RECURRENCE FOR COSINE SERIES WITH BOUNDED GAPS

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(Degree)
博士 (理学)
(Date of Degree)
2012-03-25
(Date of Publication)
2012-09-10
(Resource Type)
doctoral thesis
(Report Number)
甲5575
(URL)
https://hdl.handle.net/20.500.14094/D1005575

Doctoral Dissertation

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(有界間隙余弦列の再帰性)

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Abstract. Ullrich, Grubb and Moore proved that a lacunary trigonometric series satisfying Hadamard's gap condition is recurrent a.e. We prove the existence of a recurrent trigonometric series with bounded gaps.

1 Introduction.

If we regard the sequence $\{\cos 2\pi n_k x\}$ as a sequence of random variables on the unit interval equipped with the Lebesgue measure, it behaves like a sequence of independent random variables when n_k diverges rapidly. For example, by assuming Hadamard's gap condition

$$n_{k+1}/n_k > q > 1 \ (k = 1, 2, \dots),$$

the central limit theorem for $\sum \cos 2\pi n_k x$ was proved by Salem and Zygmund [9], the law of the iterated logarithm by Erdős and Gál [4], and the almost sure invariance principles by Philipp and Stout [8].

As to recurrence, Hawkes [7] proved that $\{\sum_{k=1}^{N} \exp(2\pi i n_k x)\}_{N\in\mathbb{N}}$ is dense in the complex plane for a.e. x assuming the very strong gap condition to $\sum n_k/n_{k+1} < \infty$. Anderson and Pitt [1] weakened the gap condition to $n_{k+1}/n_k \to \infty$ or $n_k = a^k$, where $a \geq 2$ is an integer. These results imply the recurrence of $\sum_{k=1}^{N} \cos(2\pi n_k x)$. For this one-dimensional recurrence, Ullrich, Grubb and Moore [11, 5] succeeded in weakening the condition to Hadamard's gap condition.

It is very natural to ask if the gap condition can be replaced by a weaker one. For the central limit theorem, Erdős [3] relaxed the gap condition to $n_{k+1}/n_k > 1+c_k/\sqrt{k}$ with $c_k \to \infty$. This condition is best possible. Actually Erdős [3] and Takahashi [10] constructed counter examples to the central limit theorem satisfying $n_{k+1}/n_k > 1 + c/\sqrt{k}$ with $c > 0$. But there still remains the possibility that some series having smaller gaps may obey the central limit theorem. Indeed, for any $\phi(k) \uparrow \infty$, Berkes [2] proved the existence of $\sum \cos 2\pi n_k x$ with small gaps $n_{k+1} - n_k = O(\phi(k))$ which obeys the central limit theorem. And it was a longstanding problem whether some trigonometric series with bounded gaps $n_{k+1} - n_k = O(1)$ can obey the central limit theorem. Recently the existence of such series was proved in [6] and the problem was solved.

In this paper, we consider the same problem for recurrence, and prove the existence of recurrent series with bounded gaps.

Theorem 1. *Suppose that* $\{n_k\}$ *satisfies the Hadamard's gap condition and* $\{m_j\}$ *is an arrangement in increasing arrangement of* $\mathbb{N}\setminus\{n_k\}$*. If we put*

$$S_N(x)=\sum_{j=1}^{N}\cos 2\pi m_j x,$$

then $\{S_N(x)\}$ *is recurrent a.e.* x *.*

The sequence $\{n_k\}$ satisfying the Hadamard's gap condition has null density $\lim_{k\to\infty} n_k/k=0$, and its complement sequence $\{m_k\}$ defined above has full density $\lim_{k\to\infty} m_k/k=1$. Both of these define recurrent trigonometric series. We can also construct a sequence with bounded gaps and intermediate density defining recurrent trigonometric series.

Theorem 2. *Let* p/q $(p,q\in\mathbb{N})$ *be an arbitrary rational number in* $(0,1)$*. Put* $I_{p,q}=\{lq+j|\ l=0,1,2,...;\ j=1,2,...,p\}$ *and suppose that* $\{n_k\}$ *is a sequence satisfying Hadamard's gap condition and* $\{n_k\}\bigcap I_{p,q}=\phi$*. Let* $\{m_j\}$ *be an increasing arrangement order of* $\{n_k\}\bigcup I_{p,q}$*. Then* $\sum\cos 2\pi m_k x$ *is recurrent a.e.* x*, and* $\{m_j\}$ *has density* $\lim_{k\to\infty} m_k/k=p/q$ *.*

The proofs are modifications of those in Grubb and Moore [5]. We use the properties of the Dirichlet kernel.

2 Proof.

We use a lemma which is a modification of that in Grubb and Moore [5].

Lemma 3. *Let* I *be a non-empty open interval* $E_N,F_N\subset I(N\in\mathbb{N}),c>0$ *and* $0<\delta_N\downarrow 0$*. Assume that for any* $x\in E_N$*, there exists* N_0 *such that for* $N\geq N_0$*, there exists an interval* J_N *with* $x\in J_N,|J_N|=\delta_N$ *and* $|F_N\cap J_N|\geq c|J_N|$*. If* $x\in E_N$ *infinitely often almost every* $x\in I$*, then* $x\in F_N$ *infinitely often for almost every* $x\in I$ *.*

Proof of Theorem 1: Take $\rho>0$ arbitrarily and take an open interval $I\subset[0,1]$ such that $2\sin\pi qx>\rho$ on I. Since ρ is arbitrary, it is sufficient to prove the recurrence for a.e.$x\in I$.

Put $\Delta = 2\pi\left(\frac{q}{q-1} + \frac{4}{\rho^2}\right)$ and take an arbitrary $\epsilon \in \left(0, \frac{\Delta}{2}\right)$. We have

$$S_N(x) = D_{m_N}(x) - \frac{1}{2} - \sum_{j:n_j \le m_N} \cos 2\pi n_j x,$$

where $D_n(x)$ is the Dirichlet kernel given by

$$D_n(x) = \frac{1}{2} + \sum_{j=1}^{n} \cos 2\pi j x = \frac{\sin \pi(2n+1)x}{2 \sin \pi x}.$$

It is easily verified that $|D'_n(x)| \le 2\pi(2+2n)/\rho^2 \le 8\pi n/\rho^2$ on I and $|T'_j(x)| \le 2\pi(n_1 + n_2 + ... + n_j) \le 2\pi n_j q/(q-1)$ where $T_j(x) = \cos 2\pi n_1 x + \cos 2\pi n_2 x + \cdots + \cos 2\pi n_j x$. Hence $S'_N(x) \le \Delta m_N$ on I. Take an arbitrary $a \in \mathbb{R}$ and put

$$E_N = \{x \in I : S_N(x) \ge a, S_{N+1}(x) < a\},$$

$$F_N = \{x \in I : |S_N(x) - a| < \epsilon \text{ or } |S_{N+1}(x) - a| < \epsilon\}.$$

By noting $|D_n(x)| \le \frac{1}{\rho}$ and the properties $\sup_j T_j(x) = \infty$ and $\inf_j T_j(x) = -\infty$ a.e. of lacunary trigonometric series (205pp of Zygmund [12]), we have $\sup_N S_N(x) = \infty$ and $\inf_N S_N(x) = -\infty$, for a.e. $x \in I$. Hence $x \in E_N$ infinitely often for a.e. $x \in I$.

Pick an arbitrary $x \in E_N$. Put $\delta_N = 1/m_{N+1}$ and $J_N = (x - \delta_N/2, x + \delta_N/2)$. We have $J_N \subset I$ for large N. We divide the proof into two cases:

Case I: *there exists an $x_0 \in J_N$ such that $S_N(x_0) = a$.* Then we have $|S_N(x) - a| < \epsilon$ on $(x_0 - |J_N|\epsilon/\Delta, x_0 + |J_N|\epsilon/\Delta)$. Since $J_N\epsilon/\Delta \le |J_N|/2$, either $(x_0 - |J_N|\epsilon/\Delta, x_0)$ or $(x_0, x_0 + |J_N|\epsilon/\Delta)$ is contained in J_N and hence on $F_N \cap J_N$. Therefore $|F_N \cap J_N| \ge |J_N|\epsilon/\Delta$.

Case II: *$S_N(x) > a$ on J_N.* As $x \in E_N$, we have $S_N(x) \ge a$ and $S_{N+1}(x) < a$. Since $|J_N| = 1/m_{N+1}$, there exists an $x_1 \in J_N$ such that $\cos 2\pi m_{N+1} x_1 = 0$. Hence $S_{N+1}(x_1) = S_N(x_1) \ge a$, and therefore we can find $x_2 \in J_N$ such that $S_{N+1}(x_2) = a$. In the same way as in the previous case, we can see that $|F_N \cap J_N| \ge |J_N|\epsilon/\Delta$.

Applying the lemma, we see that $x \in F_N$ infinitely often for a.e. $s \in I$.

Theorem 2 can be proved in the same way by noting that

$$D_n(x) = \sum_{l=1}^{n} \cos 2\pi(lq + j)x = \frac{\sin \pi((2n+1)q + 2j)x - \sin \pi(q + 2j)x}{2 \sin \pi qx}.$$

3 Differentiation of Monotone functions

3.1 Vitali Covering

Let J be a collection of intervals, then the collection J covers a set E in the *sense of Vitali*, if for each $\epsilon > 0$ and any $x \in E$ there is an interval $I \in J$ such that $x \in I$ and $l(I) < \epsilon$.

Lemma 4. *Vitali Lemma: Let E be a set of finite outer measure and J a collection of intervals which covers E in the sense of Vitali. Then given $\epsilon > 0$ there is a finite disjoint collection $\{I_1, \dots, I_N\}$ of intervals in J such that*

$$m^*\Big(E \sim \bigcup_{n=1}^{N} I_n\Big) < \epsilon.$$

.

Proof: Suppose each interval in J is closed.
Let O be an open set of finite outer measure containing E. Since J is a Vitali covering of E, without loss of generality we may assume that each I of J is contained in O. We choose a sequence (I_n) of disjoint intervals of J by induction as follows: Let I_1 be any interval in J.
Let $k_1 = \sup\Big\{l(I) : I \in J, I \cap I_1 = \phi\Big\}$. Since $I \subset O$, we have $k_1 \leq mO < \infty$. If $E \subset I_1$, then there is nothing to prove. If $E \subset I_1$ is not true, we can find an interval $I_2 \in J$ with $l(I_2) > k_1/2$ and $I_2 \cap I_1 = \phi$.
Let it holds for $p = n$, i.e., disjoint intervals $I_1, I_2, \dots, I_n$ have chosen by induction. For $p = n + 1$,
let

$$k_n = \sup\Big\{l(I) : I \in J, I \cap I_1 = \phi, I \cap I_2 = \phi, \dots, I \cap I_n = \phi\Big\}. \tag{1}$$

Since $I \subset O$, we have $k_n \leq mO < \infty$. If $E \subset \bigcup_{i=1}^{n} I_i$, then there is nothing to prove. If $E \subset \bigcup_{i=1}^{n} I_i$ is not true, we can find an interval $I_{n+1} \in J$ with $l(I_{n+1}) > k_n/2$ and $I_{n+1} \cap \left(\bigcup_{i=1}^{n} I_i\right) = \phi$.

Since from the definition of suprema, there exists an $I_{n+1} \in J$ such that $I_{n+1} \cap \left(\bigcup_{i=1}^{n} I_i\right) = \phi$ and

$$l(I_{n+1}) > k_n/2. \tag{2}$$

Thus we have a sequence (I_n) of disjoint intervals of J, and $\bigcup l(I_n) \leq mO < \infty$. Hence, there exists N such that

$$\sum_{N+1}^{\infty} l(I_n) < \epsilon/5. \tag{3}$$

Let

$$R = E \sim \bigcup_{n=1}^{N} I_n.$$

We prove $m^*R < \epsilon$.

Let x be an arbitrary point of R. Since $\bigcup_{n=1}^{N} I_n$ is a closed set not containing x, by using the definition of Vitali covering, there exists an interval $I \in J$ such that $x \in I$, and whose length is so small that I does not meet any of the intervals $I_1, I_2, \ldots, I_N$. From Eqs. (1) and (2), we have $l(I_n) \leq k_n \leq 2l(I_{n+1})$. Since $\lim l(I_n) = 0$, $l(I_n) \leq k_n \leq 2l(I_{n+1}) \to 0$ as $n \to \infty$ the interval I must meet at least one of the intervals I_n. Let n be the smallest integer such that I meets I_n. We have $n > N$, and $l(I) \leq k_{n-1} \leq 2l(I_n)$. Since $x \in I$, and I has a point in common with I_n, it follows that the distance from x to the midpoint of I_n is at most $l(I_n) + \frac{1}{2}l(I_n) \leq \frac{5}{2}l(I_n)$. Thus x belongs to the interval P_n having the same midpoint as I_n and

$$l(P_n) = 5l(I_n). \tag{4}$$

Thus we have shown that

$$R \subset \bigcup_{N+1}^{\infty} P_n. \tag{5}$$

Hence from Eqs. (3), (4) and (5),

$$m^*R \le \sum_{N+1}^{\infty} l(P_n) = 5\sum_{N+1}^{\infty} l(I_n) < \epsilon.$$

Suppose each interval in J is not closed.
Let $I_1, I_2, ..., I_N$ is not closed interval of J. We have

$$E \sim \bigcup_{n=1}^{N} I_n \subset \Big(E \sim \bigcup_{n=1}^{N} clo(I_n)\Big) \cup \Big(\bigcup_{n=1}^{N} \{\text{ end points of } I_n\}\Big),$$

where $clo(I_n)$ =closure of (I_n). Hence

$$m^*\Big(E \sim \bigcup_{n=1}^{N} I_n\Big) \le m^*\Big(E \sim \bigcup_{n=1}^{N} clo(I_n)\Big) + m^*\Big(\bigcup_{n=1}^{N} \{\text{ end points of } I_n\}\Big).$$

Since the measure of the set of endpoint of $I_1, I_2, ..., I_N$ is equal to 0, which implies

$$m^*\Big(E \sim \bigcup_{n=1}^{N} I_n\Big) \le m^*\Big(E \sim \bigcup_{n=1}^{N} clo(I_n)\Big) = m^*R < \epsilon.$$

Therefore we have the conclusion.

3.2 Differentiability of functions on the real line

A function f defined in a neighborhood of a point $x \in \mathbf{R}$ is called differentiable at this point if there exists a finite limit

$$\lim_{h\to\infty} \frac{f(x+h) - f(x)}{h},$$

which is called the derivative of f at the point x and denoted by $f'(x)$.

In the study of derivatives it is useful to consider the derivate of a function f that take values on the extended real line and are define by the following equalities:

$$D^+f(x) = \overline{\lim_{h\to 0_+}} \frac{f(x+h) - f(x)}{h}; \qquad (6)$$

$$D^{-}f(x) = \overline{\lim_{h\to 0_+}} \frac{f(x) - f(x-h)}{h}; \tag{7}$$

$$D_{+}f(x) = \underline{\lim_{h\to 0_+}} \frac{f(x+h) - f(x)}{h}; \tag{8}$$

$$D_{-}f(x) = \underline{\lim_{h\to 0_+}} \frac{f(x) - f(x-h)}{h}. \tag{9}$$

If $D^{+}f(x) = D_{+}f(x)$ then we say that the function f has the right derivative $f'_{+}(x) := D^{f}(x) = D_{+}f(x)$ at a point x; and if $D^{-}f(x) = D_{-}f(x)$, then we say that f has the left derivative $f'_{-}(x) := D^{-}f(x) = D_{-}f(x)$ at the point x. Clearly we have $D^{+}f(x) \geq D_{+}f(x)$ and $D^{-}f(x) \geq D_{-}f(x)$. If

$$D^{+}f(x) = D_{+}f(x) = D^{-}f(x) = D_{-}f(x) \neq \pm\infty; \tag{10}$$

we say that f is differentiable at x and $f'(x)$ to be the common value of the derivatives at x.

Theorem 5. *Let f be an increasing real-valued function on the interval $[a, b]$. Then f is differentiable almost everywhere. The derivative f' is measurable, and*

$$\int_a^b f'(x)dx \leq f(b) - f(a). \tag{11}$$

.

Proof: Let us show that the set where any two derivatives are unequal have measure zero. Consider a set $E = \{x : D^{+}f(x) > D_{-}f(x)\}$. For every pair of positive rational numbers u and v such that $u > v$. Let

$$E_{u,v} = \{x : D^{+}f(x) > u > v > D_{-}f(x)\}. \tag{12}$$

It is clear that

$$E = \bigcup_{u,v\in\mathbb{Q};0<u<v} E_{u,v}. \tag{13}$$

We first prove that $m^{*}E_{u,v} = 0$, for all $u, v \in \mathbb{Q}$ such that $0 < v < u$. Let us assume a contrary that there exist positive rational numbers u and v such that $u > v$ and $m^{*}E_{u,v} = s$. Take arbitrary $\epsilon > 0$, there is an open set O such that $E_{u,v} \subset O$ with

$$m^{*}O \leq m^{*}E_{u,v} + \epsilon = s + \epsilon. \tag{14}$$

If $x \in E_{u,v}$ then $x \in O$ and $D_{-}f(x) < v$. Since $x \in O$, there exists a δ' such that $(x-\delta', x+\delta') \subset O$, and $D_{-}f(x) < v$, from Equation (9) $\underline{\lim}_{h \to 0_+} \frac{f(x)-f(x-h)}{h} < v$, there exists an h' such that for all $0 < h <$ min $\{h', \delta'\}$ such that $h < \epsilon$ and $f(x) - f(x-h) < vh$ and $[x-h, x] \subset O$,
there is an arbitrary small interval $[x-h, x]$ contained in O such that $f(x)-f(x-h) < vh$. Denote $[x_i - h_i, x_i] = I_i$. From Vitali lemma, there exists a finite disjoint collection $\Big\{[x_i - h_i, x_i]\Big\}_{i=1}^{N}$ such that

$$m^*\Big(E_{u,v} \sim \bigcup_{i=1}^{N} I_i^o\Big) = m^*\Big(E_{u,v} \sim \bigcup_{i=1}^{N} I_i\Big) < \epsilon, \tag{15}$$

where

$$I_i^o = \text{ interior of } I_i.$$

Define

$$A = E_{u,v} \cap \bigcup_{i=1}^{N} I_i^o. \tag{16}$$

We have

$$A = E_{u,v} \cap \bigcup_{i=1}^{N} I_i^o = E_{u,v} \sim \Big(E_{u,v} \sim \bigcup_{i=1}^{N} I_i^o\Big).$$

$$m^*A = m^*E_{u,v} - m^*\big(E_{u,v} \sim \bigcup_{i=1}^{N} I_i^o\big) > s - \epsilon,$$

since by Equation (15). Then summing over these intervals, we have,

$$\sum_{n=1}^{N}\Big(f(x_n) - f(x_n - h_n)\Big) < v\sum_{n=1}^{N} h_n < vmO < v(s+\epsilon). \tag{17}$$

Since $\bigcup_{n=1}^{N} I_n \subset O$ implies $m\big(\bigcup_{n=1}^{N} I_n\big) = \sum_{n=1}^{N} l(I_n) = \sum_{n=1}^{N} h_n \leq mO$.
Now, for each point $y \in A$, there exists an arbitrary small interval $[y, y+k]$ which is contained in some I_n for $n \leq N$, and for which $f(y+k)-f(y) > uk$. Again from Vitali lemma , there is a finite disjoint collection $\{J_1, J_2, ..., J_M\}$ where $J_M = [y_M, y_M+k_M]$

such that $m^*\big(A \sim \bigcup_{i=1}^{M} J_i\big) < \epsilon$.

Define

$$B = A \cap \bigcup_{i=1}^{M} J_i. \tag{18}$$

Then

$$m^*B = m^*A - m^*\Big(A \sim \big(A \sim \bigcup_{i=1}^{M} J_i\big)\Big) > s - \epsilon - \epsilon = s - 2\epsilon. \tag{19}$$

Then summing over these intervals, we have

$$\sum_{i=1}^{M}\Big(f(y_i + k_i) - f(y_i)\Big) > u\sum_{i=1}^{M} k_i > u(s - 2\epsilon), \tag{20}$$

$$\text{since } s - 2\epsilon < m^*B \leq m\Big(\bigcup_{i=1}^{M} J_i\Big) = \sum_{i=1}^{M} k_i.$$

Each interval J_i $(i = 1, 2, ..., M)$ is contained in some interval I_n, and if we sum over those i for which $J_i \subset I_n$. We have

$$\sum\Big(f(y_i + k_i) - f(y_i)\Big) \leq f(x_n) - f(x_n - h_n), \tag{21}$$

since f is increasing. Thus

$$\sum_{n=1}^{N}\Big(f(x_n) - f(x_n - h_n)\Big) \geq \sum_{i=1}^{M}\Big(f(y_i + k_i) - f(y_i)\Big), \tag{22}$$

and so

$$v(s + \epsilon) > u(s - 2\epsilon). \tag{23}$$

Since this is true for each positive ϵ, we have $vs \geq us$. But $u > v$, and so s must be zero. Therefore $m^*E_{u,v} = 0$, for all $u, v \in \mathbb{Q}$. Then

$$m^*E = m^*\Big(\bigcup_{u,v\in\mathbf{Q}} E_{u,v}\Big) = 0. \tag{24}$$

Hence

$$m\Big\{x : D^+f(x) > D_-f(x)\Big\} = 0 \text{ implies } D^+f(x) \leq D_-f(x) \text{ a.e.} \tag{25}$$

And similarly

$$m\Big\{x : D^- f(x) > D_+ f(x)\Big\} = 0 \text{ implies } D_+ f(x) \geq D^- f(x) \text{ a.e.} \tag{26}$$

We know

$$D^+ f(x) \geq D_+ f(x) \text{ and } D^- f(x) \geq D_- f(x). \tag{27}$$

Combining Eqs. (25), (26) and (27), we get,

$$D^+ f(x) = D_+ f(x) = D^- f(x) = D_- f(x) \text{ a.e.} \tag{28}$$

This shows that

$$g(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h}$$

is defined a.e. and that f is differentiable whenever g is finite.
For each n, define

$$g_n(x) = n\Big(f(x + 1/n) - f(x)\Big), \tag{29}$$

where we set $f(x) = f(b)$ for $x \geq b$.
We have $\lim\limits_{h\to 0} \dfrac{f(x+h) - f(x)}{h}$ a.e. Therefore

$$\lim_{n\to\infty} g_n(x) = g(x) \text{ a.e.} \tag{30}$$

Then $g_n(x)$ tends to $g(x)$ for almost all x, and so g is measurable. Since f is increasing, i.e., $f(x + 1/n) \geq f(x)$, we have $g_n(x) \geq 0$. Hence by Fatou's lemma,

$$\begin{aligned}
\int_a^b g(x) &\leq \varliminf_{n\to\infty} \int_a^b g_n(x) \\
&= \varliminf_{n\to\infty} n \int_a^b \Big(f(x+1/n) - f(x)\Big)dx \\
&= \varliminf_{n\to\infty} \Big(n \int_a^b f(x+1/n)dx - n \int_a^b f(x)dx\Big) \\
&= \varliminf_{n\to\infty} \Big(n \int_b^{b+1/n} f(x)dx - n \int_a^{a+1/n} f(x)dx\Big) \\
&= \varliminf_{n\to\infty} \Big(f(b) - n \int_a^{a+1/n} f(x)dx\Big) \\
&\leq f(b) - \lim_{n\to\infty} n \int_a^{a+1/n} f(a)dx. \\
&= f(b) - f(a).
\end{aligned}$$

Hence

$$\int_a^b g(x)dx \leq f(b) - f(a). \tag{31}$$

This shows that g is integrable and hence finite almost everywhere. Thus f is differentiable *a.e.* and $g = f'$ a.e.

3.3 Function of Bounded Variation

Let f be a real-valued function defined on the interval $[a, b]$, and let $a = x_0 < x_1 < x_2 < \cdots < x_k = b$ be any subdivision of $[a, b]$. Define

$$p = \sum_{i=1}^{k} \Big(f(x_i) - f(x_{i-1})\Big)^+, \tag{32}$$

$$n = \sum_{i=1}^{k} \Big(f(x_i) - f(x_{i-1})\Big)^-, \tag{33}$$

$$t = n + p = \sum_{i=1}^{k} \Big|f(x_i) - f(x_{i-1})\Big|, \tag{34}$$

where we use r^+ to denote r, if $r \geq 0$ and 0, if $r \leq 0$ and set $r^- = |r| - r^+$. We have

$$f(b) - f(a) = p - n, \tag{35}$$

since

$$\begin{aligned} p - n &= \sum_{i=1}^{k} \Big(f(x_i) - f(x_{i-1})\Big)^+ - \sum_{i=1}^{k} \Big(f(x_i) - f(x_{i-1})\Big)^- \\ &= \sum_{i=1}^{k} \Big(\Big(f(x_i) - f(x_{i-1})\Big)^+ - \Big(f(x_i) - f(x_{i-1})\Big)^-\Big) \\ &= \sum_{i=1}^{k} \Big(f(x_i) - f(x_{i-1})\Big) \\ &= f(x_k) - f(x_0) \\ &= f(b) - f(a). \end{aligned}$$

Set

$$P = \sup\ p, \tag{36}$$

$$N = \sup\ n, \tag{37}$$

$$T = \sup\ t, \tag{38}$$

where we take these suprema over all possible subdivisions of $[a, b]$. Since $p \leq t = p + n, \sup\ p \leq \sup\ t = \sup\ (p + n) \leq \sup\ p + \sup\ n$. Hence we have

$$P \leq T \leq P + N. \tag{39}$$

We say P is the positive variation of f over $[a, b]$. Sometimes we write it by $P_a^b, P_a^b(f)$. Similarly we call N and T by negative variation of f over $[a, b]$ and total variation of f over $[a, b]$. Sometimes we denote the negative variation by $N_a^b, N_a^b(f)$ and the total variation by $T_a^b, T_a^b(f)$, it means that dependence on the interval $[a, b]$ or on the functions f. If $T \leq \infty$,we say that f is of **bounded variation** over $[a, b]$. This notation is sometimes abbreviated by writing $f \in BV$.

Lemma 6. *If f is of bounded variation on $[a, b]$, then*

$$T_a^b = P_a^b + N_a^b \tag{40}$$

and

$$f(b) - f(a) = P_a^b - N_a^b. \tag{41}$$

.

Proof: For any subdivision of $[a, b]$

$$\begin{aligned} p - n &= f(b) - f(a) \\ p &= n + f(b) - f(a) \\ &\leq N + f(b) - f(a), \end{aligned}$$

and taking suprema over all possible subdivisions, we obtain

$$P \leq N + f(b) - f(a).$$

Since $N \leq T$ and f is of bounded variation over $[a, b]$,

$$P - N \leq f(b) - f(a). \tag{42}$$

Similarly,

$$N - P \leq f(a) - f(b). \tag{43}$$

Hence from Eqs. (42) and (43),

$$P - N = f(b) - f(a). \tag{44}$$

From Eqs. (34), (35) and (38),

$$\begin{aligned} T &\geq t \\ &= p + n \\ &= p + p - \{f(b) - f(a)\}. \end{aligned}$$

From Eq. (44)

$$T \geq 2p + N - P.$$

Taking suprema over all possible subdivisions, we obtain

$$\begin{aligned} T &\geq 2P + N - P \\ &= P + N. \end{aligned}$$

Since $T \leq P + N$, hence, we have

$$T = P + N.$$

Theorem 7. *A function f is of bounded variation on $[a, b]$ if and only if f is the difference of two monotone real-valued functions on $[a.b]$.*

Proof: Let f be of bounded variation, and set

$$g(x) = P_a^x, \tag{45}$$

and

$$h(x) = N_a^x. \tag{46}$$

Let $a = x_0 < x_1 < \cdots < x_k = x \le b$ be any subdivision of $[a, x]$. Assume $x < y \le b$. From Eq. (45) and taking suprema over all the possible subdivision of $[a, x]$,

$$\begin{aligned} g(x) &= \sup \sum_{i=1}^{k} \Big(f(x_i) - f(x_{i-1})\Big)^+ \\ &\le \sup \sum_{i=1}^{k} \Big(f(x_i) - f(x_{i-1})\Big)^+ + \Big(f(y) - f(x)\Big)^+ \\ &\le P_a^y \\ &= g(y). \end{aligned}$$

Hence $g(x)$ is a monotone increasing function. Similarly $h(x)$ is also a monotone increasing function, $g(x)$ and $h(x)$ are real valued functions, since $0 \le P_a^x \le T_a^x \le T_a^b < \infty$ and $0 \le N_a^x \le T_a^x \le T_a^b < \infty$. Since f is of bounded variation, from lemma 3

$$f(x) - f(a) = P_a^x - N_a^x. \tag{47}$$

From Eqs. (45) and (46) and after calculate, we get

$$f(x) = g(x) - h(x) + f(a).$$

Since $h(x)$ is a monotone increasing real valued function and $f(a)$ is a constant, $h(x) - f(a)$ is a monotone function. Hence f is the difference of two monotone real-valued functions on $[a, b]$.

Converse Part: If $f(x) = g(x) - h(x)$ on $[a, b]$ with $g(x)$ and $h(x)$ is a monotone real-valued functions, and let $a = x_0 < x_1 < x_2 < \cdots < x_k = b$ be any sub-division of $[a, b]$, then we have

$$f(x_i) - f(x_{i-1}) = g(x_i) - g(x_{i-1}) + h(x_i) - h(x_{i-1}).$$

Taking absolute value and summing both sides from $i = 1, 2, \ldots, k$; we get

$$\begin{aligned} \sum_{i=1}^{k} |f(x_i) - f(x_{i-1})| &\le \sum_{i=1}^{k} |g(x_i) - g(x_{i-1})| + \sum_{i=1}^{k} |h(x_i) - h(x_{i-1})| \\ &= \sum_{i=1}^{k} \Big(g(x_i) - g(x_{i-1})\Big) + \sum_{i=1}^{k} \Big(h(x_i) - h(x_{i-1})\Big) \\ &= g(b) - g(a) + h(b) - h(a). \end{aligned}$$

Taking suprema over all the possible subdivisions, we obtain

$$\begin{aligned} T_a^b(f) &\leq g(b) - g(a) + h(b) - h(a) \\ &< \infty. \end{aligned}$$

Hence f is of bounded variation of $[a, b]$.

Corollary 8. *If f is of bounded variation on $[a, b]$, then $f'(x)$ exists for almost all x on $[a, b]$.*

Proof: Since f is of bounded variation, then from Theorem 7, f can be written as the difference of two increasing real-valued functions on $[a, b]$. Let suppose such functions are g and h, i.e. $f(x) = g(x) - h(x)$. Again from Theorem 5, g and h are differentiable almost everyewhere. Since

$$\pm\infty \neq \lim_{k\to 0} \frac{g(x+k) - g(x)}{k}$$

a.e. and

$$\pm\infty \neq \lim_{h\to 0} \frac{h(x+k) - h(x)}{k}$$

a.e. then

$$\begin{aligned} \pm\infty &\neq g'(x) - h'(x) \\ &= \lim_{k\to 0} \Big(\frac{g(x+k) - g(x)}{k} - \frac{h(x+k) - h(x)}{k} \Big) \\ &= \lim_{k\to 0} \frac{\big(g(x+h) - h(x+h)\big) - \big(g(x) - h(x)\big)}{k} \\ &= \lim_{k\to 0} \frac{(g-h)(x+h) - (g-h)(x)}{k} \\ &= (g-h)'(x). \end{aligned}$$

i.e. $f'(x) = g'(x) - h'(x)$ for almost all x in $[a, b]$. Hence f is differentiable almost everywhere.

3.4 Differentiation of an Integral

We use these Theorem 9, Proposition 10 and Lemma 11 from Royden[13].

Theorem 9. (Monotone Convergence Theorem): *Let (f_n) be an increasing sequence of nonnegative measurable functions, and let $f = \lim_{n\to\infty} f_n$ a.e. Then*

$$\int f = \lim_{n\to\infty} \int f_n.$$

Proposition 10. *Let f be a nonnegative function which is integrable over a set E. Then given $\epsilon > 0$ there is a $\delta > 0$ such that for every set $A \subset E$ with $mA < \delta$ we have*

$$\int_A f < \epsilon.$$

If f is integrable function on $[a, b]$, we define its indefinite integral to be the function F defined on $[a, b]$ by

$$F(x) = \int_a^x f(t)dt. \tag{47}$$

Lemma 11. *If f is integrable on $[a, b]$, then the function F defined by*

$$F(x) = \int_a^x f(t)dt$$

is a continuous function of bounded variation on $[a, b]$.

Proof: Since f is integrable over $[a, b]$. Write $f = f^+ - f^-$. Here f^+ and f^- are non-negative integrable function over $[a, b]$, then from Proposition 10, for all $\epsilon > 0$, there exists a $\delta > 0$ such that $(x, y) \subset [a, b]$ with $|x - y| < \delta$, we have $\int_x^y f^+ < \epsilon/2$ and $\int_x^y f^- < \epsilon/2$. Therefore

$$\begin{aligned}
|F(x) - F(y)| &= \left|\int_a^x f(t)dt - \int_a^y f(t)dt\right| \\
&= \left|\int_x^y f(t)dt\right| \\
&= \left|\int_x^y f^+(t)dt - \int_x^y f^-(t)dt\right| \\
&\leq \int_x^y f^+(t)dt + \int_x^y f^-(t)dt \\
&< \epsilon.
\end{aligned}$$

Hence $F(x)$ is a continuous function.
To show F is of bounded variation, let $a = x_0 < x_1 < x_2 < \cdots < x_k = b$ be any subdivision of $[a, b]$. Then

$$\begin{aligned} \sum_{i=1}^{k} \Big| F(x_i) - F(x_{i-1}) \Big| &= \sum_{i=1}^{k} \Big| \int_{x_{i-1}}^{x_i} f(t)dt \Big| \\ &\leq \sum_{i=1}^{k} \int_{x_{i-1}}^{x_i} |f(t)|dt \\ &= \int_a^b |f(t)|dt \\ &< \infty, \end{aligned}$$

since f is an integrable function over $[a, b]$. Taking the suprema over all the possible subdivisions of $[a, b]$ then

$$T_a^b(f) \leq \int_a^b |f(t)|dt.$$

Hence $F(x)$ is a continuous function of bounded variation on $[a, b]$. We take this Proposition 12 also from the theory of Lebesgue integral [13].

Proposition 12. *If E is measurable, then for all $\epsilon > 0$ there is a closed set $F \subset E$ such that $m^*(E \sim F) < \epsilon$*

Proposition 13. *Every open set of real numbers is a union of a countable collection of disjoint open intervals .*

Proposition 14. *Let f be a nonnegative measurable function and (E_i) a disjoint sequence of measurable sets. Let $E = \cup E_i$. Then*

$$\int_E f = \sum \int_{E_i} f$$

Lemma 15. *If f is integrable on $[a, b]$ and*

$$\int_a^x f(t)dt = 0$$

for all $x \in [a, b]$, then $f(t) = 0$ a.e.in $[a, b]$

Proof: If $f(t) = 0$ a.e. does not hold then $m\{x : f(x) \neq 0\} > 0$. Denote $E = \{x : f(x) \neq 0\}$. Suppose $f(x) > 0$ on a set of positive measure. Take $mE > \epsilon > 0$ then from Proposition 12, then there is a closed set $F \subset E$ such that $m^*(E \sim F) < \epsilon$ Since E and F are measurable, then $m(E \sim F) < \epsilon$. Since $F \subset E$ then $mE - mF < \epsilon$. Hence $mE - \epsilon < mF$. It implies $mF > 0$.

Let $O = (a, b) \sim F$. Since $\int_a^b f(x)dx < \infty$, then either

$$\int_a^b f \neq 0$$

or else

$$\begin{aligned} 0 &= \int_a^b f \\ &= \int_F f + \int_O f \end{aligned}$$

and

$$\int_O f = -\int_F f \neq 0.$$

Let $f = f^+ - f^-$. But from Proposition 14, O is the disjoint union of countable collection $\{(a_n, b_n)\}$ of open intervals, and so by Proposition 11,

$$\begin{aligned} \int_O f &= \int_O f^+ - \int_O f^- \\ &= \sum \int_{a_n}^{b_n} f^+ - \sum \int_{a_n}^{b_n} f^- \\ &= \sum \int_{a_n}^{b_n} \left(f^+ - f^-\right) \\ &= \sum \int_{a_n}^{b_n} f. \end{aligned}$$

Since $\int_O f \neq 0$, thus for some n, we have

$$\int_{a_n}^{b_n} f \neq 0.$$

Here

$$\int_a^{b_n} f = \int_a^{a_n} f + \int_{a_n}^{b_n} f.$$

Then either

$$\int_a^{b_n} f = 0$$

or

$$\int_a^{b_n} f \neq 0.$$

If

$$\int_a^{b_n} f = 0,$$

then

$$\int_a^{a_n} f = -\int_{a_n}^{b_n} f \neq 0.$$

Hence either

$$\int_a^{a_n} f \neq 0$$

or

$$\int_a^{b_n} f \neq 0.$$

It contradicts that

$$\forall x \in [a,b], \int_a^x f(t)dt = 0.$$

Similarly, when $f < 0$ on a set of positive measure,
i.e. $m\{x : f(x) < 0\} > 0$ or $m\{x : -f(x) > 0\} > 0$.
Put $-f(x) = g(x)$ then $m\{x : g(x) > 0\} > 0$. Again similarly, for some $x \in [a,b]$,

$$\int_a^x g(t)dt \neq 0,$$

or

$$-\int_a^x f(t)dt \neq 0,$$

implies that

$$\int_a^x f(t)dt \neq 0.$$

Therefore the lemma follows by contrapositive. We take the following Bounded convergence theorem from the theory of Lebesgue integral [13].

Theorem 16. (Bounded convergence theorem) : *Let (f_n) be a sequence of measurable functions defined on a set E of finite measure, and suppose that there is a real number K such that $|f_n(x)| \leq K$ for all n and x. If $f(x) = \lim f_n(x)$ for each x in E, then*

$$\int_E f = \lim \int_E f_n.$$

The following Lemma is taken from Royden [13].

Lemma 17. *If f is bounded and measurable on $[a, b]$ and*

$$F(x) = \int_a^x f(t)dt + F(a),$$

then $F'(x) = f(x)$ for almost all x in $[a, b]$.

Proof: Let $|f(x)| \leq K$, let $a = x_0 < x_1 < x_2 < \cdots < x_k = b$ be any subdivision of $[a, b]$. Then

$$\begin{aligned} \sum_{i=1}^{k} \Big|F(x_i) - F(x_{i-1})\Big| &= \sum_{i=1}^{k} \Big|\int_{x_{i-1}}^{x_i} f(t)dt\Big| \\ &\leq \sum_{i=1}^{k} \int_{x_{i-1}}^{x_i} \Big|f(t)\Big|dt \\ &= \int_a^b \Big|f(t)\Big|dt. \end{aligned}$$

If we take the supremum over all possible subdivision of $[a, b]$,

$$T_a^b(F) \leq \int_a^b \Big|f(t)\Big|dt.$$

Since f is bounded, therefore

$$T_a^b(F) < \infty.$$

Hence F is of bounded variation. And from Corollary 5, $F'(x)$ exists for almost all x in $[a, b]$.
Setting

$$f_n(x) = \frac{F(x+h) - F(x)}{h} \tag{48}$$

with $h = \frac{1}{n}$, we have

$$\begin{aligned} f_n(x) &= \frac{1}{h}\Big(F(x+h) - F(x)\Big) \\ &= \frac{1}{h}\Big(\int_a^{x+h} f(t)dt + F(a) - \int_a^x f(t)dt - F(a)\Big) \\ f_n(x) &= \frac{1}{h}\int_x^{x+h} f(t)dt. \end{aligned}$$

Taking absolute value on both sides,

$$\begin{aligned} |f_n(x)| &\leq \frac{1}{h}\int_x^{x+h} |f(t)|dt \\ &\leq K\frac{1}{h}h \\ &= K. \end{aligned}$$

Hence for all n and all x, $|f_n(x)| \leq K$. Since $f_n(x) \to F'(x)$ a.e., the bounded convergence theorem implies that

$$\begin{aligned} \int_a^c F'(x)dx &= \lim_{n\to\infty}\int_a^c f_n(x)dx \\ &= \lim_{h\to 0}\frac{1}{h}\int_a^c \Big(F(x+h) - F(x)\Big)dx \\ &= \lim_{h\to 0}\Big(\frac{1}{h}\int_c^{c+h} F(x)dx - \frac{1}{h}\int_a^{a+h} F(x)dx\Big) \\ &= F(c) - F(a) \\ &= \int_a^c f(x)dx, \end{aligned}$$

since F is continuous. Hence

$$\int_a^c \Big(F'(x) - f(x)\Big)dx = 0 \tag{49}$$

for all $c \in [a, b]$, and so from lemma 12,

$$F'(x) = f(x) \tag{50}$$

a.e. We will use the following Proposition from the theory of Lebesgue integral.

Proposition 18. *Let f and g be an integrable over E. Then if $f \leq g$ a.e., then*

$$\int_E f \leq \int_E g.$$

Here we show that the derivative of the indefinite integral of an integrable function is equal to the integrand almost everywhere, which is taken from Royden [13].

Theorem 19. *Let f be an integrable function on $[a, b]$, and suppose*

$$F(x) = F(a) + \int_a^x f(t)dt.$$

Then $F'(x) = f(x)$ for all most all x in $[a, b]$

Proof : It is sufficient to prove for $f \geq 0$.
Let f_n be define by $f_n(x) = f(x)$ when $f(x) \leq n$, and $f_n(x) = n$ when $f(x) > n$. Then $f_n(x) \leq f(x)$. i.e.,

$$f(x) - f_n(x) \geq 0. \tag{51}$$

Define

$$G_n(x) = \int_a^x \Big(f - f_n\Big)(t)dt. \tag{52}$$

To show $G_n(x)$ is an increasing function of x, suppose $x < y$,

$$\begin{aligned} G_n(x) &= \int_a^x \Big(f - f_n\Big)(t)dt \\ &\leq \int_a^x \Big(f - f_n\Big)(t)dt + \int_x^y \Big(f - f_n\Big)(t)dt \\ &= \int_a^y \Big(f - f_n\Big)(t)dt \\ &= G_n(y) \\ G_n(x) &\leq G_n(y), \end{aligned}$$

since $f - f_n \geq 0$ implies $\int_x^y \Big(f - f_n\Big) \geq 0$. Hence $G_n(x)$ is an increasing function of x. Then from Theorem 5, $G_n(x)$ is differentiable a.e.

$$G_n'(x) = \lim_{h \to 0} \frac{G_n(x+h) - G_n(x)}{h}.$$

When $h > 0$,

$$\begin{aligned} G_n(x+h) &\geq G_n(x) \\ G_n(x+h) - G_n(x) &\geq 0 \\ \frac{G_n(x+h) - G_n(x)}{h} &\geq 0 \\ \lim_{h \to 0^+} \frac{G_n(x+h) - G_n(x)}{h} &\geq 0. \end{aligned}$$

Therefore

$$G_n'(x) = \lim_{h \to 0} \frac{G_n(x+h) - G_n(x)}{h} > 0.$$

Here $f_n(x)$ is bounded measurable function and if we put $\int_a^x f_n(t)dt + F_n(a) = F_n(x)$ then from Lemma 17, $F_n'(x) = f_n(x)$ for almost all $x \in [a, b]$.
i.e.

$$\frac{d}{dx}\Big(\int_a^x f_n(t)dt + F_n(a)\Big) = f_n(x)$$

for almost all $x \in [a, b]$. Hence

$$\frac{d}{dx} \int_a^x f_n(t)dt = f_n(x) \tag{53}$$

for almost all $x \in [a, b]$. And so

$$\begin{aligned} \frac{d}{dx}F(x) &= \frac{d}{dx}F(a) + \frac{d}{dx}\int_a^x f(t)dt \\ &= 0 + \frac{d}{dx}\Big(G_n(x) + \int_a^x f_n(t)dt\Big) \\ &= G_n'(x) + \frac{d}{dx}\int_a^x f_n(t)dt \\ &\geq f_n(x) \end{aligned}$$

a.e. Since n is arbitrary,

$$F'(x) \geq f(x) \tag{54}$$

a.e. Consequently,

$$\int_a^b F'(x)dx \geq \int_a^b f(x)dx = F(b) - F(a).$$

Thus by Theorem 5, we have

$$\int_a^b F'(x)dx = F(b) - F(a) = \int_a^b f(x)dx,$$

and

$$\int_a^b \Big(F'(x) - f(x)\Big)dx = 0.$$

Since $F'(x) - f(x) \geq 0$, this implies that $F'(x) - f(x) = 0$ a.e., and so $F'(x) = f(x)$ a.e.

3.5 Absolute Continuity

A real valued function f defined on $[a, b]$ is said to be **absolutely continuous** on $[a, b]$ if, given $\epsilon > 0$, there is a $\delta > 0$ such that

$$\sum_{i=1}^{n} \left| f(x_i') - f(x_i) \right| \leq \epsilon$$

for every pairwise disjoint family $\left\{(x_i, x_i')\right\}_{i=1}^{n}$ of open intervals of $[a, b]$ with

$$\sum_{i=1}^{n} \left| x_i' - x_i \right| < \delta.$$

Proposition 20. *An absolute continuous function is continuous.*

Proof: Let $f : [a, b] \to \mathbb{R}$ ba an absolutely continuous function. Take arbitrary $\epsilon > 0$. And $\delta > 0$ is defined same as in the definition of absolute continuity. Take $x \in [a, b]$, for $y \in [a, b]$ such that $|x - y| < \delta$ implies $|f(x) - f(y)| < \epsilon$, since from the definition of absolute continuity. Hence $f(x)$ is an continuous function.

Proposition 21. *Every indefinite integral is absolutely continuous .*

Proof: Let

$$F(x) = \int_a^x f(t)dt,$$

for all $x \in [a, b]$, is an indefinite integral where $f(x)$ is integrable on $[a, b]$. Since $f(x)$ is an integrable function, $|f(x)|$ is also an integrable function. Let $\epsilon > 0$, for all non overlapping intervals $\{(x_i, x_i')\}_{i=1}^n \subset [a, b]$ with $m(\bigcup_{i=1}^n (x_i, x_i')) < \delta$, where δ is the positive number corresponding to ϵ in the definition of the absolute continuity of f. Applying Proposition 10, we get

$$\int_{\bigcup_{i=1}^n (x_i, x_i')} |f(t)| dt < \epsilon. \tag{55}$$

Since

$$\begin{aligned} \sum_{i=1}^n |F(x_i') - F(x_i)| &= \sum_{i=1}^n |\int_{x_i}^{x_i'} f(t) dt| \\ &\le \sum_{i=1}^n \int_{x_i}^{x_i'} |f(t)| dt \\ &= \int_{\bigcup_{i=1}^n (x_i, x_i')} |f(t)| dt. \end{aligned}$$

i.e.

$$\sum_{i=1}^n |F(x_i') - F(x_i)| \le \int_{\bigcup_{i=1}^n (x_i, x_i')} |f(t)| dt. \tag{56}$$

From Eqs (55) and (56) gives that $F(x)$ is an absolutely continuous function.

Lemma 22. *If f is absolutely continuous on $[a, b]$, then it is of bounded variation on $[a, b]$.*

Proof: Since f is an absolutely continuous on $[a, b]$, there is a $\delta > 0$ such that for every finite pairwise disjoint family $\{(a_k, b_k)\}_{i=1}^n$ of open intervals of $[a, b]$ of total length

$$\sum_{k=1}^n (b_k - a_k) < \delta$$

implies

$$\sum_{k=1}^n |f(b_k) - f(a_k)| < 1.$$

Let $a < c_0 < c_1 < \cdots < c_m < b$ be any subdivision of $[a, b]$ such that $c_{k+1} - c_k < \delta$ for $k = 0, 1, 2, \ldots m - 1$. Here $c_{k+1} - c_k < \delta$ for $k = 0, 1, 2, \ldots m - 1$. Again for every finite pairwise disjoint family $\{(c_i, c_{i+1})\}_{i=1}^n$ of open intervals of (c_k, c_{k+1}),

$$\sum_{i=1}^n |c_{i+1} - c_i| < \delta$$

implies

$$\sum_{i=1}^{n} |f(c_{i+1}) - f(c_i)| < 1,$$

since f is an absolutely continuous function. Hence

$$t < 1$$

on the interval (c_k, c_{k+1}) for $k = 0, 1, 2, \ldots m - 1$. Therefore on the interval $[a, b]$,

$$t < M.$$

If we take the suprema over all the possible subdivision of $[a, b]$, we have

$$T_a^b < \infty.$$

Hence f is of bounded variation of $[a, b]$.

Corollary 23. *If f is absolutely continuous, then f has derivative almost everywhere* .

Proof : Since f is an absolutely continuous then from Lemma 22, f is of bounded variation. Again from Corollary 8, f has a derivative almost every where.

Lemma 24. *If f is absolutely continuous on $[a, b]$ and $f'(x) = 0$ a.e., then f is a constant.*

Proof: We want to show that $f(a) = f(c)$ for any $c \in [a, b]$. Let

$$E = \Big\{x \in (a, c) : f'(x) = 0\Big\}.$$

Put $A = \Big\{x \in (a, c) : f'(x) \neq 0\Big\}$. We have $f'(x) = 0$ a.e. $x \in [a, b]$, then

$$A \subset \Big\{x \in [a, b] : f'(x) \neq 0\Big\}.$$

It implies,

$$mA \leq m\Big\{x \in [a, b] : f'(x) \neq 0\Big\} = 0.$$

Hence $mA = 0$. Here $E \cap A = \phi$ and $E \cup A = (a, c)$. Therefore $mE + mA = c - a$. Hence $mE = c - a$. Let $\epsilon > 0$ be an arbitrary number. Take $\eta > 0$ arbitrarily. For each $x \in E$ there is a small interval $[x, x + h]$ contained in $[a, c]$ such that

$$|f(x + h) - f(x)| < \eta h. \tag{57}$$

By Lemma 1, we can find a finite disjoint collection $\Big\{[x_k, x_k + h_k]\Big\}_{k=1}^{n}$ of intervals such that

$$m\Big(E \cap \big(\bigcup_{k=1}^{n}[x_k, x_k + h_k]\big)^c\Big) = m\Big(E \sim \big(\bigcup_{k=1}^{n}[x_k, x_k + h_k]\big)\Big) < \delta, \tag{58}$$

where δ is the positive number corresponding to ϵ in the definition of the absolute continuity of f. We can assume that $a = x_0 + h_0 \leq x_1 < x_1 + h_1 < x_2 < x_2 + h_2 < \cdots < x_n < x_n + h_n < x_{n+1} = c$. Since

$$(a, x_1) \cup (x_1 + h_1, x_2) \cup \cdots \cup (x_n + h_n, c) = \Big(\bigcup_{k=1}^{n}[x_k, x_k + h_k]\Big)^c$$

$$E \cap \Big((a, x_1) \cup (x_1 + h_1, x_2) \cup \cdots \cup (x_n + h_n, c)\Big) = E \cap \Big(\bigcup_{k=1}^{n}[x_k, x_k + h_k]\Big)^c.$$

Taking measure on both sides and using Eq. (58), we get,

$$|x_1 - a| + |x_2 - (x_1 + h_1)| + \cdots + |c - (x_n + h_n)| < \delta.$$

Then from definition of absolute continuity,

$$\sum_{k=0}^{n} |f(x_{k+1}) - f(x_k + h_k)| < \epsilon. \tag{59}$$

From Eq. (57)

$$\sum_{k=1}^{n} |f(x_k + h_k) - f(x_k)| < \eta \sum_{k=1}^{n} h_k \leq \eta(c - a).$$

Hence

$$|f(c) - f(a)| = \sum_{k=0}^{n} |f(x_{k+1}) - f(x_k + h_k)| + \sum_{k=1}^{n} |f(x_k + h_k) - f(x_k)| < \epsilon + \eta(c - a).$$

Since ϵ and η are an arbitrary positive numbers, $f(c) = f(a)$.

Theorem 25. *A function F is an indefinite integral if and only if it is absolutely continuous .*

Proof: Suppose F is an absolutely continuous on $[a, b]$. From Lemma 22, it is of bounded variation on $[a, b]$. Again from Theorem 7, F is the difference of two monotone real-valued functions on $[a, b]$. We may write

$$F(x) = F_1(x) - F_2(x),$$

where F_1, F_2 are monotone increasing real-valued functions. Using Theorem 5, F_1 and F_2 are differentiable almost everywhere, F_1' and F_2' are measurable and

$$\int_a^b F_1'(x)dx \leq F_1(b) - F_1(a). \tag{60}$$

$$\int_a^b F_2'(x)dx \leq F_2(b) - F_2(a). \tag{61}$$

Hence $F'(x)$ exists almost everywhere and

$$|F'(x)| \leq F_1'(x) + F_2'(x).$$

Integrate both sides from a to b, we get,

$$\begin{aligned} \int_a^b |F'(x)| &\leq \int_a^b F_1'(x) + \int_a^b F_2'(x) \\ &\leq F_1(b) + F_2(b) - F_1(a) - F_2(a) \\ &< \infty. \end{aligned}$$

From Eqs (60) and (61), hence $F'(x)$ is integrable. Let

$$G(x) = \int_a^x F'(t)dt.$$

By Lemma 21, G is absolutely continuous and so is the function $f = F - G$. It follows from Theorem 19 that $f'(x) = F'(x) - G'(x) = 0$ a.e., and so f is constant by Lemma 24. Thus

$$F(x) = \int_a^x F'(t)dt + F(a).$$

Corollary 26. *Every absolutely continuous function is the indefinite integral of its derivative .*

3.6 Lebesgue Density Theorem

Theorem 27. *Let $E \subset \mathbb{R}$ be a measurable set. Then* $\lim_{h\to 0} \frac{1}{2h} m\Big(E \cap (x-h, x+h)\Big)$ *is equal to* 1 *for a.e.* $x \in E$ *and equal to* 0 *for a.e.* $x \in E^c$.

Proof: For each $n \in N$, define

$$G_n(x) = \int_{-n}^{x} \chi_E(t)dt, \; x \in (-n, n).$$

Here $\chi_E(t)$ is integrable over $(-n, n)$. Every indefinite integral is absolutely continuous function. Therefore $G_n(x)$ is an absolutely continuous function. From Corollary 23, $G'(x)$ exists almost every $x \in (-n, n)$. Again from Theorem 19, $G_n'(x) = \chi_{E\cap(-n,n)}(x)$ a.e. We prove

$$G_n'(x) = \lim_{h\to 0} \frac{m\Big(E \cap (x-h, x+h)\Big)}{2h}.$$

Since,

$$\begin{aligned} 2G_n'(x) &= \lim_{h\to 0} \frac{1}{h}\Big(G_n(x+h) - G_n(x)\Big) + \lim_{h\to 0} \frac{1}{h}\Big(G_n(x) - G_n(x-h)\Big) \\ G_n'(x) &= \lim_{h\to 0} \frac{1}{2h}\Big(G_n(x+h) - G_n(x-h)\Big). \end{aligned}$$

Put $\delta = \min\{n - x, n + x\}$. Suppose $0 < h < \delta$,

$$G_n(x+h) - G_n(x) = \int_{x}^{x+h} \chi_E(t)dt,$$

and

$$G_n(x) - G_n(x-h) = \int_{x-h}^{x} \chi_E(t)dt.$$

Hence

$$G_n(x+h) - G_n(x-h) = \int_{x-h}^{x+h} \chi_E(t)dt.$$

Hence for $x \in (-n, n)$

$$\begin{aligned} G_n'(x) &= \lim_{h\to 0} \frac{1}{h} \int_{x-h}^{x+h} \chi_E(t)dt \\ &= \lim_{h\to 0} \frac{m\Big(E \cap (x-h, x+h)\Big)}{2h}. \end{aligned}$$

But we have,

$$G'_n(x) = \chi_{E\cap(-n,n)}(x)$$

a.e. Since n is an arbitrary, we have the result.

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