text{Res}_{\beta=\beta_{j,k}} F_{1,k} = \frac{(m_k - m_{k-1})(n_k - n_{k+1})(n_j - n_k)(m_{j-1} - m_j)}{(m_k - m_j)^2} \quad (3.4.7)$$

$$\times \prod_{i=1}^{j-1} \frac{(n_i - n_k)(m_k - m_j) - (n_j - n_k)(m_k - m_{i-1})}{(n_i - n_k)(m_k - m_j) - (n_j - n_k)(m_k - m_i)} \quad (3.4.8)$$

$$\times \prod_{i=j+1}^{k-2} \frac{(n_i - n_k)(m_k - m_j) - (n_j - n_k)(m_k - m_{i-1})}{(n_i - n_k)(m_k - m_j) - (n_j - n_k)(m_k - m_i)} \quad (3.4.9)$$

$$\times \prod_{i=k+1}^l \frac{(n_k - n_{i+1})(m_k - m_j) - (n_j - n_k)(m_i - m_k)}{(n_k - n_i)(m_k - m_j) - (n_j - n_k)(m_i - m_k)}. \quad (3.4.10)$$

Using the identity $(a-b)(x-y) - (c-b)(x-z) = (a-c)(x-y) - (c-b)(y-z)$, one finds that the factors (3.4.4) and (3.4.8) are equal. Similarly (3.4.6) and (3.4.10) are equal. Shifting the index i in (3.4.5) and using the above identity, one also finds that

$$\frac{(3.4.5)}{(3.4.9)} = -\frac{(n_j - n_k)(m_k - m_{k-1})}{(n_{j+1} - n_j)(m_k - m_j)}.$$

Thus we have

$$\frac{\text{Res}_{\beta=\beta_{j,k}} F_{1,j}}{\text{Res}_{\beta=\beta_{j,k}} F_{1,k}} = -\frac{(n_j - n_k)(m_k - m_{k-1})}{(n_{j+1} - n_j)(m_k - m_j)} \frac{(3.4.4)}{(3.4.8)} = -1.$$

Therefore we have

$$\text{Res}_{\beta=\beta_{j,k}} F_1(\{m_k\}, \{n_k\}, \beta) = 0,$$

so that F_1 is a polynomial of β .

Then, from the behavior F_1 in the limit $\beta \rightarrow \infty$, we find that F_1 is a constant as a function of β . This constant can be calculated by setting $\beta = 0$, and the result is

$$F_1(\{m_k\}, \{n_k\}, \beta) = \sum_{k=1}^l (m_k - m_{k-1}) n_k.$$

It equals to $|\lambda|$ if λ is given by (3.4.2) and $m_k = j_1 + \dots + j_k$. This is the desired consequence. $\square$

PROPOSITION 3.4.2. Using the same notation as in Proposition 3.4.1, we have

$$\begin{aligned} \sum_{(I, \lambda_I) \in C(\lambda)} \prod_{i=1}^{I-1} \frac{\lambda_i - \lambda_I + \beta(I - i + 1)}{\lambda_i - \lambda_I + \beta(I - i)} \times \prod_{i=1}^{\lambda_I - 1} \frac{\lambda_I - i + 1 + \beta(\lambda'_i - I)}{\lambda_I - i + \beta(\lambda'_i - I)} \\ \times (\lambda_I - (I + 1)\beta) = \sum_i (\lambda_i^2 - 2i\lambda_i\beta). \end{aligned} \quad (3.4.11)$$

PROOF. As in the proof of Proposition 3.4.1, set

$$\lambda = (\overbrace{n_1, \dots, n_1}^{j_1}, \overbrace{n_2, \dots, n_2}^{j_2}, \dots, \overbrace{n_l, \dots, n_l}^{j_l})$$

and $m_k := j_1 + j_2 + \dots + j_k$ ($k = 1, 2, \dots, l$). We can write the left hand side of (3.4.11) as

$$\begin{aligned} F_2(\{m_k\}, \{n_k\}, \beta) &= \sum_{k=1}^l F_{2,k}(\{m_k\}, \{n_k\}, \beta), \\ F_{2,k}(\{m_k\}, \{n_k\}, \beta) &:= (n_k - (m_k + 1)\beta)(m_k - m_{k-1})(n_k - n_{k+1}) \\ &\times \prod_{i=1}^{k-1} \frac{(n_i - n_k) + \beta(m_k - m_{i-1})}{(n_i - n_k) + \beta(m_k - m_i)} \prod_{j=k+1}^l \frac{(n_k - n_{j+1}) + \beta(m_j - m_k)}{(n_k - n_j) + \beta(m_j - m_k)}. \end{aligned}$$

The residues of F_2 are the same as those of F_1 , and by the similar calculation as in Proposition 3.4.1, one can find that F_2 is a polynomial of β . The behavior of F_2 in the limit $\beta \rightarrow \infty$ shows that F_2 is a linear function of β .

Using the original expression (3.4.11), we find that

$$F_2(\{m_k\}, \{n_k\}, 0) = \sum_i \lambda_i^2.$$

In order to determine the coefficient of β in F_2 , we rewrite F_2 as the rational function of β^{-1} , and take the limit $\beta^{-1} \rightarrow \infty$. The result is

$$\begin{aligned} \lim_{\beta \rightarrow \infty} (\beta^{-1} F_2(\{m_k\}, \{n_k\}, \beta)) \\ &= - \sum_{k=1}^l (m_k + 1)(m_k - m_{k-1})(n_k - n_{k+1}) \frac{m_k}{m_k - m_{k-1}} \\ &= - \sum_{k=1}^l n_k (m_k - m_{k-1})(m_k + m_{k-1} + 1). \end{aligned}$$

A moment thought shows that this becomes $-\sum_i 2i\lambda_i$ if $\{m_k\}$ and $\{n_k\}$ correspond to λ . Thus the proof is completed. $\square$

CHAPTER 4

$\mathcal{W}$ -algebra

In this chapter, we introduce the $\mathcal{W}$ -algebra by the quantum Drinfeld-Sokolov reduction. This theory is initiated by Feigin-Frenkel [34], and we basically follow the presentation of [60]. For details on Lie superalgebra, we refer [87, Chapter 2], although one can read through this thesis neglecting the super case.

4.1. Vertex algebra

Here we recall the basic notions on vertex algebra following [57, § 1.3]. See also [39].

4.1.1. Fundamentals. Let V be a *superspace*, i.e., a $\mathbb{Z}/2\mathbb{Z}$ -graded vector space.

$$V = V_{\bar{0}} + V_{\bar{1}}.$$

Hereafter $\bar{0}$ and $\bar{1}$ stand for the cosets in $\mathbb{Z}/2\mathbb{Z}$ of 0 and 1. The *super-dimension* of the superspace V is defined to be $\text{sdim } V := \dim V_{\bar{0}} - \dim V_{\bar{1}}$.

An element a of V is called of *parity* $p(a) \in \mathbb{Z}/2\mathbb{Z}$ if $a \in V_{p(a)}$. Hereafter, whenever $p(a)$ is written, it is to be understood that $a \in V_{p(a)}$. An element of $V_{\bar{0}}$ (resp. of $V_{\bar{1}}$) is called even (resp. odd).

A *field* is a series of the form

$$a(z) = \sum_{n \in \mathbb{Z}} a_{(n)} z^{-n-1}, \quad a_{(n)} \in \text{End}(V)$$

with the condition that for each $v \in V$ one has

$$a_{(n)} v = 0 \quad \text{for } n \gg 0.$$

We say that a field $a(z)$ has *parity* $p(a) \in \mathbb{Z}/2\mathbb{Z}$ if

$$a_{(n)} V_{\alpha} \subset V_{\alpha+p(a)} \quad \text{for all } \alpha \in \mathbb{Z}/2\mathbb{Z}, \quad n \in \mathbb{Z}.$$

DEFINITION 4.1.1. A *vertex algebra* consists of the following data:

- (1) a superspace V , called the *space of states*,
- (2) a vector $|0\rangle \in V$, called the *vacuum vector*,
- (3) a parity preserving linear map $Y(\cdot, z)$ from V to the space of the fields

$$a \mapsto Y(a, z) = \sum_{n \in \mathbb{Z}} a_{(n)} z^{-n-1},$$

called the *state-field correspondence*,

- (4) a linear map $T \in \text{End}(V)$ defined by

$$T(a) = a_{(2)} |0\rangle,$$

called the *infinitesimal translation operator*,

satisfying the following axioms:

translation covariance: For each $a \in V$,

$$[T, Y(a, z)] = \partial_z Y(a, z),$$

vacuum:

$$Y(|0\rangle, z) = \text{id}_V, \quad Y(|0\rangle, z)|_{z=0} = a,$$

locality: For each $a, b \in V$,

$$(z - w)^N Y(a, z) Y(b, w) = (-1)^{p(a)p(b)} (z - w)^N Y(b, w) Y(a, z) \quad \text{for } N \gg 0.$$

Here in the translation covariance axiom, the partial derivative ∂_z is the formal derivative:

$$\partial_z \sum_{n \in \mathbb{Z}} a_{(n)} z^{-n-1} := \sum_{n \in \mathbb{Z}} (-n-1) a_{(n)} z^{-n-2},$$

and the bracket is the usual commutator bracket: $[T, Y] := TY - YT$. The locality axiom is to be understood as a coefficient-wise equality of two series in z and w of the form $\sum_{m,n \in \mathbb{Z}} a_{m,n} z^m w^n$.

A *subalgebra* of a vertex algebra V is a subspace U of V containing $|0\rangle$ such that

$$a_{(n)}U \subset U \quad \text{for all } a \in U,$$

It has a structure of vertex algebra by defining the fields as $Y(a, z) = \sum_n a_{(n)} |U z^{-n-1}$.

A *homomorphism* of a vertex algebra V to a vertex algebra V' is a linear parity preserving map $\varphi : V \rightarrow V'$ such that

$$\varphi(a_{(n)}b) = \varphi(a)_{(n)}\varphi(b)$$

for all $a, b \in V$ and $n \in \mathbb{Z}$.

The *tensor product* of two vertex algebras U and V is defined as follows:

- The space of states is $U \otimes V$.
- The vacuum vector is $|0\rangle \otimes |0\rangle$.
- The infinitesimal translation operator is $T \otimes 1 + 1 \otimes T$.
- The fields are

$$Y(u \otimes v, z) = Y(u, z) \otimes Y(v, z) := \sum_{m,n \in \mathbb{Z}} u_{(m)} \otimes v_{(n)} z^{-m-n-2}.$$

4.1.2. Distributions. We will give several examples of vertex algebra in the next section. In order to state it, several manipulations on distributions are necessary. We follow [57, Chap. 2]

Let U be a $\mathbb{C}$ -vector space. A formal expression

$$\sum_{n_1, n_2, \dots, n_k \in \mathbb{Z}} a_{n_1, n_2, \dots, n_k} z_1^{n_1} z_2^{n_2} \dots z_k^{n_k}, \quad a_{n_1, n_2, \dots, n_k} \in U$$

is called a *formal distribution* in the indeterminates $z_1, z_2, \dots, z_k$ with values in U . They form a vector $\mathbb{C}$ -space, which is denoted by $U[[z_1^{\pm 1}, z_2^{\pm 1}, \dots, z_k^{\pm 1}]]$.

Given a rational function $R(z, w)$ with poles only at $z = 0$, $w = 0$ and $|z| = |w|$, we denote by $\iota_{z,w}R$ (resp. $\iota_{w,z}R$) the power series expansion of R in the domain $|z| > |w|$ (resp. $|w| > |z|$). For example, we have

$$\begin{aligned} \iota_{z,w} \frac{1}{(z-w)^{j+1}} &= \sum_{m=0}^{\infty} \binom{m}{j} z^{-m-1} w^{m-j}, \\ \iota_{w,z} \frac{1}{(z-w)^{j+1}} &= - \sum_{m=-1}^{-\infty} \binom{m}{j} z^{-m-1} w^{m-j} \end{aligned}$$

for $j \in \mathbb{Z}_{\geq 0}$.

Suppose U has a structure of *associative superalgebra*, that is, $U = U_{\bar{0}} + U_{\bar{1}}$ as a superspace, and it has a $\mathbb{Z}/2\mathbb{Z}$ -graded associative algebra structure: $U_{\alpha}U_{\beta} \subset U_{\alpha+\beta}$ for $\alpha, \beta \in \mathbb{Z}/2\mathbb{Z}$.

Then the (*super*)*bracket* on U is defined by

$$[a, b] := ab - (-1)^{p(a)p(b)}ba$$

for $a \in U_{p(a)}$ and $b \in U_{p(b)}$. Thus the bracket of an even element with any other element is the usual commutator, and the bracket of two odd elements is the anti-commutator.

Two formal distributions $a(z)$ and $b(z)$ with values in an associative superalgebra U are called *mutually local* if

$$(z - w)^N [a(z), b(w)] = 0 \quad \text{for } N \gg 0. \quad (4.1.1)$$

Here the bracket is the natural extension of the superbracket defined above to the formal distributions. Hereafter we always assume that all the coefficients of a formal distribution $a(z)$ have the same parity, which will be denoted by $p(a)$.

There are known several equivalent statements to (4.1.1). Here we only show one of them.

Several notations are in order. Given a formal distribution $a(z) = \sum_{n \in \mathbb{Z}} a_{(n)} z^{-n-1}$, let

$$a(z)_- := \sum_{n \in \mathbb{Z}_{\geq 0}} a_{(n)} z^{-n-1}, \quad a(z)_+ := \sum_{n \in \mathbb{Z}_{< 0}} a_{(n)} z^{-n-1}.$$

Then we have $(\partial_z a(z))_{\pm} = \partial_z (a(z)_{\pm})$.

Given formal distributions $a(z)$ and $b(z)$, define the following distribution in z and w with values in U :

$${}^{\circ}a(z)b(w)_{\circ} := a(z)_+ b(w) + (-1)^{p(a)p(b)} b(w) a(z)_-$$

FACT 4.1.2. The condition (4.1.1) is equivalent to

$$\begin{aligned} a(z)b(w) &= \sum_{j=0}^{N-1} \left(\iota_{z,w} \frac{1}{(z-w)^{j+1}} \right) c^j(w) + {}^{\circ}a(z)b(w)_{\circ}, \\ (-1)^{p(a)p(b)} b(w)a(z) &= \sum_{j=0}^{N-1} \left(\iota_{w,z} \frac{1}{(z-w)^{j+1}} \right) c^j(w) + {}^{\circ}a(z)b(w)_{\circ} \end{aligned}$$

with $c^j(w) \in U[[w^{\pm 1}]]$.

For the proof, see [57, Theorem 2.3 (iii)].

In physics literatures, the above expression is written as

$$a(z)b(w) = \sum_{j=0}^{N-1} \frac{c^j(w)}{(z-w)^{j+1}} + {}^{\circ}a(z)b(w)_{\circ},$$

or just as

$$a(z)b(w) \sim \sum_{j=0}^{N-1} \frac{c^j(w)}{(z-w)^{j+1}}. \quad (4.1.2)$$

These are called the *operator product expansion* (OPE). Hereafter we will use the expression (4.1.2) for convenience of typography.

4.2. Vertex algebra associated to Lie superalgebra

Now we explain the construction of vertex algebra from Lie (super)algebra following [57, § 4.7].

A Lie superalgebra $\mathfrak{g}$ over $\mathbb{C}$ is called a *formal distribution Lie superalgebra* if it is spanned over $\mathbb{C}$ by coefficients of a family F of $\mathfrak{g}$ -valued mutually local formal distributions.

EXAMPLE 4.2.1. (1) Current algebra $\widehat{\mathfrak{g}}$: Let $\mathfrak{g}$ be a Lie superalgebra with an invariant supersymmetric bilinear form $(\cdot|\cdot)$. The word *invariant* means

$$([a, b]|c) = (a|[b, c]), \quad \text{for all } a, b, c \in \mathfrak{g},$$

and *supersymmetric* means

$$(a|b) = (-1)^{p(a)}(b|a), \quad \text{for all } a \in \mathfrak{g}_{p(a)}, b \in \mathfrak{g}$$

The *loop algebra* $L\mathfrak{g}$ of $\mathfrak{g}$ is the Lie superalgebra

$$L\mathfrak{g} := \mathfrak{g}[t^{\pm 1}]$$

over $\mathbb{C}$ with $\mathbb{Z}/2\mathbb{Z}$ -grading extending that of $\mathfrak{g}$ by $p(t) = 0$, and commutation relations

$$[a_m, b_n] = [a, b]_{m+n} \quad \text{for all } a, b \in \mathfrak{g}, \quad m, n \in \mathbb{Z}.$$

Hereafter a_m stands for $a \otimes t_m$.

The *affinization* of the pair $(\mathfrak{g}, (\cdot|\cdot))$ is a central extension of $L\mathfrak{g}$ by a 1-dimensional even center $\mathbb{C}K$:

$$\widehat{\mathfrak{g}} := L\mathfrak{g} + \mathbb{C}K$$

defined by the commutation relations

$$[a_m, b_n] = [a, b]_{m+n} = m(a|b)\delta_{m,-n}K, \quad \text{for all } a, b \in \mathfrak{g}, \quad m, n \in \mathbb{Z}.$$

$\widehat{\mathfrak{g}}$ is also called *current algebra*. As is well known, if $\mathfrak{g}$ is a simple finite-dimensional Lie algebra with $(\cdot|\cdot)$ the Killing form, then $\widehat{\mathfrak{g}}$ is the affine Kac-Moody algebra.

For $a \in \mathfrak{g}$, define the formal distribution with values in $\widehat{\mathfrak{g}}$ by

$$a(z) := \sum_{n \in \mathbb{Z}} a_n z^{-n-1}.$$

Then we have

$$a(z)b(w) \sim \frac{[a, b](w)}{z-w} + \frac{(a|b)K}{(z-w)^2}$$

for any $a, b \in \mathfrak{g}$. (For the proof, see [57, § 2.5]).

Now one can see that the current algebra $\widehat{\mathfrak{g}}$ with $F := \{a(z) \mid a \in \mathfrak{g}\} \cup \{K\}$ is a formal distribution Lie superalgebra.

- (2) Let $\mathfrak{h}$ be a finite-dimensional superspace with a non-degenerate supersymmetric bilinear form $(\cdot|\cdot)$. Viewing $\mathfrak{h}$ as a commutative Lie superalgebra, we may consider its affinization

$$\widehat{\mathfrak{h}} = \mathfrak{h}[t, t^{-1}] + \mathbb{C}K$$

with commutation relations

$$[a_m, b_n] = m(a|b)\delta_{m,-n}K, \quad [K, \widehat{\mathfrak{h}}] = 0$$

for $a, b \in \mathfrak{h}$, $m, n \in \mathbb{Z}$. Then the currents

$$a(z) = \sum_{n \in \mathbb{Z}} a_n z^{-n-1}$$

are mutually local with the OPE

$$a(z)b(w) \sim \frac{(a|b)K}{(z-w)^2}.$$

These fields are called *free bosons*.

- (3) Let A be a superspace with a skew-supersymmetric bilinear form $(\cdot|\cdot)$, i.e.,

$$(\varphi|\psi) = -(-1)^{p(\varphi)}(\psi|\varphi)$$

for all $\varphi, \psi \in A$. In particular, $(A_0|A_1) = 0$. The *Clifford affinization* of $(A, (\cdot|\cdot))$ is the Lie superalgebra

$$C_A = A[t, t^{-1}] + \mathbb{C}K$$

with commutation relations

$$[\varphi_m, \psi_n] = (\varphi|\psi)\delta_{m,-n}K, \quad [K, C_A] = 0,$$

where $\varphi_m := \varphi \otimes t^{m-1/2}$ ($\varphi, \psi \in A$, $m, n \in \mathbb{Z} + 1/2$). The formal distributions

$$\varphi(z) = \sum_{n \in \mathbb{Z}} \varphi_{n+\frac{1}{2}} z^{-n-1}$$

are mutually local with the OPE

$$\varphi(z)\psi(w) \sim \frac{(\varphi|\psi)K}{z-w}.$$

These fields are called *free fermions*.

(4) Virasoro algebra Vir : Let us set

$$L(z) := \sum_{n \in \mathbb{Z}} L_n z^{-n-2}.$$

Then Vir with $F := \{L(z), C\}$ is a formal distribution Lie algebra. To see that, we only need to note that

$$L(z)L(w) \sim \frac{C/2}{(z-w)^4} + \frac{2L(w)}{(z-w)^2} + \frac{\partial_w L(w)}{(z-w)^4}.$$

Let $\mathfrak{g}$ be a Lie superalgebra spanned by mutually local formal distributions $a^\alpha(z)$ ($\alpha \in A$), and suppose that there exists an endomorphism T of the $\mathbb{C}$ -vector space $\mathfrak{g}$ such that

$$Ta^\alpha(z) = \partial a^\alpha(z). \quad (4.2.1)$$

Then $\mathfrak{g}$ is called a *regular* formal distribution Lie superalgebra.

Let

$$\mathfrak{g}_- := \{a \in \mathfrak{g} \mid T^k a = 0 \text{ for } k \gg 0\}.$$

This is a T -invariant subalgebra of $\mathfrak{g}$. Let

$$\lambda : \mathfrak{g}_{--} \longrightarrow \mathbb{C}$$

be a 1-dimensional $\mathfrak{g}_{--}$ -module such that

$$\lambda(T\mathfrak{g}_{--}) = 0.$$

Consider the induced $\mathfrak{g}$ -module

$$\tilde{V}_\lambda := \text{Ind}_{\mathfrak{g}_{--}}^{\mathfrak{g}} \lambda = U(\mathfrak{g})/U(\mathfrak{g})\langle a - \lambda(a) \mid a \in \mathfrak{g}_{--} \rangle,$$

and let $|0\rangle \in \tilde{V}_\lambda$ be the image of $1 \in U(\mathfrak{g})$.

Now we can state

FACT 4.2.2 ([57, Theorem 4.7]). Let $\mathfrak{g}$ be a regular formal distribution Lie superalgebra. Then the $\mathfrak{g}$ -module $\tilde{V}_\lambda$ has a unique vertex algebra structure with $|0\rangle$ the vacuum vector and generated by the fields $a^\alpha(z)$.

The vertex algebra $\tilde{V}_\lambda$ is called the *universal vertex algebra* associated to $\mathfrak{g}$.

EXAMPLE 4.2.3. (1) Consider the example of the current algebra $\hat{\mathfrak{g}}$ associated to a Lie superalgebra $\mathfrak{g}$. This is a Lie superalgebra spanned by formal distributions $a(z)$ ($a \in \mathfrak{g}$) and K . $T = -\partial_t$ satisfies (4.2.1), so that $\hat{\mathfrak{g}}$ is regular. We also have

$$\hat{\mathfrak{g}}_{--} = \mathfrak{g}[t] + \mathbb{C}K, \quad T\hat{\mathfrak{g}}_{--} = \mathfrak{g}[t].$$

So we have the following possibilities for λ :

$$\lambda(\mathfrak{g}[t]) = 0, \quad \lambda(K) = k \in \mathbb{C}.$$

The corresponding $\hat{\mathfrak{g}}$ -module by $\tilde{V}_k(\hat{\mathfrak{g}})$. By the above fact, it has a unique vertex algebra structure, and is called the *universal affine vertex algebra*.

Let $\mathfrak{h}$ be a finite-dimensional superspace with a non-degenerate supersymmetric bilinear form $(\cdot|\cdot)$. Viewing $\mathfrak{h}$ as a commutative Lie superalgebra, we may consider its affinization

$$\hat{\mathfrak{h}} = \mathfrak{h}[t, t^{-1}] + \mathbb{C}K$$

with commutation relations

$$[a_m, b_n] = m(a|b)\delta_{m,-n}K, \quad [K, \hat{\mathfrak{h}}] = 0$$

for $a, b \in \mathfrak{h}$, $m, n \in \mathbb{Z}$. Then the currents

$$a(z) = \sum_{n \in \mathbb{Z}} a_n z^{-n-1}$$

are mutually local with the OPE

$$a(z)b(w) \sim \frac{(a|b)K}{(z-w)^2}.$$

- (2) If $\mathfrak{g}$ is a commutative Lie superalgebra, then the universal affine vertex algebra is simple (i.e., have no non-zero ideals), provided that $k \neq 0$. In this case, the universal affine vertex algebra $\tilde{V}_k(\widehat{\mathfrak{g}})$ is called the *free bosonic vertex algebra* and is denoted by $B_k(\mathfrak{g})$.
- (3) Let A be a superspace with a skew-supersymmetric bilinear form $(\cdot|\cdot)$, and C_A be its Clifford affinization. Then the universal affine vertex algebra is simple provided that $k \neq 0$. In this case, the universal affine vertex algebra $\tilde{V}_k(C_A)$ is called the *free fermionic vertex algebra* and is denoted by $F_k(A)$.
- (4) Virasoro Lie algebra Vir is spanned by formal distribution $L(z) = \sum_{n \in \mathbb{Z}} L_n z^{-n-2}$ and C . The derivation $T = \text{ad}L_{-1}$ satisfies (4.2.1). We have

$$\text{Vir}_{--} = \sum_{n \geq -1} \mathbb{C}L_n + \mathbb{C}C, \quad T\text{Vir}_{--} = \sum_{n \geq -1} \mathbb{C}L_n.$$

Hence the associated universal vertex algebra is parametrized by a complex number $c = \lambda(C)$. We denote this vertex algebra by $\tilde{V}_c(\text{Vir})$. By the above fact, it has a unique vertex algebra structure, and is called the *universal Virasoro vertex algebra*.

4.3. Graded and conformal vertex algebras

Following [57, §§ 4.9–10], we introduce the important classes of vertex algebras $V_k(\widehat{\mathfrak{g}})$ and $V_c(\text{Vir})$.

4.3.1. Graded vertex algebra. A vertex algebra V is called *graded* if there is an even diagonalizable endomorphism H on V such that

$$[H, Y(a, z)] = z\partial_z Y(a, z) + Y(Ha, z) \quad (4.3.1)$$

for any $a \in V$. Such an endomorphism H is called a *Hamiltonian* of V .

Assume that $a \in V$ satisfies $Ha = \Delta a$ with $\Delta \in \mathbb{Z}$. Then $\text{ad}HY(a, z) = (z\partial_z + \Delta)Y(a, z)$. Such a field $Y(a, z)$ is called of *conformal dimension* Δ .

It is convenient to write a field $Y(a, z)$ of conformal dimension Δ in the form

$$Y(a, z) = \sum_{n \in \Delta + \mathbb{Z}} a_n z^{-n-\Delta}.$$

Then we have

$$a_{(n)} = a_{n-\Delta+1},$$

and (4.3.1) is equivalent to

$$[H, a_n] = -na_n. \quad (4.3.2)$$

The transformation covariance axiom $[T, Y(a, z)] = \partial_z Y(a, z)$ is equivalent to

$$[T, a_n] = (-n - \Delta + 1)a_{n-1}. \quad (4.3.3)$$

Applying (4.3.1) to $a = |0\rangle$ we have $H|0\rangle = 0$. Then it holds that

$$[H, T] = T,$$

since both sides commute in the same way with all a_n and both annihilate $|0\rangle$.

By the assumption of H , V has the eigenspace decomposition with respect to H :

$$V = \bigoplus_j V^{(j)}.$$

By (4.3.3) and (4.3.2), we have

$$a_n V^{(j)} \subset V^{(j-n)}, \quad TV^{(j)} \subset V^{(j+1)}.$$

A graded vertex algebra has a unique maximal graded ideal, and the corresponding quotient vertex algebra is *simple*, that is, without non-zero ideals.

EXAMPLE 4.3.1. (1) Consider the current algebra $\widehat{\mathfrak{g}}$ associated to a Lie superalgebra $\mathfrak{g}$. Set $H := -t\partial_t$. Then this is a Hamiltonian for $\widetilde{V}_k(\widehat{\mathfrak{g}})$, and the currents $a(z)$ ($a \in \mathfrak{g}$) have conformal weight 1.

Then, by the argument above, the universal affine vertex algebra $\widetilde{V}_k(\widehat{\mathfrak{g}})$ is graded. Its simple graded quotient is called the *affine vertex algebra*, and denoted by $V_k(\widehat{\mathfrak{g}})$.

(2) The universal Virasoro algebra $\widetilde{V}_c(\text{Vir})$ is graded with the Hamiltonian L_0 . Its simple graded quotient is called the *Virasoro vertex algebra*, and denoted by $V_c(\text{Vir})$.

4.3.2. Conformal vertex algebra. A *conformal vector* of a vertex algebra V is an even vector ν such that

$$Y(\nu, z) = \sum_{n \in \mathbb{Z}} L_n z^{-n-2}$$

is a Virasoro field with central charge c which has the following properties:

- (1) $L_{-1} = T$,
- (2) L_0 is diagonalizable on V .

A vertex algebra endowed with a conformal vector is called a *conformal vertex algebra*. Thus a conformal vertex algebra is a vertex algebra which contains a Virasoro field.

EXAMPLE 4.3.2. (1) Let $\mathfrak{g}$ be a simple finite-dimensional Lie superalgebra with a non-degenerate invariant supersymmetric bilinear form $(\cdot|\cdot)$, and let $\{a^i\}, \{b^i\}$ be dual bases of $\mathfrak{g}$, that is, $(b^i|a^j) = \delta_{i,j}$.

Then the Casimir operator $\overline{\Omega} := \sum_i a^i b^i \in U(\mathfrak{g})$ is central in $U(\mathfrak{g})$, and independent of the choice of dual bases. Since $\mathfrak{g}$ is simple, its adjoint representation is irreducible, so that any central element of $U(\mathfrak{g})$ acts on $\mathfrak{g}$ as a scalar. Set the scalar by

$$\overline{\Omega} \cdot g = 2h^\vee g$$

for any $g \in \mathfrak{g}$. The number $h^\vee$ is called the dual Coxeter number of the pair $(\mathfrak{g}, (\cdot|\cdot))$.

Suppose $k \neq -h^\vee$. Then the vector

$$\nu := \frac{1}{2(k + h^\vee)} \sum_i a_{-1}^i b_{-1}^i |0\rangle$$

is a conformal vector of the vertex algebra $V_k(\mathfrak{g})$ with central charge

$$c_k = \frac{k \text{ sdim } \mathfrak{g}}{k + h^\vee}.$$

Here $\text{sdim } \mathfrak{g} := \dim \mathfrak{g}_{\bar{0}} - \dim \mathfrak{g}_{\bar{1}}$ is the superdimension of the superspace $\mathfrak{g} = \mathfrak{g}_{\bar{0}} + \mathfrak{g}_{\bar{1}}$. See [57, § 5.7] for the proof. This construction is a generalization of the Sugawara construction.

(2) The Virasoro vertex algebra $V_c(\text{Vir})$ is conformal with the conformal vector

$$\nu = L_{-2} |0\rangle,$$

so that $Y(\nu, z) = L(z)$.

4.3.3. λ -bracket. Before starting the study of quantum Drinfeld-Sokolov reduction, we introduce the notion of λ -bracket, which is an economical expression of the operator product expansion, λ -bracket is convenient in the calculation of various fields appearing in the reduction. This notion is introduced in [57, § 2.6].

DEFINITION 4.3.3. Let $a(z), b(z)$ be two formal distributions with values in a Lie superalgebra $\mathfrak{g}$ over $\mathbb{C}$, and assume that they are mutually local. By Fact 4.1.2, one can write $[a(z), b(w)]$ in the form

$$[a(z), b(w)] = (\iota_{z,w} - \iota_{w,z}) \sum_{j=0}^N \frac{c^j(w)}{(z-w)^{j+1}}$$

with $c^j(w) \in \mathfrak{g}[[w, w^{-1}]]$. Then define a field $[a_\lambda b](z)$ with values in $\mathbb{C}[\lambda] \otimes_{\mathbb{C}} \mathfrak{g}$ by

$$[a_\lambda b](z) := \sum_{j=0}^N \frac{\lambda^j}{j!} c^j(z).$$

EXAMPLE 4.3.4. (1) Let $\mathfrak{g}$ be a Lie superalgebra with invariant supersymmetric bilinear form $(\cdot|\cdot)$, and $\widehat{\mathfrak{g}} = \mathfrak{g}[t, t^{-1}] + \mathbb{C}K$ be its affinization. Then for $a, b \in \mathfrak{g}$,

$$[a_\lambda b](z) = [a, b](z) + \lambda K(a|b).$$

(2) For the Virasoro Lie algebra Vir ,

$$[L_\lambda L](z) = (\partial_z + 2\lambda)L(z) + \frac{\lambda^3}{12}C.$$

λ -bracket satisfies the following properties:

LEMMA 4.3.5.

(1)

$$[\partial a_\lambda b] = -\lambda [a_\lambda b], \quad [a_\lambda \partial b] = (\partial + \lambda) [a_\lambda b].$$

(2)

$$[a_\lambda b] = -(-1)^{p(a)p(b)} [b_{-\partial-\lambda} a].$$

(3)

$$[a_\lambda [b_\mu c]] - (-1)^{p(a)p(b)} [b_\mu [a_\lambda c]] = [[a_\lambda b]_{\lambda+\mu} c].$$

The proof is straightforward.

4.4. Quantum Drinfeld-Sokolov reduction

We recall here, following [59], the construction of a vertex algebra $W_k(\mathfrak{g}, x, f)$ depending on a complex parameter k , via a differential complex $C_k(\mathfrak{g}, x, f)$.

We need the following data:

- (1) $(\mathfrak{g}, (\cdot|\cdot))$: a simple finite dimensional Lie superalgebra with a non-degenerate even supersymmetric invariant bilinear form. It has the root space decomposition $\mathfrak{g} = \mathfrak{h} + \sum_{\alpha \in \Delta} \mathfrak{g}_\alpha$.
- (2) even elements x, e, f of $\mathfrak{g}$ such that
 - (i) $x \in \mathfrak{h}$, $e \in \sum_{\alpha \in \Delta_+} \mathfrak{g}_\alpha$, $f \in \sum_{\alpha \in \Delta_-} \mathfrak{g}_\alpha$.
 - (ii) $[x, e] = -e$, $[x, f] = -f$, $[e, f] = x$.

Then by the $\mathfrak{sl}(2, \mathbb{C})$ -representation theory, the eigenvalues of $\text{ad}(x)$ are in $\frac{1}{2}\mathbb{Z}$. Write the eigenspace decomposition of $\mathfrak{g}$ with respect to $\text{ad}x$ as

$$\mathfrak{g} = \bigoplus_{j \in \frac{1}{2}\mathbb{Z}} \mathfrak{g}_j. \quad (4.4.1)$$

Let us set

$$\mathfrak{g}_+ := \bigoplus_{j>0} \mathfrak{g}_j, \quad \mathfrak{g}_- := \bigoplus_{j<0} \mathfrak{g}_j, \quad \mathfrak{g}_{\leq 0} := \mathfrak{g}_0 \oplus \mathfrak{g}_-.$$

Denote by $\mathfrak{g}^f$ the centralizer of f in $\mathfrak{g}$:

$$\mathfrak{g}^f := \{v \in \mathfrak{g} \mid [f, v] = 0\}.$$

It is included in $\mathfrak{g}_{\leq 0}$, and decomposed into

$$\mathfrak{g}^f = \sum_{j \leq 0} (\mathfrak{g}^f \cap \mathfrak{g}_j)$$

Take a basis $\{u_i\}_{i \in S_j}$ of each $\mathfrak{g}_j$ in (4.4.1), and set

$$S := \bigsqcup_{j \in \frac{1}{2}\mathbb{Z}} S_j, \quad S_+ := \bigsqcup_{j>0} S_j.$$

Then for $i \in S$, define m_i by $u_i \in \mathfrak{g}_{m_i}$.

Define three families of free fields as follows:

- (1) For each $i \in S^+$, take two free fields

$$\varphi_i(z) = \sum_{n \in \mathbb{Z}} \varphi_{i(n)} z^{-n-1}, \quad \varphi_i^*(z) = \sum_{n \in \mathbb{Z}} \varphi_{i(n)}^* z^{-n-1},$$

with reversed parity of u_i . The commutation relation is

$$[\varphi_i \lambda \varphi_j] = [\varphi_i^* \lambda \varphi_j^*] = 0, \quad [\varphi_i \lambda \varphi_j^*] = \delta_{i,j}.$$

- (2) For each $i \in S_{1/2}$, take a free field

$$\phi_i(z) = \sum_{n \in \mathbb{Z}} \phi_{i(n)} z^{-n-1}.$$

The commutation relation is

$$[\phi_i \lambda \phi_j] = (f|[u_i, u_j]).$$

Denote by A_{ch} the Clifford algebra generated by φ_i, φ_i^* ($i \in S_+$), and by A_{ne} the Clifford algebra generated by ϕ_i ($i \in S_{1/2}$).

Now introduce the differential complex $(C_k(\mathfrak{g}, x, f), d_0)$ as follows. Set the vertex algebra

$$C_k(\mathfrak{g}, x, f) := V_k(\mathfrak{g}) \otimes F(A_{\text{ch}}) \otimes F(A_{\text{ne}}),$$

where $V_k(\mathfrak{g})$ is the universal affine vertex algebra of level k associated to $\mathfrak{g}$, and $F(A_{\text{ch}})$ (resp. $F(A_{\text{ne}})$) is the vertex algebra of free fermions based on A_{ch} (resp. A_{ne}).

The degree of this complex is given by the *charge*, which is defined by

$$\text{charge } \varphi_i(z) := 1, \quad \text{charge } \varphi_i^*(z) := -1, \quad \text{charge } V_k(\mathfrak{g}) = \text{charge } F(A_{\text{ne}}) := 0.$$

It induces the charge decomposition:

$$C_k(\mathfrak{g}, x, f) = \bigoplus_{m \in \mathbb{Z}} C_m.$$

For the boundary operator d_0 , introduce the field

$$\begin{aligned} d(z) := & \sum_{i \in S_+} (-1)^{p(u_i)} u_i(z) \otimes \varphi_i^*(z) \otimes 1 - \frac{1}{2} \sum_{i,j,k \in S_+} (-1)^{p(u_i)p(u_k)} c_{ij}^k \otimes \varphi_k(z) \varphi_i^*(z) \varphi_j^*(z) \otimes 1 \\ & + \sum_{i \in S_+} (f|u_i) \otimes \varphi_i^*(z) \otimes 1 + \sum_{i \in S_{1/2}} 1 \otimes \varphi_i^*(z) \otimes \phi_i(z). \end{aligned}$$

Here c_{ij}^k is the structure constant of $\mathfrak{g}$ with respect to the basis $\{u_i\}_{i \in S}$:

$$[u_i, u_j] = \sum_{k \in S} c_{ij}^k u_k.$$

Then

$$d_0 := \text{Res}_z d(z)$$

is an odd derivation of $C_k(\mathfrak{g}, x, f)$ and $d_0^2 = 0$, since we have $[d(z), d(w)] = 0$ (see [59, Theorem 2.1]). We also have $d_0 C_m \subset C_{m-1}$.

DEFINITION 4.4.1. The 0-th homology group of $(C_k(\mathfrak{g}, x, f) = \bigoplus_{m \in \mathbb{Z}} C_m, d_0)$ is called the $\mathcal{W}$ -algebra, and denoted by $W_k(\mathfrak{g}, x, f)$.

FACT 4.4.2. If $k \neq -h^\vee$, then $W_k(\mathfrak{g}, x, f)$ is a conformal vertex algebra with the Virasoro field

$$L(z) := L^{\mathfrak{g}}(z) + \partial_z x(z) + L^{\text{ch}}(z) + L^{\text{ne}}(z)$$

with

$$L^{\mathfrak{g}}(z) := \frac{1}{2(k + h^\vee)} \sum_{i \in S} \circ u^i u_i \circ,$$

$$L^{\text{ch}}(z) := \sum_{i \in S_+} \left\{ (-1)^{p(u_i)} m_i \circ (\partial \varphi_i) \varphi_i^* \circ + (1 - m_i) \circ (\partial \varphi_i^*) \varphi_i \circ \right\},$$

$$L^{\text{ne}}(z) := \frac{1}{2} \sum_{i \in S_{1/2}} \circ (\partial \phi_i^*) \phi_i \circ.$$

Here $\{u^i\}_{i \in S}$ is the dual basis of $\{u_i\}_{i \in S}$ with respect to $(\cdot | \cdot)$. Its central charge $c(k)$ is given by

$$c(k) = \frac{k}{k + h^\vee} \text{sdim } \mathfrak{g} - 12k(x|x) - 2 \sum_{i \in S_+} (-1)^{p(u_i)} (6m_i^2 - m_i + 1) - \frac{1}{2} \text{sdim } \mathfrak{g}_{1/2}.$$

For the proof, see [59, Theorem 2.2].

4.5. Structure of $\mathcal{W}$ -algebra

4.5.1. Basic properties. For each $v \in \mathfrak{g}$, set

$$J^{(v)}(z) := a(z) + \sum_{j, k \in S_+} (-1)^{p(u_k)} c_k^j(v) \circ \varphi_j(z) \varphi_j^*(z) \circ.$$

Here $c_k^j(v) \in \mathbb{C}$ is given by $[v, u_k] = \sum_{i \in S} c_k^j(v) u_j$. Then the complex $C_k(\mathfrak{g}, x, f)$ is strongly generated by $J^{(v)}(z)$ ($v \in \mathfrak{g}$) and $\varphi_i(z), \varphi_i^*(z), \phi_i(z)$ ($i \in S$).

Then we have

- FACT 4.5.1. (1) For each $i \in S_{1/2}$, $J^{(u_i)}(z) + (-1)^{p(u_i)} \phi_i(z)$ is d_0 -closed.
 (2) For each $i \in S_{\geq 1}$, $J^{(u_i)}(z) + (f|u_i)$ is d_0 -closed.

For the proof, see [60, Theorem 2.1].

Let C^+ be the vertex subalgebra of the vertex algebra $C_k(\mathfrak{g}, x, f)$ generated by the fields

$$J^{(u_i)}(z) + (-1)^{p(u_i)} \phi_i(z) \ (i \in S_{1/2}), \quad J^{(u_i)}(z) \ (i \in S_{\geq 1}), \quad \varphi_i(z) \ (i \in S_+).$$

Similarly, let C^- be the vertex subalgebra generated by the fields

$$J^{(v)}(z) \ (v \in \mathfrak{g}_{\leq 0}), \quad \varphi_i^*(z) \ (i \in S_+), \quad \phi_i(z) \ (i \in S_{1/2}).$$

Then we have

$$C_k(\mathfrak{g}, x, f) = C^+ \otimes C^-. \quad (4.5.1)$$

It also holds that

$$H_0(C^+) = \mathbb{C}, \quad H_j(C^+) = 0 \ (j \neq 0),$$

which yields

$$H_*(C_k(\mathfrak{g}, x, f)) \cong H_*(C^-)$$

Since C^- does not contain φ_i , it has the charge decomposition

$$C^- = \bigoplus_{j \geq 0} C_j^-.$$

Since d_0 decreases the charge, we have

FACT 4.5.2.

$$H_0(C_k(\mathfrak{g}, x, f)) \cong H_0(C^-) = \{\xi(z) \in C_0^+ \mid d_0 \xi(z) = 0\}.$$

Thus the $\mathcal{W}$ -algebra can be realized as a subspace of C^- . More detailed study on the action of d_0 shows

- FACT 4.5.3. (1) For each element a of $\mathfrak{g}_{-j}^f := \mathfrak{g}^f \cap \mathfrak{g}_{-j}$ ($j \geq 0$) there exists a d_0 -closed field $J^{\{a\}}(z)$ in C^- of conformal weight $1 + j$ (with respect to $L(z)$) such that $J^{\{a\}}(z) - J^{(a)}(z)$ is a linear combination of normal ordered products of the fields $J^{(b)}(z)$, ($b \in \mathfrak{g}_{-s}$, $0 \leq s < j$), the fields $\phi_i(z)$ ($i \in S_{1/2}$) and their derivatives.
 (2) The homology classes of $J^{\{a\}}(z)$ ($a \in \mathfrak{g}^f$) strongly generates $W_k(\mathfrak{g}, x, f)$.

For the proof, see [60, Theorem 4.1].

EXAMPLE 4.5.4. Take $\mathfrak{g} = \mathfrak{sl}(2, \mathbb{C})$ and let (e, f, h) be the standard basis. Consider the $\mathcal{W}$ -algebra $C_k(\mathfrak{sl}(2, \mathbb{C}), h, f)$. The Virasoro field is given by

$$L(z) = \frac{1}{2(k+2)} \left\{ \frac{1}{2} \circ h(z)^2 \circ + \circ e(z)f(z) \circ + \circ f(z)e(z) \circ \right\} + \frac{1}{2} \partial_z h(z) + \circ (\partial_z \varphi(z)) \varphi^*(z) \circ,$$

and the central charge is

$$c(k) = \frac{3k}{k+2} - 6k - 2 = 1 - \frac{6(k+1)^2}{k+2}.$$

The fields $J^{(v)}(z)$ are

$$\begin{aligned} J^{(e)}(z) &= e(z), \\ J^{(h)}(z) &= h(z) + 2 \circ \varphi(z) \varphi^*(z) \circ, \\ J^{(f)}(z) &= f(z). \end{aligned}$$

By Fact (4.5.3), it is enough to consider $J^{\{f\}}(z)$. The choice

$$J^{\{f\}}(z) := J^{(f)}(z) - \frac{1}{4} \circ J^{(h)}(z) J^{(h)}(z) \circ - (k+1) \partial_z J^{(h)}(z)$$

satisfies

$$\left[d_\lambda J^{\{f\}}(z) \right] = 0$$

and

$$L(z) = \frac{-1}{k+2} J^{\{f\}}(z).$$

Thus $C_k(\mathfrak{sl}(2, \mathbb{C}), h, f) \cong V_{c(k)}(\text{Vir})$.

4.5.2. Good pair and list of $\mathcal{W}$ -algebras. The explanation of this chapter depends on the choice of the $\mathfrak{sl}_2$ -triple (x, e, f) . However, in the construction of $\mathcal{W}$ -algebra, the element e does not appear. Actually, one can execute the quantum Drinfeld-Sokolov reduction in a more general situation.

DEFINITION 4.5.5. Let $\mathfrak{g}$ be a simple finite dimensional Lie superalgebra with a non-degenerate even supersymmetric invariant bilinear form.

- (1) A pair (x, f) of elements in $\mathfrak{g}_0$ is called a *good $\mathbb{R}$ -pair* if the following conditions are satisfied:
 - (a) f is nilpotent.
 - (b) $\text{ad}(x)$ is diagonalizable on $\mathfrak{g}$, with eigenvalues in $\mathbb{R}$.
 - (c) $[x, f] = -f$.
 - (d) $\text{ad}(f) : \mathfrak{g}_j \rightarrow \mathfrak{g}_{j-1}$ is injective if $j \geq 1/2$, and is surjective if $j \leq 1/2$.
 - (e) $\mathfrak{g}^f \subset \mathfrak{g}_{<1/2}$.
- (2) A pair (x, f) of elements in $\mathfrak{g}$ is called a *good pair* if it is a good $\mathbb{R}$ -pair and the eigenvalue of $\text{ad}(x)$ are in $\frac{1}{2}\mathbb{Z}$.

Thus if (x, e, f) is a $\mathfrak{sl}_2$ -triple, then (x, f) is a good pair.

For a good $\mathbb{R}$ -pair (x, f) , one can apply the quantum Drinfeld-Sokolov reduction and obtains the vertex algebra $W_k(\mathfrak{g}, x, f)$. There may not exist a decomposition like (4.5.1) in this general situation. However [22] implies

FACT 4.5.6. $W(\mathfrak{g}, x, f)$ depends only on the conjugate class of f in $\text{Ad}(G)$, and is independent of the choice of (x, f) .

$\mathfrak{g}$	f	$W_k(\mathfrak{g}, x, f)$
$\mathfrak{sl}(2, \mathbb{C})$	$e_{-\theta} = f_{\text{prin}}$	the Virasoro vertex algebra
$\mathfrak{sl}(3, \mathbb{C})$	$e_{-\theta}$	the Bershadsky-Polyakov algebra [15]
$\mathfrak{sl}(3, \mathbb{C})$	f_{prin}	the Zamolodchikov $\mathcal{W}$ -algebra [98]
$\mathfrak{sl}(n, \mathbb{C})$	f_{prin}	the Fateev-Lykyanov $\mathcal{W}$ -algebra [31]
$\mathfrak{osp}(2 1)$	$e_{-\theta}$	the Neveu-Schwarz algebra,
$\mathfrak{osp}(2m+1 2)$	$e_{-\theta}$	the Bershadsky-Knizhnik algebras [16]
$\mathfrak{sl}(2 1) = \mathfrak{osp}(2 2)$	$e_{-\theta}$	the $N = 2$ superconformal algebra
$\mathfrak{sl}(2 2)/\text{center}$	$e_{-\theta}$	the $N = 4$ superconformal algebra
$\mathfrak{osp}(3 2)$	$e_{-\theta}$	the $N = 3$ superconformal algebra

TABLE 4.5.1. $\mathcal{W}$ -algebras via quantum Drinfeld-Sokolov reduction

As a conclusion of this section, we list several $\mathcal{W}$ -algebras which have been known since the emergence of conformal field theory. This list is given in [59].

Here θ is the highest root of $\mathfrak{g}$ (the Dynkin diagram of $\mathfrak{g}$ is chosen so that the highest root is even). Then the $\text{Ad}(G)$ -orbit of the root vector $e_{-\theta}$ is of minimal dimension among the orbits of nilpotent elements. Thus $e_{-\theta}$ is called the minimal nilpotent element. f_{prin} is the principal nilpotent element, whose $\text{Ad}(G)$ -orbit is of maximal dimension.

CHAPTER 5

Verma module of $\mathcal{W}$ -algebra

The aim of this chapter is to explain the Verma modules of $\mathcal{W}$ -algebras and its contravariant form.

5.1. $\mathcal{W}$ -module

Let V be a vertex algebra with a conformal vector ν , that is the field $Y(\nu, z)$, corresponding to ν , is of the form

$$Y(\nu, z) = \sum_{n \in \mathbb{Z}} L_n z^{-n-2},$$

where L_n form a representation of the Virasoro algebra with central charge c , L_{-1} coincides with the infinitesimal translation operator, and L_0 is diagonalizable on V .

If $a \in V$ is an eigenvector of L_0 with the eigenvalue equal to conformal weight $\Delta(a)$, one writes

$$Y(a, z) = \sum_{n \in \mathbb{Z} - \Delta(a)} a_n z^{-n - \Delta(a)}$$

Then it holds that

$$[a_m, b_n] + \sum_{j \in \mathbb{Z}_+} \binom{\Delta(a) + m - 1}{j} (a_{(j)} b)_{m+n},$$

where

$$a_{(j)} b := \text{Res}_z [a(z), b(w)] (z - w)^n$$

is the n -th product in V .

Recall that a V -module is a vector space M and a collection of $\text{End}(M)$ -valued fields

$$\{Y^M(a, z) = \sum_{n \in \mathbb{Z} - \Delta(a)} a_n^M z^{-n - \Delta(a)}\}_{a \in V},$$

such that the following properties hold:

- (M1) $Y^M(|0\rangle, z) = \text{id}_M$.
- (M2) $Y^M(L_{-1}a, z) = \partial_z Y^M(a, z)$.
- (M3) $Y^M(a_{(-1)}b, z) = {}^\circ Y^M(a, z) Y^M(b, z) {}^\circ$.
- (M4) $[a_m^M, b_n^M] = \sum_{j \in \mathbb{Z}_+} \binom{\Delta(a) + m - 1}{j} (a_{(j)} b)_{m+n}^M$.
- (M5) L_0^M is diagonalizable on M with eigenvalues bounded below.

Now we recall the functorial construction of representation of $\mathcal{W}$ -algebra, Given a $\mathfrak{g}$ -module P of level k (i.e., $K = \text{kid}_P$), one constructs the associated $C_k(\mathfrak{g}, x, f)$ -module

$$C(P) := P \otimes F(A_{\text{ch}}) \otimes F(A_{\text{ne}})$$

with the differential $d_0^P = \text{Res}_z d^{C(P)}(z)$. The homology $H_*(P)$ of the ocmplex $(C(P), d_0^P)$ with the induced charge decomposition (by setting charge $P = 0$) is a direct sum of $W_k(\mathfrak{g}, x, f)$ -modules

$$H(P) = \bigoplus_{j \in \mathbb{Z}} H_j(P).$$

Thus we get a functor from the category of $\widehat{\mathfrak{g}}$ -modules to the category of $\mathbb{Z}$ -graded $W_k(\mathfrak{g}, x, f)$ -modules.

Let P_0 be a $\mathfrak{g}$ -module on which x is diagonalizable and $U(\mathfrak{g}_+)$ is locally finite. Extend P_0 to a $g[t] + \mathbb{C}K$ -module by letting $K = \text{id}_{P_0}$. The induced $\widehat{\mathfrak{g}}$ -module

$$P := \text{Ind}_{g[t] + \mathbb{C}K}^{\widehat{\mathfrak{g}}} P_0 \quad (5.1.1)$$

is called a *generalized Verma module* over $\widehat{\mathfrak{g}}$.

FACT 5.1.1 ([60, Theorem 6.2]). Suppose that (x, f) is a good pair and that P is a generalized Verma module over $\widehat{\mathfrak{g}}$. Then

$$H_j(P) = 0 \quad \text{if } j \neq 0.$$

5.2. Verma module

Hereafter we assume (x, f) is a good pair. Recall the eigenspace decomposition $\mathfrak{g} = \bigoplus_{j \in \frac{1}{2}\mathbb{Z}} g_j$ with respect to $\text{ad}(x)$. Let

$$\mathfrak{g}^b := \{a \in \mathfrak{g}_0 \mid [a, f] = 0\}$$

be the centralizer of f in $\mathfrak{g}_0$.

Let h^b be a maximal diagonalizable subalgebra of the even part of $\mathfrak{g}^b$. and embed h^b in a Cartan subalgebra h of the even part of $\mathfrak{g}_0$. Let $\Delta_j \subset \mathfrak{h}^*$ be the set of roots of $\mathfrak{h}$ in $\mathfrak{g}_j$, and let Δ_{0+} be a subset of positive roots of Δ_0 .

Then $\mathfrak{h}$ is a Cartan subalgebra of the even part of $\mathfrak{g}$, and $\Delta = \sqcup_j \Delta_j$ is the set of roots of $\mathfrak{h}$ in $\mathfrak{g}$, $\Delta_+ := \Delta_{0+} \sqcup (\sqcup_{j>0} \Delta_j)$ being the subset of positive roots. Define Δ_- similarly. Then, denoting by

$$\mathfrak{n}_{0+}, \quad \mathfrak{n}_{0-}, \quad \mathfrak{n}_+, \quad \mathfrak{n}_-$$

the sum of all root spaces corresponding to

$$\Delta_{0+}, \quad \Delta_{0-}, \quad \Delta_+, \quad \Delta_-,$$

one obtains the triangular decompositions

$$\mathfrak{g} = \mathfrak{n}_- \oplus \mathfrak{h} \oplus \mathfrak{n}_+, \quad \mathfrak{g}_0 = \mathfrak{n}_{0-} \oplus \mathfrak{h} \oplus \mathfrak{n}_{0+}. \quad (5.2.1)$$

Recall that a Verma module over $\mathfrak{g}$ with highest weight $\lambda \in \mathfrak{h}^*$ is the module

$$U(\mathfrak{g}) \otimes_{U(\mathfrak{h} + \mathfrak{n}_+)} \mathbb{C}v_\lambda,$$

where $\mathbb{C}v_\lambda$ is a 1-dimensional $\mathfrak{h} + \mathfrak{n}_+$ -module such that $\mathfrak{n}_+ v_\lambda = 0$ and $h v_\lambda = \lambda(h) v_\lambda$ for all $h \in \mathfrak{h}$.

Let $\widehat{\mathfrak{h}} = \mathfrak{h} + \mathbb{C}K$ be the Cartan subalgebra of $\widehat{\mathfrak{g}}$. We extend the bilinear form $(\cdot | \cdot)$ from $\mathfrak{h}$ to $\widehat{\mathfrak{h}}$ by letting

$$(\mathfrak{h} | \mathbb{C}K) := 0, \quad (K | K) := 0,$$

and identify $\mathfrak{h}$ with $\widehat{\mathfrak{h}}$ using this form. Given $k \in \mathbb{C}$, we extend $\lambda \in \mathfrak{h}^*$ to $\widehat{\mathfrak{h}}^*$ letting

$$\widehat{\lambda}|_{\mathfrak{h}} := \lambda, \quad \widehat{\lambda}(K) := k.$$

A Verma module over $\widehat{\mathfrak{g}}$ with highest weight $\widehat{\lambda}$ of level k is a generalized Verma module (5.1.1) for which P_0 is a Verma $\mathfrak{g}$ -module with highest weight λ .

Now we will introduce a Verma module over the vertex algebra $W_k(\mathfrak{g}, x, f)$, assuming that the pair (x, f) is good. In this case the $\mathfrak{g}$ -modules $\mathfrak{g}_{-1/2}$ and $\mathfrak{g}_{1/2}$ are isomorphic. Furthermore, we have the exact sequence of $\mathfrak{g}$ -modules:

$$0 \rightarrow \mathfrak{g}^f \rightarrow \mathfrak{g}_- + \mathfrak{g}_0 + \mathfrak{g}_{1/2} \xrightarrow{\text{adf}} \mathfrak{g}_- \rightarrow 0.$$

It follows that adf induces the following isomorphism of $\mathfrak{g}$ -modules:

$$\mathfrak{g}^f \xrightarrow{\sim} \mathfrak{g}_0 \oplus \mathfrak{g}_{1/2}. \quad (5.2.2)$$

The isomorphism (5.2.2) and the triangular decomposition (5.2.1) of $\mathfrak{g}_0$ induce the following decomposition of $\mathfrak{g}^f$ as $\mathfrak{h}$ -modules:

$$\mathfrak{g}^f = \mathfrak{n}_{0-} + \mathfrak{h} + \mathfrak{n}_{0+} + \mathfrak{g}_{1/2}. \quad (5.2.3)$$

Choose a basis $\{a_i \mid i = 1, \dots, s\}$ of $\mathfrak{g}^f$ compatible with the decomposition (5.2.3), the root space decomposition with respect to $\mathfrak{h}^b$ of (5.2.3) and the gradation

$$\mathfrak{g}^f = \bigoplus_{\leq 0} \mathfrak{g}_j^f, \quad \mathfrak{g}_j^f := \mathfrak{g}^f \cap \mathfrak{g}_j.$$

Let

$$J^{(a_i)}(z) = \sum_{n \in \mathbb{Z} - \Delta(a_i)} J_n^{(i)} z^{-n - \Delta(a_i)}$$

be the field of $W_k(\mathfrak{g}, x, f)$ corresponding to a_i in Fact 4.5.3. We have $\Delta(a_i) = 1 + j$ if $a_i \in \mathfrak{g}_{-j}$ ($j \in \frac{1}{2}\mathbb{Z}$). We order a_i in such a way that $a_1, \dots, a_r = x$ form a basis of $\mathfrak{h}$.

DEFINITION 5.2.1. A $W_k(\mathfrak{g}, x, f)$ -module M is called a *highest weight module* with highest weight $\lambda = (\lambda(a_1), \dots, \lambda(a_r)) \in \mathbb{C}^r$, if there exists a non-zero vector $v \in M$ such that

(HW1) $W_k(\mathfrak{g}, x, f)v = M$.

(HW2) $J_0^{\{a_i\}}v = \lambda(a_i)v$ if $a_i \in \mathfrak{h}$.

(HW3) $J_m^{\{a_i\}}v = 0$ if $m > 0$, or $m = 0$ and $a_i \in \mathfrak{n}_{0+}$.

Note that the following vectors span a highest weight $W_k(g, x, f)$ -module M :

$$\left(J_{-m_1}^{\{a_1\}}\right)^{b_1} \left(J_{-m_2}^{\{a_2\}}\right)^{b_2} \dots \left(J_{-m_s}^{\{a_s\}}\right)^{b_s} v, \quad (5.2.4)$$

with $b_i \in \mathbb{Z}_{\geq 0}$. $b_i \leq 1$ if a_i is odd, $m_i > 0$, or $m_i = 0$ and $a_i \in \mathfrak{n}_{0-}$.

DEFINITION 5.2.2. A highest weight $W_k(\mathfrak{g}, x, f)$ -module M is called a *Verma module* if the vectors (5.2.4), where the sequence of pairs $(i_1, m_1), (i_2, m_2), \dots$ decreases in the lexicographical order, form a basis of M .

5.3. Contravariant form

In this section, the contravariant form on a Verma module will be introduced. We begin with the anti-involution on a Verma module.

We construct an anti-involution ω of the associative superalgebra A and the corresponding contravariant form on any highest weight module over $W_k(\mathfrak{g}, x, f)$. We call an anti-involution of an associative superalgebra (resp. Lie superalgebra) a vector superspace automorphism ω such that $\omega^2 = 1$ and $\omega(ab) = \omega(b)\omega(a)$ (resp. $\omega([a, b]) = [\omega(b), \omega(a)]$).

First, we construct an anti-involution ω of the Lie superalgebra $\mathfrak{g}$ such that the following properties hold: $\omega(x) = -x$, $\omega(a) = a$ for $a \in \mathfrak{h}^b$, $\omega(f) = -f$, $\omega(\mathfrak{g}_\alpha) = \mathfrak{g}_{-\alpha}$ for $\alpha \in \Delta_0$ and $(\omega(a)|\omega(b)) = (b|a)$ for $a, b \in \mathfrak{g}$.

Next, we lift ω to an anti-involution of the affine Lie superalgebra $\widehat{\mathfrak{g}}$ by letting

$$\omega(a_n) = \omega(a)_{-n} + 2K(a|x)\delta_{0,n}, \quad \omega(K) = K$$

(recall the notation $a(z) = \sum_{n \in \mathbb{Z}} a_n z^{-n - \Delta(a)}$, where $\Delta(a)$ is the conformal weight $a(z)$.)

Furthermore, ω induces superspace automorphisms of A_{ne} and A_{ch} .

In order to explain it, let us recall these superspaces. A_{ne} is identical to the vector superspace $\mathfrak{g}_{1/2}$ with the bilinear form

$$\langle a, b \rangle_{\text{ne}} := (f|[a, b]).$$

Let us denote by A (resp. A^*) the vector superspace $\mathfrak{g}_+$ (resp. $\mathfrak{g}_+^*$) with the reversed parity. Then we have $A_{\text{ch}} = A \oplus A^*$ with the even skew-symmetric non-degenerate bilinear form $\langle \cdot, \cdot \rangle$ defined by

$$\langle A, A \rangle_{\text{ch}} = \langle A^*, A^* \rangle_{\text{ch}} = 0$$

$$\langle a, b^* \rangle_{\text{ch}} = (-1)^{p(a)p(b^*)} \langle b^*, a \rangle_{\text{ch}} = b^*(a) \quad a \in A, \quad b^* \in A^*.$$

such that

$$\langle \omega(a), \omega(b) \rangle_{\text{ne}} = \langle b, a \rangle_{\text{ne}}.$$

Then ω on $\mathfrak{g}$ induces a super vector space automorphism of A_{ne} such that

$$\langle \omega(a), \omega(b) \rangle_{\text{ne}} = \langle b, a \rangle_{\text{ne}}.$$

Moreover it induces an automorphism of A^* by

$$\omega(b^*)(a) = -(-1)^{p(a)p(b)}b^*(\omega(a)),$$

so that we have

$$\langle \omega(a), \omega(b) \rangle_{\text{ch}} = \langle b, a \rangle_{\text{ch}}$$

on A_{ch} . Now we can extend the anti-involution ω on the vertex algebras $C_k(\mathfrak{g}, x, f)$ and $W_k(\mathfrak{g}, x, f)$.

Denote by $\mathcal{A}$ the unital associative superalgebra on generators $J_m^{\{i\}}$ ($i \in S$, $m \in \mathbb{Z} - \Delta(a_i)$) of $W_k(\mathfrak{g}, x, f)$. Let $\tilde{\mathcal{A}}_-$, $\tilde{\mathcal{A}}_+$ and $\tilde{\mathcal{A}}_0$ be the subalgebras of $\mathcal{A}$ generated by the $J_m^{\{i\}}$ with $m < 0$, $m > 0$ and $m = 0$, respectively. It follows from (M4) that any $W_k(\mathfrak{g}, x, f)$ -module M is an $\mathcal{A}$ -module. It holds that

$$\mathcal{A} = \tilde{\mathcal{A}}_- \tilde{\mathcal{A}}_0 \tilde{\mathcal{A}}_+. \quad (5.3.1)$$

Now let M be a Verma module over $W_k(\mathfrak{g}, x, f)$ with highest weight vector v_λ . By (5.3.1) any vector $v \in M$ can be written uniquely in the form

$$v = \langle v \rangle v_\lambda + v',$$

where $\langle v \rangle \in \mathbb{C}$ is called the expectation value of v , and v' is a linear combination of weight vectors v_μ with $\mu \neq \lambda$. By the decomposition (5.3.1) we have

$$\langle \omega(a) v_\lambda \rangle = \langle a v_\lambda \rangle$$

for any a . We define the *contravariant bilinear form* $B(\cdot, \cdot)$ on M by the formula

$$B(av_\lambda, bv_\lambda) = \langle \omega(a) bv_\lambda \rangle$$

for $a, b \in \mathcal{A}$.

By the definition, this bilinear form on M satisfies

- (1) $B(au, v) = \langle u, \omega(a)v \rangle$ for $u, v \in M$, $a \in \mathcal{A}$.
- (2) $B(v_\lambda, v_\lambda) = 1$.
- (3) $B(u, v) = B(v, w)$ for $u, v \in M$.

Conversely, these properties uniquely determine the form $B(\cdot, \cdot)$.

Note finally that the maximal submodule of the Verma module M coincides with the kernel of the form $B(\cdot, \cdot)$.

Part 2

Geometry

CHAPTER 6

Instanton moduli space

We turn to the geometric preparation for the AGT conjecture. The fundamental object is the instanton moduli space, as mentioned in Introduction. We start with the differentiable geometric setting.

6.1. Moduli space of anti-self-dual connections

We want to explain a gauge theoretic construction of moduli space of connections. The main reference here is [26].

Let X be a four-dimensional hyper-Kähler manifold, and E be a hermitian vector bundle of rank r on X . Denote by $\mathcal{A}$ the (infinite dimensional) vector space of connections on E . The tangent space at $A \in \mathcal{A}$ can be identified with $T_A \mathcal{A} \cong \Omega^1(\mathfrak{u}(E))$. Denote by $\mathcal{G}$ the (infinite dimensional) Lie group of gauge transformations of connections on P . Then one has a hyper-Kähler moment map $\mu = (\mu_1, \mu_2, \mu_3)$ of the action of $\mathcal{G}$ on $\mathcal{A}$ defined by

$$\begin{aligned} \mu_i : \mathcal{A} &\longrightarrow \text{Lie } \mathcal{G}^* \cong \Omega^4(\mathfrak{u}(E)) \\ A &\longmapsto F_A \wedge \omega_i \end{aligned}$$

for $i = 1, 2, 3$. Here F_A is the curvature 2-form of A , and ω_i is the Kähler form associated with the complex structure I, J, K on X . Then we have

$$\mu^{-1}(0)/\mathcal{G} \cong \{A \in \mathcal{A} \mid F_A^+ = 0\}/\mathcal{G}.$$

Here F_A^+ is the anti-self dual part of F_A . Thus the resulting space $\mu^{-1}(0)/\mathcal{G}$ parametrizes the *instantons* on X . Although $\mu^{-1}(0)$ and $\mathcal{G}$ are both infinite dimensional, the quotient is finite dimensional. This space is called the *moduli space of the anti-self-dual connections*. It is a singular space in general.

This construction works in the case $X = \mathbb{R}^4$ with the standard hyper-Kähler structure. Since the base space is non-compact in this case, one should consider the framed moduli space. *Framed* means taking the quotient by a group of gauge transformations converging to the identity at the end of $\mathbb{R}^4$. That is to say, considering the compactification $S^4 = \mathbb{R}^4 \cup \{\infty\}$, the framed moduli space is the space of pairs

$$M_{\text{ASD}}(E) = \left\{ (A, \varphi) \left| \begin{array}{l} A: \text{anti-self-dual connection on } E, \\ \varphi : E|_{\infty} \xrightarrow{\sim} \mathbb{C}^r. \end{array} \right. \right\} / \text{isom.},$$

There is another description of this moduli space, called the *ADHM description*. Let V, W be hermitian vector spaces of dimension n, r . Define a complex vector space $\mathbf{M}$ by

$$\mathbf{M} := \{(B_1, B_2, i, j) \mid B_1, B_2 \in \text{Hom}(V, V), i \in \text{Hom}(W, V), j \in \text{Hom}(V, W)\}.$$

Then $U(V)$ acts on $\mathbf{M}$ by

$$g \cdot (B_1, B_2, i, j) = (g^{-1}B_1g, g^{-1}B_2g, i, j)$$

for $g \in U(V)$. Introduce the moment map $\mu_1 : V \rightarrow \mathfrak{u}(V)$ by

$$\mu_1(B_1, B_2, i, j) := \frac{\sqrt{-1}}{2} \left([B_1, B_1^\dagger] + [B_2, B_2^\dagger] + ii^\dagger - j^\dagger j \right)$$

We need another map $\mu_{\mathbb{C}} : \mathbf{M} \rightarrow \mathfrak{gl}$ defined by

$$\mu_{\mathbb{C}}(B_1, B_2, i, j) = [B_1, B_2] + ij.$$

Let us set

$$M_0(r, n) := \mu_1^{-1}(0) \cap \mu_{\mathbb{C}}^{-1}(0)/U(V)$$

the hyper-Kähler quotient. Its non-singular locus

$$M_0^{\text{reg}}(r, n) := \{[(B_1, B_2, i, j)] \in M_0(r, n) \mid \text{the stabilizer in } U(V) \text{ of } (B_1, B_2, i, j) \text{ is trivial}\}.$$

FACT 6.1.1 ([5]). There is a bijective correspondence

$$M_{\text{ASD}}(E) \cong M_0^{\text{reg}}(r, n)$$

with $r = \text{rank } E$, $n = c_2(E)$.

6.2. Framed moduli space of torsion-free sheaves on $\mathbb{CP}^2$.

Here we will introduce an algebro-geometric counterpart of the moduli spaces given in the last section.

Let $M(r, n)$ be the framed moduli space of torsion free sheaves on $\mathbb{CP}^2$ with rank r and $c_2 = n$:

$$M(r, n) = \left\{ (E, \varphi) \left| \begin{array}{l} E: \text{a torsion free sheaf of rank } E = \\ r, c_2(E) = n, \text{ which is locally} \\ \text{free in a neighborhood of } \ell_{\infty}, \\ \varphi : E|_{\ell_{\infty}} \xrightarrow{\sim} \mathcal{O}_{\ell_{\infty}}^{\oplus r}. \end{array} \right. \right\} / \text{isom.},$$

where $\ell_{\infty} = \{[0 : z_1 : z_2] \in \mathbb{CP}^2\} \subset \mathbb{CP}^2$ is the line at infinity. The isomorphism φ is called a *framing at infinity*. By the existence of the framing φ , we have $c_1(E) = 0$.

Here we only give a set-theoretical description of the moduli space. Actually there is a natural functor $\mathcal{M}(r, n)$ from the category of $\mathbb{C}$ -schemes to the category of sets, and it is represented by a fine moduli scheme. See [50] for the proof.

This space is smooth of dimension $2rn$. By [25], it is known that $M_0^{\text{reg}}(r, n)$ coincides with the open subset of $M(r, n)$ consisting of locally free sheaves.

Let $M_0(r, n)$ be the Uhlenbeck partial compactification of $M_0^{\text{reg}}(r, n)$. Set theoretically it has the structure

$$M_0(r, n) = \bigsqcup_{k=0}^n M_0^{\text{reg}}(r, n-k) \times S^k \mathbb{C}^2,$$

where $S^k \mathbb{C}^2$ is the k th symmetric product of $\mathbb{C}^2$. $M_0(r, n)$ has a structure of (singular) affine algebraic variety, and there is a projective morphism

$$\pi : M(r, n) \rightarrow M_0(r, n). \quad (6.2.1)$$

The corresponding map between closed points is identified with

$$(E, \varphi) \mapsto ((E^{\vee\vee}, \varphi), \text{Supp}(E^{\vee\vee}/E)) \in M_0^{\text{reg}}(r, \ell) \times S^{n-\ell} \mathbb{C}^2.$$

Here $\bullet^{\vee} := \text{Hom}(\bullet, \mathcal{O}_{\mathbb{CP}^2})$ means taking dual of a coherent sheaf on $\mathbb{CP}^2$, and $\text{Supp}(\bullet)$ is the support of a sheaf (counted with multiplicities). Note that $E^{\vee\vee}$ is a locally free sheaf, since E is a torsion-free coherent sheaf on a projective surface.

6.3. Quiver description

In this section we explain the monad description of the framed moduli space $M(r, n)$, following [69, Chap. 2].

THEOREM 6.3.1 ([69, Theorem 2.1]). There exists a bijection

$$M(r, n) \cong \left\{ (B_1, B_2, i, j) \left| \begin{array}{l} \text{(i) } [B_1, B_2] + ij = 0, \\ \text{(ii) There exists no proper subspace} \\ S \subsetneq \mathbb{C}^n \text{ such that } B_{\alpha}(S) \subset S \\ (\alpha = 1, 2) \text{ and } \text{im } i \subset S. \end{array} \right. \right\} / \text{GL}(n, \mathbb{C}).$$

Here $B_1, B_2 \in \text{End}(\mathbb{C}^n)$, $i \in \text{Hom}(\mathbb{C}^r, \mathbb{C}^n)$, $j \in \text{Hom}(\mathbb{C}^n, \mathbb{C}^r)$. The action of $\text{GL}(n, \mathbb{C})$ is given by

$$g \cdot (B_1, B_2, i, j) = (gB_1g^{-1}, gB_2g^{-1}, gi, jg^{-1})$$

for $g \in \text{GL}(n, \mathbb{C})$.

The condition (ii) is called the *stability condition*. The data (B_1, B_2, i, j) is associated to the following quiver (directed graph):

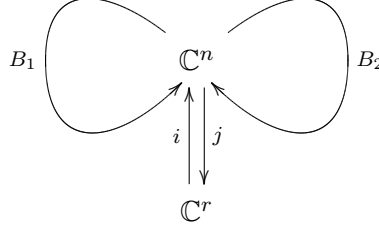


FIGURE 6.3.1. Quiver for framed moduli space

6.3.1. Beilinson spectral sequence. Let Δ be the diagonal in $\mathbb{CP}^2 \times \mathbb{CP}^2$. This subsection explains a resolution of $\mathcal{O}_\Delta$ given by Beilinson.

Let $p_i : \mathbb{CP}^2 \times \mathbb{CP}^2 \rightarrow \mathbb{CP}^2$ be the projection to the i th factor ($i = 1, 2$). Denote by $\mathcal{Q}$ the locally free sheaf on $\mathbb{CP}^2$ defined by the Euler exact sequence as

$$0 \rightarrow \mathcal{O}_{\mathbb{CP}^2}(-1) \rightarrow \mathcal{O}_{\mathbb{CP}^2}^{\oplus 3} \rightarrow \mathcal{Q} \rightarrow 0.$$

Consider the vector bundle $\mathcal{O}_{\mathbb{CP}^2}(1) \boxtimes \mathcal{Q} := p_1^* \mathcal{O}_{\mathbb{CP}^2}(1) \otimes p_2^* \mathcal{Q}$. The fiber of $\mathcal{O}_{\mathbb{CP}^2}(1) \boxtimes \mathcal{Q}$ at $(x, y) \in \mathbb{CP}^2$ is given by

$$\mathcal{O}_{\mathbb{CP}^2}(1)_x \otimes \mathcal{Q}_y \cong \text{Hom}(\mathcal{O}_{\mathbb{CP}^2}(-1)_x, \mathcal{Q}_y) \cong \text{Hom}(\mathbb{C}x, \mathbb{C}^3/\mathbb{C}y).$$

Define the section s of $\mathcal{O}_{\mathbb{CP}^2}(1) \boxtimes \mathcal{Q}$ at $(x, y) \in \mathbb{CP}^2$ by

$$s(x, y) : x \mapsto x \pmod{\mathbb{C}y}.$$

Then one has $s^{-1}(0) = \Delta$. Thus we have the associated Koszul complex

$$0 \rightarrow \wedge^2(\mathcal{O}_{\mathbb{CP}^2}(-1) \boxtimes \mathcal{Q}^\vee) \xrightarrow{t(\wedge s)} \mathcal{O}_{\mathbb{CP}^2}(-1) \boxtimes \mathcal{Q}^\vee \xrightarrow{ts} \mathcal{O}_{\mathbb{CP}^2 \times \mathbb{CP}^2} \rightarrow \mathcal{O}_\Delta \rightarrow 0.$$

Let us set

$$C^i := \wedge^{-i}(\mathcal{O}_{\mathbb{CP}^2}(-1) \boxtimes \mathcal{Q}^\vee)$$

The Beilinson spectral sequence is defined as follows. For a coherent sheaf E on $\mathbb{CP}^2$, we have the obvious isomorphism

$$p_{1*}(p_2^* E \otimes \mathcal{O}_{\mathbb{CP}^2}) = p_{1*}(p_2^* E|_\Delta) \cong E.$$

Replacing $\mathcal{O}_\Delta$ with the complex $C^\bullet$, we get the double complex $R^\bullet p_{1*}(p_2^* E \otimes C^\bullet)$. If we take the cohomology of $C^\bullet$ first, we have a spectral sequence $(E_r^{p,q})$ whose E_2 -term is given by

$$E_2^{p,q} = R^q p_{q*}(H^p(p_2^* E \otimes C^\bullet)) = \begin{cases} E & \text{for } (p, q) = (0, 0), \\ 0 & \text{for } (p, q) \neq (0, 0). \end{cases}$$

Thus the spectral sequence degenerates at the E_2 -term. On the other hand, if we take the direct image first, we have another spectral sequence $(E_r'^{p,q})$ whose E_1 -term is given by

$$E_1'^{p,q} = R^q p_{1*}(p_2^* E \otimes C^p).$$

Each C^p is of the form $C^p = \mathcal{F}_1 \boxtimes \mathcal{F}_2$, so that the E_1 -term becomes

$$E_1'^{p,q} = \mathcal{F}_1 \otimes H^q(\mathbb{CP}^2, E \otimes \mathcal{F}_2).$$

This second spectral sequence is called the *Beilinson spectral sequence*.

6.3.2. Monad description.

Now we cite
FACT 6.3.2 ([69, Lemma 2.4]). Let E be a torsion free sheaf on $\mathbb{CP}^2$ which is trivial on ℓ_∞ . Then

$$\begin{aligned} H^q(\mathbb{CP}^2, E(-p)) &= 0 \quad \text{for } p = 1, 2, \quad q = 0, 2, \\ H^q(\mathbb{CP}^2, E(-1) \otimes \mathcal{Q}^\vee) &= 0 \quad \text{for } q = 0, 2. \end{aligned}$$

Then the complex $p_2^*E(-1) \otimes \mathbb{C}^\bullet$ is written as

$$0 \rightarrow \mathcal{O}_{\mathbb{CP}^2}(-2) \boxtimes (E(-1) \otimes \wedge^2 \mathcal{Q}^\vee) \rightarrow \mathcal{O}_{\mathbb{CP}^2}(-1) \boxtimes (E(-1) \otimes \mathcal{Q}^\vee) \rightarrow \mathcal{O}_{\mathbb{CP}^2} \boxtimes E(-1) \rightarrow 0.$$

Since $\wedge^2 \mathcal{Q}^\vee \cong \mathcal{O}_{\mathbb{CP}^2}(-1)$, the E_1 -term of the Beilinson spectral sequence becomes

$$\begin{aligned} 0 \rightarrow \mathcal{O}_{\mathbb{CP}^2}(-2) \otimes H^q(\mathbb{CP}^2, E(-2)) \\ \xrightarrow{a'} \mathcal{O}_{\mathbb{CP}^2}(-1) \otimes H^q(\mathbb{CP}^2, E(-1) \otimes \mathcal{Q}^\vee) \xrightarrow{b'} \mathcal{O}_{\mathbb{CP}^2} \otimes H^q(\mathbb{CP}^2, E(-1)) \rightarrow 0. \end{aligned}$$

Thus the E_1 -term vanishes unless $q = 1$, so that the spectral sequence degenerates at the E_2 -term. Since the spectral sequence converges to $E(-1)$, we have $E(-1) \cong \ker b' / \operatorname{im} a'$. Tensoring $\mathcal{O}(-1)$, we obtain the complex

$$\mathcal{O}_{\mathbb{CP}^2}(-1) \otimes H^1(\mathbb{CP}^2, E(-2)) \xrightarrow{a} \mathcal{O}_{\mathbb{CP}^2} \otimes H^1(\mathbb{CP}^2, E(-1) \otimes \mathcal{Q}^\vee) \xrightarrow{b} \mathcal{O}_{\mathbb{CP}^2}(1) \otimes H^1(\mathbb{CP}^2, E(-1))$$

with

$$\ker a = 0, \quad \operatorname{coker} b = 0, \quad \ker b / \operatorname{im} a \cong E.$$

This complex is called the *monad* for E .

6.3.3. Quiver description. The framing φ enables us to make the monad description into a simpler form.

Set

$$\widetilde{W} := H^1(\mathbb{CP}^2, E(-1) \otimes \mathcal{Q}^\vee), \quad V := H^1(\mathbb{CP}^2, E(-2)), \quad V' := H^1(\mathbb{CP}^2, E(-1)).$$

By the Riemann-Roch theorem, we have

$$\dim V = \dim V' = c_2(E), \quad \dim \widetilde{W} = 2c_2(E) + \operatorname{rank} E.$$

The monad of E is written as

$$\mathcal{O}_{\mathbb{CP}^2}(-1) \otimes V \xrightarrow{a} \mathcal{O}_{\mathbb{CP}^2} \otimes \widetilde{W} \xrightarrow{b} \mathcal{O}_{\mathbb{CP}^2}(1) \otimes V',$$

with

$$\ker a = 0, \quad \operatorname{coker} b = 0, \quad \ker b / \operatorname{im} a \cong E.$$

Since $a \in \operatorname{Hom}(\mathcal{O}_{\mathbb{CP}^2}(-1) \otimes V, \mathcal{O}_{\mathbb{CP}^2} \otimes \widetilde{W}) \cong H^0(\mathbb{CP}^2, \mathcal{O}_{\mathbb{CP}^2}(1)) \otimes \operatorname{Hom}(V, \widetilde{W})$, we can write

$$a = z_0 a_0 + z_1 a_1 + z_2 a_2, \quad a_i \in \operatorname{Hom}(V, \widetilde{W}).$$

Similarly b can be written as

$$b = z_0 b_0 + z_1 b_1 + z_2 b_2, \quad b_i \in \operatorname{Hom}(\widetilde{W}, V').$$

(Recall the homogeneous coordinate $[z_0 : z_1 : z_2]$ of $\mathbb{CP}^2$.)

Restricting the complex to ℓ_∞ , we have

$$\mathcal{O}_{\ell_\infty}(-1) \otimes V \xrightarrow{a|_{\ell_\infty}} \mathcal{O}_{\ell_\infty} \otimes \widetilde{W} \xrightarrow{b|_{\ell_\infty}} \mathcal{O}_{\ell_\infty}(1) \otimes V'$$

with $a|_{\ell_\infty} = z_1 a_1 + z_2 a_2$ and $b|_{\ell_\infty} = z_1 b_1 + z_2 b_2$. Then taking the cohomology of the exact sequence

$$0 \rightarrow \mathcal{O}_{\ell_\infty}(-1) \otimes V \xrightarrow{a|_{\ell_\infty}} \ker b|_{\ell_\infty} \rightarrow E|_{\ell_\infty} \rightarrow 0,$$

we have

$$\begin{aligned} 0 \rightarrow H^0(\ell_\infty, \mathcal{O}_{\ell_\infty}(-1)) \otimes V &\rightarrow H^0(\ell_\infty, \ker b|_{\ell_\infty}) \rightarrow H^0(\ell_\infty, E|_{\ell_\infty}) \\ &\rightarrow H^1(\ell_\infty, \mathcal{O}_{\ell_\infty}(-1)) \otimes V \rightarrow H^1(\ell_\infty, \ker b|_{\ell_\infty}) \rightarrow H^1(\ell_\infty, E|_{\ell_\infty}) \rightarrow 0. \end{aligned}$$

Since $H^i(\ell_\infty, \mathcal{O}_{\ell_\infty}(-1)) = 0$ for $i = -1, 1$ and $E|_{\ell_\infty} \cong \mathcal{O}_{\ell_\infty}$, we have

$$H^1(\ell_\infty, \ker b|_{\ell_\infty}) = 0, \quad H^0(\ell_\infty, \ker b|_{\ell_\infty}) \cong E_p,$$

where E_x is the fiber of E at $p \in \ell_\infty$.

Similarly from the exact sequence

$$0 \rightarrow \ker b|_{\ell_\infty} \rightarrow \mathcal{O}_{\ell_\infty} \otimes \widetilde{W} \xrightarrow{b|_{\ell_\infty}} \mathcal{O}_{\ell_\infty}(1) \otimes V' \rightarrow 0,$$

we have

$$0 \rightarrow H^0(\ell_\infty, \ker b|_{\ell_\infty}) \rightarrow \widetilde{W} \xrightarrow{b|_{\ell_\infty}} H^0(\ell_\infty, \mathcal{O}_{\ell_\infty}(1)) \otimes V' \rightarrow 0,$$

Identifying $H^0(\ell_\infty, \mathcal{O}_{\ell_\infty}(1)) \otimes V' = (\mathbb{C}z_1 \oplus \mathbb{C}z_2) \otimes V' \cong V' \oplus V'$, we can rewrite the last exact sequence as

$$0 \rightarrow H^0(\ell_\infty, \ker b|_{\ell_\infty}) \rightarrow \widetilde{W} \xrightarrow{\begin{pmatrix} b_1 \\ b_2 \end{pmatrix}} V' \oplus V' \rightarrow 0,$$

We put

$$W := H^0(\ell_\infty, \ker b|_{\ell_\infty}) \cong \ker b_1 \cap \ker b_2.$$

Applying the same argument to the dual complex

$$0 \rightarrow \mathcal{O}_{\ell_\infty}(-1) \otimes V'^* \xrightarrow{^t b|_{\ell_\infty}} \mathcal{O}_{\ell_\infty} \otimes \widetilde{W}^* \xrightarrow{^t a|_{\ell_\infty}} \mathcal{O}_{\ell_\infty}(1) \otimes V^* \rightarrow 0,$$

we have the exact sequence

$$0 \rightarrow H^0(\ell_\infty, \ker ^t a|_{\ell_\infty}) \rightarrow \widetilde{W}^* \xrightarrow{\begin{pmatrix} a_1 \\ a_2 \end{pmatrix}} V^* \oplus V^* \rightarrow 0.$$

Therefore the linear map $(a_1, a_2) : V \otimes V \rightarrow \widetilde{W}$ is injective, so that $a|_{\ell_\infty}$ is injective on each fiber. Considering the fiber at $p := [0 : 1 : 0] \in \ell_\infty$, we have

$$V \xrightarrow{a_1} \widetilde{W} \xrightarrow{b_1} V'$$

with $\ker b_1 / \text{im } a_1 = E_p$. Since $W = \ker b_1 \cap \ker b_2 \cong E_p$, we have $\text{im } a_1 \cap \ker b_2 = \text{im } a_2 \cap W = 0$. Then from

$\dim V = \dim V'$, it holds that $b_2 a_1 : V \rightarrow V'$ is bijective.

Now we can identify $\widetilde{W}$ with $V \otimes V \otimes W$ by the sequence

$$V \xrightarrow{(a_1, a_2)} \widetilde{W} \xrightarrow{\begin{pmatrix} b_1 \\ b_2 \end{pmatrix}} V'.$$

We can also identify V' with V via $b_1 a_2 = -b_2 a_1$. (The identity $b_1 a_2 = -b_2 a_1$ comes from $ba = 0$.) Then we have

$$a_1 = \begin{pmatrix} -\text{id}_V \\ 0 \\ 0 \end{pmatrix}, \quad a_2 = \begin{pmatrix} 0 \\ -\text{id}_V \\ 0 \end{pmatrix}, \quad b_1 = (0, -\text{id}_V, 0), \quad b_2 = (\text{id}_V, 0, 0),$$

The condition $ba = 0$ yields

$$a_0 = \begin{pmatrix} B_1 \\ B_2 \\ j \end{pmatrix}, \quad b_0 = (-B_2, B_1, i)$$

with

$$[B_1, B_2] + ij = 0. \tag{6.3.1}$$

Summarizing the arguments, the monad is rewritten in to a simpler form

$$\begin{array}{ccccc} & & V \otimes \mathcal{O}_{\mathbb{CP}^2} & & \\ & & \oplus & & \\ V \otimes \mathcal{O}_{\mathbb{CP}^2}(-1) & \xrightarrow{a} & V \otimes \mathcal{O}_{\mathbb{CP}^2} & \xrightarrow{b} & V \otimes \mathcal{O}_{\mathbb{CP}^2}(1) \\ & & \oplus & & \\ & & W \otimes \mathcal{O}_{\mathbb{CP}^2} & & \end{array}$$

with

$$a = \begin{pmatrix} z_0 B_1 - z_1 \\ z_0 B_2 - z_2 \\ z_0 j \end{pmatrix}, \quad b = (-(z_0 b_2 - z_2), z_0 B_1 - z_1, z_0 i). \quad (6.3.2)$$

Conversely we have

FACT 6.3.3 ([69, Lemma 2.7]). Suppose that (B_1, B_2, i, j) satisfies (6.3.1). Then definig a, b as (6.3.2), one has

- (1) $\ker a = 0$.
- (2) b is surjective if and only if there exists no proper subspace $S \subsetneq V$ such that $B_\alpha(S) \subset S$ ($\alpha = 1, 2$) and $\text{im } i \subset S$.

Thus we obtain Theorem 6.3.1.

6.3.4. Tangent space. As an application of the quiver description, we give a presentation of the tangent space of the framed moduli space.

Let (B_1, B_2, i, j) be a datum as above. We consider a complex

$$\begin{array}{ccc} & \text{Hom}(V, V) \oplus \text{Hom}(V, V) & \\ \text{Hom}(V, V) \xrightarrow{\sigma} & \oplus & \xrightarrow{\tau} \text{Hom}(V, V), \\ & \text{Hom}(W, V) \oplus \text{Hom}(V, W) & \end{array} \quad (6.3.3)$$

with

$$\sigma(\xi) := \begin{pmatrix} [\xi, B_1] \\ [\xi, B_2] \\ \xi i \\ -j\xi \end{pmatrix}, \quad \tau \begin{pmatrix} C_1 \\ C_2 \\ I \\ J \end{pmatrix} := [B_1, C_2] + [C_1, B_2] + iJ + Ij.$$

Note that σ is the differential of $\text{GL}(V)$ -action, and τ is the differential of the map $(B_1, B_2, i, j) \mapsto [B_1, B_2] + ij$. By the stability condition, it is easy to see that σ is injective and τ is surjective. Then the tangent space of $M(r, n)$ at $\text{GL}(V) \cdot (B_1, B_2, i, j)$ is isomorphic to the middle cohomology group of the above complex.

CHAPTER 7

Equivariant homology theory

In this chapter we briefly recall the equivariant homology and the equivariant K -theory. These are necessary for the definition of Nekrasov's partition function. as mentioned in Introduction.

7.1. Borel-Moore homology

The chief object of this section is the review of the equivariant Borel-Moore homology. We start with the recollection of the ordinary Borel-Moore homology.

7.1.1. Ordinary Borel-Moore homology. This section gives a brief recollections of the ordinary Borel-Moore homology, following [41, §B.2] and [69, §8.2].

Denote by $H^*(X, Y)$ the singular cohomology groups for a topological space X and its subspace Y . They are defined to be the cohomology group of the complex $C^*(X, Y)$ of singular cochains ($\mathbb{Z}$ -valued functions on singular chains) on X that vanish on the chains in Y . If $Z \subset Y \subset X$, then there is a long exact sequence

$$\cdots \rightarrow H^i(X, Y) \rightarrow H^i(X, Z) \rightarrow H^i(Y, Z) \rightarrow H^{i+1}(X, Y) \rightarrow H^{i+1}(X, Z) \rightarrow \cdots. \quad (7.1.1)$$

Let us recall two fundamental notions. The first is the excision. Let Y be an open subset of X , and A be a closed subset of X contained in Y . Then the natural map

$$H^i(X, Y) \rightarrow H^i(X \setminus A, Y \setminus A)$$

is an isomorphism, called the excision.

The second fundamental notion is the Thom class. Let $\pi : E \rightarrow X$ be an oriented real vector bundle of rank r on a topological space X . Denote by $\{0\} \subset E$ the image of the zero section. The Thom class γ_E is an element of $H^r(E, E \setminus \{0\})$. It satisfies that for any closed subset $A \subset X$, the map

$$H^i(X, X \setminus A) \rightarrow H^{i+r}(E, E \setminus A), \quad \alpha \mapsto \pi^*(\alpha) \cup \gamma_E$$

is an isomorphism.

Suppose X is a closed submanifold of a differentiable manifold M . Let N be the normal bundle of X in M . If N is oriented, then for any closed subset $A \subset X$ we have a canonical isomorphism

$$H^i(X, X \setminus A) \cong H^{i+\dim X - \dim M}(M, M \setminus A). \quad (7.1.2)$$

This is given as follows. Choose a tubular neighborhood U of X in M .

$$H^i(X, X \setminus A) \cong H^i(N, N \setminus A) \cong H^{i+\dim X - \dim M}(U, U \setminus A) \cong H^{i+\dim X - \dim M}(M, M \setminus A),$$

where the first isomorphism is the one induced by the Thom class, the second is the identification of N with U , and the third is the excision of $M \setminus U$.

Now for a topological space X that can be embedded as a closed subspace of $\mathbb{R}^m$, we define a version of Borel-Moore homology groups by

$$H_*^{\text{lf}}(X) := H^{m-i}(\mathbb{R}^m, \mathbb{R}^m \setminus X)$$

The definition of $H_*^{\text{lf}}(X)$ is independent of choices of the embedding $X \hookrightarrow \mathbb{R}^m$.

Below we list several fundamental properties of the Borel-Moore homology.

- (1) If X is embedded as a closed subspace of an oriented differentiable manifold M , then we have

$$H_i^{\text{lf}}(X) = H^{\dim M - i}(M, M \setminus X)$$

by (7.1.2). In particular, if X itself is an oriented differentiable manifold, then

$$H_i^{\text{lf}}(X) = H^{\dim X - i}(X).$$

- (2) For a proper continuous map $f : X \rightarrow Y$, there is a push-forward homomorphism $f_* : H_*^{\text{lf}}(X) \rightarrow H_*^{\text{lf}}(Y)$. For another proper continuous map $g : Y \rightarrow Z$, we have $(g \circ f)_* = g_* \circ f_*$.
- (3) Suppose that F is a closed subset of a space X that admits a closed embedding in an oriented manifold M . Denote by $i : F \hookrightarrow X$ the embedding. Set $U := X \setminus F$, and denote by $j : U \hookrightarrow X$ the inclusion. The restriction map in relative cohomology induces

$$j^* : H_i^{\text{lf}}(X) = H^{\dim M - i}(M, M \setminus X) \longrightarrow H^{\dim M - i}(M \setminus F, M \setminus U) = H_i^{\text{lf}}(U).$$

Then (7.1.1) induces the natural long exact sequence

$$\cdots \rightarrow H_i^{\text{lf}}(Y) \xrightarrow{\iota_*} H_i^{\text{lf}}(X) \xrightarrow{j^*} H_i(U)^{\text{lf}} \xrightarrow{\delta_*} H_{i-1}(Y)^{\text{lf}} \rightarrow \cdots,$$

where δ_* is the boundary homomorphism.

- (4) If X is an N -dimensional complex subvariety of a non-singular complex manifold M , then

$$H_i^{\text{lf}}(X) = 0 \quad \text{for } i > 2n,$$

and the fundamental classes of n -dimensional irreducible components form a basis of $H_{2n}^{\text{lf}}(X)$. If X is irreducible, then the fundamental class is well-defined, and we denote it by $[X]$.

7.1.2. Equivariant Borel-Moore homology. Let us recall the definition of the equivariant Borel-Moore homology briefly. We follow the description in [71, Appendix C].

In this subsection, only the Borel-Moore homology is used, so we denote it simply by $H_*(\bullet)$, instead of $H_*^{\text{lf}}(X)$. The coefficient is the complex one unless otherwise stated. We also assume that everything is defined over $\mathbb{C}$.

Let G be a linear algebraic group acting on an algebraic variety X . We have a finite dimensional approximation of the classifying space $EG \rightarrow BG$, that is, for any $n \in \mathbb{Z}_{\geq 1}$, there exists a smooth irreducible variety U with G -action such that

- The quotient $U \rightarrow U/G$ exists and is a principal G -bundle.
- $H^i(U) = 0$ for $i = 1, \dots, n$.

We then define

$$H_n^G(X) := H_{n - 2 \dim G + 2 \dim U}(X \times_G U).$$

This definition is in fact independent of the choice of U .

Note that

$$H_n^G(X) = 0 \quad \text{for } n > 2 \dim X$$

by the property (4) of the ordinary Borel-Moore homology, but $H_n^G(X)$ may be nonzero for $n < 0$.

Since a projection $X \times_G U \rightarrow U/G$ is flat, $H_*^G(X)$ has a structure of a $H_G^*(\text{pt})$ -module.

Suppose that G is reductive. Then

$$H_G^*(\text{pt}) \cong S^*(\mathfrak{h}^*)^W,$$

where $\mathfrak{h}$ is a Cartan subalgebra of the Lie algebra $\mathfrak{g} := \text{Lie}(G)$, $S^*(\mathfrak{h}^*)$ is the symmetric algebra of its dual, and W is the Weyl group of G . We denote this by S or $S(G)$.

Let T be a torus acting on X . Let X^T be the fixed point set and $\iota: X^T \rightarrow X$ be the inclusion. By the property (1) of the ordinary Borel-Moore homology, we have the push-forward homomorphism $\iota_*: H_*^T(X^T) \rightarrow H_*(X)$. Since T acts trivially on X^T , we have

$$H_*^T(X^T) = H_*(X^T) \otimes_{\mathbb{C}} S$$

By the localization theorem [4], ι_* becomes an isomorphism after tensoring the quotient field $Q(S)$ of S .

If X is nonsingular, the inverse of ι_* can be explicitly given. Let $X^T = \bigsqcup F_i$ be the decomposition to irreducible components. Let N_i be the normal bundle of F_i in X . Then we have

$$(\iota_*)^{-1} = \sum_i \frac{1}{e_T(N_i)} \iota_i^*,$$

where $e_T(N_i)$ is the equivariant Euler class, and ι_i^* is the pull-back homomorphism for the inclusion $\iota_i: F_i \hookrightarrow X$.

7.2. K -theory

We will quickly review the equivariant K -theory of a quasi-projective variety with a group action. We assume some knowledge for the ordinary K -theory in the algebro-geometric settings. See [23, Chapter 5] and [70, §6] for further details.

In this section, X is a quasi-projective variety over $\mathbb{C}$ with algebraic action of a linear algebraic group G . Let $K^G(X)$ be the Grothendieck group of the abelian category of G -equivariant coherent sheaves on X . It is a module over $R(G)$, the representation ring of G .

Let $K_G^0(X)$ be the Grothendieck group of the abelian category of G -equivariant locally free sheaves on X . This is also an $R(G)$ -module.

Below we list several operations on the equivariant K -theory.

- (1) Suppose that Y is a G -invariant closed subvariety of X , and let $U := X \setminus Y$. Two inclusions

$$Y \xrightarrow{i} X \xleftarrow{j} U.$$

induce an exact sequence

$$K^G(Y) \xrightarrow{i_*} K^G(X) \xrightarrow{j^*} K^G(U) \rightarrow 0,$$

where i_* is given by $[E] \mapsto [i_*E]$, and j^* is given by $[E] \mapsto [E|_U]$.

- (2) If E is a G -equivariant vector bundle, its rank and dual vector bundle will be denoted by $\text{rank } E$ and E^* respectively. We extend rank and $*$ to operations on $K_G^0(X)$:

$$\text{rank}: K_G^0(X) \rightarrow \mathbb{Z}^{\pi_0(X)}, \quad *: K_G^0(X) \rightarrow K_G^0(X),$$

where $\pi_0(X)$ is the set of the connected components of X .

- (3) If L is a G -equivariant line bundle, we define $L^{\otimes r} = (L^*)^{\otimes(-r)}$ for $r < 0$. Thus we have $L^{\otimes r} \otimes L^{\otimes s} = L^{\otimes(r+s)}$ for $r, s \in \mathbb{Z}$.

- (4) If E is locally free, then we define

$$\det E := \wedge^{\text{rank } E} E, \quad \wedge_u E := \sum_{i=0}^{\text{rank } E} u^i \wedge^i E.$$

These operations are extended to $K_G^0(X)$ as

$$\det: K_G^0(X) \rightarrow K_G^0(X), \quad \wedge_u: K_G^0(X) \rightarrow [\mathcal{O}_X] + K_G^0(X) \otimes u\mathbb{Z}[[u]].$$

This is well-defined since we have $\det F = \det E \otimes \det G$, $\wedge_u F = \wedge_u E \otimes \wedge_u G$ for an exact sequence $0 \rightarrow E \rightarrow F \rightarrow G \rightarrow 0$.

- (5) Let $f : Y \rightarrow X$ be a G -equivariant morphism between nonsingular G -varieties. Suppose that X' and Y' are G -invariant closed subvarieties of X and Y respectively satisfying $f^{-1}(X') \subset Y'$. Then we have the pull-back homomorphism

$$f^* : K^G(X') \rightarrow K^G(Y').$$

Note that this depends on the ambient spaces X, Y .

- (6) Let $f : X \rightarrow Y$ be a proper G -equivariant morphism between G -varieties (not necessarily nonsingular). Then we have a push-forward homomorphism $f_* : K^G(X) \rightarrow K^G(Y)$ defined by

$$f_*[E] := \sum (-1)^i [R^i f_* E].$$

Suppose further that X and Y are nonsingular. If $X' \subset X$, $Y' \subset Y$ are G -invariant closed subvarieties, the projection formula $f_*(E \otimes_X^L f^* F) = f_* E \otimes_Y^L F$ holds for $E \in K^G(X')$, $F \in K^G(Y')$.

We conclude this section by stating the localization theorem of equivariant K -theory due to Thomason [85].

Let T be a torus acting on a non-singular variety X . Let X^T be the fixed point set and $\iota : X^T \rightarrow X$ be the inclusion. Then the push-forward homomorphism $\iota_* : K^T(X^T) \rightarrow K^T(X)$ becomes an isomorphism after tensoring the quotient field $Q(R)$ of the representation ring $R = R(T) \cong K^T(\text{pt})$.

CHAPTER 8

The Nekrasov partition function

In this chapter we will introduce the main object in the four dimensional side of the AGT conjecture, i.e., the Nekrasov partition function.

8.1. Torus action and fixed points

Let $T \cong (\mathbb{C}^*)^r$ be the maximal torus of $\mathrm{GL}(r, \mathbb{C})$ consisting of diagonal matrices, and let $\tilde{T} := (\mathbb{C}^*)^2 \times T$. We define an action of $\tilde{T}$ on $M(r, n)$ as follows: For $(t_1, t_2) \in (\mathbb{C}^*)^2$, let $F_{(t_1, t_2)}$ be an automorphism of $\mathbb{CP}^2$ defined by

$$F_{(t_1, t_2)}([z_0 : z_1 : z_2]) = [z_0 : t_1 z_1 : t_2 z_2].$$

For $e = \mathrm{diag}(e_1, e_2, \dots, e_r) \in T$, let G_e denotes the isomorphism of $\mathcal{O}_{\ell_\infty}^{\oplus r}$ given by

$$\mathcal{O}_{\ell_\infty}^{\oplus r} \ni (s_1, s_2, \dots, s_r) \mapsto (e_1 s_1, e_2 s_2, \dots, e_r s_r).$$

Then for $(E, \varphi) \in M(r, n)$, we define

$$(t_1, t_2, e) \cdot (E, \varphi) := \left((F_{(t_1, t_2)}^{-1})^* E, \varphi' \right),$$

where φ' is the composite of homomorphisms

$$(F_{(t_1, t_2)}^{-1})^* E|_{\ell_\infty} \xrightarrow{(F_{(t_1, t_2)}^{-1})^* \varphi} (F_{\vec{t}}^{-1})^* \mathcal{O}_{\ell_\infty}^{\oplus r} \longrightarrow \mathcal{O}_{\ell_\infty}^{\oplus r} \xrightarrow{G_e} \mathcal{O}_{\ell_\infty}^{\oplus r}.$$

Here the middle arrow is the homomorphism given by the action of $F_{\vec{t}}$.

In a similar way, we have a $\tilde{T}$ -action on $M_0(r, n)$. The map $\pi : M(r, n) \rightarrow M_0(r, n)$ is $\tilde{T}$ -equivariant.

Recall the quiver description given in § 6.3. In terms of this description, the $\tilde{T}$ -action can be written as follows:

LEMMA 8.1.1 ([72, Lemma 2.8]). The $\tilde{T}$ -action on $M(r, n)$ can be identified with the actions on the linear data defined by

$$(B_1, B_2, i, j) \mapsto (t_1 B_1, t_2 B_2, i e^{-1}, t_1 t_2 e j)$$

for $t_1, t_2 \in \mathbb{C}^*$ and $e = \mathrm{diag}(e_1, \dots, e_r) \in T$.

PROOF. Recall the monad for $E \in M(r, n)$ explained in § 6.3. E is identified with the middle cohomology group of the complex

$$\begin{array}{ccccc} & & V \otimes \mathcal{O}_{\mathbb{CP}^2}(-1) & & \\ & & \oplus & & \\ V \otimes \mathcal{O}_{\mathbb{CP}^2} & \xrightarrow{a} & V \otimes \mathcal{O}_{\mathbb{CP}^2} & \xrightarrow{b} & V \otimes \mathcal{O}_{\mathbb{CP}^2} \\ & & \oplus & & \\ & & W \otimes \mathcal{O}_{\mathbb{CP}^2}(1) & & \end{array}$$

with

$$a = \begin{pmatrix} z_0 B_1 - z_1 \\ z_0 B_2 - z_2 \\ z_0 j \end{pmatrix}, \quad b = (-(z_0 b_2 - z_2), z_0 B_1 - z_1, z_0 i).$$

The pull back of this complex by $F_{(t_1, t_2)}^{-1}$ gives

$$\begin{array}{ccccc} & & V \otimes \mathcal{O}_{\mathbb{CP}^2}(-1) & & \\ & & \oplus & & \\ V \otimes \mathcal{O}_{\mathbb{CP}^2} & \xrightarrow{a'} & V \otimes \mathcal{O}_{\mathbb{CP}^2} & \xrightarrow{b'} & V \otimes \mathcal{O}_{\mathbb{CP}^2} \\ & & \oplus & & \\ & & W \otimes \mathcal{O}_{\mathbb{CP}^2} & & \end{array}$$

with

$$a' = \begin{pmatrix} z_0 B_1 - t_1^{-1} z_1 \\ z_0 t_2^{-1} B_2 - z_2 \\ z_0 j \end{pmatrix}, \quad b' = \begin{pmatrix} -(z_0 b_2 - t_2^{-1} z_2), z_0 B_1 - t_1^{-1} z_1, z_0 i \end{pmatrix}.$$

By the isomorphism

$${}^t(v_1, v_2, w) \mapsto {}^t(t_2^{-1} v_1, t_1^{-1} v_2, w)$$

of $(V \otimes \mathcal{O}_{\mathbb{CP}^2}) \oplus (V \otimes \mathcal{O}_{\mathbb{CP}^2}) \oplus (W \otimes \mathcal{O}_{\mathbb{CP}^2})$, $\ker b'$ is mapped to the kernel of

$$(-(z_0 t_2 b_2 - z_2), z_0 t_1 B_1 - z_1, z_0 i),$$

and $\ker a'$ is mapped to the kernel of

$$a' = \frac{1}{t_1 t_2} \begin{pmatrix} z_0 t_1 B_1 - z_1 \\ z_0 t_2 B_2 - z_2 \\ z_0 t_1 t_2 j \end{pmatrix},$$

Thus the pull-back $(F_{(t_1, t_2)}^{-1})^* E$ is identified with the middle cohomology of the monad associated to the data $(t_1 B_1, t_2 B_2, i, t_1, t_2 j)$. Composing the change of the framing by G_e , we get the claim. $\square$

Note that this action preserves the equation $[B_1, B_2] + ij = 0$ and the stability condition, and commutes with the action of $GL_n(\mathbb{C})$. Hence it induces an action on $M(r, n)$ and $M_0(r, n)$.

Next we turn to the description of the fixed point set $M(r, n)^{\tilde{T}}$.

LEMMA 8.1.2 ([72, Proposition 2.9 (1)]). $M(r, n)^{\tilde{T}}$ consist of

$$(E, \varphi) = (I_1, \varphi_1) \oplus (I_2, \varphi_2) \oplus \cdots \oplus (I_r, \varphi_r)$$

such that

- (i) I_α is an ideal sheaf of 0-dimensional subscheme Z_α contained in $\mathbb{C}^2 = \mathbb{CP}^2 \setminus \ell_\infty$.
- (ii) φ_α is an isomorphism from $(I_\alpha)_{\ell_\infty}$ to the α th factor of $\mathcal{O}_{\ell_\infty}^{\oplus r}$.
- (iii) I_α is fixed by the action of $(\mathbb{C}^*)^2$, coming from that on $\mathbb{CP}^2$.

PROOF. $(E, \varphi) \in M(r, n)$ is fixed by the $\tilde{T}$ -action if and only if E decomposes as

$$E = I_1 \oplus \cdots \oplus I_r \quad (I_\alpha \in M(1, n_\alpha))$$

such that $I_\alpha|_{\ell_\infty}$ is mapped to the α th factor $\mathcal{O}_{\ell_\infty}$ of $\mathcal{O}_{\ell_\infty}^{\oplus r}$ under φ . Since I_α is a torsion-free sheaf with rank $I_\alpha = 1$ and $c_1(I_\alpha) = 0$, it is an ideal sheaf of 0-dimensional subscheme Z_α contained in $\mathbb{C}^2$. Thus the condition (i) and (ii) is obtained. Since E is fixed by the $(\mathbb{C}^*)^2$ -action, we have the condition (iii). $\square$

The fixed point set $M_0(r, n)^{\tilde{T}}$ is described as follows:

LEMMA 8.1.3 ([72, Proposition 2.9 (2)]). $M_0(r, n)^{\tilde{T}}$ consist of the single point $n[0] \in S^n \mathbb{C}^2 \subset M_0(r, n)$, where $[0] = (0, 0) \in \mathbb{C}^2$.

PROOF. It is convenient to use the quiver description. Suppose that the equivalence class of (B_1, B_2, i, j) is fixed by the $(\mathbb{C}^*)^2$ -action. We may assume that (B_1, B_2, i, j) has a closed $GL(n, \mathbb{C})$ -orbit. By Lemma 8.1.1, the equivalence class is fixed if and only if $(t_1 B_1, t_2 B_2, i, t_1 t_2 j)$ lies in the same $GL(n, \mathbb{C})$ -orbit. Since $(t_1 B_1, t_2 B_2, i, t_1 t_2 j)$ converges to $(0, 0, i, 0)$ under $t_1, t_2 \rightarrow 0$, $(0, 0, i, 0)$ lies in the closure of the orbit. But the orbit is closed, so $(0, 0, i, 0)$ must be in the orbit. But $GL(n, \mathbb{C}) \cdot (0, 0, i, 0)$ is closed if and only if $i = 0$. Hence we have the claim. $\square$

We denote

$$\iota_0 : \{n[0]\} \hookrightarrow M_0(r, n) \quad (8.1.1)$$

the inclusion map of this fixed point.

By Lemma 8.1.2, one can parametrize the fixed point by Young diagrams. This parametrization is an extension of [28] for the rank one case. The fixed point set $M(r, n)^{\tilde{T}}$ is indexed by an r -tuple of partitions

$$\vec{\lambda} = (\lambda^{(1)}, \lambda^{(2)}, \dots, \lambda^{(r)}). \quad (8.1.2)$$

The ideal I_α is spanned by monomials $x^i y^j$ placed at $(i-1, j-1)$ outside the Young diagram corresponding to $\lambda^{(\alpha)}$. (See §A.2 for the notation of Young diagram.) $\vec{\lambda}$ satisfies the condition that the total number of boxes $|\vec{\lambda}| := \sum_\alpha |\lambda_\alpha|$ is equal to n .

Now we can compute the $\tilde{T}$ -character of the tangent space of instanton moduli at the fixed point. We will denote by e_α ($\alpha = 1, 2, \dots, r$) the 1-dimensional $\tilde{T}$ -module given by

$$\tilde{T} \ni (t_1, t_2, e) \mapsto e_\alpha,$$

and similarly by t_i ($i = 1, 2$) the 1-dimensional $\tilde{T}$ -modules given by

$$\tilde{T} \ni (t_1, t_2, e) \mapsto t_i.$$

Thus the representation ring $R(\tilde{T})$ is isomorphic to

$$R(\tilde{T}) \cong \mathbb{Z}[t_1^{\pm 1}, t_2^{\pm 1}, e_1^{\pm 1}, e_2^{\pm 1}, \dots, e_r^{\pm 1}], \quad (8.1.3)$$

where e_α^{-1}, t_i^{-1} are the duals of e_α, t_i .

PROPOSITION 8.1.4 ([72, Theorem 2.11]). Let (E, φ) be a fixed point of $\tilde{T}$ -action corresponding to $\vec{\lambda} = (\lambda^{(1)}, \lambda^{(2)}, \dots, \lambda^{(r)})$. Then the $\tilde{T}$ -module structure of $T_{(E, \varphi)} M(r, n)$ is given by

$$T_{(E, \varphi)} M(r, n) = \sum_{\alpha, \beta=1}^r N_{\alpha, \beta}^{\vec{\lambda}}(t_1, t_2, e)$$

with

$$N_{\alpha, \beta}^{\vec{\lambda}}(t_1, t_2, e) := e_\beta e_\alpha^{-1} \times \left\{ \sum_{\square \in \lambda_\alpha} t_1^{-\ell_{\lambda_\beta}(\square)} t_2^{a_{\lambda_\alpha}(\square)+1} + \sum_{\blacksquare \in \lambda_\beta} t_1^{\ell_{\lambda_\alpha}(\blacksquare)+1} t_2^{-a_{\lambda_\beta}(\blacksquare)} \right\}. \quad (8.1.4)$$

PROOF. Suppose $\mathrm{GL}(V) \cdot (B_1, B_2, i, j)$ is fixed by the $\tilde{T}$ -action. Then for any $(t_1, t_2, e) \in \tilde{T}$ there is an element $g(t_1, t_2, e) \in \mathrm{GL}(V)$ such that

$$(t_1 B_1, t_2 B_2, i e^{-1}, t_1 t_2 e j) = g(t_1, t_2, e)^{-1} \cdot (B_1, B_2, i, j).$$

Such $g(t_1, t_2, e)$ is unique, since the $\mathrm{GL}(V)$ -action on the set of stable points is free. Therefore the map

$$(t_1, t_2, e) \mapsto g(t_1, t_2, e)$$

is a group homomorphism. One can regard V as a $\tilde{T}$ -module via it. W is also a $\tilde{T}$ -module via $(t_1, t_2, e) \mapsto e \in \mathrm{GL}(W)$.

Now we want to treat the complex (6.3.3), but it is not $\tilde{T}$ -equivariant. We modify it as

$$\begin{array}{ccc} \mathrm{Hom}(V, V) & \xrightarrow{\sigma} & t_1 \mathrm{Hom}(V, V) \oplus t_2 \mathrm{Hom}(V, V) \\ & & \oplus \\ & & \mathrm{Hom}(W, V) \oplus t_1 t_2 \mathrm{Hom}(V, W) \end{array} \xrightarrow{\tau} t_1 t_2 \mathrm{Hom}(V, V).$$

We have a decomposition

$$W = \bigoplus_{\alpha=1}^r e_\alpha$$

as $\tilde{T}$ -modules. By the stability condition, V decomposes as

$$V = \bigoplus V_\alpha e_\alpha,$$

where V_α is a $(\mathbb{C}^*)$ -module. Thus $\ker \tau / \operatorname{im} \sigma$ decomposes as

$$\bigoplus_{\alpha, \beta} (\ker \tau_{\beta\alpha} / \operatorname{im} \sigma_{\beta\alpha}) e_\beta e_\alpha^{-1}$$

with

$$\operatorname{Hom}(V_\alpha, V_\beta) \xrightarrow{\sigma_{\beta\alpha}} \frac{t_1 \operatorname{Hom}(V_\alpha, V_\beta) \oplus t_2 \operatorname{Hom}(V_\alpha, V_\beta)}{\operatorname{Hom}(W_\alpha, V_\beta) \oplus t_1 t_2 \operatorname{Hom}(V_\alpha, W_\beta)} \xrightarrow{\tau_{\beta\alpha}} t_1 t_2 \operatorname{Hom}(V_\alpha, V_\beta).$$

Each summand has the weight $e_\beta e_\alpha^{-1}$ with respect to $(\mathbb{C}^*)^r$, so we suppress this factor, and only consider the $(\mathbb{C}^*)^2$ -weight hereafter.

By the description of the fixed point (Lemma 8.1.2), V_α has a basis $\{x^{i-1}y^{j-1} \mid (i, j) \in \lambda^{(\alpha)}\}$. Then, denoting $\lambda^{(\alpha)} = (\lambda^{(\alpha)}_1, \lambda^{(\alpha)}_2, \dots)$, we have

$$V_\alpha = \sum_{j=1}^{\lambda^{(\alpha)}_1} \sum_{i=1}^{\lambda^{(\alpha)}_j^\vee} t_1^{-i+1} t_2^{-j+1} = \sum_{i=1}^{\lambda^{(\alpha)}_1^\vee} \sum_{j=1}^{\lambda^{(\alpha)}_i} t_1^{-i+1} t_2^{-j+1}.$$

Since $\lambda^{(\alpha)}_1^\vee = \ell(\lambda^{(\alpha)})$, we get

$$\begin{aligned} & (t_1 + t_2 - 1 - t_1 t_2) V_\alpha^* \otimes V_\beta \\ &= \sum_{i=1}^{\ell(\lambda^{(\alpha)})} \sum_{j'=1}^{\lambda^{(\alpha)}_i} t_1^{-i-1} t_2^{j'-1} (t_2 - 1) \times \sum_{j=1}^{\lambda^{(\beta)}_1} \sum_{i'=1}^{\lambda^{(\beta)}_j^\vee} t_1^{-i'+1} (1 - t_1) t_2^{-j+1} \\ &= \sum_{i=1}^{\ell(\lambda^{(\alpha)})} \sum_{j=1}^{\lambda^{(\beta)}_1} \left(t_1^{i-\lambda^{(\beta)}_j^\vee} - t_1^i \right) \left(t_2^{-j+\lambda^{(\alpha)}_i+1} - t_2^{-j+1} \right) \\ &= \sum_{i=1}^{\ell(\lambda^{(\alpha)})} \sum_{j=1}^{\lambda^{(\beta)}_1} \left[t_1^{i-\lambda^{(\beta)}_j^\vee} t_2^{1-j+\lambda^{(\alpha)}_i} - t_1^i t_2^{1-j} - \left(t_1^{i-\lambda^{(\beta)}_j^\vee} - t_1^i \right) t_2^{1-j} - t_1^i \left(t_2^{1-j+\lambda^{(\alpha)}_i} - t_2^{1-j} \right) \right]. \end{aligned}$$

Noticing that

$$\begin{aligned} \sum_{i=1}^{\ell(\lambda^{(\alpha)})} \sum_{j=1}^{\lambda^{(\beta)}_1} \left(t_1^{i-\lambda^{(\beta)}_j^\vee} - t_1^i \right) t_2^{-j+1} &= \sum_{j=1}^{\lambda^{(\beta)}_1} \sum_{i=1}^{\lambda^{(\beta)}_j^\vee} \left(t_1^{1-i} - t_1^{\lambda^{(\alpha)}_1^\vee - i + 1} \right) t_2^{-j+1} \\ &= V_\beta - \sum_{j=1}^{\lambda^{(\beta)}_1} \sum_{i=1}^{\lambda^{(\beta)}_j^\vee} t_1^{\lambda^{(\alpha)}_1^\vee - i + 1} t_2^{-j+1}, \\ \sum_{i=1}^{\ell(\lambda^{(\alpha)})} \sum_{j=1}^{\lambda^{(\beta)}_1} t_1^i (t_2^{-j+\lambda^{(\alpha)}_i+1} - t_2^{-j+1}) &= t_1 t_2 V_\alpha^* - \sum_{i=1}^{\lambda^{(\alpha)}_1^\vee} \sum_{j=1}^{\lambda^{(\alpha)}_i} t_1^i t_2^{-\lambda^{(\beta)}_1+j}, \end{aligned}$$

we have

$$\begin{aligned} N_{\alpha, \beta}(t_1, t_2) &:= \ker \tau_{\beta\alpha} / \operatorname{im} \sigma_{\beta\alpha} \\ &= (t_1 + t_2 - 1 - t_1 t_2) V_\alpha^* \otimes V_\beta + V_\beta + t_1 t_2 V_\alpha^* \\ &= \sum_{i=1}^{\ell(\lambda^{(\alpha)})} \sum_{j=1}^{\lambda^{(\beta)}_1} \left(t_1^{i-\lambda^{(\beta)}_j^\vee} t_2^{1-j+\lambda^{(\alpha)}_i} - t_1^i t_2^{1-j} \right) + \sum_{j=1}^{\lambda^{(\beta)}_1} \sum_{i=1}^{\lambda^{(\beta)}_j^\vee} t_1^{\lambda^{(\alpha)}_1^\vee - i + 1} t_2^{1-j} + \sum_{i=1}^{\ell(\lambda^{(\alpha)})} \sum_{j=1}^{\lambda^{(\alpha)}_i} t_1^i t_2^{-\lambda^{(\beta)}_1+j}. \end{aligned} \tag{8.1.5}$$

Let us decompose it as

$$N_{\alpha,\beta}(t_1, t_2) = N_{\alpha,\beta}^{>0}(t_1, t_2) + N_{\alpha,\beta}^{\leq 0}(t_1, t_2).$$

Then

$$\begin{aligned} N_{\alpha,\beta}^{>0}(t_1, t_2) &= \sum_{i=1}^{\ell(\lambda^{(\alpha)})} \sum_{j=1}^{\min(\lambda^{\beta_1}, \lambda^{(\alpha)}_i)} t_1^{i-\lambda^{(\beta)}_j} t_2^{-j+\lambda^{(\alpha)}_i+1} + \sum_{i=1}^{\ell(\lambda^{(\alpha)})} \sum_{j=\lambda^{(\beta)}_1+1}^{\lambda^{(\alpha)}_i} t_1^i t_2^{-\lambda^{(\beta)}_1+j} \\ &= \sum_{\square \in \heartsuit \lambda^{(\alpha)}} t_1^{-\ell_{\lambda^{(\beta)}}(\square)} t_2^{a_{\lambda^{(\alpha)}}(\square)+1} + \sum_{\blacksquare \in \spadesuit \lambda^{(\alpha)}} t_1^{i(\blacksquare)} t_2^{j(\blacksquare)-\ell(Y'_\beta)+1}. \end{aligned} \quad (8.1.6)$$

Here $\heartsuit \lambda^{(\alpha)}$ and $\spadesuit \lambda^{(\alpha)}$ are defined in the appendix § A.2.

By the symmetry $N_{\alpha,\beta}(t_1, t_2) = N_{\beta,\alpha}(t_1^{-1}, t_2^{-1})t_1 t_2$, we have

$$N_{\alpha,\beta}^{\leq 0}(t_1, t_2) = \sum_{\square \in \heartsuit \lambda^{(\beta)}} t_1^{\ell_{\lambda^{(\alpha)}}(\square)+1} t_2^{-a_{\lambda^{(\beta)}}(\square)} + \sum_{\blacksquare \in \spadesuit \lambda^{(\beta)}} t_1^{-i(\blacksquare)+1} t_2^{-j(\blacksquare)+1+\lambda^{(\alpha)}_1}. \quad (8.1.7)$$

Combining (8.1.6) with (8.1.7), and using $-\ell_{\lambda^{(\beta)}}(\blacksquare) = i(\blacksquare)$ for $\blacksquare \in \spadesuit Y_\alpha$, we get the assertion. $\square$

8.2. Equivariant integration

Before giving the definition of the partition function, we explain a general setting for the equivariant integration via the localization theorem. We use the equivariant Borel-Moore homology, which was briefly reviewed in §7.1.

Let T be a torus acting on an algebraic variety X defined over $\mathbb{C}$. Suppose that the fixed point set X^T consists of a single point x . Then the push-forward homomorphism for the inclusion $\iota : \{x\} \rightarrow X$ induces an isomorphism between localized equivariant homology groups

$$\iota_* : Q(S) \cong H_*^T(x) \otimes_S Q(S) \xrightarrow{\sim} H_*^T(X) \otimes_S Q(S),$$

where $S := H_T^*(\text{pt})$ and $Q(S)$ is its quotient field.

In the next subsection, we take as T the group $\tilde{T}$. Recall that S is isomorphic to $S(\text{Lie}(\tilde{T}^*))$, the symmetric algebra of the dual of the Lie algebra of $\tilde{T}$. We choose its generators $\varepsilon_1, \varepsilon_2, a_1, a_2, \dots, a_r$ corresponding to $t_1, t_2, e_1, e_2, \dots, e_r$ respectively (see (8.1.3)). We use the vector notation $\vec{a}$ for $(a_1, a_2, \dots, a_r)$.

If $f : Y \rightarrow X$ is a T -equivariant proper morphism, we can define a map $H_*^T(Y) \rightarrow Q(S)$ by

$$\alpha \mapsto \iota_*^{-1} f_* \alpha.$$

We denote this map by $\int_Y$.

Suppose Y is nonsingular. Let $Y^T = \bigsqcup_i F_i$ be the decomposition of the fixed point set Y^T to irreducible components. Let N_i be the normal bundle of F_i . Note that we only have finitely many components, and each F_i is compact since f is proper and $X^T = \{x\}$. Then we have

$$\int_X \alpha = \sum_i \int_{F_i} \frac{1}{e_T(F_i)} \iota_i^* \alpha, \quad (8.2.1)$$

where $e_T(N_i)$ is the equivariant Euler class and ι_i^* is the pull-back homomorphism for the inclusion $\iota_i : F_i \rightarrow Y$.

8.3. Definition of partition function

DEFINITION 8.3.1 ([72, §6]). The Nekrasov partition function is defined as the following generating function:

$$Z(\varepsilon_1, \varepsilon_2, \vec{a}; x) = \sum_{n=0}^{\infty} x^n Z_n(\varepsilon_1, \varepsilon_2, \vec{a}) = \sum_{n=0}^{\infty} x^n (\iota_{0*})^{-1} \pi_* [M(r, n)],$$

where $[M(r, n)]$ denotes the fundamental class of $H_*^{\tilde{T}}(M(r, n))$, ι_0 is given by (8.1.1), and π is given by (6.2.1).

REMARK 8.3.2. In physics literatures, this function is called the pure $SU(r)$ Nekrasov partition function. The word *pure* means that in terms of the equivariant integral the integrand is the fundamental class. If the integrand is the class of the tangent bundle of the moduli, then the partition function is called *Nekrasov partition function with adjoint matter*.

By (8.1.4) and (8.2.1), we have

$$Z(\varepsilon_1, \varepsilon_2, \vec{a}; x) = \sum_{\vec{\lambda}} \frac{x^{|\vec{\lambda}|}}{e(T_{\vec{\lambda}})} = \sum_{\vec{\lambda}} \frac{x^{|\vec{\lambda}|}}{\prod_{\alpha, \beta} n_{\alpha, \beta}^{\vec{\lambda}}(\varepsilon_1, \varepsilon_2, \vec{a})},$$

where

$$\begin{aligned} n_{\alpha, \beta}^{\vec{\lambda}}(\varepsilon_1, \varepsilon_2, \vec{a}) := & \prod_{\square \in Y_{\alpha}} (-\ell_{\lambda_{\beta}}(\square) \varepsilon_1 + (a_{\lambda_{\alpha}}(\square) + 1) \varepsilon_2 + a_{\beta} - a_{\alpha}) \\ & \times \prod_{\blacksquare \in \lambda_{\beta}} ((\ell_{\lambda_{\alpha}}(\blacksquare) + 1) \varepsilon_1 - a_{\lambda_{\beta}}(\blacksquare) \varepsilon_2 + a_{\beta} - a_{\alpha}). \end{aligned}$$

This is nothing but Nekrasov's definition of the partition function [74, (1.6), (3.20)].

8.4. K -theoretic partition function

In this section, we introduce a K -theoretic partition function, following [73]. For the brief introduction to equivariant K -theory, see §7.2

Let $K^{\tilde{T}}(M(r, n))$ be the Grothendieck group of $\tilde{T}$ -equivariant coherent sheaves on $M(r, n)$, and similarly let us write $K^{\tilde{T}}(M_0(r, n))$. These are modules over the representation ring $R(\tilde{T})$ of the torus $\tilde{T}$. As in (8.1.3), we identify it with the Laurent polynomial ring $\mathbb{Z}[t_1^{\pm}, t_2^{\pm}, e_1^{\pm}, \dots, e_r^{\pm}]$.

Since $M(r, n)$ is nonsingular, $K^{\tilde{T}}(M(r, n))$ is isomorphic to the Grothendieck group of $\tilde{T}$ -equivariant vector bundles. The proper equivariant morphism π induces a homomorphism

$$\pi_* : K^{\tilde{T}}(M(r, n)) \longrightarrow K^{\tilde{T}}(M_0(r, n))$$

by taking the alternating sum of higher direct image sheaves $\sum_i (-1)^i R^i \pi_*$.

Recall that the fixed points $M_0(r, n)^{\tilde{T}}$ consist of the single point $n[0] \in S^n \mathbb{C}^2 \subset M_0(r, n)$. Let ι_0 denote the inclusion map of the fixed point. By the localization theorem for the equivariant K -theory [85], the homomorphism ι_{0*} is an isomorphism after the localization:

$$\iota_{0*} : Q(R) \cong K^{\tilde{T}}(M_0(r, n)^{\tilde{T}}) \otimes_R Q(R) \xrightarrow{\sim} K^{\tilde{T}}(M_0(r, n)) \otimes_R Q(R),$$

where $Q(R) = \mathbb{Q}(t_1, t_2, e_1, e_2, \dots, e_r)$ is the quotient field of $R(\tilde{T})$.

DEFINITION 8.4.1. The K -theoretic Nekrasov partition function is defined as

$$\begin{aligned} Z^K(t_1, t_2, e; x) &:= \sum_{n \in \mathbb{Z}_{\geq 0}} \left(x(t_1 t_2)^{-r/2} \right)^n Z_n(t_1, t_2, e), \\ Z_n^K(t_1, t_2, e) &:= (\iota_{0*})^{-1} \pi_* ([\mathcal{O}_{M(r, n)}]). \end{aligned}$$

Here $[\mathcal{O}_{M(r, n)}]$ is the element in $K^{\tilde{T}}(M(r, n))$ corresponding to $\mathcal{O}_{M(r, n)}$.

Recall the parametrization (8.1.2) of the fixed point set $M(r, n)^{\tilde{T}}$ by an r -tuple of partitions $\vec{\lambda} = (\lambda^{(1)}, \lambda^{(2)}, \dots, \lambda^{(r)})$. The constraint is that the total number of boxes $|\vec{\lambda}| := \sum_{\alpha} |\lambda^{(\alpha)}|$ is equal to n . Let ι denote the inclusion map of the fixed point set. By the localization theorem, we have

$$\iota_*: \bigoplus_{\vec{\lambda}} Q(R) \cong K^{\tilde{T}}(M(r, n)^{\tilde{T}}) \otimes_R Q(R) \xrightarrow{\sim} K^{\tilde{T}}(M(r, n)) \otimes_R Q(R).$$

Therefore we have

$$Z_n^K(\varepsilon_1, \varepsilon_2, \vec{a}; x) = \sum_{\vec{\lambda}} (\iota_*)^{-1}(\mathcal{O}_{M(r, n)}).$$

As $M(r, n)$ is nonsingular, $(\iota_*)^{-1}$ can be explicitly given by Atiyah-Bott Lefschetz fixed points formula.

$$\iota_*^{-1}(\bullet) = \bigoplus_{\vec{\lambda}} \frac{\iota_{\vec{\lambda}}^*(\bullet)}{\wedge_{-1} T_{\vec{\lambda}}^* M(r, n)},$$

where $T_{\vec{\lambda}}^* M(r, n)$ is the cotangent bundle of $M(r, n)$ at a fixed point of $\vec{\lambda}$ considered as a $\tilde{T}$ -module, $\wedge_{-1}$ is the alternating sum of exterior powers, and $\iota_{\vec{\lambda}}^*$ is the pull-back homomorphism induced by the inclusion $\iota_{\vec{\lambda}}: \{\vec{\lambda}\} \hookrightarrow M(r, n)$.

The term $\wedge_{-1} T_{\vec{\lambda}}^*(M(r, n))$ can be computed by Proposition . The result is

$$Z^K(t_1, t_2, e; x) = \sum_{\vec{\lambda}} \frac{(x(t_1 t_2)^{-r/2})^{|\vec{\lambda}|}}{\wedge_{-1} T_{\vec{\lambda}}^* M(r, n)} = \sum_{\vec{\lambda}} \frac{x^{|\vec{\lambda}|}}{\prod_{\alpha, \beta} n_{\alpha, \beta}^{K, \vec{\lambda}}(t_1, t_2, e)}, \quad (8.4.1)$$

where

$$\begin{aligned} n_{\alpha, \beta}^{K, \vec{\lambda}}(t_1, t_2, e) &= \prod_{\square \in \lambda^{(\alpha)}} \left(1 - t_1^{\ell_{\lambda^{(\beta)}}(\square)} t_2^{-(a_{\lambda^{(\alpha)}}(\square)+1)} e_{\beta}^{-1} e_{\alpha} \right) \\ &\quad \times \prod_{\blacksquare \in \lambda_{\beta}} \left(1 - t_1^{-(\ell_{\lambda^{(\alpha)}}(\blacksquare)+1)} t_2^{a_{\lambda^{(\beta)}}(\blacksquare)} e_{\beta}^{-1} e_{\alpha} \right). \end{aligned}$$

We close this section by adding a few remarks.

REMARK 8.4.2. (1) The K -theoretic partition function was written down by Nekrasov [74]. It also appeared in [75]

(2) In [73], the additional factor $(t_1 t_2)^{-r/2}$ appears in the partition function. The factor is the half of the canonical bundle of $M(r, n)$, so that Z_n^K in [73] coincides with the index of the Dirac operator.

(3) This K -theoretic partition function converges to the homological partition function in the last section under the limit

$$t_i = e^{\beta \varepsilon_i} \ (i = 1, 2), \quad e_{\alpha} = e^{\beta a_{\alpha}} \ (\alpha = 1, 2, \dots, r), \quad \beta \rightarrow 0.$$

Part 3

AGT conjectures

CHAPTER 9

SU(2) AGT conjecture

We give a detailed description of the AGT conjecture for the case $G = \text{SU}(2)$. In this case, the two-dimensional side is described by Virasoro algebra, and the four-dimensional side is described by the SU(2) Nekrasov partition function.

9.1. Whittaker vector of Virasoro algebra

Let us recall the definition of Whittaker vector in Verma module of Virasoro algebra, Fix a non-zero complex number Λ . Consider the completed Verma module $\widehat{M}(c, h)$ of $M(c, h)$, where the completion is done with respect to the L_0 -weight gradation $M(c, h) = \bigoplus_{n \in \mathbb{Z}_{\geq 0}} M(c, h)_n$. An element $|G\rangle \in \widehat{M}(c, h)$ is called a Gaiotto state if

$$L_1 |G\rangle = \Lambda^2 |G\rangle, \quad L_n |G\rangle = 0 \quad (n > 1).$$

We normalize $|G\rangle$ by the condition

$$|G\rangle = |c, h\rangle + \cdots,$$

which means that the homogeneous component of $|G\rangle$ in $M(c, h)_0$ is $|c, h\rangle$.

Recall that by Fact 3.3.2 a Gaiotto state exists uniquely for generic c, h . In Chapter 3, we expressed the bosonized Gaiotto state via Jack symmetric functions. Here we give an expression of the Gaiotto state in terms of the Virasoro generators $\{L_n\}$:

$$|G\rangle = |c, h\rangle + \Lambda^2 v_1 + \Lambda^4 v_2 + \Lambda^6 v_3 + \cdots, \quad v_n \in M(c, h)_n$$

with

$$\begin{aligned} v_1 &= \frac{1}{2h} L_{-1} |c, h\rangle, \\ v_2 &= \frac{(8h + 8c)L_{-1}^2 - 12hL_{-2}}{4h(16h^2 + 2ch - 10h + c)} |c, h\rangle, \\ v_3 &= \frac{1}{24h(3h^2 + ch - 7h + 2)(16h^2 + 2ch - 10h + c)} \left[12h(7h - c - 3)L_{-3} \right. \\ &\quad \left. - 12(9h^2 + 3ch - 7h + c)L_{-2}L_{-1} + (24h^2 - 26h + 11ch + 8c + c^2)L_{-1}^3 \right] |c, h\rangle. \end{aligned}$$

Note that in the denominators of v_i , the Kac determinant formula appears (see (1.3.6)).

Using the expression of $|G\rangle$ and the matrix K_i (1.3.5), one can compute the norm of Gaiotto state:

$$\langle G | G \rangle = \sum_{n=0}^{\infty} \Lambda^{4n} (v_n^\dagger \cdot v_n).$$

Let us also mention the work of [67], which shows

$$\langle G | G \rangle = \sum_{n=0}^{\infty} \Lambda^{4n} (K_n^{-1})_{(1^n), (1^n)}, \tag{9.1.1}$$

where K_n^{-1} is the inverse matrix of (1.3.4), and the index $'(1^n), (1^n)'$ indicates the position of the element of this inverse matrix (recall that the matrix K_n is indexed by partitions of n).

9.2. SU(2) Nekrasov partition function

On the other hand, the pure SU(2) gauge Nekrasov partition function has the next combinatorial expression (which can be considered as the definition of the partition function). Let $x, \epsilon_1, \epsilon_2, \vec{a} = (a_1, a_2)$ be indeterminates. Then, by the argument in § 8.3,

$$\begin{aligned} Z^{\text{rank}=2}(\epsilon_1, \epsilon_2, \vec{a}; x) &= \sum_{\vec{\lambda}} \frac{x^{|\vec{\lambda}|}}{\prod_{1 \leq \alpha, \beta \leq 2} n_{\alpha, \beta}^{\vec{\lambda}}(\epsilon_1, \epsilon_2, \vec{a})}, \\ n_{\alpha, \beta}^{\vec{\lambda}}(\epsilon_1, \epsilon_2, \vec{a}) &:= \prod_{\square \in \lambda^{(\alpha)}} [-\ell_{\lambda^{(\beta)}}(\square) \epsilon_1 + (a_{\lambda^{(\alpha)}}(\square) + 1) \epsilon_2 + a_\beta - a_\alpha] \\ &\quad \times \prod_{\blacksquare \in \lambda^{(\beta)}} [(\ell_{\lambda^{(\alpha)}}(\blacksquare) + 1) \epsilon_1 - a_{\lambda^{(\beta)}}(\blacksquare) \epsilon_2 + a_\beta - a_\alpha]. \end{aligned} \quad (9.2.1)$$

Here $\vec{\lambda} = (\lambda^{(1)}, \lambda^{(2)})$ is a 2-tuple of partitions, $|\vec{\lambda}| := |\lambda^{(1)}| + |\lambda^{(2)}|$, and $a_{\lambda^{(\alpha)}}(\square)$, $\ell_{\lambda^{(\alpha)}}(\square)$ are the arm and the leg of the box $\square$ with respect to $\lambda^{(\alpha)}$ as explained in (A.2.1).

Below we show the first few terms of the partition function:

$$Z^{\text{rank}=2}(\epsilon_1, \epsilon_2, \vec{a}; x) = \sum_{n=0}^{\infty} x^n Z_n(\epsilon_1, \epsilon_2, \vec{a})$$

with

$$\begin{aligned} Z_0(\epsilon_1, \epsilon_2, \vec{a}) &= 1, \\ Z_1(\epsilon_1, \epsilon_2, \vec{a}) &= Z_{(1), \emptyset} + Z_{\emptyset, (1)} \\ &= \frac{1}{\epsilon_1 \epsilon_2 (a_1 - a_2) (\epsilon_1 + \epsilon_2 + a_2 - a_1)} + \frac{1}{(\epsilon_1 + \epsilon_2 + a_1 - a_2) (a_2 - a_1) \epsilon_2 \epsilon_1}, \\ Z_2(\epsilon_1, \epsilon_2, \vec{a}) &= Z_{(2), \emptyset} + Z_{(1,1), \emptyset} + Z_{(1), (1)} + Z_{\emptyset, (1,1)} + Z_{\emptyset, (2)}, \\ Z_3(\epsilon_1, \epsilon_2, \vec{a}) &= Z_{(3), \emptyset} + Z_{(2,1), \emptyset} + Z_{(1,1,1), \emptyset} + Z_{(2), (1)} + Z_{(1,1), (1)} \\ &\quad + Z_{(1), (1,1)} + Z_{(1), (2)} + Z_{\emptyset, (1,1,1)} + Z_{\emptyset, (2,1)} + Z_{\emptyset, (3)}. \end{aligned}$$

9.3. The conjecture

Now the statement of the Gaiotto conjectures is

$$\langle G | G \rangle \stackrel{?}{=} Z^{\text{rank}=2}(x; \epsilon_1, \epsilon_2, \vec{a}). \quad (9.3.1)$$

The parameters in both hand sides are related as in Table 9.3.1.

Virasoro	Nekrasov
c	$13 + 6(\epsilon_1/\epsilon_2 + \epsilon_2/\epsilon_1)$
h	$((\epsilon_1 + \epsilon_2)^2 - (a_2 - a_1)^2)/4\epsilon_1\epsilon_2$
Λ	$x^{1/4}/(\epsilon_1\epsilon_2)^{1/2}$

TABLE 9.3.1. Parameter correspondence

Note that by (9.1.1), the conjecture (9.3.1) is equivalent to

$$(K_n^{-1})_{(1^n), (1^n)}(c, h) \stackrel{?}{=} (\epsilon_1 \epsilon_2)^{4n} Z_n(\epsilon_1, \epsilon_2, \vec{a}) \quad (9.3.2)$$

for each $n \in \mathbb{Z}_{\geq 0}$.

9.4. Appendix: comparison of the inner products

Recall the formula (3.1.1) in Chapter 3, which describes the Gaiotto state $|G\rangle$ by Jack symmetric functions. In order to calculate the norm $\langle G|G\rangle$, we should compare the contravariant form $\cdot : M(c, h)^* \times M(c, h) \rightarrow \mathbb{C}$ and the inner product on the space Λ of symmetric functions.

Let us recall the isomorphism ι_β :

$$\iota_\beta : \mathcal{F}_\alpha \rightarrow \Lambda_{\mathbb{C}}, \quad a_{-n} |\alpha\rangle_{\mathcal{F}} \mapsto p_n \cdot \beta^{1/2} / \sqrt{2} \quad (n > 0).$$

In order to give the consistency between the bilinear form $\cdot : \mathcal{F}_\alpha^* \times \mathcal{F}_\alpha \rightarrow \mathbb{C}$ on the Heisenberg Fock space and the contravariant form $\cdot : M_h^* \times M_h \rightarrow \mathbb{C}$ on the Verma module of the Virasoro algebra, we need to give the anti-homomorphism

$$\omega : \mathcal{F}_\alpha \rightarrow \mathcal{F}_\alpha^*$$

so that

$$\langle c, h | u_1^\dagger \cdot u_2 | c, h \rangle =_{\mathcal{F}} \langle \alpha | \omega(\varphi(u_1)) \cdot \varphi(u_2) | \alpha \rangle_{\mathcal{F}}$$

holds for any $u_1, u_2 \in U(\text{Vir}_+)$, where $\varphi : U(\text{Vir}) \rightarrow \widehat{U}(\mathcal{H})$ is the bosonization map.

The consistent definition of ω is given as follows [62], [86]:

$$\omega(a_n) = a_{-n} - 2\rho\delta_{n,0}, \quad \omega(\rho) = -\rho.$$

Then under the parametrization $\rho = -(\beta^{1/2} - \beta^{-1/2})/\sqrt{2}$ we have $\omega(\beta^{1/2}) = -\beta^{1/2}$. Then we can spell out the inner product on Λ which is consistent with the contravariant form on the Verma module M_h :

$$\langle p_n, p_m \rangle =_{\mathcal{F}} \langle \alpha | \omega(\sqrt{2}\beta^{-1/2} a_{-n}) \cdot \sqrt{2}\beta^{-1/2} a_{-m} | \alpha \rangle_{\mathcal{F}} = -2n/\beta \cdot \delta_{n,m}.$$

This is the inner product $\langle \cdot, \cdot \rangle_{-2/\beta}$ defined at (2.4.1) in §2.4. But the Jack symmetric function orthogonal with respect to it is $P_\lambda^{(-2/\beta)}$, not $P_\lambda^{(1/\beta)}$ which is used in our expansion.

This calculation implies that the SU(2) AGT conjecture (9.3.1) is equivalent to

$$\sum_{\lambda, \mu \in \mathcal{P}} \Lambda^{2|\mu|+2|\lambda|} c_\lambda(\alpha, \beta) c_\mu(\alpha, \beta) \langle P_\lambda^{(1/\beta)}, P_\mu^{(1/\beta)} \rangle_{-2/\beta} \stackrel{?}{=} Z^{\text{rank}=2}(\epsilon_1, \epsilon_2, \vec{a}; x). \quad (9.4.1)$$

Now one may easily find that

$$\langle P_\lambda^{(1/\beta)}, P_\mu^{(1/\beta)} \rangle_{-2/\beta} = 0 \quad \text{unless} \quad |\lambda| = |\mu|.$$

Using this fact and comparing the homogeneous parts (the coefficients of Λ^{4d} and those of x^d) of both sides, one finds that (9.4.1) is equivalent to

$$\begin{aligned} & \sum_{\lambda, \mu \vdash d} c_\lambda(\alpha, \beta) c_\mu(\alpha, \beta) \langle P_\lambda^{(1/\beta)}, P_\mu^{(1/\beta)} \rangle_{-2/\beta} \\ & \stackrel{?}{=} (\epsilon_1 \epsilon_2)^{2d} \sum_{\substack{\lambda, \mu \in \mathcal{P} \\ |\lambda|+|\mu|=d}} \frac{1}{\prod_{1 \leq \alpha, \beta \leq 2} n_{\alpha, \beta}^{(\lambda, \mu)}(\epsilon_1, \epsilon_2, \vec{a})} \end{aligned} \quad (9.4.2)$$

for each $d \in \mathbb{Z}_{\geq 0}$. In the right hand side we changed the notation $\vec{\lambda} \in \mathcal{P}^2$ to the pair $(\lambda, \mu) \in \mathcal{P}^2$. Note that the ranges of running indexes in the left and right sides are different. The equation (9.4.2) seems to contain non-trivial relations among the ‘non-diagonal’ pairings $\langle P_\lambda^{(1/\beta)}, P_\mu^{(1/\beta)} \rangle_{-2/\beta}$. According to the computer experiment, these pairings have complicated looks (in particular, no factored expressions) in general, although each summand in the right hand side of (9.4.2) is factored. A combinatorial proof of (9.4.2) would be another justification of the AGT relation (9.3.1), but we have no clue to show it directly at this moment.

We have another combinatorial restatement of (9.3.1). If $\lambda \vdash d$, then one can expand $P_\lambda^{(1/\beta)} \in \Lambda_{\mathbb{C}}^d$ by the basis $\{P_\nu^{(-2/\beta)} \mid \nu \vdash d\}$ of $\Lambda_{\mathbb{C}}^d$. Let us express it as

$$P_\lambda^{(1/\beta)} = \sum_{\nu \vdash d} \gamma_\lambda^\nu(\beta) P_\nu^{(-2/\beta)}, \quad \gamma_\lambda^\nu(\beta) \in \mathbb{C}. \quad (9.4.3)$$

Then by an elementary calculation one finds that (9.3.1) is equivalent to

$$\begin{aligned} & \sum_{\lambda, \mu, \nu \vdash d} c_\lambda(\alpha, \beta) c_\mu(\alpha, \beta) \gamma_\lambda^\nu(\beta) \gamma_\mu^\nu(\beta) N_\nu(-2/\beta) \\ & \stackrel{?}{=} (\epsilon_1 \epsilon_2)^{2d} \sum_{\substack{\lambda, \mu \in \mathcal{P} \\ |\lambda| + |\mu| = d}} \frac{1}{\prod_{1 \leq \alpha, \beta \leq 2} n_{\alpha, \beta}^{(\lambda, \mu)}(\epsilon_1, \epsilon_2, \vec{a})}. \end{aligned} \quad (9.4.4)$$

Here we used the norm of Jack symmetric function

$$N_\nu(b) := \langle P_\nu^{(b)}, P_\nu^{(b)} \rangle_b = \prod_{\square \in \nu} \frac{a_\nu(\square) + b \ell_\nu(\square) + 1}{a_\nu(\square) + b \ell_\nu(\square) + b}.$$

According to the computer experiment, the coefficient $\gamma_\lambda^\mu(\beta)$ in (9.4.3) doesn't have a factored expression in general, although it looks a little simpler than the pairing $\langle P_\lambda^{(1/\beta)}, P_\mu^{(1/\beta)} \rangle_{-2/\beta}$. One might find an explicit formula of $\gamma_\lambda^\mu(\beta)$. However a direct proof of (9.4.4) will require a manipulation on the changes of indexes from $\lambda, \mu, \nu \vdash d$ to $\lambda, \mu \in \mathcal{P}$ with $|\lambda| + |\mu| = d$, which seems to be hard at this moment.

Proof of SU(2) AGT conjecture

We give an explicit formula on norm of logarithmic primary of Virasoro algebra, which was proposed by Al. Zamolodchikov. This formula appears in the recursion formula for norm of Gaiotto state, and it guarantees the AGT relation for the four-dimensional SU(2) pure gauge theory.

This chapter is based on the author's paper [94].

10.1. Norm of logarithmic primary of Virasoro algebra

This chapter discusses the norm of certain element in the Verma module of Virasoro algebra. Our main result is Theorem 10.1.2, which gives a rigorous proof of the formula on norm of logarithmic primary of Virasoro algebra. This formula was originally proposed in [97], and the proof [47] of the AGT relation for pure SU(2) gauge theory [42] depends on it. Our result is the last piece of the proof of the AGT relation.

In §10.1.1, we introduce the norm of logarithmic primary, which is the main topic of this chapter and appears directly in the statement of Theorem 10.1.2.

In §10.1.2, we mention to AGT conjectures/relations and its connection to our main theorem. These subsections are a detour, but it will be interesting for those working on AGT conjectures.

In the last subsection §10.1.3 of this introduction, we state our main theorem and the contents of the main part of this chapter.

In the late 1980s and the early 1990s, a series of works tried to write down $P_{r,s}(t)$ explicitly. [14] gave an explicit formula for $P_{1,s}(t)$. [9] gave an algorithm for constructing general $P_{r,s}(t)$ from $P_{1,s}(t)$. See also [24, §8.A] for this algorithm. [63] gave a formula of $P_{r,s}(t)$ using ‘analytic continuation’ of $P_{1,s}(t)$. See also [40] for a mathematically rigorous treatment of this ‘analytic continuation’. Although these formulas for $P_{r,s}(t)$ were enough for some studies on representation theory (see [64] for example), an ‘explicit’ formula for $P_{r,s}(t)$ could not be obtained.

The paper [68] shed a new light on this problem. It discovered that the following two objects coincide up to normalization: the integral expression of the Jack symmetric function $J_{(s^r)}^{(t)}$ [66, Chap.VI §10], and the expression of $|\chi_{r,s}\rangle$ in terms of the screening operators for the Feigin-Fuchs bosonization of Virasoro algebra [35]. See Fact 2.4.2 in §2.4 for the precise statement.

10.1.1. Norm of logarithmic primary. Let us recall the anti-homomorphism:

$$\dagger : U(\text{Vir}_-) \rightarrow U(\text{Vir}_+), \quad L_{-n} \mapsto L_n.$$

We will denote this map as $L_{-n}^\dagger = L_n$. Note that $(L_{-\lambda})^\dagger = L_\lambda$ under the notations (1.2.1) and (1.2.2). The anti-homomorphism $\dagger$ naturally induces a linear map $M(c, h) \rightarrow M^*(c, h)$, which is also written by $\dagger$. Then $(|c, h\rangle)^\dagger = \langle c, h|$ holds. We define $\langle \chi_{r,s} | := (|\chi_{r,s}\rangle)^\dagger$.

The norm of element $v \in M(c, h)$ is defined to be

$$v^\dagger \cdot v,$$

where $\cdot$ is the contravariant form (1.3.1). For example, by the definition of singular vector (1.4.1), it is obvious that

$$\langle \chi_{r,s} | \chi_{r,s} \rangle = \langle c(t), h_{r,s}(t) | P_{r,s}^\dagger(t) P_{r,s}(t) | c(t), h_{r,s}(t) \rangle = 0.$$

In [51], a curious observation was given on the norm

$$N_{1,s}(t, h) := \langle c(t), h | P_{1,s}^\dagger(t) P_{1,s}(t) | c(t), h \rangle$$

of the vector $P_{1,s}(t) |c(t), h\rangle$. They obtained a formula

$$\begin{aligned} N_{1,s}(t, h) &= (h - h_{1,s}(t)) \cdot R_{1,s}(t) + O((h - h_{1,s}(t))^2), \\ R_{1,s}(t) &:= 2s!(s-1)! \prod_{k=1}^{s-1} (k^2 t^2 - 1). \end{aligned} \quad (10.1.1)$$

A proof (containing some physical discussion) of this factor $R_{1,s}(t)$ was given in [51].

There was a several year gap between the studies of this kind of norm of $P_{r,s}(t) |c(t), h\rangle$. One of the reasons why such a calculation didn't attract so much interest may be, as a simple matter, that there was no necessity.

In the first 2000s, a revival of the Liouville field theory occurred. Among several important papers, [97, §6] proposed a generalization of the formula (10.1.1). Let us denote

$$N_{r,s}(t, h) := \langle c(t), h | P_{r,s}^\dagger(t) P_{r,s}(t) |c(t), h\rangle. \quad (10.1.2)$$

Using (1.4.3), one can calculate some examples:

$$\begin{aligned} N_{1,1}(t, h) &= 2(h - h_{1,1}(t)), \\ N_{1,2}(t, h) &= 4(t^2 - 1)(h - h_{1,2}(t)) + 8(h - h_{1,2}(t))^2, \\ N_{1,3}(t, h) &= 24(t^2 - 1)(4t^2 - 1)(h - h_{1,3}(t)) \\ &\quad + 8(16t^2 - 9)(h - h_{1,3}(t))^2 + 48(h - h_{1,3}(t))^3, \\ N_{1,4}(t, h) &= 288(t^2 - 1)(4t^2 - 1)(9t^2 - 1)(h - h_{1,4}(t)) \\ &\quad + 16(594t^4 - 481t^2 + 66)(h - h_{1,4}(t))^2 \\ &\quad + 128(25t^2 - 9)(h - h_{1,4}(t))^3 + 384(h - h_{1,4}(t))^4 \\ N_{2,2}(t, h) &= -8(t^2 - 1)(t^2 - 4)(t^{-2} - 1)(t^{-2} - 4)(h - h_{2,2}(t)) \\ &\quad + 16(2t^{-4} - 33t^{-2} + 91 - 33t^2 + 2t^4)(h - h_{2,2}(t))^2 \\ &\quad + 128(t^2 - 7 + t^{-2})(h - h_{2,2}(t))^3 + 384(h - h_{2,2}(t))^4. \end{aligned}$$

Then the following conjecture was proposed:

$$\begin{aligned} N_{r,s}(t, h) &\stackrel{?}{=} (h - h_{r,s}(t)) \cdot R_{r,s}(t) + O((h - h_{r,s}(t))^2), \\ R_{r,s}(t) &:= 2 \prod_{\substack{(k,l) \in \mathbb{Z}^2, \\ 1-r \leq k \leq r, \ 1-s \leq l \leq s, \\ (k,l) \neq (0,0), (r,s)}} (kt^{-1/2} + lt^{1/2}). \end{aligned} \quad (10.1.3)$$

REMARK 10.1.1. In [97], $N_{r,s}(t, h)$ is not expanded with respect to h , but α given in (2.2.3), the Heisenberg counterpart of the highest weight.

The element $P_{r,s}(t) |c(t), h\rangle$ is named the logarithmic primary in [97], so that it is natural to call $N_{r,s}(t)$ the norm of logarithmic primary. This norm is the main object in this chapter. The expression (10.1.3) is a generalization of (10.1.1). A physical derivation of the factor $R_{r,s}(t)$ was shown in [97] based on the theory of Liouville field on the Poincaré disk [101], but it seems lack of mathematically rigorous arguments. An analogous explanation for the supersymmetric Liouville field theory was given in [13], but no mathematical proof seems to exist.

As indicated in the last line of [97, §6], the factor $R_{r,s}(t)$ resembles the denominator of the factor appearing in the recursive formula of conformal block given in [95], [96] and [100].

10.1.2. Proving AGT relation via recursive formula. There exist several strategies for proving AGT conjectures. As for the pure gauge version (9.3.2), one of the strategies is to show that both sides satisfy the same recursive formula with respect to n . This strategy was first proposed by [78].

Later the paper [30] executed the procedure in case of the adjoint matter theory. They used an integral expression of the Nekrasov partition function and showed that it satisfies a recursive formula. The same recursive formula for the Virasoro side was then derived from the Zamolodchikov recursive formula for conformal block by limiting procedure.

Similar arguments were given in [47], where the cases of $N_f = 0, 1, 2$ (the number of matter fields) were treated. Here we only recall the case $N_f = 0$, that is, the pure gauge theory case. It was shown that $z_n := (\epsilon_1 \epsilon_2)^{4n} Z_n(\epsilon_1, \epsilon_2; \vec{a})$ satisfies the recursive formula:

$$z_n(t, h) = \delta_{0,n} + \sum_{\substack{(r,s) \in \mathbb{Z}_{\geq 0}^2, \\ 1 \leq rs \leq n}} \frac{R_{r,s}(t)^{-1} z_{n-rs}(t, h_{r,s}(t) + rs)}{h - h_{r,s}(t)}. \quad (10.1.4)$$

Here the formula is written in the Virasoro parameter (t, h) . In order to see it in the Nekrasov parameter, one needs to rewrite parameters by Table 9.3.1 and $c = c(t)$ defined in (1.3.9).

On the Virasoro side, it was stated that $f_n(t, h) := (K_n^{-1})_{(1^n), (1^n)}(c(t), h)$ satisfies the following recursive formula:

$$f_n(t, h) = \delta_{0,n} + \sum_{\substack{(r,s) \in \mathbb{Z}_{\geq 0}^2, \\ 1 \leq rs \leq n}} \left[\lim_{h \rightarrow h_{r,s}(t)} \frac{N_{r,s}(t, h)}{h - h_{r,s}(t)} \right]^{-1} \frac{f_{n-rs}(t, h_{r,s}(t) + rs)}{h - h_{r,s}(t)}. \quad (10.1.5)$$

Actually one can obtain this formula by considering the Jantzen filtration (see [54], [36] and [29]) on $M(c, h)$ and by some calculation on K_n . Thus from the comparison of (10.1.4) and (10.1.5), the verification of the conjecture (9.3.2) is reduced to the conjecture (10.1.3).

However it seems that the justification of (10.1.3) has not been discussed so far. Note that in the argument of AGT relations, the formula of $R_{r,s}(t)$ (10.1.3) is cited without proof, only with the remark to the paper [97]. See [30, (1.13a)] and [47, p. 7].

This kind of ‘normalization factor’ always appears in the recursive formula of conformal blocks and its degenerated versions. See, for example, [99, (5.40)] (Liouville conformal block), [13, (35)] (SUSY Liouville case), [43], [44] and [45] (also SUSY Liouville case).

Therefore we consider that it is valuable to give a proof of the formula (10.1.3) for $R_{r,s}$.

10.1.3. Main result. The main result of this chapter is a rigorous proof of (10.1.3). Let us rephrase the statement.

THEOREM 10.1.2. Let

$$P_{r,s}(t) = L_{-1}^{rs} + \cdots \in U(\text{Vir}_-) \otimes \mathbb{C}[t, t^{-1}]$$

be the element which generates the singular vector

$$|\chi_{r,s}\rangle = P_{r,s}(t) |c(t), h_{r,s}(t)\rangle$$

in $M(c(t), h_{r,s}(t))$. Define

$$N_{r,s}(t, h) := \langle c(t), h | P_{r,s}^\dagger(t) P_{r,s}(t) |c(t), h\rangle.$$

Then $N_{r,s}(t, h)$ has the form

$$N_{r,s}(t, h) = (h - h_{r,s}(t)) \cdot R_{r,s}(t) + O((h - h_{r,s}(t))^2),$$

$$R_{r,s}(t) := 2 \prod_{\substack{(k,l) \in \mathbb{Z}^2, \\ 1-r \leq k \leq r, \ 1-s \leq l \leq s, \\ (k,l) \neq (0,0), (r,s)}} (kt^{-1/2} + lt^{1/2}).$$

Let us explain the content of this chapter. §2 is devoted to the preliminaries on bosonization and symmetric functions, which are crucial tools in our discussion. In §3 we give the proof of the main Theorem 10.1.2. Since our argument is rather complicated, the outline is explained in the beginning of this section.

10.2. Proof of the main theorem

In this section we will show our main theorem following the strategy of [51]. Let us set

$$A_{r,s}(t) := \lim_{h \rightarrow h_{r,s}(t)} R_{r,s}(t) / (h - h_{r,s}(t)). \quad (10.2.1)$$

Our proof consists of the following steps.

Step 1: Estimate the degree of $A_{r,s}(t)$ as the Laurent polynomial of t . This step is executed with the help of asymptotic behavior of $P_{r,s}(t)$ studied in [3]. We show these arguments in §10.2.1.

Step 2: Determine the set S of the zeros of $A_{r,s}(t)$. This step is done in §10.2.2, and it is divided into three sub-steps:

- Upper bound of $\#S$. The degree estimate in Step 1 gives the upper limit of the number of zeros.
- Lower bound of $\#S$. Next we show that S includes a certain subset S' using bosonization. S' is determined from a coefficient of Jack symmetric functions appearing in the bosonization of singular vector. Then we also show that S is invariant under the action $t \mapsto -t$. Thus S contains $S' \cup -S'$, where $-S' := \{-t \mid t \in S'\}$.
- Third sub-step. Since $\#(S' \cup -S')$ is equal to the upper limit, S should be equal to $S' \cup -S'$.

As the result of Step 1 and Step 2, $A_{r,s}(t)$ is determined up to a numerical factor.

Step 3: Determine the numerical factor of $A_{r,s}(t)$. We can determine this factor from the asymptotic behavior of $P_{r,s}(t)$ with respect to t . This step is done by direct calculation so that we omit the detail.

10.2.1. Step 1. Degree estimate. Let us recall the next fact:

FACT 10.2.1 ([3]). If one expands $P_{r,s}(t)$ as a Laurent polynomial of t , then

$$P_{r,s}(t) = [(r-1)!]^{2s} L_{-r}^s t^{-(r-1)s} + \dots + [(s-1)!]^{2r} L_{-s}^r t^{r(s-1)},$$

where ‘ $\dots$ ’ denotes the intermediate degrees in t .

Now, for a Laurent series $a(t) = \sum_k a_k t^k$ of t , we define the maximum and minimum degrees of $a(t)$ by

$$\max \deg a(t) := \max\{k \mid a_k \neq 0\}, \quad \min \deg a(t) := \min\{k \mid a_k \neq 0\}. \quad (10.2.2)$$

LEMMA 10.2.2. The maximum and minimum degrees of $A_{r,s}(t)$ are estimated as

$$\max \deg A_{r,s}(t) \leq 2r(s-1), \quad \min \deg A_{r,s}(t) \geq -2(r-1)s.$$

PROOF. First we treat $\max \deg A_{r,s}(t)$. Expand $\langle c(t), h \mid L_s^r L_{-s}^r \mid c(t), h \rangle$ with respect to h around $h_{r,s}(t)$ as

$$\langle c(t), h \mid L_s^r L_{-s}^r \mid c(t), h \rangle = \sum_{k=0}^{rs-1} e_k^{(r,s)}(t) (h - h_{r,s}(t))^k.$$

By Fact 10.2.1, we have

$$\max \deg A_{r,s}(t) \leq \max \deg (e_1^{(r,s)}(t) t^{2r(s-1)}).$$

Since $\max \deg e_1^{(r,s)}(t) = 0$ by Lemma 10.2.13, we have the result.

The estimate for $\min \deg A_{r,s}(t)$ is obtained from that of $\max \deg A_{r,s}(t)$ by using the symmetry $t \mapsto t^{-1}$, $(r, s) \mapsto (s, r)$ (1.4.4). $\square$

COROLLARY 10.2.3. The number of zeros of $A_{r,s}(t)$ is at most $4rs - 2r - 2s$.

10.2.2. Step 2. Zero counting. Recall the bosonization morphism φ (2.2.1), the correspondence (2.2.3) and the highest weight (2.2.4) of Heisenberg algebra at which a singular vector exists. We will use the highest weight shift of Heisenberg algebra:

$$\varepsilon_{r,s}(t, \alpha) := \alpha - \alpha_{r,s}(t), \quad \varepsilon_{r,s}^\dagger(t, \alpha) := \alpha - \alpha_{-r,-s}(t).$$

Note that we have

$$h - h_{r,s}(t) = \frac{1}{2} \varepsilon_{r,s}(t, \alpha) \varepsilon_{r,s}^\dagger(t, \alpha). \quad (10.2.3)$$

under the correspondence (2.2.3).

For a partition λ of n , define the degree of $a_{-\lambda} \in \mathbb{C}[a_{-1}, a_{-2}, \dots, a_{-m}]$ by $\deg a_{-\lambda} := |\lambda| = n$, and denote by $\mathbb{C}[a_{-1}, a_{-2}, \dots, a_{-m}]_n$ the subspace of homogeneous elements of degree n . Then $\mathcal{F}_{\alpha, n}$ defined in (2.1.1) is isomorphic to $\mathbb{C}[a_{-1}, a_{-2}, \dots, a_{-n}]_n |\alpha\rangle_{\mathcal{F}}$. Similarly, we define the degree of a_{λ} by $\deg a_{\lambda} := |\lambda|$ and denote the homogeneous subspace by $\mathbb{C}[a_1, a_2, \dots, a_m]_n$.

LEMMA 10.2.4. (1) We have

$$\varphi(P_{r,s}(c(t), h)) |\alpha\rangle_{\mathcal{F}} = \varepsilon_{r,s}^{\dagger}(t, \alpha) \left[g_0(t) + \sum_{k=1}^{rs-1} (\varepsilon_{r,s}(t, \alpha))^k g_k(t) \right] |\alpha\rangle_{\mathcal{F}},$$

with

$$\begin{aligned} g_0(t) &\in \mathbb{C}[t^{\pm 1/2}] \otimes \mathbb{C}[a_{-1}, a_{-2}, \dots, a_{-rs}]_{rs}, \\ g_k(t) &\in \mathbb{C}[t^{\pm 1/2}] \otimes \mathbb{C}[a_{-1}, a_{-2}, \dots, a_{-rs+1}]_{rs} \quad (k \neq 0). \end{aligned}$$

(2) We also have

$${}_{\mathcal{F}} \langle \alpha | \varphi(P_{r,s}^{\dagger}(c(t), h)) = {}_{\mathcal{F}} \langle \alpha | \varepsilon_{r,s}(\alpha, t) \left[g_0^{\dagger}(t) + \sum_{k=1}^{rs-1} (\varepsilon_{r,s}^{\dagger}(t, \alpha))^k g_k^{\dagger}(t) \right],$$

with

$$\begin{aligned} g_0^{\dagger}(t) &\in \mathbb{C}[t^{\pm 1/2}] \otimes \mathbb{C}[a_1, a_2, \dots, a_{rs}]_{rs}, \\ g_k^{\dagger}(t) &\in \mathbb{C}[t^{\pm 1/2}] \otimes \mathbb{C}[a_1, a_2, \dots, a_{rs-1}]_{rs} \quad (k \neq 0). \end{aligned}$$

The point is that a_{-rs} (resp. a_{rs}) only appears in $g_0(t)$ (resp. $g_0^{\dagger}(t)$). Before starting the proof, we show some examples.

EXAMPLE 10.2.5.

$$\begin{aligned} \varphi(P_{1,1}(c(t), h)) |\alpha\rangle_{\mathcal{F}} &= \varepsilon_{1,1}^{\dagger}(t, \alpha) |\alpha\rangle_{\mathcal{F}}, \\ \varphi(P_{2,1}(c(t), h)) |\alpha\rangle_{\mathcal{F}} &= \varepsilon_{2,1}^{\dagger}(t, \alpha) \left[(1-t)t^{-1}(\sqrt{2}t^{1/2}a_{-1}^2 - a_{-2}) + \varepsilon_{2,1}(t, \alpha)a_{-1}^2 \right] |\alpha\rangle_{\mathcal{F}}, \\ \varphi(P_{3,1}(c(t), h)) |\alpha\rangle_{\mathcal{F}} &= \varepsilon_{3,1}^{\dagger}(t, \alpha) \left[(1-t)(2-t)t^{-2}(2ta_{-1}^3 - 3\sqrt{2}t^{1/2}a_{-2}a_{-1} + 2a_{-3}) \right. \\ &\quad \left. + \varepsilon_{3,1}(t, \alpha)((3-2t)\sqrt{2}t^{-1/2}a_{-1}^3 - (4t^{-1} - 3)a_{-2}a_{-1}) \right. \\ &\quad \left. + (\varepsilon_{3,1}(t, \alpha))^2 a_{-1}^3 \right] |\alpha\rangle_{\mathcal{F}}, \\ \varphi(P_{4,1}(c(t), h)) |\alpha\rangle_{\mathcal{F}} &= \varepsilon_{4,1}^{\dagger}(t, \alpha) \left[(1-t)(2-t)(3-t)t^{-3}(2\sqrt{2}t^{3/2}a_{-1}^4 + 8\sqrt{2}t^{1/2}a_{-3}a_{-1} \right. \\ &\quad \left. + 3\sqrt{2}t^{1/2}a_{-2}^2 - 12ta_{-2}a_{-1}^2 - 6a_{-4}) \right. \\ &\quad \left. + \varepsilon_{4,1}(t, \alpha)(2t^{-1}(11 - 12t + 3t^2)a_{-1}^4 \right. \\ &\quad \left. - 2\sqrt{2}t^{-3/2}(5 - 2t)(4 - 3t)a_{-2}a_{-1}^2 \right. \\ &\quad \left. + t^{-2}(9 - 10t + 3t^2)a_{-2}^2 \right. \\ &\quad \left. + 2t^{-2}(12 - 15t + 4t^2)a_{-3}a_{-1}) \right. \\ &\quad \left. + (\varepsilon_{4,1}(t, \alpha))^2(3\sqrt{2}t^{-1/2}(2-t)a_{-1}^4 - 2t^{-1}(5 - 3t)a_{-2}a_{-1}^2) \right. \\ &\quad \left. + (\varepsilon_{4,1}(t, \alpha))^3 a_{-1}^4 \right] |\alpha\rangle_{\mathcal{F}}. \end{aligned}$$

PROOF OF LEMMA 10.2.4. (1) $\varphi(P_{r,s}(c(t), h)) |\alpha\rangle_{\mathcal{F}}$ is a polynomial of degree rs in terms of α . It has at least one zero at $\alpha = \alpha_{-r,-s}$ by the discussion of [35] (see also [86, 62]). Thus the polynomial considered has the form

$$(\alpha - \alpha_{-r,-s}(t)) \sum_{k=0}^{rs-1} (\alpha - \alpha_{r,s}(t))^k g_k(t) |\alpha\rangle_{\mathcal{F}},$$

where $g_k(t) \in \mathbb{C}[t^{\pm 1/2}] \otimes \mathbb{C}[a_{-1}, a_{-2}, \dots, a_{-rs}]_{rs}$ for any k . Note that the coefficients are in $\mathbb{C}[t^{\pm 1/2}]$ by Fact 1.4.2 and the definition of φ .

Now Lemma 10.2.14 proved later means that the coefficient of a_{-rs} has degree one as a polynomial of α . Therefore a_{-rs} cannot appear in $g_k(t)$ for $k > 1$. Thus we have the consequence.

(2) is similarly shown so that we omit the detail. $\square$

LEMMA 10.2.6. The set of zeros in $A_{r,s}(t) = 0$ includes the set of zeros in the coefficient of a_{-rs} in $g_0(t)$.

PROOF. Precisely speaking, if the condition holds, then the state $\varphi(P_{r,s}(c(t), h)) |\alpha\rangle_{\mathcal{F}}$ vanishes identically. In order to show it, it is enough to show by induction that

$$a_n \cdot \varphi(P_{r,s}(c(t), h)) |\alpha\rangle_{\mathcal{F}} = 0$$

for any n . This is proved in [51, Appendix B]. $\square$

LEMMA 10.2.7. The coefficient $c_\lambda(t)$ of a_{-rs} in $g_0(t)$ is

$$c_\lambda(t) = \prod_{\substack{(k,l) \in \mathbb{Z}^2 \setminus \{(r,s)\} \\ 1 \leq k \leq r, 1 \leq l \leq s}} (kt^{-1} - l) \cdot \prod_{\substack{(k,l) \in \mathbb{Z}^2 \setminus \{(0,0)\} \\ 0 \leq k \leq r-1, 0 \leq l \leq s-1}} (lt - k).$$

PROOF. Note that by Fact 2.4.2 and Lemma 10.2.4 we have

$$(\alpha_{r,s}(t) - \alpha_{-r,-s}(t)) \cdot \iota(g_0(t) |\alpha\rangle_{\mathcal{F}}) = B_{r,s}(t) J_{(s^r)}^{(t)}.$$

On the other hand, by Corollary 10.2.16 in §10.2.3 we have

$$J_{(s^r)}^{(t)} = \sum_{\mu \vdash rs} \theta_{(s^r)}^\mu(t) p_\mu, \quad \theta_{(s^r)}^{(rs)}(t) = \prod_{\substack{(k,l) \in \mathbb{Z}^2 \setminus \{(0,0)\} \\ 0 \leq k \leq r-1, 0 \leq l \leq s-1}} (lt - k).$$

Since $\iota(a_{-rs} |\alpha\rangle_{\mathcal{F}}) = p_{rs}/\sqrt{2t}$, we have

$$(\alpha_{r,s}(t) - \alpha_{-r,-s}(t)) c_\lambda(t) / \sqrt{2t} = B_{r,s}(t) \cdot \theta_\lambda^{(n)}(t).$$

Then an easy calculation shows the statement. $\square$

COROLLARY 10.2.8. $A_{r,s}(t, h)$ has zeros at

$$\begin{aligned} S' := & \{t = k/l \mid 1 \leq k \leq r, 1 \leq l \leq s, (k, l) \neq (r, s)\} \\ & \cup \{t = k/l \mid 1 \leq k \leq r-1, 1 \leq l \leq s-1\} \quad (\text{multiplicities included}). \end{aligned}$$

In particular, $A_{r,s}(t, h)$ has at least $\#S' = 2rs - r - s$ zeros.

PROOF. This is the consequence of Lemma 10.2.6 and Lemma 10.2.7. $\square$

LEMMA 10.2.9. $A_{r,s}(t) = A_{r,s}(-t)$.

PROOF. In fact one can show a stronger statement: if one expands

$$N_{r,s}(t, h) = \sum_{k=1}^{rs} (h - h_{r,s}(t))^k n_{r,s,k}(t), \quad (10.2.4)$$

then we have $n_{r,s,k}(t) = n_{r,s,k}(-t)$. $A_{r,s}(t)$ is nothing but $n_{r,s,1}(t)$. This is the consequence of the following Fact 10.2.10 stated in [36, Theorem 1.13] (for a proof, see also [64]). $\square$

FACT 10.2.10 ([36, Thm. 1.13]). The anti-automorphism on $U(\text{Vir}_-)[t^{\pm 1}]$ defined by

$$t \mapsto -t, \quad L_{-i} \mapsto (-1)^{i-1} L_{-i}$$

takes $P_{r,s}(t)$ into itself.

COROLLARY 10.2.11. Let us denote $-S' := \{-t \mid t \in S'\}$. Then $A_{r,s}(t)$ has zeros at $S' \cup (-S')$ (multiplicities included). Thus $A_{r,s}(t)$ has at least $2 \cdot \#S = 4rs - 2r - 2s$ zeros.

PROOF. This is the consequence of Corollary 10.2.8 and Lemma 10.2.9. $\square$

COROLLARY 10.2.12. $A_{r,s}(t)$ is equal to

$$A'_{r,s}(t) := \prod_{\substack{(k,l) \in \mathbb{Z}^2, \\ 1-r \leq k \leq r, \ 1-s \leq l \leq s, \\ (k,l) \neq (0,0), (r,s)}} (kt^{-1/2} + lt^{1/2})$$

up to a numerical factor in $\mathbb{C}$.

PROOF. By Corollary 10.2.3 and Corollary 10.2.11, we have known that the set S of zeros of $A_{r,s}(t)$ is equal to $S' \cup (-S')$. Then the inequalities in Lemma 10.2.2 should be equalities: $\max \deg A_{r,s}(t) = 2r(s-1)$, $\min \deg A_{r,s}(t) = -2(r-1)s$.

Next note that $A'_{r,s}(t) \in \mathbb{C}[t, t^{-1}]$, that the set of its zeros is equal to $S' \cup (-S')$, and that $\max \deg A'_{r,s}(t) = 2r(s-1)$, $\min \deg A'_{r,s}(t) = -2(r-1)s$. Thus $A_{r,s}(t)$ and $A'_{r,s}(t)$ should be proportional. $\square$

10.2.3. Lemmas for the proof.

10.2.3.1. A lemma on Shapovalov form.

LEMMA 10.2.13. (1)

$$\langle c, h \mid L_s^r L_{-s}^r \mid c, h \rangle = s^r r! \prod_{k=1}^r \left[2h + s(k-1) + \frac{s^2-1}{12} c \right]$$

(2) In the expansion

$$\langle c(t), h \mid L_s^r L_{-s}^r \mid c(t), h \rangle = \sum_{k=0}^s (h - h_{r,s}(t))^k e_k^{(r,s)}(t),$$

the maximum and the minimum degrees (see (10.2.2) for the definitions) of $e_1^{(r,s)}(t)$ are

$$\max \deg e_1^{(r,s)}(t) = 0, \quad \min \deg e_1^{(r,s)}(t) = \begin{cases} 0 & r = s \\ 1 - r & r \neq s \end{cases}.$$

PROOF. By the defining relations of Vir, we have

$$\langle c, h \mid L_s^r L_{-s}^r \mid c, h \rangle = \langle c, h \mid L_s^{r-1} L_{-s}^{r-1} \mid c, h \rangle \cdot \sum_{k=1}^r \left[\frac{s^3-s}{12} c + 2s(sk+h) \right].$$

Then an easy calculation gives the result of (1).

(2) is an immediate consequence of (1). $\square$

10.2.3.2. A lemma on bosonization.

LEMMA 10.2.14. Let λ be a partition of n . In the expansion

$$\varphi(L_{-\lambda}) \mid \alpha \rangle_{\mathcal{F}} = \sum_{\mu \vdash n} c_{\lambda}^{\mu}(t, \alpha) a_{-\mu} \mid \alpha \rangle_{\mathcal{F}}, \quad c_{\lambda}^{\mu}(t, \alpha) \in \mathbb{C}[\alpha, t^{\pm 1/2}],$$

the coefficient $c_{\lambda}^{(n)}(t, \alpha)$ of $a_{-(n)} \mid \alpha \rangle_{\mathcal{F}}$ is of degree one in terms of α .

PROOF. This is an easy consequence of the bosonization φ :

$$\begin{aligned} \varphi(L_{-2n}) &= (2n-1)\rho(t)a_{-2n} + \frac{a_{-n}^2}{2} + a_{-n-1}a_{-n+1} + a_{-n-2}a_{-n+2} + \\ &\quad \cdots + a_{-2n+1}a_{-1} + a_{-2n}a_0 + a_{-2n-1}a_1 + \cdots, \\ \varphi(L_{-2n-1}) &= 2n\rho(t)a_{-2n} + a_{-n-1}a_{-n} + a_{-n-2}a_{-n+1} + \\ &\quad \cdots + a_{-2n}a_{-1} + a_{-2n-1}a_0 + a_{-2n-2}a_1 + \cdots. \end{aligned}$$

In fact, for the case $\ell(\lambda) = 1$, i.e., $\lambda = (n)$, the above expression of $\varphi(L_{-n})$ gives

$$\varphi(L_{-n})|\alpha\rangle_{\mathcal{F}} = (\alpha + (n-1)\rho(t))a_n|\alpha\rangle_{\mathcal{F}} + \cdots,$$

which is the desired consequence.

For the case $k := \ell(\lambda) > 1$, set $\nu := (\lambda_2, \lambda_3, \dots, \lambda_k)$. Since

$$\varphi(L_{-\nu})|\alpha\rangle_{\mathcal{F}} = \sum_{\mu \vdash n - \lambda_1} c_{\nu}^{\mu}(t, \alpha) a_{-\mu}|\alpha\rangle_{\mathcal{F}},$$

we find that the term $a_{-n}|\alpha\rangle_{\mathcal{F}}$ in $\varphi(L_{-\lambda})|\alpha\rangle_{\mathcal{F}} = \varphi(L_{-\lambda_1})\varphi(L_{-\nu})|\alpha\rangle_{\mathcal{F}}$ appears as

$$a_{-n}a_{n-\lambda_1} \cdot c_{\nu}^{(n-\lambda_1)}(t, \alpha) a_{-n+\lambda_1}|\alpha\rangle_{\mathcal{F}} = (n - \lambda_1) c_{\nu}^{(n-\lambda_1)}(t, \alpha) a_{-n}|\alpha\rangle_{\mathcal{F}}.$$

By the induction hypothesis, we know that $c_{\nu}^{(n-\lambda_1)}(t, \alpha)$ is of degree one as a polynomial of α . Thus the desired consequence holds. $\square$

10.2.3.3. A coefficient of Jack symmetric function.

FACT 10.2.15. If one expands the power symmetric function p_n by the family of monic Jack symmetric functions $\{P_{\lambda}^{(t)}\}$, then

$$p_n = nt \sum_{\lambda \vdash n} \prod_{\square \in \lambda} \frac{1}{ta_{\lambda}(\square) + \ell_{\lambda}(\square) + t} \prod_{\substack{(i,j) \in \lambda \\ (i,j) \neq (1,1)}} (t(j-1) - (i-1)) \cdot P_{\lambda}^{(t)}.$$

(For the notations of Young diagrams, see §2.4.)

PROOF. See [48]. $\square$

COROLLARY 10.2.16. Let λ be a partition of n . If one expands the integral Jack symmetric function $J_{\lambda}^{(t)}$ by the family of power symmetric function $\{p_{\mu}\}$ as

$$J_{\lambda}^{(t)} = \sum_{\mu \vdash n} \theta_{\lambda}^{\mu}(t) p_{\mu}$$

then the following holds:

$$\theta_{\lambda}^{(n)}(t) = \prod_{\substack{(i,j) \in \lambda \\ (i,j) \neq (1,1)}} ((j-1)t - (i-1)).$$

PROOF. The coefficient c_{λ} can be calculated by Fact 10.2.15, the ratio (2.4.3) of $P_{\lambda}^{(t)}$ and $J_{\lambda}^{(t)}$, and the norm (2.4.2) of $P_{\lambda}^{(t)}$. $\square$

K-theoretic AGT conjecture

This chapter deals with the five-dimensional pure SU(2) AGT conjecture proposed by Awata and Yamada. We give a conjecture on a recursive formula for the inner product of the deformed Gaiotto state. We also show that the *K*-theoretic pure SU(2) Nekrasov partition function satisfies the same recursion relation. Therefore the five-dimensional AGT conjecture is reduced to our conjectural recursive formula.

This chapter is based on the author's paper [92].

11.1. Introduction

In [7], Awata and Yamada proposed a conjecture which relates Nekrasov's five-dimensional pure SU(2) partition function $Z(\Lambda, Q, q, t)$ [74, 73] to the deformed Virasoro algebra [82]. The conjecture claims that $Z(\Lambda, Q, q, t)$ coincides with the inner product $\langle G|G \rangle$ of the so-called deformed Gaiotto state $|G\rangle$. This state is a natural q -deformed analogue of the state proposed by Gaiotto in [42]. These states are sorts of Whittaker vectors [65].

This conjecture is a natural extension of the degenerate version [42] of the four-dimensional AGT conjecture [2]. The original AGT conjecture claims that the instanton part of Nekrasov's four-dimensional SU(2) partition function with $N_f = 4$ anti-fundamental hypermultiplets [74] coincides with the conformal block of the Virasoro algebra.

In [30], a strategy was proposed for the proof of the four-dimensional conjecture in the case of the theory with adjoint matter multiplet. Its main idea is to show that both the Nekrasov partition function and the one-point conformal block on the torus satisfy the same recursion formula. On the conformal field theory side the recursion formula was conjectured by Poghossian [78] and derived in [46]. On the Nekrasov side Fateev and Litvinov [30] used the integral expression of the four-dimensional partition function to analyze its poles and residues, and they obtained the same recursion formula.

In [47], the same strategy was used for the proof of four-dimensional conjectures with $N_f = 0, 1, 2$ anti-fundamental hypermultiplets. In these cases, the Nekrasov partition functions were again analytically analyzed with the help of the integral expressions which is similar to that of [30]. And the CFT recursion formulas were derived through degeneration (decoupling) limit technique [67] of the Zamolodchikov elliptic recursive formula for the four-point conformal block on the sphere [95].

Note that the strategy of the proofs of the AGT conjectures using recursive formulas are restricted to the case of $N_f \leq 2$. This is because there are some difficulties in the asymptotic analysis on the integral expression of the Nekrasov partition function for the theory with $N_f \geq 3$, as mentioned in [30, 47].

What we may stress here is that in our consideration and in those proofs only the instanton part of the partition function is considered, although the full partition function consists of three contributions : classical, instanton and one-loop parts. As mentioned in [2, §4], the one-loop factor reproduces the product of the DOZZ three-point functions [27, 100] of the Liouville theory. And there is a general proposal that Nekrasov's full partition function, integrated over the vev with a natural measure, is the $n + 3$ point correlation function on a sphere, and is also related to the n point function on a torus. This point of view is out of the scope of this thesis.

The aim of this chapter is two-fold. First we propose Conjecture 11.2.3 on the recursion formula for the inner product $\langle G|G \rangle$ of the deformed Gaiotto state. This is an analogue of the recursive formulas appearing in [95, 78, 67, 46]. Our conjecture suggests a new resemblance

between the Fock representation of the Virasoro algebra and that of the deformed Virasoro algebra.

The second aim is to prove Theorem 11.4.1 which states that the K -theoretic Nekrasov partition function $Z(\Lambda, Q, q, t)$ satisfies the same recursion formula proposed in Conjecture 11.2.3. In the proof we utilize the integral form of the five-dimensional partition function, which in itself seems to be new.

As a consequence, the conjecture of Awata and Yamada [7] is reduced to Conjecture 11.2.3. At this point, not enough is known on the representation of the deformed Virasoro algebra, so that we do not have a clue to solve our conjecture.

This chapter is organized as follows. In §11.2 we recall the deformed Virasoro algebra and the deformed Gaiotto state. We will state the main conjecture on the inner product of the deformed Gaiotto state in this section. In §11.3 we introduce the integral expression of the K -theoretic Nekrasov partition function $Z(\Lambda, Q, q, t)$. In §11.4 we prove Theorem 11.4.1.

11.2. Deformed Virasoro algebra $\text{Vir}_{q,t}$

11.2.1. Definition. Let q, t be two generic complex parameters. Set $p := q/t$ and assume that p is not a root of -1 . The deformed Virasoro algebra $\text{Vir}_{q,t}$ [82, 17] is defined to be the associative algebra generated by $\{T_n \mid n \in \mathbb{Z}\}$ with relations

$$[T_n, T_m] = - \sum_{l=1}^{\infty} f_l (T_{n-l} T_{m+l} - T_{m-l} T_{n+l}) - \frac{(1-q)(1-t^{-1})}{1-p} (p^n - p^{-n}) \delta_{m+n,0},$$

where the coefficients f_l 's are determined through

$$f(x) = \sum_{l \geq 0} f_l x^l = \exp \left(\sum_{n \geq 1} \frac{(1-q)(1-t^{-1})}{1+p^n} \frac{x^n}{n} \right).$$

$\text{Vir}_{q,t}$ has a $\mathbb{C}$ -linear anti-involution ι defined by

$$\iota(T_n) = -T_n. \quad (11.2.1)$$

For a partition $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_\ell)$ ($\lambda_1 \geq \lambda_2 \geq \dots > 0$), let us introduce the symbols $T_\lambda := T_{\lambda_\ell} T_{\lambda_{\ell-1}} \cdots T_{\lambda_1}$ and $T_{-\lambda} := \iota(T_\lambda) = T_{-\lambda_1} T_{-\lambda_2} \cdots T_{-\lambda_\ell}$.

11.2.2. Verma module and Deformed Gaiotto state. This subsection follows [7, §3.2].

For $h \in \mathbb{C}$, let $|h\rangle$ be a vector and define the actions of the generators by $T_0 |h\rangle = h |h\rangle$ and $T_n |h\rangle = 0$ for any $n \in \mathbb{Z}_{>0}$. Then the Verma module M_h for $\text{Vir}_{q,t}$ is defined to be the $\text{Vir}_{q,t}$ -module generated by $|h\rangle$.

Let us introduce the outer grading operator d satisfying $[d, T_n] = -n T_n$. Defining the action of d on M_h by $d |h\rangle = 0$, we have the direct decomposition $M_h = \bigoplus_{n \in \mathbb{Z}_{\geq 0}} M_{h,n}$. $M_{h,n}$ has a basis consisting of the vectors

$$T_{-\lambda} |h\rangle = T_{-\lambda_1} T_{-\lambda_2} \cdots T_{-\lambda_\ell} |h\rangle,$$

each of which is indexed by a partition $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_\ell)$, $\lambda_1 \geq \lambda_2 \geq \dots > 0$, with $|\lambda| = n$.

The dual module M_h^* is similarly defined. It is generated by the highest weight vector $\langle h|$ satisfying $\langle h| T_n = 0$ for any $n \in \mathbb{Z}_{<0}$ and $\langle h| T_0 = h \langle h|$. M_h^* has a basis consisting of the vectors

$$\langle h| T_\lambda = \langle h| T_{\lambda_\ell} T_{\lambda_{\ell-1}} \cdots T_{\lambda_1},$$

each of which is indexed by a partition $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_\ell)$.

Also recall the bilinear contravariant form S on $M_h \otimes M_h \rightarrow \mathbb{C}$:

$$S(T_\lambda |h\rangle, T_\mu |h\rangle) := \langle h| T_\mu T_{-\lambda} |h\rangle,$$

which is uniquely determined by $\langle h|h\rangle = 1$. By the outer grading operator d defined by $[d, T_n] = n T_n$, we find that $S(T_\lambda |h\rangle, T_\mu |h\rangle) = 0$ unless $|\lambda| = |\mu|$. The determinant

$$D_n(h, q, t) := (\det S(T_\lambda |h\rangle, T_\mu |h\rangle))_{\lambda, \mu \in \mathcal{P}_n}$$

is called the Kac determinant for $\text{Vir}_{q,t}$.

DEFINITION 11.2.1. Deformed Gaiotto state $|G\rangle \in M_h$ is defined to be a vector satisfying the condition:

$$T_1 |G\rangle = \Lambda^2 |G\rangle, \quad T_n |G\rangle = 0 \quad (n \geq 2),$$

where Λ^2 is a (non-zero) complex number. The dual vector $\langle G| \in M_h^*$ is defined similarly:

$$\langle G| T_{-1} = \Lambda^2 \langle G|, \quad \langle G| T_n = 0 \quad (n \leq -2).$$

LEMMA 11.2.2. If q, t and h are generic complex numbers, then the deformed Gaiotto state $|G\rangle$ uniquely exists.

PROOF. This is the consequence of the non-vanishing of the Kac determinant for $\text{Vir}_{q,t}$, which follows from the factorized formula [17]:

$$D_n(h, q, t) = C_n \prod_{\substack{r, s \in \mathbb{Z}_{\geq 1} \\ rs \leq n}} (h^2 - h_{r,s}^2)^{p(n-rs)} \left(\frac{(1-q^r)(1-t^{-s})}{1+(q/t)^r} \right)^{p(n-rs)}. \quad (11.2.2)$$

Here $h_{r,s} := t^{r/2} q^{-s/2} + t^{-r/2} q^{s/2}$, and C_n is a constant independent of q, t and h . $\square$

11.2.3. Conjecture on the inner product of the deformed Gaiotto state. In [7], Awata and Yamada conjectured that the inner product $\langle G|G\rangle$ is equal to the five-dimensional pure $\text{SU}(2)$ Nekrasov partition function. To express $\langle G|G\rangle$, it is convenient to introduce the parameter Q by

$$h = Q^{1/2} + Q^{-1/2}.$$

From the Kac determinant formula (11.2.2), $\langle G|G\rangle$ is a rational function of q, t and h^2 . Thus it is a rational function of q, t and Q .

We propose a recursive formula on the inner product $\langle G|G\rangle$ of the deformed Gaiotto state $|G\rangle$.

CONJECTURE 11.2.3. Assume that q and t are generic complex numbers and that Λ is a non-zero complex number. Let $|G\rangle$ be the deformed Gaiotto state of $\text{Vir}_{q,t}$ (see §11.2.2 for the definition). Then $\langle G|G\rangle$ is of the form $F(\Lambda, Q, q, t) = \sum_{n=0}^{\infty} (\Lambda^4 t/q)^n F_n(Q, q, t)$, where $F_n(Q, q, t)$ is a rational function in terms of the variables Q, q, t . Moreover it satisfies the following recursive relation :

$$F_n(Q, q, t) = \delta_{n,0} + \sum_{\substack{r, s \in \mathbb{Z}, \\ 1 \leq rs \leq n}} \frac{G(r, s; q, t) F_{n-rs}(q^r t^s, q, t)}{Q - q^r t^{-s}}. \quad (11.2.3)$$

Here we introduced the rational function

$$G(r, s; q, t) := -\text{sgn}(r) q^r t^{-s} \prod_{\substack{-|r| \leq i \leq |r|-1, \\ -|s| \leq j \leq |s|-1, \\ (i,j) \neq (0,0)}} \frac{1}{1 - q^i t^{-j}} \quad (11.2.4)$$

with $\text{sgn}(r) = 1$ if $r > 0$ and $\text{sgn}(r) = -1$ if $r < 0$.

REMARK 11.2.4. (1) Recall that in the limit $q = t^\beta$, $t \rightarrow 1$, the deformed algebra $\text{Vir}_{q,t}$ reduces to the Virasoro algebra with the central charge $c = 1 - 6(\sqrt{\beta} - 1/\sqrt{\beta})^2$. In this limit the deformed Gaiotto state $|G\rangle$ reduces to the Gaiotto state $|\Delta, \Lambda^2\rangle$ for the $N_f = 0$ case [42, (2.3)], and our conjecture gives the formula appearing in [78, (4.11)] and [47, (20)].

(2) Let us write $S_{q,t}^{(n)}$ the matrix of size $p(n)$ defined to be

$$S_{q,t}^{(n)} := (S(T_\lambda |h\rangle, T_\mu |h\rangle))_{\lambda, \mu \in \mathcal{P}_n}.$$

Note that we have

$$\langle G|G\rangle = (S_{q,t}^{(n)})^{-1} (1^n, 1^n),$$

where the right hand side is the $(1^n, 1^n)$ -element of the inverse matrix of $S_{q,t}^{(n)}$. Our conjecture implies that all the poles of this element with respect to the variable Q are simple. It is worth noting that in the Virasoro case this simple-pole phenomenon is a non-trivial fact. By [67] the inner product $\langle \Delta, \Lambda^2 | \Delta, \Lambda^2 \rangle$ is equal to $\sum_n \Lambda^{4n} Q_{\Delta}^{-1}(1^n, 1^n)$, where $Q_{\Delta}^{-1}(1^n, 1^n)$ is the $(1^n, 1^n)$ -element of the inverse matrix of the contravariant form Q_{Δ} of the Virasoro algebra. In the mathematical literature, the simple-pole phenomenon of the inverse contravariant form is proved in [21] for the Virasoro case and in [77] for the finite Lie algebra case. See also [49, Chapter 5].

11.3. Integral expression of K -theoretic Nekrasov partition function

11.3.1. Combinatorial definition of the K -theoretic Nekrasov partition. As reviewed in §8.4, the instanton part of the K -theoretic Nekrasov partition function was defined using the equivariant K -theory on the framed moduli space of torsion free sheaves on $\mathbb{CP}^2$ in [74] and [73]. By the Atiyah-Bott-Lefschetz localization theorem it becomes a summation over the fixed point contribution. These fixed points are parametrized by r -tuples of Young diagrams, and one obtains a combinatorial form of the partition function.

Following the notation in [7], we introduce the K -theoretic Nekrasov partition function $Z(\Lambda, Q, q, t)$ as follows.

$$Z(\Lambda, Q, q, t) := \sum_{\lambda, \mu \in \mathcal{P}} (\Lambda^4 t / q)^{|\lambda| + |\mu|} Z_{\lambda, \mu}(Q, q, t), \quad (11.3.1)$$

$$Z_{\lambda, \mu}(Q, q, t) := \frac{1}{N_{\lambda, \lambda}(1) N_{\mu, \mu}(1) N_{\lambda, \mu}(Q) N_{\mu, \lambda}(Q^{-1})}, \quad (11.3.2)$$

$$N_{\lambda, \mu}(Q) := \prod_{(i,j) \in \mu} (1 - Q q^{\lambda_i - j} t^{\mu_j^{\vee} - i + 1}) \prod_{(i,j) \in \lambda} (1 - Q q^{-\mu_i + j - 1} t^{-\lambda_j^{\vee} + i}). \quad (11.3.3)$$

REMARK 11.3.1. In the previous §8.4, we used the notation $Z^K(t_1, t_2, e; x)$. The parameters are corresponded as

$$x = \Lambda^4, \quad t_1 = q, \quad t_2 = t^{-1}, \quad e_1 / e_2 = Q.$$

For the convenience of the discussion, we introduce the graded part $Z_n(Q, q, t)$ of $Z(\Lambda, Q, q, t)$ by

$$Z_n(Q, q, t) := \sum_{\substack{\lambda, \mu \in \mathcal{P}, \\ |\lambda| + |\mu| = n}} Z_{\lambda, \mu}(Q, q, t).$$

Then we have $Z(\Lambda, Q, q, t) = \sum_{n=0}^{\infty} (\Lambda^4 t / q)^n Z_n(Q, q, t)$.

Note also that $Z_{\lambda, \mu}$ enjoys the following relation :

$$Z_{\lambda, \mu}(Q, q, t) = Z_{\mu, \lambda}(Q^{-1}, q, t). \quad (11.3.4)$$

11.3.2. Integral expression. We have the next integral expression of the K -theoretic Nekrasov partition function. Note that the four-dimensional counterpart appeared in [74] and was investigated in [30, 47].

PROPOSITION 11.3.2. Let Q, q, t be generic complex numbers. Then the following integral formula holds.

$$Z_n(Q, q, t) = \frac{1}{n!} \left(\frac{1 - qt^{-1}}{(1 - q)(1 - t^{-1})} \right)^n \oint_{C_n} \frac{dx_n}{2\pi\sqrt{-1}} \cdots \oint_{C_1} \frac{dx_1}{2\pi\sqrt{-1}} \left(\prod_{k=1}^n P(x_k; Q^{1/2}, q^{-1}t) \prod_{1 \leq i < j \leq n} \omega(x_j / x_i; q, t^{-1}, q^{-1}t) \right), \quad (11.3.5)$$

Here we introduced rational functions

$$P(x; a, p) := \frac{x}{(x - a)(x - a^{-1})(x - pa)(x - pa^{-1})},$$

$$\omega(y; q_1, q_2, q_3) := \frac{(y-1)^2(y-q_3)(y-q_3^{-1})}{(y-q_1)(y-q_1^{-1})(y-q_2)(y-q_2^{-1})}.$$

and the contour C_k is chosen so that it surrounds only the poles at $Q^{1/2}$, $Q^{-1/2}$, $x_j q$ and $x_j t^{-1}$ for $j \neq k$.

Before starting the proof, we show some examples of the integration. See [30, §2] for the four-dimensional counterparts.

EXAMPLE 11.3.3. The case $n = 0$ holds trivially. Let us denote this situation by the pair of empty partitions $(\emptyset, \emptyset)$.

For the case $n = 1$, note that the poles with respect to the variable x_1 are at $Q^{1/2}$ and $Q^{-1/2}$. If the pole $Q^{1/2}$ is chosen, then let us associate the pair of partitions $((1), \emptyset)$. In this situation the integral is calculated as

$$\begin{aligned} & \frac{1 - qt^{-1}}{(1-q)(1-t^{-1})} \frac{Q^{1/2}}{(Q^{1/2} - Q^{-1/2})(Q^{1/2} - q^{-1}tQ^{1/2})(Q^{1/2} - q^{-1}tQ^{-1/2})} \\ &= \frac{1}{(t-1)(q^{-1}-1)(1-Q)(1-q^{-1}tQ^{-1})}, \end{aligned}$$

which is nothing but $Z_{(1), \emptyset}(Q, q, t)$. Similarly for the choice $Q^{-1/2}$ we associate the pair of Young diagrams $(\emptyset, (1))$. The corresponding integral yields $Z_{\emptyset, (1)}(Q, q, t)$. Thus we have shown that the right hand side of (11.3.5) is $Z_1(Q, q, t)$.

In the case $n = 2$, the integral over the variable x_1 surrounds two poles $Q^{1/2}$ and $Q^{-1/2}$. Let us choose $Q^{1/2}$ and associate the pair $((1), \emptyset)$. Then the integral becomes (excluding the factor $(2!)^{-1}$)

$$\begin{aligned} & \frac{(1 - q^{-1}t)}{(q^{-1} - 1)^2(t - 1)^2} \cdot \frac{Q^2}{(Q - 1)(Q - q^{-1}t)} \\ & \times \frac{(x_2 - qt^{-1}Q^{1/2})(x_2 - Q^{1/2})x_2}{(x_2 - Q^{-1/2})(x_2 - qQ^{1/2})(x_2 - q^{-1}Q^{1/2})} \\ & \times \frac{1}{(x_2 - t^{-1}Q^{1/2})(x_2 - tQ^{1/2})(x_2 - tq^{-1}Q^{-1/2})}. \end{aligned}$$

By the condition of the contour C_2 , the choices of the poles with respect to x_2 are $Q^{-1/2}$, $qQ^{1/2}$, $t^{-1}Q^{1/2}$. Note that the term $x_2 - Q^{1/2}$ cancels with the numerator $x_2/x_1 - 1$ in the function ω , so that the pole does not exist at $x_2 = Q$. To each case we associate the pair of partitions $((1), (1))$, $((2), \emptyset)$, $((1, 1), \emptyset)$. One can show that each integral coincides with $Z_{(1), (1)}$, $Z_{(2), \emptyset}$, $Z_{(1, 1), \emptyset}$ by direct calculations. If we choose $Q^{-1/2}$ instead, then the possible choice of the poles with respect to x_2 are $Q^{1/2}$, $qQ^{-1/2}$ and $t^{-1}Q^{-1/2}$. To each case we associate the pair of partitions $((1), (1))$, $(\emptyset, (2))$, $(\emptyset, (1, 1))$. One can show that each integral coincides with $Z_{(1), (1)}$, $Z_{\emptyset, (2)}$, $Z_{\emptyset, (1, 1)}$. We also have the choices $(x_1, x_2) = (q^{1/2}Q^{1/2}, Q^{1/2})$, $(t^{-1/2}Q^{1/2}, Q^{1/2})$, $(q^{1/2}Q^{-1/2}, Q^{-1/2})$, $(t^{-1/2}Q^{1/2}, Q^{1/2})$. They yield $Z_{(2), \emptyset}$, $Z_{(1, 1), \emptyset}$, $Z_{\emptyset, (2)}$, $Z_{\emptyset, (1, 1)}$ respectively. Noting the combinatorial factor $1/2!$ in the integral expression, we find that in the case $n = 2$ the right hand side of (11.3.5) is $Z_2(Q, q, t)$.

Now we start the proof.

PROOF OF PROPOSITION 11.3.2. Let us assume $n \geq 2$. By the choice of contours, the poles of the integral are located at

$$\begin{aligned} \{x_1, x_2, \dots, x_n\} = & \{Q^{1/2}q^{j-1}t^{-(i-1)} \mid 1 \leq i \leq \ell(\lambda), 1 \leq j \leq \lambda_i\} \\ & \bigcup \{Q^{-1/2}q^{i-1}t^{-(j-1)} \mid 1 \leq i \leq \ell(\mu), 1 \leq j \leq \mu_i\} \end{aligned} \quad (11.3.6)$$

with some pair of partitions (λ, μ) such that $|\lambda| + |\mu| = n$. Let us associate this pair (λ, μ) to the choice of poles (11.3.6).

We will show the residue at the pole parametrized by (λ, μ) coincides with $Z_{\lambda, \mu}$. The proof will be done by the induction on $n = |\lambda| + |\mu|$. The case $\lambda = \mu = \emptyset$ is trivial. For the general

λ and μ , let us set $L := \ell(\lambda)$, $\ell := \ell(\mu)$, $B := \lambda_\ell$, $b := \mu_\ell$, $m := |\lambda|$. We may fix the ordering in (11.3.6) such that

$$\begin{aligned} x_{i+\sum_{a=1}^{j-1} \lambda_a} &= Q^{1/2} q^{j-1} t^{-(i-1)} & (1 \leq i \leq \ell(\lambda) = L, 1 \leq j \leq \lambda_i), \\ x_{m+j+\sum_{a=1}^{j-1} \mu_a} &= Q^{-1/2} q^{j-1} t^{-(i-1)} & (1 \leq i \leq \ell(\mu) = \ell, 1 \leq j \leq \lambda_i). \end{aligned}$$

Then we can remove the factor $(n!)^{-1}$ from the integral. Let us define

$$\begin{aligned} I_{\lambda,\mu}(Q, q, t) &:= \left[\frac{1 - qt^{-1}}{(1-q)(1-t^{-1})} \right]^n \times \\ &\quad \text{Res}_{x_n=Q^{-1/2}q^{b-1}t^{-(\ell-1)}} \cdots \text{Res}_{x_{m+j+\sum_{a=1}^{j-1} \mu_a}=Q^{-1/2}q^{j-1}t^{-(i-1)}} \cdots \text{Res}_{x_{m+1}=Q^{-1/2}} \\ &\quad \cdot \text{Res}_{x_m=Q^{1/2}q^{B-1}t^{-(L-1)}} \cdots \text{Res}_{x_{j+\sum_{a=1}^{j-1} \lambda_a}=Q^{1/2}q^{j-1}t^{-(i-1)}} \cdots \text{Res}_{x_1=Q^{1/2}} \left[\right. \\ &\quad \left. \prod_{k=1}^n P(x_k; Q^{1/2}) \prod_{1 \leq i < j \leq n} \omega(x_j/x_i) \right]. \end{aligned} \quad (11.3.7)$$

Here and afterwards in this proof we use abbreviations

$$P(x; Q) := P(x; Q, q^{-1}t), \quad \omega(y) := \omega(y; q, t^{-1}, q^{-1}t).$$

Then $I_{\lambda,\mu}(Q, q, t)$ satisfies the relation

$$I_{\lambda,\mu}(Q, q, t) = I_{\mu,\lambda}(Q^{-1}, q, t), \quad (11.3.8)$$

which follows from the definition of the rational functions P and ω . Note that this is the same as (11.3.4).

Now what we must proof is

$$Z_{\lambda,\mu}(Q, q, t) = I_{\lambda,\mu}(Q, q, t). \quad (11.3.9)$$

Let us also define $\tilde{\mu}$ to be the partition made from μ by moving away the last one box. In other words we have

$$\tilde{\mu} := (\mu_1, \dots, \mu_{\ell-1}, b-1).$$

We will show that

$$Z_{\lambda,\mu}(Q, q, t)/Z_{\lambda,\tilde{\mu}}(Q, q, t) = I_{\lambda,\mu}(Q, q, t)/I_{\lambda,\tilde{\mu}}(Q, q, t). \quad (11.3.10)$$

Then by repeating the use of (11.3.10), the desired consequence (11.3.9) reduces to $Z_{\lambda,\emptyset}(Q, q, t) = I_{\lambda,\emptyset}(Q, q, t)$, and by the relations (11.3.4) and (11.3.8), it reduces to $Z_{\emptyset,\lambda}(Q, q, t) = I_{\emptyset,\lambda}(Q, q, t)$, which again by (11.3.10) reduces to the trivial case $Z_{\emptyset,\emptyset}(Q, q, t) = I_{\emptyset,\emptyset}(Q, q, t)$.

For the proof of (11.3.10), we set

$$\begin{aligned} \lambda &= (\overbrace{m_1, \dots, m_1}^{n_1}, \overbrace{m_2, \dots, m_2}^{n_2}, \dots, \overbrace{m_r, \dots, m_r}^{n_r}), \\ \mu &= (\overbrace{\tilde{m}_1, \dots, \tilde{m}_1}^{\tilde{n}_1}, \overbrace{\tilde{m}_2, \dots, \tilde{m}_2}^{\tilde{n}_2}, \dots, \overbrace{\tilde{m}_r, \dots, \tilde{m}_r}^{\tilde{n}_s}). \end{aligned} \quad (11.3.11)$$

Thus we have $L = n_1 + \dots + n_r$, $\ell = \tilde{n}_1 + \dots + \tilde{n}_s$, $B = m_r$, $b = \tilde{m}_s$. We also define the integer k by the condition

$$n_1 + \dots + n_{k-1} < \ell \leq n_1 + \dots + n_k$$

if $\ell \leq L$. Otherwise we set $k := r + 1$. In the following calculation we set $n_0 = \tilde{n}_0 = m_{r+1} = \tilde{m}_{s+1} := 0$.

First we treat $Z_{\lambda,\mu}/Z_{\lambda,\tilde{\mu}}$. By (11.3.3) and the direct calculation we have

$$\frac{N_{\lambda,\mu}(Q, q, t)}{N_{\lambda,\tilde{\mu}}(Q, q, t)}$$

$$\begin{aligned}
&= (1 - Qq^{a_\lambda(b,\ell)}t) \left[\prod_{\substack{\square \in \lambda \\ i(\square)=\ell}} \frac{1 - Qq^{-a_{\tilde{\mu}}(\square)-2}t^{-l_\lambda(\square)}}{1 - Qq^{-a_{\tilde{\mu}}(\square)-1}t^{-l_\lambda(\square)}} \right] \\
&\times \left[\prod_{\substack{\square \in \mu \\ j(\square)=b}} \frac{1 - Qq^{a_{\tilde{\mu}}(\square)}t^{-l_\lambda(\square)+2}}{1 - Qq^{a_{\tilde{\mu}}(\square)}t^{-l_\lambda(\square)+1}} \right] \\
&= (1 - Qq^{a_\lambda(b,\ell)}t) \left[\prod_{i=k}^r \frac{1 - Qq^{m_i-b}t^{\ell-(n_1+\dots+n_i)}}{1 - Qq^{m_{i+1}-b}t^{\ell-(n_1+\dots+n_i)}} \right] \\
&\times \left[\prod_{i=1}^{k-1} \frac{1 - Qq^{m_i-b}t^{\ell-(n_1+\dots+n_{i-1})}}{1 - Qq^{m_i-b}t^{\ell-(n_1+\dots+n_i)}} \right] \frac{1 - Qq^{m_k-b}t^{\ell-(n_1+\dots+n_{k-1})}}{1 - Qq^{m_k-b}t} \\
&= (1 - Qq^{-b}t^{\ell-L}) \prod_{i=1}^r \frac{1 - Qq^{m_i-b}t^{\ell-(n_1+\dots+n_i)}}{1 - Qq^{m_i-b}t^{\ell-(n_1+\dots+n_i)}}.
\end{aligned}$$

In the second equality we used the convention (11.3.11). Similar treatments yield

$$\begin{aligned}
\frac{N_{\mu,\lambda}(Q^{-1}, q, t)}{N_{\tilde{\mu},\lambda}(Q^{-1}, q, t)} &= (1 - Q^{-1}q^{b-1}t^{L-\ell+1}) \\
&\times \prod_{i=1}^r \frac{1 - Q^{-1}q^{b-1-m_i}t^{(n_1+\dots+n_{i-1})-\ell+1}}{1 - Q^{-1}q^{b-1-m_i}t^{(n_1+\dots+n_i)-\ell+1}}, \\
\frac{N_{\mu,\mu}(1, q, t)}{N_{\tilde{\mu},\tilde{\mu}}(1, q, t)} &= (1 - q^{-b})(1 - q^{b-1}t) \left[\prod_{i=1}^s \frac{1 - q^{b-\tilde{m}_i-1}t^{(\tilde{n}_1+\dots+\tilde{n}_{i-1})-\ell+1}}{1 - q^{b-\tilde{m}_i-1}t^{(\tilde{n}_1+\dots+\tilde{n}_i)-\ell+1}} \right] \\
&\times \left[\prod_{i=1}^s \frac{1 - q^{\tilde{m}_i-b}t^{\ell-(\tilde{n}_1+\dots+\tilde{n}_{i-1})}}{1 - q^{\tilde{m}_i-b}t^{\ell-(\tilde{n}_1+\dots+\tilde{n}_i)}} \right]
\end{aligned}$$

Thus by (11.3.2) and by some elementary calculation we have

$$\begin{aligned}
\frac{Z_{\lambda,\mu}(Q, q, t)}{Z_{\lambda,\tilde{\mu}}(Q, q, t)} &= \frac{Qq^{2(1-b)}t^{2(\ell-1)}}{(1 - Q^{-1}q^{b-1}t^{L-\ell+1})(1 - Qq^{a_\lambda(b,\ell)}t)(1 - q^{-b})(1 - q^{b-1}t)} \\
&\times \left[\prod_{i=1}^r \frac{1 - Qq^{m_i-b}t^{\ell-(n_1+\dots+n_i)}}{1 - Qq^{m_i-b}t^{\ell-(n_1+\dots+n_{i-1})}} \right] \left[\prod_{i=1}^r \frac{1 - Qq^{m_i-b+1}t^{\ell-(n_1+\dots+n_i)-1}}{1 - Qq^{m_i-b+1}t^{\ell-(n_1+\dots+n_{i-1})-1}} \right] \\
&\times \left[\prod_{i=1}^s \frac{1 - q^{\tilde{m}_i-b+1}t^{\ell-(\tilde{n}_1+\dots+\tilde{n}_{i-1})-1}}{1 - q^{\tilde{m}_i-b+1}t^{\ell-(\tilde{n}_1+\dots+\tilde{n}_{i-1})-1}} \right] \left[\prod_{i=1}^{s-1} \frac{1 - q^{\tilde{m}_i-b}t^{\ell-(\tilde{n}_1+\dots+\tilde{n}_i)}}{1 - q^{\tilde{m}_i-b}t^{\ell-(\tilde{n}_1+\dots+\tilde{n}_{i-1})}} \right] \\
&\times \frac{1 - t^1}{1 - t^{\ell-(\tilde{n}_1+\dots+\tilde{n}_{s-1})}}. \tag{11.3.12}
\end{aligned}$$

Next we calculate $I_{\lambda,\mu}/I_{\lambda,\tilde{\mu}}$. By the definition (11.3.7) we have

$$\begin{aligned}
\frac{I_{\lambda,\mu}(Q, q, t)}{I_{\lambda,\tilde{\mu}}(Q, q, t)} &= \frac{1 - qt^{-1}}{(1 - q)(1 - t^{-1})} \times \\
&P(Q^{-1/2}q^{b-1}t^{-\ell-1}; Q^{1/2}, q^{-1}t) \times Q^{-1/2}q^{b-1}t^{-\ell-1} \times \\
&\left[\prod_{(i,j) \in \lambda} \omega\left(\frac{Q^{-1/2}q^{b-1}t^{-(\ell-1)}}{Q^{1/2}q^{j-1}t^{-(i-1)}}; q, t^{-1}, q^{-1}t\right) \right] \times \\
&\left[\prod_{\substack{(i,j) \in \tilde{\mu} \\ (i,j) \neq (\ell,b-1)}} \omega\left(\frac{Q^{-1/2}q^{b-1}t^{-(\ell-1)}}{Q^{-1/2}q^{j-1}t^{-(i-1)}}; q, t^{-1}, q^{-1}t\right) \right] \frac{(q-1)^2(q-qt^{-1})(q-q^{-1}t)}{(q-q^{-1})(q-t)(q-t^{-1})}. \tag{11.3.13}
\end{aligned}$$

An elementary calculation yields

$$\begin{aligned}
& \prod_{(i,j) \in \lambda} \omega\left(\frac{Q^{-1/2}q^{b-1}t^{-(\ell-1)}}{Q^{1/2}q^{j-1}t^{-(i-1)}}; q, t^{-1}, q^{-1}t\right) \\
&= \frac{1 - Qq^{1-b}t^{\ell-1}}{1 - Q^{1-b}t^{\ell-L-1}} \frac{1 - Qq^{-b}t^{\ell}}{1 - Q^{-b}t^{\ell-L}} \\
&\times \left[\prod_{i=1}^L \frac{1 - Qq^{\lambda_i+1-b}t^{\ell-i-1}}{1 - Qq^{\lambda_i+1-b}t^{\ell-i}} \right] \left[\prod_{i=1}^L \frac{1 - Qq^{\lambda_i-b}t^{\ell-i}}{1 - Qq^{\lambda_i-b}t^{\ell-i+1}} \right] \\
&= \frac{1 - Qq^{1-b}t^{\ell-1}}{1 - Q^{1-b}t^{\ell-L-1}} \frac{1 - Qq^{-b}t^{\ell}}{1 - Q^{-b}t^{\ell-L}} \\
&\times \left[\prod_{i=1}^r \frac{1 - Qq^{m_i-b}t^{\ell-(n_1+\dots+n_i)}}{1 - Qq^{m_i-b}t^{\ell-(n_1+\dots+n_{i-1})}} \right] \left[\prod_{i=1}^r \frac{1 - Qq^{m_i-b+1}t^{\ell-(n_1+\dots+n_i)-1}}{1 - Qq^{m_i-b+1}t^{\ell-(n_1+\dots+n_{i-1})-1}} \right],
\end{aligned}$$

where at the second equality we used the convention (11.3.11). A similar consideration give

$$\begin{aligned}
& \left[\prod_{\substack{(i,j) \in \tilde{\mu} \\ (i,j) \neq (\ell, b-1)}} \omega\left(\frac{Q^{-1/2}q^{b-1}t^{-(\ell-1)}}{Q^{-1/2}q^{j-1}t^{-(i-1)}}; q, t^{-1}, q^{-1}t\right) \right] \frac{(q-1)^2(q-qt^{-1})(q-q^{-1}t)}{(q-q^{-1})(q-t)(q-t^{-1})} \\
&= \frac{1 - q^{1-b}t^{\ell-1}}{1 - q^{1-b}t^{-1}} \frac{1 - q^{-b}t^{\ell}}{1 - q^{-b}} \\
&\times \left[\prod_{i=1}^s \frac{1 - q^{\tilde{m}_i-b+1}t^{\ell-(\tilde{n}_1+\dots+\tilde{n}_i)-1}}{1 - q^{\tilde{m}_i-b+1}t^{\ell-(\tilde{n}_1+\dots+\tilde{n}_{i-1})-1}} \right] \left[\prod_{i=1}^{s-1} \frac{1 - q^{\tilde{m}_i-b}t^{\ell-(\tilde{n}_1+\dots+\tilde{n}_i)}}{1 - q^{\tilde{m}_i-b}t^{\ell-(\tilde{n}_1+\dots+\tilde{n}_{i-1})}} \right] \\
&\times \frac{1-t}{1 - t^{\ell-(\tilde{n}_1+\dots+\tilde{n}_{s-1})}}.
\end{aligned}$$

Substituting these terms, one finds that (11.3.13) is equal to (11.3.12). This is the desired consequence. $\square$

REMARK 11.3.4. In the above integral expression one may find a reminiscence of the Feigin-Odesskii algebra with three parameters, which was investigated in [32, 37, 80]. Note also that the specialization (11.3.6) also appeared in the Gordon filtration introduced in [32].

11.4. Recursion relation of K -theoretic partition function

The main theorem of this chapter is

THEOREM 11.4.1. The five-dimensional Nekrasov partition function

$$Z(\Lambda, Q, q, t) = \sum_{n=0}^{\infty} (\Lambda^4 t/q)^n Z_n(Q, q, t)$$

satisfies the same recursive formula (11.2.3) as in Conjecture 11.2.3:

$$Z_n(Q, q, t) = \delta_{n,0} + \sum_{\substack{r,s \in \mathbb{Z}, \\ 1 \leq rs \leq n}} \frac{G(r, s; q, t) Z_{n-rs}(q^r t^s, q, t)}{Q - q^r t^{-s}}.$$

The proof is divided in two steps (Propositions 11.4.2 and 11.4.4). We first show that each pole of Z_n is simple when Z_n is viewed as a rational function of the variable Q .

PROPOSITION 11.4.2. As a rational function of the variable Q , $Z_n(Q, q, t)$ has poles at $Q = q^r t^{-s}$ ($r, s \in \mathbb{Z}$, $1 \leq rs \leq n$). Moreover each pole is simple.

Before starting the proof of this proposition, we prepare the next lemma, which is inspired by [30, §A] and [47, Appendix].

LEMMA 11.4.3. Let $f(x_1, \dots, x_n)$ be a symmetric polynomial function of all its arguments. Assume that the integral

$$I_n(Q, q, t) = \oint_{C_n} \frac{dx_n}{2\pi\sqrt{-1}} \cdots \oint_{C_1} \frac{dx_1}{2\pi\sqrt{-1}} \left(\frac{f(x_1, \dots, x_n)}{\prod_{k=1}^n (x_k - Q^{1/2})(x_k - Q^{-1/2})} \right. \\ \left. \times \prod_{1 \leq i < j \leq n} \frac{(x_j/x_i - 1)^2 (x_j/x_i - qt^{-1})(x_j/x_i - q^{-1}t)}{(x_j/x_i - q)(x_j/x_i - q^{-1})(x_j/x_i - t)(x_j/x_i - t^{-1})} \right) \quad (11.4.1)$$

converges, where the contour C_k is chosen so that it surrounds only the poles at $Q^{1/2}$, $Q^{-1/2}$, x_jq and x_jt^{-1} for $j \neq k$. Then $I_n(Q, q, t)$ is a holomorphic function of Q .

PROOF. Let us integrate over the variable x_n first. By the choice of C_n , the poles at $x_n = Q^{\pm 1/2}$, $x_n = x_jq$ and $x_n = x_jt^{-1}$ ($j < n$) are inside the contour, while the poles at $x_n = x_jq^{-1}$ and $x_n = x_jt$ are outside.

If we vary Q then some of these poles move. When two of the poles coincide, a higher order singularity of the integrand may occur. If this happens for the poles on the same side of the contour, then one can move the contour away from the colliding pair and the integral contains no singularity. If the pair approach from the opposite sides of the contour, then the integral becomes singular.

It is convenient to express the contour C_n as $C_n = C_n^a - \sum_{1 \leq k \leq n-1} C_{n,k}^d$, where C_n^a is the contour surrounding all possible poles of x_n , and $C_{n,k}^d$ is the contour surrounding the point x_kq^{-1} and x_kt . By the observation above, the integral over C_n^a does not contribute to the singularity. Thus we only have to consider the integrals over $C_{n,k}^d$. The factors depending on x_n are expressed as follows:

$$J_n := \sum_{1 \leq k \leq n-1} \oint_{-C_{n,k}^d} \frac{dx_n}{2\pi\sqrt{-1}} \left(\frac{f(x_1, \dots, x_n)}{(x_n - Q^{1/2})(x_n - Q^{-1/2})} \right. \\ \left. \times \prod_{1 \leq i \leq n-1} \frac{(x_n/x_i - 1)^2 (x_n/x_i - qt^{-1})(x_n/x_i - q^{-1}t)}{(x_n/x_i - q)(x_n/x_i - q^{-1})(x_n/x_i - t)(x_n/x_i - t^{-1})} \right) \\ = - \sum_{1 \leq k \leq n-1} \left[\frac{f(x_1, \dots, x_{n-1}, x_kq^{-1})}{(x_kq^{-1} - Q^{1/2})(x_kq^{-1} - Q^{-1/2})} \frac{(1-q)^2(1-q^2t^{-1})(1-t)}{(1-q^2)(1-qt)(1-qt^{-1})} \right. \\ \times \prod_{\substack{1 \leq i \leq n-1 \\ i \neq k}} \frac{(x_k/x_i - q)^2 (x_k/x_i - q^2t^{-1})(x_k/x_i - t)}{(x_k/x_i - q^2)(x_k/x_i - 1)(x_k/x_i - qt)(x_k/x_i - qt^{-1})} \\ + \frac{f(x_1, \dots, x_{n-1}, x_kt)}{(x_kt - Q^{1/2})(x_kt - Q^{-1/2})} \frac{(1-t^{-1})^2(1-qt^{-2})(1-q^{-1})}{(1-t^{-2})(1-qt^{-1})(1-q^{-1}t^{-1})} \\ \left. \times \prod_{\substack{1 \leq i \leq n-1 \\ i \neq k}} \frac{(x_k/x_i - t^{-1})^2 (x_k/x_i - qt^{-2})(x_k/x_i - q^{-1})}{(x_k/x_i - qt^{-1})(x_k/x_i - q^{-1}t^{-1})(x_k/x_i - 1)(x_k/x_i - t^{-2})} \right].$$

Now we have several apparent poles in the integrand. Since we have assumed that $f(x_1, \dots, x_n)$ is symmetric, we find that $x_k = x_i$ is not a genuine pole. Moreover two types of poles $x_k = x_iqt$ and $x_k = x_iq^{-1}t^{-1}$ are not poles. In fact if we exchange i and k in the second line of J_n then the terms containing $x_k = x_iqt$ can be calculated as

$$\frac{f(x_1, \dots, x_{n-1}, x_kq^{-1})}{(x_kq^{-1} - Q^{1/2})(x_kq^{-1} - Q^{-1/2})} \frac{(1-q)(1-q^2t^{-1})(1-t)}{(1+q)(1-qt)(1-qt^{-1})} \\ \times \prod_{\substack{1 \leq j \leq n-1 \\ j \neq k}} \frac{(x_k/x_j - q)^2 (x_k/x_j - q^2t^{-1})(x_k/x_j - t)}{(x_k/x_j - q^2)(x_k/x_j - 1)(x_k/x_j - qt)(x_k/x_j - qt^{-1})} \\ + \frac{f(x_1, \dots, x_{n-1}, x_it)}{(x_it - Q^{1/2})(x_it - Q^{-1/2})} \frac{(1-t^{-1})(1-qt^{-2})(1-q^{-1})}{(1+t^{-1})(1-qt^{-1})(1-q^{-1}t^{-1})}$$

$$\begin{aligned}
& \times \prod_{\substack{1 \leq j \leq n-1 \\ j \neq k}} \frac{(x_i/x_j - t^{-1})^2 (x_i/x_j - qt^{-2})(x_i/x_j - q^{-1})}{(x_i/x_j - qt^{-1})(x_i/x_j - q^{-1}t^{-1})(x_i/x_j - 1)(x_i/x_j - t^{-2})} \\
& = \frac{(1-q)(1-t)}{(1-qt)(1-qt^{-1})} \frac{(x_k/x_i - q)(x_k/x_i - t)}{(x_k/x_i - 1)(x_k/x_i - qt)} H(x_1, \dots, x_{n-1}, Q, q, t)
\end{aligned} \tag{11.4.2}$$

where we have introduced

$$\begin{aligned}
H(x_1, \dots, x_{n-1}, Q, q, t) &:= \\
& \frac{f(x_1, \dots, x_{n-1}, x_k q^{-1})}{(x_k q^{-1} - Q^{1/2})(x_k q^{-1} - Q^{-1/2})} \frac{1 - q^2 t^{-1}}{1 + q} \frac{(x_k/x_i - q)(x_k/x_i - q^2 t^{-1})}{(x_k/x_i - q^2)(x_k/x_i - qt^{-1})} \\
& \times \prod_{\substack{1 \leq j \leq n-1 \\ j \neq i, k}} \frac{(x_k/x_j - q)^2 (x_k/x_j - q^2 t^{-1})(x_k/x_j - t)}{(x_k/x_j - q^2)(x_k/x_j - 1)(x_k/x_j - qt)(x_k/x_j - qt^{-1})} \\
& - \frac{f(x_1, \dots, x_{n-1}, x_i t)}{(x_i t - Q^{1/2})(x_i t - Q^{-1/2})} \frac{1 - qt^{-2}}{1 + t^{-1}} \frac{(x_k/x_i - t)(x_k/x_i - q^{-1} t^2)}{(x_k/x_i - t^2)(x_k/x_i - q^{-1} t)} \\
& \times \frac{(x_i/x_j - t^{-1})^2 (x_i/x_j - q^2 t^{-1})(x_i/x_j - q^{-1})}{(x_i/x_j - qt^{-1})(x_i/x_j - 1)(x_i/x_j - q^{-1} t^{-1})(x_i/x_j - t^{-2})}.
\end{aligned}$$

Then we can calculate the residue of (11.4.2) at $x_k = x_i q t$ as

$$\begin{aligned}
& H(x_1, \dots, x_{n-1}, Q, q, t)|_{x_k = x_i q t} \\
& = f(x_1, \dots, x_{n-1}, x_i t) \\
& \times \prod_{\substack{1 \leq j \leq n-1 \\ j \neq i, k}} \frac{(x_i/x_j - t^{-1})^2 (x_i/x_j - q^2 t^{-1})(x_i/x_j - q^{-1})}{(x_i/x_j - qt^{-1})(x_i/x_j - 1)(x_i/x_j - q^{-1} t^{-1})(x_i/x_j - t^{-2})} \\
& \times \left(\frac{1 - q^2 t^{-1}}{1 + q} \frac{(qt - q)(qt - q^2 t^{-1})}{(qt - q^2)(qt - qt^{-1})} - \frac{1 - qt^{-2}}{1 + t^{-1}} \frac{(qt - t)(qt - q^{-1} t^2)}{(qt - t^2)(qt - q^{-1} t)} \right).
\end{aligned}$$

Since the last line is zero, J_n contains no pole at $x_k = x_i q t$. The case $x_k = x_i q^{-1} t^{-1}$ is similar and we omit the detail.

Thus after performing the integral over x_n , I_n becomes an integral over $x_1, x_2, \dots, x_{n-1}$ with the function f replaced by some holomorphic function. The parameters $(Q^{\pm 1}, q, t^{-1})$ are replaced by $(Q^{\pm 1} q^i t^{-j}, q^a, t^{-b})$ with some $i, j, a, b \in \mathbb{Z}_{\geq 1}$. At last the integral becomes the form as

$$I_n(Q, q, t) = \oint_{C_1} \frac{dx_1}{2\pi\sqrt{-1}} \frac{\tilde{f}(x_1)}{\prod_{\alpha, \beta} (x_1 - Q^{1/2} q^{\alpha} t^{-\beta})(x_1 - Q^{-1/2} q^{\alpha} t^{-\beta})}. \tag{11.4.3}$$

Here α and β run over a finite set of non-negative integers, and $\tilde{f}$ is a polynomial of x_1 . We also find that no higher poles occur in this integral, i.e., the terms $x_1 - Q^{\pm 1/2} q^{\alpha} t^{-\beta}$ are different from each other. By the choice of C_1 all the poles of x_1 are inside the contour, so that the result of the integral contains no singularity with respect to Q . This is the desired consequence. $\square$

Now we turn to the proof of Proposition 11.4.2

PROOF OF PROPOSITION 11.4.2. The integral expression (11.3.5) of Z_n is the same form as I_n (11.4.1) except for the term $f(x_1, \dots, x_n)$ which is not holomorphic. Thus by the same argument of the proof of Lemma 11.4.3, The pole of Z_n as a rational function of Q may only come from the collision of poles at $x_i = Q^{\pm 1/2} q^{\alpha} t^{-\beta}$ and $x_i = Q^{\mp 1/2} q^{-1} t$. Note that the singularity with respect to Q are all single poles by the argument after (11.4.3) and by the assumption that the parameters are generic and . Then by the original definition (11.3.1) of Z_n , the poles of Q are at $Q = q^r t^{-s}$ ($r, s \in \mathbb{Z}, 1 \leq rs \leq n$). This is the desired consequence. $\square$

The second step calculates the residue of $Z_n(Q, q, t)$ at $Q = q^r t^{-s}$.

PROPOSITION 11.4.4.

$$\operatorname{Res}_{Q=q^r t^{-s}} Z_n(Q, q, t) = G(r, s; q, t) Z_{n-rs}(q^r t^s, q, t). \quad (11.4.4)$$

PROOF. We assume that $r, s \geq 1$. The case $r, s \leq -1$ can be treated similarly.

From the combinatorial expression (11.3.1), the pole at $Q = q^r t^{-s}$ only comes from the pair (λ, μ) of partitions such that μ contains (if displayed by the Young diagram) the rectangle r^s .

The residue of the pole at $Q = q^r t^{-s}$ can be evaluated by using the integral form (11.3.5). The condition of (λ, μ) corresponds to taking the residues of the poles with respect to x_i at $Q^{-1/2}, Q^{-1/2}q, \dots, Q^{-1/2}q^r t^{-s}$. By relabelling the indices of x_i , we have

$$\begin{aligned} \operatorname{Res}_{Q=q^r t^{-s}} Z_n(Q, q, t) = & \frac{1}{(n-rs)!} \left(\frac{1-qt^{-1}}{(1-q)(1-t^{-1})} \right)^{n-rs} \oint_{C_n} \frac{dx_n}{2\pi\sqrt{-1}} \cdots \oint_{C_{rs+1}} \frac{dx_{rs+1}}{2\pi\sqrt{-1}} \left(\right. \\ & \left. \prod_{k=rs+1}^n P(x_k; Q^{1/2}, q^{-1}t) \prod_{k \leq l} \omega(x_l/x_k; q, t^{-1}, q^{-1}t) K_{r,s} \right). \end{aligned}$$

Here the indices i, j, k, l runs as $1 \leq i, j \leq rs$ and $rs < k, l \leq n$. The function $K_{r,s}$ is defined to be

$$\begin{aligned} K_{r,s} := & \operatorname{Res}_{Q=q^r t^{-s}} \frac{1}{(rs)!} \left(\frac{1-qt^{-1}}{(1-q)(1-t^{-1})} \right)^{rs} \oint_{C_{rs}} \frac{dx_{rs}}{2\pi\sqrt{-1}} \cdots \oint_{C_1} \frac{dx_1}{2\pi\sqrt{-1}} \left(\right. \\ & \left. \prod_{i,k} \omega(x_k/x_i; q, t^{-1}, q^{-1}t) \prod_{i=1}^{rs} P(x_i; Q^{1/2}, q^{-1}t) \prod_{i < j} \omega(x_j/x_i; q, t^{-1}, q^{-1}t) \right) \\ = & \operatorname{Res}_{Q=q^r t^{-s}} Z_{\emptyset, (rs)}(Q, q, t) \prod_{k=rs+1}^N \prod_{\alpha=1}^r \prod_{\beta=1}^s \omega(x_k/(Q^{-1/2}q^\alpha t^{-\beta}); q, t^{-1}, q^{-1}t). \end{aligned}$$

Note that

$$\begin{aligned} & P(x; Q^{1/2}, q^{-1}t) \prod_{\alpha=1}^r \prod_{\beta=1}^s \omega(x/(Q^{-1/2}q^\alpha t^{-\beta}); q, t^{-1}, q^{-1}t) \\ = & \frac{x}{(x - Q^{-1/2}q^r)(x - Q^{-1/2}t^{-s})(x - Q^{-1/2}q^r t^{-s})(x - Q^{-1/2}t^{-s}q^{-1}t)} \\ & \times \frac{(x - Q^{-1/2}q^r t^{-s})(x - Q^{-1/2}q^{r-1}t^{-s+1})}{(x - Q^{1/2})(x - Q^{1/2}q^{-1}t)}. \end{aligned} \quad (11.4.5)$$

If $Q = q^r t^{-s}$, then the last line of (11.4.5) is equal to one and we have

$$\begin{aligned} & P(x; Q^{1/2}, q^{-1}t) \prod_{\alpha=1}^r \prod_{\beta=1}^s \omega(x/(Q^{1/2}q^\alpha t^{-\beta}); q, t^{-1}, q^{-1}t) \Big|_{Q=q^r t^{-s}} \\ = & \frac{x}{(x - q^{r/2}t^{s/2})(x - q^{-r/2}t^{-s/2})(x - q^{r/2}t^{s/2}q^{-1}t)(x - q^{-r/2}t^{-s/2}q^{-1}t)} \\ = & P(x; (q^r t^s)^{1/2}, q^{-1}t). \end{aligned}$$

Thus we obtain

$$\operatorname{Res}_{Q=q^r t^{-s}} Z_n(Q, q, t) = Z_{n-rs}(q^r t^s, q, t) \cdot \operatorname{Res}_{Q=q^r t^{-s}} Z_{\emptyset, (rs)}(Q, q, t). \quad (11.4.6)$$

Comparing (11.4.6) and the statement (11.4.4), we find that it is enough to show $\operatorname{Res}_{Q=q^r t^{-s}} Z_{\emptyset, (rs)}(Q, q, t) = G(r, s; q, t)$. By the direct calculation we have

$$\begin{aligned} N_{\emptyset, \emptyset}(1) &= 1, \\ N_{(rs), (rs)}(1) &= \prod_{\substack{-r \leq i \leq -1 \\ 0 \leq j \leq s-1}} (1 - q^i t^{-j}) \times \prod_{\substack{0 \leq i \leq r-1 \\ -s \leq j \leq -1}} (1 - q^i t^{-j}), \end{aligned}$$

$$\begin{aligned} \operatorname{Res}_{Q=q^r t^{-s}} N_{(\emptyset), (rs)}(Q) &= (-q^r t^{-s}) \prod_{\substack{0 \leq i \leq r-1 \\ 0 \leq j \leq s-1 \\ (i,j) \neq (0,0)}} (1 - q^i t^{-j}), \\ N_{(rs), (\emptyset)}(Q^{-1}) \Big|_{Q=q^r t^{-s}} &= \prod_{\substack{-r \leq i \leq -1 \\ -s \leq j \leq -1}} (1 - q^i t^{-j}). \end{aligned}$$

Multiplying these factors and recalling the definition (11.2.4) of $G(r, s; q, t)$, we obtain

$$\operatorname{Res}_{Q=q^r t^{-s}} Z_{\emptyset, (rs)}(Q, q, t) = -q^r t^{-s} \prod_{\substack{-r \leq i \leq r-1 \\ -s \leq j \leq s-1 \\ (i,j) \neq (0,0)}} (1 - q^i t^{-j}) = G(r, s; q, t).$$

This is the desired consequence. □

APPENDIX A

Combinatorial Notations

Here we list the notations used throughout this thesis.

A.1. Partition

A partition of positive integer n means a sequence $(\lambda_1, \lambda_2, \dots, \lambda_k)$ of positive integers such that $\lambda_1 \geq \lambda_2 \geq \dots$ and $\sum_{i=1}^k \lambda_i = n$.

We also consider the empty sequence $\emptyset = ()$ as a partition of the number 0.

The symbol $\lambda \vdash n$ means that λ is a partition of n .

For a partition λ of n , we also denote $|\lambda| := n = \sum_{i=1}^k \lambda_i$.

In addition we denote by $\mathcal{P}$ the set of all the partitions of natural numbers including the empty partition $\emptyset$. So that we have

$$\mathcal{P} = \{\emptyset, (1), (2), (1, 1), (3), (2, 1), (1, 1, 1), \dots\}.$$

As usual, $p(n) := \#\{\lambda \in \mathcal{P} \mid |\lambda| = n\} = \#\{\lambda \mid \lambda \vdash n\}$ denotes the number of partitions of n .

In the main text, we sometimes use the dominance semi-ordering on the partitions: $\lambda \geq \mu$ if and only if $|\lambda| = |\mu|$ and $\sum_{k=1}^i \lambda_k \geq \sum_{k=1}^i \mu_k$ ($i = 1, 2, \dots$).

EXAMPLE A.1.1. For $n \geq 5$, the dominance ordering on $\mathcal{P}_n$ is a total ordering:

$$\begin{aligned} (2) &> (1, 1), \\ (3) &> (2, 1) > (1, 1, 1), \\ (4) &> (3, 1) > (2, 2) > (2, 1, 1) > (1, 1, 1, 1), \\ (5) &> (4, 1) > (3, 2) > (3, 1^2) > (2^2, 1) > (2, 1^3) > (1^5). \end{aligned}$$

For the partitions of 6, the dominance ordering is not total:

$$(6) > (5, 1) > (4, 2) \begin{matrix} > (4, 1^2) > \\ > (3^2) > \end{matrix} (3, 2, 1) \begin{matrix} > (3, 1^3) > \\ > (2^3) > \end{matrix} (2^2, 1^2) > (2, 1^4) > (1^6).$$

In Figure 1.1.1, we show the Hasse diagrams of partitions of 6 and 7 in dominance ordering.

A.2. Young diagram

Basically following [66], we prepare several notations for Young diagrams.

To a partition $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_k)$, we associate the Young diagram, which is the set of boxes located at $\{(i, j) \in \mathbb{Z}^2 \mid 1 \leq i \leq k, 1 \leq j \leq \lambda_i\}$. The coordinate (i, j) is taken so that the index i increases if one reads from top to bottom, and the index j increases if one reads from left to right. (See Figure 1.2.1 for example.) We will often identify a partition and its associated Young diagram.

Next we define the arm and leg of a box $\square$ located at $(i, j) \in \mathbb{Z}_{\geq 1}^2$ with respect to λ by

$$a_\lambda(\square) := \lambda_i - j, \quad \ell_\lambda(\square) := \lambda_j^\vee - i. \tag{A.2.1}$$

Here $\lambda^\vee$ is the conjugate partition of λ , which is obtained by transposing the Young diagram of λ . We also used the convention $\lambda_i = 0$ for $i > \ell(\lambda)$ and $\lambda_j^\vee = 0$ for $j > \lambda_1$. Thus $a_\lambda(\square)$ and $\ell_\lambda(\square)$ could be minus in general.

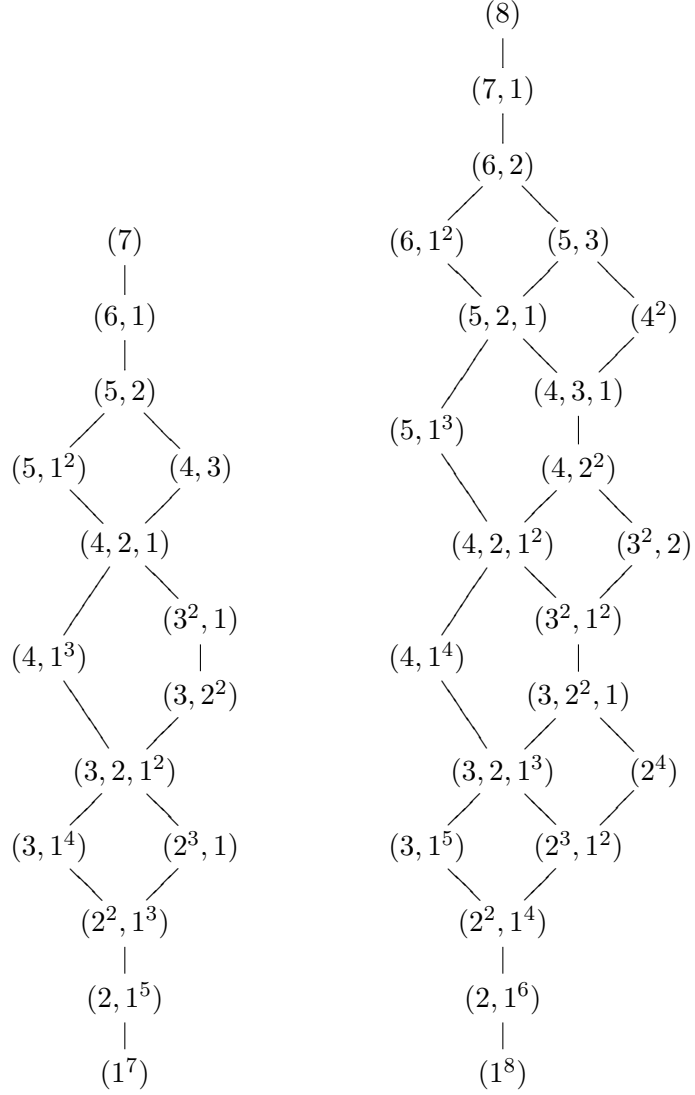


FIGURE 1.1.1. Hasse diagram of partitions of 7 and 8

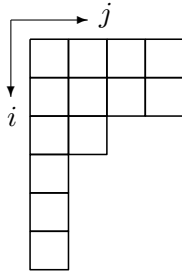
Let us also use the notation $(i, j) \in \lambda$, which means that $i, j \in \mathbb{Z}_{\geq 1}$, $1 \leq i \leq \ell(\lambda)$ and $1 \leq j \leq \lambda_i$. For the Young diagram of λ , the symbol $(i, j) \in \lambda$ corresponds to the box located at the coordinate (i, j) .

EXAMPLE A.2.1. For $\lambda = (4, 4, 2, 1, 1, 1)$ as in Figure 1.2.1, we have $\lambda^\vee = (6, 3, 2, 2)$.

For $\square = (3, 1)$, we have $a_\lambda(\square) = 1$ and $\ell_\lambda(\square) = 3$.

For $\blacksquare = (4, 3)$, we have $a_\lambda(\blacksquare) = -2$ and $\ell_\lambda(\blacksquare) = -2$.

Note also that $(2, 3) \in \lambda := (4, 4, 2, 1, 1, 1)$ but $(4, 3) \notin \lambda$.

FIGURE 1.2.1. The Young diagram for $(4, 4, 2, 1, 1, 1)$

Following [72], we introduce the symbols $\heartsuit \lambda^{(\alpha)}$ and $\spadesuit \lambda^{(\alpha)}$ as follows.

If two partitions $\lambda^{(\alpha)}$ and $\lambda^{(\beta)}$ are given, we separate $\lambda^{(\alpha)}$ into two regions $\heartsuit\lambda^{(\alpha)}$ and $\spadesuit\lambda^{(\alpha)}$ as

$$\heartsuit\lambda^{(\alpha)} := \left\{ (i, j) \in \lambda^{(\alpha)} \mid j \leq \lambda^{(\beta)}_1 \right\}, \quad \spadesuit\lambda^{(\alpha)} := \left\{ (i, j) \in \lambda^{(\alpha)} \mid j > \lambda^{(\beta)}_1 \right\}.$$

In other words, we separate $\lambda^{(\alpha)}$ at the line $j = \lambda^{(\beta)}_1$.

If $\lambda^{(\alpha)}_1 \leq \lambda^{(\beta)}_1$, then $\spadesuit\lambda^{(\alpha)} = \emptyset$. Switching the roles of α and β , we divide $\lambda^{(\beta)}$ into $\heartsuit\lambda^{(\beta)}$ and $\spadesuit\lambda^{(\beta)}$. Note that either $\spadesuit\lambda^{(\alpha)}$ or $\spadesuit\lambda^{(\beta)}$ is the empty set.

A.3. Symmetric function

In this thesis, the term *symmetric function* means the infinite-variable symmetric polynomial. To treat such an object rigorously, we follow the argument of [66, Chap.I §2].

Let us denote by Λ_n the space of n -variable symmetric polynomials over $\mathbb{Z}$, and by Λ_n^d the space of homogeneous symmetric polynomials of degree d . The space of symmetric functions Λ is defined as the inverse limit of the Λ_n in the category of graded rings (with respect to the gradation by the degree d).

We denote by $\Lambda_K := \Lambda \otimes_{\mathbb{Z}} K$ the coefficient extension to the ring K .

Among several bases of Λ , the power sum symmetric function

$$p_n = p_n(x) := \sum_{i \in \mathbb{Z}_{\geq 1}} x_i^n, \quad p_\lambda := p_{\lambda_1} p_{\lambda_2} \cdots p_{\lambda_k},$$

plays an important role. It is known that $\{p_\lambda \mid \lambda \vdash d\}$ is a basis of $\Lambda_{\mathbb{Q}}^d$, the space of homogeneous symmetric functions of degree d .

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