Direct and inverse problems of scattering theory in time-dependent electric fields

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博士論文

Direct and inverse problems of scattering theory in time-dependent electric fields

(時間変動電場の下での散乱理論の順問題と逆問題について)

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Contents

1 Direct Problem **3**
- 1.1 Introduction . . . 3
- 1.2 Propagation estimates . . . 12
- 1.3 Proof of Theorem 1.1.7 . . . 26
- 1.4 Charge Transfer Model . . . 29

2 Inverse Problem **42**
- 2.1 Introduction . . . 42
- 2.2 Preliminaries . . . 47
- 2.3 Short-range Case . . . 50
- 2.4 Long-range Case . . . 66

Preface. We study scattering problems in time-dependent electric fields asymptotically constant in time. In the first section, we report the result obtained by Adachi-Ishida [5]. We show the asymptotic completeness for one-body quantum systems in an external electric field converging on non-zero constant. In addition, this result is applied to a charge transfer model considered in Ishida [25]. In the second section, according to Adachi-Fujiwara-Ishida [4], we discuss one of the multidimensional inverse scattering problems based on the time-dependent method of Enss-Weder [13]. We allow that the electric field converges on zero and show that the high velocity limit of the scattering operator determines uniquely the short-range part of the potential. This work is an improvement of Adachi-Kamada-Kazuno-Toratani [6].

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1 Direct Problem

1.1 Introduction

Throughout this paper, we consider a quantum system in a time-dependent electric field $E(t) \in \mathbb{R}^d$ asymptotically constant in time. The free Hamiltonian under the consideration is given by

$$H_0(t) = p^2/(2m) - q_0 E(t) \cdot x \tag{1.1.1}$$

acting on $L^2(\mathbb{R}^d)$, where $m > 0$, $q_0 \in \mathbb{R} \setminus \{0\}$ and $p = -i\nabla$ stand for the mass, the charge and the momentum of the particle under consideration. The mass m of the particle can be put as 1 by a suitable scale transformation. Except in subsection 1.4, we put $q_0 = 1$ for simplicity's sake.

$$H_0(t) = p^2/2 - E(t) \cdot x. \tag{1.1.2}$$

Let $U_0(t, s)$ be the unitary propagator generated by $H_0(t)$.

In the first section, we assume that the external electric field converges on non-zero as in Assumption 1.1.1 mentioned below. We will prove the asymptotic completeness a one-body system and a charge transfer model. All of the arguments in one-body case are originated in Adachi-Ishida [5]. As an application of the one-body case, we discuss the scattering theory for the charge transfer model. We will obtain an improvement of the previous result of Ishida [25].

Assumption 1.1.1. *For some $\eta > 0$, let $e(t) \in C(\mathbb{R}, \mathbb{R}^d)$ be the perturbation of the constant electric field $E \in \mathbb{R}^d \setminus \{0\}$ such that $E(t) = E + e(t)$ and $e(t) = O(|t|^{-\eta})$ as $|t| \to \infty$.*

Until the end of subsection 1.3, we suppose that the potential V satisfies the following assumption and the full Hamiltonian is given by

$$H(t) = H_0(t) + V, \tag{1.1.3}$$

where $V = V^{\mathrm{s}} + V^{\mathrm{l}} \in \mathscr{V}_{\rho_{\mathrm{s}}} + \mathscr{V}_{\rho_{\mathrm{l}}}$ with some $\rho_{\mathrm{s}} > 1/2$ and $0 < \rho_{\mathrm{l}} \leqslant 1/2$.

Assumption 1.1.2. *Let $\rho > 0$. $\mathscr{V}_\rho$ is the class of the real-valued multiplication operators V satisfying $V \in C^\infty(\mathbb{R}^d)$ and*

$$|\partial_x^\beta V(x)| \leqslant C_\beta \langle x \rangle^{-\rho - |\beta|/2}, \quad \langle x \rangle = (1 + x^2)^{1/2}. \tag{1.1.4}$$

It is known that the self-adjointness of $H(t)$ on $L^2(\mathbb{R}^d)$ for each $t \in \mathbb{R}$ can be guaranteed. By virtue of results of Yajima [44] and the Avron-Herbst formula (see Cycon-Froese-Kirsch-Simon [10]), the existence and uniqueness of the propagator $U(t,s)$ generated by $H(t)$ can be also guaranteed as below. We introduce a strongly continuous family of unitary operators $\{\tilde{\mathscr{T}}(t)\}_{t\in\mathbb{R}}$ by

$$\tilde{\mathscr{T}}(t) = e^{-i\tilde{a}(t)} e^{i\tilde{b}(t)\cdot x} e^{-i\tilde{c}(t)\cdot p} \tag{1.1.5}$$

where

$$\tilde{b}(t) = \int_0^t E(\tau)d\tau, \quad \tilde{c}(t) = \int_0^t \tilde{b}(\tau)d\tau, \quad \tilde{a}(t) = \int_0^t \tilde{b}(\tau)^2/2d\tau. \tag{1.1.6}$$

We also introduce the time-dependent Hamiltonian

$$H^{Sc}(t) = p^2/2 + V(x + \tilde{c}(t)). \tag{1.1.7}$$

Since the propagator generated by $H^{Sc}(t)$ exists uniquely by virtue of results of [44], we write it as $U^{Sc}(t,s)$. Then we see that the propagator $U(t,s)$ generated by $H(t)$ also exists uniquely because of the Avron-Herbst formula

$$U(t,s) = \tilde{\mathscr{T}}(t) U^{Sc}(t,s) \tilde{\mathscr{T}}(s)^*. \tag{1.1.8}$$

In the case where $0 < \eta \leqslant 2$, we need an additional assumption.

Assumption 1.1.3. *When $0 < \eta \leqslant 2$, we suppose*

$$e_0 = |E| - \sup_{x\in\mathbb{R}^d} \omega \cdot (\nabla_x V)(x) > 0, \quad \omega = E/|E|. \tag{1.1.9}$$

We first consider the case where $V^1 = 0$ and we can report the following theorem obtained by [5].

Theorem 1.1.4. (Asymptotic Completeness [5]) *In the case where $V^1 = 0$, the wave operators*

$$W^{\pm} = \operatorname*{s-lim}_{t\to\pm\infty} U(t,0)^* U_0(t,0). \tag{1.1.10}$$

exist and are unitary on $L^2(\mathbb{R}^d)$.

We next consider the case where $V^1 \neq 0$. In this case, it is known that the wave operators (1.1.10) do not exist generally (see e.g. Ozawa [33]) and we have to discuss some modifications. Throughout this section, we often use the following convention for smooth cut-off functions $0 \leqslant F_\delta \leqslant 1$ with $\delta > 0$ satisfying

$$F_\delta(\lambda \leqslant \mu) = \begin{cases} 1 & \lambda \leqslant \mu - \delta \\ 0 & \lambda \geqslant \mu, \end{cases} \quad F_\delta(\lambda \geqslant \mu) = \begin{cases} 1 & \lambda \geqslant \mu + \delta \\ 0 & \lambda \leqslant \mu \end{cases} \tag{1.1.11}$$

and put $F_\delta(\mu_1 \leqslant \lambda \leqslant \mu_2) = F_\delta(\lambda \geqslant \mu_1)F_\delta(\lambda \leqslant \mu_2)$. Introduce well behaved potential V_l as

$$V_l(t,x) = V^l(x)F_{\epsilon_0}(z/\langle t\rangle^2 \geqslant \epsilon_0), \quad z = \omega \cdot x, \tag{1.1.12}$$

where

$$\epsilon_0 = \begin{cases} e_0/14 & 0 < \eta \leqslant 2 \\ |E|/14 & \eta > 2. \end{cases} \tag{1.1.13}$$

An approximate solution of the Hamilton-Jacobi equation

$$(\partial_t K)(t,\xi) = (\xi + \tilde{b}(t))^2/2 + V_l(t, (\nabla_\xi K)(t,\xi)) \tag{1.1.14}$$

can be constructed quite similarly in Adachi [2] and Adachi-Tamura [9], by the decaying property

$$|\partial_x^\beta V_l(t,x)| \leqslant C_\beta \langle t\rangle^{-2\rho_l - |\beta|}. \tag{1.1.15}$$

[5] could obtain the asymptotic completeness of the modified wave operators.

Theorem 1.1.5. (Asymptotic Completeness [5]) *In the case where $V^l \neq 0$, the modified wave operators*

$$W_D^\pm = \operatorname*{s-lim}_{t\to\pm\infty} U(t,0)^* U_0(t,0) e^{-i\int_0^t V^l((\nabla_\xi K)(\tau,p))d\tau} \tag{1.1.16}$$

exist and are unitary on $L^2(\mathbb{R}^d)$.

In particular, when $1/4 < \rho \leqslant 1/2$, one can take $K(t,\xi)$ as $K_0(t,\xi) = t\xi^2/2 + \tilde{c}(t)\cdot\xi + \tilde{a}(t)$, which is an unique solution of (1.1.14) with $V^l = 0$. This modifier

$$e^{-i\int_0^t V^l((\nabla_\xi K_0)(\tau,p))d\tau} = e^{-i\int_0^t V^l(p\tau + \tilde{c}(\tau))d\tau} \tag{1.1.17}$$

is the so-called Dollard-type one (cf. Jensen-Ozawa [26], Jensen-Yajima [27] and White [40]). As for a modifier in the position representation, see Adachi-Kimura-Shimizu [7], in which a time-periodic $E(t)$ is treated. Moreover, let $\tilde{\mathscr{V}}_\rho$ be the class of real-valued potentials V satisfying $V \in C^\infty(\mathbb{R}^d)$ and

$$|\partial_x^\beta V(x)| \leqslant C_\beta \langle x\rangle^{-\rho-|\beta|}. \tag{1.1.18}$$

[5] could also obtain the asymptotic completeness of the modified wave operators.

Theorem 1.1.6. (Asymptotic Completeness [5]) *If $V^l \neq 0$ satisfies a stronger condition $V^l \in \tilde{\mathscr{V}}_{\rho_l}$ with $0 < \rho_l \leqslant 1/2$, then the modified wave operators*

$$W_G^\pm = \operatorname*{s-lim}_{t\to\pm\infty} U(t,0)^* U_0(t,0) e^{-i\int_0^t V^l(\tilde{c}(\tau))d\tau} \tag{1.1.19}$$

exist and are unitary on $L^2(\mathbb{R}^d)$.

This modifier $e^{-i\int_0^t V^1(\tilde{c}(\tau))d\tau}$ is the so-called Graf-type (or Zorbas-type) one (cf. Graf [17], [26] and Zorbas [49]).

[5] claims that the threshold of $-\eta$, which is the power of the decaying order of $e(t)$, is -2 in terms of the boundedness of a certain energy in time. We first move the effect arising from $e(t)$ into the potential V, and reduce the present problem to the one for a so-called Stark Hamiltonian with a certain time-dependent potential, by using a version of the Avron-Herbst formula. This treatment was initiated by Møller [30] in the case where $e(t)$ is periodic in time. When $0 < \eta \leqslant 2$, we put

$$b(t) = \int_0^t e(\tau)d\tau, \quad c(t) = \int_0^t b(\tau)d\tau \tag{1.1.20}$$

On the other hand, when $\eta > 2$, by noting that $e(t)$ is integrable on $\mathbb{R}$, we put

$$b(t) = -\int_t^\infty e(\tau)ds, \quad c(t) = -\int_t^\infty b(\tau)d\tau \tag{1.1.21}$$

if we treat the case where $t \to \infty$, since $b(t) = O(t^{1-\eta})$ with $1 - \eta < -1$. If we treat the case where $t \to -\infty$, similarly we put

$$b(t) = \int_{-\infty}^t e(\tau)d\tau, \quad c(t) = -\int_{-\infty}^t b(\tau)d\tau. \tag{1.1.22}$$

Here we note that $c(t) = o(|t|^2)$ as $|t| \to \infty$, more precisely,

$$c(t) = \begin{cases} O(|t|^{2-\eta}) & \eta < 1 \text{ or } \eta > 2 \\ O(|t| \log |t|) & \eta = 1 \\ O(|t|) & 1 < \eta \leqslant 2. \end{cases} \tag{1.1.23}$$

We also introduce the time-dependent Hamiltonian $H^S(t)$ by

$$H^S(t) = H_0^S + V(x + c(t)), \quad H_0^S = p^2/2 - E \cdot x, \tag{1.1.24}$$

and strongly continuous family of unitary operators $\{\mathscr{T}(t)\}_{t\in\mathbb{R}}$ by

$$\mathscr{T}(t) = e^{-ia(t)}e^{ib(t)\cdot x}e^{-ic(t)\cdot p}, \quad a(t) = \int_0^t (b(\tau)^2/2 - E \cdot c(\tau))d\tau. \tag{1.1.25}$$

H_0^S is called the free Stark Hamiltonian. Write the unitary propagator generated by $H^S(t)$ as $U^S(t,s)$, then the Avron-Herbst formula

$$U(t,s) = \mathscr{T}(t)U^S(t,s)\mathscr{T}(s)^* \tag{1.1.26}$$

holds. We note the domain invariance property $U^S(t,0)\mathscr{D}(H_0^S) \subset \mathscr{D}(H_0^S)$ (see e.g. Theorem X. 70 in Reed-Simon [34]). The Heisenberg derivative of $\Phi(t)$ associated with $H(t)$ is denoted by $\mathbb{D}_{H(t)}\Phi(t) = \partial_t \Phi(t) + i[H(t), \Phi(t)]$. Then we obtain

$$\mathbb{D}_{H^S(t)} H^S(t) = b(t) \cdot (\nabla_x V)(x + c(t)). \tag{1.1.27}$$

When $\eta > 2$, $\mathbb{D}_{H^S(t)} H^S(t)$ is integrable on I because $b(t)$ is integrable on I and $\nabla_x V$ is bounded, where $I = [0, \infty)$ if the case $t \to \infty$ is treated. In the case where $t \to -\infty$ we just put $I = (-\infty, 0]$. This implies the boundedness of $H^S(t)U^S(t,0)f(H_0^S)$ in $t \in I$ for $f \in C_0^\infty(\mathbb{R})$, which is one of the keys in this section. Let $U_1^S(t,s)$ be the unitary propagator generated by the time-dependent Hamiltonian

$$H_1^S(t) = H_0^S + V_1(t, x + c(t)). \tag{1.1.28}$$

The following theorem yields the proof of Theorems 1.1.4, 1.1.5 and 1.1.6.

Theorem 1.1.7. (Asymptotic Clustering [5]) *Under the above assumptions,*

$$W_1^{S,\pm} = \operatorname*{s-lim}_{t \to \pm\infty} U^S(t,0)^* U_1^S(t,0) \tag{1.1.29}$$

exist and are unitary on $L^2(\mathbb{R}^d)$.

It follows from this theorem and the Avron-Herbst formula (1.1.26) that

$$W_1^{\pm} = \operatorname*{s-lim}_{t \to \pm\infty} U(t,0)^* U_1(t,0) \tag{1.1.30}$$

exist and are unitary on $L^2(\mathbb{R}^d)$, where $U_1(t,s)$ is the unitary propagator generated by the time-dependent Hamiltonian

$$H_1(t) = H_0(t) + V_1(t,x). \tag{1.1.31}$$

If $V^1 = 0$, that is, V is a Stark short-range potential, then this means Theorem 1.1.4 itself. For a Stark short-range potential $V \in \tilde{\mathscr{V}}_\rho$ with $\rho_s > 1/2$, Yokoyama [46] showed this theorem, that is, asymptotic completeness of $W^\pm$, in the following two cases. In the first case, it is supposed that $\eta > 1$ with the additional condition (1.1.9). In the second case, it is supposed that $\eta > 7/2$ without (1.1.9). The condition $\eta > 1$ implies the integrability of $e(t)$ on $\mathbb{R}$. [5] was be able to relax the decaying conditions on $e(t)$ and the derivatives of V, and also deal with Stark long-range case. The notion of the asymptotic clustering has played an important role particularly in the study of the N-body long-range scattering in an external electric field (see e.g. [9], Herbst-Møller-Skibsted [21], [2] and Adachi [3]).

In the case where $V_1 \neq 0$, we will firstly prove Theorem 1.1.6 by assuming that Theorem 1.1.7 holds. In the argument below, we will consider the case where $t \to \infty$ only. The case where $t \to -\infty$ can be dealt with quite similarly.

Proof of Theorem 1.1.6. By virtue of Theorem 1.1.7, it is sufficient to show the existence and the unitarity of

$$\operatorname*{s-lim}_{t\to\infty} U_1(t,0)^* U_0(t,0) e^{-i\int_0^t V^1(\tilde{c}(\tau))d\tau} \tag{1.1.32}$$

We will prove the existence of

$$\operatorname*{s-lim}_{t\to\infty} e^{i\int_0^t V^1(\tilde{c}(\tau))d\tau} U_0(t,0)^* U_1(t,0) \tag{1.1.33}$$

only. The proof of the existence of (1.1.32) are quite same. We compute

$$\begin{aligned}&\partial_t(e^{i\int_0^t V^1(\tilde{c}(\tau))d\tau} U_0(t,0)^* U_1(t,0))\\ &\quad= ie^{i\int_0^t V^1(\tilde{c}(\tau))d\tau} U_0(t,0)^*(V^1(\tilde{c}(t)) - V_1(t,x))U_1(t,0).\end{aligned} \tag{1.1.34}$$

Noting that $V^1 \in \tilde{\mathscr{V}}_{\rho_1}$ and the decaying property (see (1.1.15)), we have

$$|\partial_x^\beta V_1(t,x)| \leqslant C_\beta \langle t\rangle^{-2\rho_1-2|\beta|}. \tag{1.1.35}$$

Then we can obtain following propagation estimate for $U_1(t,0)$

$$\||x-\tilde{c}(t)|U_1(t,0)\phi\| = O(t) \tag{1.1.36}$$

for $\phi \in C_0^\infty(\mathbb{R})$ because

$$\mathbb{D}_{H_1(t)}(p-\tilde{b}(t)) = -(\nabla_x V_1)(t,x), \quad \mathbb{D}_{H_1(t)}(x-\tilde{c}(t)) = p-\tilde{b}(t). \tag{1.1.37}$$

Noting that $c(t) = o(t^2)$ and (1.1.13), we can see that

$$\tilde{c}(t)\cdot\omega/\langle t\rangle^2 \geqslant |E|t^2/(2\langle t\rangle^2) - |c(t)|/\langle t\rangle^2 \geqslant 2\epsilon_0 \tag{1.1.38}$$

as $t\to\infty$ holds. We thus obtain $V^1(\tilde{c}(t)) = V_1(t,\tilde{c}(t))$ and

$$V^1(\tilde{c}(t)) - V_1(t,x) = \int_0^t (\nabla_x V_1)(t,\theta(x-\tilde{c}(t))+\tilde{c}(t))d\theta\cdot(x-\tilde{c}(t)). \tag{1.1.39}$$

Using (1.1.35) and (1.1.36), we can estimate

$$\|\partial_t(e^{i\int_0^t V^1(\tilde{c}(\tau))d\tau} U_0(t,0)^* U_1(t,0))\phi\| = O(t^{-2\rho_1-1}). \tag{1.1.40}$$

This implies the exisitence of (1.1.33) by the Cook-Kuroda method and a density argument. □

In order to lead Theorem 1.1.5 from Theorem 1.1.7, we prepare some notation and propositions. For simplicity's sake, we assume that $1/4 < \rho_l \leqslant 1/2$. The case where $0 < \rho_l \leqslant 1/4$ can be shown on the analogy (see [3] and [9] for the details). Recall

$$K_0(t,\xi) = \int_0^t (\xi + \tilde{b}(\tau))^2 d\tau/2 = \xi^2 t/2 + \tilde{c}(t)\cdot\xi + \tilde{a}(t) \tag{1.1.41}$$

as mentioned above and take $K_1(t,\xi)$ as an unique solution of the first approximate Hamiltom-Jacobi equation (1.1.14)

$$(\partial_t K_1)(t,\xi) = (\xi + \tilde{b}(t))^2/2 + V_l(t, (\nabla_\xi K_0)(t,\xi)), \tag{1.1.42}$$

that is,

$$K_1(t,\xi) = K_0(t,\xi) + \int_0^t V_l(\tau, (\nabla_\xi K_0)(\tau,\xi)) d\tau. \tag{1.1.43}$$

We here note that

$$\sup_{\xi\in\mathbb{R}^d} |\partial_\xi^\beta (K_1(t,\xi) - K_0(t,\xi))| = O(t^{1-2\rho_l}) \tag{1.1.44}$$

holds by virtue of (1.1.15). Putting $S_0(t,\xi) = K_0(t,\xi - \tilde{b}(t))$ and $S_1(t,\xi) = K_1(t,\xi - \tilde{b}(t))$, $S_1(t,\xi)$ satisfies

$$(\partial_t S_1)(t,\xi) = \xi^2/2 - E(t)\cdot(\nabla_\xi S_1)(t,\xi) + V_l(t,(\nabla_\xi S_0)(t,\xi)). \tag{1.1.45}$$

We will write $V_{l,0}(t,\xi) = V_l(t,(\nabla_\xi S_0)(t,\xi))$ and $V_{l,1}(t,\xi) = V_l(t,(\nabla_\xi S_1)(t,\xi))$ below and define the Hamiltonian $\hat{H}_l(t)$ by

$$\hat{H}_l(t) = H_0(t) + V_{l,0}(t,\xi) \tag{1.1.46}$$

and let $\hat{U}_l(t,s)$ be the propagator generated by $\hat{H}_l(t)$.

Proposition 1.1.8. *The following propagation estimates hold for* $\phi \in C_0^\infty(\mathbb{R}^d)$.

$$\| |x - (\nabla_\xi S_0)(t,p)| U_l(t,0)\phi \| = O(t^{1-2\rho_l}), \tag{1.1.47}$$

$$\| |x - (\nabla_\xi S_0)(t,p)| \hat{U}_l(t,0)\phi \| = O(t^{1-2\rho_l}). \tag{1.1.48}$$

Proof. Put $\Phi(t) = (x - (\nabla_\xi S_1)(t,p))$. We first note that

$$\mathbb{D}_{H_0(t)}\Phi(t) = p - (\partial_t \nabla_\xi S_1)(t,p) - E(t)\cdot\nabla_\xi^2 S_1(t,p) = -(\nabla_\xi V_{l,0})(t,p) \tag{1.1.49}$$

holds by (1.1.45). We thus obtain

$$\mathbb{D}_{\hat{H}_l(t)}\Phi(t) = -\mathbb{D}_{H_0(t)}\Phi(t) - (\nabla_\xi V_{l,0})(t,p) = 0. \tag{1.1.50}$$

(1.1.48) can be obtained by this and (1.1.44). We next note that $H_l(t) = \hat{H}_l(t) + V_l(t,x) - V_{l,0}(t,p)$. We put

$$g(t,x,\xi) = \int_0^1 (\nabla_x V_l)(t, \theta(x - (\nabla_\xi S_1)(t,p)) + (\nabla_\xi S_1)(t,p)) d\theta \tag{1.1.51}$$

By the Baker-Campbell-Hausdorff formula with the Weyl symbol of the Pseudo-differential calculation (see e.g. Dereziński-Gérard [11]),

$$V_l(t,x) - V_{l,1}(t,p) = g(t,x,\xi) \cdot \Phi(t) + i \operatorname{div}_\xi g(t,x,\xi)/2. \tag{1.1.52}$$

We thus can compute

$$\begin{aligned}
\mathbb{D}_{H_l(t)}\Phi(t) &= \mathbb{D}_{\hat{H}_l(t)}\Phi(t) + i[V_l(t,p) - V_{l,0}(t,p), \Phi(t)] \\
&= i[g(t,x,p), \Phi(t)]\Phi(t) - [(\operatorname{div}_\xi g)(t,x,p), \Phi(t)]/2 \\
&\quad + i[V_{l,1}(t,p) - V_{l,0}(t,p), \Phi(t)] \\
&= O(t^{-2\rho_l - 1})\Phi(t) + O(t^{-2\rho_l - 2}) + O(t^{-4\rho_l}),
\end{aligned} \tag{1.1.53}$$

where we used (1.1.15) and (1.1.44). Since $-4\rho_l < -1$ by assumption, we have

$$\||\Phi(t)|U_l(t,0)\phi\| \leqslant C\left(1 + \int_0^t \tau^{-2\rho_l}\||\Phi(\tau)|U_l(\tau,0)\phi\| d\tau\right) \tag{1.1.54}$$

with $C > 0$, which implies $\||\Phi(t)|U_l(t,0)\phi\| = O(1)$ by virtue of the Gronwall inequality. Thus (1.1.47) follows from this and (1.1.44). □

Proposition 1.1.9. *The strong limit*

$$\operatorname*{s-lim}_{t\to\infty} \hat{U}_l(t,0)^* U_l(t,0) \tag{1.1.55}$$

exists and is unitary on $L^2(\mathbb{R}^d)$.

Proof. We decompose

$$\begin{aligned}
V_l(t,x) - V_{l,0}(t,p) &= g(t,x,p) \cdot (x - (\nabla_\xi S_0(t,p)) + i(\operatorname{div}_\xi g)(t,x,p)/2 \\
&\quad + g(t,x,p) \cdot ((\nabla_\xi S_0(t,p)) - (\nabla_\xi S_1(t,p))) + V_{l,1}(t,p) - V_{l,0}(t,p),
\end{aligned} \tag{1.1.56}$$

using (1.1.52). Since $g(t,x,p) = O(t^{-2\rho_l - 1})$, $(\operatorname{div}_\xi g)(t,x,p) = O(t^{-2\rho_l - 1})$, $V_{l,1}(t,p) - V_{l,0}(t,p) = O(t^{-4\rho_l})$ and (1.1.44),

$$V_l(t,x) - V_{l,0}(t,p) = O(t^{-2\rho_l - 1})(x - (\nabla_\xi S_0(t,p)) + O(t^{-2\rho_l - 1}) + O(t^{-4\rho_l}) \tag{1.1.57}$$

holds. By virtue of Proposition 1.1.8 and the Cook-Kuroda method, we obtain this proposition because $-\rho_l - 1 + 1 - 2\rho_l = -4\rho_l < -1$ by assumption. □

Proposition 1.1.10. *The strong limit*

$$\operatorname*{s-lim}_{t\to\infty} e^{i\int_0^t V^1((\nabla_\xi K_0)(\tau,p))} e^{-i\int_0^t V_1(\tau,(\nabla_\xi K_0)(\tau,p))} \tag{1.1.58}$$

exists and is unitary on $L^2(\mathbb{R}^d)$.

Proof. Let $\phi \in \mathscr{S}(\mathbb{R}^d)$ be such that $\hat{\phi} \in C_0^\infty(\mathbb{R}^d)$. Here $\hat{\phi}$ stands for the Fourier transform of ϕ. We can estimate

$$(\nabla_\xi K_0)(t,\xi)\cdot\omega/\langle t\rangle^2 = (\xi t+\tilde{c}(t))\cdot\omega/\langle t\rangle^2 \geqslant |E|t^2/(2\langle t\rangle^2) - (|\xi|t+|c(t)|)/\langle t\rangle^2 \geqslant 2\epsilon_0 \tag{1.1.59}$$

as $t \to \infty$ for $\xi \in \operatorname{supp}\hat{\phi}$, here we used $c(t) = o(t^2)$. We thus obtain

$$(V^1((\nabla_\xi K_0)(t,p)) - V_1(t,(\nabla_\xi K_0)(t,p)))\phi = 0. \tag{1.1.60}$$

This implies the proposition by the Cook-Kuroda method and a density argument. □

Now we prove Theorem 1.1.5 assuming that Theorem 1.1.7 holds.

Proof of Theorem 1.1.5. By virtue of Theorem 1.1.7, Propositions 1.1.8, 1.1.9 and 1.1.10, it is sufficient to see the existence and the unitarity of

$$\operatorname*{s-lim}_{t\to\infty} \hat{U}_1(t,0)^* U_0(t,0) e^{-i\int_0^t V_1(\tau,(\nabla_\xi K)(\tau,p))}. \tag{1.1.61}$$

However, this follows immediately from

$$\partial_t(\hat{U}_1(t,0)^* U_0(t,0) e^{-i\int_0^t V_1(\tau,(\nabla_\xi K)(\tau,p))}) = 0, \tag{1.1.62}$$

$$\partial_t(e^{i\int_0^t V_1(\tau,(\nabla_\xi K)(\tau,p))} U_0(t,0)^* \hat{U}_1(t,0)) = 0 \tag{1.1.63}$$

by the Avron-Herbst formula (1.1.8) for $U_0(t,0)$ and the Cook-Kuroda method. □

Thus we are devoted to the proof of Theorem 1.1.7.

For time-dependent Hamiltonians, the lack of energy conservation is an obstruction in studying this scattering problem. Howland [23] proposed the stationary scattering theory for time-dependent Hamiltonians by introducing a new Hamiltonian which is formally given by $-i\partial_t + H(t)$ acting on an appropriate grand Hilbert space like $L^2(\mathbb{R}, L^2(\mathbb{R}^d))$. This formulation is the quantum analogue to the procedure in the classical mechanics for the sake of recovering the conservation of energy. As is well known, it is shown by Yajima [42] that this method does work well for the Hamiltonians which govern two-body quantum systems with time-periodic short-range potentials, also by virtue of the Floquet theory (see also Howland [24], Yokoyama [45], [30] , [2] and so on). Nowadays the method is called the Howland-Yajima method. However, when one deals with the long-range case, it seems necessary to obtain propagation properties of the physical propagator directly (see Kitada-Yajima [29], [3] and [7]). Hence, [5] derives some useful propagation estimates for the physical propagator $U^S(t,0)$ directly for smooth potentials. Some of them were already obtained in [46] under the assumptions stronger than [5] (see also Adachi [1] as for the case where $E(t)$ is a non-zero constant vector, and [3] and [7] as for the case where $E(t)$ is periodic in time with non-zero mean). We will watch the influence of the decaying condition on $e(t)$ upon propagation estimates of $U^S(t,0)$ later.

1.2 Propagation estimates

We first prepare the notations by following Skibsted [35]. For given β, $\alpha \geqslant 0$ and $\epsilon > 0$, take a function $\psi_{\alpha,\epsilon}(\lambda) = F_\epsilon(\lambda \leqslant -\epsilon)$ such that $\psi'_{\alpha,\epsilon}(\lambda) \leqslant 0$, $\alpha\psi_{\alpha,\epsilon}(\lambda) + \lambda\psi'_{\alpha,\epsilon}(\lambda) = \kappa(\lambda)^2$ for some $\kappa \in C^\infty(\mathbb{R})$ with $\kappa \geqslant 0$. Put $g_{\beta,\alpha,\epsilon}(\lambda, t) = -t^{-\beta}(-\lambda)^\alpha\psi_{\alpha,\epsilon}(\lambda/t)$ for $\lambda \in \mathbb{R}$ and $t > 0$. Write $g^{(k)}_{\beta,\alpha,\epsilon}(\lambda, t) = \partial^k_\lambda g_{\beta,\alpha,\epsilon}(\lambda, t)$ for $k \in \mathbb{N} \cup \{0\}$.

We recall the almost analytic extension method due to Helffer-Sjöstrand[22], which is useful in analyzing operators given by functions of self-adjoint operators. For $m \in \mathbb{R}$, let S^m be the set of functions $f \in C^\infty(\mathbb{R})$ such that $|f^{(k)}(\lambda)| \leqslant C_k\langle\lambda\rangle^{m-k}$ for $k \geqslant 0$. If $f \in S^m$ with $m \in \mathbb{R}$, then there exists $\tilde{f} \in C^\infty(\mathbb{C})$ such that $\tilde{f}(\lambda) = f(\lambda)$ for $\lambda \in \mathbb{R}$, $\operatorname{supp}\tilde{f}(\zeta) \subset \{\zeta \in \mathbb{C} \bigm| |\operatorname{Im}\zeta| \leqslant C(1 + |\operatorname{Re}\zeta|)\}$ and $|\bar{\partial}_\zeta\tilde{f}(\zeta)| \leqslant C_M\langle\zeta\rangle^{m-1-M}|\operatorname{Im}\zeta|^M$ for $M \in \mathbb{N}\bigcup\{0\}$. Such a function $\tilde{f}$ is called an almost analytic extension of f. Let B be a self-adjoint operator. If $f \in S^{-m}$ with $m > 0$, then $f(B)$ is represented by

$$f(B) = \int_{\mathbb{C}} \bar{\partial}_\zeta\tilde{f}(\zeta)(\zeta - B)^{-1}d\zeta \wedge d\bar{\zeta}/(2\pi i). \tag{1.2.1}$$

For $f \in S^m$ with $m \in \mathbb{R}$ and self-adjoint operators B_1 and B_2, we have the following formulas of asymptotic expansion of the commutator (see Gérard[15], [11] and Remark 1.2.6). Define $\operatorname{ad}^k_{B_1}(B_2)$ by $\operatorname{ad}^0_{B_1}(B_2) = B_2$, and $\operatorname{ad}^k_{B_1}(B_2) = [\operatorname{ad}^{k-1}_{B_1}(B_2), B_1]$ for $k \geqslant 1$, then

$$\begin{aligned}[B_2, f(B_1)] &= \sum_{k=1}^{M-1}(-1)^{k-1}\operatorname{ad}^k_{B_1}(B_2)f^{(k)}(B_1)/k! + R_M &(1.2.2)\\ &= \sum_{k=1}^{M-1} f^{(k)}(B_1)\operatorname{ad}^k_{B_1}(B_2)/k! + R'_M, &(1.2.3)\end{aligned}$$

where

$$R_M = (-1)^{M+1}\int_{\mathbb{C}} \bar{\partial}_\zeta \tilde{f}(\zeta)(\zeta - B_1)^{-1}\,\mathrm{ad}_{B_1}^M(B_2)(\zeta - B_1)^{-M} d\zeta \wedge d\bar{\zeta}/(2\pi i), \quad (1.2.4)$$

$$R'_M = \int_{\mathbb{C}} \bar{\partial}_\zeta \tilde{f}(\zeta)(\zeta - B_1)^{-M}\,\mathrm{ad}_{B_1}^M(B_2)(\zeta - B_1)^{-1} d\zeta \wedge d\bar{\zeta}/(2\pi i). \quad (1.2.5)$$

We first consider the case where $0 < \eta \leqslant 2$, which can be treated easier than the case where $\eta > 2$ by virtue of the additional condition (1.1.9). We set $A = \omega \cdot p$, which is known as a conjugate operator for the Stark Hamiltonian $H^S = H_0^S + V$ (see Theorem 1.1.5 below due to Herbst-Møller-Skibsted [20]). One of the main purposes in this subsection is obtaining the following propagation estimates for $U^S(t,0)$.

Proposition 1.2.1. (Minimal Acceleration Bound [5]) *Let $f \in C_0^\infty(\mathbb{R})$, $e_1 \leqslant e_0$, $0 \leqslant s_0 \leqslant s_1$ and $\epsilon > 0$. Put $\zeta_{A,s_1}(t) = U^S(t,0)f(H_0^S)\langle A\rangle^{-s_1/2}$. Then the following estimate holds as $t \to \infty$.*

$$\|(e_1 - A/t)^{s_0/2}F_\epsilon(A/t \leqslant e_1 - \epsilon)\zeta_{A,s_1}(t)\|_{\mathscr{B}(L^2)} = O(t^{-s_1/2}). \quad (1.2.6)$$

Proof. This proposition was proved essentially in [46] under the assumption that $\eta > 1$ by the abstract theory due to [35]. Here, we sketch the proof by following the argument slightly modified by [3].

Let us abbreviate $\mathbb{D}_{H^S(\tau)}$ as $\mathbb{D}$. Set $A(t) = A - e_1 t$. Then we have

$$\mathbb{D}A(t) = i[H^S(t), A] - e_1 = |E| - \omega\cdot(\nabla_x V)(x + \tilde{c}(t)) - e_1 \geqslant e_0 - e_1 > 0, \quad (1.2.7)$$

which is one of the keys in the proof. Take $0 < \beta_0 < 1$, and set $B(t) = A(t)/t$. Let $(\beta,\alpha) = (\beta_0,\alpha_0)$ or $(\beta,\alpha) = (0,\alpha_0 - 1) \neq (0,0)$ with $\alpha_0 \in \mathbb{N}$. For $\phi \in L^2(\mathbb{R}^d)$, we put $\phi(t) = \zeta_{A,\alpha}(t)\phi$ and use the convention $\langle P(t)\rangle_t = (\phi(t), P(t)\phi(t))$. Putting $\sigma_{\alpha,\epsilon}(\lambda) = (-\lambda)^\alpha \chi_{\alpha,\epsilon}(\lambda)$ as in [3], $g_{\beta,\alpha,\epsilon}(\lambda,t)$ is written as $g_{\beta,\alpha,\epsilon}(\lambda,t) = -t^{\alpha-\beta}\sigma_{\alpha,\epsilon}(\lambda/t)$. Hence we have

$$g_{\beta,\alpha,\epsilon}(A(t),t) = -t^{\alpha-\beta}\sigma_{\alpha,\epsilon}(B(t)). \quad (1.2.8)$$

We abbreviate $g_{\beta,\alpha,\epsilon}$ and $\sigma_{\alpha,\epsilon}$ as g and σ, respectively. Note that an almost analytic extension $\tilde{\sigma}$ of $\sigma \in S^\alpha$ satisfies $|\bar{\partial}_\zeta\tilde{\sigma}(\zeta)| \leqslant C_M\langle\zeta\rangle^{\alpha-1-M}|\operatorname{Im}\zeta|^M$ for $M \geqslant 0$. Since

$$\langle -g(A(t),t)\rangle_t = \langle -g(A(1),1)\rangle_1 - \int_1^t \langle \mathbb{D}g(A(\tau),\tau)\rangle_\tau d\tau, \quad (1.2.9)$$

let us watch

$$\mathbb{D}g(A(\tau),\tau) = -\tau^{\alpha-\beta}\mathbb{D}\sigma(B(\tau)) - (\alpha-\beta)\tau^{\alpha-\beta-1}\sigma(B(\tau)). \quad (1.2.10)$$

By virtue of the almost analytic extension method, one can write

$$\mathbb{D}g(A(\tau),\tau)=\sum_{j=1}^{7}E_j(\tau) \tag{1.2.11}$$

with

$$\begin{aligned}
E_1(\tau) &= \tau^{\alpha-\beta-1}B(\tau)\sigma^{(1)}(B(\tau))-(\alpha-\beta)\tau^{\alpha-\beta-1}\sigma(B(\tau)), && (1.2.12)\\
E_2(\tau) &= \tau^{\alpha-\beta-1}\gamma(B(\tau))(\mathbb{D}A(\tau))\gamma(B(\tau)), && (1.2.13)\\
E_3(\tau) &= \tau^{\alpha-\beta-1}\gamma(B(\tau))\sum_{m=1}^{\alpha_1}(-1)^m\tau^{-m}\operatorname{ad}^m_{A(\tau)}(\mathbb{D}A(\tau))\gamma^{(m)}(B(\tau))/m!, && (1.2.14)\\
E_4(\tau) &= \tau^{\alpha-\beta-1}\gamma(B(\tau))R_1(\tau), && (1.2.15)\\
E_5(\tau) &= -\sum_{m=2}^{\alpha_0}\tau^{\alpha-\beta-m}g_m(B(\tau))h_m(B(\tau))/m!\\
&\quad\times\sum_{m_1=0}^{\alpha-m}(-1)^{m_1}\tau^{-m_1}\operatorname{ad}^{m_1+m-1}_{A(\tau)}(\mathbb{D}A(\tau))g_m^{(m_1)}(B(\tau))/m_1!, && (1.2.16)\\
E_6(\tau) &= -\sum_{m=2}^{\alpha_0}\tau^{\alpha-\beta-m}g_m(B(\tau))h_m(B(\tau))R_m(\tau)/m!, && (1.2.17)\\
E_7(\tau) &= -\tau^{\alpha-\alpha_0-\beta-1}\int_{\mathbb{C}}\bar\partial_\zeta\tilde\sigma(\zeta)(\zeta-B(\tau))^{-\alpha_0-1}\operatorname{ad}^{\alpha_0}_{A(\tau)}(\mathbb{D}A(\tau))\\
&\quad\times(\zeta-B(\tau))^{-1}d\zeta\wedge d\bar\zeta/(2\pi i), && (1.2.18)
\end{aligned}$$

where $\alpha_1=(\alpha-1)/2$ if α is odd, and $\alpha_1=\alpha/2$ if α is even,

$$\begin{aligned}
R_1(\tau) &= (-1)^{\alpha_1+1}\tau^{-\alpha_1-1}\int_{\mathbb{C}}\bar\partial_\zeta\tilde\gamma(\zeta)(\zeta-B(\tau))^{-1}\operatorname{ad}^{\alpha_1+1}_{A(\tau)}(\mathbb{D}A(\tau))\\
&\quad\times(\zeta-B(\tau))^{-\alpha_1-1}d\zeta\wedge d\bar\zeta/(2\pi i), && (1.2.19)\\
R_m(\tau) &= (-1)^{\alpha-m+1}\tau^{-\alpha+m-1}\int_{\mathbb{C}}\bar\partial_\zeta\tilde g_m(\zeta)(\zeta-B(\tau))^{-1}\operatorname{ad}^{\alpha}_{A(\tau)}(\mathbb{D}A(\tau))\\
&\quad\times(\zeta-B(\tau))^{-\alpha+m-1}d\zeta\wedge d\bar\zeta/(2\pi i), && (1.2.20)
\end{aligned}$$

$\gamma(\lambda)=(-\sigma^{(1)}(\lambda))^{1/2}=(-\lambda)^{(\alpha-1)/2}\kappa(\lambda)\in S^{(\alpha-1)/2}$, $g_m(\lambda)=\sigma_{(\alpha-m)_+/2,\epsilon/2}(\lambda)\in S^{(\alpha-m)_+/2}$ and $h_m(\lambda)=(-\lambda)^{-(\alpha-m)_+}\sigma^{(m)}(\lambda)\in S^0$ with $(s)_+=\max\{0,s\}$ for $s\in\mathbb{R}$, and $\tilde\gamma$ and $\tilde g_m$ are almost analytic extensions of γ and g_m respectively (see [3] as for the details). Since

$$\sigma^{(1)}(\lambda)=-\alpha(-\lambda)^{\alpha-1}\psi_{\alpha,\epsilon}(\lambda)+(-\lambda)^{\alpha}\psi^{(1)}_{\alpha,\epsilon}(\lambda)=-(-\lambda)^{\alpha-1}\kappa(\lambda)^2, \tag{1.2.21}$$

we have

$$E_1(\tau)=-\tau^{\alpha-\beta-1}(-B(\tau))^{\alpha+1}\psi^{(1)}_{\alpha,\epsilon}(B(\tau))+\beta\tau^{\alpha-\beta-1}\sigma(B(\tau))\geqslant 0 \tag{1.2.22}$$

because of $\psi^{(1)}_{\alpha,\epsilon} \leqslant 0$ and $\beta \geqslant 0$. Since $\mathbb{D}A(\tau) \geqslant 0$ as mentioned above,

$$E_2(\tau) \geqslant 0 \tag{1.2.23}$$

holds. Noting that $\mathrm{ad}^n_{A(\tau)}(\mathbb{D}A(\tau)) = O(1)$ for $0 \leqslant n \leqslant \alpha_0$, we have

$$R_1(\tau) = O(\tau^{-\alpha_1 - 1}) \tag{1.2.24}$$

because of $(\alpha - 1)/2 < \alpha_1 + 1$, which is satisfied obviously. Similarly, we have

$$R_m(\tau) = O(\tau^{-\alpha + m - 1}) \tag{1.2.25}$$

if $(\alpha - m)/2 < \alpha - m + 1$, that is, $\alpha > m - 2$ for $2 \leqslant m \leqslant \alpha_0$, which is satisfied because $\alpha = \alpha_0 - 1$ or $\alpha = \alpha_0$.

By the non-negativity of $E_1(\tau)$ and $E_2(\tau)$, we have

$$\langle -g(A(t), t)\rangle_t \leqslant \langle -g(A(1), 1)\rangle_1 - \sum_{j=3}^{7} \int_1^t \langle E_j(\tau)\rangle_\tau d\tau. \tag{1.2.26}$$

Now we would like to show

$$\langle -g(A(t), t)\rangle_t = O(1)\|\phi\|^2 \tag{1.2.27}$$

for $(\beta, \alpha) = (\beta_0, \alpha_0)$, $(0, \alpha_0 - 1) \neq (0, 0)$ with $\alpha_0 \in \mathbb{N}$ by an induction on α_0. We first consider the case where $\alpha_0 = 1$. Let $(\beta, \alpha) = (\beta_0, 1)$. Since $E_3(\tau) = E_5(\tau) = E_6(\tau) = 0$ and

$$E_4(\tau) = O(\tau^{-\beta_0 - 1}), \quad E_7(\tau) = O(\tau^{-\beta_0 - 1}), \tag{1.2.28}$$

we have (1.2.27) for $(\beta, \alpha) = (\beta_0, 1)$ because of $\beta_0 > 0$.

Suppose that (1.2.27) in the case where $\alpha_0 = n \in \mathbb{N}$ is true. We now consider the case where $\alpha_0 = n + 1$. We first let $(\beta, \alpha) = (0, \alpha_0 - 1) = (0, n)$. By the assumption of induction, for any $r \in \mathbb{R}$ such that $0 \leqslant r \leqslant n$

$$(-g_{0,n-r,\epsilon}(A(\tau), \tau))^{1/2}\zeta_{A,n}(\tau) = O(\tau^{(\beta_0 - r)/2}) \tag{1.2.29}$$

holds because of $-g_{\beta_0,n,\epsilon}(\lambda, \tau) \geqslant -\tau^{-\beta_0}(\epsilon\tau)^r g_{0,n-r,\epsilon}(\lambda, \tau)$. By this, we have

$$\sigma_{(n-r)/2,\epsilon}(B(\tau))\zeta_{A,n}(\tau) = O(\tau^{(\beta_0 - n)/2}). \tag{1.2.30}$$

It follows from (1.2.30) that

$$\|\gamma^{(m)}(B(\tau))\phi(\tau)\| = O(\tau^{(\beta_0 - \alpha)/2})\|\phi\| \tag{1.2.31}$$

for $0 \leqslant m \leqslant \alpha_1$,

$$\|g^{(m_1)}_m(B(\tau))\phi(\tau)\| = O(\tau^{(\beta_0 - \alpha)/2})\|\phi\| \tag{1.2.32}$$

for $m \geqslant 2$ and $0 \leqslant m_1 \leqslant \alpha - m$, and

$$h_m(B(\tau)) = O(1) \tag{1.2.33}$$

for $2 \leqslant m \leqslant \alpha_0$. Then we have $E_3(\tau) = 0$ if $n = 1$,

$$\langle E_3(\tau)\rangle_\tau = O(\tau^{\beta_0 - 2})\|\phi\|^2 \tag{1.2.34}$$

if $n \geqslant 2$, and

$$\langle E_4(\tau)\rangle_\tau = \begin{cases} O(\tau^{\beta_0/2-3/2})\|\phi\|^2 & n \text{ is odd} \\ O(\tau^{\beta_0/2-2})\|\phi\|^2 & n \text{ is even,} \end{cases} \tag{1.2.35}$$

$$\langle E_5(\tau)\rangle_\tau = O(\tau^{\beta_0-2})\|\phi\|^2, \quad \langle E_6(\tau)\rangle_\tau = O(\tau^{(\beta_0-\alpha)/2-1})\|\phi\|^2, \tag{1.2.36}$$

$$\langle E_7(\tau)\rangle_\tau = O(\tau^{-2})\|\phi\|^2, \tag{1.2.37}$$

which are all integrable on $[1, \infty)$ by virtue of $\beta_0 < 1$. Thus we obtain (1.2.27) for $(\beta, \alpha) = (0, n)$.

Next let $(\beta, \alpha) = (\beta_0, \alpha_0) = (\beta_0, n+1)$. It follows from (1.2.27) for $(\beta, \alpha) = (0, n)$, which is shown above, that

$$\|\gamma^{(m)}(B(\tau))\phi(\tau)\| = O(\tau^{-(\alpha_0-1)/2})\|\phi\| \tag{1.2.38}$$

for $0 \leqslant m \leqslant \alpha_1$,

$$\|g_m^{(m_1)}(B(\tau))\phi(\tau)\| = O(\tau^{-(\alpha_0-1)/2})\|\phi\| \tag{1.2.39}$$

for $m \geqslant 2$ and $0 \leqslant m_1 \leqslant \alpha - m$, in the same way as above. Then we have

$$\langle E_3(\tau)\rangle_\tau = O(\tau^{-\beta_0-1})\|\phi\|^2 \tag{1.2.40}$$

$$\langle E_4(\tau)\rangle_\tau = \begin{cases} O(\tau^{-\beta_0-3/2})\|\phi\|^2 & n+1 \text{ is odd} \\ O(\tau^{-\beta_0/-1})\|\phi\|^2 & n+1 \text{ is even,} \end{cases} \tag{1.2.41}$$

$$\langle E_5(\tau)\rangle_\tau = O(\tau^{-\beta_0-1})\|\phi\|^2, \quad \langle E_6(\tau)\rangle_\tau = O(\tau^{-\beta_0-1-(\alpha_0-1)/2})\|\phi\|^2, \tag{1.2.42}$$

$$\langle E_7(\tau)\rangle_\tau = O(\tau^{-\beta_0-1})\|\phi\|^2, \tag{1.2.43}$$

which are all integrable on $[1, \infty)$ by virtue of $\beta_0 > 0$. Thus we obtain (1.2.27) for $(\beta, \alpha) = (\beta_0, n+1)$. This complete the proof. □

Proposition 1.2.2. (Minimal Acceleration Bound [5]) *Let $f \in C_0^\infty(\mathbb{R})$, $0 < \beta_0 < 1$, $0 \leqslant s_1 \leqslant 3$, $e_1 < e_0$ and $\epsilon > 0$. Put $\zeta_z(t) = U^S(t,0)f(H_0^S)\langle z\rangle^{-1}$. Then the following estimate holds as $t \to \infty$.*

$$\|(e_1/2 - z/t^2)^{s_1/4} F_\epsilon(z/t^2 \leqslant e_1/2 - \epsilon)\zeta_z(t)\|_{\mathscr{B}(L^2)} = O(t^{(\beta_0-3)/2}). \tag{1.2.44}$$

Proof. The proof is quite similar to the one of Proposition 1.2.1 (see also [1] and [3]). We sketch the proof by using the notations in the proof of Proposition 1.2.1.

We set $M(t) = -(e_1/2 - t/z)^{1/2}$, $B(t) = -\sigma_{1,\epsilon/2}(M(t))$ and $A(t) = tB(t)$. Since

$$\partial_\tau \sigma_{1,\epsilon/2}(M(\tau)) = -\sigma^{(1)}_{1,\epsilon/2}(M(\tau))(M(\tau)/\tau - e_1/(2\tau M(\tau))), \quad (1.2.45)$$

$$\nabla_x \sigma_{1,\epsilon/2}(M(\tau)) = -\sigma^{(1)}_{1,\epsilon/2}(M(\tau))\omega/(2\tau^2 M(\tau)), \quad (1.2.46)$$

we have

$$\begin{aligned}\mathbb{D}A(\tau) = {} & \sigma^{(1)}_{1,\epsilon/2}(M(\tau))M(\tau) - \sigma_{1,\epsilon/2}(M(\tau)) - e_1\sigma^{(1)}_{1,\epsilon/2}(M(\tau))/(2M(\tau)) \\ & +(\sigma^{(1)}_{1,\epsilon/2}(M(\tau))/M(\tau))^{1/2}(A/\tau)(\sigma^{(1)}_{1,\epsilon/2}(M(\tau))/M(\tau))^{1/2}/2. \quad (1.2.47)\end{aligned}$$

Then we obtain one of the keys in the proof by a straightforward computation,

$$\mathrm{ad}^1_{A(\tau)}(\mathbb{D}A(\tau)) = -i(\sigma^{(1)}_{1,\epsilon/2}(M(\tau))/M(\tau))^2/(4\tau^2). \quad (1.2.48)$$

For $\phi \in L^2(\mathbb{R}^d)$, we put $\phi(t) = \zeta_z(t)\phi$. As in the proof of Proposition 1.2.1, we compute $\mathbb{D}g(A(\tau), \tau) = \sum_{j=1}^{7} E_j(\tau)$. Here we note that

$$E_1(\tau) \geqslant 0. \quad (1.2.49)$$

Now we watch $E_2(\tau)$. Let δ be such that $2\delta = e_0 - e_1 > 0$. Put $A'(t) = A - e_0 t$, $B'(t) = A'(t)/t = A/t - e_0$ and $C(t) = (\sigma^{(1)}_{1,\epsilon/2}(M(t))/M(t))^{1/2}\gamma(B(t))$. Note that

$$A/\tau - e_1 = B'(\tau) + e_0 - e_1 \geqslant B'(\tau)F_\delta(B'(\tau) \leqslant -\delta). \quad (1.2.50)$$

Let us set $G'_2(\tau) = (-B'(\tau))F_\delta(B'(\tau) \leqslant -\delta)$ for the sake of simplicity. Since

$$C(\tau)[G'_2(\tau), C(\tau)] = O(\tau^{-3}) \quad (1.2.51)$$

if $\alpha \leqslant 4$ and

$$\sigma^{(1)}_{1,\epsilon/2}(M(\tau))M(\tau) - \sigma_{1,\epsilon/2}(M(\tau)) = -(-M(\tau))^2\chi^{(1)}_{1,\epsilon/2}(M(\tau)) \geqslant 0 \quad (1.2.52)$$

because of $\chi^{(1)}_{1,\epsilon/2} \leqslant 0$, we have

$$E_2(\tau) \geqslant E'_2(\tau) + O(\tau^{\alpha-\beta-4}), \quad (1.2.53)$$

$$E'_2(\tau) = -\tau^{\alpha-\beta-1}(C(\tau)^2 G'_2(\tau) + G'_2(\tau)C(\tau)^2)/4. \quad (1.2.54)$$

Although $E'_2(\tau)$ is slightly different from the ones in [1] and [3], it is a device for dealing with the case where $H^S(t)\zeta_z(t)$ may be unbounded in t. Let $f_1 \in C_0^\infty(\mathbb{R})$

be such that $f_1 f = f$. Then we note that $\langle A\rangle^2 f_1(H_0^S)\langle z\rangle^{-1}$ is bounded on $L^2(\mathbb{R}^d)$. Thus it follows from Proposition 1.2.1 that

$$G_2'(t)\zeta_z(t) = O(t^{-2}). \tag{1.2.55}$$

We first consider the case where $(\beta, \alpha) = (0, 1)$. Put $\alpha_0 = 2$. Here we note that $C(\tau) = O(1)$ if $\alpha \leqslant 2$ because of $\gamma \in S^{(\alpha-1)/2}$. We also note that $E_3(\tau) = E_5(\tau) = E_7(\tau) = 0$ and that

$$E_4(\tau) = O(\tau^{-3}), \quad E_6(\tau) = O(\tau^{-3}), \tag{1.2.56}$$

$$-\langle E_2(\tau)\rangle_\tau \leqslant (O(\tau^{-2}) + O(\tau^{-3}))\|\phi\|^2 \tag{1.2.57}$$

by (1.2.48) and (1.2.55). These imply

$$\langle -g(A(t), t)\rangle_t = O(1)\|\phi\|^2 \tag{1.2.58}$$

for $(\beta, \alpha) = (0, 1)$. Then we have

$$(-B(t))^{s_0/2} F_\epsilon(B(t) \leqslant -\epsilon)\zeta_z(t) = O(t^{-1/2}) \tag{1.2.59}$$

with $0 \leqslant s_0 \leqslant 1$ temporarily. We next consider the case where $(\beta, \alpha) = (0, 2)$. Put $\alpha_0 = 3$, Here we note that $E_4(\tau) = E_6(\tau) = E_7(\tau) = 0$ if $\alpha \geqslant 2$ and that

$$\langle E_3(\tau)\rangle_\tau = O(\tau^{-3})\|\phi\|^2, \quad \langle E_6(\tau)\rangle_\tau = O(\tau^{-3})\|\phi\|^2, \tag{1.2.60}$$

$$-\langle E_2(\tau)\rangle_\tau \leqslant (O(\tau^{-3/2}) + O(\tau^{-2}))\|\phi\|^2 \tag{1.2.61}$$

by (1.2.48), (1.2.55) and (1.2.59). Thus, (1.2.58) is obtained for $(\beta, \alpha) = (0, 2)$. Then we have a temporary estimate

$$(-B(t))^{s_0/2} F_\epsilon(B(t) \leqslant -\epsilon)\zeta_z(t) = O(t^{-1}) \tag{1.2.62}$$

with $0 \leqslant s_0 \leqslant 2$. We finally consider the case where $(\beta, \alpha) = (\beta_0, \alpha_0) = (\beta_0, 3)$ with $0 < \beta_0 < 1$. Here we note that

$$\langle E_3(\tau)\rangle_\tau = O(\tau^{-3-\beta_0})\|\phi\|^2, \quad \langle E_6(\tau)\rangle_\tau = O(\tau^{-3-\beta_0})\|\phi\|^2, \tag{1.2.63}$$

$$-\langle E_2(\tau)\rangle_\tau \leqslant O(\tau^{-1-\beta_0})\|\phi\|^2 \tag{1.2.64}$$

by (1.2.48), (1.2.55) and (1.2.62). These imply (1.2.58) for $(\beta, \alpha) = (\beta_0, 3)$. □

We next consider the case where $\eta > 2$. Now we recall the Mourre estimate for the Stark Hamiltonian $H^S = H_0^S + V$ (see [20]).

Theorem 1.2.3. (Mourre Estimate [20]) *The pure point spectrum of H^S is empty. And let $0 < \nu < |E|$. Then one can take $\delta > 0$ so small uniformly in $\lambda \in \mathbb{R}$ that*

$$\chi_\delta(H^S - \lambda) i [H^S, A] \chi_\delta(H^S - \lambda) \geqslant \nu \chi_\delta(H^S - \lambda)^2 \tag{1.2.65}$$

holds, where $\chi_\delta(\lambda) = F_\delta(|\lambda| \leqslant 2\delta)$. In particular, the spectrum of H^S is absolutely continuous.

Then the following propagation estimates for $U^S(t, 0)$ can be obtained.

Proposition 1.2.4. (Minimal Acceleration Bound [5]) *Let $f \in C_0^\infty(\mathbb{R})$, $\nu_1 < |E|$, $0 \leqslant s_0 \leqslant 1$, $\max\{0, 3 - \eta\} < \beta_0 < 1$ and $\epsilon > 0$. Put $\chi_{\delta,\lambda}(H^S(t)) = \chi_\delta(H^S - \lambda)$ and $\zeta_{A,\lambda}(t) = \chi_{\delta,\lambda}(H^S(t)) U^S(t, 0) f(H_0^S) \langle A \rangle^{-1}$. Then there exists a $\delta > 0$ uniformly in $\lambda \in \mathbb{R}$ such that the following estimate holds as $t \to \infty$.*

$$\|(\nu_1 - A/t)^{s_0/2} F_\epsilon(A/t \leqslant \nu_1 - \epsilon) \zeta_{A,\lambda}(t)\|_{\mathscr{B}(L^2)} = O(t^{(\beta_0 - 1)/2}). \tag{1.2.66}$$

Proof. This proposition was proved essentially in [46] under a stronger assumption that $\eta > 7/2 > 3$. We sketch the proof by using the notations in the proof of Proposition 1.2.1.

Set $A(t) = A - \nu_1 t$. Then it follows from Theorem 1.1.5 that there exists a $\delta > 0$ uniformly in $\lambda \in \mathbb{R}$ such that

$$\chi_{2\delta,\lambda}(H^S(t))(\mathbb{D}A(t))\chi_{2\delta,\lambda}(H^S(t)) \geqslant 0 \tag{1.2.67}$$

holds, which is one of the keys in the proof. In fact, it can be obtained by using

$$\chi_{2\delta,\lambda}(H^S(t)) = e^{ic(t)\cdot p} \chi_{2\delta,\lambda}(H^S(t) - \lambda + E \cdot c(t)) e^{-ic(t)\cdot p}, \tag{1.2.68}$$

$$e^{-ic(t)\cdot p}(\mathbb{D}A(t)) e^{ic(t)\cdot p} = i[H^S, A] - \nu_1. \tag{1.2.69}$$

Let $\max\{0, 3 - \eta\} < \beta_0 < 1$, and set $B(t) = A(t)/t$. Let $(\beta, \alpha) = (\beta_0, \alpha_0) = (\beta_0, 1)$. For $\phi \in L^2(\mathbb{R}^d)$, we put $\phi(t) = U^S(t, 0) f(H_0^S) \langle A \rangle^{-1} \phi$. Put $\hat{g}(A(t), t) = \chi_{\delta,\lambda}(H^S(t)) g(A(t), t) \chi_{\delta,\lambda}(H^S(t))$. By a computation similar to the one in the proof of Proposition 1.2.1, we obtain

$$\mathbb{D}\hat{g}(A(\tau), \tau) = \sum_{j=1}^{8} \hat{E}(\tau), \tag{1.2.70}$$

where $\hat{E}_j(\tau) = \chi_{\delta,\lambda}(H^S(\tau)) E_j(\tau) \chi_{\delta,\lambda}(H^S(\tau))$ for $1 \leqslant j \leqslant 7$, and

$$\hat{E}_8(\tau) = G_8(\tau) g(A(\tau), \tau) \chi_{\delta,\lambda}(H^S(\tau)) + \chi_{\delta,\lambda}(H^S(\tau)) g(A(\tau), \tau) G_8(\tau), \tag{1.2.71}$$

$$G_8(\tau) = \int_{\mathbb{C}} \bar{\partial}_\zeta \tilde{\chi}_\delta(\zeta - \lambda)(\zeta - H^S(\tau))^{-1} b(\tau) \cdot (\nabla_x V)(x + c(\tau)) \times (\zeta - H^S(\tau))^{-1} d\zeta \wedge d\bar{\zeta}/(2\pi i). \tag{1.2.72}$$

Note that

$$\hat{E}_1(\tau) \geqslant 0. \tag{1.2.73}$$

Since $\alpha_0 = 1$, we see that $\hat{E}_3(\tau) = \hat{E}_5(\tau) = \hat{E}_6(\tau) = 0$ and that

$$\hat{E}_4(\tau) = O(\tau^{-1-\beta_0}), \quad \hat{E}_7(\tau) = O(\tau^{-1-\beta_0}), \tag{1.2.74}$$

as in the proof of Proposition 1.2.1.

Since $\hat{E}_2(\tau) = \chi_{\delta,\lambda}(H^S(\tau))\chi_{2\delta,\lambda}(H^S(\tau))E_2(\tau)\chi_{2\delta,\lambda}(H^S(\tau))\chi_{\delta,\lambda}(H^S(\tau))$, let us replace $E_2(\tau)$ by

$$\chi_{2\delta,\lambda}(H^S(\tau))E_2(\tau)\chi_{2\delta,\lambda}(H^S(\tau)) = \sum_{j=0}^{2} E_{2,j}(\tau), \tag{1.2.75}$$

where

$$E_{2,0}(\tau) = \tau^{\alpha-\beta-1}\gamma(B(\tau))\chi_{2\delta,\lambda}(H^S(\tau))(\mathbb{D}A(\tau))\chi_{2\delta,\lambda}(H^S(\tau))\gamma(B(\tau)), \tag{1.2.76}$$
$$E_{2,1}(\tau) = \tau^{\alpha-\beta-1}\chi_{2\delta,\lambda}(H^S(\tau))\gamma(B(\tau))(\mathbb{D}A(\tau))R_{1,1}(\tau), \tag{1.2.77}$$
$$E_{2,2}(\tau) = \tau^{\alpha-\beta-1}R_{1,2}(\tau)(\mathbb{D}A(\tau))\chi_{2\delta,\lambda}(H^S(\tau))\gamma(B(\tau)) \tag{1.2.78}$$

with

$$\begin{aligned} R_{1,k}(\tau) = (-1)^k\tau^{-1}\int_{\mathbb{C}} \bar{\partial}_\zeta\tilde{\gamma}(\zeta)(\zeta - B(\tau))^{-1}\,\mathrm{ad}^1_{A(\tau)}(\chi_{2\delta,\lambda}(H^S(\tau))) \\ \times(\zeta - B(\tau))^{-1}d\zeta\wedge d\bar{\zeta}/(2\pi i) \end{aligned} \tag{1.2.79}$$

for $k = 1,2$. Noting that $\mathrm{ad}^1_{A(\tau)}(\chi_{2\delta,\lambda}(H^S(\tau))) = O(1)$ and $E_{2,0}(\tau) \geqslant 0$ by virtue of Theorem 1.2.3, we have

$$\hat{E}_2(\tau) \geqslant O(\tau^{-1-\beta_0}). \tag{1.2.80}$$

By using

$$\begin{aligned} &AU^S(t,0)f(H_0^S)\langle A\rangle^{-1} - U^S(t,0)Af(H_0^S)\langle A\rangle^{-1} \\ &= \int_0^t U^S(t,\tau)i[H^S(\tau),A]U^S(\tau,0)f(H_0^S)\langle A\rangle^{-1}d\tau, \end{aligned} \tag{1.2.81}$$

$$i[H^S(\tau),A] = |E| - \omega\cdot(\nabla_x V)(x + c(\tau)) = O(1), \tag{1.2.82}$$

we obtain as a priori estimate

$$AU^S(t,0)f(H_0^S)\langle A\rangle^{-1} = O(t), \tag{1.2.83}$$

which leads to

$$(A(\tau) + i)U^S(\tau,0)f(H_0^S)\langle A\rangle^{-1} = O(\tau). \tag{1.2.84}$$

Note that

$$(A(\tau)+i)\chi_{\delta,\lambda}(H^S(\tau))(A(\tau)+i)^{-1} = O(1) \tag{1.2.85}$$

because $i[H^S(\tau), A(\tau)] = i[H^S(\tau), A] = O(1)$. Since

$$g(A(\tau),\tau)(A(\tau)+i)^{-1} = O(\tau^{-\beta_0}), \tag{1.2.86}$$

we have

$$g(A(\tau),\tau)\zeta_{A,z}(\tau) = O(\tau^{1-\beta_0}). \tag{1.2.87}$$

Also note that

$$G_8(\tau) = O(\tau^{1-\eta}) \tag{1.2.88}$$

because of $b(\tau) = O(\tau^{1-\eta})$. Then we obtain

$$\langle \hat{E}_8(\tau)\rangle_\tau = O(\tau^{2-\eta-\beta_0})\|\phi\|^2, \tag{1.2.89}$$

which is integrable on $[1,\infty)$ by $\beta_0 > 3-\eta$. This implies the proposition. □

Proposition 1.2.5. (Minimal Acceleration Bound [5]) *Let $f \in C_0^\infty(\mathbb{R})$, $\nu_1 < |E|$, $\max\{3-\eta\} < \beta_0 < \alpha_2 < 1$, $0 \leqslant s_1 \leqslant \alpha_2$ and $\epsilon > 0$. Put $\zeta_{z,\lambda}(t) = \chi_{\delta,\lambda}(H^S(t))U^S(t,0)f(H_0^S)\langle z\rangle^{-1}$. Then there exists a $\delta > 0$ uniformly in $\lambda \in \mathbb{R}$ such that the following estimate holds as $t \to \infty$.*

$$\|(\nu_1/2 - z/t^2)^{s_1/4}F_\epsilon(z/t^2 \leqslant \nu_1/2-\epsilon)\zeta_{z,\lambda}(t)\|_{\mathscr{B}(L^2)} = O(t^{(\beta_0-\alpha_2)/2}). \tag{1.2.90}$$

Proof. The proof is quite similar to the one of Proposition 1.2.4 (see also the proof of Proposition 1.2.2). Hence we sketch it.

set $M(t) = -(\nu_1/2 - z/t^2)^{1/2}$, $B(t) = -\sigma_{1,\epsilon/2}(M(t))$ and $A(t) = tB(t)$. In the same way as in the proof of Proposition 1.2.2, we have

$$\begin{aligned}\mathbb{D}A(\tau) =\ & \sigma_{1,\epsilon/2}^{(1)}(M(\tau))M(\tau) - \sigma_{1,\epsilon/2}(M(\tau)) - \nu_1\sigma_{1,\epsilon/2}^{(1)}(M(\tau))/(2M(\tau)) \\ & + (\sigma_{1,\epsilon/2}^{(1)}(M(\tau))/M(\tau))^{1/2}(A/\tau)(\sigma_{1,\epsilon/2}^{(1)}(M(\tau))/M(\tau))^{1/2}/2. \end{aligned} \tag{1.2.91}$$

and the key fact

$$\mathrm{ad}^1_{A(\tau)}(\mathbb{D}A(\tau)) = -i(\sigma_{1,\epsilon/2}^{(1)}(M(\tau))/M(\tau))^2/(4\tau^2). \tag{1.2.92}$$

For $\phi \in L^2(\mathbb{R}^d)$, we put $\phi(t) = U^S(t,0)f(H_0^S)\langle z\rangle^{-1}\phi$. As in the proof of Proposition 1.2.4, we obtain

$$\mathbb{D}\hat{g}(A(\tau),\tau) = \sum_{j=1}^{8}\hat{E}(\tau). \tag{1.2.93}$$

We consider the case where $(\beta,\alpha) = (\beta_0,\alpha_2)$ with $\max\{0, 3-\eta\} < \beta_0 < \alpha_2 < 1$. Put $\alpha_0 = 1$ and $\alpha_1 = 0$. Here we note that

$$\hat{E}_1(\tau) \geqslant 0, \tag{1.2.94}$$

$\hat{E}_3(\tau) = \hat{E}_5(\tau) = \hat{E}_6(\tau) = 0$ and that

$$\hat{E}_4(\tau) = O(\tau^{\alpha_2-\beta_0-4}), \quad \hat{E}_7(\tau) = O(\tau^{\alpha_2-\beta_0-4}), \tag{1.2.95}$$

by (1.2.92). Now we consider $\hat{E}_2(\tau)$. Let μ be such that $3\mu = |E| - \nu_1 > 0$. Put $\nu_0 = \nu_1 + 2\mu < |E|$, $B'(t) = A/t - \nu_0$ and $C(t) = (\sigma^{(1)}_{1,\epsilon/2}(M(t))/M(t))^{1/2}\gamma(B(t))$. Since $\alpha \leqslant 2$, we have $C(t) = O(1)$. Note that

$$A/\tau - \nu_1 = B'(\tau) + \nu_0 - \nu_1 \geqslant B'(\tau)F_\mu(B'(\tau) \leqslant -\mu)^2. \tag{1.2.96}$$

Let us set $G''_2(\tau) = (-B'(\tau))^{1/2}F_\mu(B'(\tau) \leqslant -\mu)$ for the sake of simplicity. Since

$$G''_2(\tau)[G''_2(\tau), C(\tau)] = O(\tau^{-3}), \tag{1.2.97}$$

we have

$$\hat{E}_2(\tau) \geqslant \hat{E}''_2(\tau) + O(\tau^{\alpha_2-\beta_0-4}), \tag{1.2.98}$$

$$E''_2(\tau) = -\tau^{\alpha_2-\beta_0-1}\chi_{\delta,\lambda}(H^S(\tau))G''_2(\tau)C(\tau)^2G''_2(\tau)\chi_{\delta,\lambda}(H^S(\tau))/2. \tag{1.2.99}$$

Since $\langle A\rangle f_1(H^S_0)\langle z\rangle^{-1}$ is bounded on $L^2(\mathbb{R}^d)$ for $f_1 \in C^\infty_0(\mathbb{R})$ such that $f_1 f = f$,

$$G''_2(\tau)\zeta_{z,\lambda}(\tau) = O(\tau^{(\beta_0-1)/2}) \tag{1.2.100}$$

is obtained by Proposition 1.2.4. Thus, we have

$$\langle \hat{E}''_2(\tau)\rangle_\tau = O(\tau^{\alpha_2-2})\|\phi\|^2. \tag{1.2.101}$$

Here we note that $\alpha_2 - \beta_0 - 4 < \alpha_2 - 2 < -1$. We finally consider $\hat{E}''_8(\tau)$. By using

$$\begin{aligned} &zU^S(t,0)f(H^S_0)\langle z\rangle^{-1} - U^S(t,0)zf(H^S_0)\langle z\rangle^{-1} \\ &= \int_0^t U^S(t,\tau)i[H^S(\tau), z]U^S(\tau,0)f(H^S_0)\langle z\rangle^{-1}d\tau \\ &= \int_0^t U^S(t,\tau)AU^S(\tau,0)f(H^S_0)\langle z\rangle^{-1}d\tau \end{aligned} \tag{1.2.102}$$

and (1.2.83), we have an a priori estimate

$$\langle z - \nu_1 t^2/2\rangle U^S(t,0)f(H^S_0)\langle z\rangle^{-1} = O(t^2). \tag{1.2.103}$$

Since $U^S(t,0)f(H^S_0)\langle z\rangle^{-1} = O(1)$, by a complex interpolation, we obtain

$$\langle z - \nu_1 t^2/2\rangle^\kappa U^S(t,0)f(H^S_0)\langle z\rangle^{-1} = O(t^{2\kappa}) \tag{1.2.104}$$

for $0 \leqslant \kappa \leqslant 1$. Now we will claim

$$\langle z-\nu_1 t^2/2\rangle^{\alpha_2/2}\chi_{\delta,\lambda}(H^S(t))\langle z-\nu_1 t^2/2\rangle^{-\kappa}=O(1) \tag{1.2.105}$$

for $\kappa>1/2$. Note that $z-\nu_1 t^2/2=e^{-i\nu_1 t^2 A/2} z e^{i\nu_1 t^2 A/2}$. Then one can write

$$\chi_{\delta,\lambda+|E|\nu_1 t^2/2}(\hat{H}^S(t))=e^{-i\nu_1 t^2 A/2}\chi_{\delta,\lambda}(H^S(t))e^{i\nu_1 t^2 A/2}, \tag{1.2.106}$$

$$\hat{H}^S(t)=H_0^S+V(x+c(t)+\nu_1\omega t^2/2). \tag{1.2.107}$$

Then we have only to obtain a constant C such that

$$\|\langle z\rangle^{\alpha_2/2}\chi_{\delta,\lambda}(\hat{H}^S(t))\langle z\rangle^{-\kappa}\|_{\mathscr{B}(L^2)}\leqslant C \tag{1.2.108}$$

holds for any t, $\lambda\in\mathbb{R}$. It follows from Proposition 1.2.7 below that

$$\|\langle z\rangle^{-\gamma_1}\langle A\rangle R_0(\zeta)\langle z\rangle^{-\gamma_1}\|_{\mathscr{B}(L^2)}\leqslant C\max\{1,|\operatorname{Im}\zeta|^{-\sigma_0}\} \tag{1.2.109}$$

holds with $\gamma_1>1/2$ and $\sigma_0>2$, where $R_0(\zeta)=(H_0^S-\zeta)^{-1}$ for $\zeta\in\mathbb{C}\setminus\mathbb{R}$. By the almost analytic extension method for $\langle\cdot\rangle^{1/2}\in S^{1/2}$,

$$\operatorname{ad}_A^1(R_0(\zeta))=-R_0(\zeta)\operatorname{ad}_A^1(H_0^S)R_0(\zeta), \tag{1.2.110}$$

$$\operatorname{ad}_A^2(R_0(\zeta))=2R_0(\zeta)\operatorname{ad}_A^1(H_0^S)R_0(\zeta)\operatorname{ad}_A^1(H_0^S)R_0(\zeta)-R_0(\zeta)\operatorname{ad}_A^2(H_0^S)R_0(\zeta) \tag{1.2.111}$$

and $i\operatorname{ad}_A^1(H_0^S)=|E|$, we have

$$\|\langle A\rangle^{1/2}[\langle A\rangle^{1/2},R_0(\zeta)]\|_{\mathscr{B}(L^2)}\leqslant C\max\{|\operatorname{Im}\zeta|^{-2},|\operatorname{Im}\zeta|^{-3}\}. \tag{1.2.112}$$

Combining these, we obtain

$$\|\langle z\rangle^{-\gamma_1}\langle A\rangle^{1/2}R_0(\zeta)\langle A\rangle^{1/2}\langle z\rangle^{-\gamma_1}\|_{\mathscr{B}(L^2)}\leqslant C\max\{1,|\operatorname{Im}\zeta|^{-3}\} \tag{1.2.113}$$

because one can take σ_0 as $2<\sigma_0\leqslant 3$. By $R_0(\zeta)-R_0(\bar{\zeta})=2i\operatorname{Im}\zeta R_0(\zeta)R_0(\bar{\zeta})$, we obtain

$$\|\langle z\rangle^{-\gamma_1}\langle A\rangle^{1/2}R_0(\zeta)\|_{\mathscr{B}(L^2)}\leqslant C\max\{|\operatorname{Im}\zeta|^{-1/2},|\operatorname{Im}\zeta|^{-2}\}. \tag{1.2.114}$$

Note

$$\|(H_0^S-\zeta)(\hat{H}^S(t)-\zeta)^{-1}\|_{\mathscr{B}(L^2)}\leqslant C\max\{1,|\operatorname{Im}\zeta|^{-1}\} \tag{1.2.115}$$

because of the boundedness of $V(x+c(t)+\nu_1\omega t^2/2)$, and

$$\|[\langle A\rangle^{1/2},(\hat{H}^S(t)-\zeta)^{-1}]\|_{\mathscr{B}(L^2)}\leqslant C\max\{|\operatorname{Im}\zeta|^{-2},|\operatorname{Im}\zeta|^{-3}\}, \tag{1.2.116}$$

which can be obtained by

$$i\,\mathrm{ad}_A^1(\hat{H}^S(t)) = |E| - \omega\cdot(\nabla_x V)(x + c(t) + \nu_1\omega t^2/2) = O(1) \tag{1.2.117}$$

in the same way as above. Then we have

$$\|\langle z\rangle^{-\gamma_1}\langle A\rangle^{1/2}(\hat{H}^S(t)-\zeta)^{-1}\|_{\mathscr{B}(L^2)} \leqslant C\max\{|\operatorname{Im}\zeta|^{-1/2}, |\operatorname{Im}\zeta|^{-3}\}, \tag{1.2.118}$$
$$\|\langle z\rangle^{-\gamma_1}(\hat{H}^S(t)-\zeta)^{-1}\langle A\rangle^{1/2}\|_{\mathscr{B}(L^2)} \leqslant C\max\{|\operatorname{Im}\zeta|^{-1/2}, |\operatorname{Im}\zeta|^{-3}\}. \tag{1.2.119}$$

Noting that an almost analytic extension $\tilde{\chi}_\delta \in C^\infty(\mathbb{C})$ of $\chi_\delta \in C^\infty(\mathbb{R})$ satisfies $|\bar{\partial}_\zeta\tilde{\chi}_\delta(\zeta)| \leqslant C_M|\operatorname{Im}|^M$ for $M \geqslant 0$ and that

$$i[\hat{H}^S(t), \langle z\rangle^{\alpha_2/2}] = (\langle z\rangle^{\alpha_2/2})'A - i(\langle z\rangle^{\alpha_2/2})''/2, \tag{1.2.120}$$
$$|(\langle z\rangle^{\alpha_2/2})'| \leqslant (\alpha_2/2)\langle z\rangle^{\alpha_2/2-1}, \quad |(\langle z\rangle^{\alpha_2/2})''| \leqslant (\alpha_2/2)\langle z\rangle^{\alpha_2/2-2}, \tag{1.2.121}$$

we obtain (1.2.108) by virtue of the almost analytic extension method because $\alpha_2/2-1 < -1/2$ and $-\kappa < -1/2$. Noe we put $3\mu' = \beta_0 - \max\{0, 3-\eta\} > 0$ and $\kappa = 1/2 + \mu' > 1/2$. By (1.2.105) and

$$g(A(\tau),\tau)\langle z - \nu_1 t^2/2\rangle^{-\alpha_2/2} = O(\tau^{-\beta_0}), \quad G_8(\tau) = O(\tau^{1-\eta}), \tag{1.2.122}$$

we finally obtain

$$\langle \hat{E}_8(\tau)\rangle_\tau = O(\tau^{1-\eta+2\kappa-\beta_0})\|\phi\|^2 = O(\tau^{-\max\{\eta-2,1\}-\mu'})\|\phi\|^2 \tag{1.2.123}$$

which is integrable on $[1,\infty)$ by $-\max\{\eta-2,1\}-\mu' < -1$. □

Remark 1.2.6. *In the proofs of Propositions 1.2.1, 1.2.2, 1.2.4 and 1.2.5, we have used the formulas of asymptotic expansions of commutators in connection with the almost analytic extension method due to Helffer-Sjöstrand. For avoiding the problem on the domains of the iterated commutators there, it is necessary to apply the formulas to not objective unbounded operators but the ones truncated appropriately once. After that, one has only to remove the truncation by taking some limiting procedure (see e.g. [35]).*

Now let us prove the following proposition, which was used in the above proof.

Proposition 1.2.7. *Suppose γ_1, $\gamma_2 > 1/2$, $\sigma_0 > 2$ and $\mu \in \mathbb{R}\setminus\{0\}$. Then there exists a constant $C > 0$ such that for any $\phi \in C_0^\infty(\mathbb{R}^d)$ and $\lambda \in \mathbb{R}$ the following holds.*

$$\|\langle z\rangle^{-\gamma_1}A\phi\| \leqslant C\max\{1, |\mu|^{-\sigma_0}\}\|\langle z\rangle^{\gamma_2}(H_0^S - \lambda - i\mu)\phi\|. \tag{1.2.124}$$

Proof. In the case where $|\mu| \geqslant \mu_0$ with some $\mu_0 > 0$, this was already given in Lemmas 2.3 and 2.4 of Herbst [19]. Hence we sketch the proof only for sufficiently small $|\mu| > 0$ by following the argument of [19]. We first consider the case where $d = 1$ and H_0^S is written as $A^2/2 - |E|z$. One can suppose that $1/2 < \gamma_1 < 1$ without loss of generality. In the same way as in [19], we have

$$\|\langle z\rangle^{-\gamma_1} A\phi\| \leqslant \|\langle z\rangle^{-\gamma_1}\|(2\|\langle z\rangle^{-\gamma_2}\|\|\langle z\rangle^{\gamma_2}(H_0^S - \lambda - i\mu)\phi\| + 2^{1/2}\|\langle z\rangle^{-\epsilon} g'\|\|\langle z\rangle^{\epsilon}\phi\|) \tag{1.2.125}$$

for $\epsilon > 0$, where $g(z) = (|E|z + \lambda + i\mu)^{1/2}$ whose branch is chosen so that $\operatorname{Im} g(z)$ has constant sign. By the Hölder inequality and $(1 + |z|)/2^{1/2} \leqslant \langle z\rangle$, $\|\langle z\rangle^{-\epsilon} g'\|$ can be estimated as

$$\begin{aligned}
\|\langle z\rangle^{-\epsilon} g'\| &\leqslant (|E|/2)^2 \left(\int_{\mathbb{R}} ((|E|z + \lambda)^2 + \mu^2)^{-(1+\epsilon)/2} dz\right)^{1/(1+\epsilon)} \\
&\qquad \times \left(\int_{\mathbb{R}} \langle z\rangle^{-2(1+\epsilon)} dz\right)^{\epsilon/(1+\epsilon)} \\
&\leqslant 2^{-2+\epsilon/(1+\epsilon)+\epsilon}|E|^{2-1/(1+\epsilon)}|\mu|^{-1+1/(1+\epsilon)}(1+2\epsilon)^{-\epsilon/(1+\epsilon)} \left(\int_{\mathbb{R}} \langle z\rangle^{-(1+\epsilon)}\right)^{1/(1+\epsilon)} \\
&\leqslant 2^{-1/2+\epsilon}|E|^{2-1/(1+\epsilon)}|\mu|^{-1+1/(1+\epsilon)}\epsilon^{-1/(1+\epsilon)}.
\end{aligned} \tag{1.2.126}$$

In the same way as in [19], we see that for $0 < \epsilon < \min\{1, 2|\mu|\}$

$$\|\langle z\rangle^{\epsilon}\phi\| \leqslant (\|\langle z\rangle^{\epsilon}(H_0^S - \lambda - i\mu)\phi\| + \epsilon\|\langle z\rangle^{\epsilon-1}A\phi\|)/(|\mu| - \epsilon/2) \tag{1.2.127}$$

holds, since $|(\langle z\rangle^{\epsilon})'| \leqslant \epsilon\langle z\rangle^{\epsilon-1}$ and $|(\langle z\rangle^{\epsilon})''| \leqslant \epsilon\langle z\rangle^{\epsilon-2}$. Combining these, for $0 < \epsilon < \min\{1-\gamma_1, 2|\mu|\}$

$$\|\langle z\rangle^{-\gamma_1}A\phi\| \leqslant (c_0 + c_{\mu,\epsilon})\|\langle z\rangle^{\gamma_2}(H_0^S - \lambda - i\mu)\phi\| + \epsilon c_{\mu,\epsilon}\|\langle z\rangle^{-\gamma_1}A\phi\| \tag{1.2.128}$$

with

$$c_0 = 2\|\langle z\rangle^{-\gamma_1}\|\|\langle z\rangle^{-\gamma_2}\|, \quad c_{\mu,\epsilon} = c_1|\mu|^{-\epsilon/(2(1+\epsilon))}\epsilon^{-1/(2(1+\epsilon))}/(|\mu| - \epsilon/2), \tag{1.2.129}$$

$$c_1 = 2^{3/4-\gamma_1/2}\max\{1, |E|\}^{1-1/(2(2-\gamma_1))}\|\langle z\rangle^{-\gamma_1}\|, \tag{1.2.130}$$

since $\epsilon < 1 - \gamma_1 < 1/2 < \gamma_2$. Now we put $\epsilon = |\mu|^{\sigma_1}$ with $\sigma_1 > 2$. For $0 < |\mu| < (1-\gamma_1)^{1/\sigma_1}$,

$$\begin{aligned}
|\mu|^{\sigma_1} c_{\mu,|\mu|^{\sigma_1}} &\leqslant c_1|\mu|^{\sigma_1(1-1/(2(1+|\mu|^{\sigma_1}))) - |\mu|^{\sigma_1}/(2(1+|\mu|^{\sigma_1}))-1}/(1 - (1-\gamma_1)^{1-1/\sigma_1}/2) \\
&\leqslant c_1|\mu|^{(\sigma_1-2)/2}/(1 - (1-\gamma_1)^{1-1/\sigma_1}/2)
\end{aligned} \tag{1.2.131}$$

holds because

$$\begin{aligned}
&\sigma_1(1 - 1/(2(1+|\mu|^{\sigma_1}))) - |\mu|^{\sigma_1}/(2(1+|\mu|^{\sigma_1})) - 1 \\
&\quad = (\sigma_1-2)/2 + |\mu|^{\sigma_1}(\sigma_1-1)/(2(1+|\mu|^{\sigma_1})) > (\sigma_1-2)/2.
\end{aligned} \tag{1.2.132}$$

Thus, for

$$0 < |\mu| < \min\{(1-\gamma_1)^{1/\sigma_1}, ((1-(1-\gamma_1)^{1-1/\sigma_1}/2)/(2c_1))^{2/(\sigma_1-2)}\}, \qquad (1.2.133)$$

we see that

$$\|\langle z\rangle^{-\gamma_1} A\phi\| \leqslant 2(c_0 + c_2|\mu|^{-\sigma_0})\|\langle z\rangle^{\gamma_2}(H_0^S - \lambda - i\mu)\phi\| \qquad (1.2.134)$$

holds with $c_2 = c_1/(1-(1-\gamma_1)^{1-1/\sigma_1}/2)$ and $\sigma_0 = (\sigma_1+2)/2 > 2$. This implies the proposition in the case where $d = 1$. If $d \geqslant 2$, we introduce $p_\perp = p - A\omega$ Then H_0^S is written as

$$H_0^S = A^2/2 - |E|z + p_\perp^2/2. \qquad (1.2.135)$$

Thus, by using a direct integral method and above result, we obtain the proposition. □

1.3 Proof of Theorem 1.1.7

We first show the existence of $W_{\mathrm{l}}^{S,\pm}$. The following proposition is useful.

Proposition 1.3.1. *Let $\phi \in C_0^\infty(\mathbb{R}^d)$. Then*

$$\|(A - |E|t)^k U_{\mathrm{l}}^S(t,0)\phi\| = O(1), \qquad (1.3.1)$$
$$\|(z - |E|t^2/2)^k U_{\mathrm{l}}^S(t,0)\phi\| = O(t^k) \qquad (1.3.2)$$

with $k = 1, 2$ hold as $t \to \infty$.

Proof. Set $A_0(t) = A - |E|t$ and $z_0(t) = z - |E|t^2/2$. Since

$$\mathbb{D}_{H_{\mathrm{l}}^S(\tau)} A_0(\tau) = -\omega\cdot(\nabla_x V_{\mathrm{l}})(\tau, x + \tilde{c}(\tau)) = O(\tau^{-2\rho_{\mathrm{l}}-1}), \qquad (1.3.3)$$
$$i[A_0(\tau), \mathbb{D}_{H_{\mathrm{l}}^S(\tau)} A_0(\tau)] = -((\omega\cdot\nabla_x)^2 V_{\mathrm{l}})(\tau, x + \tilde{c}(\tau)) = O(\tau^{-2\rho_{\mathrm{l}}-2}) \qquad (1.3.4)$$

by (1.1.15), we obtain (1.3.1) with $k = 1, 2$ because of $-2\rho_{\mathrm{l}} - 1 < -1$. (1.3.2) with $k = 1$ can be obtained easily by (1.3.1) with $k = 1$, since

$$\mathbb{D}_{H_{\mathrm{l}}^S(\tau)} z_0(\tau) = A_0(\tau). \qquad (1.3.5)$$

Similarly, we see that

$$\|A_0(\tau) z_0(\tau) U_{\mathrm{l}}^S(\tau,0)\phi\| = O(\tau), \quad \|z_0(\tau) A_0(\tau) U_{\mathrm{l}}^S(\tau,0)\phi\| = O(\tau) \qquad (1.3.6)$$

by (1.3.1), (1.3.2) with $k = 1$, (1.3.3) and (1.3.5). These imply (1.3.2) with $k = 2$. □

Proof of Theorem 1.1.7. By virtue of (1.3.2) with $k = 2$, we see that for $\phi \in C_0^\infty(\mathbb{R}^d)$

$$\begin{aligned} Ct^2 &\geqslant \|(z - |E|t^2/2)^2 U_1^S(t,0)\phi\| \\ &\geqslant \|F_\epsilon(z/t^2 \leqslant |E|/2 - \epsilon)(z - |E|t^2/2)^2 U_1^S(t,0)\phi\| \\ &\geqslant \epsilon^2 t^4 \|F_\epsilon(z/t^2 \leqslant |E|/2 - \epsilon) U_1^S(t,0)\phi\| \end{aligned} \tag{1.3.7}$$

holds with $\epsilon > 0$, since $\epsilon \leqslant |E|/2 - z/t^2 = -(z - |E|t^2/2)/t^2$ holds on the support of $F_\epsilon(z/t^2 \leqslant |E|/2 - \epsilon)$. This leads to

$$\|F_\epsilon(z/t^2 \leqslant |E|/2 - \epsilon) U_1^S(t,0)\phi\| \leqslant C/(\epsilon t)^2. \tag{1.3.8}$$

By virtue of (1.3.8), $c(t) = o(t^2)$ and the boundedness of V, we have

$$\begin{aligned} &\|\partial_t (U^S(t,0)^* U_1^S(t,0))\phi\| \\ &= \|(V(x + c(t)) - V_1(t, x + c(t))) U_1^S(t,0)\phi\| = O(t^{-\min\{2, 2\rho_\mathrm{s}\}}). \end{aligned} \tag{1.3.9}$$

In fact, recall (1.1.13) and note that $\epsilon_0 \leqslant |E|/14$. We have

$$V^\mathrm{s}(x + c(t))(1 - F_{\epsilon_0}(z/t^2 \leqslant |E|/2 - \epsilon_0)) = O(t^{-2\rho_\mathrm{s}}) \tag{1.3.10}$$

because of $|E|/2 - 2\epsilon_0 \geqslant 5\epsilon_0 > 0$. On the other hand, we can wirte

$$V^\mathrm{l}(x+c(t)) - V_1(t, x+c(t)) = V^\mathrm{l}(x+c(t))(1 - F_{\epsilon_0}((z+\omega\cdot c(t))/\langle t\rangle^2 \geqslant \epsilon_0)). \tag{1.3.11}$$

On the support of $1 - F_{\epsilon_0}((z + \omega \cdot c(t))/\langle t\rangle^2 \geqslant \epsilon_0)$, $(z + \omega \cdot c(t))/\langle t\rangle^2 \leqslant 2\epsilon_0$ holds. If $t > 0$ is so large that $|\omega \cdot c(t)|/\langle t\rangle^2 \leqslant \epsilon_0$ and $\langle t\rangle^2/t^2 \leqslant 4/3$, then we obtain

$$z/t^2 = z/\langle t\rangle^2 \times \langle t\rangle^2/t^2 \leqslant (2\epsilon_0 + |\omega \cdot c(t)|/\langle t\rangle^2) \times 4/3 \leqslant 4\epsilon_0. \tag{1.3.12}$$

Thus,

$$(1 - F_{\epsilon_0}((z + \omega \cdot c(t))/\langle t\rangle^2 \geqslant \epsilon_0))(1 - F_{\epsilon_0}(z/t^2 \leqslant |E|/2 - \epsilon_0)) = 0 \tag{1.3.13}$$

holds for sufficiently large $t > 0$, because $|E|/2 - 2\epsilon_0 \geqslant 5\epsilon_0 > 4\epsilon_0$. Since $\rho_\mathrm{s} > 1/2$, the existence of $W_1^{S,+}\phi$ can be shown by the Cook-Kuroda method, which implies the existence of $W_1^{S,+}$ by a density argument.

We next show the unitary of $W_1^{S,+}$. Let $\phi \in \mathscr{D}(\langle z\rangle)$ be such that $\phi = f(H_0^S)\phi$ for some $f \in C_0^\infty(\mathbb{R})$. We first consider the case where $0 < \eta \leqslant 2$ with the additional condition (1.1.9). Then, by virtue of Proposition 1.2.2,

$$\|F_{\epsilon_0}(z/t^2 \leqslant e_1/2 - \epsilon_0) U^S(t,0)\phi\| = O(t^{(\beta_0 - 3)/2}) \tag{1.3.14}$$

holds with $e_1 = e_0 - 2\epsilon_0 = 12\epsilon_0 < e_0$ and $0 < \beta_0 < 1$ (cf. (1.3.8)). Thus, in the same way as above, the existence of

$$\lim_{t\to\infty} U_1^S(t,0)^* U^S(t,0)\phi \tag{1.3.15}$$

can be shown, which implies the unitary of $W_1^{S,+}$ by a density argument. We finally consider the case where $\eta > 2$. Set $h_R(\lambda) = F_1(-2 \leqslant \lambda/R \leqslant 2) \in C_0^\infty(\mathbb{R})$ for $R > 0$. Note that $h_1(0) = 1$, one has

$$\begin{aligned} 1 - h_R(H^S(t)) &= \int_{\mathbb{C}} \bar{\partial}_\zeta \tilde{h}_1(\zeta)(\zeta^{-1} - (\zeta - H^S(t)/R)^{-1}) d\zeta \wedge d\bar{\zeta}/(2\pi i) \\ &= -\int_{\mathbb{C}} \bar{\partial}_\zeta \tilde{h}_1(\zeta)\zeta^{-1} H^S(t)(\zeta - H^S(t)/R)^{-1} d\zeta \wedge d\bar{\zeta}/(2\pi i R), \end{aligned} \tag{1.3.16}$$

where $\tilde{h}_1 \in C_0^\infty(\mathbb{C})$ is an almost analytic extension of h_1. Then we see that there exists a constant $C > 0$ such that

$$\|(1 - h_R(H^S(t)))U^S(t,0)\phi\| \leqslant C/R \tag{1.3.17}$$

holds for any $t \geqslant 0$, since $H^S(t)U^S(t,0)f(H_0^S)$ is bounded in t (see (1.1.27)). Hence, instead of showing the existence of $\lim_{t\to\infty} U_1^S(t,0)^* U^S(t,0)\phi$, one has only to prove the existence of

$$\lim_{t\to\infty} U_1^S(t,0)^* h_{R_0}(H^S(t)) U^S(t,0)\phi \tag{1.3.18}$$

for a sufficiently large $R_0 > 0$. Since, by virtue of Proposition 1.2.4,

$$\|F_{\epsilon_0}(z/t^2 \leqslant \nu_1/2 - \epsilon_0) h_R(H^S(t)) U^S(t,0)\phi\| = O(t^{(\beta_0 - \alpha_2)/2}) \tag{1.3.19}$$

holds with $\max\{0, 3-\eta\} < \beta_0 < \alpha_2 < 1$ and $\nu_1 = |E| - 2\epsilon_0 = 12\epsilon_0 < |E|$, one has only to prove the existence of

$$\lim_{t\to\infty} U_1^S(t,0)^* (1 - F_{\epsilon_0}(z/t^2 \leqslant \nu_1/2 - \epsilon_0)) h_{R_0}(H^S(t)) U^S(t,0)\phi \tag{1.3.20}$$

because $(\beta_0 - \alpha_2)/2 < 0$. Here we note that

$$(V_1(t, x + c(t)) - V(x + c(t)))(1 - F_{\epsilon_0}(z/t^2 \leqslant \nu_1/2 - \epsilon_0)) = O(t^{-2\rho_s}), \tag{1.3.21}$$

which can be shown in the same way as above,

$$\mathbb{D}_{H^S(t)} h_{R_0}(H^S(t)) = O(t^{1-\eta}) \tag{1.3.22}$$

(cf. $G_8(t)$ in subsection 1.2), and

$$\begin{aligned} &\mathbb{D}_{H^S(t)}(1 - F_{\epsilon_0}(z/t^2 \leqslant \nu_1/2 - \epsilon_0)) \\ &\quad = 2zF'_{\epsilon_0}(z/t^2 \leqslant \nu_1/2 - \epsilon_0)/t^3 - AF'_{\epsilon_0}(z/t^2 \leqslant \nu_1/2 - \epsilon_0)/t^2 \\ &\qquad\qquad - iF''_{\epsilon_0}(z/t^2 \leqslant \nu_1/2 - \epsilon_0)/(2t^4). \end{aligned} \tag{1.3.23}$$

Now let us watch of the second term of (1.3.23). By using

$$\begin{aligned} &A^2U^S(t,0)f(H_0^S)\langle A\rangle^{-2} - U^S(t,0)A^2f(H_0^S)\langle A\rangle^{-2} \\ &= \int_0^t U^S(t,\tau)i[H^S(\tau),A^2]U^S(\tau,0)f(H_0^S)\langle A\rangle^{-1}d\tau, \qquad (1.3.24) \\ &i[H^S(\tau),A^2] = 2i\,\mathrm{ad}_A^1(H^S(\tau))A - i\,\mathrm{ad}_A^2(H^S(\tau)) \qquad (1.3.25) \end{aligned}$$

and (1.2.83), we obtain an a priori estimate

$$\langle A\rangle^2U^S(t,0)f(H_0^S)\langle z\rangle^{-1} = O(t^2), \qquad (1.3.26)$$

because $\langle A\rangle^2 f_1(H_0^S)\langle z\rangle^{-1}$ is bounded for $f_1 \in C_0^\infty(\mathbb{R})$ such that $f_1 f = f$. Since

$$\langle A\rangle^2 h_R(H^S(t))\langle A\rangle^{-2} = O(1), \quad \langle A\rangle^2 F'_{\epsilon_0}(z/t^2 \leqslant \nu_1/2-\epsilon_0)\langle A\rangle^{-2} = O(1), \quad (1.3.27)$$

that can be shown in the same way as above, we have

$$\langle A\rangle^2 F'_{\epsilon_0}(z/t^2 \leqslant \nu_1/2-\epsilon_0)h_R(H^S(t))U^S(t,0)f(H_0^S)\langle z\rangle^{-1} = O(t^2). \qquad (1.3.28)$$

Since

$$F'_{\epsilon_0}(z/t^2 \leqslant \nu_1/2-\epsilon_0)h_R(H^S(t))U^S(t,0)f(H_0^S)\langle z\rangle^{-1} = O(t^{(\beta_0-\alpha_2)/2}) \qquad (1.3.29)$$

in virtue of Proposition 1.2.5, by a complex interpolation, we obtain

$$\langle A\rangle F'_{\epsilon_0}(z/t^2 \leqslant \nu_1/2-\epsilon_0)h_R(H^S(t))U^S(t,0)f(H_0^S)\langle z\rangle^{-1} = O(t^{1+(\beta_0-\alpha_2)/4}). \qquad (1.3.30)$$

Combining these, we finally get

$$\begin{aligned} &\|\partial_t(U_1^S(t,0)^*(1-F_{\epsilon_0}(z/t^2 \leqslant \nu_1/2-\epsilon_0))h_{R_0}(H^S(t))U^S(t,0)\phi)\| \\ &\qquad = O(t^{\max\{-2\rho_s,1-\eta,-1+(\beta_0-\alpha_2)/4\}}) \qquad (1.3.31) \end{aligned}$$

because $4\epsilon_0 t^2 \leqslant z \leqslant 5\epsilon_0 t^2$ on the support of $F'_{\epsilon_0}(z/t^2 \leqslant \nu_1/2-\epsilon_0)$ and $-1+(\beta_0-\alpha_2)/2 < -1+(\beta_0-\alpha_2)/4$ by $\beta_0 < \alpha_2$. Since $\max\{-2\rho_s, 1-\eta, -1+(\beta_0-\alpha_2)/4\} < -1$ by assumption, one can shoe the existence of (1.3.20) by the Cook-Kuroda method. This completes the proof of the unitary of $W_1^{S,+}$ by virtue of a density argument. □

1.4 Charge Transfer Model

In this subsection, we will apply the propagation estimates obtained by Adachi-Ishida [5] to a charge transfer model. The charge transfer model describes a quantum dynamics of a light particle in collisions with heavy N-particles obeying the laws of classical dynamics. Only the light particle is regarded as a quantum particle while the heavy particles follow free classical trajectories. Recall the free Hamiltonian is given by (1.1.1) with $m = 1$ in subsection 1.1. We denote the specific charge of the k-th heavy particle by q_k, $1 \leqslant k \leqslant N$. Now we assume

$$q_k \neq q_l, \quad 0 \leqslant k < l \leqslant N \tag{1.4.1}$$

and put $\tilde{q}_k = q_k - q_0$. (1.4.1) plays an important role. The classical trajectories $\chi_k(t) \in C^2(\mathbb{R}, \mathbb{R}^d)$ satisfy the free Newton equation

$$\ddot{\chi}_k(t) = q_k E(t), \quad \chi_k(0) = w_k, \quad \dot{\chi}_k(0) = v_k, \quad 1 \leqslant k \leqslant N \tag{1.4.2}$$

where $v_k, w_k \in \mathbb{R}^d$ are constant vectors. Therefore χ_k is given by

$$\chi_k(t) = q_k \tilde{c}(t) + v_k t + w_k. \tag{1.4.3}$$

Let $V_k^{\mathrm{s}} \in \mathscr{V}_{\rho_{\mathrm{s}}}$ with $\rho_{\mathrm{s}} > 1/2$ be the short-rage interaction potential between the quantum particle and the k-th classical particle. As in the previous subsections, when $0 < \eta \leqslant 2$, we impose (1.1.9) in Assumption 1.1.3 on V_k^{s}, that is,

$$|\tilde{q}_k E| - \sup_{x \in \mathbb{R}^d} (-(\tilde{q}_k / |\tilde{q}_k|))\omega \cdot (\nabla_x V_k^{\mathrm{s}})(x) > 0. \tag{1.4.4}$$

The full Hamiltonian which governs the system of charge transfer is given by

$$H(t) = H_0(t) + V^{\mathrm{s}}(t, x), \quad V^{\mathrm{s}}(t, x) = \sum_{k=1}^{N} V_k^{\mathrm{s}}(x - \chi_k(t)) \tag{1.4.5}$$

and we denote the propagator generated by (1.4.5) as $U(t, s)$. The existence and uniqueness of $U(t, s)$ can be also guaranteed by Yajima [44] as in subsection 1.1. The result we want to claim in this subsection is following theorem.

Theorem 1.4.1. (Asymptotic Completeness) *Under these assumptions, the wave operators*

$$W^{\pm} = \operatorname*{s-lim}_{t \to \pm\infty} U(t, 0)^* U_0(t, 0) \tag{1.4.6}$$

exist and are unitary on $L^2(\mathbb{R}^d)$.

The study of the scattering problem for the charge transfer model without external electromagnetic field was initiated by Yajima [43] and continued by Graf [16] in the short-range case and by Wüller [41] and Zieliński [47] in the long-range case. Afterwards the similar problems were studied by Zieliński [48] in the case where a system is set in a constant electric field and by Kayama [28] in a time-periodic electric field whose mean in time is nonzero. In these two studies, the assumption about specific charges (1.4.1) has played an important role too. This assumption implies one has only to consider a single channel. Ishida [25] applied the propagation estimates obtained by Yokoyama [46] and treated the case where the time-dependent electric field tends to a constant electric field asymptotically supposing $\eta > 7/2$ without (1.4.4) and $V_k^{\mathrm{s}} \in \tilde{\mathscr{V}}_{\rho_{\mathrm{s}}}$ with $\rho_{\mathrm{s}} > 1/2$. Theorem 1.1.5 says that we can relax the condition such that $\eta > 2$ without (1.4.4) and $V_k^{\mathrm{s}} \in \mathscr{V}_{\rho_{\mathrm{s}}}$ with $\rho_{\mathrm{s}} > 1/2$ by virtue of [5], in addition we can treat the case where $0 < \eta \leqslant 2$ as long as we assume (1.4.4).

We here prepare some notations which will be used in this subsection. As in the previous subsections, we consider $t \to \infty$ only. For $1 \leqslant k \leqslant N$ we denote by $U_k(t,s)$ the evolutional propagator generated by the Hamiltonian

$$H_k(t) = H_0(t) + V_k^{\mathrm{s}}(x - \chi_k(t)). \tag{1.4.7}$$

Put

$$v_0(t) = p - q_0\tilde{b}(t), \quad x_0(t) = x - q_0\tilde{c}(t). \tag{1.4.8}$$

We also consider only $t \to \infty$ below argument. Inspired by the discussion in [48] and [28] we introduce some self-adjoint operators for $t \geqslant 1$,

$$y_t^{(0)} = (v_0(t)/t - 2x_0(t)/t^2)^2/2 + (x_0(t)/t^2)^2 + r, \tag{1.4.9}$$

$$y_t = y_t^{(0)} + V^{\mathrm{s}}(t,x)/t^2, \tag{1.4.10}$$

$$y_t^{(k)} = y_t^{(0)} + V_k^{\mathrm{s}}(x - \chi_k(t))/t^2, \quad 1 \leqslant k \leqslant N, \tag{1.4.11}$$

where we have set $r = \sum_{k=1}^N \sup_{x\in\mathbb{R}^d} |V_k^{\mathrm{s}}(x)|$. By virtue of $t \geqslant 1$ and $y_t^{(0)} \geqslant r$ we can define

$$\tilde{y}_{t,n} = (1 + y_t/n)^{-1}, \quad \tilde{y}_{t,n}^{(k)} = (1 + y_t^{(k)}/n)^{-1}, \quad 0 \leqslant k \leqslant N \tag{1.4.12}$$

for $n \in \mathbb{N}$ as bounded operators and $0 \leqslant \tilde{y}_{t,n},\, \tilde{y}_{t,n}^{(k)} \leqslant 1$ is satisfied.

We give some propagation estimates to prove the existence of the Deift-Simon wave operators. Then we can reduce the problem to the one-body case by the partition of identity.

Proposition 1.4.2.

$$\lim_{n\to\infty} \sup_{t\geqslant 1} \|(1 - \tilde{y}_{t,n})U(t,0)\phi\| = 0, \quad \lim_{n\to\infty} \sup_{t\geqslant 1} \|(1 - \tilde{y}_{t,n}^{(0)})U_0(t,0)\phi\| = 0 \tag{1.4.13}$$

hold for $\phi \in L^2(\mathbb{R}^d)$.

Proof. This proposition was proved essentially in [25] under the assumption that $\eta > 7/2$ without (1.4.4) and $V_k^{\mathrm{s}} \in \mathscr{V}_{\rho_{\mathrm{s}}}$ with $\rho_{\mathrm{s}} > 1/2$ (see also [28]). We will sketch the proof.

We compute

$$\mathbb{D}_{H(t)}y_t = \mathbb{D}_{H_0(t)}y_0^{(0)} + \partial_t(V^{\mathrm{s}}(t,x)/t^2) + (q_0\tilde{b}(t)/t^2 + 2x_0(t)/t^3)\cdot\nabla_x V^{\mathrm{s}}(t,x). \quad (1.4.14)$$

Note that $\mathbb{D}_{H_0(t)}(A(t)B(t)) = (\mathbb{D}_{H_0(t)}A(t))B(t) + A(t)(\mathbb{D}_{H_0(t)}B(t))$. It follows from $\mathbb{D}_{H_0(t)}v_0(t) = 0$, $\mathbb{D}_{H_0(t)}x_0(t) = v_0(t)$ and

$$\mathbb{D}_{H_0(t)}(v_0(t)/t - 2x_0(t)/t^2) = 4x_0(t)/t^3 - 3v_0(t)/t^2, \quad (1.4.15)$$

$$\mathbb{D}_{H_0(t)}x_0(t)/t^2 = v_0(t)/t - 2x_0(t)/t^2 \quad (1.4.16)$$

that

$$\mathbb{D}_{H_0(t)}y_t^{(0)} = -3(v_0(t)/t - 2x_0(t)/t^2)^2/t. \quad (1.4.17)$$

Recall (1.1.23) and note that $2\tilde{c}(t) - t\tilde{b}(t) = 2c(t) + tb(t) = O(c(t))$. Since V_k^{s}, $\nabla_x V_k^{\mathrm{s}}$ and $x\cdot\nabla_x V_k^{\mathrm{s}}(x)$ are bounded, we can compute

$$\begin{aligned}
&\partial_t(V_k^{\mathrm{s}}(x-\chi_k(t))/t^2) + (q_0\tilde{b}(t)/t^2 + 2x_0(t)/t^3)\cdot(\nabla_x V_k^{\mathrm{s}})(x-\chi_k(t)) \\
&= 2(x-\chi_k(t))\cdot(\nabla_x V_k^{\mathrm{s}})(x-\chi_k(t))/t^3 - 2V_k^{\mathrm{s}}(x-\chi_k(t))/t^3 \\
&\quad +(\tilde{q}_k(2\tilde{c}(t) - t\tilde{b}(t)) + v_k t + 2w_k)\cdot(\nabla_x V_k^{\mathrm{s}})(x-\chi_k(t))/t^3 \\
&= O(\theta(t))
\end{aligned} \quad (1.4.18)$$

for $1 \leqslant k \leqslant N$, where $\theta(t)$ is

$$\theta(t) = \begin{cases} O(t^{-1-\eta}) & \eta < 1 \\ O(t^{-2}\log t) & \eta = 1 \\ O(t^{-2}) & \eta > 1, \end{cases} \quad (1.4.19)$$

and is integrable on $[2,\infty)$. Combining these, we obtain

$$\mathbb{D}_{H(t)}y_t = -3(v_0(t)/t - 2x_0(t)/t^2)^2/t + O(\theta(t)). \quad (1.4.20)$$

Then we have

$$\begin{aligned}
&\mathbb{D}_{H(t)}(1-\tilde{y}_{t,n}) = \tilde{y}_{t,n}(\mathbb{D}_{H(t)}y_t)\tilde{y}_{t,n}/n \\
&\quad = -3\tilde{y}_{t,n}(v_0(t)/t - 2x_0(t)/t^2)^2\tilde{y}_{t,n}/(nt) + O(n^{-1}\theta(t)) \\
&\quad \leqslant O(n^{-1}\theta(t)).
\end{aligned} \quad (1.4.21)$$

By this estimate, there exists $C > 0$ such that

$$((1-\tilde{y}_{t,n})\phi_t, \phi_t) \leqslant C\|\phi\|^2/n + ((1-\tilde{y}_{2,n})\phi_2, \phi_2) \quad (1.4.22)$$

holds for any $t \geqslant 2$, where $\phi \in \mathscr{D}(p^2 + x^2)$ and $\phi_t = U(t,0)\phi$. This implies the first equality of (1.4.13) by a density argument because $\text{s-lim}_{n\to\infty}\tilde{y}_{2,n} = 1$ and $\|(1-\tilde{y}_{t,n})\phi_t\|^2 \leqslant 2((1-\tilde{y}_{t,n})\phi_t, \phi_t)$. The other can be shown quite similarly. □

Proposition 1.4.3. *Fix $n \in \mathbb{N}$. Then there exists $C > 0$ such that for $\phi \in L^2(\mathbb{R}^d)$,*

$$\int_2^\infty \|((v_0(t)/t - 2x_0(t)/t^2)^2 + 1/t^2)^{1/2}\tilde{y}_{t,n}U(t,0)\phi\|^2/t \; dt \leqslant C\|\phi\|^2, \quad (1.4.23)$$

$$\int_2^\infty \|((v_0(t)/t - 2x_0(t)/t^2)^2 + 1/t^2)^{1/2}\tilde{y}_{t,n}^{(k)}U_k(t,0)\phi\|^2/t \; dt \leqslant C\|\phi\|^2 \quad (1.4.24)$$

with $0 \leqslant k \leqslant N$ hold.

Proof. This proposition can be proved similarly with [25] (see also [28]). We first note that $0 \leqslant \tilde{y}_{t,n} \leqslant 1$, especially $\tilde{y}_{t,n} = O(1)$. On the other hand, there exists a constant C such that

$$n\mathbb{D}_{H(t)}\tilde{y}_{t,n} = -\tilde{y}_{t,n}(\mathbb{D}_{H(t)}y_t)\tilde{y}_{t,n} \geqslant 3\tilde{y}_{t,n}\Lambda(t)\tilde{y}_{t,n}/t - C(1/t^2 + \theta(t)) \qquad (1.4.25)$$

by virtue of the computation (1.4.20), where we have put

$$\Lambda(t) = (v_0(t)/t - 2x_0(t)/t^2)^2 + 1/t^2. \qquad (1.4.26)$$

We thus obtain

$$\|\Lambda(t)^{1/2}\tilde{y}_{t,n}\phi_t\|^2/t \leqslant n((\mathbb{D}_{H(t)}\tilde{y}_{t,n})\phi_t, \phi_t)/3 + O(\theta(t))\|\phi\|^2, \qquad (1.4.27)$$

since $O(\theta(t)) \geqslant O(t^{-2})$. This implies (1.4.23) because the right hand side of this inequality is integrable on $[2, T]$ uniformly in $T \geqslant 2$. (1.4.24) can be shown quite similarly. □

Take

$$0 < \epsilon \leqslant |E|/16 \times \min_{0 \leqslant k < l \leqslant N} |q_k - q_l| \qquad (1.4.28)$$

and we have following key estimate.

Proposition 1.4.4. *Fix $n \in \mathbb{N}$. Then there exists $C > 0$ such that for $\phi \in L^2(\mathbb{R}^d)$,*

$$\int_2^\infty \|F_\epsilon(\epsilon \leqslant |\zeta_k(t)| \leqslant 5\epsilon)\tilde{y}_{t,n}U(t,0)\phi\|^2/t \; dt \leqslant C\|u\|^2 \qquad (1.4.29)$$

with $1 \leqslant k \leqslant N$ hold, where $\zeta_k(t) = (x_0(t)/t^2 - \tilde{q}_k E/2) \cdot \omega$.

Proof. This proposition was also proved essentially in [25] under the assumption that $\eta > 7/2$ without (1.4.4) and $V_k^{\mathrm{s}} \in \mathscr{V}_{\rho_{\mathrm{s}}}$ with $\rho_{\mathrm{s}} > 1/2$ (see also [28]).

We put the smooth cut-off

$$\Phi(\lambda) = \int_0^\lambda F_\epsilon(\epsilon \leqslant |\mu| \leqslant 5\epsilon)^2 d\mu \qquad (1.4.30)$$

and define

$$M(t) = 6\Phi(\zeta_k(t)) + (v_0(t)/t - 2x_0(t)/t^2) \cdot \omega\Phi'(\zeta_k(t)) \\ +\Phi'(\zeta_k(t))\omega \cdot (v_0(t)/t - 2x_0(t)/t^2). \quad (1.4.31)$$

Since

$$\mathbb{D}_{H_0(t)}\zeta_k(t) = (v_0(t)/t - 2x_0(t)/t^2) \cdot \omega/t, \quad (1.4.32)$$

we obtain

$$\mathbb{D}_{H(t)}\Phi(\zeta_k(t)) = (v_0(t)/t - 2x_0(t)/t^2) \cdot \omega\Phi'(\zeta_k(t))/(2t) \\ +\Phi'(\zeta_k(t))\omega \cdot (v_0(t)/t - 2x_0(t)/t^2)/(2t). \quad (1.4.33)$$

We also obtain

$$\mathbb{D}_{H(t)}(v_0(t)/t - 2x_0(t)/t^2) = 4x_0(t)/t^3 - 3v_0(t)/t^2 - \nabla_x V^{\mathrm{s}}(t,x)/t. \quad (1.4.34)$$

Combining these, we can compute

$$\mathbb{D}_{H(t)}M(t) = -4x_0(t) \cdot \omega\Phi'(\zeta_k(t))/t^3 - 2\nabla_x V^{\mathrm{s}}(t,x) \cdot \omega\Phi'(\zeta_k(t))/t \\ +2(v_0(t)/t - 2x_0(t)/t^2) \cdot \omega(\Phi''(\zeta_k(t))/t)\omega \cdot (v_0(t)/t - 2x_0(t)/t^2) \\ -\Phi''''(\zeta_k(t))/(2t^7). \quad (1.4.35)$$

Note that $|\zeta_k(t)| \leqslant 5\epsilon$ holds on the support of $\Phi'(\zeta_k(t))$ and we can estimate

$$-(\tilde{q}_k/|\tilde{q}_k|)\tilde{y}_{t,n}(-4x_0(t) \cdot \omega\Phi'(\zeta_k(t))/t^3)\tilde{y}_{t,n} \\ = 4(\tilde{q}_k/|\tilde{q}_k|)\tilde{y}_{t,n}(\tilde{q}_k|E|/2 + \zeta_k(t))\Phi'(\zeta_k(t))\tilde{y}_{t,n}/t \\ \geqslant 4\tilde{y}_{t,n}(|\tilde{q}_k||E|/2 - |\zeta_k(t)|)\Phi'(\zeta_k(t))\tilde{y}_{t,n}/t \\ \geqslant 12\epsilon\tilde{y}_{t,n}\Phi'(\zeta_k(t))\tilde{y}_{t,n}/t. \quad (1.4.36)$$

$x - \chi_k(t)$ is represented by

$$x - \chi_k(t) = x_0(t) - \tilde{q}_k t^2 E/2 - \tilde{q}_k c(t) - v_k t - w_k \quad (1.4.37)$$

and we obtain

$$|(x - \chi_k(t)) \cdot \omega| \geqslant |\zeta_k(t)|t^2 - |\tilde{q}_k||c(t)| - |v_k t + w_k| \geqslant \epsilon t^2/2, \quad (1.4.38)$$

as $t \to \infty$, if $|\zeta_k(t)| \geqslant \epsilon$ holds, noting $c(t) = o(t^2)$. Let l satisfy $l \neq k$. $x - \chi_l(t)$ is represented by

$$x - \chi_l(t) = (q_k - q_l)t^2 E/2 + x_0(t) - \tilde{q}_k t^2 E/2 - \tilde{q}_l c(t) - v_l t - w_l \quad (1.4.39)$$

and we also obtain

$$|(x-\chi_l(t))\cdot\omega| \geqslant |q_k-q_l|t^2E/2-|\zeta_k(t)|t^2-|\tilde{q}_l||c(t)|-|v_lt+w_l| \geqslant 2\epsilon t^2, \quad (1.4.40)$$

as $t\to\infty$, if $|\zeta_k(t)|\leqslant 5\epsilon$ holds, noting $c(t)=o(t^2)$ again. It follows from these estimates and the assumption for the potential that

$$F_\epsilon(|\zeta_k(t)|\geqslant\epsilon)(\nabla_x V_k^{\mathrm{s}})(x-\chi_k(t)) = O(t^{-1-2\rho_{\mathrm{s}}}), \quad (1.4.41)$$

$$F_\epsilon(|\zeta_k(t)|\leqslant 5\epsilon)(\nabla_x V_l^{\mathrm{s}})(x-\chi_l(t)) = O(t^{-1-2\rho_{\mathrm{s}}}). \quad (1.4.42)$$

Thus, we have

$$\begin{aligned}\nabla_x V^{\mathrm{s}}(t,x)\cdot\omega\Phi'(\zeta_k(t))/t &= (\nabla_x V_k^{\mathrm{s}})(x-\chi_k(t))\cdot\omega\Phi'(\zeta_k(t))/t \\ &\quad+\sum_{1\leqslant l\leqslant N, l\neq k}(\nabla_x V_l^{\mathrm{s}})(x-\chi_l(t))\cdot\omega\Phi'(\zeta_k(t))/t \\ &= O(t^{-2-2\rho_{\mathrm{s}}}). \quad (1.4.43)\end{aligned}$$

Since

$$\Lambda(t)^{-1/2}(v_0(t)/t-2x_0(t)/t^2)^2\Lambda(t)^{-1/2}\leqslant 1, \quad (1.4.44)$$

we obtain

$$(v_0(t)/t-2x_0(t)/t^2)\Lambda(t)^{-1/2}=O(1). \quad (1.4.45)$$

Then there exists $C>0$ such that

$$\begin{aligned}-(\tilde{q}_k/|\tilde{q}_k|)\tilde{y}_{t,n}(v_0(t)/t-2x_0(t)/t^2)\cdot\omega(\Phi''(\zeta_k(t))/t)\omega\cdot(v_0(t)/t-2x_0(t)/t^2)\tilde{y}_{t,n} \\ \geqslant C\tilde{y}_{t,n}\Lambda(t)\tilde{y}_{t,n}/t. \quad (1.4.46)\end{aligned}$$

Thus, we obtain

$$\begin{aligned}&-(\tilde{q}_k/|\tilde{q}_k|)\tilde{y}_{t,n}(\mathbb{D}_{H(t)}M(t))\tilde{y}_{t,n} \\ &\quad\geqslant 12\epsilon\tilde{y}_{t,n}\Phi'(\zeta_k(t))\tilde{y}_{t,n}/t-C_1\tilde{y}_{t,n}\Lambda(t)\tilde{y}_{t,n}/t-C_2/t^{\min\{7,2+2\rho_{\mathrm{s}}\}} \quad (1.4.47)\end{aligned}$$

with some C_1 and $C_2>0$. On the other hand, we compute by (1.4.20) (also (1.4.25))

$$\begin{aligned}n(\mathbb{D}_{H(t)}\tilde{y}_{t,n})M(t)\tilde{y}_{t,n} &= -3\tilde{y}_{t,n}\Lambda(t)\tilde{y}_{t,n}M(t)\tilde{y}_{t,n}/t \\ &\quad+3\tilde{y}_{t,n}^2M(t)\tilde{y}_{t,n}/t^3+\tilde{y}_{t,n}O(\theta(t))\tilde{y}_{t,n}M(t)\tilde{y}_{t,n}. \quad (1.4.48)\end{aligned}$$

Since

$$\tilde{y}_{t,n}(x_0(t)/t^2)^2\tilde{y}_{t,n}\leqslant n, \quad \tilde{y}_{t,n}(v_0(t)/t-2x_0(t)/t^2)^2\tilde{y}_{t,n}\leqslant 2n, \quad (1.4.49)$$

we have

$$(x_0(t)/t^2)\tilde{y}_{t,n}=O(1), \quad (v_0(t)/t-2x_0(t)/t^2)\tilde{y}_{t,n}=O(1). \quad (1.4.50)$$

By this computation, we can see the commutator

$$\begin{aligned}[\Lambda(t), \tilde{y}_{t,n}] &= -\tilde{y}_{t,n}[(v_0(t)/t - 2x_0(t)/t^2)^2, (x_0(t)/t^2)^2 + V^{\mathrm{s}}(t,x)/t^2]\tilde{y}_{t,n}/n. \\ &= O(t^{-3}) \end{aligned} \tag{1.4.51}$$

This implies

$$\Lambda(t)\tilde{y}_{t,n}\Lambda(t)^{-1} = O(1). \tag{1.4.52}$$

Similarly

$$\Lambda(t)\Phi(\zeta_k(t))\Lambda(t)^{-1} = O(1) \tag{1.4.53}$$

follows from

$$[\Lambda(t), \Phi(\zeta_k(t)] = -\Phi''(\zeta_k(t))/t^6 - 2i\Phi'(\zeta_k(t))\omega \cdot (v_0(t)/t - 2x_0(t)/t^2)/t^3, \tag{1.4.54}$$

because of $\Lambda(t)^{-1} = O(t^2)$, $\Lambda(t)^{-1/2} = O(t)$ and (1.4.38). Applying the complex interpolation to (1.4.52) and (1.4.53), we have

$$\Lambda(t)^{1/2}\tilde{y}_{t,n}\Lambda(t)^{-1/2} = O(1), \quad \Lambda(t)^{1/2}\Phi(\zeta_k(t))\Lambda(t)^{-1/2} = O(1). \tag{1.4.55}$$

Noting estimates (1.4.45), (1.4.55) and $\Lambda(t)^{1/2}\tilde{y}_{t,n} = O(1)$ since $\tilde{y}_{t,n}\Lambda(t)\tilde{y}_{t,n} \leqslant 2n+1$, we decompose

$$\begin{aligned} &\tilde{y}_{t,n}\Lambda(t)\tilde{y}_{t,n}M(t)\tilde{y}_{t,n} \\ &= 6\tilde{y}_{t,n}\Lambda(t)^{1/2} \times \Lambda(t)^{1/2}\tilde{y}_{t,n}\Lambda(t)^{-1/2} \times \Lambda(t)^{1/2}\Phi(\zeta_k(t))\Lambda(t)^{-1/2} \times \Lambda(t)^{1/2}\tilde{y}_{t,n} \\ &\quad +\tilde{y}_{t,n}\Lambda(t)^{1/2} \times \Lambda(t)^{1/2}\tilde{y}_{t,n} \times (v_0(t)/t - 2x_0(t)/t^2) \cdot \omega\Lambda(t)^{-1/2} \\ &\qquad \times\Lambda(t)^{1/2}\Phi'(\zeta_k(t))\Lambda(t)^{-1/2} \times \Lambda(t)^{1/2}\tilde{y}_{t,n} \\ &\quad +\tilde{y}_{t,n}\Lambda(t)^{1/2} \times \Lambda(t)^{1/2}\tilde{y}_{t,n} \times \Phi'(\zeta_k(t)) \\ &\qquad \times(v_0(t)/t - 2x_0(t)/t^2) \cdot \omega\Lambda(t)^{-1/2} \times \Lambda(t)^{1/2}\tilde{y}_{t,n}. \end{aligned} \tag{1.4.56}$$

Therefore, there exists $C > 0$ such that

$$\begin{aligned} -(\tilde{q}_k/|\tilde{q}_k|)(-3\tilde{y}_{t,n}\Lambda(t)\tilde{y}_{t,n}M(t)\tilde{y}_{t,n}/t - 3\tilde{y}_{t,n}M(t)\tilde{y}_{t,n}\Lambda(t)\tilde{y}_{t,n}/t) \\ \geqslant -C\tilde{y}_{t,n}\Lambda(t)\tilde{y}_{t,n}/t. \end{aligned} \tag{1.4.57}$$

Since

$$\tilde{y}_{t,n}M(t)\tilde{y}_{t,n} = O(1) \tag{1.4.58}$$

by (1.4.50), we obtain

$$\begin{aligned} -(\tilde{q}_k/|\tilde{q}_k|)((\mathbb{D}_{H(t)}\tilde{y}_{t,n})M(t)\tilde{y}_{t,n} + \tilde{y}_{t,n}M(t)(\mathbb{D}_{H(t)}\tilde{y}_{t,n})) \\ \geqslant -C_3\tilde{y}_{t,n}\Lambda(t)\tilde{y}_{t,n}/t - C_4/t^3 - C_5\theta(t) \end{aligned} \tag{1.4.59}$$

with some C_3, C_4 and $C_5 > 0$. Combining (1.4.47) and (1.4.59),

$$\begin{aligned} &-(\tilde{q}_k/|\tilde{q}_k|)\mathbb{D}_{H(t)}(\tilde{y}_{t,n}M(t)\tilde{y}_{t,n}) \\ &\quad \geqslant 12\epsilon\tilde{y}_{t,n}\Phi'(\zeta_k(t))\tilde{y}_{t,n}/t - (C_1 + C_3)\tilde{y}_{t,n}\Lambda(t)\tilde{y}_{t,n}/t - C_6\theta(t) \end{aligned} \tag{1.4.60}$$

with $C_6 > 0$ is obtained. This implies the proposition by virtue of Proposition 1.4.3 and the integrability of $\theta(t)$. □

Now we can show the existence of the Deift-Simon wave operators. We set

$$J_k(t,x) = \begin{cases} F_{2\epsilon}(|\zeta_k(t)| \leqslant 4\epsilon), & 1 \leqslant k \leqslant N \\ 1 - \sum_{1\leqslant l\leqslant N} F_{2\epsilon}(|\zeta_l(t)| \leqslant 4\epsilon), & k = 0. \end{cases} \tag{1.4.61}$$

Proposition 1.4.5. *The Deift-Simon wave operators*

$$\mathscr{W}_k = \operatorname*{s-lim}_{t\to\infty} U_k(t,0)^* J_k(t,x) U(t,0) \tag{1.4.62}$$

with $0 \leqslant k \leqslant N$ all exist.

Proof. This proposition can be also proved similarly with [25] (see also [28]). Put

$$V^{\mathrm{s},k}(t,x) = \begin{cases} V^{\mathrm{s}}(t,x) - V^{\mathrm{s}}_k(x - \chi_k(t)) & 1 \leqslant k \leqslant N \\ V^{\mathrm{s}}(t,x) & k = 0. \end{cases} \tag{1.4.63}$$

We first claim

$$V^{\mathrm{s},k}(t,x) J_k(t,x) = O(t^{-2\rho_{\mathrm{s}}}) \tag{1.4.64}$$

for $0 \leqslant k \leqslant N$. In fact, let l satisfy $l \neq k$ for $1 \leqslant k \leqslant N$. If $|\zeta_k(t)| \leqslant 4\epsilon$ holds, we have $|(x - \chi_k(t)) \cdot \omega| \geqslant 3\epsilon t^2$ as in (1.4.40). We thus obtain

$$V^{\mathrm{s},k}(t,x) J_k(t,x) = \sum_{1\leqslant l\leqslant N, l\neq k} V^{\mathrm{s}}_l(x - \chi_l(t)) F_{2\epsilon}(|\zeta_k(t)| \leqslant 4\epsilon) = O(t^{-2\rho_{\mathrm{s}}}). \tag{1.4.65}$$

On the support of $J_0(t,x)$, $|\zeta_l(t)| \geqslant 2\epsilon$ holds for any $1 \leqslant l \leqslant N$. Then we have $|(x - \chi_l(t)) \cdot \omega| \geqslant \epsilon t^2$ with $1 \leqslant l \leqslant N$ as in (1.4.38). This implies

$$V^{\mathrm{s},0}(t,x) J_0(t,x) = O(t^{-2\rho_{\mathrm{s}}}) \tag{1.4.66}$$

and (1.4.64) have otained.

Noting (1.4.50) and $(\nabla_x J_k)(t,x) = O(t^{-2})$, we can compute

$$\tilde{y}^{(k)}_{t,n}[(v_0(t)/t - 2x_0(t)/t^2)^2, J_k(t,x)]\tilde{y}_{t,n} = O(t^{-3}). \tag{1.4.67}$$

It follows from this and (1.4.64) that

$$\begin{aligned}J_k(t,x)\tilde{y}_{t,n}^2 - \tilde{y}_{t,n}^{(k)} J_k(t,x)\tilde{y}_{t,n} &= \tilde{y}_{t,n}^{(k)}(y_k^{(k)} J_k(t,x) - J_k(t,x) y_t)\tilde{y}_{t,n}^2/n \\ &= \tilde{y}_{t,n}^{(k)}[(v_0(t)/t - 2x_0(t)/t^2)^2, J_k(t,x)]\tilde{y}_{t,n}^2/(2n) \\ &\qquad -\tilde{y}_{t,n}^{(k)} V^{\mathrm{s},k}(t,x) J_k(t,x)\tilde{y}_{t,n}^2/(nt^2) \\ &= O(t^{-3}) \end{aligned} \tag{1.4.68}$$

with $0 \leqslant k \leqslant N$. Then we have only to prove that the strong limits

$$\operatorname*{s-lim}_{t\to\infty} U_k(t,0)^* \tilde{y}_{t,n}^{(k)} J_k(t,x)\tilde{y}_{t,n} U(t,0) \tag{1.4.69}$$

with $0 \leqslant k \leqslant N$ all exist for any $n \in \mathbb{N}$ because of the decomposition

$$\begin{aligned}U_k(t,0)^* J_k(t,x) U(t,0) &= U_k(t,0)^* J_k(t,x)(1 - \tilde{y}_{t,n} + \tilde{y}_{t,n}(1-\tilde{y}_{t,n})))U(t,0) \\ &\quad + U_k(t,0)^*(J_k(t,x)\tilde{y}_{t,n}^2 - \tilde{y}_{t,n}^{(k)} J_k(t,x)\tilde{y}_{t,n})U(t,0) \\ &\quad + U_k(t,0)^* \tilde{y}_{t,n}^{(k)} J_k(t,x)\tilde{y}_{t,n} U(t,0)\end{aligned} \tag{1.4.70}$$

and Proposition 1.4.2. Compute

$$\begin{aligned}&\partial_t(U_k(t,0)^* \tilde{y}_{t,n}^{(k)} J_k(t,x)\tilde{y}_{t,n} U(t,0)) \\ &= U_k(t,0)^*(\mathbb{D}_{H_k(t)}\tilde{y}_{t,n}^{(k)}) J_k(t,x)\tilde{y}_{t,n} U(t,0) + U_k(t,0)^* \tilde{y}_{t,n}^{(k)} J_k(t,x)(\mathbb{D}_{H(t)}\tilde{y}_{t,n})U(t,0) \\ &\quad + U_k(t,0)^* \tilde{y}_{t,n}^{(k)}(\mathbb{D}_{H_k(t)} J_k(t,x))\tilde{y}_{t,n} U(t,0) \\ &\qquad - iU_k(t,0)^* \tilde{y}_{t,n}^{(k)} V^{\mathrm{s},k}(t,x) J_k(t,x)\tilde{y}_{t,n} U(t,0).\end{aligned} \tag{1.4.71}$$

Use the computation (1.4.20) and (1.4.25) (see also (1.4.17) if $k=0$) and we have

$$\begin{aligned}&nU_k(t,0)^*(\mathbb{D}_{H_k(t)}\tilde{y}_{t,n}^{(k)}) J_k(t,x)\tilde{y}_{t,n} U(t,0) \\ &= 3U_k(t,0)^* \tilde{y}_{t,n}^{(k)} \Lambda(t)^{1/2} \times \Lambda(t)^{1/2} \tilde{y}_{t,n}^{(k)} \Lambda(t)^{-1/2} \times \Lambda(t)^{1/2} J_k(t,x) \Lambda(t)^{-1/2} \\ &\quad \times \Lambda(t)^{1/2} \tilde{y}_{t,n} U(t,0)/t + O(\theta_k(t)),\end{aligned} \tag{1.4.72}$$

where

$$\theta_k(t) = \begin{cases} \theta(t) & 1 \leqslant k \leqslant N \\ t^{-2} & k = 0. \end{cases} \tag{1.4.73}$$

By virtue of (1.4.59), there exists $C > 0$ such that

$$\begin{aligned}&|(U_k(t,0)^*(\mathbb{D}_{H_k(t)}\tilde{y}_{t,n}^{(k)}) J_k(t,x)\tilde{y}_{t,n} U(t,0)\phi, \varphi)| \\ &\leqslant C\|\Lambda(t)^{1/2}\tilde{y}_{t,n} U(t,0)\phi\| \|\Lambda(t)^{1/2}\tilde{y}_{t,n}^{(k)} U_k(t,0)\varphi\|/t + O(\theta_k(t))\|\phi\|\|\varphi\|.\end{aligned} \tag{1.4.74}$$

For $1 \leqslant k \leqslant N$, we compute

$$\mathbb{D}_{H_k(t)} J_k(t,x) = (v_0(t)/t - 2x_0(t)/t^2) \cdot \omega J'(\zeta_k(t))/t + iJ''(\zeta_k(t))/(2t^4), \quad (1.4.75)$$

where we have put $J(\lambda) = F_{2\epsilon}(|\lambda| \leqslant 4\epsilon)$. By virtue of (1.4.52) and

$$J'(\zeta_k(t)) = J'(\zeta_k(t)) F_\epsilon(\epsilon \leqslant |\zeta_k(t)| \leqslant 5\epsilon), \quad (1.4.76)$$

there exists $C > 0$ such that

$$\begin{aligned} &|(U_k(t,0)^* \tilde{y}_{t,n}^{(k)} (\mathbb{D}_{H_k(t)} J_k(t,x)) \tilde{y}_{t,n} U(t,0)\phi, \varphi)| \\ &= C \|F_\epsilon(\epsilon \leqslant |\zeta_k(t)| \leqslant 5\epsilon) \tilde{y}_{t,n} U(t,0)\phi\| \|\Lambda(t)^{1/2} \tilde{y}_{t,n}^{(k)} U_k(t,0)\varphi\|/t \\ &\quad + O(t^{-4})) \|\phi\| \|\varphi\| \end{aligned} \quad (1.4.77)$$

The case where $k = 0$ is also obtained similarly because

$$\mathbb{D}_{H_0(t)} J_0(t,x) = -\sum_{k=1}^{N} \mathbb{D}_{H_0(t)} J_k(t,x). \quad (1.4.78)$$

Combining these estimates and (1.4.64), we obtain

$$\begin{aligned} &|\partial_t (U_k(t,0)^* \tilde{y}_{t,n}^{(k)} J_k(t,x) \tilde{y}_{t,n} U(t,0)\phi, \varphi)| \\ &\leqslant C_1 \|\Lambda(t)^{1/2} \tilde{y}_{t,n} U(t,0)\phi\| \|\Lambda(t)^{1/2} \tilde{y}_{t,n}^{(k)} U_k(t,0)\varphi\|/t \\ &\quad + C_2 \|F_\epsilon(\epsilon \leqslant |\zeta_k(t)| \leqslant 5\epsilon) \tilde{y}_{t,n} U(t,0)\phi\| \|\Lambda(t)^{1/2} \tilde{y}_{t,n}^{(k)} U_k(t,0)\varphi\|/t \\ &\quad + O(\theta_k(t)) \|\phi\| \|\varphi\| \end{aligned} \quad (1.4.79)$$

with some C_1 and $C_2 > 0$.This implies the existence of (1.4.69) by Propositions 1.4.3, 1.4.4 and the integrability of $\theta_k(t)$. □

Proof of Theorem 1.4.1. We only prove the asymptotic completeness, that is, the existence of

$$\operatorname*{s-lim}_{t \to \infty} U_0(t,0)^* U(t,0). \quad (1.4.80)$$

The existence of the wave operator can be shown quite similarly in [25] (see also [28]). If we can prove

$$\operatorname*{s-lim}_{t \to \infty} F_\epsilon(-(\tilde{q}_k/|\tilde{q}_k|)\zeta_k(t) \leqslant 5\epsilon) U_k(t,0) = 0 \quad (1.4.81)$$

for $1 \leqslant k \leqslant N$, (1.4.80) is obtained as follows. Since

$$J_k(t,x) = F_\epsilon(-(\tilde{q}_k/|\tilde{q}_k|)\zeta_k(t) \leqslant 5\epsilon) J_k(t,x), \quad (1.4.82)$$

we have

$$
\begin{aligned}
&J_k(t,x)U(t,0)\phi \\
&= F_\epsilon(-(\tilde{q}_k/|\tilde{q}_k|)\zeta_k(t) \leqslant 5\epsilon)U_k(t,0)U_k(t,0)^* J_k(t,x)U(t,0)\phi \\
&= F_\epsilon(-(\tilde{q}_k/|\tilde{q}_k|)\zeta_k(t) \leqslant 5\epsilon)U_k(t,0)\mathscr{W}_k\phi + o(1) \\
&= o(1)
\end{aligned}
\tag{1.4.83}
$$

as $t \to \infty$ for $\phi \in L^2(\mathbb{R}^d)$ by Proposition 1.4.5. Then by the partition of identity (1.4.61) and Proposition 1.4.5 again,

$$
\begin{aligned}
U(t,0)\phi &= J_0(t,x)U(t,0) + \sum_{k=1}^{N} J_k(t,x)U(t,0)\phi \\
&= U_0(t,0)\mathscr{W}_0\phi + o(1)
\end{aligned}
\tag{1.4.84}
$$

holds as $t \to \infty$. This implies that

$$
\operatorname*{s-lim}_{t\to\infty} U_0(t,0)^* U(t,0) = \mathscr{W}_0. \tag{1.4.85}
$$

Thus the proof is completed. The duty left for us is showing (1.4.81). For the proof, we try to rewrite $U_k(t,0)$ into what we can pursue the time evolution. For $1 \leqslant k \leqslant N$, we define the time-dependent Hamiltonians

$$
H_k^{(1)}(t) = p^2/2 + \tilde{q}_k E(t)\cdot x + V_k^{\mathrm{s}}(x), \tag{1.4.86}
$$

$$
H_k^{(2)}(t) = p^2/2 + \tilde{q}_k E\cdot x + V_k^{\mathrm{s}}(x - \tilde{q}_k c(t)) \tag{1.4.87}
$$

and let $U_k^{(1)}(t,s)$ and $U_k^{(2)}(t,s)$ be the propagators generated by (1.4.86) and (1.4.87) respectively. We introduce unitary operators

$$
\mathscr{G}_k^{(1)}(t) = e^{-i\int_0^t (\dot{\chi}_k(s)^2/2 + \tilde{q}_k E(s)\cdot\chi_k(s))ds} e^{i\dot{\chi}_k(t)\cdot x} e^{-i\chi_k(t)\cdot p}, \tag{1.4.88}
$$

$$
\mathscr{G}_k^{(2)}(t) = e^{-i\tilde{q}_k^2 a(t)} e^{-i\tilde{q}_k b(t)\cdot x} e^{i\tilde{q}_k c(t)\cdot p}. \tag{1.4.89}
$$

Then the Avron-Herbst formulas

$$
U_k(t,s) = \mathscr{G}_k^{(1)}(t)U_k^{(1)}(t,s)\mathscr{G}_k^{(1)}(s)^*, \quad U_k^{(1)}(t,s) = \mathscr{G}_k^{(2)}(t)U_k^{(2)}(t,s)\mathscr{G}_k^{(2)}(s)^* \tag{1.4.90}
$$

hold (for the proof see [25] or [28]). Since $|v_k t + w_k|/t^2 \leqslant \epsilon$ as $t \to \infty$, we obtain

$$
\omega_k\cdot(x - \chi_k(t) + \tilde{q}_k c(t)) \leqslant -(\tilde{q}_k/|\tilde{q}_k|)\zeta_k(t) - (v_k t + w_k)\cdot\omega_k/t^2 \leqslant 6\epsilon \tag{1.4.91}
$$

if $-(\tilde{q}_k/|\tilde{q}_k|)\zeta_k(t) \leqslant 5\epsilon$, where $\omega_k = -(\tilde{q}_k/|\tilde{q}_k|)\omega$. This implies

$$
F_\epsilon(-(\tilde{q}_k|/\tilde{q}_k)|\zeta_k(t) \leqslant 5\epsilon) = F_\epsilon(-(\tilde{q}_k/|\tilde{q}_k)|\zeta_k(t) \leqslant 5\epsilon)F_\epsilon(\omega_k\cdot(x - \chi_k(t) + \tilde{q}_k c(t)) \leqslant 7\epsilon). \tag{1.4.92}
$$

In the case where $0 < \eta \leqslant 2$ with (1.4.4), if necessary retake $\epsilon > 0$ sufficiently small, then we obtain

$$\operatorname*{s-lim}_{t\to\infty} F_\epsilon(z_k/t^2 \leqslant 7\epsilon)U_k^{(2)}(t,0) = 0, \tag{1.4.93}$$

where $z_k = \omega_k \cdot x$, as we have seen (1.3.14) in subsection 1.3 by Proposition 1.2.2 and a density argument. In the case where $\eta > 2$, we also obtain (1.4.93) in the same way of (1.3.17) and (1.3.19) in subsection 1.3 by virtue of the boundedness of the energy (1.1.27) and Proposition 1.2.5. It follows from (1.4.90), (1.4.92) and (1.4.93) that

$$\begin{aligned}
&F_\epsilon(-(\tilde{q}_k/|\tilde{q}_k|)\zeta_k(t) \leqslant 5\epsilon)U_k(t,0)\phi \\
&= F_\epsilon(-(\tilde{q}_k/|\tilde{q}_k|)\zeta_k(t) \leqslant 5\epsilon)F_\epsilon(\omega_k\cdot(x-\chi_k(t)+\tilde{q}_k c(t)) \leqslant 7\epsilon) \\
&\qquad \times \mathscr{G}_k^{(1)}(t)\mathscr{G}_k^{(2)}(t)U_k^{(2)}(t,0)\mathscr{G}_k^{(2)}(0)^*\mathscr{G}_k^{(1)}(0)^*\phi \\
&= F_\epsilon(-(\tilde{q}_k/|\tilde{q}_k|)\zeta_k(t) \leqslant 5\epsilon)\mathscr{G}_k^{(1)}(t)\mathscr{G}_k^{(2)}(t)F_\epsilon(z_k/t^2 \leqslant 7\epsilon) \\
&\qquad \times U_k^{(2)}(t,0)\mathscr{G}_k^{(2)}(0)^*\mathscr{G}_k^{(1)}(0)^*\phi \\
&= o(1)
\end{aligned} \tag{1.4.94}$$

as $t \to \infty$ for $\phi \in L^2(\mathbb{R}^d)$. Thus we have obtained (1.4.81). □

2 Inverse Problem

2.1 Introduction

In the second section, we study one of the inverse scattering problems for quantum systems in a time-dependent electric field, which was obtained in Adachi-Fujiwara-Ishida [4]. As in the first section, we consider that the external electric field converging on constant in time, but the most important point of this section is that we can also consider the case where $E(t) \to 0$ as $t \to \pm\infty$. Throughout this section, we assume the space dimension $d \geqslant 2$. The free Hamiltonian was given by (1.1.2) in the first section. We firstly state the assumption of $E(t)$.

Assumption 2.1.1. *The time-dependent electric field $E(t) \in \mathbb{R}^d$ is represented as*

$$E(t) = E_0(1 + |t|)^{-\mu} + E_1(t), \tag{2.1.1}$$

where $0 \leqslant \mu < 1$, $E_0 \in \mathbb{R}^d \setminus \{0\}$ and $E_1(t) \in C(\mathbb{R}, \mathbb{R}^d)$ such that

$$\left| \int_0^t \int_0^s E_1(\tau) d\tau ds \right| \leqslant C \max\{|t|, |t|^{2-\mu_1}\} \tag{2.1.2}$$

with $\mu < \mu_1 \leqslant 1$.

Roughly speaking about the perturbation part $E_1(t)$, we assume that $|E_1(t)| \leqslant C(1 + |t|)^{-\mu_2}$ for some $\mu_2 > \mu$ and take μ_1 as follows,

$$\begin{cases} \mu_1 = \mu_2 & \mu < \mu_2 < 1 \\ \mu < \mu_1 < \mu_2 & \mu_2 = 1 \\ \mu_1 = 1 & \mu_2 > 1. \end{cases} \tag{2.1.3}$$

Such $E(t)$ was first dealt with in Adachi-Kamada-Kazuno-Toratani [6]. For brevity's sake, we suppose that $E_0 = e_1 = (1, 0, \ldots, 0) \in \mathbb{R}^d$. We next assume that the potential V is represented as $V = V^{\mathrm{vs}} + V^{\mathrm{s}} + V^{\mathrm{l}} \in \mathscr{V}^{\mathrm{vs}} + \mathscr{V}^{\mathrm{s}}_{\mu,\alpha_\mu} + \mathscr{V}^{\mathrm{l}}_{\mu,\gamma_\mu}$, where the classes $\mathscr{V}^{\mathrm{vs}}$, $\mathscr{V}^{\mathrm{s}}_{\mu,\alpha_\mu}$ and $\mathscr{V}^{\mathrm{l}}_{\mu,\gamma_\mu}$satisfy following assumption.

Assumption 2.1.2. *$\mathscr{V}^{\mathrm{vs}}$ is the class of real-valued multiplication operators V^{vs} is satisfying that V^{vs} is decomposed into a sum of a singular part V_1^{vs} and a regular part V_2^{vs}. V_1^{vs} is compactly supported, belongs to $L^{q_1}(\mathbb{R}^d)$ and satisfies $|\nabla V_1^{\mathrm{vs}}| \in L^{q_2}(\mathbb{R}^d)$. $V_2^{\mathrm{vs}} \in C^1(\mathbb{R}^d)$ satisfies that V_2^{vs} and its first derivatives are all bounded in $\mathbb{R}^d$ and that*

$$\int_0^\infty \|F(|x| \geqslant R) V_2^{\mathrm{vs}}(x)\|_{\mathscr{B}(L^2)} dR < \infty. \tag{2.1.4}$$

Here q_1 satisfies that $q_1 > d/2$ and $q_1 \geqslant 2$, q_2 satisfies

$$\begin{cases} 1/q_2 = 1/(2q_1) + 2/d & d \geqslant 5 \\ 1/q_2 < 1/(2q_1) + 1/2 & d = 4 \\ 1/q_2 = 1/(2q_1) + 1/2 & d \leqslant 3, \end{cases} \tag{2.1.5}$$

and $F(|x| \geqslant R)$ is the characteristic function of $\{x \in \mathbb{R}^d \mid |x| \geqslant R\}$.

$\mathscr{V}^{\mathrm{s}}_{\mu,\alpha_\mu}$ with some $\alpha_\mu > 0$ is the class of real-valued multiplication operators V^{s} is satisfying that V^{s} belongs to $C^1(\mathbb{R}^d)$ and satisfies

$$|V^{\mathrm{s}}(x)| \leqslant C\langle x\rangle^{-\gamma}, \quad |\partial_x^\beta V^{\mathrm{s}}(x)| \leqslant C_\beta \langle x\rangle^{-1-\alpha}, \quad |\beta| = 1 \tag{2.1.6}$$

with some γ and α such that $1/(2-\mu) < \gamma \leqslant 1$ and $\alpha_\mu < \alpha \leqslant \gamma$.

Finally, $\mathscr{V}^{\mathrm{l}}_{\mu,\gamma_\mu}$ with some $\gamma_\mu \geqslant 1/(2(2-\mu))$ is the class of real-valued multiplication operators V^{l} is satisfying that V^{l} belongs to $C^2(\mathbb{R}^d)$ and satisfies

$$|\partial_x^\beta V^{\mathrm{l}}(x)| \leqslant C\langle x\rangle^{-\gamma_D - |\beta|/(2-\mu)}, \quad |\beta| \leqslant 2, \tag{2.1.7}$$

with some γ_D such that $\gamma_\mu < \gamma_D \leqslant 1/(2-\mu)$.

We note that one can obtain

$$\int_0^\infty \|F(|x| \geqslant R)V^{\mathrm{vs}}(x)\langle p\rangle^{-2}\|_{\mathscr{B}(L^2)} dR < \infty \tag{2.1.8}$$

by this assumption and it is equivalent to

$$\int_0^\infty \|V^{\mathrm{vs}}(x)\langle p\rangle^{-2}F(|x| \geqslant R)\|_{\mathscr{B}(L^2)} dR < \infty \tag{2.1.9}$$

because V^{vs} is a multiplication operator (see e.g. Reed-Simon [34]).

As for the class $\mathscr{V}^{\mathrm{s}}_{\mu,\alpha_\mu}$, we also note that by virtue of $\alpha \leqslant \gamma$, we can treat an oscillation part. For example, the following function belongs to $\mathscr{V}^{\mathrm{s}}_{\mu,\alpha_\mu}$.

$$V^{\mathrm{s}}(x) = \langle x\rangle^{-\gamma}\cos\langle x\rangle^{\gamma-\alpha}. \tag{2.1.10}$$

In fact, we can verify easily that $|\nabla_x V^{\mathrm{s}}(x)| \leqslant C(\langle x\rangle^{-1-\gamma} + \langle x\rangle^{-1-\alpha}) \leqslant C\langle x\rangle^{-1-\alpha}$ holds with some $C > 0$.

For $V \in \mathscr{V}^{\mathrm{vs}} + \mathscr{V}^{\mathrm{s}}_{\mu,\alpha_\mu} + \mathscr{V}^{\mathrm{l}}_{\mu,\gamma_\mu}$, it is known that the full Hamiltonian

$$H(t) = H_0(t) + V \tag{2.1.11}$$

is self-adjoint on $L^2(\mathbb{R}^d)$ and that the unitary propagator $U(t,s)$ generated by $H(t)$ exists uniquely by the results of Yajima [44] and the Avron-Herbst formula (1.1.8), defining $\tilde{b(t)}$, $\tilde{c(t)}$ and $\tilde{a(t)}$ as in (1.1.6).

We first consider the case where $V^{\mathrm{l}} = 0$. Then the wave operators

$$W^{\pm} = \operatorname*{s\text{-}lim}_{t \to \pm\infty} U(t,0)^* U_0(t,0) \tag{2.1.12}$$

exist as this fact was shown in [6] (see subsection 2.2). In the case where $\mu = 0$, the asymptotic completeness of $W^{\pm}$ was discussed in Yokoyama [46] and Adachi-Ishida [5] (see the first section), although that in the case $0 < \mu < 1$ has never been discussed, to our knowledge. Then the scattering operator $S = S(V)$ is defined by

$$S = (W^+)^* W^-. \tag{2.1.13}$$

The following obtained in [4] is one of those which we would like to report in this section.

Theorem 2.1.3. (Uniqueness of Short-range Potentials [4]) *Put*

$$\tilde{\alpha}_\mu = \begin{cases} \dfrac{7 - 3\mu - \sqrt{(1-\mu)(17-9\mu)}}{4(2-\mu)} & 0 \leqslant \mu \leqslant 1/2 \\ \dfrac{1+\mu}{2(2-\mu)} & 1/2 < \mu < 1. \end{cases} \tag{2.1.14}$$

Let $V_1, V_2 \in \mathscr{V}^{\mathrm{vs}} + \mathscr{V}^{\mathrm{s}}_{\mu,\tilde{\alpha}_\mu}$*. If* $S(V_1) = S(V_2)$*, then* $V_1 = V_2$*.*

In fact, the result corresponding to this theorem was already obtained under the assumption that $V_1, V_2 \in \mathscr{V}^{\mathrm{vs}} + \mathscr{V}^{\mathrm{s}}_{\mu,1/(2-\mu)}$ (see Theorem 1.1 of [6]). $\tilde{\alpha}_\mu < 1/(2-\mu)$ implies that above result is finer than the previous one. Here we note that if $a < b$, then $\mathscr{V}^{\mathrm{s}}_{\mu,b} \subsetneq \mathscr{V}^{\mathrm{s}}_{\mu,a}$.

The scattering operators can determine uniquely the potentials which belong to a certain class of short-range potentials. It is well-known that V^{vs} is short-range and V^{s} is long-range in the absence of the external electric field. Even in the absence of the external electric field, a potential belonging to $\mathscr{V}^{\mathrm{vs}}$ can be determined by the associated scattering operator (see e.g. Enss-Weder [13]). Thus we are interested in the influence of external electric fields $E(t)$ in the unique determination of V^{s} by the scattering operator S.

In the case where $E(t) \equiv E_0$, that is, the case of the Stark effect, this theorem was first proved by Weder [38] under the condition $V^{\mathrm{s}} \in \mathscr{V}^{\mathrm{s}}_{0,0}$ and the additional assumption $\gamma > 3/4$. However, as it is well-known, the short-range condition on V under the Stark effect is $\gamma > 1/2$. Later Nicoleau [31] proved this theorem for real-valued $V \in C^\infty(\mathbb{R}^d)$ satisfying $|\partial_x^\beta V(x)| \leqslant C_\beta \langle x \rangle^{-\gamma - |\beta|}$ with $\gamma > 1/2$, under the condition $d \geqslant 3$. After that, this theorem was obtained by Adachi-Maehara [8] under the condition that $V^{\mathrm{s}} \in \mathscr{V}^{\mathrm{s}}_{0,1/2}$. Moreover, recently Valencia-Weder [37] obtained an extention of the results of the two-body case to the N-body case (see also [38]) by using the method similar to the one in [8]

In dealing with the Stark short-range part V^{s} of V for the sake of improving the result of [38], the Dollard-type modifier $e^{-i\int_0^t V^{\mathrm{s}}(p_\perp\tau+e_1\tau^2/2)d\tau}$ due to White [40] (see also Adachi-Tamura [9] and Jensen-Yajima [27]) was introduced in [31], where $p_\perp=(p_1,p_\perp)$. This modifier was also used in Nicoleau [32] in the study of the case where $E(t)$ is periodic in t with non-zero mean E_0 in t. The assumption $d-1\geqslant 2$ is needed for the method of Nicoleau. On the other hand, in [8], in stead of $e^{-i\int_0^t V^{\mathrm{s}}(p_\perp\tau+e_1\tau^2/2)d\tau}$, the v-dependent Graf-type (or Zorbas-type) modifier $e^{-i\int_0^t V^{\mathrm{s}}(v\tau+e_1\tau^2/2)d\tau}$ was introduced (in Graf [17] and Zorbas [49], it is supposed that $v=0$). By virtue of this device, one can deal with the case where $d\geqslant 2$ and relax the smoothness condition on potentials supposed in [31].

The method of [8] has been used for the study of the cases where $|E(t)-E_0|\leqslant C(1+|t|)^{-\mu_2}$ with $\mu_2>1$ (see Toyoda [36]), where $E(t)$ is periodic in t with non-zero mean E_0 as in [32] (see Remark 2.1.5, Theorem 2.1.6 and Fujiwara [14]), and where $E(t)$ is given by (2.1.1)(see [6]). The present work obtained by [4] is a continuation of [14] and [6].

As for the case of the time-dependent potentials in the absence of external electric field, see Weder [39], which is the first work where the Enss-Weder time-dependent method [13] has been applied to time-dependent potentials.

We next consider the case where $V^{\mathrm{l}}\neq 0$. If $V^{\mathrm{l}}\in\mathscr{V}^{\mathrm{l}}_{\mu,\gamma_\mu}$, the Dollard-type modified wave opearators

$$W_D^{\pm}=\operatorname*{s-lim}_{t\to\pm\infty}U(t,0)^*U_0(t,0)M_D(t),\quad M_D(t)=e^{-i\int_0^t V^{\mathrm{l}}(p\tau+\tilde{c}(\tau))d\tau} \tag{2.1.15}$$

exist by virtue of the condition $\gamma_D>1/(2(2-\mu))$ (see subsection 2.2 and also [6]). Then the Dollard-type modified scattering operator $S_D=S_D(V^{\mathrm{l}},V^{\mathrm{vs}}+V^{\mathrm{s}})$ is defined by

$$S_D=(W_D^+)^*W_D^-. \tag{2.1.16}$$

Then we also report the following result.

Theorem 2.1.4. (Uniqueness of Short-range Potentials [4]) *Suppose that a given V^{l} satisfies $V^{\mathrm{l}}\in\mathscr{V}^{\mathrm{l}}_{\mu,\tilde{\gamma}_\mu}$ with*

$$\tilde{\gamma_\mu}=\frac{1}{2(2-\mu)}+\frac{1-\mu}{2(2-\mu)}. \tag{2.1.17}$$

Put

$$\tilde{\alpha}_{\mu,D}=\begin{cases}\dfrac{15-5\mu-\sqrt{(1-\mu)(41-25\mu)}}{8(2-\mu)} & 0\leqslant\mu\leqslant 5/7\\[2ex] \dfrac{1+\mu}{2(2-\mu)} & 5/7<\mu<1.\end{cases} \tag{2.1.18}$$

Let $V_1,V_2\in\mathscr{V}^{\mathrm{vs}}+\mathscr{V}^{\mathrm{s}}_{\mu,\tilde{\alpha}_{\mu,D}}$. If $S_D(V^{\mathrm{l}},V_1)=S(V^{\mathrm{l}},V_2)$, then $V_1=V_2$. Moreover, any one of the Dollard-type modified scattering operators S_D determines uniquely the total potential V.

It is obvious that this theorem is an improvement of Theorem 1.2 of [6]. In [6], if $0 < \mu \leqslant (7-\sqrt{3}-\sqrt{60-22\sqrt{3}})/4$, then $\tilde{\gamma}_\mu$ was replaced by $(-1/2+\sqrt{1/4+(1-\mu)^2/(1+\mu)})/(2-\mu)$, while, if $(7-\sqrt{3}-\sqrt{60-22\sqrt{3}})/4 < \mu < 1$, then $\tilde{\gamma}_\mu$ was replaced by $-(3-\mu)/8+\sqrt{(3-\mu)^2/64+(2\mu^2-7\mu+7)/(4(2-\mu)^2)}$. $\tilde{\alpha}_{\mu,D}$ was replaced by $1/(2-\mu)$, which is strictly greater than $\tilde{\alpha}_{\mu,D}$ in (2.1.18). In particular, there was no result for the case where $\mu = 0$.

Here we emphasize that if $5/7 \leqslant \mu < 1$, then $\tilde{\alpha}_{\mu,D} = \tilde{\alpha}_\mu$ holds, although if $0 \leqslant \mu < 5/7$, then $\tilde{\alpha}_{\mu,D} > \tilde{\alpha}_\mu$ holds.

Remark 2.1.5. *We assume that $E(t) \in C(\mathbb{R},\mathbb{R}^d)$ is T-periodic in time with non-zero mean E_0, that is,*

$$E_0 = \int_0^T E(\tau)d\tau/T \neq 0, \tag{2.1.19}$$

which was treated by Nicoleau [32] and Fujiwara [14]. In this case, the method in the proofs of Theorems 2.1.3 and 2.1.4 does work well also, because we have

$$|\tilde{b}(t) - tE_0| \leqslant \int_0^T |E(\tau) - E_0| d\tau, \tag{2.1.20}$$

$$|\tilde{c}(t) - t^2E_0/2| \leqslant \int_0^{|t|} |\tilde{b}(\tau) - \tau E_0| d\tau \leqslant C|t|, \tag{2.1.21}$$

with $C = \int_0^T |E(\tau) - E_0| d\tau$ by the periodicity of $E(t)$. (2.1.21) *implies $\mu = 0$ in* (2.1.1) *and $\mu_1 = 1$ in* (2.1.2)*.*

By virtue of this fact, one can obtain an improvement of the results of [32] and [14].

Theorem 2.1.6. *Suppose that $E(t) \in C(\mathbb{R},\mathbb{R}^d)$ is T-periodic in time with non-zero mean E_0. Then the followings hold.*

1. *Let $V_1, V_2 \in \mathscr{V}^{\mathrm{vs}} + \mathscr{V}^{\mathrm{s}}_{0,\tilde{\alpha}_0}$. If $S(V_1) = S(V_2)$, then $V_1 = V_2$.*
2. *Suppose that a given V^1 satisfies $V^1 \in \mathscr{V}^1_{0,\tilde{\gamma}_0}$. Let $V_1, V_2 \in \mathscr{V}^{\mathrm{vs}} + \mathscr{V}^{\mathrm{s}}_{0,\tilde{\alpha}_{0,D}}$. If $S_D(V^1, V_1) = S_D(V^1, V_2)$, then $V_1 = V_2$. Moreover, any one of the Dollard-type modified scattering operators S_D determines uniquely the total potential V.*

The plan of this section is as follows. In subsection 2.2, we recall some useful properties of $U_0(t,0)$ obtained by [6]. In subsection 2.3, we consider the case where $V^1 = 0$. In subsection 2.4, we consider the general case.

In the following subsections, we always suppose $\alpha \leqslant \gamma < 1$ without loss of generality, for the sake of simplicity.

2.2 Preliminaries

In this subsection, we will recall some propagation estimates for the free propagator $U_0(t,0)$ and the existence of the wave operators $W^{\pm}$ and the modified wave operators $W_D^{\pm}$, which were already obtained in [6].

Proposition 2.2.1. *Let $\phi \in \mathscr{S}(\mathbb{R}^d)$. Then*

$$\|(p-\tilde{b}(t))_j U_0(t,0)\phi\| = O(1), \tag{2.2.1}$$

$$\|(x-\tilde{c}(t))_j U_0(t,0)\phi\| = O(|t|), \tag{2.2.2}$$

$$\|(x-\tilde{c}(t)-t(p-\tilde{b}(t)))_j U_0(t,0)\phi\| = O(1) \tag{2.2.3}$$

hold as $|t| \to \infty$ for $1 \leqslant j \leqslant d$.

Proof. This proposition was proved in [6]. Recall the Avron-Herbst formula (1.1.8) for the free propagator, that is,

$$U_0(t,0) = e^{-i\tilde{a}(t)}e^{i\tilde{b}(t)}e^{-i\tilde{c}(t)}e^{-itp^2/2}. \tag{2.2.4}$$

Then we only have to use

$$U_0(t,0)^*(p-\tilde{b}(t))U_0(t,0) = p, \quad U_0(t,0)^*(x-\tilde{c}(t))U_0(t,0) = x+pt \tag{2.2.5}$$

because $\phi \in \mathscr{D}(p)\bigcap\mathscr{D}(x)$. $\square$

According to the argument of [4] and [6], we decompose $\tilde{c}(t)$ into sum of $\tilde{c}_0(t)$ and $\tilde{c}_1(t)$. We note that for $t \geqslant 0$,

$$\int_0^t\int_0^s (1+|\tau|)^{-\mu}d\tau ds = ((1+t)^{2-\mu}-1)/((1-\mu)(2-\mu)) - t/(1-\mu) \tag{2.2.6}$$

holds, and that for $t<0$,

$$\int_0^t\int_0^s (1+|\tau|)^{-\mu}d\tau ds = ((1-t)^{2-\mu}-1)/((1-\mu)(2-\mu)) + t/(1-\mu) \tag{2.2.7}$$

holds. By taking account of these, we put $\tilde{c}_0(t)$ and $\tilde{c}_1(t)$ as

$$\tilde{c}_0(t) = ((1+|t|)^{2-\mu}-1)/((1-\mu)(2-\mu))e_1, \tag{2.2.8}$$

$$\tilde{c}_1(t) = -|t|/(1-\mu)e_1 + \int_0^t\int_0^s E_1(\tau)d\tau ds, \tag{2.2.9}$$

with $E_0 = e_1$. This decomposition will be used frequently in the argument bellow. We will write $1/((1-\mu)(2-\mu))$ as a_μ for brevity's sake. Here we note that

$$|\tilde{c}_0(t)| = a_\mu((1+|t|)^{2-\mu}-1) \geqslant a_\mu|t|^{2-\mu}, \tag{2.2.10}$$

$$|\tilde{c}_1(t)| \leqslant M_1\max\{|t|,|t|^{2-\mu_1}\} \tag{2.2.11}$$

hold with some $M_1>0$ for any $t\in\mathbb{R}$ by $2-\mu>1$ and assumption.

Proposition 2.2.2. *Let $\phi \in \mathscr{S}(\mathbb{R}^d)$ be such that $\hat{\phi} \in C_0^\infty$. Then*

$$\|(p - \tilde{b}(t))_j U_0(t,0) M_D(t)\phi\| = O(1), \tag{2.2.12}$$

$$\|(x - \tilde{c}(t))_j U_0(t,0) M_D(t)\phi\| = O(|t|), \tag{2.2.13}$$

$$\|(x - \tilde{c}(t) - t(p - \tilde{b}(t)))_j U_0(t,0) M_D(t)\phi\| = O(|t|^{1-\gamma_D(2-\mu)}) \tag{2.2.14}$$

hold as $|t| \to \infty$ for $1 \leqslant j \leqslant d$.

Proof. Noting $M_D(t)^* p M_D(t) = p$, one can obtain (2.2.12) by Proposition 2.2.1. As for (2.2.13) and (2.2.14), we note that

$$M_D(t)^* x M_D(t) = x + \int_0^t \tau(\nabla_x V^{\mathrm{l}})(p\tau + \tilde{c}(\tau))d\tau. \tag{2.2.15}$$

By (2.2.10) and (2.2.11), there exists a constant $C > 0$ such that

$$|\tau\xi + \tilde{c}(\tau)| \geqslant |\tilde{c}_0(\tau)| - |\tilde{c}_1(\tau)| - |\tau||\xi| \geqslant a_\mu|\tau|^{2-\mu} - (M_1 + C)\max\{|\tau|, |\tau|^{2-\mu_1}\} \tag{2.2.16}$$

holds for $\xi \in \operatorname{supp}\hat{\phi}$. Since, in particular,

$$|\tau\xi + \tilde{c}(\tau)| \geqslant a_\mu|\tau|^{2-\mu}/2 \tag{2.2.17}$$

holds $\xi \in \operatorname{supp}\hat{\phi}$ and $|\tau| \geqslant (2(M_1 + C)/a_\mu)^{1/(\mu_1-\mu)} \geqslant 1$ by $\mu < \mu_1 \leqslant 1$, one has

$$\|\int_0^t \tau(\nabla_x V^{\mathrm{l}})(p\tau + \tilde{c}(\tau))d\tau\phi\| = O(|t|^{1-\gamma_D(2-\mu)}), \tag{2.2.18}$$

noting $\gamma_D(2-\mu) < 1$. This implies the proposition. □

Proposition 2.2.3. *1. If $V \in \mathscr{V}^{\mathrm{vs}} + \mathscr{V}^{\mathrm{s}}_{\mu,0}$, then $W^\pm$ exist.*

2. If $V \in \mathscr{V}^{\mathrm{vs}} + \mathscr{V}^{\mathrm{s}}_{\mu,0} + \mathscr{V}^{\mathrm{l}}_{\mu,1/(2(2-\mu))}$, then $W_D^\pm$ exist.

Proof. We will prove the existence of $W_D^\pm$ for $V \in \mathscr{V}^{\mathrm{vs}} + \mathscr{V}^{\mathrm{s}}_{\mu,0} + \mathscr{V}^{\mathrm{l}}_{\mu,1/(2(2-\mu))}$, since the existence of $W^\pm$ can be shown quite similarly. We dealt with $t \to \infty$ only. Let $J \in C_0^\infty(\mathbb{R}^d)$ be such that $0 \leqslant J \leqslant 1$,

$$J(x) = \begin{cases} 1 & |x| \leqslant a_\mu/8 \\ 0 & |x| \geqslant a_\mu/4. \end{cases} \tag{2.2.19}$$

Put $\tilde{p}(t) = p - \tilde{b}(t)$, $\tilde{x}(t) = x - \tilde{c}(t)$, and write $J(\tilde{x}(t)/t^{2-\mu})$ as $J(t,x)$ for brevity's sake. Let $\phi \in \mathscr{S}(\mathbb{R}^d)$ be such that $\hat{\phi} \in C_0^\infty(\mathbb{R}^d)$. Noting that $|\tilde{x}(t)| \geqslant a_\mu/(8t^{2-\mu})$ holds on the supprt of $1 - J(t,x)$, we have

$$a_\mu t^{2-\mu}\|(1 - J(t,x))U_0(t,0)M_D(t)\phi\|/8 \leqslant \|(1 - J(t,x))|\tilde{x}(t)|U_0(t,0)M_D(t)\phi\| \leqslant Ct \tag{2.2.20}$$

by Proposition 2.2.2. This implies

$$\lim_{t\to\infty} U(t,0)^*(1-J(t,x))U_0(t,0)M_D(t)\phi = 0 \tag{2.2.21}$$

because of $2-\mu>1$. Then we only have to prove the existence of

$$\lim_{t\to\infty} U(t,0)^*J(t,x)U_0(t,0)M_D(t)\phi \tag{2.2.22}$$

by a density argument. Let $\hat{J}\in C_0^\infty(\mathbb{R}^d)$ be such that $0\leqslant \hat{J}\leqslant 1$,

$$\hat{J}(x) = \begin{cases} 1 & |x|\leqslant a_\mu/4 \\ 0 & |x|\geqslant a_\mu/2. \end{cases} \tag{2.2.23}$$

Put $V^{\mathrm{l}}(t,x) = V^{\mathrm{l}}(x)\hat{J}(\tilde{x}(t)/t^{2-\mu})$. Then we have $J(t,x)V^{\mathrm{l}}(x) = J(t,x)V^{\mathrm{l}}(t,x)$ because $J\hat{J}=J$. Moreover, we see that

$$V^{\mathrm{l}}(pt+\tilde{c}(t))M_D(t)\phi = V^{\mathrm{l}}(t,pt+\tilde{c}(t))M_D(t)\phi \tag{2.2.24}$$

as $t\to\infty$ holds because by $\mu<1$, $t|\xi|/t^{2-\mu}\leqslant a_\mu/4$ for $\xi\in\operatorname{supp}\hat{\phi}$. Then we compute

$$\begin{aligned}\partial_t(U(t,0)^*J(t,x)U_0(t,0)M_D(t))\phi &= U(t,0)^*(\mathbb{D}_{H_0(t)}J(t,x))U_0(t,0)M_D(t)\phi \\ &+iU(t,0)^*J(t,x)(V^{\mathrm{vs}}(x)+V^{\mathrm{s}}(x))U_0(t,0)M_D(t)\phi \\ &+iU(t,0)^*J(t,x)(V^{\mathrm{l}}(t,x)-V^{\mathrm{l}}(t,t\tilde{p}(t)+\tilde{c}(t)))U_0(t,0)M_D(t)\phi, \end{aligned}\tag{2.2.25}$$

where we used $U_0(t,0)V^{\mathrm{l}}(t,pt+\tilde{c}(t)) = V^{\mathrm{l}}(t,t\tilde{p}(t)+\tilde{c}(t))U_0(t,0)$ by the Avron-Herbst formula (2.2.4). It follows from

$$\begin{aligned}\mathbb{D}_{H_0(t)}J(t,x) &= (\nabla_x J)(t,x)\cdot(\tilde{p}(t))/t^{2-\mu} \\ &-(2-\mu)(\nabla_x J)(t,x)\cdot(\tilde{x}(t))/t^{3-\mu} - i(\nabla_x J)(t,x)/(2t^{2(2-\mu)})\end{aligned} \tag{2.2.26}$$

that

$$\|(\mathbb{D}_{H_0(t)}J(t,x))U_0(t,0)M_D(t)\phi\| \leqslant C_1/t^{2-\mu} \tag{2.2.27}$$

with some $C_1>0$ by Proposition 2.2.2. If $|\tilde{x}(t)|\leqslant a_\mu t^{2-\mu}/4$ holds, then we can estimate

$$\begin{aligned}|x| &\geqslant |\tilde{c}(t)| - |\tilde{x}(t)| \geqslant |\tilde{c}_0(t)| - |\tilde{c}_1(t)| - a_\mu t^{2-\mu}/4 \\ &\geqslant 3a_\mu t^{2-\mu}/4 - M_1\max\{t, t^{2-\mu_1}\} \geqslant a_\mu t^{2-\mu}/2\end{aligned} \tag{2.2.28}$$

as $t\to\infty$ by (2.2.10), (2.2.11) and $\mu<\mu_1\leqslant 1$. This implies

$$J(t,x)V^{\mathrm{s}}(x) = O(t^{-\gamma(2-\mu)}), \quad J(t,x)V_1^{\mathrm{vs}}(x) = 0, \tag{2.2.29}$$

$$\|J(t,x)V_2^{\mathrm{vs}}(x)U_0(t,0)M_D(t)\phi\| \leqslant \|F(|x|\geqslant a_\mu t^{2-\mu}/2)V_2^{\mathrm{vs}}(x)\|_{\mathscr{B}(L^2)}\|\phi\|. \tag{2.2.30}$$

We thus obtain

$$\begin{aligned}&\|(V^{\mathrm{vs}}(x)+V^{\mathrm{s}}(x))J(t,x)U_0(t,0)\phi\| \\ &\quad \leqslant C_2(t^{-\gamma(2-\mu)}+\|F(|x|\geqslant a_\mu t^{2-\mu}/2)V_2^{\mathrm{vs}}(x)\|_{\mathscr{B}(L^2)}) \end{aligned} \tag{2.2.31}$$

with some C_2. Finally, by virtue of the Baker-Campbell-Hausdorff formula, we have

$$\begin{aligned}&V^{\mathrm{l}}(t,x)-V^{\mathrm{l}}(t,t\tilde{p}(t)+\tilde{c}(t)) \\ &\quad =\int_0^1(\nabla_x V^{\mathrm{l}})(t,t\tilde{p}(t)+\tilde{c}(t)+\theta(\tilde{x}(t)-t\tilde{p}(t)))d\theta\cdot(\tilde{x}(t)-t\tilde{p}(t)) \\ &\quad\quad -it\int_0^1(1-\theta)(\triangle_x V^{\mathrm{l}})(t,t\tilde{p}(t)+\tilde{c}(t)+\theta(\tilde{x}(t)-t\tilde{p}(t)))d\theta/2. \end{aligned} \tag{2.2.32}$$

Since

$$\sup_{x\in\mathbb{R}^d}|\nabla_x V^{\mathrm{l}}(t,x)|=O(t^{-1-\gamma_D(2-\mu)}),\quad \sup_{x\in\mathbb{R}^d}|\triangle_x V^{\mathrm{l}}(t,x)|=O(t^{-2-\gamma_D(2-\mu)}) \tag{2.2.33}$$

hold (cf. (2.2.28)), we obtain

$$\|(V^{\mathrm{l}}(t,x)-V^{\mathrm{l}}(t,t\tilde{p}(t)+\tilde{c}(t)))U_0(t,0)M_D(t)\phi\|\leqslant C_3(t^{-2\gamma_D(2-\mu)}+t^{-1-\gamma_D(2-\mu)}) \tag{2.2.34}$$

with some $C_3>0$ by Proposition 2.2.2. Combine (2.2.27), (2.2.31) and (2.2.34). Then we have

$$\begin{aligned}&\|\partial_t(U(t,0)^*J(t,x)U_0(t,0))M_D(t)\phi\| \\ &\quad \leqslant (C_1+C_2)t^{-\gamma(2-\mu)}+C_3t^{-2\gamma_D(2-\mu)}+C_2\|F(|x|\geqslant a_\mu t^{2-\mu}/2)V_2^{\mathrm{vs}}(x)\|, \end{aligned} \tag{2.2.35}$$

here used $\gamma(2-\mu)\leqslant 2-\mu$ and $\gamma_D(2-\mu)\leqslant 1$. This implies the existence of (2.2.22) by the Cook-Kuroda method, because $-\gamma(2-\mu)<-1$, $-2\gamma_D(2-\mu)<-1$ and (2.1.4). □

To our knowledge, the problem of the asymptotic completeness of the wave operators is still open in this case. However, it is well known that utilizing the Enss-Weder method, one needs their existence only.

2.3 Short-range Case

The main purpose of this subsection is showing the following reconstruction formula, which yeilds the proof of Theorem 2.1.3.

Theorem 2.3.1. (Reconstruction Formula [4]) *Let $\hat{v} \in \mathbb{R}^d$ be given such that $|\hat{v} \cdot e_1| < 1$. Put $v = |v|\hat{v}$. Let $\eta > 0$ be given, and $\Phi_0, \Psi_0 \in \mathscr{S}(\mathbb{R}^d)$ be such that $\hat{\Phi}_0, \hat{\Psi}_0 \in C_0^\infty(\mathbb{R}^d)$ with $\operatorname{supp}\hat{\Phi}_0, \operatorname{supp}\hat{\Psi}_0 \subset \{\xi \in \mathbb{R}^d \bigm| |\xi| \leqslant \eta\}$. Put $\Phi_v = e^{iv\cdot x}\Phi_0, \Psi_v = e^{iv\cdot x}\Psi_0$. Let $V^{\mathrm{vs}} \in \mathscr{V}^{\mathrm{vs}}$ and $V^{\mathrm{s}} \in \mathscr{V}^{\mathrm{s}}_{\mu,\tilde{\alpha}_\mu}$, where $\tilde{\alpha}_\mu$ is the same as in Theorem 2.1.3. Then*

$$|v|(i[S,p_j]\Phi_v,\Psi_v) = \int_{-\infty}^{\infty}((V^{\mathrm{vs}}(x+\hat{v}t)p_j\Phi_0,\Psi_0) - (V^{\mathrm{vs}}(x+\hat{v}t)\Phi_0,p_j\Psi_0) + (i(\partial_{x_j}V^{\mathrm{s}})(x+\hat{v}t)\Phi_0,\Psi_0))dt + o(1) \tag{2.3.1}$$

holds as $|v| \to \infty$ for $1 \leqslant j \leqslant d$.

We will make preparations for the proof of Theorem 1.1.7. We first need the following proposition due to Enss [12] (see Proposition 2.10 in [12]).

Proposition 2.3.2. *For any $f \in C_0^\infty(\mathbb{R}^d)$ with $\operatorname{supp} f \subset \{x \in \mathbb{R}^d \bigm| |x| \leqslant \eta\}$ for some $\eta > 0$ and any $N \in \mathbb{N}$, there exists a constant C_N dependent on f only such that*

$$\|F(x \in \mathscr{M}')e^{-itp^2/2}f(p)F(x \in \mathscr{M})\| \leqslant C_N\langle r + |t|\rangle^{-N} \tag{2.3.2}$$

for $t \in \mathbb{R}$ and measurable sets $\mathscr{M}'$, $\mathscr{M}$ with the property that $r = \operatorname{dist}(\mathscr{M}',\mathscr{M}) - \eta|t| \geqslant 0$. Here $F(x \in \mathscr{M})$ stands for the characteristic function of $\mathscr{M}$.

The following proposition, which was already obtained in [6] (see Lemma 3.6 of [6]), can be proved as in [38].

Proposition 2.3.3. *Let v and Φ_v be as in Theorem 2.3.1, then*

$$\int_{-\infty}^{\infty}\|V^{\mathrm{vs}}(x)U_0(t,0)\Phi_v\|dt = O(|v|^{-1}) \tag{2.3.3}$$

holds as $|v| \to \infty$ for $V^{\mathrm{vs}} \in \mathscr{V}^{\mathrm{vs}}$.

Proof. We will sketch the proof. Take $f \in C_0^\infty(\mathbb{R}^d)$ such that $f\hat{\Phi}_0 = \hat{\Phi}_0$ and $\operatorname{supp} f \subset \{\xi \in \mathbb{R}^d \bigm| |\xi| \leqslant \eta\}$. By the Avron-Herbst formula (2.2.4) and the relation that

$$e^{-iv\cdot x}e^{-itp^2/2}e^{iv\cdot x} = e^{-itv^2/2}e^{-itp\cdot v}e^{-itp^2/2}, \tag{2.3.4}$$

we compute

$$\|V^{\mathrm{vs}}(x)U_0(t,0)\Phi_v\| = \|V^{\mathrm{vs}}(x+vt+\tilde{c}(t))e^{-itp^2/2}\Phi_0\| \leqslant I_1 + I_2 + I_3. \tag{2.3.5}$$

We have set I_1, I_2 and I_3 in (2.3.5) as

$$I_1 = \|V^{\mathrm{vs}}(x+vt+\tilde{c}(t))\langle p\rangle^{-2}\|_{\mathscr{B}(L^2)}\|\langle p\rangle^2\Phi_0\| \times\|F(|x| \geqslant 3\lambda|v||t|)e^{-itp^2/2}f(p)F(|x| < \lambda|v||t|)\|_{\mathscr{B}(L^2)}, \tag{2.3.6}$$

$$I_2 = \|V^{\mathrm{vs}}(x+vt+\tilde{c}(t))\langle p\rangle^{-2}\|_{\mathscr{B}(L^2)}\|\langle x\rangle^2\langle p\rangle^2\Phi_0\| \times\|F(|x| \geqslant 3\lambda|v||t|)e^{-itp^2/2}f(p)F(|x| \geqslant \lambda|v||t|)\langle x\rangle^{-2}\|_{\mathscr{B}(L^2)}, \tag{2.3.7}$$

$$I_3 = \|V^{\mathrm{vs}}(x+vt+\tilde{c}(t))\langle p\rangle^{-2}F(|x| < 3\lambda|v||t|)\|_{\mathscr{B}(L^2)}\|e^{-itp^2/2}f(p)\langle p\rangle^2\Phi_0\|, \tag{2.3.8}$$

with $\lambda > 0$. According to Weder[38], λ is independent of $|v|$, will be determined below. Since $\|V^{\rm vs}(x+vt+\tilde{c}(t))\langle p\rangle^{-2}\| = \|V^{\rm vs}(x)\langle p\rangle^{-2}\|$ is bounded by assumption, we have

$$I_1 + I_2 \leqslant C\langle |v||t|\rangle^{-2} \tag{2.3.9}$$

for $\lambda|v| \geqslant \eta$, using Proposition 2.3.2 for estimating I_1. As for I_3, we note that

$$\begin{aligned} &\|V^{\rm vs}(x+vt+\tilde{c}(t))\langle p\rangle^{-2}F(|x| < 3\lambda|v||t|)\|_{\mathscr{B}(L^2)} \\ &= \|V^{\rm vs}(x)\langle p\rangle^{-2}F(|x-(vt+\tilde{c}(t))| < 3\lambda|v||t|)\|_{\mathscr{B}(L^2)}. \end{aligned} \tag{2.3.10}$$

Put $\delta = |\hat{v}\cdot e_1| < 1$. It follows from (2.2.10) and (2.2.11) that

$$|\tilde{c}_1(t)|/|\tilde{c}_0(t)| \leqslant M_1|t|^{\mu-\mu_1}/a_\mu \leqslant (1-\delta)/3 \tag{2.3.11}$$

for $|t| \geqslant T_\delta$ by $\mu < \mu_1$, where $T_\delta = \max\{1, (3M_1/(a_\mu(1-\delta))^{1/(\mu_1-\mu)}\}$. Then we have

$$\begin{aligned} |vt+\tilde{c}(t)|^2 &= |v|^2|t|^2 + |\tilde{c}_0(t)+\tilde{c}_1(t)|^2 + 2vt\cdot(\tilde{c}_0(t)+\tilde{c}_1(t)) \\ &\geqslant |v|^2|t|^2 + (1-(1-\delta)/3)^2|\tilde{c}_0(t)|^2 - 2(\delta+(1-\delta)/3)|v||t||\tilde{c}_0(t)| \\ &= ((2+\delta)|\tilde{c}_0(t)|/3 - (1+2\delta)|v||t|/(2+\delta))^2 + 3(1-\delta^2)|v|^2|t|^2/(2+\delta)^2 \\ &\geqslant 3(1-\delta^2)|v|^2|t|^2/(2+\delta)^2 \geqslant (1-\delta^2)|v|^2|t|^2/3 \end{aligned} \tag{2.3.12}$$

for $|t| \geqslant T_\delta$. On the other hand, for $|t| < T_\delta$, $|\tilde{c}_1(t)| \leqslant M_2|t|$ holds with $M_2 = M_1T_\delta^{1-\mu_1}$. Hence, we have or $|t| < T_\delta$ and $M_2/|v| \leqslant (1-\delta)/3$

$$\begin{aligned} |vt+\tilde{c}(t)|^2 &= |vt+\tilde{c}_1(t)|^2 + |\tilde{c}_0(t)|^2 + 2(vt+\tilde{c}_1(t))\cdot\tilde{c}_0(t) \\ &\geqslant (1-(1-\delta)/3)^2|v|^2|t|^2 + |\tilde{c}_0(t)|^2 - 2(\delta+(1-\delta)/3)|v||t||\tilde{c}_0(t)| \\ &= (|\tilde{c}_0(t)| - (1+2\delta)|v||t|/3)^2 + (1-\delta^2)|v|^2|t|^2/3 \geqslant (1-\delta^2)|v|^2|t|^2/3. \end{aligned} \tag{2.3.13}$$

Summing up, for $|v| \geqslant 3M_2/(1-\delta)$,

$$|vt+\tilde{c}(t)| \geqslant ((1-\delta^2)/3)^{1/2}|v||t| \tag{2.3.14}$$

holds. Take λ as $4\lambda = ((1-\delta^2)/3)^{1/2} > 0$. Then we have

$$F(|x-(vt+\tilde{c}(t))| < 3\lambda|v||t|) = F(|x-(vt+\tilde{c}(t))| < 3\lambda|v||t|)F(|x| \geqslant \lambda|v||t|) \tag{2.3.15}$$

and

$$I_3 \leqslant \|V^{\rm vs}(x)\langle p\rangle^{-2}F(|x| \geqslant \lambda|v||t|)\|_{\mathscr{B}(L^2)}\|\langle p\rangle^2\Phi_0\|. \tag{2.3.16}$$

Therefore, we obtain

$$\int_{-\infty}^{\infty}(I_1+I_2+I_3)dt = O(|v|^{-1}) \tag{2.3.17}$$

by assumption, which implies the proposition. □

The following proposition is an improvement of Lemma 3.4 of [6], which is the key in this subsection.

Proposition 2.3.4. *Let v and Φ_v be as in Theorem 2.3.1 and $\epsilon > 0$. Put*

$$\Theta_0(\alpha) = \begin{cases} -\alpha - \dfrac{(\alpha-\mu)(1-\alpha)}{(1-\mu)(2-\alpha)} & \alpha > \mu \\ -\alpha + \dfrac{\mu-\alpha}{1-\mu} & \mu/(2-\mu) < \alpha \leqslant \mu. \end{cases} \tag{2.3.18}$$

Then

$$\int_{-\infty}^{\infty} \|(V^{\mathrm{s}}(x) - V^{\mathrm{s}}(vt + \tilde{c}(t)))U_0(t,0)\Phi_v\| dt = O(|v|^{\Theta_0(\alpha)+\epsilon}) \tag{2.3.19}$$

holds as $|v| \to \infty$ for $V^{\mathrm{s}} \in \mathscr{V}^{\mathrm{s}}_{\mu,\mu/(2-\mu)}$.

Proof. Take $f \in C_0^\infty(\mathbb{R}^d)$ such that $f\hat{\Phi}_0 = \hat{\Phi}_0$ and $\operatorname{supp} f \subset \{\xi \in \mathbb{R}^d \mid |\xi| \leqslant \eta\}$. Then one has

$$\begin{aligned} &\|(V^{\mathrm{s}}(x) - V^{\mathrm{s}}(vt + \tilde{c}(t)))U_0(t,0)\Phi_v\| \\ &\quad = \|(V^{\mathrm{s}}(x + vt + \tilde{c}(t)) - V^{\mathrm{s}}(vt + \tilde{c}(t)))e^{-itp^2/2} f(p)\Phi_0\| \end{aligned} \tag{2.3.20}$$

by virtue of the Avron-Herst formula (2.2.4) and (2.3.4). This can be estimated as

$$\|(V^{\mathrm{s}}(x) - V^{\mathrm{s}}(vt + \tilde{c}(t)))U_0(t,0)\Phi_v\| \leqslant I_0, \quad I_0 = 2\sup_{y\in\mathbb{R}^d} |V^{\mathrm{s}}(y)\|\Phi_0\|. \tag{2.3.21}$$

This can be also estimated as

$$\|(V^{\mathrm{s}}(x) - V^{\mathrm{s}}(vt + \tilde{c}(t)))U_0(t,0)\Phi_v\| \leqslant I_{1,\rho} + I_{2,\rho} + I_{3,\rho}, \tag{2.3.22}$$

with

$$\begin{aligned} I_{1,\rho} &= 2\sup_{y\in\mathbb{R}^d} |V^{\mathrm{s}}(y)|\|\Phi_0\| \\ &\quad \times \|F(|x| \geqslant 3\lambda_1|v|^\rho|t|)e^{-itp^2/2} f(p) F(|x| < \lambda_1|v|^\rho|t|)\|_{\mathscr{B}(L^2)}, \end{aligned} \tag{2.3.23}$$

$$\begin{aligned} I_{2,\rho} &= 2\sup_{y\in\mathbb{R}^d} |V^{\mathrm{s}}(y)|\|\langle x\rangle^{N+1}\Phi_0\| \\ &\quad \times \|F(|x| \geqslant 3\lambda_1|v|^\rho|t|)e^{-itp^2/2} f(p) F(|x| \geqslant \lambda_1|v|^\rho|t|)\langle x\rangle^{-N-1}\|_{\mathscr{B}(L^2)}, \end{aligned} \tag{2.3.24}$$

$$\begin{aligned} I_{3,\rho} &= \|(V^{\mathrm{s}}(x + vt + \tilde{c}(t)) - V^{\mathrm{s}}(vt + \tilde{c}(t)))F(|x| < 3\lambda_1|v|^\rho|t|)\|_{\mathscr{B}(L^2)} \\ &\quad \times \|e^{-itp^2/2} f(p)\Phi_0\|, \end{aligned} \tag{2.3.25}$$

by $F(|x| \geqslant 3\lambda_1|v|^\rho|t|) + F(|x| < 3\lambda_1|v|^\rho|t|) = 1$ and $F(|x| \geqslant \lambda_1|v|^\rho|t|) + F(|x| < \lambda_1|v|^\rho|t|) = 1$, where $3\lambda_1 = (1-\delta)/12$, $N \in \mathbb{N}$ and $0 < \rho \leqslant 1$.

One of the key points in this section is the way how to get better estimates of the term like $\int(I_{1,*} + I_{2,*})dt$ and $\int I_{3,*}dt$ than those in the previous works. We suppose $\mu = 0$ and $E_1(t) = 0$ temporarily for ease of explanation of our idea. In [38], Weder essentially obtained the estimate

$$\int_{-\infty}^{\infty}(I_{1,1} + I_{2,1})dt = O(|v|^{-1}), \quad \int_{-\infty}^{\infty} I_{3,1} = O(|v|^{1-2\alpha}), \tag{2.3.26}$$

although in [38], these estimates with $V^{\mathrm{s}}(vt + \tilde{c}(t)) = 0$ and $\alpha = \gamma$ were really used. After that, in [8], Adachi and Maehara obtained the estimate

$$\int_{-\infty}^{\infty}(I_{1,\rho_1} + I_{2,\rho_1})dt = O(|v|^{-\rho_1}), \quad \int_{-\infty}^{\infty} I_{3,\rho_1} = O(|v|^{\rho_1-2\alpha}) \tag{2.3.27}$$

with $0 < \rho_1 < 1$. As is seen, obviously, the estimate of $\int(I_{1,*} + I_{2,*})dt$ in [38] is better than that in [8], while the estimate of $\int I_{3,*}dt$ in [8] is better than that in [38]. In [8], by taking ρ_1 as $-\rho_1 = \rho_1 - 2\alpha$, that is, $\rho_1 = \alpha$, they obtained an optimal estimate

$$\int_{-\infty}^{\infty}(I_{1,\alpha} + I_{2,\alpha} + I_{3,\alpha})dt = O(|v|^{-\alpha}). \tag{2.3.28}$$

By taking account of the advantage of both these estimations, we will introduce a new parameter $\sigma_1 \in \mathbb{R}$ for the sake of getting better estimates of the term like $\int(I_{1,*} + I_{2,*} + I_{3,*})dt$. More concretely, by dividing the integral interval $\mathbb{R}$ into $\{t \in \mathbb{R} \mid |t| \leqslant |v|^{\sigma_1}\}$ and $\{t \in \mathbb{R} \mid |t| > |v|^{\sigma_1}\}$, we will obtain better estimates

$$\int_{|t|\leqslant|v|^{\sigma_1}}(I_{1,1} + I_{2,1})dt = O(|v|^{-1}), \quad \int_{|t|\leqslant|v|^{\sigma_1}} I_{3,1}dt = O(|v|^{\Theta_1(\alpha,\sigma_1)}), \tag{2.3.29}$$

$$\int_{|t|>|v|^{\sigma_1}}(I_{1,\rho_1} + I_{2,\rho_1})dt = O(|v|^{-1}), \quad \int_{|t|>|v|^{\sigma_1}} I_{3,\rho_1}dt = O(|v|^{\Theta_2(\alpha,\sigma_1)}) \tag{2.3.30}$$

with negative $\Theta_1(\alpha, \sigma_1)$ and $\Theta_1(\alpha, \sigma_2)$. In order to get the optimal estimate (2.3.19), after taking $\sigma_1(\alpha, \rho_1)$ as $\Theta_1(\alpha, \sigma_1(\alpha, \rho_1)) = \Theta_2(\alpha, \sigma_1(\alpha, \rho_1))$, we will put $\Theta_0(\alpha)$ as the infimum of $\Theta_1(\alpha, \sigma_1(\alpha, \rho_1))$ when ρ_1 varies in the interval $(0, 1)$. This is the essence of our idea.

We first have

$$\int_{|t|\leqslant|v|^{-1}} I_0dt = O(|v|^{-1}). \tag{2.3.31}$$

Let ρ_1 be given as $0 < \rho_1 < 1$. Take a new parameter $\sigma_1 \in \mathbb{R}$ such that $\sigma_1 > -\rho_1$. By using Proposition 2.3.2 for bestimating $I_{1,1}$ under the condition $\lambda_1|v| \geqslant \eta$, we have

$$I_{1,1} + I_{2,1} \leqslant C\langle|v||t|\rangle^{-N-1} \tag{2.3.32}$$

for $\lambda_1|v| \geqslant \eta$, which implies

$$\int_{|v|^{-1}\leqslant|t|\leqslant|v|^{\sigma_1}} (I_{1,1}+I_{2,1})dt = O(|v|^{-1}) \tag{2.3.33}$$

By simple calculation. Put $\delta = |\hat{v}\cdot e_1| < 1$. Let $|x| < 3\lambda_1|v||t|$. Then,

$$\begin{aligned}|x+vt+\tilde{c}(t)|^2 &= |x+vt|^2 + |\tilde{c}_0(t)+\tilde{c}_1(t)|^2 + 2(x+vt)\cdot(\tilde{c}_0(t)+\tilde{c}_1(t))\\ &\geqslant (1-(1-\delta)/12)^2|v|^2|t|^2 + ((2+\delta)/3)^2|\tilde{c}_0(t)|^2\\ &\quad -2((1+2\delta)/3+((1-\delta)12)(1+(1-\delta)/3))|v||t||\tilde{c}_0(t)|\end{aligned} \tag{2.3.34}$$

for $|t| \geqslant T_\delta$ can be obtained as in the proof of Proposition 2.3.3. It follows from this that for $|t| \geqslant T_\delta$

$$\begin{aligned}|x+vt+\tilde{c}(t)|^2 &\geqslant k_0\max\{k_1|v|^2|t|^2/(2+\delta)^2, 16k_1|\tilde{c}_0(t)|^2/(11+\delta)^2\}\\ &\geqslant k_0\max\{k_1|v|^2|t|^2/9, k_1|\tilde{c}_0(t)|^2/9\}\end{aligned} \tag{2.3.35}$$

holds with $k_0 = (1-\delta)/6$ and $k_1 = (\delta^2+16\delta+19)/2$, by a straightforward computation. On the other hand, for $|t| < T_\delta$ and $M_2/|v| \leqslant (1-\delta)/3$

$$\begin{aligned}|x+vt+\tilde{c}(t)|^2 &= |x+vt+\tilde{c}_1(t)|^2 + |\tilde{c}_0(t)|^2 + 2(x+vt+\tilde{c}_1(t))\cdot\tilde{c}_0(t)\\ &\geqslant ((2+\delta)/3-(1-\delta)/12)^2|v|^2|t|^2 + |\tilde{c}_0(t)|^2\\ &\quad -2((1+2\delta)/3+(1-\delta)/12)|v||t||\tilde{c}_0(t)|\end{aligned} \tag{2.3.36}$$

holds. In the same way as above, we have

$$\begin{aligned}|x+vt+\tilde{c}(t)|^2 &\geqslant k_0\max\{(1+\delta)|v|^2|t|^2/(2+\delta)^2, 144(1+\delta)|\tilde{c}_0(t)|^2/(7+5\delta)^2\}\\ &\geqslant k_0\max\{(1+\delta)|v|^2|t|^2, (1+\delta)|\tilde{c}_0(t)|^2\}.\end{aligned} \tag{2.3.37}$$

Since

$$k_1/9-(1+\delta) = (\delta^2+16\delta+19)/18-(1+\delta) = (1-\delta)^2/18 > 0, \tag{2.3.38}$$

we finally see that $|x| < 3\lambda_1|v||t|$ and $|v| \geqslant 3M_2/(1-\delta)$

$$|x+vt+\tilde{c}(t)| \geqslant d_1\max\{|v||t|, |\tilde{c}_0(t)|\} \tag{2.3.39}$$

holds with $d_1 = ((1-\delta)k_0)^{1/2} = ((1-\delta^2)/6)^{1/2}$. This estimate was obtained by [6]. Since $|\tilde{c}_0(t)| \geqslant a_\mu|t|^{2-\mu}$ by (2.2.10), we have

$$|x+vt+\tilde{c}(t)| \geqslant d_\kappa|v|^\kappa|t|^{\tilde{\sigma}_\kappa} \tag{2.3.40}$$

with $d_\kappa = d_1a_\mu^{1-\kappa}$ and

$$\tilde{\sigma}_\kappa = \kappa+(2-\mu)(1-\kappa) = 2-\mu-\kappa(1-\mu), \tag{2.3.41}$$

where $0 \leqslant \kappa \leqslant 1$. Now, by using this new parameter κ, we introduce $V^{\mathrm{s}}_{v,t,\kappa}(x)$ as

$$V^{\mathrm{s}}_{v,t,\kappa}(x) = V^{\mathrm{s}}(x) g_\kappa(x/(|v|^\kappa |t|^{\tilde{\sigma}_\kappa})), \tag{2.3.42}$$

where $g_\kappa \in C^\infty(\mathbb{R}^d)$ such that $0 \leqslant g_\kappa \leqslant 1$,

$$g_\kappa(x) = \begin{cases} 1 & |x| \geqslant d_\kappa \\ 0 & |x| \leqslant d_\kappa/2. \end{cases} \tag{2.3.43}$$

We note that the potential like $V^{\mathrm{s}}_{v,t,\kappa}(x)$ with parameter κ was already introduce [6] and [8] for the sake of dealing with the long-range part V^{l} of V. Then $I_{3,1}$ is estimated as

$$I_{3,1} \leqslant \|(V^{\mathrm{s}}_{v,t,1}(x+vt+\tilde{c}(t)) - V^{\mathrm{s}}_{v,t,1}(vt+\tilde{c}(t)))F(|x| < 3\lambda_1|v||t|)\|_{\mathscr{B}(L^2)}\|\Phi_0\| \tag{2.3.44}$$

if $|v| \geqslant 3M_2/(1-\delta)$. By using

$$V^{\mathrm{s}}_{v,t,\kappa}(x+vt+\tilde{c}(t)) - V^{\mathrm{s}}_{v,t,\kappa}(vt+\tilde{c}(t)) = \int_0^1 (\nabla_x V^{\mathrm{s}}_{v,t,\kappa})(\theta x + vt + \tilde{c}(t)) \cdot x d\theta, \tag{2.3.45}$$

we have

$$I_{3,1} \leqslant 3\lambda_1 \|\Phi_0\| |v||t| \sup_{y \in \mathbb{R}^d} |(\nabla_x V^{\mathrm{s}}_{v,t,1})(y)|. \tag{2.3.46}$$

Here we note that

$$\sup_{y \in \mathbb{R}^d} |(\nabla_x V^{\mathrm{s}}_{v,t,1})(y)| \leqslant C(\langle |v|^\kappa |t|^{\tilde{\sigma}_\kappa} \rangle^{-1-\alpha} + (|v|^\kappa |t|^{\tilde{\sigma}_\kappa})^{-1} \langle |v|^\kappa |t|^{\tilde{\sigma}_\kappa} \rangle^{-\gamma}). \tag{2.3.47}$$

holds. Thus

$$\begin{aligned} \int_{|v|^{-1} \leqslant |t| \leqslant |v|^{\sigma_1}} I_{3,1} dt &\leqslant C \int_{|v|^{-1} \leqslant |t| \leqslant |v|^{\sigma_1}} (|v||t| \langle |v|^\kappa |t|^{\tilde{\sigma}_\kappa} \rangle^{-1-\alpha} + \langle |v|^\kappa |t|^{\tilde{\sigma}_\kappa} \rangle^{-\gamma}) dt \\ &= O(|v|^{\sigma_1 - \alpha(1+\sigma_1)}) + O(|v|^{\sigma_1 - \gamma(1+\sigma_1)}) = O(|v|^{\sigma_1 - \alpha(1+\sigma_1)}) \end{aligned} \tag{2.3.48}$$

is obtained under the conditions $\alpha \leqslant \gamma < 1$ and $1 + \sigma_1 > 1 - \rho_1 > 0$. Now we put

$$\Theta_1(\alpha, \sigma_1) = \sigma_1 - \alpha(1+\sigma_1) = -\alpha + \sigma_1(1-\alpha). \tag{2.3.49}$$

We note that $\Theta_1(\alpha, \sigma_1)$ is monotonically increasing in σ_1 because of $1 - \alpha > 0$.

By using Proposition 2.3.2 for estimating I_{1,ρ_1} under the condition $\lambda_1 |v|^{\rho_1} \geqslant \eta$, we have

$$I_{1,\rho_1} + I_{2,\rho_1} \leqslant C \langle |v|^{\rho_1} |t| \rangle^{-N-1} \tag{2.3.50}$$

for $\lambda_1 |v|^{\rho_1} \geqslant \eta$, which implies

$$\int_{|t| \geqslant |v|^{\sigma_1}} (I_{1,\rho_1} + I_{2,\rho_1}) dt = O(|v|^{-\rho_1 - (\rho_1 + \sigma_1)N}) \tag{2.3.51}$$

by simple calculation. Since $\rho_1+\sigma_1>0$, for sufficiently large $N\in\mathbb{N}$, $-\rho_1-(\rho_1+\sigma_1)N\leqslant -1$ holds.

Let $|x|<3\lambda_1|v|^{\rho_1}|t|$ and $|v|\geqslant \max\{1,3M_2/(1-\delta)\}$. Then, in the same way as in obtaining the estimate (2.3.39), one can obtain the estimate

$$|x+vt+\tilde{c}(t)|\geqslant d_1\max\{|v||t|,|\tilde{c}_0(t)|\}. \tag{2.3.52}$$

Then I_{3,ρ_1} is estimated as

$$I_{3,\rho_1}\leqslant \|(V^{\mathrm{s}}_{v,t,\kappa}(x+vt+\tilde{c}(t))-V^{\mathrm{s}}_{v,t,\kappa}(vt+\tilde{c}(t)))F(|x|<3\lambda_1|v|^{\rho_1}|t|)\|_{\mathscr{B}(L^2)}\|\Phi_0\| \tag{2.3.53}$$

if $|v|\geqslant \max\{1,3M_2/(1-\delta)\}$. By using (2.3.45), we have

$$I_{3,\rho_1}\leqslant 3\lambda_1\|\Phi_0\||v|^{\rho_1}|t|\sup_{y\in\mathbb{R}^d}|(\nabla_x V^{\mathrm{s}}_{v,t,\kappa})(y)|. \tag{2.3.54}$$

By (2.3.47),

$$\begin{aligned}\int_{|t|\geqslant|v|^{\sigma_1}} I_{3,\rho_1}dt &\leqslant C\int_{|t|\geqslant|v|^{\sigma_1}}(|v|^{\rho_1}|t|\langle|v|^{\kappa}|t|^{\tilde{\sigma}_\kappa}\rangle^{-1-\alpha}+|v|^{\rho_1-\kappa}|t|^{1-\tilde{\sigma}_\kappa}\langle|v|^{\kappa}|t|^{\tilde{\sigma}_\kappa}\rangle^{-\gamma})dt\\ &=O(|v|^{\rho_1-\kappa(1+\alpha)+\sigma_1(2-\tilde{\sigma}_\kappa(1+\alpha))})+O(|v|^{\rho_1-\kappa(1+\gamma)+\sigma_1(2-\tilde{\sigma}_\kappa(1+\gamma))})\end{aligned} \tag{2.3.55}$$

is obtained under the integrable conditions

$$2-\tilde{\sigma}_\kappa(1+\alpha)<0,\quad 2-\tilde{\sigma}_\kappa(1+\gamma))<0, \tag{2.3.56}$$

which imply the condition on κ

$$0\leqslant\kappa<\frac{1}{1-\mu}\left(2-\mu-\frac{2}{1+\alpha}\right)=\frac{(2-\mu)(1+\alpha)-2}{(1-\mu)(1+\alpha)}=:\kappa_\alpha \tag{2.3.57}$$

since $\alpha\leqslant\gamma$. The reason why we need the additional condition

$$\alpha>\mu/(2-\mu) \tag{2.3.58}$$

is that we have to guarantee the positivity of κ_α. Moreover, we see that $\kappa_\alpha<1$ hods because of $\alpha<1$. If σ_1 satisfies

$$\kappa+\sigma_1\tilde{\sigma}_\kappa>0, \tag{2.3.59}$$

then we have

$$\int_{|t|\geqslant|v|^{\sigma_1}} I_{3,\rho_1}dt=O(|v|^{\rho_1-\kappa(1+\alpha)+\sigma_1(2-\tilde{\sigma}_\kappa(1+\alpha))}) \tag{2.3.60}$$

because $\alpha\leqslant\gamma$. Now we put

$$\Theta_2(\alpha,\rho_1,\kappa,\sigma_1)=\rho_1-\kappa(1+\alpha)+\sigma_1(2-\tilde{\sigma}_\kappa(1+\alpha)) \tag{2.3.61}$$

We note that $\Theta_2(\alpha, \rho_1, \kappa, \sigma_1)$ is monotonically decreasing in σ_1 because of $2 - \tilde{\sigma}_\kappa(1+\alpha) < 0$.

Step 1.

Now we will obtain the optimal estimate of the left-hand side of (2.3.19) for given α, ρ_1 and κ. To this end, we have only to obtain the condition under which there exists σ_1 such that $\sigma_1 > -\rho_1$, (2.3.59) and

$$\Theta_1(\alpha, \sigma_1) < 0, \quad \Theta_2(\alpha, \rho_1, \kappa, \sigma_1) < 0 \tag{2.3.62}$$

hold. The conditions (2.3.59) and (2.3.62) can be written as

$$\sigma_1 > -\frac{\kappa}{2-\mu-\kappa(1-\mu)} =: h(\kappa), \tag{2.3.63}$$

$$\sigma_1 < \frac{\alpha}{1-\alpha} \tag{2.3.64}$$

$$\sigma_1 > \frac{\rho_1 - \kappa(1+\alpha)}{(1+\alpha)(2-\mu-\kappa(1-\mu))-2} =: G(\alpha, \rho_1, \kappa), \tag{2.3.65}$$

respectively, since $1-\alpha$ and $(1+\alpha)(2-\mu-\kappa(1-\mu))-2 = \tilde{\sigma}_1(1+\alpha)-2$ are positive. By simple calculation, we see that

$$G(\alpha, \rho_1, \kappa) \geqslant -\rho_1 \iff \kappa \leqslant \frac{\rho_1((2-\mu)(1+\alpha)-1)}{(1+\alpha)(1+\rho_1(1-\mu))} =: \kappa_1(\alpha, \rho_1), \tag{2.3.66}$$

$$G(\alpha, \rho_1, \kappa) \geqslant h(\kappa) \iff \kappa \leqslant \frac{\rho_1(2-\mu)}{2+\rho_1(1-\mu)} =: \kappa_2(\rho_1), \tag{2.3.67}$$

$$h(\kappa) \geqslant -\rho_1 \iff \kappa \leqslant \frac{\rho_1(2-\mu)}{1+\rho_1(1-\mu)} =: \kappa_3(\rho_1). \tag{2.3.68}$$

We also note that $\kappa_3(\rho_1) > \kappa_1(\alpha, \rho_1)$, $\kappa_3(\rho_1) > \kappa_2(\rho_1)$,

$$\kappa_1(\alpha, \rho_1) > \kappa_2(\rho_1) \iff \rho_1 < \frac{(2-\mu)(1+\alpha)-2}{1-\mu} =: \tilde{\rho}_\alpha, \tag{2.3.69}$$

$$\kappa_\alpha > \kappa_1(\alpha, \rho_1) \iff \rho_1 < \tilde{\rho}_\alpha, \tag{2.3.70}$$

$$\kappa_\alpha > \kappa_3(\rho_1) \iff \rho_1 < \tilde{\rho}_\alpha/2, \tag{2.3.71}$$

$$\tilde{\rho}_\alpha < 1 \iff 1/(2-\mu), \tag{2.3.72}$$

$$\tilde{\rho}_\alpha/2 < 1 \iff \alpha < 1, \tag{2.3.73}$$

by simple calculation. We will divide situation into cases below for ease of consideration.

Case A. $0 < \rho_1 < \tilde{\rho}_\alpha/2$.

Then $\kappa_2(\rho_1) < \kappa_1(\alpha, \rho_1) < \kappa_3(\rho_1) < \kappa_\alpha$ holds. We first suppose $0 \leqslant \kappa \leqslant \kappa_2(\rho_1)$. Then it is necessary that

$$\max\{-\rho_1, h(\kappa), G(\alpha, \rho_1, \kappa)\} = G(\alpha, \rho_1, \kappa) < \alpha/(1-\alpha) \tag{2.3.74}$$

is satisfied. Note

$$\begin{aligned}&\frac{\alpha}{1-\alpha}-G(\alpha,\rho_1,\kappa)\\&\quad=\frac{\alpha((2-\mu)(1+\alpha)-2)-\rho_1(1-\alpha)+\kappa(1+\alpha)(1-\alpha(2-\mu))}{(1-\alpha)((1+\alpha)((2-\mu)-\kappa(1-\mu))-2)}.\end{aligned}\tag{2.3.75}$$

If $1-\alpha(2-\mu)<0$, that is, $\alpha>1/(2-\mu)$, then we obtain an additional condition on κ

$$\kappa<\frac{\alpha((2-\mu)(1+\alpha)-2)-\rho_1(1-\alpha)}{(1+\alpha)(\alpha(2-\mu)-1)}=:\kappa_4(\alpha,\rho_1).\tag{2.3.76}$$

We note that

$$\kappa_4(\alpha,\rho_1)-\kappa_\alpha=\frac{(1-\alpha)((2-\mu)(1+\alpha)-2-\rho_1(1-\mu))}{(1-\mu)(1+\alpha)(\alpha(2-\mu)-1)},\tag{2.3.77}$$

which implies that when $\rho_1<\tilde{\rho}_\alpha$,

$$\kappa_\alpha<\kappa_4(\alpha,\rho_1)\tag{2.3.78}$$

holds. Hence the condition (2.3.76) is satisfied by $0\leqslant\kappa\leqslant\kappa_2(\rho_1)<\kappa_\alpha$ and (2.3.78). When σ_1 varies in interval $(G(\alpha,\rho_1,\kappa),\alpha/(1-\alpha))$, we have

$$R(\Theta_1(\alpha,\cdot))=(\Theta_1(\alpha,G(\alpha,\rho_1,\kappa)),0),\tag{2.3.79}$$

$$R(\Theta_2(\alpha,\rho_1,\kappa,\cdot))=(\Theta_2(\alpha,\rho_1,\kappa,\alpha/(1-\alpha)),0),\tag{2.3.80}$$

since $\Theta_1(\alpha,\alpha/(1-\alpha))=\Theta_2(\alpha,\rho_1,\kappa,G(\alpha,\rho_1,\kappa))=0$, where $R(\Theta_1(\alpha,\cdot))$ and $R(\Theta_2(\alpha,\rho_1,\kappa,\cdot))$ stand for the ranges of $\Theta_1(\alpha,\sigma_1)$ and $\Theta_2(\alpha,\rho_1,\kappa,\sigma_1)$ when σ_1 varies in the interval under the consideration. Then it follows from

$$R(\Theta_1(\alpha,\cdot))\bigcap R(\Theta_2(\alpha,\rho_1,\kappa,\cdot))\neq\emptyset\tag{2.3.81}$$

that there exists a unique σ_1 such that $\Theta_1(\alpha,\sigma_1)=\Theta_2(\alpha,\rho_1,\kappa,\sigma_1)$, by virtue of the monotonicity of Θ_1 and Θ_2. In fact, by putting

$$\sigma_1(\alpha,\rho_1,\kappa)=\frac{1}{1-\mu}-\frac{1-\rho_1}{(1-\mu)(1+\alpha)(1-\kappa)},\tag{2.3.82}$$

we have

$$\begin{aligned}\Theta_1(\alpha,\sigma_1(\alpha,\rho_1,\kappa))&=\Theta_2(\alpha,\rho_1,\kappa,\sigma_1(\alpha,\rho_1,\kappa))\\&=-\alpha+\sigma_1(\alpha,\rho_1,\kappa)(1-\alpha)=:\Theta_3(\alpha,\rho_1,\kappa).\end{aligned}\tag{2.3.83}$$

Then we obtain the optimal estimate

$$\int_{-\infty}^{\infty}\|(V^{\mathrm{s}}(x)-V^{\mathrm{s}}(vt+\tilde{c}(t)))U_0(t,0)\Phi_v\|dt=O(|v|^{\Theta_3(\alpha,\rho_1,\kappa)})\tag{2.3.84}$$

for given α, ρ_1 and κ. Here we note that $\Theta_3(\alpha,\rho_1,\kappa) > -1$. In fact, we see that

$$\Theta_3(\alpha,\rho_1,\kappa) > -1 \Longleftrightarrow \kappa < 1 - \frac{1-\rho_1}{(2-\mu)(1+\alpha)} =: \kappa_c(\alpha,\rho_1) \tag{2.3.85}$$

by simple calculation. Noting that for $\rho_1 < 1$,

$$\kappa_c(\alpha,\rho_1) - \kappa_3(\rho_1) = (1-\rho_1)\left(\frac{1}{1+\rho_1(1-\mu)} - \frac{1}{(2-\mu)(1+\alpha)}\right) > 0 \tag{2.3.86}$$

since $1+\rho_1(1-\mu) < 2-\mu < (2-\mu)(1+\alpha)$, we have $\Theta_3(\alpha,\rho_1,\kappa) > -1$.

If $1-\alpha(2-\mu) = 0$, that is, $\alpha = 1/(2-\mu)$, then it follows from (2.3.75) that (2.3.74) is equivalent to $\rho_1 < \tilde{\rho}_\alpha$ with $\alpha = 1/(2-\mu)$, by using $(1-1/(2-\mu))/(1/(2-\mu)) = 1-\mu$. Hence, the optimal estimate (2.3.80) can be obtained in the same way as above.

If $1-\alpha(2-\mu) > 0$, that is, $\alpha < 1/(2-\mu)$, then we obtain an additional condition on κ

$$\kappa > \kappa_4(\alpha,\rho_1). \tag{2.3.87}$$

Since $\kappa_4(\alpha,\rho_1) < 0$ in this case, (2.3.87) is automatically satisfied by $0 \leqslant \kappa \leqslant \kappa_2(\rho_1)$. Hence, the optimal estimate (2.3.84) can be obtained in the same way as above.

We next suppose $\kappa_2(\rho_1) < \kappa \leqslant \kappa_3(\rho_1)$. Then it is necessary that

$$\max\{-\rho_1, h(\kappa), G(\alpha,\rho_1,\kappa)\} = h(\kappa) < \alpha/(1-\alpha) \tag{2.3.88}$$

is satisfied, but this holds obviously because $h(\kappa) \leqslant 0$. When σ_1 varies in the interval $(h(\kappa), \alpha/(1-\alpha))$, we have

$$R(\Theta_1(\alpha,\cdot)) = (\Theta_1(\alpha,h(\kappa)), 0), \tag{2.3.89}$$

$$R(\Theta_2(\alpha,\rho_1,\kappa,\cdot)) = (\Theta_2(\alpha,\rho_1,\kappa,\alpha/(1-\alpha)), \Theta_2(\alpha,\rho_1,\kappa,h(\kappa))). \tag{2.3.90}$$

We will show (2.3.81) with (2.3.89) and (2.3.90) hold. By

$$\Theta_2(\alpha,\rho_1,\kappa,h(\kappa)) - \Theta_1(\alpha,h(\kappa)) = \rho_1 + \alpha - \frac{\kappa(1+\alpha)}{2-\mu-\kappa(1-\mu)}, \tag{2.3.91}$$

we see that

$$\begin{aligned} \Theta_2(\alpha,\rho_1,\kappa,h(\kappa)) \quad &> \quad \Theta_1(\alpha,h(\kappa)) \\ &\Longleftrightarrow \kappa < \frac{(2-\mu)(\rho_1+\alpha)}{1+\alpha+(1-\mu)(\rho_1+\alpha)} =: \kappa_5(\alpha,\rho_1). \end{aligned} \tag{2.3.92}$$

Note that

$$\kappa_5(\alpha,\rho_1) - \kappa_3(\rho_1) = \frac{\alpha(2-\mu)(1-\rho_1)}{(1+\alpha+(1-\mu)(\rho_1+\alpha))(1+\rho_1(1-\mu))} > 0 \tag{2.3.93}$$

holds. Hence we see that (2.3.81) with (2.3.89) and (2.3.90) is satisfied. In the same way as above, the optimal estimate (2.3.84) is obtained for given α, ρ_1 and κ.

We finally suppose $\kappa_3(\rho_1) < \kappa < \kappa_\alpha$. Then it is necessary that

$$\max\{-\rho_1, h(\kappa), G(\alpha, \rho_1, \kappa)\} = -\rho_1 < \alpha/(1-\alpha) \tag{2.3.94}$$

is satisfied, but this holds obviously because $-\rho_1 < 0$. When σ_1 varies in the interval $(-\rho_1, \alpha/(1-\alpha))$, we have

$$R(\Theta_1(\alpha, \cdot)) = (\Theta_1(\alpha, -\rho_1), 0), \tag{2.3.95}$$

$$R(\Theta_2(\alpha, \rho_1, \kappa, \cdot)) = (\Theta_2(\alpha, \rho_1, \kappa, \alpha/(1-\alpha)), \Theta_2(\alpha, \rho_1, \kappa, -\rho_1)). \tag{2.3.96}$$

We will consider the condition under which (2.3.81) with (2.3.95) and (2.3.96) can be satisfied. By

$$\begin{aligned}\Theta_2(\alpha, \rho_1, \kappa, -\rho_1) - \Theta_1(\alpha, -\rho_1)\\ = \rho_1((2-\mu)(1+\alpha) - \alpha) + \alpha - \kappa(1+\alpha)(1+\rho_1(1-\mu)),\end{aligned} \tag{2.3.97}$$

We see that

$$\begin{aligned}\Theta_2(\alpha, \rho_1, \kappa, -\rho_1) \quad &> \quad \Theta_1(\alpha, -\rho_1)\\ &\Longleftrightarrow \kappa < \frac{\rho_1((2-\mu)(1+\alpha) - \alpha) + \alpha}{(1+\alpha)(1+\rho_1(1-\mu))} =: \kappa_6(\alpha, \rho_1).\end{aligned} \tag{2.3.98}$$

By

$$\kappa_6(\alpha, \rho_1) - \kappa_\alpha = \frac{\rho_1(1-\mu)(2-\alpha) - (\alpha-\mu)}{(1-\mu)(1+\alpha)(1+\rho_1(1-\mu))}, \tag{2.3.99}$$

we have

$$\kappa_6(\alpha, \rho_1) \geqslant \kappa_\alpha \Longleftrightarrow \rho_1 \geqslant \frac{\alpha-\mu}{(1-\mu)(2-\alpha)} =: \hat{\rho}_\alpha. \tag{2.3.100}$$

Hence, if $\alpha > \mu$, $0 < \rho_1 \leqslant \hat{\rho}_\alpha$ and $\kappa_6(\alpha, \rho_1) \leqslant \kappa < \kappa_\alpha$, then

$$R(\Theta_1(\alpha, \cdot)) \bigcap R(\Theta_2(\alpha, \rho_1, \kappa, \cdot)) = \emptyset \tag{2.3.101}$$

with (2.3.95) and (2.3.96) holds, otherwise (2.3.81) with (2.3.95) and (2.3.96) is satisfied, which yields the optimal estimate (2.3.84). Noting that for $\rho_1 < 1$,

$$\kappa_c(\alpha, \rho_1) - \kappa_6(\alpha, \rho_1) = \frac{1-\rho_1}{1+\alpha}\left(\frac{1}{1+\rho_1(1-\mu)} - \frac{1}{2-\mu}\right) > 0 \tag{2.3.102}$$

since $1 + \rho_1(1-\mu) < 2 - \mu$, we have $\Theta_3(\alpha, \rho_1, \kappa) > -1$. On the other hand, when $\alpha > \mu$, $0 < \rho_1 < \hat{\rho}_\alpha$ and $\kappa_6(\alpha, \rho_1) \leqslant \kappa < \kappa_\alpha$, the estimate

$$\int_{-\infty}^{\infty} \|(V^{\mathrm{s}}(x) - V^{\mathrm{s}}(vt + \tilde{c}(t)))U_0(t, 0)\Phi_v\| dt = O(|v|^{\Theta_1(\alpha, -\rho_1)+\epsilon}) \tag{2.3.103}$$

is obtained for any $\epsilon > 0$.
Case B. $\tilde{\rho}_\alpha/2 \leqslant \rho_1 < \min\{1, \tilde{\rho}_\alpha\}$.

Then $\kappa_2(\rho_1) < \kappa_1(\alpha, \rho_1) < \kappa_\alpha \leqslant \kappa_3(\rho_1)$ holds. In the same way as in the Case A, we first suppose $0 \leqslant \kappa \leqslant \kappa_2(\rho_1)$. Then it is necessary that (2.3.74) is satisfied. Thus we have only to show that (2.3.81) with (2.3.79) and (2.3.80) is satisfied. This can be done in the same way in the Case A. Therefore we have (2.3.84).

We next suppose $\kappa_2(\rho_1) < \kappa < \kappa_\alpha$. Then we have only to show that (2.3.81) with (2.3.89) and (2.3.90) is satisfied. We note that when $\rho_1 \geqslant \tilde{\rho}_\alpha/2$,

$$\begin{aligned}\kappa_5(\alpha, \rho_1) - \kappa_\alpha &= \frac{2(1-\mu)(\rho_1+\alpha) - (1+\alpha)((2-\mu)(1+\alpha)-2)}{(1-\mu)(1+\alpha)(1+\alpha+(1-\mu)(\rho_1+\alpha))} \\ &= \frac{2(\rho_1+\alpha) - (1+\alpha)\tilde{\rho}_\alpha}{(1+\alpha)(1+\alpha+(1-\mu)(\rho_1+\alpha))} \\ &\geqslant \frac{\alpha(2-\tilde{\rho}_\alpha)}{(1+\alpha)(1+\alpha+(1-\mu)(\rho_1+\alpha))} > 0. \qquad (2.3.104)\end{aligned}$$

Hence we used $(2-\mu)(1+\alpha) - 2 = \tilde{\rho}_\alpha(1-\mu)$. Hence, in the same way in the Case A, we obtain (2.3.81) with (2.3.89) and (2.3.90), which yields (2.3.84).
Case C. $\tilde{\rho}_\alpha \leqslant \rho_1 < 1$.

Then we assume that $\alpha < 1/(2-\mu)$ necessarily, and need an additional condition (2.3.87). Here we note that $\kappa_\alpha \leqslant \kappa_1(\alpha, \rho_1) \leqslant \kappa_2(\rho_1) < \kappa_3(\rho_1)$ in this case. Since $\kappa_\alpha \leqslant \kappa_4(\alpha, \rho_1)$ by (2.3.77) in this case, we see that there is no κ satisfying both $0 \leqslant \kappa < \kappa_\alpha$ and (2.3.87).

Step 2.

Now we will obtain the optimal estimate of the left-hand side of (2.3.19) for given α. We will divide the situation into the cases below for ease of consideration.
Case 1. $\alpha > \mu$.

First of all, we note that $\Theta_3(\alpha, \rho_1, \kappa)$ is monotonically decreasing in κ. In fact, we have

$$\partial_\kappa \Theta_3(\alpha, \rho_1, \kappa) = -\frac{(1-\alpha)(1-\rho_1)}{(1-\mu)(1+\alpha)(1-\kappa)^2} < 0. \qquad (2.3.105)$$

We first suppose that ρ_1 is given as $0 < \rho_1 < \hat{\rho}_\alpha$. By virtue of the result in the Case A, for $0 \leqslant \kappa < \kappa_6(\alpha, \rho_1)$, the estimate (2.3.84) holds. Here we note that

$$\begin{aligned}\inf_{0 \leqslant \kappa < \kappa_6(\alpha,\rho_1)} \Theta_3(\alpha, \rho_1, \kappa) &= \Theta_3(\alpha, \rho_1, \kappa_6(\alpha, \rho_1)) \\ &= -\alpha - \rho_1(1-\alpha) = \Theta_1(\alpha, -\rho_1), \qquad (2.3.106)\end{aligned}$$

by simple calculation. Here we see that the optimal estimate of the left-hand side of (2.3.19) when κ varies in the interval $[0, \kappa_6(\alpha, \rho_1))$ is just the estimate (2.3.103). On the other hand, for $\kappa_6(\alpha, \rho_1) \leqslant \kappa < \kappa_\alpha$, the estimate (2.3.103) holds. Summing up, for given ρ_1 such that $0 < \rho_1 < \hat{\rho}_\alpha$, the optimal estimate of the left-hand side of (2.3.19) is just the estimate (2.3.103). Here we note that $\Theta_1(\alpha, -\rho_1)$ is monotonically decreasing in ρ_1 by $-(1-\alpha) < 0$.

We next suppose that ρ_1 is given as $\hat{\rho}_\alpha \leqslant \rho_1 < \min\{1, \tilde{\rho}_\alpha\}$. By virtue of the result in the Cases A and B, for $0 \leqslant \kappa < \kappa_\alpha$, the estimate (2.3.84) holds. Now we put

$$\begin{aligned}\Theta_4(\alpha, \rho_1) &= \inf_{0 \leqslant \kappa < \kappa_\alpha} \Theta_3(\alpha, \rho_1, \kappa) \\ &= \Theta_3(\alpha, \rho_1, \kappa_\alpha) = \rho_1 - \frac{(2-\mu)(1+\alpha)-2}{1-\mu}. \end{aligned} \tag{2.3.107}$$

Then we obtain the estimate

$$\int_{-\infty}^{\infty} \|(V^{\mathrm{s}}(x) - V^{\mathrm{s}}(vt + \tilde{c}(t)))U_0(t,0)\Phi_v\| dt = O(|v|^{\Theta_4(\alpha,\rho_1)+\epsilon}) \tag{2.3.108}$$

as the optimal estimate of the left-hand side of (2.3.19) for given ρ_1 such that $\hat{\rho}_\alpha \leqslant \rho_1 < \min\{1, \tilde{\rho}_\alpha\}$, by the monotonicity of $\Theta_3(\alpha, \rho_1, \kappa)$ in κ. Here we note that $\Theta_4(\alpha, \rho_1)$ is monotonically increasing in ρ_1, as is seen obviously.

Now we will obtain the optimal estimate of the left-hand side of (2.3.19) when ρ_1 varies in $(0, \min\{1, \tilde{\rho}_\alpha\})$. Here we note that

$$\inf_{0<\rho_1<\hat{\rho}_\alpha} \Theta_1(\alpha, -\rho_1) = \Theta_1(\alpha, -\hat{\rho}_\alpha) = -\alpha - \frac{(\alpha-\mu)(1-\alpha)}{(1-\mu)(2-\alpha)}, \tag{2.3.109}$$

$$\inf_{\hat{\rho}_\alpha \leqslant \rho_1 < \min\{1,\tilde{\rho}_\alpha\}} \Theta_4(\alpha, \rho_1) = \Theta_4(\alpha, \hat{\rho}_\alpha) = -\alpha - \frac{(\alpha-\mu)(1-\alpha)}{(1-\mu)(2-\alpha)}, \tag{2.3.110}$$

by simple calculation. Putting

$$\Theta_0(\alpha) = -\alpha - \frac{(\alpha-\mu)(1-\alpha)}{(1-\mu)(2-\alpha)}, \tag{2.3.111}$$

we obtain the optimal estimate (2.3.19) by the monotonicity of $\Theta_1(\alpha, -\rho_1)$ and $\Theta_4(\alpha, \rho_1)$.

Case 2. $\mu/(2-\mu) < \alpha \leqslant \mu$.

We first note that $\min\{1, \tilde{\rho}_\alpha\} = \tilde{\rho}_\alpha$ holds in this case by $\mu < 1/(2-\mu)$. We suppose that ρ_1 is given as $0 < \rho_1 < \tilde{\rho}_\alpha$. By virtue of the result in the Cases A and B, for $0 \leqslant \kappa < \kappa_\alpha$, the estimate (2.3.84) holds. Then, for given ρ_1, the optimal estimate (2.3.108) is obtained as in the Case 1. Now we will obtain the optimal estimate of left-hand side of (2.3.19) when ρ_1 varies in $(0, \tilde{\rho}_\alpha)$. Putting

$$\begin{aligned}\Theta_0(\alpha) &= \inf_{0<\rho_1<\tilde{\rho}_\alpha} \Theta_4(\alpha, \rho_1) = \Theta_4(\alpha, 0) \\ &= -\frac{(2-\mu)(1+\alpha)-2}{1-\mu} = -\alpha + \frac{\mu-\alpha}{1-\mu}, \end{aligned} \tag{2.3.112}$$

we obtain the optimal estimate (2.3.19) by the monotonicity of $\Theta_4(\alpha, \rho_1)$. □

Remark 2.3.5. *In Lemma 3.4 of [6], the power* $\Theta_0(\alpha)+\epsilon$ *of* $|v|$ *in the estimate* (2.3.19) *is replaced by* $-((2-\mu)(1+\alpha)-2)/(2(1-\mu))$. *The inequality* $\Theta_0(\alpha) < -((2-\mu)(1+\alpha)-2)/(2(1-\mu))$, *which can be verified easily, shows that the above proposition is an improvement of Lemma 3.4 of [6].*

We now introduce auxiliary wave operators

$$\Omega^{\pm}_{G,v} = \operatorname*{s-lim}_{t\to\pm\infty} U(t,0)^* U_{G,v}(t), \quad U_{G,v}(t) = U_0(t,0)M_{G,v}(t) \tag{2.3.113}$$

with

$$M_{G,v}(t) = e^{-i\int_0^t V^{\mathrm{s}}(v\tau+\tilde{c}(\tau))d\tau} \tag{2.3.114}$$

as in [8] and [6]. We know that

$$\Omega^{\pm}_{G,v} = W^{\pm} I^{\pm}_{G,v}, \quad I^{\pm}_{G,v} = \lim_{t\to\pm\infty} M_{G,v}(t), \tag{2.3.115}$$

by virtue of the estimate

$$|vt+\tilde{c}(t)| \geqslant ((1-\delta^2)/3)^{1/2}|\tilde{c}_0(t)| \geqslant a_\mu((1-\delta^2)/3)^{1/2}|t|^{2-\mu} \tag{2.3.116}$$

for $|v| \geqslant 3M_2/(1-\delta)$, which can be obtained in the same way as in obtaining the estimate (2.3.14). Here we used (2.2.10). Then the assumption $\gamma > 1/(2-\mu)$ yields that $I^{\pm}_{G,v}$ exist. As emphasized in [8] and [6], the Graf-type modifier $M_{G,v}(t)$ commutes with any operators. This fact will be used frequently. Then the following can be obtained as in [8] and [6].

Proposition 2.3.6. *Let* v *and* Φ_v *be as in Theorem 2.3.1,* $\Theta_0(\alpha)$ *be as in Proposition 2.3.4, and* $\epsilon > 0$. *Then*

$$\sup_{t\in\mathbb{R}} \|(U(t,0)\Omega^{-}_{G,v} - U_{G,v}(t))\Phi_v\| = O(|v|^{\Theta_0(\alpha)+\epsilon}) \tag{2.3.117}$$

holds as $|v|\to\infty$ *for* $V^{\mathrm{vs}} \in \mathscr{V}^{\mathrm{vs}}$ *and* $V^{\mathrm{s}} \in \mathscr{V}^{\mathrm{s}}_{\mu,\mu/(2-\mu)}$.

Proof. Compute as following,

$$\begin{aligned} U(t,0)^*U_{G,v}(t) - \Omega^{-}_{G,v} &= \int_{-\infty}^{t} \partial_\tau(U(\tau,0)^*U_{G,v}(\tau))d\tau \\ &= i\int_{-\infty}^{t} U(\tau,0)^* V^{\mathrm{s}}_v(\tau,x)U_{G,v}(\tau)d\tau \end{aligned} \tag{2.3.118}$$

with $V^s_v(t,x) = V^{\mathrm{vs}}(x) + V^{\mathrm{s}}(x) - V^{\mathrm{s}}(vt+\tilde{c}(t))$. Noting the unitarity of $U(t,0)$, we obtain

$$\begin{aligned} &\|(U(t,0)\Omega^{-}_{G,v} - U_{G,v}(t))\Phi_v\| = \|(\Omega^{-}_{G,v} - U(t,0)^*U_{G,v}(t))\Phi_v\| \\ &\leqslant \int_{-\infty}^{\infty} (\|V^{\mathrm{vs}}(x)U_0(\tau,0)\Phi_v\| + \|(V^{\mathrm{s}}(x) - V^{\mathrm{s}}(v\tau+\tilde{c}(\tau)))U_0(\tau,0)\Phi_v\|)d\tau, \end{aligned} \tag{2.3.119}$$

which yields the proposition by virtue of Propositions 2.3.3 and 2.3.4. We here used the commutativity of $M_{G,v}(t)$ mentioned above. □

Proof of Theorem 2.3.1. The proof is quite similar to the one of Theorem 2.4 in [38] (see also [8] and [6]). Suppose that $V^{\mathrm{vs}} \in \mathscr{V}^{\mathrm{vs}}$ and $V^{\mathrm{s}} \in \mathscr{V}^{\mathrm{s}}_{\mu,\tilde{\alpha}_\mu}$. We first note that S is represented as

$$S = (W^+)^* W^- = I_{G,v}(\Omega^+_{G,v})^*\Omega^-_{G,v}, \quad I_{G,v} = I^+_{G,v}\overline{I^-_{G,v}} = e^{-i\int_{-\infty}^{\infty} V^{\mathrm{s}}(v\tau+\tilde{c}(\tau))d\tau}. \tag{2.3.120}$$

By virtue of (2.3.14) and (2.3.116),

$$|vt+\tilde{c}(t)| \geqslant \max\{((1-\delta^2)/3)^{1/2}|v||t|, a_\mu((1-\delta^2)/3)^{1/2}|t|^{2-\mu}\} \tag{2.3.121}$$

is obtained for $|v| \geqslant 3M_2/(1-\delta)$. Therefore it follows by $\gamma > 1/(2-\mu)$ and the Lebesgue-dominated convergence theorem that $I_{G,v} \to 1$ as $|v| \to \infty$. Noting $[S, p_j] = [S - I_{G,v}, p_j - v_j]$, $(p_j - v_j)\Phi_v = (p_j\Phi_0)_v$ and

$$\begin{aligned} S - I_{G,v} &= I_{G,v}(\Omega^+_{G,v} - \Omega^-_{G,v})^*\Omega^-_{G,v} \\ &= -iI_{G,v}\int_{-\infty}^{\infty} U_{G,v}(t)^* V^s_v(t,x)U(t)\Omega^-_{G,v}dt \end{aligned} \tag{2.3.122}$$

we have

$$|v|(i[S,p_j]\Phi_v, \Psi_v) = I_{G,v}(I(v)+R(v)) \tag{2.3.123}$$

with

$$\begin{aligned} I(v) &= |v|\int_{-\infty}^{\infty}((V^s_v(t,x)U_{G,v}(t)(p_j\Phi_0)_v, U_{G,v}(t)\Psi_v) \\ &\qquad -(V^s_v(t,x)U_{G,v}(t)\Phi_v, U_{G,v}(t)(p_j\Psi_0)_v))dt, \end{aligned} \tag{2.3.124}$$

$$\begin{aligned} R(v) &= |v|\int_{-\infty}^{\infty}(((U(t,0)\Omega^-_{G,v} - U_{G,v}(t))(p_j\Phi_0)_v, V^s_v(t,x)U_{G,v}(t)\Phi_v) \\ &\qquad -((U(t,0)\Omega^-_{G,v} - U_{G,v}(t))\Phi_v, V^s_v(t,x)U_{G,v}(t)(p_j\Psi_0)_v))dt. \end{aligned} \tag{2.3.125}$$

By Propositions 2.3.3, 2.3.4 and 2.3.6, one has

$$R(v) = O(|v|^{1+(2(\Theta_0(\alpha)+\epsilon))}) \tag{2.3.126}$$

noting the commutativity of the Graf-modifier $M_{G,v}(t)$. Then we need the condition

$$1 + (2(\Theta_0(\alpha)+\epsilon)) < 0 \tag{2.3.127}$$

in order to get $R(v) \to 0$ as $|v| \to \infty$. This is equivalent to

$$\Theta_0(\alpha) < -1/2, \tag{2.3.128}$$

since one can take $\epsilon > 0$ so small that (2.3.127) holds for $\Theta_0(\alpha)$ satisfying (2.3.128). By simple calculation, we see that (2.3.128) is equivalent to

$$\alpha > \tilde{\alpha}_\mu. \tag{2.3.129}$$

The rest of argument is going on in the same way as in [38], [6] and [8]. Using the commutativity of $M_{G,v}(t)$ again, by the Avron-Herbst formula (2.2.4) and (2.3.4), we can compute

$$\begin{aligned}
&(V_v^s(t,x)U_{G,v}(t)(p_j\Phi_0)_v, U_{G,v}(t)\Psi_v) - (V_v^s(t,x)U_{G,v}(t)\Phi_0, U_{G,v}(t)(p_j\Psi_0)) \\
&= (V^{\mathrm{vs}}(x+vt+\tilde{c}(t))e^{-itp^2/2}p_j\Phi_0, e^{-itp^2/2}\Psi_0) \\
&\quad -(V^{\mathrm{vs}}(x+vt+\tilde{c}(t))e^{-itp^2/2}\Phi_0, e^{-itp^2/2}p_j\Psi_0) \\
&\quad +(i(\partial_{x_j}V^{\mathrm{s}})(x+vt+\tilde{c}(t))e^{-itp^2/2}\Phi_0, e^{-itp^2/2}\Psi_0). \qquad (2.3.130)
\end{aligned}$$

Then we write $I(v) = \int_{-\infty}^{\infty} I_v(\tau)d\tau$ where

$$\begin{aligned}
I_v(\tau) &= (V^{\mathrm{vs}}(x+\hat{v}\tau+\tilde{c}(\tau/|v|))e^{-i(\tau/|v|)p^2/2}p_j\Phi_0, e^{-i(\tau/|v|)p^2/2}\Psi_0) \\
&\quad -(V^{\mathrm{vs}}(x+\hat{v}\tau+\tilde{c}(\tau/|v|))e^{-i(\tau/|v|)p^2/2}\Phi_0, e^{-i(\tau/|v|)p^2/2}p_j\Psi_0) \\
&\quad +(i(\partial_{x_j}V^{\mathrm{s}})(x+\hat{v}\tau+\tilde{c}(\tau/|v|))e^{-i(\tau/|v|)p^2/2}\Phi_0, e^{-i(\tau/|v|)p^2/2}\Psi_0) \qquad (2.3.131)
\end{aligned}$$

by changing the integral variable such that $t = \tau/|v|$. This implies

$$\begin{aligned}
\lim_{|v|\to\infty} I(v) = \int_{-\infty}^{\infty} &((V^{\mathrm{vs}}(x+\hat{v}\tau)p_j\Phi_0, \Psi_0) - (V^{\mathrm{vs}}(x+\hat{v}\tau)\Phi_0, p_j\Psi_0) \\
&+(i(\partial_{x_j}V^{\mathrm{s}})(x+\hat{v}\tau)\Phi_0, \Psi_0))d\tau \qquad (2.3.132)
\end{aligned}$$

by the Lebesgue-dominated convergence theorem. In fact, noting that $\partial_{x_j}V^{\mathrm{s}} \in \mathscr{V}^{\mathrm{vs}}$ by assumption,

$$|I_v(\tau)| \leqslant C(\|V^{\mathrm{vs}}(x)\langle p\rangle^{-2}F(|x| \geqslant \lambda|\tau|)\|_{\mathscr{B}(L^2)} + \langle|\tau|\rangle^{-2} + \langle|\tau|\rangle^{-1-\alpha}) \qquad (2.3.133)$$

can be obtained as in the proof of Proposition 2.3.3 and $|I_v(\tau)|$ is integrable independently of $|v|$. □

By virtue of Theorem 2.3.1 and the Plancherel formula associated with the Radon transform (see Helgason[18]), Theorem 2.1.3 can be shown in the quite same way as in the proof of Theorem 1.2 in [38] (see also Enss-Weder [13]). We thus can omit the proof of Theorem 2.1.3.

2.4 Long-range Case

The main purpose of this subsection is showing the following reconstruction formula, which yields the proof of Theorem 2.1.4.

Theorem 2.4.1. (Reconstruction Formula [4]) *Let* $\hat{v} \in \mathbb{R}^d$ *be given such that* $|\hat{v}\cdot e_1| < 1$. *Put* $v = |v|\hat{v}$. *Let* $\eta > 0$ *be given, and* $\Phi_0, \Psi_0 \in \mathscr{S}(\mathbb{R}^d)$ *be such that* $\hat{\Phi}_0, \hat{\Psi}_0 \in C_0^\infty(\mathbb{R}^d)$ *with* $\operatorname{supp}\hat{\Phi}_0, \operatorname{supp}\hat{\Psi}_0 \subset \{\xi \in \mathbb{R}^d \mid |\xi| \leqslant \eta\}$. *Put*

$\Phi_v = e^{iv\cdot x}\Phi_0, \Psi_v = e^{iv\cdot x}\Psi_0$. *Let* $V^{\mathrm{vs}} \in \mathscr{V}^{\mathrm{vs}}$, $V^{\mathrm{s}} \in \mathscr{V}^{\mathrm{s}}_{\mu,\tilde{\alpha}_{\mu,D}}$ *and* $V^{\mathrm{l}} \in \mathscr{V}^{\mathrm{l}}_{\mu,\tilde{\gamma}_\mu}$, *where* $\tilde{\alpha}_{\mu,D}$ *and* $\tilde{\gamma}_\mu$ *are the same as in Theorem 2.1.4. Then*

$$\begin{aligned}|v|(i[S_D,p_j]\Phi_v,\Psi_v) = \int_{-\infty}^{\infty}((V^{\mathrm{vs}}(x+\hat{v}t)p_j\Phi_0,\Psi_0) - (V^{\mathrm{vs}}(x+\hat{v}t)\Phi_0,p_j\Psi_0)\\ +(i(\partial_{x_j}V^{\mathrm{s}})(x+\hat{v}t)\Phi_0,\Psi_0) + (i(\partial_{x_j}V^{\mathrm{l}})(x+\hat{v}t)\Phi_0,\Psi_0))dt + o(1) \quad (2.4.1)\end{aligned}$$

holds as $|v| \to \infty$ *for* $1 \leqslant j \leqslant d$.

We first need the following proposition.

Proposition 2.4.2. *Let* v *and* Φ_v *be as in Theorem 2.4.1, and* $V^{\mathrm{l}} \in \mathscr{V}^{\mathrm{l}}_{\mu,1/(2(2-\mu))}$. *Let* κ_j *be such that* $0 \leqslant \kappa_j \leqslant 1$ *for* $1 \leqslant j \leqslant 3$. *Then there exists a positive constant* C *such that*

$$\begin{aligned}\|\langle x\rangle^2 M_{D,v}(t)\Phi_0\| \leqslant C(1 + |v|^{-\kappa_1\gamma_{D,1}}|t|^{2-\tilde{\sigma}_{\kappa_1}\gamma_{D,1}}\\ +|v|^{-2\kappa_2\gamma_{D,1}}|t|^{2(2-\tilde{\sigma}_{\kappa_2}\gamma_{D,1})} + |v|^{-\kappa_3\gamma_{D,2}}|t|^{3-\tilde{\sigma}_{\kappa_3}\gamma_{D,2}}) \qquad (2.4.2)\end{aligned}$$

holds as $|v| \to \infty$, *where* $M_{D,v} = e^{-i\int_0^t V^{\mathrm{l}}(p\tau+v\tau+\tilde{c}(\tau))d\tau}$, *and* $\gamma_{D,l} = \gamma_D + l/(2-\mu)$ *for* $1 \leqslant l \leqslant 2$.

Proof. Take $f \in C_0^\infty(\mathbb{R}^d)$ such that $f\hat{\Phi}_0 = \hat{\Phi}_0$ and $\operatorname{supp} f \subset \{\xi \in \mathbb{R}^d \mid |\xi| \leqslant \eta\}$. In the same way as in [8] and [6], one can obtain

$$\begin{aligned}&\|x^2 M_{D,v}(t)f(p)\langle x\rangle^{-2}\|_{\mathscr{B}(L^2)} \leqslant \|M_{D,v}(t)f(p)x^2\langle x\rangle^{-2}\|_{\mathscr{B}(L^2)}\\ &+2\|M_{D,v}(t)\int_0^t \tau(\nabla_x V^{\mathrm{l}})(p\tau+v\tau+\tilde{c}(\tau))d\tau f(p)\cdot x\langle x\rangle^{-2}\|_{\mathscr{B}(L^2)}\\ &+2\|M_{D,v}(t)(\nabla_x f)(p)\cdot x\langle x\rangle^{-2}\|_{\mathscr{B}(L^2)}\\ &+2\|M_{D,v}(t)\int_0^t \tau(\nabla_x V^{\mathrm{l}})(p\tau+v\tau+\tilde{c}(\tau))d\tau\cdot(\nabla_x f)(p)\langle x\rangle^{-2}\|_{\mathscr{B}(L^2)}\\ &+\|M_{D,v}(t)(\triangle_x f)(p)\langle x\rangle^{-2}\|_{\mathscr{B}(L^2)}\\ &+\|M_{D,v}(t)\left(\int_0^t \tau(\nabla_x V^{\mathrm{l}})(p\tau+v\tau+\tilde{c}(\tau))d\tau\right)^2 f(p)\langle x\rangle^{-2}\|_{\mathscr{B}(L^2)}\\ &+\|M_{D,v}(t)\int_0^t \tau^2(\triangle_x V^{\mathrm{l}})(p\tau+v\tau+\tilde{c}(\tau))d\tau f(p)\langle x\rangle^{-2}\|_{\mathscr{B}(L^2)}. \qquad (2.4.3)\end{aligned}$$

Put $\delta = |\hat{v}\cdot e_1| < 1$. If $|\xi| \leqslant \eta$ and $|v| \geqslant \max\{3M_2/(1-\delta), 12\eta/(1-\delta)\}$, then we have the inequality

$$|\xi t + vt + \tilde{c}(t)| \geqslant d_1 \max\{|v||t|, |\tilde{c}_0(t)|\}, \qquad (2.4.4)$$

which yields the inequality

$$|\xi t + vt + \tilde{c}(t)| \geqslant d_\kappa |v|^\kappa |t|^{\tilde{\sigma}_\kappa} \qquad (2.4.5)$$

for $0 \leqslant \kappa \leqslant 1$ because $|\tilde{c}_0(t) \geqslant a_\mu |t|^{2-\mu}$ by (2.2.10), as in the proof of Proposition 2.3.4 (see (2.3.39) and (2.3.40)). By using (2.4.5), the straightforward calculation leads to

$$\sup_{\xi \in \mathrm{supp}\, f} \left| \int_0^t \tau (\nabla_x V^1)(p\tau + v\tau + \tilde{c}(\tau)) d\tau \right| \leqslant C|v|^{-\kappa\gamma_{D,1}} |t|^{2-\tilde{\sigma}_\kappa \gamma_{D,1}}, \tag{2.4.6}$$

$$\sup_{\xi \in \mathrm{supp}\, f} \left| \int_0^t \tau^2 (\triangle_x V^1)(p\tau + v\tau + \tilde{c}(\tau)) d\tau \right| \leqslant C|v|^{-\kappa\gamma_{D,2}} |t|^{3-\tilde{\sigma}_\kappa \gamma_{D,1}} \tag{2.4.7}$$

with $0 \leqslant \kappa \leqslant 1$. Here we used $1 - \tilde{\sigma}_\kappa \gamma_{D,1} \geqslant -\gamma_D(2-\mu) > -1$ and $2 - \tilde{\sigma}_\kappa \gamma_{D,2} \geqslant -\gamma_D(2-\mu) > -1$ by assumption. These estimates yield the proposition. □

Then the following proposition can be shown as in [8] and [6].

Proposition 2.4.3. *Let v and Φ_v be as in Theorem 2.4.1, and $V^1 \in \mathscr{V}^1_{\mu,1/(2(2-\mu))}$. Then*

$$\int_{-\infty}^{\infty} \|V^{\mathrm{vs}}(x) U_D(t) \Phi_v\| dt = O(|v|^{-1}) \tag{2.4.8}$$

holds as $|v| \to \infty$ for $V^{\mathrm{vs}} \in \mathscr{V}^{\mathrm{vs}}$, where $U_D(t) = U_0(t,0) M_D(t)$ and $M_D(t)$ is the same as in (2.1.15)*.*

Proof. One has to only note that

$$\|V^{\mathrm{vs}}(x) U_D(t) \Phi_v\| = \|V^{\mathrm{vs}}(x + vt + \tilde{c}(t)) e^{-itp^2/2} M_{D,v}(t) \Phi_0\| \tag{2.4.9}$$

holds by (2.2.4) and (2.3.4). Then This proposition can be proved in the same way as in the proof of Proposition 2.3.3 by virtue of Proposition 2.4.2. One should also take account of the estimate (2.4.2) with $\kappa_1 = \kappa_2 = \kappa_3 = 0$, that is,

$$\|\langle x \rangle^2 M_{D,v}(t) \Phi_0\| \leqslant C(1 + |t|^{2(1-\gamma_D(2-\mu))}) \tag{2.4.10}$$

and

$$\int_{-\infty}^{\infty} \langle |v||t| \rangle^{-2} |t|^{2(1-\gamma_D(2-\mu))} dt = O(|t|^{-1-2(1-\gamma_D(2-\mu))}) \tag{2.4.11}$$

by $-2 + 2(1-\gamma_D(2-\mu)) = -2\gamma_D(2-\mu) < -1$. □

The following proposition can be also proved in the same way as in the proof of Proposition 2.3.4.

Proposition 2.4.4. *Let v and Φ_v be as in Theorem 2.4.1, $\epsilon > 0$ and $V^1 \in \mathscr{V}^1_{\mu,1/(2(2-\mu))}$. Put*

$$\Theta_{0,D}(\alpha, \gamma_D) = \begin{cases} -\dfrac{\alpha(3-2\alpha)}{4-3\alpha} - \dfrac{2(\alpha-\mu)(1-\alpha)}{(1-\mu)(4-3\alpha)} & \gamma_D > \Gamma_1(\alpha) \\ -\dfrac{2\alpha(1-\alpha)}{4-3\alpha} - \dfrac{2(\alpha-\mu)(1-\alpha)}{(1-\mu)(4-3\alpha)} & \\ \qquad -\dfrac{2\gamma_D(2-\mu)-1}{1-\mu} & \gamma_D \leqslant \Gamma_1(\alpha) \end{cases} \tag{2.4.12}$$

in the case where $\alpha > \alpha_-(\mu)$*, while, in the case where* $\mu/(2-\mu) < \alpha \leqslant \alpha_-(\mu)$*, put*

$$\Theta_{0,D}(\alpha,\gamma_D) = \begin{cases} -\alpha - \dfrac{\alpha-\mu}{1-\mu} & \gamma_D > \Gamma_2(\alpha) \\ -\dfrac{2\alpha}{3} - \dfrac{2(\alpha-\mu)}{3(1-\mu)} - \dfrac{2\gamma_D(2-\mu)-1}{3(1-\mu)} & \gamma_D \leqslant \Gamma_2(\alpha), \end{cases} \tag{2.4.13}$$

where

$$\alpha_-(\mu) = \frac{3-\sqrt{9-8\mu(2-\mu)}}{2(2-\mu)}, \tag{2.4.14}$$

$$\Gamma_1(\alpha) = \frac{1}{2(2-\mu)} + \frac{2\mu-\alpha(1+\mu)}{2(2-\mu)(4-3\alpha)}, \tag{2.4.15}$$

$$\Gamma_2(\alpha) = \frac{1}{2(2-\mu)} + \frac{\alpha(2-\mu)-\mu}{2(2-\mu)}. \tag{2.4.16}$$

Then

$$\int_{-\infty}^{\infty} \|(V^{\mathrm{s}}(x) - V^{\mathrm{s}}(vt+\tilde{c}(t)))U_D(t)\Phi_v\| dt = O(|v|^{\Theta_{0,D}(\alpha,\gamma_D)+\epsilon}) \tag{2.4.17}$$

holds as $|v| \to \infty$ *for* $V^{\mathrm{s}} \in \mathscr{V}^{\mathrm{s}}_{\mu,\mu/(2-\mu)}$.

Proof. We borrow some notation used in Proposition 2.3.4. Take $f \in C_0^\infty(\mathbb{R}^d)$ such that $f\hat{\Phi}_0 = \hat{\Phi}_0$ and $\operatorname{supp} f \subset \{\xi \in \mathbb{R}^d \bigm| |\xi| \leqslant \eta\}$. Then one has

$$\begin{aligned} &\|(V^{\mathrm{s}}(x) - V^{\mathrm{s}}(vt+\tilde{c}(t)))U_D(t)\Phi_v\| \\ &\quad = \|(V^{\mathrm{s}}(x+vt+\tilde{c}(t)) - V^{\mathrm{s}}(vt+\tilde{c}(t)))e^{-itp^2/2}f(p)M_{D,v}(t)\Phi_0\| \end{aligned} \tag{2.4.18}$$

as in the proof of Proposition 2.3.4. This can be estimated as

$$\|(V^{\mathrm{s}}(x) - V^{\mathrm{s}}(vt+\tilde{c}(t)))U_D(t)\Phi_v\| \leqslant I_0, \quad I_0 = 2\sup_{y\in\mathbb{R}^d}|V^{\mathrm{s}}(y)|\|\Phi_0\|. \tag{2.4.19}$$

This can be also estimated as

$$\|(V^{\mathrm{s}}(x) - V^{\mathrm{s}}(vt+\tilde{c}(t)))U_D(t)\Phi_v\| \leqslant I_{1,\rho} + \tilde{I}_{2,\rho} + I_{3,\rho}, \tag{2.4.20}$$

with

$$\begin{aligned} I_{1,\rho} &= 2\sup_{y\in\mathbb{R}^d}|V^{\mathrm{s}}(y)|\|M_{D,v}(t)\Phi_0\| \\ &\qquad \times \|F(|x| \geqslant 3\lambda_1|v|^\rho|t|)e^{-itp^2/2}f(p)F(|x| < \lambda_1|v|^\rho|t|)\|_{\mathscr{B}(L^2)}, && (2.4.21) \\ \tilde{I}_{2,\rho} &= 2\sup_{y\in\mathbb{R}^d}|V^{\mathrm{s}}(y)|\|\langle x\rangle^2 M_{D,v}(t)\Phi_0\| \\ &\qquad \times \|F(|x| \geqslant 3\lambda_1|v|^\rho|t|)e^{-itp^2/2}f(p)F(|x| \geqslant \lambda_1|v|^\rho|t|)\langle x\rangle^{-2}\|_{\mathscr{B}(L^2)}, && (2.4.22) \\ I_{3,\rho} &= \|(V^{\mathrm{s}}(x+vt+\tilde{c}(t)) - V^{\mathrm{s}}(vt+\tilde{c}(t)))F(|x| < 3\lambda_1|v|^\rho|t|)\|_{\mathscr{B}(L^2)} \\ &\qquad \times \|e^{-itp^2/2}f(p)M_{D,v}(t)\Phi_0\|, && (2.4.23) \end{aligned}$$

where $3\lambda_1 = (1-\delta)/12$ and $0 < \rho \leqslant 1$. Here we note that I_0, $I_{1,\rho}$ and $I_{3,\rho}$ are the same as those in the proof of Proposition 2.3.4. Hence, by Step 1 of the proof of Proposition 2.3.4, we have

$$\int_{|t|\leqslant|v|^{-1}} I_0 dt = O(|v|^{-1}), \quad \int_{|v|^{-1}\leqslant|t|\leqslant|v|^{\sigma_1(\alpha,\rho_1,\kappa)}} I_{1,1} dt = O(|v|^{-1}), \tag{2.4.24}$$

$$\int_{|t|\geqslant|v|^{\sigma_1(\alpha,\rho_1,\kappa)}} I_{1,\rho_1} dt = O(|v|^{-\rho_1-(\rho_1+\sigma_1(\alpha,\rho_1,\kappa))N}), \tag{2.4.25}$$

$$\int_{|v|^{-1}\leqslant|t|\leqslant|v|^{\sigma_1(\alpha,\rho_1,\kappa)}} I_{3,1} dt + \int_{|t|\geqslant|v|^{\sigma_1(\alpha,\rho_1,\kappa)}} I_{3,\rho_1} dt = O(|v|^{\Theta_3(\alpha,\rho_1,\kappa)}), \tag{2.4.26}$$

with $N \in \mathbb{N}$, where $0 \leqslant \kappa < \kappa_6(\alpha, \rho_1)$ and $0 < \rho_1 < \hat{\rho}_\alpha$ if $\alpha > \mu$, while $0 \leqslant \kappa < \kappa_\alpha$ and $0 < \rho_1 < \tilde{\rho}_\alpha$ if $\mu/(2-\mu) < \alpha \leqslant \mu$. On the other hand, by virtue of Proposition 2.4.2, we have

$$\begin{aligned}\int_{|v|^{-1}\leqslant|t|\leqslant|v|^{\sigma_1(\alpha,\rho_1,\kappa)}} \tilde{I}_{2,1} dt = O(|v|^{-1}) \\ +O(|v|^{-1-(1-\gamma_D(2-\mu))}) + O(|v|^{-1-2(1-\gamma_D(2-\mu))}) = O(|v|^{-1})\end{aligned} \tag{2.4.27}$$

and

$$\begin{aligned}\int_{|t|\geqslant|v|^{\sigma_1(\alpha,\rho_1,\kappa)}} \tilde{I}_{2,\rho_1} dt &= O(|v|^{\theta_0(\alpha,\rho_1,\kappa)}) + O(|v|^{\tilde{\theta}_1(\alpha,\rho_1,\kappa,\tilde{\kappa}_1)}) \\ &\quad +O(|v|^{\tilde{\theta}_2(\alpha,\rho_1,\kappa,\tilde{\kappa}_2)}) + O(|v|^{\tilde{\theta}_3(\alpha,\rho_1,\kappa,\tilde{\kappa}_3)}),\end{aligned} \tag{2.4.28}$$

where

$$\theta_0(\alpha,\rho_1,\kappa) = -2\rho_1 - \sigma_1(\alpha,\rho_1,\kappa), \tag{2.4.29}$$
$$\tilde{\theta}_1(\alpha,\rho_1,\kappa,\tilde{\kappa}_1) = -2\rho_1 - \tilde{\kappa}_1\gamma_{D,1} + \sigma_1(\alpha,\rho_1,\kappa)(1-\tilde{\sigma}_{\tilde{\kappa}_1}\gamma_{D,1}), \tag{2.4.30}$$
$$\tilde{\theta}_2(\alpha,\rho_1,\kappa,\tilde{\kappa}_2) = -2\rho_1 - 2\tilde{\kappa}_2\gamma_{D,1} + \sigma_1(\alpha,\rho_1,\kappa)(3-2\tilde{\sigma}_{\tilde{\kappa}_2}\gamma_{D,1}), \tag{2.4.31}$$
$$\tilde{\theta}_3(\alpha,\rho_1,\kappa,\tilde{\kappa}_3) = -2\rho_1 - \tilde{\kappa}_3\gamma_{D,2} + \sigma_1(\alpha,\rho_1,\kappa)(2-\tilde{\sigma}_{\tilde{\kappa}_3}\gamma_{D,2}) \tag{2.4.32}$$

for $\tilde{\kappa}_1$, $\tilde{\kappa}_2$ and $\tilde{\kappa}_3$ satisfying

$$0 \leqslant \tilde{\kappa}_1 \leqslant 1, \quad 0 \leqslant \tilde{\kappa}_2 \leqslant 1, \quad 0 \leqslant \tilde{\kappa}_3 \leqslant 1, \tag{2.4.33}$$
$$1-\tilde{\sigma}_{\tilde{\kappa}_1}\gamma_{D,1} < 0 \quad ,3-2\tilde{\sigma}_{\tilde{\kappa}_2}\gamma_{D,1} < 0, \quad 2-\tilde{\sigma}_{\tilde{\kappa}_3}\gamma_{D,2} < 0. \tag{2.4.34}$$

Since $-1+\sigma_1(\alpha,\rho_1,\kappa)(1-\mu) < 0$, each $\tilde{\theta}_j(\alpha,\rho_1,\kappa,\tilde{\kappa}_j)$ is monotonically decreasing in $\tilde{\kappa}_j$. Therefore we obtain a better estimate for $\int_{|t|\geqslant|v|^{\sigma_1(\alpha,\rho_1,\kappa)}} \tilde{I}_{2,\rho_1} dt$ as follows.

$$\begin{aligned}\int_{|t|\geqslant|v|^{\sigma_1(\alpha,\rho_1,\kappa)}} \tilde{I}_{2,\rho_1} dt &= O(|v|^{\theta_0(\alpha,\rho_1,\kappa)}) + O(|v|^{\theta_1(\alpha,\rho_1,\kappa)}) \\ &\quad +O(|v|^{\theta_2(\alpha,\rho_1,\kappa)}) + O(|v|^{\theta_3(\alpha,\rho_1,\kappa)}),\end{aligned} \tag{2.4.35}$$

where

$$\theta_1(\alpha,\rho_1,\kappa) = \begin{cases} \theta_{1,D}(\rho_1)+\epsilon & \gamma_{D,1}\leqslant 1 \\ -2\rho_1-\gamma_{D,1}+\sigma_1(\alpha,\rho_1,\kappa)(1-\gamma_{D,1}) & \gamma_{D,1}>1, \end{cases} \tag{2.4.36}$$

$$\theta_2(\alpha,\rho_1,\kappa) = \begin{cases} \theta_{2,D}(\rho_1)+\epsilon & \gamma_{D,1}\leqslant 3/2 \\ -2\rho_1-2\gamma_{D,1}+\sigma_1(\alpha,\rho_1,\kappa)(3-2\gamma_{D,1}) & \gamma_{D,1}>3/2, \end{cases} \tag{2.4.37}$$

$$\theta_3(\alpha,\rho_1,\kappa) = \begin{cases} \theta_{1,D}(\rho_1)+\epsilon & \gamma_{D,2}\leqslant 2 \\ -2\rho_1-\gamma_{D,2}+\sigma_1(\alpha,\rho_1,\kappa)(2-\gamma_{D,2}) & \gamma_{D,2}>2, \end{cases} \tag{2.4.38}$$

with

$$\theta_{1,D}(\rho_1) = -2\rho_1-\frac{\gamma_D(2-\mu)}{1-\mu}, \quad \theta_{2,D}(\rho_1) = -2\rho_1-\frac{2\gamma_D(2-\mu)-1}{1-\mu}, \tag{2.4.39}$$

and $\epsilon>0$. As is seen easily, $\theta_1(\alpha,\rho_1,\kappa)$ with $\gamma_{D,1}>1$, $\theta_2(\alpha,\rho_1,\kappa)$ with $\gamma_{D,1}>3/2$ and $\theta_3(\alpha,\rho_1,\kappa)$ with $\gamma_{D,2}>2$ are all less than $\Theta_3(\alpha,\rho_1,\kappa)=-\alpha+\sigma_1(\alpha,\rho_1,\kappa)(1-\alpha)$ by $-1-\sigma_1(\alpha,\rho_1,\kappa)<0$ and $\alpha<1$. Here we used

$$\begin{aligned} &-2\gamma_{D,1}+\sigma_1(\alpha,\rho_1,\kappa)(3-2\gamma_{D,1}) \\ &\quad = -2-(2\gamma_{D,1}-2)+\sigma_1(\alpha,\rho_1,\kappa)(1-(2\gamma_{D,1}-2)), \end{aligned} \tag{2.4.40}$$

$$\begin{aligned} &-\gamma_{D,2}+\sigma_1(\alpha,\rho_1,\kappa)(2-\gamma_{D,2}) \\ &\quad = -1-(\gamma_{D,2}-1)+\sigma_1(\alpha,\rho_1,\kappa)(1-(\gamma_{D,2}-1)), \end{aligned} \tag{2.4.41}$$

$2\gamma_{D,1}-2>1$ if $\gamma_{D,1}>3/2$, and $\gamma_{D,2}-1>1$ if $\gamma_{D,2}>2$. We note that $\theta_{1,D}(\rho_1)<\theta_{2,D}(\rho_1)$ holds by $\gamma_D(2-\mu)-1<0$. Thus we obtain optimal estimate

$$\begin{aligned} &\int_{-\infty}^{\infty}\|(V^{\mathrm{s}}(x)-V^{\mathrm{s}}(vt+\tilde{c}(t)))U_D(t)\Phi_v\|dt \\ &\quad = \begin{cases} O(|v|^{\max\{\Theta_3(\alpha,\rho_1,\kappa),\theta_0(\alpha,\rho_1,\kappa)\}})+O(|v|^{\theta_{2,D}(\rho_1)+\epsilon}) & \gamma_{D,1}\leqslant 3/2 \\ O(|v|^{\max\{\Theta_3(\alpha,\rho_1,\kappa),\theta_0(\alpha,\rho_1,\kappa)\}}) & \gamma_{D,1}>3/2 \end{cases} \end{aligned} \tag{2.4.42}$$

for given α,ρ_1 and κ by (2.4.24), (2.4.25), (2.4.26), (2.4.27), (2.4.28) and (2.4.35). Here we emphasize that $\theta_{2,D}(\rho_1)$ is monotonically decreasing in ρ_1. Now we will divide the situation into cases for ease of consideration.

Case 1. $\alpha>\mu$.

We first note that $\theta_0(\alpha,\rho_1,\kappa)$ is monotonically increasing in κ, and that

$$\begin{aligned} &\theta_0(\alpha,\rho_1,\kappa)<\Theta_3(\alpha,\rho_1,\kappa) \\ &\quad \Longleftrightarrow \kappa<1-\frac{(2-\alpha)(1-\rho_1)}{(1+\alpha)(2+2\rho_1(1-\mu)-\alpha(2-\mu))} =: \tilde{\kappa}_0(\alpha,\rho_1) \end{aligned} \tag{2.4.43}$$

holds. If $\tilde{\kappa}_0(\alpha, \rho_1) \leqslant 0$, then $\theta_0(\alpha, \rho_1, \kappa) \geqslant \Theta_3(\alpha, \rho_1, \kappa)$ holds for any $\kappa > 0$. We recall that if $0 < \rho_1 < \hat{\rho}_\alpha$, then κ can vary in the interval $[0, \kappa_6(\alpha, \rho_1))$, while if $\hat{\rho}_\alpha \leqslant \rho_1 < \min\{1, \tilde{\rho}_\alpha\}$, then κ can vary in the interval $[0, \kappa_\alpha)$. We also see that

$$\tilde{\kappa}_0(\alpha, \rho_1) < \kappa_6(\alpha, \rho_1) \iff \rho_1 < 1, \tag{2.4.44}$$

$$\tilde{\kappa}_0(\alpha, \rho_1) < \kappa_\alpha \iff \rho_1 < -\frac{(2-\mu)\alpha^2 - 3\alpha + 2\mu}{(1-\mu)(4-3\alpha)} =: \hat{\rho}_{\alpha,D}, \tag{2.4.45}$$

$$\tilde{\kappa}_0(\alpha, \rho_1) > 0 \iff \rho_1 > \frac{\alpha((1+\alpha)(2-\mu) - 3)}{4 + \alpha - 2\mu(1+\alpha)} =: \check{\rho}_{\alpha,D}, \tag{2.4.46}$$

$$\check{\rho}_{\alpha,D} \leqslant 0 \iff \alpha \leqslant \frac{1+\mu}{2-\mu}, \tag{2.4.47}$$

hold, by simple calculation. Here we note that $(1+\mu)/(2-\mu) > \mu$ because of $(1+\mu)/(2-\mu) - \mu = (\mu^2 - \mu + 1)/(2-\mu) > 0$. If $(1+\mu)/(2-\mu) < \alpha \leqslant \gamma < 1$, then one has $\check{\rho}_{\alpha,D} < \hat{\rho}_\alpha$ besides $\check{\rho}_{\alpha,D} > 0$. In fact, one can see that

$$\check{\rho}_{\alpha,D} < \hat{\rho}_\alpha \iff 0 < (1-\mu)\alpha^3 - (2-\mu)\alpha^2 + 3\alpha - 2\mu =: \eta_0(\alpha) \tag{2.4.48}$$

by straightforward computation. Since $\eta_0'(\alpha) = 3(1-\mu)\alpha^2 - 2(2-\mu)\alpha + 3 > 0$ by $(2-\mu)^2 - 9(1-\mu) = \mu^2 + 5\mu - 5 < 0$ when $0 \leqslant \mu < 2 - 3/(1+\gamma) < 1/2$, $\eta_0((1+\mu)/(2-\mu)) = (1-\mu)(((1+\mu)/(2-\mu))^3 + 1) > 0$ leads to $\eta_0(\alpha) > 0$ for $\alpha > (1+\mu)/(2-\mu)$. Here we take account of that $(1+\mu)/(2-\mu) < \gamma$ with $0 \leqslant \mu < 1$ is equivalent to $0 \leqslant \mu < 2 - 3/(1+\gamma)$. It can be verified easily that $\hat{\rho}_\alpha < \hat{\rho}_{\alpha,D} < 1$ holds. Noting that

$$\check{\rho}_{\alpha,D} < \tilde{\rho}_\alpha \iff 0 > 2(2-\mu)\alpha^2 - (5-\mu)\alpha + 2\mu =: \eta_1(\alpha), \tag{2.4.49}$$

one can also verify easily that $\hat{\rho}_{\alpha,D} < \tilde{\rho}_\alpha$ holds when $\mu < \alpha \leqslant \gamma < 1$, by $\eta_1(\mu) = -2\mu(1-\mu)(3-2\mu) < 0$ and $\eta_1(1) = -1 + \mu < 0$. Therefore we have

$$\begin{aligned} \theta_0(\alpha, \rho_1, \tilde{\kappa}_0(\alpha, \rho_1)) &= \Theta_3(\alpha, \rho_1, \tilde{\kappa}_0(\alpha, \rho_1)) \\ &= -\frac{\alpha}{2-\alpha} - \frac{2\rho_1(1-\alpha)}{2-\alpha} =: \hat{\theta}_0(\alpha, \rho_1) \end{aligned} \tag{2.4.50}$$

for $\max\{0, \check{\rho}_{\alpha,D}\} < \rho_1 < \hat{\rho}_{\alpha,D}$. We note that $\hat{\theta}_0(\alpha, \rho_1)$ is monotonically decreasing in ρ_1 by $\alpha \leqslant \gamma < 1$. If $\gamma_{D,1} > 3/2$, then we have the optimal estimate (2.4.17) with

$$\begin{aligned} \Theta_{0,D}(\alpha, \gamma_D) &= \inf_{\max\{0, \check{\rho}_{\alpha,D}\} < \rho_1 < \hat{\rho}_{\alpha,D}} \hat{\theta}_0(\alpha, \rho_1) = \hat{\theta}_0(\alpha, \hat{\rho}_{\alpha,D}) \\ &= -\frac{\alpha(3-2\alpha)}{4-3\alpha} - \frac{2(\alpha-\mu)(1-\alpha)}{(1-\mu)(4-3\alpha)}. \end{aligned} \tag{2.4.51}$$

Here we note that $\hat{\theta}_0(\alpha, \hat{\rho}_{\alpha,D}) = \Theta_4(\alpha, \hat{\rho}_{\alpha,D})$ holds. On the other hand, if $\gamma_{D,1} \leqslant 3/2$, then we have the optimal estimate (2.4.17) with

$$\Theta_{0,D}(\alpha, \gamma_D) = \max\{\hat{\theta}_0(\alpha, \hat{\rho}_{\alpha,D}), \theta_{2,D}(\hat{\rho}_{\alpha,D})\}, \tag{2.4.52}$$

by virtue of the fact that both $\hat{\theta}_0(\alpha, \rho_1)$ and $\theta_{2,D}(\rho_1)$ are monotonically decreasing in ρ_1. By simple calculation, we see that $\hat{\theta}_0(\alpha, \hat{\rho}_{\alpha,D}) > \theta_{2,D}(\hat{\rho}_{\alpha,D})$ is equivalent to

$$\gamma_D > \frac{1}{2(2-\mu)} + \frac{2\mu - \alpha(1+\mu)}{2(2-\mu)(4-3\alpha)} =: \Gamma_1(\alpha). \tag{2.4.53}$$

Here we note that $\Gamma_1(\alpha) + 1/(2-\mu) < 3/2$ holds. In fact, $\Gamma_1(\alpha) + 1/(2-\mu) \geqslant 3/2$ is equivalent to $(8 - 10\mu)\alpha \geqslant 12 - 14\mu$. When $\mu = 4/5$, this inequality does not hold as can be seen easily. When $0 \leqslant \mu < 4/5$, this inequality is equivalent to $\alpha \geqslant (6 - 7\mu)/(4 - 5\mu)$, but there is no α satisfying both this inequality and $\alpha \leqslant \gamma < 1$, because $(6 - 7\mu)/(4 - 5\mu) = 1 + 2(1 - \mu)/(4 - 5\mu) > 1$. When $4/5 < \mu < 1$, the above inequality is equivalent to $\alpha \leqslant (7\mu - 6)/(5\mu - 4) - \mu = -(1 - \mu)(6 - 5\mu)/(5\mu - 4) < 0$. Hence, if $\gamma_D > \Gamma_1(\alpha)$, then we have the optimal estimate (2.4.4) with (2.4.51), while, if $\gamma_D \leqslant \Gamma_1(\alpha)$, then we have the optimal estimate (2.4.4) with

$$\Theta_{0,D}(\alpha, \gamma_D) = -\frac{2\alpha(1-\alpha)}{4-3\alpha} - \frac{2(\alpha-\mu)(1-\alpha)}{(1-\mu)(4-3\alpha)} - \frac{2\gamma_D(2-\mu)-1}{1-\mu}. \tag{2.4.54}$$

Case 2. $\mu/(2-\mu) < \alpha \leqslant \mu$.

We note that $\tilde{\kappa}_0(\alpha, \rho_1) > 0$ holds for $0 < \rho_1 < \tilde{\rho}_\alpha$, but $\hat{\rho}_{\alpha,D}$ is not always positive, in this case. In fact, putting

$$\alpha_-(\mu) = \frac{3 - \sqrt{9 - 8\mu(2-\mu)}}{2(2-\mu)}, \tag{2.4.55}$$

if $\mu/(2 - \mu) < \alpha \leqslant \alpha_-(\mu)$, then $\hat{\rho}_{\alpha,D} \leqslant 0$, while, if $\alpha_-(\mu) < \alpha \leqslant \mu$, then $0 < \hat{\rho}_{\alpha,D} < \tilde{\rho}_\alpha$. Here we used $\eta_1(\alpha_-(\mu)) = (1 + \mu)\alpha_-(\mu) - 2\mu < 0$ because of $\alpha_-(\mu) < \mu < 2\mu/(1 + \mu)$.

We first consider the case where $\alpha_-(\mu) < \alpha \leqslant \mu$. Then (2.4.50) holds for $0 < \rho_1 < \hat{\rho}_{\alpha,D}$. Therefore we obtain the same result as in Case 1.

We next consider the case where $\mu/(2 - \mu) < \alpha \leqslant \alpha_-(\mu)$. Then $\theta_0(\alpha, \rho_1, \kappa) < \Theta_3(\alpha, \rho_1, \kappa)$ holds for $0 < \rho_1 < \tilde{\rho}_\alpha$ and $0 \leqslant \kappa < \kappa_\alpha$. In order to get a better estimate for $\int_{-\infty}^{\infty} \|(V^{\mathrm{s}}(x) - V^{\mathrm{s}}(vt + \tilde{c}(t)))U_D(t)\Phi_v\|dt$, we have only to replace $\Theta_3(\alpha, \rho_1, \kappa)$ by $\Theta_4(\alpha, \rho_1) + \epsilon$. Hence, if $\gamma_{D,1} > 3/2$, then one can obtain the optimal estimate (2.4.17) with

$$\Theta_{0,D}(\alpha, \gamma_D) = \Theta_4(\alpha, 0) = -\alpha + \frac{\mu - \alpha}{1 - \mu}, \tag{2.4.56}$$

as in the proof of Proposition 2.3.4. We next suppose that $\gamma_{D,1} \leqslant 3/2$. We note that

$$\begin{aligned} &\theta_{2,D}(\rho_1) < \Theta_4(\alpha, \rho_1) \\ &\qquad \Longleftrightarrow \rho_1 > \frac{(2-\mu)(1+\alpha) - 2 - 2\gamma_D(2-\mu) + 1}{3(1-\mu)} =: \rho_{\alpha,D,-} \end{aligned} \tag{2.4.57}$$

holds. Here we took account of that $\Theta_4(\alpha, \rho_1)$ is monotonically increasing in ρ_1, while $\theta_{2,D}(\rho_1)$ is monotonically decreasing in ρ_1. Hence, if

$$\gamma_D > \frac{1}{2(2-\mu)} + \frac{\alpha(2-\mu) - \mu}{2(2-\mu)} =: \Gamma_2(\alpha), \tag{2.4.58}$$

then one can also obtain the optimal estimate (2.4.4) with (2.4.56), since $\rho_{\alpha,D,-} < 0$. Here we note that $\Gamma_2(\alpha) + 1/(2-\mu) < 3/2$ can be seen easily. On the other hand, if $\gamma_D \leqslant \Gamma_2(\alpha)$, then one can also obtain the optimal estimate (2.4.4) with

$$\Theta_{0,D}(\alpha, \gamma_D) = \Theta_4(\alpha, \rho_{\alpha,D,-}) = -\frac{2\alpha}{3} - \frac{2(\alpha-\mu)}{3(1-\mu)} - \frac{2\gamma_D(2-\mu) - 1}{3(1-\mu)}. \tag{2.4.59}$$

This completes the proof of this proposition. □

The following proposition is the key in this subsection.

Proposition 2.4.5. *Let v and Φ_v be as in Theorem 2.4.1, and $V^1 \in \mathscr{V}^1_{\mu,1/(2(2-\mu))}$. Put*

$$\Theta_{1,D}(\gamma_D) = \begin{cases} -1 & \gamma_D > 1/2 \\ -\dfrac{2\gamma_D(2-\mu) - 1}{1-\mu} & \gamma_D \leqslant 1/2. \end{cases} \tag{2.4.60}$$

Then

$$\int_{-\infty}^{\infty} \|(V^1(x) - V^1(t(p - \tilde{b}(t)) + \tilde{c}(t)))U_D(t)\Phi_v\| dt = O(|v|^{\Theta_{1,D}(\gamma_D)+\epsilon}) \tag{2.4.61}$$

holds as $|v| \to \infty$.

Proof. One should note that

$$U_0(t,0)V^1(pt + \tilde{c}(t)) = V^1(t(p - \tilde{b}(t)) + \tilde{c}(t))U_0(t,0) \tag{2.4.62}$$

holds by virtue of (2.2.4). Take $f \in C_0^\infty(\mathbb{R}^d)$ such that $f\hat{\Phi}_0 = \hat{\Phi}_0$ and $\operatorname{supp} f \subset \{\xi \in \mathbb{R}^d \mid |\xi| \leqslant \eta\}$. By virtue of (2.2.4) and (2.3.4), one has

$$\begin{aligned} &\|(V^1(x) - V^1(t(p - \tilde{b}(t)) + \tilde{c}(t)))U_D(t)\Phi_v\| \\ &\quad = \|(V^1(x + vt + \tilde{c}(t)) - V^1(pt + vt + \tilde{c}(t)))e^{-itp^2/2} f(p) M_{D,v}(t)\Phi_0\|. \end{aligned} \tag{2.4.63}$$

Put $\delta = |\hat{v} \cdot e_1| < 1$. Let $|\xi| \leqslant \eta$ and $|v| \geqslant \max\{3M_2/(1-\delta), 12\eta/(1-\delta)\}$. Then (2.4.5) holds as mentioned above. Here we introduce $V^{\mathrm{l}}_{v,t,\kappa}(x)$ as

$$V^{\mathrm{l}}_{v,t,\kappa}(x) = V^{\mathrm{l}}(x) g_\kappa(x/(|v|^\kappa |t|^{\tilde{\sigma}_\kappa})), \tag{2.4.64}$$

where $g_\kappa \in C^\infty(\mathbb{R}^d)$ such that $0 \leqslant g_\kappa \leqslant 1$ and

$$g_\kappa(x) = \begin{cases} 1 & |x| \geqslant d_\kappa \\ 0 & |x| \leqslant d_\kappa/2. \end{cases} \tag{2.4.65}$$

Then we have

$$\begin{aligned} &\|(V^{\mathrm{l}}(x) - V^{\mathrm{l}}(t(p-\tilde{b}(t)) + \tilde{c}(t)))U_D(t)\Phi_v\| \\ &= \|(V^{\mathrm{l}}(x+vt+\tilde{c}(t)) - V^{\mathrm{l}}_{v,t,\kappa}(pt+vt+\tilde{c}(t)))e^{-itp^2/2} f(p) M_{D,v}(t)\Phi_0\| \\ &\leqslant \|(V^{\mathrm{l}}(x+vt+\tilde{c}(t)) - V^{\mathrm{l}}_{v,t,\kappa}(x+vt+\tilde{c}(t)))e^{-itp^2/2} f(p) M_{D,v}(t)\Phi_0\| \\ &\quad + \|(V^{\mathrm{l}}_{v,t,\kappa}(x+pt+vt+\tilde{c}(t)) - V^{\mathrm{l}}_{v,t,\kappa}(pt+vt+\tilde{c}(t)))M_{D,v}(t)\Phi_0\| \end{aligned} \tag{2.4.66}$$

by $e^{itp^2/2} x e^{-itp^2/2} = x + pt$. This can be estimated as

$$\|(V^{\mathrm{l}}(x) - V^{\mathrm{l}}(t(p-\tilde{b}(t)) + \tilde{c}(t)))U_D(t)\Phi_v\| \leqslant I_0, \quad I_0 = 2\sup_{y\in\mathbb{R}^d} |V^{\mathrm{s}}(y)\|\Phi_0\|. \tag{2.4.67}$$

Therefore we also have

$$\|(V^{\mathrm{l}}(x) - V^{\mathrm{l}}(t(p-\tilde{b}(t)) + \tilde{c}(t)))U_D(t)\Phi_v\| \leqslant I_{1,\rho} + \tilde{I}_{2,\rho} + \tilde{I}_{3,\kappa}, \tag{2.4.68}$$

with

$$\begin{aligned} I_{1,\rho} &= 2\sup_{y\in\mathbb{R}^d} |V^{\mathrm{s}}(y)|\|M_{D,v}(t)\Phi_0\| \\ &\qquad \times \|F(|x| \geqslant 3\lambda_1 |v|^\rho |t|) e^{-itp^2/2} f(p) F(|x| < \lambda_1 |v|^\rho |t|)\|_{\mathscr{B}(L^2)}, \end{aligned} \tag{2.4.69}$$

$$\begin{aligned} \tilde{I}_{2,\rho} &= 2\sup_{y\in\mathbb{R}^d} |V^{\mathrm{s}}(y)|\|\langle x\rangle^2 M_{D,v}(t)\Phi_0\| \\ &\qquad \times \|F(|x| \geqslant 3\lambda_1 |v|^\rho |t|) e^{-itp^2/2} f(p) F(|x| \geqslant \lambda_1 |v|^\rho |t|)\langle x\rangle^{-2}\|_{\mathscr{B}(L^2)}, \end{aligned} \tag{2.4.70}$$

$$\tilde{I}_{3,\kappa} = \|(V^{\mathrm{l}}_{v,t,\kappa}(x+pt+vt+\tilde{c}(t)) - V^{\mathrm{l}}_{v,t,\kappa}(pt+vt+\tilde{c}(t)))M_{D,v}(t)\Phi_0\| \tag{2.4.71}$$

when $|v| \geqslant \max\{1, 3M_2/(1-\delta), 12\eta/(1-\delta)\}$, where $3\lambda_1 = (1-\delta)/12$, $N \in \mathbb{N}$, $0 < \rho \leqslant 1$ and $0 \leqslant \kappa \leqslant 1$. Here we used the fact that

$$(V^{\mathrm{l}}(x+vt+\tilde{c}(t)) - V^{\mathrm{l}}_{v,t,\kappa}(x+vt+\tilde{c}(t)))F(|x| < 3\lambda_1 |v|^\rho |t|) = 0 \tag{2.4.72}$$

holds when $|v| \geqslant \max\{1, 3M_2/(1-\delta), 12\eta/(1-\delta)\}$ because of $3\lambda_1|v|^{\rho-1} \leqslant 3\lambda_1$. We note that by virtue of the Baker-Campbell-Hausdorff formula,

$$\begin{aligned} &V^{\mathrm{l}}_{v,t,\kappa}(x+pt+vt+\tilde{c}(t)) - V^{\mathrm{l}}_{v,t,\kappa}(pt+vt+\tilde{c}(t)) \\ &\quad = \int_0^1 (\nabla_x V^{\mathrm{l}}_{v,t,\kappa})(x\theta+pt+vt+\tilde{c}(t)) \cdot x d\theta \\ &\qquad + it\int_0^1 (\triangle_x V^{\mathrm{l}}_{v,t,\kappa})(x\theta+pt+vt+\tilde{c}(t)) d\theta/2 \end{aligned} \tag{2.4.73}$$

hold. We also note that

$$\sup_{y\in\mathbb{R}^d} |(\nabla_x V^{\mathrm{l}}_{v,t,\kappa})(y)| \leqslant C(\langle |v|^\kappa |t|^{\tilde{\sigma}_\kappa}\rangle^{-\gamma_{D,1}} + (|v|^\kappa|t|^{\tilde{\sigma}_\kappa})^{-1}\langle |v|^\kappa |t|^{\tilde{\sigma}_\kappa}\rangle^{-\gamma_D}), \tag{2.4.74}$$

$$\begin{aligned} \sup_{y\in\mathbb{R}^d} |(\triangle_x V^{\mathrm{l}}_{v,t,\kappa})(y)| \leqslant C(&\langle |v|^\kappa |t|^{\tilde{\sigma}_\kappa}\rangle^{-\gamma_{D,2}} + (|v|^\kappa|t|^{\tilde{\sigma}_\kappa})^{-1}\langle |v|^\kappa |t|^{\tilde{\sigma}_\kappa}\rangle^{-\gamma_{D,1}} \\ &+ (|v|^\kappa|t|^{\tilde{\sigma}_\kappa})^{-2}\langle |v|^\kappa |t|^{\tilde{\sigma}_\kappa}\rangle^{-\gamma_D}) \end{aligned} \tag{2.4.75}$$

holds. Now we take a parameter $\sigma_1 \in \mathbb{R}$ such that $\sigma_1 > -1$. Then, in the same way as in the proof of Proposition 2.4.4, we have

$$\int_{|t|\leqslant|v|^{-1}} I_0 dt = O(|v|^{-1}), \quad \int_{|v|^{-1}\leqslant|t|\leqslant|v|^{\sigma_1}} I_{1,1} dt = O(|v|^{-1}), \tag{2.4.76}$$

$$\int_{|t|\geqslant|v|^{\sigma_1}} I_{1,1} dt = O(|v|^{-1-(1+\sigma_1)N}), \quad \int_{|v|^{-1}\leqslant|t|\leqslant|v|^{\sigma_1}} \tilde{I}_{2,1} dt = O(|v|^{-1}), \tag{2.4.77}$$

$$\int_{|t|\geqslant|v|^{\sigma_1}} \tilde{I}_{2,1} dt = O(|v|^{\theta_0(\sigma_1)}) + O(|v|^{\theta_1(\sigma_1)}) + O(|v|^{\theta_2(\sigma_1)}) + O(|v|^{\theta_3(\sigma_1)}), \tag{2.4.78}$$

where

$$\theta_0(\sigma_1) = -2-\sigma_1 \tag{2.4.79}$$

$$\theta_1(\sigma_1) = \begin{cases} -2 - \dfrac{\gamma_D(2-\mu)}{1-\mu} + \epsilon & \gamma_{D,1} \leqslant 1 \\ -2-\gamma_{D,1}+\sigma_1(1-\gamma_{D,1}) & \gamma_{D,1} > 1, \end{cases} \tag{2.4.80}$$

$$\theta_2(\sigma_1) = \begin{cases} -2 - \dfrac{\gamma_D(2-\mu)}{1-\mu} + \epsilon & \gamma_{D,1} \leqslant 3/2 \\ -2-2\gamma_{D,1}+\sigma_1(3-2\gamma_{D,1}) & \gamma_{D,1} > 3/2, \end{cases} \tag{2.4.81}$$

$$\theta_3(\sigma_1) = \begin{cases} -2 - \dfrac{\gamma_D(2-\mu)}{1-\mu} + \epsilon & \gamma_{D,2} \leqslant 2 \\ -2-\gamma_{D,2}+\sigma_1(2-\gamma_{D,2}) & \gamma_{D,2} > 2, \end{cases} \tag{2.4.82}$$

with $\epsilon > 0$. Because for $\sigma_1 > -1$, every $\theta_j(\sigma_1)$ is less than -1, we finally obtain

$$\int_{|t|\leqslant|v|^{-1}} I_0 dt = O(|v|^{-1}), \quad \int_{|v|^{-1}\leqslant|t|\leqslant|v|^{\sigma_1}} I_{1,1} dt = O(|v|^{-1}), \tag{2.4.83}$$

$$\int_{|t|\geqslant|v|^{\sigma_1}} I_{1,1} dt = O(|v|^{-1}), \quad \int_{|v|^{-1}\leqslant|t|\leqslant|v|^{\sigma_1}} \tilde{I}_{2,1} dt = O(|v|^{-1}), \tag{2.4.84}$$

$$\int_{|t|\geqslant|v|^{\sigma_1}} \tilde{I}_{2,1} dt = O(|v|^{-1}). \tag{2.4.85}$$

Since

$$\||x|M_{D,v}(t)\Phi_0\| \leqslant C(1 + |v|^{-\kappa_1\gamma_{D,1}}|t|^{2-\tilde{\sigma}_{\kappa_1}\gamma_{D,1}}) \tag{2.4.86}$$

holds with $0 \leqslant \kappa_1 \leqslant 1$ by the proof of Proposition 2.4.2, we obtain

$$\begin{aligned}\int_{|v|^{-1}\leqslant|t|\leqslant|v|^{\sigma_1}} \tilde{I}_{3,\kappa} dt = O(|v|^{\theta_{\mathrm{l},1}(\gamma_{D,1},\sigma_1,\kappa)}) + O(|v|^{\theta_{\mathrm{l},1}(1+\gamma_D,\sigma_1,\kappa)}) \\ +O(|v|^{\theta_{\mathrm{l},2}(\gamma_{D,1},\gamma_{D,1},\sigma_1,\kappa,\kappa_1)}) + O(|v|^{\theta_{\mathrm{l},2}(1+\gamma_D,\gamma_{D,1},\sigma_1,\kappa,\kappa_1)}) \\ +O(|v|^{\theta_{\mathrm{l},3}(\gamma_{D,2},\sigma_1,\kappa)}) + O(|v|^{\theta_{\mathrm{l},3}(1+\gamma_{D,1},\sigma_1,\kappa)}) + O(|v|^{\theta_{\mathrm{l},3}(2+\gamma_D,\sigma_1,\kappa)}),\end{aligned} \tag{2.4.87}$$

where

$$\theta_{\mathrm{l},1}(c_0,\sigma_1,\kappa) = \begin{cases} \theta_1^{\mathrm{l}}(c_0,\sigma_1,\kappa) & \tilde{\sigma}_\kappa c_0 < 1 \\ \theta_1^{\mathrm{l}}(c_0,-1,\kappa) & \tilde{\sigma}_\kappa c_0 > 1, \end{cases} \tag{2.4.88}$$

$$\theta_{\mathrm{l},2}(c_0,c_1,\sigma_1,\kappa,\kappa_1) = \begin{cases} \theta_2^{\mathrm{l}}(c_0,c_1,\sigma_1,\kappa,\kappa_1) & \tilde{\sigma}_\kappa c_0 + \tilde{\sigma}_{\kappa_1} c_1 < 3 \\ \theta_2^{\mathrm{l}}(c_0,c_1,-1,\kappa,\kappa_1) & \tilde{\sigma}_\kappa c_0 + \tilde{\sigma}_{\kappa_1} c_1 > 3, \end{cases} \tag{2.4.89}$$

$$\theta_{\mathrm{l},3}(c_0,\sigma_1,\kappa) = \begin{cases} \theta_3^{\mathrm{l}}(c_0,\sigma_1,\kappa) & \tilde{\sigma}_\kappa c_0 < 2 \\ \theta_3^{\mathrm{l}}(c_0,-1,\kappa) & \tilde{\sigma}_\kappa c_0 > 2, \end{cases} \tag{2.4.90}$$

with $0 \leqslant \kappa \leqslant 1$ and $0 \leqslant \kappa_1 \leqslant 1$, by simple calculation. Here

$$\theta_1^{\mathrm{l}}(c_0,\sigma_1,\kappa) = -\kappa c_0 + \sigma_1(1 - \tilde{\sigma}_\kappa c_0), \tag{2.4.91}$$

$$\theta_2^{\mathrm{l}}(c_0,c_1,\sigma_1,\kappa,\kappa_1) = -\kappa c_0 - \kappa_1 c_1 + \sigma_1(3 - \tilde{\sigma}_\kappa c_0 - \tilde{\sigma}_{\kappa_1} c_1), \tag{2.4.92}$$

$$\theta_3^{\mathrm{l}}(c_0,\sigma_1,\kappa) = -\kappa c_0 + \sigma_1(2 - \tilde{\sigma}_\kappa c_0). \tag{2.4.93}$$

We omitted the formulas of $\theta_1^{\mathrm{l}}(c_0,\sigma_1,\kappa)$ when $\tilde{\sigma}_\kappa c_0 = 1$, of $\theta_2^{\mathrm{l}}(c_0,c_1,\sigma_1,\kappa,\kappa_1)$ when $\tilde{\sigma}_\kappa c_0 + \tilde{\sigma}_{\kappa_1} c_1 = 3$, and $\theta_3^{\mathrm{l}}(c_0,\sigma_1,\kappa)$ when $\tilde{\sigma}_\kappa c_0 = 2$, for ease of representation. Noting that $-\kappa - \sigma_1\tilde{\sigma}_\kappa = -(2-\mu)\sigma_1 + (-1 + (1-\mu)\sigma_1)\kappa$, the following can be verified easily. For $c_0 > 0$ and $c_1 > 0$, $\theta_1^{\mathrm{l}}(c_0,-1,\kappa)$ and $\theta_3^{\mathrm{l}}(c_0,-1,\kappa)$ are monotonically decreasing in κ, and $\theta_2^{\mathrm{l}}(c_0,c_1,-1,\kappa,\kappa_1)$ is monotonically decreasing in both κ and κ_1, because $-1 - (1-\mu) = -2 + \mu < 0$. Moreover, if $\sigma_1 <$

$1/(1-\mu)$, then $\theta_1^{\rm l}(c_0,\sigma_1,\kappa)$ and $\theta_3^{\rm l}(c_0,\sigma_1,\kappa)$ are monotonically decreasing in κ, and $\theta_2^{\rm l}(c_0,c_1,\sigma_1,\kappa,\kappa_1)$ is monotonically decreasing in both κ and κ_1, because $-1+(1-\mu)\sigma_1<0$. Hence, for given σ_1 such that $-1<\sigma_1<1/(1-\mu)$, in order to obtain a better estimate on $\int_{|v|^{-1}\leqslant|t|\leqslant|v|^{\sigma_1}}\tilde{I}_{3,*}dt$, we have only to put $\kappa=\kappa_1=1$. We note that $\tilde{\sigma}_1=1$. Since $-1-\sigma_1<0$, one can verify easily that $\theta_{\rm l,1}(\gamma_{D,1},\sigma_1,1)>\theta_{\rm l,1}(1+\gamma_D,\sigma_1,1)$, $\theta_{\rm l,2}(\gamma_{D,1},\gamma_{D,1},\sigma_1,1,1)>\theta_{\rm l,2}(1+\gamma_D,\gamma_{D,1},\sigma_1,1,1)$ and $\theta_{\rm l,3}(\gamma_{D,2},\sigma_1,1)>\theta_{\rm l,3}(1+\gamma_{D,1},\sigma_1,1)>\theta_{\rm l,3}(2+\gamma_D,\sigma_1,1)$. Since $-1/(2-\mu)+\sigma_1(1-1/(2-\mu))<0$ by $\sigma_1<1/(1-\mu)$, $\theta_{\rm l,1}(\gamma_{D,1},\sigma_1,1)>\theta_{\rm l,3}(\gamma_{D,2},\sigma_1,1)$ can be verified easily. For $1<\gamma_{D,1}<3/2$, $\theta_{\rm l,2}(\gamma_{D,1},\gamma_{D,1},\sigma_1,1,1)<\theta_{\rm l,1}(\gamma_{D,1},\sigma_1,1)=-1$ holds because of $\sigma_1<1/(1-\mu)\leqslant 1$. Moreover, if

$$\sigma_1<\gamma_{D,1}/(2-\gamma_{D,1}), \tag{2.4.94}$$

then $\theta_{\rm l,2}(\gamma_{D,1},\gamma_{D,1},\sigma_1,1,1)<\theta_{\rm l,1}(\gamma_{D,1},\sigma_1,1)$ holds even when $\gamma_{D,1}<1$. Here we note that $\gamma_{D,1}/(2-\gamma_{D,1})<1/(1-\mu)$ holds since $\gamma_D<1/(2-\mu)$. As a consequence, we obtain a better estimate

$$\int_{|v|^{-1}\leqslant|t|\leqslant|v|^{\sigma_1}}\tilde{I}_{3,1}dt=O(|v|^{\theta_{\rm l,1}(\gamma_{D,1},\sigma_1,1)}) \tag{2.4.95}$$

with

$$\theta_{\rm l,1}(c_0,\sigma_1,1)=\begin{cases}-c_0+\sigma_1(1-c_0) & c_0<1\\ -1 & c_0>1,\end{cases} \tag{2.4.96}$$

under the additional assumption (2.4.94) besides $\sigma_1>-1$.

As for $\int_{|t|\geqslant|v|^{\sigma_1}}\tilde{I}_{3,\kappa}dt$, we have the estimate

$$\begin{aligned}\int_{|t|\geqslant|v|^{\sigma_1}}\tilde{I}_{3,\kappa}dt&=O(|v|^{\theta_1^{\rm l}(\gamma_{D,1},\sigma_1,\kappa)})+O(|v|^{\theta_1^{\rm l}(1+\gamma_D,\sigma_1,\kappa)})\\ &+O(|v|^{\theta_2^{\rm l}(\gamma_{D,1},\gamma_{D,1},\sigma_1,\kappa,\kappa_1)})+O(|v|^{\theta_2^{\rm l}(1+\gamma_D,\gamma_{D,1},\sigma_1,\kappa,\kappa_1)})\\ +O(|v|^{\theta_3^{\rm l}(\gamma_{D,2},\sigma_1,\kappa)})&+O(|v|^{\theta_3^{\rm l}(1+\gamma_{D,1},\sigma_1,\kappa)})+O(|v|^{\theta_3^{\rm l}(2+\gamma_D,\sigma_1,\kappa)})\end{aligned} \tag{2.4.97}$$

under the integrable conditions

$$\tilde{\sigma}_\kappa\gamma_{D,1}>1,\quad \tilde{\sigma}_\kappa\gamma_{D,1}+\tilde{\sigma}_{\kappa_1}\gamma_{D,1}>3,\quad \tilde{\sigma}_\kappa\gamma_{D,2}>2. \tag{2.4.98}$$

Here we used $1+\gamma_D>\gamma_{D,1}$ and $2+\gamma_D>1+\gamma_{D,1}>\gamma_{D,2}$. These integrable conditions can be rewritten as

$$\kappa<\frac{(2-\mu)-2/\gamma_{D,2}}{1-\mu}=:\kappa_{D,1},\quad \kappa+\kappa_1<\frac{2(2-\mu)-3/\gamma_{D,1}}{1-\mu}=:\kappa_{D,2}. \tag{2.4.99}$$

Here we note that

$$\kappa_{D,1} > 1 \Longleftrightarrow \gamma_{D,2} > 2, \quad \kappa_{D,2} > 1 \Longleftrightarrow \gamma_{D,1} > 3/(3-\mu), \tag{2.4.100}$$

$$\kappa_{D,2} > 2 \Longleftrightarrow \gamma_{D,1} > 3/2, \quad \kappa_{D,1} < \kappa_{D,2} \Longleftrightarrow \gamma_D > (\sqrt{3}-1)/(2-\mu), \tag{2.4.101}$$

$$\kappa_{D,1} + 1 < \kappa_{D,2} \Longleftrightarrow \gamma_{D,2} > \frac{3-\mu+\sqrt{\mu^2-14\mu+25}}{2(2-\mu)}. \tag{2.4.102}$$

$\sqrt{3}-1$ is so called the Enss number. Now we will divide the situation into cases for ease of consideration.

Case I. $\kappa_{D,1} > 1$, that is, $\kappa_{D,2} > 2$.

Here we note that $\gamma_{D,1} > 2 - 1/(2-\mu) > 1$ holds by $0 \leqslant \mu < 1$.

Subcase i. $\kappa_{D,2} > 2$, that is, $\gamma_{D,1} > 3/2$.

As mentioned above, for $c_0 > 0$, $\theta_1^{\mathrm{l}}(c_0, \sigma_1, \kappa)$, $\theta_2^{\mathrm{l}}(c_0, \gamma_{D,1}, \sigma_1, \kappa, \kappa_1)$ and $\theta_3^{\mathrm{l}}(c_0, \sigma_1, \kappa)$ are all monotonically decreasing in κ, by $-1+(1-\mu)\sigma_1 < 0$. Hence, in order to a better estimate on $\int_{|t|\geqslant|v|^{\sigma_1}} \tilde{I}_{3,*}dt$, we have only to put $\kappa = 1$. Here we note that

$$\theta_1^{\mathrm{l}}(\gamma_{D,1}, \sigma_1, 1) > \theta_1^{\mathrm{l}}(1+\gamma_D, \sigma_1, 1), \tag{2.4.103}$$

$$\theta_2^{\mathrm{l}}(\gamma_{D,1}, \gamma_{D,1}, \sigma_1, 1, \kappa_1) > \theta_2^{\mathrm{l}}(1+\gamma_D, \gamma_{D,1}, \sigma_1, 1, \kappa_1), \tag{2.4.104}$$

$$\theta_3^{\mathrm{l}}(\gamma_{D,2}, \sigma_1, 1) > \theta_3^{\mathrm{l}}(1+\gamma_{D,1}, \sigma_1, 1) > \theta_3^{\mathrm{l}}(2+\gamma_D, \sigma_1, 1) \tag{2.4.105}$$

hold by $\sigma_1 > -1$. Since $\theta_2^{\mathrm{l}}(\gamma_{D,1}, \gamma_{D,1}, \sigma_1, 1, \kappa_1)$ is also monotonically decreasing in κ_1, in order to a better estimate on $\int_{|t|\geqslant|v|^{\sigma_1}} \tilde{I}_{3,*}dt$, we shloud put $\kappa_1 = 1 < \kappa_{D,2} - 1$. By $\sigma_1 < 1/(1-\mu)$, $\theta_1^{\mathrm{l}}(\gamma_{D,1}, \sigma_1, 1) > \theta_3^{\mathrm{l}}(\gamma_{D,2}, \sigma_1, 1)$ holds, while, under the additional condition (2.4.94), $\theta_1^{\mathrm{l}}(\gamma_{D,1}, \sigma_1, 1) > \theta_2^{\mathrm{l}}(\gamma_{D,1}, \gamma_{D,1}, \sigma_1, 1, 1)$ holds. By $\sigma_1 > -1$ and $\gamma_{D,1} > 3/2$, $\theta_1^{\mathrm{l}}(\gamma_{D,1}, \sigma_1, 1) + 1 = (1+\sigma_1)(1-\gamma_{D,1}) < 0$ holds. By (2.4.95), (2.4.83),(2.4.84) and (2.4.85), we obtain the optimal estimate

$$\int_{-\infty}^{\infty} \|(V^{\mathrm{l}}(x) - V^{\mathrm{l}}(t(p-\tilde{b}(t)) + \tilde{c}(t)))U_D(t)\Phi_v\| dt = O(|v|^{-1}). \tag{2.4.106}$$

Subcase ii. $1 < \kappa_{D,2} \leqslant 2$, that is, $3/(3-\mu) < \gamma_{D,1} \leqslant 3/2$.

By the argument similar to the above one, in order to a better estimate on $\int_{|t|\geqslant|v|^{\sigma_1}} \tilde{I}_{3,*}dt$, we have only to put $\kappa = 1$ and $\kappa_1 = \kappa_{D,2} - 1$. Here we note that under the additional assumption

$$\sigma_1 < \frac{(\kappa_{D,2}-1)\gamma_{D,1}}{2-\tilde{\sigma}_{\kappa_{D,2}-1}\gamma_{D,1}} \left(\leqslant \frac{\gamma_{D,1}}{2-\gamma_{D,1}} \right), \tag{2.4.107}$$

$\theta_1^{\mathrm{l}}(\gamma_{D,1}, \sigma_1, 1) > \theta_2^{\mathrm{l}}(\gamma_{D,1}, \gamma_{D,1}, \sigma_1, 1, \kappa_{D,2}-1)$ holds. By $\sigma_1 > -1$ and $\gamma_{D,1} > 3/(3-\mu) \geqslant 1$, $\theta_1^{\mathrm{l}}(\gamma_{D,1}, \sigma_1, 1) + 1 = (1+\sigma_1)(1-\gamma_{D,1}) < 0$ holds. As a consequence,we obtain the optimal estimate (2.4.106) in the same way as in Subcase i.

Subcase iii. $0 < \kappa_{D,2} \leqslant 1$, that is, $3/(2(2-\mu)) < \gamma_{D,1} \leqslant 3/(3-\mu)$.

By the argument similar to the above one, in order to a better estimate on $\int_{|t|\geqslant|v|^{\sigma_1}} \tilde{I}_{3,*}dt$, we have only to put $\kappa = \kappa_{D,2}$ and $\kappa_1 = 0$. Here we note that under the additional assumption

$$\sigma_1 > -\frac{\kappa_{D,2}}{\tilde{\sigma}_{\kappa_{D,2}}}(\geqslant -1), \tag{2.4.108}$$

one has

$$\theta_1^{\mathrm{l}}(\gamma_{D,1}, \sigma_1, \kappa_{D,2}) > \theta_1^{\mathrm{l}}(1 + \gamma_D, \sigma_1, \kappa_{D,2}), \tag{2.4.109}$$

$$\theta_2^{\mathrm{l}}(\gamma_{D,1}, \gamma_{D,1}, \sigma_1, \kappa_{D,2}, 0) > \theta_2^{\mathrm{l}}(1 + \gamma_D, \gamma_{D,1}, \sigma_1, \kappa_{D,2}, 0), \tag{2.4.110}$$

$$\theta_3^{\mathrm{l}}(\gamma_{D,2}, \sigma_1, \kappa_{D,2}) > \theta_3^{\mathrm{l}}(1 + \gamma_{D,1}, \sigma_1, \kappa_{D,2}) > \theta_3^{\mathrm{l}}(2 + \gamma_D, \sigma_1, \kappa_{D,2}). \tag{2.4.111}$$

Moerover, under the additional assumption

$$\sigma_1 < 0, \tag{2.4.112}$$

$\theta_1^{\mathrm{l}}(\gamma_{D,1}, \sigma_1, \kappa_{D,2}) > \theta_2^{\mathrm{l}}(\gamma_{D,1}, \gamma_{D,1}, \sigma_1, \kappa_{D,2}, 0)$ holds by $2 - (2 - \mu)\gamma_{D,1} > 0$. Here we note that $\theta_1^{\mathrm{l}}(\gamma_{D,1}, \sigma_1, \kappa_{D,2}) \geqslant \theta_1^{\mathrm{l}}(\gamma_{D,1}, \sigma_1, 1)$. Since

$$\theta_1^{\mathrm{l}}(\gamma_{D,1}, \sigma_1, \kappa_{D,2}) = -\frac{2(2-\mu)\gamma_{D,1} - 3}{1-\mu} + \sigma_1((2-\mu)\gamma_{D,1} - 2) \tag{2.4.113}$$

and $\gamma_{D,1} < 2/(2-\mu)$, $\theta_1^{\mathrm{l}}(\gamma_{D,1}, \sigma_1, \kappa_{D,2})$ is monotonically decreasing in σ_1. Here we note that $\theta_1^{\mathrm{l}}(\gamma_{D,1}, 0, \kappa_{D,2}) = -(2(2-\mu)\gamma_D - 1)/(1-\mu) < -1$ is equivalent to $\gamma_D > 1/2$. Therefore we obtain the optimal estimate

$$\int_{-\infty}^{\infty} \|(V^{\mathrm{l}}(x) - V^{\mathrm{l}}(t(p - \tilde{b}(t)) + \tilde{c}(t)))U_D(t)\Phi_v\|dt = O(|v|^{\Theta_{1,D}(\gamma_D)+\epsilon}), \tag{2.4.114}$$

$$\Theta_{1,D}(\gamma_D) = \begin{cases} -1 & \gamma_D > 1/2 \\ -\dfrac{2\gamma_D(2-\mu) - 1}{1-\mu} & \gamma_D \leqslant 1/2 \end{cases} \tag{2.4.115}$$

with $\epsilon > 0$.

Case II. $\kappa_{D,1} \leqslant 1$, that is, $\kappa_{D,2} \leqslant 2$.

Subcase i. $\kappa_{D,2} > \gamma_{D,1} + 1$.

Since for $0 \leqslant \mu < 1$, $(3 - \mu + \sqrt{\mu^2 - 14\mu + 25})/(2(2-\mu)) > 2$ holds, we do not have to consider this subcase.

Subcase ii. $\kappa_{D,1} < \kappa_{D,2} \leqslant \kappa_{D,1} + 1$.

By the argument similar to the above one, in order to a better estimate on $\int_{|t|\geqslant|v|^{\sigma_1}} \tilde{I}_{3,*}dt$, we have only to put $\kappa = \kappa_{D,1}$ and $\kappa_1 = \kappa_{D,2} - \kappa_{D,1}$. Here we note that under the additional assumption

$$\sigma_1 > -\frac{\kappa_{D,1}}{\tilde{\sigma}_{\kappa_{D,1}}}(\geqslant -1), \tag{2.4.116}$$

one has

$$\theta_1^{\mathrm{l}}(\gamma_{D,1},\sigma_1,\kappa_{D,1}) > \theta_1^{\mathrm{l}}(1+\gamma_D,\sigma_1,\kappa_{D,1}), \tag{2.4.117}$$

$$\theta_2^{\mathrm{l}}(\gamma_{D,1},\gamma_{D,1},\sigma_1,\kappa_{D,1},\kappa_{D,2}-\kappa_{D,1}) > \theta_2^{\mathrm{l}}(1+\gamma_D,\gamma_{D,1},\sigma_1,\kappa_{D,1},\kappa_{D,2}-\kappa_{D,1}), \tag{2.4.118}$$

$$\theta_3^{\mathrm{l}}(\gamma_{D,2},\sigma_1,\kappa_{D,1}) > \theta_3^{\mathrm{l}}(1+\gamma_{D,1},\sigma_1,\kappa_{D,1}) > \theta_3^{\mathrm{l}}(2+\gamma_D,\sigma_1,\kappa_{D,1}). \tag{2.4.119}$$

By $\sigma_1 < 1/(1-\mu)$, $\theta_1^{\mathrm{l}}(\gamma_{D,1},\sigma_1,\kappa_{D,1}) > \theta_3^{\mathrm{l}}(\gamma_{D,2},\sigma_1,\kappa_{D,1})$ holds, while, under the additional assumption

$$\sigma_1 < \frac{(\kappa_{D,2}-\kappa_{D,1})\gamma_{D,1}}{2-\tilde{\sigma}_{\kappa_{D,2}-\kappa_{D,1}}\gamma_{D,1}} \left(\leqslant \frac{\gamma_{D,1}}{2-\gamma_{D,1}} \right), \tag{2.4.120}$$

$\theta_1^{\mathrm{l}}(\gamma_{D,1},\sigma_1,\kappa_{D,1}) > \theta_2^{\mathrm{l}}(\gamma_{D,1},\gamma_{D,1},\sigma_1,\kappa_{D,1},\kappa_{D,2}-\kappa_{D,1})$ holds. Here we note that $\theta_1^{\mathrm{l}}(\gamma_{D,1},\sigma_1,\kappa_{D,1}) \geqslant \theta_1^{\mathrm{l}}(\gamma_{D,1},\sigma_1,1)$. Since

$$\theta_1^{\mathrm{l}}(\gamma_{D,1},\sigma_1,\kappa_{D,1}) = -\frac{(2-\mu)\gamma_D\gamma_{D,1}}{(1-\mu)\gamma_{D,2}} - \frac{\gamma_D}{\gamma_{D,2}}\sigma_1 \tag{2.4.121}$$

is monotonically decreasing in σ_1, we should take σ_1 as $(\kappa_{D,2}-\kappa_{D,1})\gamma_{D,1}/(2-\tilde{\sigma}_{\kappa_{D,2}-\kappa_{D,1}}\gamma_{D,1})$ in order to obtain a better estimate on $\int_{|t|\geqslant|v|^{\sigma_1}} \tilde{I}_{3,*}dt$. We note that

$$\theta_1^{\mathrm{l}}\Big(\gamma_{D,1}, \frac{(\kappa_{D,2}-\kappa_{D,1})\gamma_{D,1}}{2-\tilde{\sigma}_{\kappa_{D,2}-\kappa_{D,1}}\gamma_{D,1}}, \kappa_{D,1}\Big) = -\frac{2\gamma_D(2-\mu)-1}{1-\mu} \tag{2.4.122}$$

holds by straightforward computation. Therefore we obtain the optimal estimate (2.4.114) in the same way as above.

Subcase iii. $0<\kappa_{D,2}\leqslant\kappa_{D,1}$.

In the same way as in the subcase iii of Case I, we obtain the optimal estimate (2.4.114). □

In the same way as in [6], we introduce auxiliary wave operators

$$\Omega_{D,G,v}^{\pm} = \operatorname*{s-lim}_{t\to\infty} U(t,0)^*U_{D,G,v}(t), \quad U_{D,G,v}(t) = U_D(t)M_{G,v}(t) \tag{2.4.123}$$

with $M_{G,v}(t) = e^{-i\int_0^t V^{\mathrm{s}}(v\tau+\tilde{c}(\tau))d\tau}$ as in subsection 2.3. Then we see that $\Omega_{D,G,v}^{\pm} = W_D^{\pm}I_{G,v}^{\pm}$ exist with $I_{G,v}^{\pm} = \lim_{t\to\pm\infty} M_{G,v}(t)$. By virtue of (2.4.62), Propositions 2.4.3, 2.4.4 and 2.4.5, the following proposition can be obtained as Proposition 2.3.6. Thus we omit the proof.

Proposition 2.4.6. *Let v and Φ_v be as in Theorem 2.4.1, $\epsilon>0$ and $\Theta_{0,D}(\alpha,\gamma_D)$ and $\Theta_{1,D}(\gamma_D)$ be as in Propositions 2.4.4 and 2.4.5, respectively. Then*

$$\sup_{t\in\mathbb{R}} \|(U(t,0)\Omega_{D,G,v}^{-} - U_{D,G,v}(t))\Phi_v\| = O(|v|^{\max\{\Theta_{0,D}(\alpha,\gamma_D),\Theta_{1,D}(\gamma_D)\}+\epsilon}) \tag{2.4.124}$$

holds as $|v|\to\infty$ for $V^{\mathrm{vs}}\in\mathscr{V}^{\mathrm{vs}}$, $V^{\mathrm{s}}\in\mathscr{V}^{\mathrm{s}}_{\mu,\mu/(2-\mu)}$ and $V^{\mathrm{l}}\in\mathscr{V}^{\mathrm{l}}_{\mu,1/(2(2-\mu))}$.

Proof of Theorem 2.4.1. Since the proof is quite similar to the one of Theorem 2.3.1, we give its sketch only.

Suppose that $V^{\rm vs} \in \mathscr{V}^{\rm vs}$ and $V^{\rm s} \in \mathscr{V}^{\rm s}_{\mu,\tilde{\alpha}_\mu}$. We first note that S is represented as

$$S = (W_D^+)^* W_D^- = I_{G,v}(\Omega^+_{D,G,v})^*\Omega^-_{D,G,v}, \quad I_{G,v} = I^+_{G,v}\overline{I^-_{G,v}} = e^{-i\int_{-\infty}^{\infty} V^{\rm s}(v\tau+\tilde{c}(\tau))d\tau}. \tag{2.4.125}$$

Noting $[S_D, p_j] = [S_D - I_{G,v}, p_j - v_j]$, $(p_j - v_j)\Phi_v = (p_j\Phi_0)_v$ and

$$\begin{aligned} S_D - I_{G,v} &= I_{G,v}(\Omega^+_{D,G,v} - \Omega^-_{D,G,v})^*\Omega^-_{D,G,v} \\ &= -iI_{G,v}\int_{-\infty}^{\infty} U_{D,G,v}(t)^* V_v^D(t,x)U(t)\Omega^-_{D,G,v}dt \end{aligned} \tag{2.4.126}$$

with

$$V_v^D(t,x) = V^{\rm vs}(x)+V^{\rm s}(x)-V^{\rm s}(vt+\tilde{c}(t))+V^{\rm l}(x)-V^{\rm l}(t(p-\tilde{b}(t))+\tilde{c}(t)), \tag{2.4.127}$$

we have

$$|v|(i[S_D,p_j]\Phi_v,\Psi_v) = I_{G,v}(I_D(v)+R_D(v)) \tag{2.4.128}$$

with

$$\begin{aligned} I_D(v) &= |v|\int_{-\infty}^{\infty}((V_v^D(t,x)U_{D,G,v}(t)(p_j\Phi_0)_v, U_{D,G,v}(t)\Psi_v) \\ &\qquad -(V_v^D(t,x)U_{D,G,v}(t)\Phi_v, U_{D,G,v}(t)(p_j\Psi_0)_v))dt, \end{aligned} \tag{2.4.129}$$

$$\begin{aligned} R_D(v) &= |v|\int_{-\infty}^{\infty}(((U(t,0)\Omega^-_{D,G,v} - U_{D,G,v}(t))(p_j\Phi_0)_v, V_v^D(t,x)U_{D,G,v}(t)\Phi_v) \\ &\qquad -((U(t,0)\Omega^-_{D,G,v} - U_{D,G,v}(t))\Phi_v, V_v^D(t,x)U_{D,G,v}(t)(p_j\Psi_0)_v))dt. \end{aligned} \tag{2.4.130}$$

By Propositions 2.4.3, 2.4.4, 2.4.5 and 2.4.6, one has

$$R_D(v) = O(|v|^{1+2(\max\{\Theta_{0,D}(\alpha,\gamma_D),\Theta_{1,D}(\gamma_D)\}+\epsilon)}). \tag{2.4.131}$$

In the same way as in the proof of Theorem 2.3.1, we need the condition

$$\max\{\Theta_{0,D}(\alpha,\gamma_D),\Theta_{1,D}(\gamma_D)\} < -1/2 \tag{2.4.132}$$

in order to get $R_D(v) \to 0$ as $|v| \to \infty$. $\Theta_{1,D}(\gamma_D) < -1/2$ implies the necesary condition

$$\gamma_D > \frac{1}{2(2-\mu)} + \frac{1-\mu}{4(2-\mu)} =: \tilde{\gamma}_\mu. \tag{2.4.133}$$

We first consider the case where $\alpha > \alpha_-(\mu)$. Under the condition $\gamma_D > \Gamma_1(\alpha)$, $\Theta_{0,D}(\alpha,\gamma_D) < -1/2$ implies

$$\alpha > \frac{15-5\mu-\sqrt{(1-\mu)(41-25\mu)}}{8(2-\mu)} =: \alpha^-(\mu). \tag{2.4.134}$$

As can be verified easily, we see that if $0 \leqslant \mu \leqslant 5/7$, then $\alpha^-(\mu) \geqslant \alpha_-(\mu)$ holds, while, if $5/7 < \mu < 1$, then $\alpha^-(\mu) < \alpha_-(\mu)$ holds. Since

$$\tilde{\gamma}_\mu - \Gamma_1(\alpha) = \frac{(5\mu-1)\alpha - 4(2\mu-1)}{4(2-\mu)(4-3\alpha)}, \tag{2.4.135}$$

one can verify easily that if $0 \leqslant \mu \leqslant 7/5$, then $\Gamma_1(\alpha) \leqslant \tilde{\gamma}_\mu$ holds. Here we note that

$$\alpha_-(\mu) \geqslant \frac{4(2\mu-1)}{5\mu-1} = 1 - \frac{3(1-\mu)}{5\mu-1} =: \alpha_0(\mu) \tag{2.4.136}$$

when $1/5 < \mu \leqslant 5/7$. On the other hand, if $5/7 < \mu < 1$ and $\alpha \geqslant \alpha_0(\mu)$, then $\Gamma_1(\alpha) \leqslant \tilde{\gamma}_\mu$ holds, while if $5/7 < \mu < 1$ and $\alpha_-(\mu) < \alpha < \alpha_0(\mu)$, then $\Gamma_1(\alpha) > \tilde{\gamma}_\mu$ holds. As a consequence, in the case where $0 \leqslant \mu \leqslant 5/7$, under the assumptions $\alpha > \alpha^-(\mu)$ and $\gamma_D > \tilde{\gamma}_\mu$, (2.4.132) holds, in the case where $5/7 < \mu < 1$, under the assumptions $\alpha \geqslant \alpha_0(\mu)$ and $\gamma_D > \tilde{\gamma}_\mu$, (2.4.132) holds. Under the assumption $\alpha_-(\mu) < \alpha < \alpha_0(\mu)$,

$$-\frac{2\alpha(1-\alpha)}{4-3\alpha} - \frac{2(\alpha-\mu)(1-\alpha)}{(1-\mu)(4-3\alpha)} - \frac{2\gamma_D(2-\mu)-1}{1-\mu} < -\frac{1}{2}, \tag{2.4.137}$$

that is,

$$\gamma_D > \tilde{\gamma}_\mu + \frac{(2-\mu)\alpha^2 - 3\alpha + 2\mu}{(2-\mu)(4-3\alpha)} \tag{2.4.138}$$

implies $\Theta_{0,D}(\alpha, \gamma_D) < -1/2$. Since $(2-\mu)\alpha^2 - 3\alpha + 2\mu)/((2-\mu)(4-3\alpha)) < 0$ when $\alpha_-(\mu) < \alpha < \alpha_0(\mu)$, (2.4.132) holds under the assumptions $\alpha_-(\mu) < \alpha < \alpha_0(\mu)$ and $\gamma_D > \tilde{\gamma}_\mu$.

We next consider the case where $\mu/(2-\mu) < \alpha \leqslant \alpha_-(\mu)$. In this case, we assume $5/7 < \mu < 1$ necessarily by the above argument. Under the condition $\gamma_D > \Gamma_2(\alpha)$, $\Theta_{0,D}(\alpha, \gamma_D) < -1/2$ implies

$$\alpha > \frac{1+\mu}{2(2-\mu)} =: \alpha_1(\mu) > \frac{\mu}{2-\mu}. \tag{2.4.139}$$

If $\alpha_1(\mu) < \alpha \leqslant \alpha_-(\mu)$, then

$$\Gamma_2(\alpha) - \tilde{\gamma}_\mu = \frac{2\alpha(2-\mu) - 1 - \mu}{4(2-\mu)} > 0 \tag{2.4.140}$$

holds. Under the assumption $\alpha_1(\mu) < \alpha \leqslant \alpha_-(\mu)$,

$$-\frac{2\alpha}{3} - \frac{2(\alpha-\mu)}{3(1-\mu)} - \frac{2\gamma_D(2-\mu)-1}{3(1-\mu)} < -\frac{1}{2}, \tag{2.4.141}$$

that is,

$$\gamma_D > \tilde{\gamma}_\mu + \frac{1+\mu-2\alpha(2-\mu)}{2(2-\mu)} \tag{2.4.142}$$

implies $\Theta_{0,D}(\alpha, \gamma_D) < -1/2$. Since $(1+\mu-2\alpha(2-\mu))/(2(2-\mu)) < 0$ when $\alpha_1(\mu) < \alpha \leqslant \alpha_-(\mu)$, (2.4.132) holds under the assumptions $\alpha_1(\mu) < \alpha \leqslant \alpha_-(\mu)$ and $\gamma_D > \tilde{\gamma}_\mu$. □

Moreover, Theorem 2.1.4 can be shown in the same way as in the proof of Theorem 1.2 in [38] (see also Enss-Weder [13]). Thus we omit the proof.

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