Cardinal Invariants and Large Continuum

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博士論文

Cardinal Invariants and Large Continuum

（大きな連続体の下での基数不変量）

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A mis padres, hermanas y abuelita Lucila.

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ABSTRACT

We use and improve some forcing techniques of iterations with finite support to construct models of inequalities between some cardinal invariants with large continuum.

The first and most elaborated forcing technique that we look at is Shelah's theory of template iterations. We present a version of this theory where non-definable forcing notions are allowed in the forcing construction. Doing this, we obtain many local properties of such an iteration to prove consistency results. One is to get models where the groupwise-density number $\mathfrak{g}$ can be arbitrarily large. Another application, that involves a more complicated construction, is forcing $\mathfrak{s} < \kappa < \mathfrak{b} < \mathfrak{a}$ where κ is a measurable cardinal in the ground model and the cardinal invariants $\mathfrak{s}$, $\mathfrak{b}$ and $\mathfrak{a}$ can take arbitrary regular uncountable values.

The second forcing technique that we use is matrix iterations of ccc posets. By including Suslin ccc posets in the construction of such an iteration, we prove that some cases where the cardinal invariants of the right hand side of Cichon's diagram take three different values are consistent.

Our last results concern gaps in quotients by F_σ ideals on ω. For (such) an ideal $\mathcal{I}$, the *Rothberger number of* $\mathcal{I}$, denoted by $\mathfrak{b}(\mathcal{I})$, is the least cardinal κ such that there is a (ω, κ)-gap in the quotient $\mathcal{P}(\omega)/\mathcal{I}$. We focus our research on the Rothberger number for *fragmented ideals*, which is a subclass of F_σ ideals, and we prove that the Rothberger number is $\aleph_1$ for a large subclass of these ideals. On the other hand, we prove that the Rothberger number is above the additivity of the null ideal for another quite large subclass of fragmented ideals. At the end, by introducing properties for preservation of (ω, κ)-gaps in quotients by fragmented ideals, we show that it is consistent that there are infinitely many (even continuum many by assuming the existence of a weakly inaccessible cardinal) fragmented ideals with pairwise different Rothberger numbers.

The main results of this dissertation are included in [Me13a], [Me13b], [Me] and [BrMe].

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INTRODUCTION

Set theory is an area of mathematical logic which studies the foundations of mathematics and the combinatorics of *infinite sets.* The father of this area, Georg Cantor, discovered that the set of real numbers cannot be enumerated by natural numbers, which means that the *infinite* that measures the size of the set of real numbers is bigger than the *infinite* that measures the size of the set of natural numbers. He introduced the notion of *cardinal number*, which is a number that measures the size of a set. For a set x, $|x|$ denotes the *cardinality of* x, which is the cardinal number that represents its size. Cantor proved that, for any set x, $|x| < |\mathcal{P}(x)|$ where $\mathcal{P}(x) := \{z \,/\, z \subseteq x\}$ is the *power set of* x. Given a cardinal κ, 2^κ denotes the cardinality of the power set of any set of size κ, so Cantor's result can be expressed as $\kappa < 2^\kappa$ for any cardinal number κ. Since $\mathfrak{c} = 2^{\aleph_0}$ is well-known, this implies that $\aleph_0 < \mathfrak{c}$ where $\aleph_0$ denotes the size of the set of natural numbers and $\mathfrak{c}$, the *size of the continuum*, is the size of the set of real numbers.

Cantor's result also gave rise to the idea of the existence of infinitely many infinite cardinals. Moreover, he established that all the infinite cardinals can be well-ordered. In fact, $\aleph_0$ is the minimal infinite cardinal and it is defined as the set of natural numbers, which is also denoted by ω. Given a cardinal κ, the immediate successor of κ is denoted by κ^+. $\aleph_1 = \omega_1$ denotes $\aleph_0^+$ and $\aleph_2 = \omega_2$ is $\aleph_1^+$.

Cantor worked in (the creation of) set theory between 1874 and 1884. He also conjectured the continuum hypothesis, which is the first problem in set theory that, decades later, was proved to be undecidable in standard mathematics. Recall ZFC *Zermelo-Fraenkel Set Theory*, which is a standard formal system in which the basic results of modern mathematics can be formalized. To fix some notation, a formal theory $\mathfrak{T}$ is said to be *consistent*, denoted by $\mathrm{Con}(\mathfrak{T})$, if it does not prove a contradiction. A mathematical statement φ (formally, a statement in the language of $\mathfrak{T}$) is said to be *consistent with* $\mathfrak{T}$ if $\mathrm{Con}(\mathfrak{T})$ implies $\mathrm{Con}(\mathfrak{T} + \varphi)$, where $\mathfrak{T} + \varphi$ is the theory that results by adding φ to the axioms of $\mathfrak{T}$. This means that, whenever $\mathfrak{T}$ is free of contradiction, $\neg\varphi$ (the negation of φ) cannot be proved in $\mathfrak{T}$. If, in addition, $\neg\varphi$ is also consistent with $\mathfrak{T}$, we say that φ *is independent from* $\mathfrak{T}$, which means that, under the consistency of $\mathfrak{T}$, φ can not be proved or refuted under the axioms of $\mathfrak{T}$.

The *continuum hypothesis* (CH) is the statement "there is no set whose size is strictly between $\aleph_0$ and $\mathfrak{c}$" or, equivalently, $\mathfrak{c} = \aleph_1$. Kurt Gödel [Go] proved in 1938 that the *Axiom of Choice* (AC) plus the *generalized continuum hypothesis* (GCH) are statements consistent with ZF (the system ZFC without AC), where

GCH**:** $2^\kappa = \kappa^+$ for any infinite cardinal κ.

Later, Paul J. Cohen [C] proved in 1963 that CH is independent from ZFC and, also, that AC is independent from ZF. Both Gödel's and Cohen's works triggered new techniques in set theory that became the cornerstone of most of the work in contemporary set theory, especially for obtaining consistency results.

The standard methods to prove, in set theory, that a statement is consistent with ZFC are derived from some basic techniques of model theory. The usual way to prove that a statement φ is consistent with ZFC is to construct a model, in ZFC, that satisfies the axioms of ZFC and that also satisfies[1] φ.

[1]In more technical detail, for any finite arbitrarily large enough amount of axioms of ZFC, one constructs a model that satisfies these axioms and φ.

For instance, Gödel's proof consisted in defining, in ZF, the *constructible universe* **L**, a model of ZFC that satisfies GCH.

The most powerful technique for obtaining consistency results in set theory is the *forcing method*, which was created by Cohen for his original proof of the independence of CH from ZFC. In brief, this method consists in extending a model V of ZFC (or a model of a finite large enough amount of axioms of ZFC), known as the *ground model*, to a *generic extension* $V^{\mathbb{P}}$ by a *forcing notion* $\mathbb{P}$, which is a partial order defined in V that has the machinery for constructing $V^{\mathbb{P}}$. The generic extension $V^{\mathbb{P}}$ is a model of ZFC with many new objects that have special properties in relation with the ground model. Cohen's proof consists in the construction of *Cohen forcing* $\mathbb{C}_\lambda$ in V where λ is a cardinal that satisfies $\lambda^\omega = \lambda$, so this forcing adds λ different *Cohen reals* in a generic extension $V^{\mathbb{C}_\lambda}$ and, as $V^{\mathbb{C}_\lambda}$ preserves the cardinals of V, $\mathfrak{c} = \lambda$ is true in this extension (see Subsection 1.3.1).

Several techniques of the forcing method have been developed throughout the years and, now, forcing is one of the main tools in the study of set theory. In particular, it became interesting in relation with *set theory of the reals* and *descriptive set theory* (see Section 1.1) as it is a tool that can be used to obtain generic extensions that have new reals with special properties in relation with the ground model. Conversely, problems about set theory of the reals motivate new research in forcing theory, so both forcing and the combinatorics of real numbers are closely related.

The main topic of this dissertation is the application of some forcing techniques to obtain models related to *cardinal invariants of the continuum*, in particular, to those that are defined in Section 1.4. These invariants describe important facts about the combinatorial structure of the real line and statements about these invariants express properties of the reals in a simple and short way. In practice, they assume values between $\aleph_1$ and $\mathfrak{c}$ and are important because of their role in applications of Set Theory to other fields of mathematics like General Topology, Group Theory, Measure Theory, among others.

Typically, a cardinal invariant $\mathfrak{x}$ assumes different values in different models of set theory, so its value is not determined in ZFC. For instance, there may be models where $\mathfrak{x}$ is $\aleph_1$ (e.g., models where CH is true) and other models where $\mathfrak{x}$ is $\aleph_2$ or some other fixed larger cardinal value. However, it is often the case that, for two cardinal invariants $\mathfrak{x}$ and $\mathfrak{y}$, $\mathfrak{x} \leq \mathfrak{y}$ is provable in ZFC. If this is not the case, one needs to exhibit a model of ZFC in which $\mathfrak{x} > \mathfrak{y}$, that is, to show the consistency of $\mathfrak{x} > \mathfrak{y}$ with ZFC. This is usually done via forcing techniques, moreover, there is a strong interplay between forcing theory and cardinal invariants of the continuum. Forcing is not only used to prove results like the consistency of $\mathfrak{x} > \mathfrak{y}$ with ZFC, but important open problems about cardinal invariants also have triggered new developments in forcing theory.

An important example is *Cichon's diagram* in Figure 1. Consider the ideal $\mathcal{M}$ of meager sets of reals and the ideal $\mathcal{N}$ of null sets of real numbers (under the Lebesgue measure). For $\mathcal{I} \in \{\mathcal{M}, \mathcal{N}\}$, the following are cardinal invariants.

add$(\mathcal{I})$**,** the *additivity of* $\mathcal{I}$, which is the least size of a subfamily of $\mathcal{I}$ whose union is not in $\mathcal{I}$.

cov$(\mathcal{I})$**,** the *covering of* $\mathcal{I}$, which is the least size of a subfamily of $\mathcal{I}$ whose union is the set of all reals.

non$(\mathcal{I})$**,** the *uniformity of* $\mathcal{I}$, which is the least size of a set of reals that is not in $\mathcal{I}$.

cof$(\mathcal{I})$**,** the *cofinality of* $\mathcal{I}$, which is the least size of a $\subseteq$-cofinal subfamily of $\mathcal{I}$.

Denote by ω^ω the set of functions from ω to ω. If ω is endowed with the discrete topology, ω^ω with the product topology is known as *the Baire space*, which is homeomorphic to the subspace of the irrational numbers (see [HarW, Ch. X]). Consider the preorder $\leq^*$ defined on ω^ω by $f \leq^* g$ iff $f(n) \leq g(n)$ for all but finitely many $n \in \omega$. Define $\mathfrak{b}$, the *(un)bounding number*, as the least size of a subset of ω^ω that is not bounded in $\langle \omega^\omega, \leq^* \rangle$. $\mathfrak{d}$, the *dominating number*, is the least size of a $\leq^*$-cofinal subset of ω^ω.

Clearly, $\mathfrak{c}$ is also considered a cardinal invariant. In Cichon's diagram (Figure 1), the lines from bottom to top and from left to right represent $\leq$-inequalities provable in ZFC. The dotted lines mean

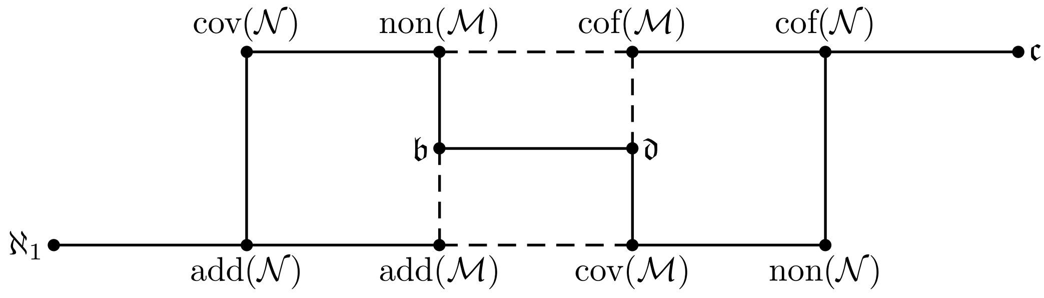

Figure 1: Cichon's diagram

$\text{add}(\mathcal{M}) = \min\{\mathfrak{b}, \text{cov}(\mathcal{M})\}$ and $\text{cof}(\mathcal{M}) = \max\{\mathfrak{d}, \text{non}(\mathcal{M})\}$. All the inequalities that are not stated in the diagram have been proved to be consistent with ZFC, see [BaJ] for proofs and references.

We say that the size of *the continuum is small* if it is $\aleph_1$ or $\aleph_2$. Otherwise, we say that it is *large*. A lot is known about cardinal invariants when the continuum is small. For example, when its size is $\aleph_1$ (that is, CH is true), all the cardinal invariants are $\aleph_1$ and so this is not interesting. If $\mathfrak{c} = \aleph_2$, for almost all pairs of cardinal invariants $\mathfrak{x}$ and $\mathfrak{y}$ for which $\mathfrak{x} \leq \mathfrak{y}$ is not provable in ZFC, the consistency of $\mathfrak{x} > \mathfrak{y}$ is known and there are few open questions. This is known, for example, for all the cardinal invariants in Cichon's diagram. However, when the continuum is large, many of these questions remain unanswered. This is so because, for $\mathfrak{c} = \aleph_2$, we have a much better understanding of forcing theory than for larger continuum. The reason is that one of the two main methods for carrying out consistency proofs, namely, *countable support iteration (csi) of proper forcing*, can only yield models with $\mathfrak{c} \leq \aleph_2$. The forcing method, in general, has been quite successful when the size of the continuum is assumed to be small.

On the other hand, when the continuum is assumed to be large, some of the classical forcing techniques do not seem to apply. Only with one technique, *the finite support iteration (fsi) of countable chain condition (ccc) forcing*, has it been possible to obtain many consistency results. However, this method also has drawbacks, for example, since a fsi of ccc forcing adds Cohen reals at limit stages, it can be used only to get models in which $\text{non}(\mathcal{M}) \leq \text{cov}(\mathcal{M})$. Therefore, if one wants to obtain models with $\text{non}(\mathcal{M}) > \text{cov}(\mathcal{M})$, one has to try something else, for example, a *large product construction with countable support*, the *random algebra* or a *novel iteration technique*. This is a difficult and challenging area of research.

Some known interesting consistency results about cardinal invariants with large continuum that are proved with fsi of ccc forcing are due to Judah and Shelah [JS88, JS90] and Brendle [Br91, Br98]. More sophisticated techniques of finite-supported type of iterations were introduced as well. Blass and Shelah [BlS84] introduced *matrix iterations*, which are fsi of ccc forcing that are constructed in a two-dimensional way. Besides, Shelah [S04] invented *iterations along a template*, which is a generalization of fsi of ccc forcing but with supports that are not necessarily well ordered. Brendle is also developing an iteration technique called *shattered iterations* [Br-1, Br-2]. A more detailed introduction to these techniques (except the last one, that is not used in this thesis), as well as the consistency results involved, are described in the following two sections of this introduction.

Large product construction with countable support is a forcing technique that has been used to generate models where, for a type of cardinal invariants that depend on a real parameter, there are many different parameters (in practice, uncountably many) that produce pairwise different cardinal invariants of that type. For example, Kellner [Kell08] and Kellner and Shelah [KellS09, KellS12] have been using large product constructions to prove the consistency of the existence of continuum-many pairwise different cardinal invariants of certain type. Recently, Hrušák, Rojas-Rebolledo and Zapletal [HrRZ] used this technique to obtain a model where there are continuum many ideals on ω that have pairwise different cofinality numbers. A general approach to this technique can be found in [RosS].

Although we do not use large product constructions in this work, we produce models, by fsi of ccc forcing, where there are infinitely many pairwise different cardinal invariants related to gaps in quotients by definable ideals on ω. We discuss this, and other problems related to gaps, in the third section of this introduction.

The main results of this dissertation are applications and improvements of certain forcing techniques that involve finite-supported type of iterations, in order to solve consistency problems about cardinal invariants with large continuum. In the following sections, we describe the specific problems that are tackled and solved in this thesis. Some of these problems were solved by the author in [Me13a], [Me13b], [Me] and [BrMe], the last one in joint work with J. Brendle.

Template iterations, almost disjointness and groupwise-density

We first introduce two cardinal invariants. Two infinite sets of natural numbers A and B are said to be *almost disjoint* if their intersection is finite. A family $\mathcal{A}$ of infinite sets of natural numbers is an *almost disjoint (a.d.) family* if any two different sets in $\mathcal{A}$ are pairwise almost disjoint. A *maximal a.d. (mad) family* is an a.d. family that cannot be extended to a larger a.d. family. The *almost disjointness number* $\mathfrak{a}$ denotes the least size of an infinite mad family.

For two infinite sets of natural numbers X and A, say that X *splits* A if $X \cap A$ and $A \smallsetminus X$ are infinite. A family $\mathcal{C}$ of infinite sets of natural numbers is a *splitting family* if any infinite set of natural numbers is splitted by some member of $\mathcal{C}$. Define $\mathfrak{s}$, the *splitting number*, as the least size of a splitting family.

Interesting knowledge about forcing and cardinal invariants has been obtained from the study of the relation between the cardinals $\mathfrak{s}$, $\mathfrak{b}$, $\mathfrak{d}$ and $\mathfrak{a}$. It is known that $\mathfrak{b} \leq \mathfrak{a}$ and $\mathfrak{s} \leq \mathfrak{d}$ are provable in ZFC, as well as $\mathfrak{b} \leq \mathfrak{d}$ (included in Cichon's diagram). Thanks to intensive research on forcing, we already know that any other inequality between these cardinals is consistent with ZFC. Cohen models (see Subsection 1.3.1) satisfy $\mathfrak{s} = \mathfrak{b} = \mathfrak{a} = \aleph_1 < \mathfrak{d}$ and, with fsi of ccc forcing, Baumgartner and Dordal [BD85] proved the consistency of $\mathfrak{s} < \mathfrak{b}$. With csi of proper forcing, Shelah [S84] proved the consistency of $\mathfrak{b} = \aleph_1 < \mathfrak{a} = \mathfrak{s} = \mathfrak{c} = \aleph_2$ and $\mathfrak{b} = \mathfrak{a} = \aleph_1 < \mathfrak{s} = \mathfrak{c} = \aleph_2$.

With these results proved in the 80's, the only remaining open problem (about those four cardinal invariants) was the consistency of $\mathfrak{d} < \mathfrak{a}$. Many related problems are still unanswered, like

Problem A. *(1) (Roitman, 1970's) Does* $\mathfrak{d} = \aleph_1$ *imply* $\mathfrak{a} = \aleph_1$*?*

(2) (Brendle and Raghavan, [BrRa14]) Does $\mathfrak{b} = \mathfrak{s} = \aleph_1$ *imply* $\mathfrak{a} = \aleph_1$*?*

Shelah [S04] proved the consistency of $\aleph_1 < \mathfrak{d} < \mathfrak{a}$ in two different ways. In the first proof, it is assumed the existence of a measurable cardinal κ and, by forcing with ultrapowers (see Section 4.2 for details), he constructed a ccc poset that forces $\kappa < \mathfrak{b} = \mathfrak{d} < \mathfrak{a} = \mathfrak{c}$. The same type of construction with ultrapowers works to get a model of $\kappa < \mathfrak{u} < \mathfrak{a} = \mathfrak{c}$, where $\mathfrak{u}$ is the ultrafilter number (see Section 1.4).

The previous technique can be used only to get consistent statements with ZFC+*"there exists a measurable cardinal"*, but recall that a measurable cardinal is inaccessible and that it cannot be proved that the existence of an inaccessible cardinal is consistent with ZFC. However, inaccessible cardinals have very strong combinatorial properties and are extensively used in set theory.

For the second proof, he invented the technique of *iterations along a template*, or just *template iterations*, which generalizes a fsi of definable ccc posets (or *Suslin ccc posets*, see Sections 1.3 and 2.2) in the sense that an iteration can be constructed along a linear order that is not necessarily well ordered (which is a requirement for a typical fsi). An additional feature of such an iteration is that it can be localized in any subset of the linear order, fact that was used by Shelah to replace the ultrapower argument in the first construction by an isomorphism-of-names argument, this to prove the consistency of $\aleph_1 < \mathfrak{d} < \mathfrak{a} = \mathfrak{c}$ with ZFC alone. We discuss this iteration technique carefully in Chapter 2.

Although Shelah's construction with ultrapowers for the consistency of $\mathfrak{d} < \mathfrak{a}$ can be seen as a sequence of template iterations, the template structure is not important here to understand the model. However, it is unclear whether the chain of iterations constructed with ultrapowers for the consistency of $\mathfrak{u} < \mathfrak{a}$ can be put into the template framework. It is an open question whether it is possible to get the consistency of $\mathfrak{u} < \mathfrak{a}$ on the basis of ZFC alone.

For this project, we are particularly interested in getting consistency results with large continuum where the values of $\mathfrak{b}$, $\mathfrak{a}$ and $\mathfrak{s}$ can be separated. Assume that $\mu < \lambda$ are uncountable regular cardinals with $\lambda^\omega = \lambda$. Some extension of the classical consistency results, with large continuum, are the consistency of $\mathfrak{b} = \mu < \mathfrak{a} = \mathfrak{s} = \mu^+$ due to Brendle [Br98] by using fsi of ccc forcing and, with matrix iterations, Brendle and V. Fischer [BrF11] proved the consistency of $\mathfrak{b} = \mathfrak{a} = \mu < \mathfrak{s} = \mathfrak{c} = \lambda$ with ZFC and the consistency of $\kappa < \mathfrak{b} = \mu < \mathfrak{a} = \mathfrak{s} = \mathfrak{c} = \lambda$ where κ is a measurable cardinal (in the ground model). In Shelah's model for the consistency of $\mathfrak{u} < \mathfrak{a}$ it is also true that $\kappa < \mathfrak{b} = \mathfrak{s} = \mathfrak{u} = \mu < \mathfrak{a} = \lambda$.

We turn to the case where $\mathfrak{s}$, $\mathfrak{b}$ and $\mathfrak{a}$ may take pairwise different values.

Problem B ([BrF11])**.** *Is it consistent that*

(1) $\mathfrak{b} < \mathfrak{a} < \mathfrak{s}$*?*

(2) $\mathfrak{b} < \mathfrak{s} < \mathfrak{a}$*?*

(3) $\mathfrak{s} < \mathfrak{b} < \mathfrak{a}$*?*

Since models for $\mathfrak{b} < \mathfrak{s}$ and $\mathfrak{b} < \mathfrak{a}$ are very difficult to construct, a very involved forcing construction may be necessary to answer (1) and (2). To handle (3), an argument to force $\mathfrak{b} < \mathfrak{a}$ while preserving splitting families of the ground model seems to work, but fsi techniques from [Br98] do not seem viable.

We analyze Problem B(3) from the point of view of template iterations. Actually, both models of Shelah (explained above) satisfy $\mathfrak{s} = \aleph_1$, so (3) is consistent with $\mathfrak{s} = \aleph_1$. We answer (3) where $\mathfrak{s}$ is allowed to be (almost) an arbitrary value below a measurable cardinal, which is one of the main results of this thesis.

Theorem C. *Assume that κ is a measurable cardinal. Then, it is consistent that* $\aleph_1 < \mathfrak{s} < \kappa < \mathfrak{b} < \mathfrak{a} = \mathfrak{c}$.

The forcing construction to prove this result is similar to Shelah's construction to get the consistency of $\mathfrak{d} < \mathfrak{a}$ modulo a measurable cardinal. However, in this case it seems to be relevant to look at the template structure of the iterations in order to ensure that $\mathfrak{s}$ will not become too big in the final extension. Moreover, as non-definable forcing notions are involved in these iterations, we need to extend Shelah's theory of template iterations by allowing these posets in the construction. By doing so, many consistency results with fsi of ccc forcing can be generalized. For example, an application of a result of Blass [Bl89] (Lemma 4.1.1) to force the *groupwise-density number* $\mathfrak{g}$ (see Section 1.4) to be equal to $\aleph_1$ can be extended to the framework of template iterations in order to get larger values for $\mathfrak{g}$. In summary:

Theorem D. *(1) Shelah's theory of template iterations is extended to allow quite arbitrary forcing notions.*

(2) Arguments to force $\mathfrak{g} = \aleph_1$ *can be extended, in view of (1), to obtain models with larger* $\mathfrak{g}$.

Theorems C and D were obtained by the author in [Me].

Matrix iterations and Cichon's diagram

We are interested in obtaining models of Cichon's diagram (Figure 1) with large continuum. With small continuum, Cichon's diagram is now fully understood, but there are still many interesting open problems concerning large continuum. Judah and Shelah [JS90] and Brendle [Br91] established techniques with fsi of ccc forcing that can be used to get many different values for the cardinal invariants in Cichon's diagram, in particular for those that appear on the left hand side. For example, these techniques can be used to get the consistency of $\mathrm{add}(\mathcal{M}) < \mathrm{cov}(\mathcal{N}) < \mathfrak{b} < \mathrm{non}(\mathcal{M}) = \mathrm{cov}(\mathcal{M}) < \mathfrak{d} = \mathrm{non}(\mathcal{N}) = \mathfrak{c}$

(see Section 4.1). Also, by adding random reals (see Subsection 1.3.2) over a model of the previous statement, we can get the consistency of $\mathrm{non}(\mathcal{N}) = \aleph_1 < \mathfrak{b} < \mathfrak{d} < \mathrm{cov}(\mathcal{N}) = \mathfrak{c}$.

However, it seems more complicated to get models where the cardinal invariants of the right hand side of Cichon's diagram can assume more than two values. With Brendle, Judah and Shelah's techniques it only seems to be possible to obtain models where invariants of the right hand side assume at most two different values. The previous example with random forcing shows how to get three different values on the right hand side, but we don't know how to get more examples like this.

Problem E. *Get models where cardinal invariants on the right hand side of Cichon's diagram assume three or more different values. For example, is it consistent that* $\mathrm{cov}(\mathcal{M}) < \mathfrak{d} < \mathrm{non}(\mathcal{N}) < \mathrm{cof}(\mathcal{N}) = \mathfrak{c}$*?*

We found out that matrix iteration constructions solve part of this problem, at least some cases for three values on the right hand side of Cichon's diagram. Blass and Shelah [BlS84] used this technique for the first time to prove the consistency of $\mathfrak{u} < \mathfrak{d}$ with large continuum. Later, Brendle and V. Fischer [BrF11] improved this type of construction to get the consistency results mentioned in the previous section.

In this work, we show how to use this technique with Suslin ccc forcing notions and to get models for some cases of Problem E.

Theorem F. *There are many consistent cases where cardinal invariants of the right hand side of Cichon's diagram can assume three different values. For example, it is consistent that* $\mathrm{cov}(\mathcal{M}) < \mathfrak{d} < \mathrm{non}(\mathcal{N}) = \mathfrak{c}$.

We also extend this result to some other classical cardinal invariants of the continuum, which are defined in Section 1.4. These results were obtained by the author in [Me13a, Me13b].

There is a work in progress by M. Goldstern, J. Kellner, S. Shelah and A. Fischer where they use a large product construction with countable support to prove the consistency of $\mathrm{cov}(\mathcal{N}) = \mathfrak{d} = \aleph_1 < \mathrm{non}(\mathcal{M}) < \mathrm{non}(\mathcal{N}) < \mathrm{cof}(\mathcal{N}) < \mathfrak{c}$, which gives an example of 5 different values on the right hand side of Cichon's diagram.

Rothberger gaps in F_σ quotients

We review some notation concerning gaps in quotients by ideals on ω. Fin denotes the ideal of finite subsets of ω. For an arbitrary ideal $\mathcal{I}$ on ω, define the relation $\subseteq_{\mathcal{I}}$ on $\mathcal{P}(\omega)$ by $A \subseteq_{\mathcal{I}} B$ iff $A \smallsetminus B \in \mathcal{I}$, and let $\sim_{\mathcal{I}}$ be the equivalence relation on $\mathcal{P}(\omega)$ given by $A \sim_{\mathcal{I}} B$ iff $A \subseteq_{\mathcal{I}} B$ and $B \subseteq_{\mathcal{I}} A$. The quotient $\mathcal{P}(\omega)/\mathcal{I} := \mathcal{P}(\omega)/\sim_{\mathcal{I}}$ is a Boolean algebra, in fact, $0 = \bar{\varnothing} = \mathcal{I}$ (the zero object in the Boolean algebra), $\bar{A} \wedge \bar{B} = \overline{A \cap B}$ and $\bar{A} \leq \bar{B}$ iff $A \subseteq_{\mathcal{I}} B$ for $A, B \in \mathcal{P}(\omega)$ where the bar on top denotes the corresponding equivalence class. Also note that $\bar{A} \wedge \bar{B} = 0$ iff $A \cap B \in \mathcal{I}$.

For two families $\mathcal{A}, \mathcal{B} \subseteq \mathcal{P}(\omega)/\mathcal{I}$, the pair $\langle \mathcal{A}, \mathcal{B} \rangle$ is a *gap in* $\mathcal{P}(\omega)/\mathcal{I}$ if

- $\bar{A} \wedge \bar{B} = 0$ for all $\bar{A} \in \mathcal{A}$ and $\bar{B} \in \mathcal{B}$, and
- there is no $\bar{C} \in \mathcal{P}(\omega)/\mathcal{I}$ such that $\bar{A} \wedge \bar{C} = 0$ for all $\bar{A} \in \mathcal{A}$ and $\bar{B} \leq \bar{C}$ for all $\bar{B} \in \mathcal{B}$.

Let $\langle \mathcal{A}, \mathcal{B} \rangle$ be a gap in $\mathcal{P}(\omega)/\mathcal{I}$. Say that $\langle \mathcal{A}, \mathcal{B} \rangle$ is a *Hausdorff gap* if both $\mathcal{A}$ and $\mathcal{B}$ are σ-directed (i.e. any countable subfamily of $\mathcal{A}$ has an upper bound in $\mathcal{A}$, likewise for $\mathcal{B}$). On the other hand, if one of $\mathcal{A}$ and $\mathcal{B}$ is countable, we say that $\langle \mathcal{A}, \mathcal{B} \rangle$ is a *Rothberger gap*. For two cardinals κ and λ, say that $\langle \mathcal{A}, \mathcal{B} \rangle$ is a *linear gap of type* (κ, λ) or a (κ, λ)*-gap* if $\mathcal{A}$ and $\mathcal{B}$ are well ordered increasing sequences of order type κ and λ, respectively. Clearly, any (ω, λ)-gap is Rothberger.

We are interested in the study of Rothberger gaps in quotients by definable ideals on ω. To understand definability in this sense, consider $\mathcal{P}(\omega)$ with the topology given by the Cantor space 2^ω ($2^\omega = \prod_{n<\omega} 2$ with the product topology where $2 = \{0, 1\}$ has the discrete topology) by associating each member of $\mathcal{P}(\omega)$ with its characteristic function. If Γ is a pointclass on $\mathcal{P}(\omega)$ (e.g. Γ is the collection of F_σ subsets, or of the analytic subsets, etc.), we say that an ideal $\mathcal{I}$ on ω is a Γ *ideal* if $\mathcal{I}$, as a subset of $\mathcal{P}(\omega)$, belongs to Γ. In practice, a definable ideal is understood as a Γ ideal where Γ is the pointclass of analytic sets, or even a pointclass higher in the projective hierarchy (see Section 1.1). Unless otherwise stated, we assume that our ideals are non-trivial, in the sense that they contain Fin but do not contain ω as an element. The simplest definable non-trivial ideals are F_σ.

Gaps in $\mathcal{P}(\omega)/\mathrm{Fin}$ have been studied since almost 100 years ago and there is still a lot of ongoing research nowadays. The first celebrated result is the construction of an (ω_1, ω_1)-gap by Hausdorff [Ha1909, Ha36]. Years later, Rothberger [Ro41] produced his $(\omega, \mathfrak{b})$-gap and, moreover, proved that there are no (ω, κ)-gaps in $\mathcal{P}(\omega)/\mathrm{Fin}$ for any $\kappa < \mathfrak{b}$. However, Todorčević [T, Thm. 8.6] proved that, under $\mathfrak{c} = \aleph_2$ and the open coloring axiom OCA, there are no gaps in $\mathcal{P}(\omega)/\mathrm{Fin}$ of type different from $(\omega, \mathfrak{b})$ and (ω_1, ω_1).

Recently, the research on gaps has been extended to quotients by definable ideals in general. First, Mazur [Ma91] found out that Hausdorff's construction also works for a class of ideals he calls *pseudosolid*, a class which contains the F_σ-ideals, so quotients by such ideals contain an (ω_1, ω_1)-gap. Later, Todorčević [T98] made a great improvement of this by showing that, when $\mathcal{I}$ is either a pseudosolid ideal or an analytic P-ideal, $\mathcal{P}(\omega)/\mathrm{Fin}$ can be embedded into $\mathcal{P}(\omega)/\mathcal{I}$ in such a way that gaps are preserved, so there is a gap in $\mathcal{P}(\omega)/\mathcal{I}$ of any of the type of gaps that exist in $\mathcal{P}(\omega)/\mathrm{Fin}$. Therefore, such quotients also contain $(\omega, \mathfrak{b})$-gaps and (ω_1, ω_1)-gaps. An important problem, addressed in [T98, Problem 2] and discussed deeply in [Fa, Section 5], is determining the gap spectrum of these quotients.

Problem G (Todorčević [T98, Problem 2]). *Determine the gap spectrum of $\mathcal{P}(\omega)/\mathcal{I}$ for every analytic ideal $\mathcal{I}$ on ω.*

One interesting example, proved by Kankaanpää [Kank13], is the existence of an $(\omega, \mathrm{add}(\mathcal{M}))$-gap in $\mathcal{P}(\mathbb{Q})/\mathrm{nwd}$ where nwd denotes the ideal of nowhere dense subsets of the rationals $\mathbb{Q}$, which is an $F_{\sigma\delta}$ ideal. Moreover, like Rothberger's result, there are no gaps in this quotient of type (ω, κ) when $\kappa < \mathrm{add}(\mathcal{M})$.

Results about Rothberger gaps can also be handled using a cardinal invariant that we define as follows. For an ideal $\mathcal{I}$ on ω define the *Rothberger number* $\mathfrak{b}(\mathcal{I})$ *of* $\mathcal{I}$ as the minimal cardinal κ such that there is an (ω, κ)-gap in $\mathcal{P}(\omega)/\mathcal{I}$ (see Lemma 6.1.1 for equivalent definitions). Rothberger's result can be interpreted, in terms of this cardinal invariant, as $\mathfrak{b}(\mathrm{Fin}) = \mathfrak{b}$ (this is why we use the letter $\mathfrak{b}$ for the Rothberger number). Also, Todorčević's result above directly implies that $\mathfrak{b}(\mathcal{I}) \leq \mathfrak{b}$ when $\mathcal{I}$ is either an F_σ ideal (or even pseudosolid) or an analytic P-ideal. Moreover, $\mathfrak{b}(\mathcal{I}) = \mathfrak{b}$ for an analytic P-ideal $\mathcal{I}$ follows from Solecki's characterization [So96, So99] of analytic P-ideals[2] (see Corollary 1.5.7 for a proof).

For our main results about gaps, we focus our interest on the Rothberger number of *fragmented ideals* (see Definition 6.1.4), which is a subclass of the F_σ ideals defined by Hrušák, Rojas-Rebolledo and Zapletal [HrRZ]. They also consider the subclass of *gradually fragmented ideals* to get an interesting dichotomy for fragmented ideals and their cofinality number, moreover, a specific type of gradually fragmented ideal is used to get, via large product constructions, the consistency of the existence of continuum many such ideals that have pairwise different cofinality numbers. Typical examples are $\mathcal{ED}_{\mathrm{fin}}$ (see [Hr11, p.42], also Example 6.1.12(2)), which is a fragmented not gradually fragmented ideal with domain the set of ordered pairs of natural numbers below the identity function and generated by the graphs of functions below the identity; the *linear growth ideal* $\mathcal{I}_L$ (see [Hr11, p.56], also Example 6.1.12(3)), given by $X \in \mathcal{I}_L$ iff the sequence $\left\{\frac{|X \cap a_i|}{i+1}\right\}_{i\in\omega}$ is bounded where $\{a_i\}_{i\in\omega}$ is a partition of ω with $|a_i| = 2^i$; this ideal is also fragmented but not gradually fragmented; and the *polynomial growth ideal* $\mathcal{I}_P$ (see [Hr11, p.56], also Example 6.1.12(1)), which is a gradually fragmented ideal given by $X \in \mathcal{I}_P$ iff $\exists_{m<\omega}\forall_{i<\omega}(|X \cap a_i| \leq \max\{i, 2\}^m)$, where $\{a_i\}_{i\in\omega}$ is the same partition as before.

Brendle proved in 2009 that there is an (ω, ω_1)-gap in $\mathcal{P}(\omega)/\mathcal{ED}_{\mathrm{fin}}$. A similar argument for this proof works to get $\mathfrak{b}(\mathcal{I}_L) = \omega_1$. In view of this, the following has been conjectured.

Problem H (Hrušák). *Is $\mathfrak{b}(\mathcal{I}) = \aleph_1$ for any fragmented not gradually fragmented ideal $\mathcal{I}$?*

[2] Although this is a well-known fact, we could not find a reference.

We solve this problem for a large class of fragmented ideals.

Theorem I. *There is an (ω, ω_1)-gap in $\mathcal{P}(\omega)/\mathcal{I}$ for a large class of fragmented, not gradually fragmented ideals $\mathcal{I}$ that include $\mathcal{ED}_{\text{fin}}$ and $\mathcal{I}_L$.*

Like in [HrRZ], we use the gradually fragmented ideals to force that there are many different such ideals with pairwise different Rothberger numbers. To prove this, we first study how to destroy gaps in gradually fragmented ideals and also how to preserve gaps in some forcing extensions. In this case, we do not use large product constructions to force such a statement because these Rothberger numbers are below $\mathfrak{b}$ and this forcing technique typically generates models with $\mathfrak{d} = \aleph_1$. However, we use fsi of ccc forcing and preservation properties inspired from [KaO14] to construct such a model. The outcome is summarized in the following

Theorem J. *(1) For all gradually fragmented ideals $\mathcal{I}$,*

- $\mathfrak{b}(\mathcal{I}) \geq \text{add}(\mathcal{N})$,
- $\mathfrak{b}(\mathcal{I}) = \mathfrak{b}$ *when $\mathcal{I}$ is nowhere tall (see Section 1.5 for this definition), and*
- *it is consistent that $\mathfrak{b}(\mathcal{I}) < \mathfrak{b}$ for all somewhere tall ideals $\mathcal{I}$ (see Section 1.5 for this definition).*

(2) For a large class of gradually fragmented ideals $\mathcal{I}$ including $\mathcal{I}_P$, $\text{add}(\mathcal{N}) < \mathfrak{b}(\mathcal{I}) < \mathfrak{b}$ is consistent.

(3) There may be many gradually fragmented ideals with pairwise different Rothberger numbers.

Theorems I and J correspond to the main results of the research of Brendle and the author in [BrMe].

Outline of the dissertation

We summarize in Chapter 1 the preliminary knowledge about set theory of the reals, forcing, cardinal invariants and gaps, which is needed to understand the main results of this work. Chapter 2 is the most technical part of this dissertation as it contains the elements to construct a template iteration with non-definable forcing notions as expected for Theorem D(1). Also, many general properties about embeddability and ccc-ness for template iterations are included. The well known general theory of preservation properties for cardinal invariants is presented in Chapter 3. These properties are known in the context of fsi of ccc forcing and of matrix iterations, but we extend this to a general context of iterations that works in the framework of template iterations. This chapter is fundamental for the proofs of the main results.

The following chapters are devoted to the proofs of the main results of this dissertation. The proofs of Theorems D(2) and C are included in Chapter 4 as applications of the theory of template iterations and its corresponding preservation properties. We explain the construction of a matrix iteration in Chapter 5 and, using preservation properties, we deal with the proof of Theorem F and related results. We introduce fragmented ideals in Chapter 6 and prove Theorems I and J. For the first theorem we present many interesting combinatorial constructions based on eventually different reals and on a notion of independent families. For the second, we present a forcing notion that is useful to destroy gaps in gradually fragmented ideals and, in a context similar to Chapter 3, we explain how to preserve some kind of gaps for fragmented ideals.

At the end, we include a chapter discussing open problems that are related to the main results of this thesis.

CHAPTER 1

PRELIMINARIES

No one shall expel us from the paradise that Cantor has created for us.
— D. Hilbert

This chapter is devoted to the basic notions and the knowledge that is needed for the main results of this thesis. The main topics discussed are: elementary concepts of descriptive set theory for Polish spaces and Borel sets; elementary forcing theory, with special attention to Suslin ccc posets and particular cases that are used in all our applications throughout this work; cardinal invariants and gaps in quotients by ideals on ω. The last topic, discussed in Section 1.5, is only necessary for the results in Chapter 6.

Our notation is quite standard, like in [Je], [Ke] and [Ku]. Given a cardinal number μ, $[X]^{<\mu}$ denotes the collection of all the subsets of X of size $<\mu$. Likewise, define $[X]^{\leq\mu}$ and $[X]^{\mu}$, the latter being the collection of all the subsets of X of size μ. Given two sets X, Y, Y^X denotes the set of functions from X into Y. For an ordinal δ, $Y^{<\delta} := \bigcup_{\alpha<\delta} Y^\alpha$. If we consider the product $\prod_{i\in I} X_i$, for a function $p \in \prod_{i\in J} X_i$ where $J \subseteq I$, denote by $[p] := \{x \in \prod_{i\in I} X_i \ / \ p \subseteq x\}$. Also, if $k \in I \smallsetminus J$ and $z \in X_k$, let $p^\frown\langle z\rangle_k$ be the function that extends p with domain $J \cup \{k\}$ and whose k-th component is z. In the case where $I = \delta$ and $J = \alpha < \delta$ are ordinals, $p^\frown\langle z\rangle = p^\frown\langle z\rangle_\alpha$.

Given a formula $\varphi(x)$ of the language of ZFC, $\forall^\infty_{n<\omega}\varphi(n)$ means that $\varphi(n)$ holds for all but finitely many $n < \omega$. $\exists^\infty_{n<\omega}\varphi(n)$ means that infinitely many $n < \omega$ satisfy $\varphi(n)$. Say that $\bar{J} = \langle J_n\rangle_{n<\omega}$ is an *interval partition of* ω if it is a partition of ω into non-empty finite intervals such that $\max(J_n) + 1 = \min(J_{n+1})$ for all $n < \omega$.

Given a set X, an *ideal on* X is a collection of subsets of X that is closed under finite unions and downwards closed under $\subseteq$. Throughout this thesis we assume, unless indicated otherwise, that an ideal on X contains all the singletons from (and thus all the finite subsets of) X and that X itself does not belong to the ideal. If $\mathcal{I}$ is an ideal on X, for $A, B \subseteq X$, $A \subseteq_{\mathcal{I}} B$ denotes $A \smallsetminus B \in \mathcal{I}$ and $\sim_{\mathcal{I}}$ denotes the equivalence relation given by $A \sim_{\mathcal{I}} B$ iff $A \subseteq_{\mathcal{I}} B$ and $B \subseteq_{\mathcal{I}} A$. If $\mathcal{B} \subseteq \mathcal{P}(X)$ is a subalgebra, $\mathcal{B}/\mathcal{I} = \mathcal{B}/((\mathcal{B}\times\mathcal{B})\cap \sim_{\mathcal{I}})$ denotes the *quotient Boolean algebra of* $\mathcal{B}$ *modulo* $\mathcal{I}$. Say that A is $\mathcal{I}$*-positive* if $A \notin \mathcal{I}$ and $\mathcal{I}^+$ denotes the collection of $\mathcal{I}$-positive sets. $\mathcal{I}{\restriction}B = \{A \in \mathcal{I} \ / \ A \subseteq B\}$ is the *restriction of the ideal* $\mathcal{I}$ *to* B. A σ*-ideal* is an ideal that is closed under countable unions. An ideal $\mathcal{I}$ is called *P-ideal* if, for any $\mathcal{C} \subseteq \mathcal{I}$ countable, there exists a $B \in \mathcal{I}$ such that $C \subseteq^* B$ for all $C \in \mathcal{C}$, where $C \subseteq^* B$ means that C *is almost contained in* B, that is, $C \smallsetminus B$ is finite.

As a dual notion, $\mathcal{F} \subseteq X$ is a *filter on* X if it is closed under finite intersections and upwards closed under $\subseteq$. We assume that all cofinite subsets of X are in a filter and that $\varnothing$ does not belong to a filter. An *ultrafilter on* X is a maximal filter. Equivalently, $\mathcal{U} \subseteq \mathcal{P}(X)$ is an ultrafilter iff it is a filter on X and, for any $A \subseteq X$, either $A \in \mathcal{U}$ or $X \smallsetminus A \in \mathcal{U}$.

A cardinal κ is *measurable* if it is uncountable and has a κ-complete (non-trivial) ultrafilter $\mathcal{U}$, where *κ-complete* means that $\mathcal{U}$ is closed under intersections of $< \kappa$ many sets. Note that, in this case, κ is an inaccessible cardinal. For a formula $\varphi(x)$ in the language of ZFC, say that $\varphi(\alpha)$ *holds for $\mathcal{D}$-many* α iff $\{\alpha < \kappa \ / \ \varphi(\alpha)\} \in \mathcal{D}$. To fix a notation about *ultraproducts* and *ultrapowers*, if $\langle X_\alpha \rangle_{\alpha<\kappa}$ is a sequence of sets, $(\prod_{\alpha<\kappa} X_\alpha)/\mathcal{D} = [\{X_\alpha\}_{\alpha<\kappa}]$ denotes the quotient of $\prod_{\alpha<\kappa} X_\alpha$ modulo the equivalence relation given by $x \sim_{\mathcal{D}} y$ iff $x_\alpha = y_\alpha$ for $\mathcal{D}$-many $\alpha < \kappa$. If $x = \langle x_\alpha \rangle_{\alpha<\kappa} \in \prod_{\alpha<\kappa} X_\alpha$, denote its equivalence class under $\sim_{\mathcal{D}}$ by $\bar{x} = \langle x_\alpha \rangle_{\alpha<\omega}/\mathcal{D}$. It is known that posets of size $< \kappa$ does not destroy the measurability of κ, that is, preserves the κ-completeness of $\mathcal{D}$. For facts about measurable cardinals (and large cardinals in general), see [Kan].

1.1 Polish spaces and Borel sets

In this section, we summarize many notions and important results about Borel sets on Polish spaces, including a breve description of the Borel and projective hierarchies, as well as coding of Borel sets. For details on the contents of this section, see [Ke] and [Je].

A topological space $\langle X, \tau \rangle$ is *Polish* if it is completely metrizable and separable. Typical examples of Polish spaces are the real line $\mathbb{R}$ with the usual topology, the Cantor space 2^ω and the Baire space ω^ω. As the structure of an uncountable Polish space X is very close to the one of real numbers, sometimes we refer to the elements of X as *reals*. It is known that the size of such a space is $\mathfrak{c}$.

Fix $n < \omega$. For $\{\eta_k\}_{k<n} \subseteq \{2, \omega\}$ and a tree $T \subseteq (\prod_{k<n} \eta_k)^{<\omega}$, denote $[T] := \{(x_0, \dots x_{n-1}) \in \prod_{k<n} \eta_k^\omega \ / \ \forall_{i<\omega}((x_0 \restriction i, \dots, x_{n+1} \restriction i) \in T)\}$, which is a closed subset of $\prod_{k<n} \eta_k^\omega$. Recall that, if $C \subseteq \prod_{k<n} \eta_k^\omega$ is closed, then there is a tree $T \subseteq (\prod_{k<n} \eta_k)^{<\omega}$ such that $C = [T]$.

For a topological space X, the *Borel hierarchy* of Borel subsets of X is defined by recursion on $0 < \alpha < \omega_1$, where $\mathbf{\Sigma}^0_1(X)$ is the collection of open subsets of X and, for $\alpha > 1$, $\mathbf{\Sigma}^0_\alpha(X)$ is the collection of countable unions of members of $\bigcup_{\xi<\alpha} \mathbf{\Pi}^0_\xi(X)$. For all $\alpha < \omega_1$, $\mathbf{\Pi}^0_\alpha(X)$ is the collection of complements of sets in $\mathbf{\Sigma}^0_\alpha(X)$. Let $\mathbf{\Delta}^0_\alpha(X) = \mathbf{\Sigma}^0_\alpha(X) \cap \mathbf{\Pi}^0_\alpha(X)$. $\mathcal{B}(X) := \bigcup_{\alpha<\omega_1} \mathbf{\Sigma}^0_\alpha(X) = \bigcup_{\alpha<\omega_1} \mathbf{\Pi}^0_\alpha(X)$ is the σ-algebra of *Borel subsets of* X. Sets in $\mathbf{\Sigma}^0_2(X)$ are typically known as *F_σ-sets* and sets in $\mathbf{\Pi}^0_2(X)$ are known as *G_δ-sets*.

The *projective hierarchy* for all Polish spaces is defined by recursion on $0 < n < \omega$. $\mathbf{\Sigma}^1_1(X)$, the collection of *analytic sets*, is the collection of sets of the form $\pi_0[F]$ where F is a closed subset of $X \times \omega^\omega$ and $\pi_0 : X \times \omega^\omega \to X$ is the projection onto the first coordinate. For $1 < n < \omega$, $\mathbf{\Sigma}^1_{n+1}(X)$ is the collection of sets of the form $\pi_0[A]$ where A is in $\mathbf{\Pi}^1_n(X \times \omega^\omega)$. $\mathbf{\Pi}^1_n(X)$ is the collection of complements of sets in $\mathbf{\Sigma}^1_n(X)$. Let $\mathbf{\Delta}^1_n(X) = \mathbf{\Sigma}^1_n(X) \cap \mathbf{\Pi}^1_n(X)$. Sets in $\mathbf{\Pi}^1_1(X)$ are known as *co-analytic sets*. It is known that $\mathcal{B}(X) = \mathbf{\Delta}^1_1(X)$. When the space X is understood, we just refer to these classes as $\mathbf{\Sigma}^0_\alpha$, $\mathbf{\Sigma}^1_n$, etc. In many cases, when a subset of X is defined by a formula $P(x)$, say, $A = \{x \in X \ / \ P(x)\}$, we say that the formula $P(x)$ *is* $\mathbf{\Sigma}^1_n$ if $A \in \mathbf{\Sigma}^1_n$, likewise for the other defined point-classes.

1.1.1 Theorem (Mostowski's absoluteness theorem [Je, Thm. 25.4])**.** *For spaces of the form* $\prod_{k<\omega} \eta_k$ *with* $\{\eta_k\}_{k<\omega} \subseteq \{2, \omega\}$*: if* M *is a model of* ZFC*, then any* $\mathbf{\Sigma}^1_1$ *statement with parameters in* M *is absolute. The same holds for* $\mathbf{\Pi}^1_1$ *statements.*

1.1.2 Theorem (Shoenfield's absoluteness theorem[Je, Thm. 25.20])**.** *For spaces of the form* $\prod_{k<\omega} \eta_k$ *with* $\{\eta_k\}_{k<\omega} \subseteq \{2, \omega\}$*: if* $M \subseteq N$ *are transitive models of* ZFC *and* $\omega_1^N \in M$*, then any* $\mathbf{\Sigma}^1_2$ *statement with parameters in* M *is absolute for* M, N*. The same holds for* $\mathbf{\Pi}^1_2$ *statements.*

For a topological space X, say that $A \subseteq X$ is *nowhere dense (nwd)* if $\operatorname{int}(\overline{A}) = 0$. A is *meager* if it is the countable union of nwd sets. Let $\mathcal{M}(X)$ be the *σ-ideal of meager subsets of* X. When the space is clear from the context, we just write $\mathcal{M}$ to denote the ideal.

1.1.3 Theorem. *Let* X *and* Y *be uncountable Polish spaces. Then*

1. *([Ke, Subsect. 8.F]) $\mathcal{B}(X)/\mathcal{M}(X)$ and $\mathcal{B}(Y)/\mathcal{M}(Y)$ are isomorphic complete Boolean algebras.*

2. *([Ke, Subsect. 15.D]) If $\Psi : \mathcal{B}(X)/\mathcal{M}(X) \to \mathcal{B}(Y)/\mathcal{M}(Y)$ is an isomorphism, then there exists a Borel isomorphism $f : Y \to X$ such that $\Psi([A]) = [f^{-1}[A]]$ for any $A \in \mathcal{B}(X)$.*

3. *There exists a Borel isomorphism $f : Y \to X$ such that, for $A \subseteq X$, $A \in \mathcal{M}(X)$ iff $f^{-1}[X] \in \mathcal{M}(Y)$.*

Let X be a set and $\mathcal{A}$ be a σ-algebra of subsets of X. A function $\mu : \mathcal{A} \to [0, +\infty]$ is a *measure* if $\mu(\varnothing) = 0$ and if it is σ*-additive*, that is, $\mu(\bigcup_{n<\omega} A_n) = \sum_{n<\omega} \mu(A_n)$ for any pairwise disjoint family $\{A_n \;/\; n < \omega\} \subseteq \mathcal{B}(X)$. μ is a *probability measure* if it is a measure such that $\mu(X) = 1$. A measure μ is σ*-finite* if X is the countable union of subsets of finite measure. A measure μ is *continuous* if $\mu(\{x\}) = 0$ for all $x \in X$. We say that $N \subseteq X$ is μ*-null* if there is a $B \in \mathcal{A}$ such that $N \subseteq B$ and $\mu(B) = 0$. Denote by $\mathcal{N}(X, \mathcal{A}, \mu)$ the σ-ideal of null subsets of X. When the space and the measure are clear from the context, we just write $\mathcal{N}$.

1.1.4 Theorem ([Ke, Thm. 17.41])**.** *Let X be a Polish space and $\mu : \mathcal{B}(X) \to [0, 1]$ a continuous probability measure. Then, there exists a Borel isomorphism $f : X \to [0, 1]$ such that $\mu(f^{-1}[A]) = \lambda(A)$ for any $A \in \mathcal{B}([0, 1])$, where λ is the Lebesgue measure on $[0, 1]$.*

There are also canonical measures for spaces of the form $\prod_{k<\omega} \eta_k$ with $\{\eta_k\}_{k<\omega} \subseteq \{2, \omega\}$, which are also called *Lebesgue measures*. They are given by the product measure where, in 2, each singleton has measure $\frac{1}{2}$ and, in ω, $\{n\}$ has measure $\frac{1}{2^{n+1}}$ for each $n < \omega$.

1.1.5 Theorem (Lebesgue density theorem [Ke, Sect. 17.B])**.** *Let $\lambda : \mathcal{B}(2^\omega) \to [0, 1]$ be the Lebesgue measure for 2^ω. If $B \in \mathcal{B}(2^\omega)$, then $\lambda(B) = \lambda(\{x \in B \;/\; \lim_{n\to\infty} \lambda(B \cap [x{\restriction}n])/\lambda([x{\restriction}n]) = 1\})$.*

There are similar versions of this theorem for the Lebesgue measures on ω^ω and on $\mathbb{R}$.

To finish this section, we introduce *Borel coding*, which is a way to code the construction of Borel sets by real numbers. $\mathrm{BC} = \bigcup_{\alpha<\omega_1} \mathrm{BC}_\alpha$ is constructed recursively in such a way that $\langle \mathrm{BC}_\alpha \rangle_{\alpha<\omega_1}$ is an increasing sequence of subsets of ω^ω and that each $a \in \mathrm{BC}$ indicates how to construct a certain Borel subset B_a of any Polish space. This is done in such a way that, for $\alpha < \omega_1$, $\mathbf{\Sigma}^0_\alpha(X) = \{B_a \;/\; a \in \mathrm{BC}_\alpha\}$ for each Polish space X. Typically, BC is a $\mathbf{\Pi}^1_1$ subset of ω^ω such that the statement "$x \in B_a$" is a Borel statement in $X \times \omega^\omega$ where X is any Polish space of the form $\prod_{k<\omega} \eta_k$ with $\{\eta_k\}_{k<\omega} \subseteq \{2, \omega\}$. Therefore, this implies that statements such as "$B_a \subseteq B_c$", "$B_a \cap B_c = \varnothing$" and "$B_a = \bigcup_{n<\omega} B_{c_n}$" are $\mathbf{\Pi}^1_1$-statements for Borel codes $a, c, c_0, \ldots$. This coding, along with Mostowski's absoluteness Theorem 1.1.1, is important in order to have absolute statements about Borel sets between ZFC models. For the construction of Borel codes, details and the following result, see [Je, Sect. 25] and [Ku84].

1.1.6 Theorem. *The statements "$B_a \in \mathcal{M}$" and "$B_a \in \mathcal{N}$" are Borel.*

For an infinite set I consider the topological space 2^I given by the product topology where 2 has the discrete topology. Note that all its *clopen* subsets (that is, open and closed sets) are of the form $\bigcup_{p\in F}[p]$ where $F \subseteq \mathrm{Fn}(I, 2)$ is finite, with $\mathrm{Fn}(I, 2)$ the set of finite partial functions from I to 2. Let $\mathcal{B}a(2^I)$ be the σ-algebra of *Baire subsets of* 2^I, which is generated by the clopen subsets of 2^I.

It is known that any Baire subset of 2^I is of the form $2^J \times B$ where $I = J \cup K$ is a disjoint union, K is countable and $B \in \mathcal{B}(2^K) = \mathcal{B}a(2^K)$. Therefore, the coding of Borel subsets of 2^ω can be extended for Baire subsets of 2^I. For example, $\mathrm{BaC}_\alpha(I) = \{(a, \Delta) \;/\; a \in \mathrm{BC}_\alpha, \Delta : \omega \to I \text{ one-to-one}\}$ for $\alpha < \omega_1$ and $\mathrm{BaC}(I) = \bigcup_{\alpha<\omega_1} \mathrm{BaC}_\alpha(I)$. If $(a, \Delta) \in \mathrm{BaC}(I)$, then $B_{a,\Delta} = \Delta_*^{-1}[B_a]$ where $\Delta_* : 2^I \to 2^\omega$ with $\Delta_*(f) = f \circ \Delta$ (a projection function). Clearly, any Baire subset of 2^I has the form $B_{a,\Delta}$ for some Baire code $(a, \Delta) \in \mathrm{BaC}(I)$. As in the case of Borel sets, we can get many absolute statements about Baire subsets, for example, "$f \in B_c$", "$B_c \subseteq B_{c'}$", "$B_c \in \mathcal{M}(2^I)$" and "$B_c \in \mathcal{N}(2^I, \mathcal{B}a(2^I), \lambda)$" are absolute between ZFC-models for $c, c' \in \mathrm{BaC}(I)$, where 2 is given the measure where each singleton has measure $\frac{1}{2}$ and λ is the corresponding product measure.

1.2 Preliminaries about forcing

This section contains many results that are known in the folklore of forcing theory. For basic knowledge about forcing, see [Je] and [Ku].

For a poset $\mathbb{P}$ and $p, q \in \mathbb{P}$, $p \perp q$ means that p and q are incompatible. $p \parallel q$ represents the compatibility of both conditions. $p \in \mathbb{P}$ is an *atom* if any pair of conditions that extend p are compatible. $\mathbb{P}$ is said to be *non-atomic* or *non-trivial* if none of its conditions is an atom. If M is a model of ZFC, $\mathbb{P} \in M$, $p \in \mathbb{P} \cap M$ and φ a formula in the forcing language of $\mathbb{P}$ with parameters in M, $p \Vdash_{\mathbb{P},M} \varphi$ denotes that "p forces φ" is true in the model M. When $\mathbb{P}$ and M are clearly understood, we ignore such subindexes in the notation.

If $\mathbb{Q}$ is another poset, recall that $g : \mathbb{P} \to \mathbb{Q}$ is a *complete embedding* if

(i) for any $p, p' \in \mathbb{P}$, $p \leq p'$ implies $g(p) \leq g(p')$,

(ii) for any $p, p' \in \mathbb{P}$, $p \perp p'$ if and only if $g(p) \perp g(p')$, and

(iii) for any $q \in \mathbb{Q}$ there is a $p \in \mathbb{P}$ which is a *reduction of* q *(with respect to* g*)*, that is, for every $p' \leq p$, $g(p')$ is compatible with q.

Conditions (ii) and (iii) can be replaced by: for any maximal antichain A in $\mathbb{P}$, $g[A]$ is a maximal antichain in $\mathbb{Q}$. When g is the identity function, we say that $\mathbb{P}$ *is a complete suborder of* $\mathbb{Q}$.

For this section, fix $M \subseteq N$ transitive models of ZFC. If $\mathbb{P} \in M$ say that $g : \mathbb{P} \to \mathbb{Q}$ is a *complete embedding with respect to* M if (i) holds and, for any $A \in M$ maximal antichain, $g[A]$ is a maximal antichain in $\mathbb{Q}$. When g is the identity function, say that $\mathbb{P}$ *is a complete suborder of* $\mathbb{Q}$ *with respect to* M. This notion is important because, if $g, \mathbb{Q} \in N$ and G is $\mathbb{Q}$-generic over N, then $g^{-1}[G]$ is $\mathbb{P}$-generic over M and $M[g^{-1}[G]] \subseteq N[G]$.

$g : \mathbb{P} \to \mathbb{Q}$ is a *dense embedding* if it satisfies (i) and (ii) and also $g[\mathbb{P}]$ is dense in $\mathbb{Q}$. It is known that, for any poset $\mathbb{P}$, there exists a complete Boolean algebra $\mathbb{B}$ and a dense embedding $g : \mathbb{P} \to \mathbb{B}$. In fact, if we associate to $\mathbb{P}$ the topology generated by the subsets of the form $[p] = \{p' \in \mathbb{P} \,/\, p' \leq p\}$ for $p \in \mathbb{P}$, then $\mathbb{B} = \mathrm{ro}(\mathbb{P})$, the *complete Boolean algebra of the regular open sets of* $\mathbb{P}$ (that is, those subsets A such that $\mathrm{int}(\overline{A}) = A$). g is defined as $g(p) = \mathrm{int}(\overline{[p]})$. This complete Boolean algebra $\mathbb{B}$ is unique under isomorphism and is known as the *completion of* $\mathbb{P}$.

If $\mathbb{A}$ and $\mathbb{B}$ are complete Boolean algebras, then any complete embedding as posets between them is an embedding as complete Boolean algebras, and any dense embedding is actually an isomorphism. Moreover, it is known that the forcing structure of a poset can be completely described by its completion.

A poset $\mathbb{P}$ *(forcing) embeds into* $\mathbb{Q}$, denoted by $\mathbb{P} \lessdot \mathbb{Q}$, if there is a complete embedding $g : \mathrm{ro}(\mathbb{P}) \to \mathrm{ro}(\mathbb{Q})$. $\mathbb{P} \simeq \mathbb{Q}$ denotes that $\mathbb{P}$ *is forcing equivalent with* $\mathbb{Q}$, which means that their completions are isomorphic. Just for notation, if $\mathbb{P} \in M$, $\mathbb{P} \lessdot_M \mathbb{Q}$ means that $\mathbb{P}$ is a complete suborder of $\mathbb{Q}$ with respect to M.

1.2.1 Lemma. *Assume that* $\mathbb{P} \in M$ *and* $\mathbb{Q} \in N$ *are posets such that* $\mathbb{P} \lessdot_M \mathbb{Q}$. *If* φ *is a* $\mathbf{\Sigma}^1_1$*-statement in the forcing language of* $\mathbb{P}$ *with parameters in* M *and* $p \in \mathbb{P}$, *then* $p \Vdash_{\mathbb{P},M} \varphi$ *iff* $p \Vdash_{\mathbb{Q},N} \varphi$. *The same holds for* $\mathbf{\Pi}^1_1$*-statements. Moreover, if* $\omega_1^M = \omega_1^N$, *this also holds for* $\mathbf{\Sigma}^1_2$ *and* $\mathbf{\Pi}^1_2$*-statements.*

Assume that $g : \mathbb{P} \to \mathbb{Q}$ is a complete embedding between posets. Define the *quotient poset* $\mathbb{Q}/\mathbb{P}$ *(with respect to* g*)* as the $\mathbb{P}$-name of the poset $\{q \in \mathbb{Q} \,/\, \exists_{p \in \dot{G}}(p \text{ is a reduction of } q)\}$ with the same order as $\mathbb{Q}$, where $\dot{G}$ is the canonical $\mathbb{P}$-name of the generic subset.

1.2.2 Lemma. $\mathbb{Q} \simeq \mathbb{P} * (\mathbb{Q}/\mathbb{P})$. *Moreover, if* $q \in \mathbb{Q}$ *and* φ *is a formula in the forcing language,* $q \Vdash_{\mathbb{Q}} \varphi$ *iff* $(p, q) \Vdash_{\mathbb{P}*(\mathbb{Q}/\mathbb{P})} \varphi$ *for all the reductions* $p \in \mathbb{P}$ *of* q.

Proof. Assume that $g : \mathbb{P} \to \mathbb{Q}$ is a complete embedding. Let $f : \mathbb{P} \to \mathbb{A}$ and $f' : \mathbb{Q} \to \mathbb{B}$ be dense embeddings into complete Boolean algebras. There is a complete embedding $g' : \mathbb{A} \to \mathbb{B}$ such that $g' \circ f = f' \circ g$, in fact,

$$g'(a) = \bigvee_{\mathbb{B}} \{f'(g(p)) \,/\, p \in \mathbb{P},\ f(p) \leq a\}.$$

Consider $h : \mathbb{B} \to \mathbb{A}$ the projection with respect to g', that is, $h(b) = \bigwedge_{\mathbb{A}}\{a \in \mathbb{A} \ / \ g'(a) \geq b\}$. Note that $\mathbb{B}/\mathbb{A} = \{b \in \mathbb{B} \ / \ h(b) \in \dot{G}\}$, where $\dot{G}$ is the $\mathbb{A}$-name of its generic subset. We claim that $\mathbb{A}$ forces that $\dot{k} : \mathbb{Q}/\mathbb{P} \to \mathbb{B}/\mathbb{A}$ is a dense embedding, where $\dot{k}$ is an $\mathbb{A}$-name for the function defined as $\dot{k}(q) = f'(q)$. Indeed, let G be $\mathbb{A}$-generic over V and $b \in \mathbb{B}/\mathbb{A}$ arbitrary. By genericity, as $\{h(f'(q)) \ / \ f'(q) \leq b, q \in \mathbb{Q}\}$ is dense below $h(b)$, we can find $q \in \mathbb{Q}$ such that $f'(q) \in \mathbb{B}/\mathbb{A}$ and $f'(q) \leq b$. As $\{f(p) \ / \ f(p) \leq h(f'(q)), p \in \mathbb{P}\}$ is dense below $h(f'(q))$, we can find a reduction $p \in f^{-1}[G]$ of q, so $q \in \mathbb{Q}/\mathbb{P}$.

Note that $\mathbb{P} * (\mathbb{Q}/\mathbb{P}) \simeq \mathbb{A} * (\mathbb{B}/\mathbb{A})$ is witnessed by the function that sends $(p, \dot{q}) \in \mathbb{P} * (\mathbb{Q}/\mathbb{P})$ to $(f(p), \dot{k}((q)))$. On the other hand, $\mathbb{B} \simeq \mathbb{A} * (\mathbb{B}/\mathbb{A})$ by the dense embedding that sends $b \in \mathbb{B}$ to $(h(b), b)$, so $\mathbb{Q} \simeq \mathbb{P} * (\mathbb{Q}/\mathbb{P})$.

Assume that $q \Vdash_{\mathbb{Q}} \varphi$ and $p \in \mathbb{P}$ is a reduction of q. Then, $f(p) \leq h(f'(q))$, so $(f(p), f'(q)) \leq_{\mathbb{A}*(\mathbb{B}/\mathbb{A})} (h(f'(q)), f'(q))$. On the other hand, $q \Vdash_{\mathbb{Q}} \varphi$ implies $f'(q) \Vdash_{\mathbb{B}} \varphi$ and, then, $(h(f'(q)), f'(q)) \Vdash_{\mathbb{A}*(\mathbb{B}/\mathbb{A})} \varphi$. Thus, $(f(p), f'(q)) \Vdash_{\mathbb{A}*(\mathbb{B}/\mathbb{A})} \varphi$, which clearly implies that $(p, q) \Vdash_{\mathbb{P}*(\mathbb{Q}/\mathbb{P})} \varphi$.

Now, assume that $q \not\Vdash_{\mathbb{Q}} \varphi$, that is, there is a $q' \leq q$ such that $q' \Vdash_{\mathbb{Q}} \neg\varphi$. By the previous argument, $(p', q') \Vdash_{\mathbb{P}*(\mathbb{Q}/\mathbb{P})} \neg\varphi$ for every reduction $p' \in \mathbb{P}$ of q'. Any such p' is a reduction of q and, as $(p', q') \leq_{\mathbb{P}*(\mathbb{Q}/\mathbb{P})} (p', q)$, then $(p', q) \not\Vdash_{\mathbb{P}*(\mathbb{Q}/\mathbb{P})} \varphi$. □

1.2.3 Lemma. *Let $\mathbb{P}$ be a complete suborder of $\mathbb{Q}$ and $\dot{\mathbb{R}}$ a $\mathbb{Q}$-name for a poset. Then, $\mathbb{P}$ forces that $(\mathbb{Q} * \dot{\mathbb{R}})/\mathbb{P} \simeq (\mathbb{Q}/\mathbb{P}) * \dot{\mathbb{R}}$.*

Proof. Let G be $\mathbb{P}$-generic over V. Work in $V[G]$. Note that, if $(q, \dot{r}) \in (\mathbb{Q} * \dot{\mathbb{R}})/\mathbb{P}$ and $p \in G$ is a reduction of $(q, \dot{r})$, then p is also a reduction of q, so $q \in \mathbb{Q}/\mathbb{P}$. Define $g : (\mathbb{Q} * \dot{\mathbb{R}})/\mathbb{P} \to (\mathbb{Q}/\mathbb{P}) * \dot{\mathbb{R}}$ as $g(q, \dot{r}) = (q, \dot{r}_G)$ where $\dot{r}_G$ is a $\mathbb{Q}/\mathbb{P}$-name (which can be defined canonically) such that $\Vdash_{\mathbb{Q}/\mathbb{P}} \dot{r} = \dot{r}_G$. Note that g is increasing and onto, so it remains to see that it preserves incompatibilities. Let $(q, \dot{r}), (q', \dot{r}') \in (\mathbb{Q} * \dot{\mathbb{R}})/\mathbb{P}$ and assume that there is a $q_2 \in \mathbb{Q}/\mathbb{P}$ extending q and q' such that $q_2 \Vdash_{\mathbb{Q}/\mathbb{P}} \dot{r} \parallel \dot{r}'$. Let $p_2 \in G$ be a reduction of q_2 such that, in V, $(p_2, q_2) \Vdash_{\mathbb{P}*(\mathbb{Q}/\mathbb{P})} \dot{r} \parallel \dot{r}'$.

1.2.4 Claim. *In V, if $q_3 \in \mathbb{Q}$ is an extension of q_2 and p_2, then $q_3 \Vdash_{\mathbb{Q}} \dot{r} \parallel \dot{r}'$.*

Proof. If $p \in \mathbb{P}$ is a reduction of q_3, then any extension of (p, q_3) in $\mathbb{P} * (\mathbb{Q}/\mathbb{P})$ is compatible with (p_2, q_2), so $(p, q_3) \Vdash_{\mathbb{P}*(\mathbb{Q}/\mathbb{P})} \dot{r} \parallel \dot{r}'$. Therefore, by Lemma 1.2.2, $q_3 \Vdash_{\mathbb{Q}} \dot{r} \parallel \dot{r}'$. □

Note that $D := \{p \in \mathbb{P} \ / \ \exists_{q_3 \in \mathbb{Q}}(p \text{ is a reduction of } q_3 \text{ and } q_3 \leq p, q_2)\}$ is dense below p_2 in $\mathbb{P}$. Then, in $V[G]$, there exists $p_3 \in D \cap G$ with $p_3 \leq p_2$. Choose q_3 a witness of $p_3 \in D$ so, by Claim 1.2.4, $q_3 \in \mathbb{Q}/\mathbb{P}$ and $q_3 \Vdash_{\mathbb{Q},V} \dot{r} \parallel \dot{r}'$. Thus, $(q, \dot{r})$ and $(q', \dot{r}')$ are compatible in $(\mathbb{Q} * \dot{\mathbb{R}})/\mathbb{P}$. □

1.2.5 Corollary. *Let $\mathbb{Q}$ be a complete suborder of $\mathbb{R}$ and $\mathbb{P}$ a complete suborder of $\mathbb{Q}$. Then, $\mathbb{R}/\mathbb{P} \simeq (\mathbb{Q}/\mathbb{P}) * (\mathbb{R}/\mathbb{Q})$.*

Proof. By Lemma 1.2.3, $\mathbb{R}/\mathbb{P} \simeq (\mathbb{Q} * (\mathbb{R}/\mathbb{Q}))/\mathbb{P} \simeq (\mathbb{Q}/\mathbb{P}) * (\mathbb{R}/\mathbb{Q})$. □

Recall that a partial order $\langle I, \leq \rangle$ is *directed* iff any two elements of I have an upper bound in I. A sequence of posets $\langle \mathbb{P}_i \rangle_{i \in I}$ is a *directed system of posets* if, for any $i \leq j$ in I, $\mathbb{P}_i$ is a complete suborder[1] of $\mathbb{P}_j$. In this case, the *direct limit of* $\langle \mathbb{P}_i \rangle_{i \in I}$ is defined as the partial order $\text{limdir}_{i \in I} \mathbb{P}_i := \bigcup_{i \in I} \mathbb{P}_i$. It is clear that, for any $i \in I$, $\mathbb{P}_i$ is a complete suborder of this direct limit.

For an infinite cardinal μ, recall that a poset $\mathbb{P}$ has the *μ-chain condition (μ-cc)* if all its antichains have size $< \mu$. In the case that $\mu = \aleph_1$, μ-cc is known as the *countable chain condition (ccc)*, a property that is relevant for preserving cardinals and cofinalities in generic extensions. We introduce stronger versions of this condition.

1.2.6 Definition. (1) For $n < \omega$, $B \subseteq \mathbb{P}$ is *n-linked* if every $F \subseteq B$ of size $\leq n$ has a common extension in $\mathbb{P}$.

[1] In a more general way, we can think of a directed system with complete embeddings $e_{i,j} : \mathbb{P}_i \to \mathbb{P}_j$ for $i < j$ in I such that, for $i < j < k$, $e_{j,k} \circ e_{i,j} = e_{i,k}$. This allows to define a direct limit of the system as well.

(2) $C \subseteq \mathbb{P}$ is *centered* if it is n-linked for every $n < \omega$.

(3) $\mathbb{P}$ is *μ-linked* if it is the union of $\leq \mu$ many 2-linked subsets of $\mathbb{P}$. In the case $\mu = \aleph_0$, we say *σ-linked.*

(4) $\mathbb{P}$ is *μ-centered* if it is the union of $\leq \mu$ many centered subsets of $\mathbb{P}$. In the case $\mu = \aleph_0$, we say *σ-centered.*

(5) $\mathbb{P}$ is *μ-Knaster* if, for every sequence $\{p_\alpha\}_{\alpha<\mu}$ of conditions in $\mathbb{P}$, there is an $A \subseteq \mu$ of size μ such that $\{p_\alpha \,/\, \alpha \in A\}$ is 2-linked. For $\mu = \aleph_1$, we just say *Knaster*.

Note that μ-centered implies μ-linked, and μ-Knaster implies μ-cc. Also, μ-linked implies μ^+-Knaster.

1.2.7 Lemma. *Let $n < \omega$ and $B \subseteq \mathbb{P}$ be n-linked. If $F \in V$ has size $\leq n$ and $\dot{a}$ is a $\mathbb{P}$-name for a member of F, then there exists a $c \in F$ such that no $p \in B$ forces $\dot{a} \neq c$.*

Proof. Assume that, for every $c \in F$, there exists a $p_c \in B$ that forces $\dot{a} \neq c$. As $\{p_c \,/\, c \in F\}$ has size $\leq n$ and it is a subset of B, it has a common extension $p \in \mathbb{P}$. Thus, p forces that $\dot{a} \notin F$, a contradiction. □

In Section 1.3 we introduce many examples of definable forcing notions. Meanwhile, consider the following two posets that will be relevant for many proofs of this text. Recall that $\mathcal{F} \subseteq [\omega]^\omega$ is a *filter base* if it nonempty and closed under finite intersections. $C \in [\omega]^\omega$ is a *pseudo-intersection of $\mathcal{F}$* if $C \subseteq^* A$ for all $A \in \mathcal{F}$.

1.2.8 Definition. Let $\mathcal{F}$ be a filter base (of infinite subsets of ω).

(1) *Mathias forcing with $\mathcal{F}$* is the poset $\mathbb{M}_\mathcal{F} = \{(s, A) \,/\, s \in [\omega]^{<\omega},\ A \in \mathcal{F},\ \sup(s + 1) \leq \min(A)\}$ (where $s + 1 = \{k + 1 \,/\, k \in s\}$) ordered as $(t, B) \leq (s, A)$ iff $s \subseteq t$, $B \subseteq A$ and $t \smallsetminus s \subseteq A$.

(2) *Laver forcing with $\mathcal{F}$* is the poset $\mathbb{L}_\mathcal{F}$ whose conditions are trees $T \subseteq \omega^{<\omega}$ such that $\{i < \omega \,/\, t^\frown\langle i\rangle \in T\} \in \mathcal{F}$ for any $t \in T$ with $\mathrm{stem}(T) \subseteq t$, where $\mathrm{stem}(T)$, the *stem of* T, is the least splitting node of T. The order of $\mathbb{L}_\mathcal{F}$ is $\subseteq$.

Notice that $\mathbb{M}_\mathcal{F}$ and $\mathbb{L}_\mathcal{F}$ are σ-centered posets, the first witnessed by the centered sets $\{(s, F) \in \mathbb{M}_\mathcal{F} \,/\, s = a\}$ for $a \in [\omega]^{<\omega}$, and the latter witnessed by $\{T \in \mathbb{L}_\mathcal{F} \,/\, \mathrm{stem}(T) = t\}$ for $t \in \omega^{<\omega}$. The generic object added by $\mathbb{M}_\mathcal{F}$ is $\dot{m}_\mathcal{F} = \bigcup \mathrm{dom}\dot{G} = \bigcup\{s \,/\, \exists_F((s, F) \in \dot{G})\}$ where $\dot{G}$ is the $\mathbb{M}_\mathcal{F}$-name of the generic subset. $\mathbb{M}_\mathcal{F}$ forces that this generic object is a pseudo-intersection of $\mathcal{F}$. On the other hand, $\mathbb{L}_\mathcal{F}$ adds generically a function $\dot{l}_\mathcal{F} := \bigcup_{T\in\dot{H}} \mathrm{stem}(T)$ where $\dot{H}$ is the $\mathbb{L}_\mathcal{F}$-name of the generic subset. $\mathbb{L}_\mathcal{F}$ forces that $f \leq^* \dot{l}_\mathcal{F}(n)$ for every ground model function f in ω^ω and that $\mathrm{ran}(\dot{l}_\mathcal{F})$ is a pseudo-intersection of $\mathcal{F}$. Laver forcing with an ultrafilter has a very special property about decision of statements.

1.2.9 Theorem (Pure decision [Bl88, Thm. 9])**.** *Let $\mathcal{U}$ be an ultrafilter on ω, $s \in [\omega]^{<\omega}$ and φ a formula in the forcing language of $\mathbb{L}_\mathcal{U}$. Then, there exists a $T \in \mathbb{L}_\mathcal{U}$ with stem s such that either $T \Vdash \varphi$ or $T \Vdash \neg\varphi$.*

The proof of pure decision for Laver forcing with an ultrafilter uses a *rank argument*, which is useful to solve many problems. Consider a dense set $D \subseteq \mathbb{L}_\mathcal{U}$ where $\mathcal{U}$ is an ultrafilter on ω. For $s \in \omega^{<\omega}$, define the ordinal $\rho_D(s)$ recursively, such that, for $\alpha > 0$,

$$\begin{aligned} \rho_D(s) = 0 \quad & \text{iff there exists a } T \in D \text{ with stem } s,\\ \rho_D(s) = \alpha \quad & \text{iff } \rho_D(s) \not< \alpha \text{ and } \{j < \omega \,/\, \rho_D(s^\frown\langle j\rangle) < \alpha\} \in \mathcal{U}. \end{aligned}$$

To see this, define $R_D(0) = \{s \in \omega^{<\omega} \,/\, \exists_{T\in D}(\mathrm{stem}(T) = s)\}$ and, for $\alpha > 0$,

$$R_D(\alpha) = \left\{ s \in \omega^{<\omega} \,/\, \left\{ j < \omega \,/\, s^\frown\langle j\rangle \in \bigcup_{\beta<\alpha} R_D(\beta) \right\} \in \mathcal{U} \right\} \smallsetminus \bigcup_{\beta<\alpha} R_D(\beta).$$

Clearly, $\rho(s) = \alpha$ iff $s \in R_D(\alpha)$ for any ordinal α.

The main feature of this rank function is that $\rho_D(s) < \omega_1$ for any $s \in \omega^{<\omega}$. To see this, note that, if $\rho_D(s) \geq \omega_1$ then $\{j < \omega \;/\; \rho_D(s^\frown\langle j\rangle) \geq \omega_1\} \in \mathcal{U}$ so, if such an s exists, then it is possible to construct a tree T by recursion such that $\mathrm{stem}(T) = s$ and $\rho_D(t) \geq \omega_1$ for any $t \in T$ above the stem. Now, as D is dense, find $S \in D$, $S \subseteq T$. But $\rho_D(\mathrm{stem}(S)) \geq \omega_1$, a contradiction.

Proof of Theorem 1.2.9. Let $D = \{T \in \mathbb{L}_{\mathcal{U}} \;/\; T \Vdash \varphi \text{ or } T \Vdash \neg\varphi\}$, which is clearly dense. We show that $\rho_D(s) = 0$ for all $s \in \omega^{<\omega}$. Assume the contrary and let $s \in \omega^\omega$ of minimal rank > 0. Then, $\{j < \omega \;/\; \rho_D(s^\frown\langle j\rangle) = 0\} \in \mathcal{U}$, that is, either $\{j < \omega \;/\; \exists_{T\in\mathbb{L}_{\mathcal{U}}}(\mathrm{stem}(T) = s^\frown\langle j\rangle \text{ and } T \Vdash \varphi)\} \in \mathcal{U}$ or $\{j < \omega \;/\; \exists_{T\in\mathbb{L}_{\mathcal{U}}}(\mathrm{stem}(T) = s^\frown\langle j\rangle \text{ and } T \Vdash \neg\varphi)\} \in \mathcal{U}$. In the first case, it is easy to get $T' \subseteq T$ in $\mathbb{L}_{\mathcal{U}}$ with stem s such that $T' \Vdash \varphi$. For the second case, we get such a T' that forces $\neg\varphi$. In each case, we get a contradiction. □

1.3 Suslin ccc posets

1.3.1 Definition (Suslin ccc poset)**.** A *Suslin ccc poset* $\mathbb{S}$ is a ccc poset, whose conditions are reals, such that the relations $\leq$ and $\perp$ are $\mathbf{\Sigma}^1_1$.

In this case, note that $\mathbb{S}$ itself is a $\mathbf{\Sigma}^1_1$-set because $x \in \mathbb{S}$ iff $x \leq x$. The main feature of this type of poset is that, thanks to its definability, many of its features, when relativized to models of ZFC, become absolute. For example, if M is a model of a large enough fragment of ZFC and the parameters of the poset $\mathbb{S}$ are in M, then $\mathbb{S}$ may be interpreted in M, say $\mathbb{S}^M$, and many statements like "$x \in \mathbb{S}$" or "$p, q \in \mathbb{S}$ and $p \perp q$" are absolute. The notion of Suslin ccc forcing is due to Judah and Shelah [JS88]. See also [Br10] and [BaJ] for the proofs of the results of this section and for more about Suslin ccc forcing notions.

Fix $\mathbb{S}$ a Suslin ccc poset.

1.3.2 Lemma. *The statement "$\{z_n \;/\; n < \omega\}$ is a maximal antichain in $\mathbb{S}$" is $\mathbf{\Sigma}^1_1 \cup \mathbf{\Pi}^1_1$. Moreover, if "$x \in \mathbb{S}$" is a Borel statement, then the previous statement is $\mathbf{\Pi}^1_1$.*

1.3.3 Corollary. *Let $M \subseteq N$ be transitive models of* ZFC*. Then:*

(i) If $\mathbb{S}$ is coded in M, then $\mathbb{S}^M \lessdot_M \mathbb{S}^N$.

(ii) Assume that $\mathbb{P} \in M$ and $\mathbb{Q} \in N$ are posets such that $\mathbb{P} \lessdot_M \mathbb{Q}$ and that $\dot{\mathbb{S}}$ is a $\mathbb{P}$-name of a Suslin ccc poset. Then, $\mathbb{P} * \dot{\mathbb{S}}^{M^{\mathbb{P}}} \lessdot_M \mathbb{Q} * \dot{\mathbb{S}}^{N^{\mathbb{Q}}}$

The following are examples of well known Suslin ccc posets that we use throughout this text.

1.3.1 Cohen forcing

1.3.4 Lemma. *Any pair of countable non-atomic posets are forcing equivalent.*

Cohen forcing, denoted by $\mathbb{C}$, is any non-atomic countable poset. If we consider $\mathbb{C} = \{z \in \omega^\omega \;/\; \forall_{i\geq z(0)}(z(i+1) = 0)\}$ ordered by $z' \leq z$ iff $z(0) \leq z'(0)$ and $z'{\restriction}[1, z(0)] = z{\restriction}[1, z(0)]$, it is an F_σ subset of ω^ω. Note also that the order and $\perp$ are Borel relations, so it is a Suslin ccc poset, moreover, it is σ-centered.

The notion of Cohen forcing is extended to $\mathbb{C}_I = \mathrm{Fn}(I, 2)$ (see at the end of Section 1.1), ordered by $\supseteq$, for any set I. Also, $\mathbb{C} \simeq \mathbb{C}_\omega$. It is clear that, if $I = J \cup K$ is a disjoint union, then $\mathbb{C}_I \simeq \mathbb{C}_J \times \mathbb{C}_K \simeq \mathbb{C}_J * \dot{\mathbb{C}}_K$. Actually, $\mathbb{C}_{I\times\omega}$ is equivalent to the finite support product of $\mathbb{C}$ with indexes in I. Also, if κ is an infinite cardinal, then $\mathbb{C}_\kappa$ is equivalent to the fsi of length κ of $\mathbb{C}$.

$\mathbb{C}_I$ is equivalent to $\mathcal{B}a(2^I)/\mathcal{M}(2^I)$ witnessed by the dense embedding $g : \mathbb{C}_I \to \mathcal{B}a(2^I)/\mathcal{M}(2^I)$ where $g(p)$ is the equivalence class of $[p]$. Therefore, by Theorem 1.1.3, Cohen forcing is also given by $\mathcal{B}(X)/\mathcal{M}(X)$ where X is an uncountable Polish space. $\mathbb{C}_I$ adds a real that evades all the Baire meager sets coded in the ground model, that is, $\Vdash_{\mathbb{C}_I} \dot{c}_I \notin B_a^{V[\dot{G}]}$ where $\dot{G}$ is the $\mathbb{C}$-name for the generic subset, $\dot{c}_I = \bigcup \dot{G}$ and a is any code for a Baire meager set that belongs to the ground model. Also, $V[c_I] = V[G]$. In this concept, if M is a model of ZFC, say that a real c is a *Cohen real over M* if $c \notin B_a$ for any code $a \in M \cap \mathrm{BC}$ of a Borel meager set. Therefore, the generic real added by $\mathbb{C}$ is a Cohen real over the ground model.

Note that a fsi of ccc posets adds Cohen reals at limit stages.

1.3.5 Lemma. *Let $\mathbb{P} = \langle \mathbb{P}_n, \dot{\mathbb{Q}}_n \rangle_{n<\omega}$ be a fsi of ccc posets such that each $\mathbb{P}_n$ forces that $\dot{\mathbb{Q}}_n$ contains a pair of incompatible conditions. Then, $\mathbb{P}$ adds a Cohen real over the ground model V, that is, $\mathbb{C} \lessdot \mathbb{P}$.*

1.3.2 Random forcing

Typically, *random forcing on I*, denoted by $\mathbb{B}_I$, is the poset $\mathcal{B}(2^I) \smallsetminus \mathcal{N}(2^I)$ ordered by $\subseteq$, but note that its completion is $\mathcal{B}(2^I)/\mathcal{N}(2^I)$. By Theorem 1.1.4, *random forcing* $\mathbb{B} := \mathbb{B}_\omega$ is equivalent to $\mathcal{B}(X)/\mathcal{N}(X, \mu)$ where X is any uncountable Polish space and μ is a continuous probability measure on X. $\mathbb{B}_I$ adds a generic function $\dot{r}_I = \bigcup\{p \in \mathrm{Fn}(I, 2) \mathbin{/} [p] \in \dot{G}\}$ that evades all the Baire null subsets of 2^I coded in the ground model. Clearly, $V[G] = V[r_I]$. If M is a model of ZFC, say that a real r is a *random real over M* if $r \notin B_a$ for any code $a \in M \cap \mathrm{BC}$ of a Borel null set.

Note that $\{[T] \mathbin{/} T \subseteq 2^{<\omega}$ is a tree such that $\forall_{t\in T}(\lambda([T] \cap [t]) > 0)\}$ is dense in $\mathcal{B}(2^\omega) \smallsetminus \mathcal{N}$, so the set of such trees, ordered by $\subseteq$, is forcing equivalent to random forcing. Therefore, as this set of trees, the order and incompatibility relations are Borel (note that such trees T, S are incompatible iff $\lambda([T \cap S]) > 0$, which is a Borel statement), random forcing is a Suslin ccc poset. Moreover, it is σ-linked as witnessed by the Borel sets $B_p = \{T \in \mathbb{B} \mathbin{/} \lambda([p] \smallsetminus [T]) < \frac{1}{2}\lambda([p])\}$ with $p \in \mathrm{Fn}(\omega, 2)$ (Lebesgue density Theorem 1.1.5 implies that the union of these sets is $\mathbb{B}$).

1.3.6 Lemma. *$\mathbb{B}_I$ does not add Cohen reals. Dually, $\mathbb{C}_I$ does not add random reals.*

1.3.7 Lemma. *σ-centered posets do not add random reals.*

Like Cohen forcing, if $I = J \cup K$ is a disjoint union, then $\mathbb{B}_I \simeq \mathbb{B}_K * \dot{\mathbb{B}}_K$. But $\mathbb{B}_I \not\simeq \mathbb{B}_J \times \mathbb{B}_K$ because $\mathbb{B} \times \mathbb{B}$ adds a Cohen real, that is, $\mathbb{C} \lessdot \mathbb{B} \times \mathbb{B}$.

1.3.8 Definition. A poset $\mathbb{P}$ is *ω^ω-bounding* if, for any $\mathbb{P}$-name $\dot{g}$ for a real in ω^ω and $p \in \mathbb{P}$, there exists a $q \leq p$ in $\mathbb{P}$ and an $f \in \omega^\omega$ (in the ground model) such that $q \Vdash \dot{g} \leq f$ (pointwise, that is, $\forall_{n<\omega}(\dot{g}(n) \leq f(n))$). Equivalently, for any $\mathbb{P}$-generic set G and $g \in \omega^\omega \cap V[G]$, there exists an $f \in \omega^\omega \cap V$ such that $g \leq f$.

$\mathbb{B}_I$ is ω^ω-bounding, but $\mathbb{C}$ is not because, if $c \in \omega^\omega$ is a Cohen real over a model M of ZFC, then $\exists^\infty_{i<\omega}(f(i) < c(i))$ for any $f \in \omega^\omega \cap M$. For more about $\mathbb{C}_I$ and $\mathbb{B}_I$, see also [Ku84].

1.3.3 Hechler forcing

Recall from the Introduction that $f \leq^* g$ denotes $\forall^\infty_{i<\omega}(f(i) \leq g(i))$, which is read as g *dominates* f. Define $f <^* g$ likewise. For a set M, say that $d \in \omega^\omega$ is *dominating over M* if d dominates all the reals in $\omega^\omega \cap M$.

Hechler forcing $\mathbb{D}$ is the poset whose conditions are pairs $(s, f) \in \omega^{<\omega} \times \omega^\omega$ where $s \subseteq f$. The order is given by $(t, g) \leq (s, f)$ iff $s \subseteq t$ and $f \leq g$. If G is $\mathbb{D}$-generic, the *Hechler real* $d = \bigcup \mathrm{dom} G$ is dominating over the ground model. Also, $V[d] = V[G]$. $\mathbb{D}$ can be coded by ω^ω with the transformation $z \in \omega^\omega \mapsto (z{\restriction}[1, 1 + z(0)), z{\restriction}[1, \omega))$, so $\mathbb{D}$, $\leq_\mathbb{D}$ and $\perp_\mathbb{D}$ are Borel, that is, $\mathbb{D}$ is Suslin ccc. Moreover, it is σ-centered as witnessed by the Borel sets $D_s = \{(t, f) \in \mathbb{D} \mathbin{/} t = s\}$ for $s \in \omega^{<\omega}$.

It is known that $\mathbb{C} \lessdot \mathbb{D}$ but, by Lemma 1.3.7, $\mathbb{D}$ does not add random reals. As $\mathbb{B}_I$ is ω^ω-bounding, it does not add dominating reals. $\mathbb{C}$ does not add dominating reals (see Lemma 3.1.7 and Example 3.2.2).

1.3.4 Eventually different real forcing

Eventually different real forcing $\mathbb{E}$ is the poset with conditions in $\omega^{<\omega}\times[\omega^\omega]^{<\omega}$ and ordered by $(t,B)\leq(s,A)$ iff $s\subseteq t$, $A\subseteq B$ and $\forall_{i\in[|s|,|t|)}\forall_{f\in A}(t(i)\neq f(i))$. If G is $\mathbb{E}$-generic, the generic real $e=\bigcup\mathrm{dom}G$ is *eventually different* from the reals in the ground model V, that is, if $f\in\omega^\omega\cap V$ then $\forall^\infty_{n<\omega}(e(i)\neq f(i))$. Here, $V[e]=V[G]$. By coding $\mathbb{E}$ by reals in ω^ω, it is clear that $\mathbb{E}$, its order and the incompatibility relation are Borel, so $\mathbb{E}$ is Suslin ccc. Moreover, it is a σ-centered poset witnessed by the Borel sets $E_s=\{(t,A)\in\mathbb{E}\ /\ s=t\}$ for $s\in\omega^{<\omega}$.

It is known that $\mathbb{C}_\mathfrak{c}\lessdot\mathbb{E}$, but $\mathbb{E}$ does not add either random (Lemma 1.3.7) or dominating reals (Lemma 3.2.4). $\mathbb{C}$ does not add eventually different reals (Lemma 3.1.7 and Example 3.2.1). Although $\mathbb{D}$ and $\mathbb{B}$ add eventually different reals over the ground model, $\mathbb{E}$ is not forcing embeddable into any of them.

1.3.5 Amoeba forcing

Given a real $\delta\in(0,1)$, *amoeba forcing* $\mathbb{A}_\delta$ is the poset whose conditions are open subsets of 2^ω that have Lebesgue measure strictly smaller than δ, ordered by $\supseteq$. Equivalently, it is the poset whose conditions are the closed subsets of 2^ω with Lebesgue measure strictly bigger than $1-\delta$, ordered by $\subseteq$. Also, we get equivalent notions by considering just Borel sets in place of open or closed sets. If G is $\mathbb{A}_\delta$-generic (for the poset given by open sets), $a=\bigcup G$ is called *amoeba real*; it is an open set of measure δ. Clearly, $V[a]=V[G]$.

By Theorem 1.1.4, an equivalent poset is obtained if we consider an arbitrary uncountable Polish space with a continuous probability measure. Also, Truss [Tr88] proved that $\mathbb{A}_\delta\simeq\mathbb{A}_{\delta'}$ for two different $\delta,\delta'\in(0,1)$. Therefore, we can denote amoeba forcing by $\mathbb{A}$. It is a σ-linked Suslin ccc forcing. To see this, put $\mathbb{A}_{1/2}$ as the set of subtrees $T\subseteq 2^{<\omega}$ such that $\lambda([T])>\frac{1}{2}$, ordered by $\subseteq$. In this way, it is clear that $\mathbb{A}_{1/2}$, the order and the incompatibility relation are Borel. To see σ-linkedness, for $C\subseteq 2^\omega$ clopen of measure $<\frac{1}{2}$ and $n<\omega$ such that $\lambda(C)+\frac{2}{n}<\frac{1}{2}$, let $A_{C,n}=\{T\subseteq 2^{<\omega}$ tree $/\ C\cap[T]=\varnothing$ and $\lambda(C\cup[T])>1-\frac{1}{n}\}$ which is a subset of $\mathbb{A}_{1/2}$, it is Borel (as C can be coded by finitely many objects in $2^{<\omega}$) and 2-linked. Also, $\mathbb{A}_{1/2}$ is the union of all such $A_{C,n}$.

$\mathbb{A}$ adds a G_δ null set that covers all the Borel null sets coded in the ground model V. To see this, consider an amoeba real a of measure $\frac{1}{2}$ and note that every ground model Borel null set is contained in $N=\bigcap_{z\in\mathbb{Q}}a+z$ where $\mathbb{Q}=\{z\in 2^{<\omega}\ /\ \forall^\infty_{i<\omega}(z(i)=0)\}$ and the sum in 2^ω is the sum by coordinates of the group $\mathbb{Z}_2$. As $z+N=N$ for any $z\in\mathbb{Q}$, the measure of N is either 0 or 1 but, as $N\subseteq a$, we get that N is null. Also, in $V[a]$, $2^\omega\smallsetminus N$ is an F_σ set of measure 1 of random reals over V.

1.3.6 Localization forcing

For $h\in\omega^\omega$, put $S(\omega,h)=\prod_{i<\omega}[\omega]^{\leq h(i)}$, $S_n(\omega,h)=\prod_{i<n}[\omega]^{\leq h(i)}$ and $S_{<\omega}(\omega,h)=\bigcup_{n<\omega}S_n(\omega,h)$. We often call the reals in $S(\omega,h)$ *slaloms*. If $x\in\omega^\omega$ and $\psi\in([\omega]^{<\omega})^\omega$, define the relation $x\in^*\psi$ by $\forall^\infty_{i<\omega}(x(i)\in\varphi(i))$, which we read ψ *localizes* x.

If h is non-decreasing and converges to infinity, the *localization forcing* $\mathbb{LOC}^h$ *at* h is the poset given by the conditions of the form (s,F) where $s\in S_{<\omega}(\omega,h)$ and $F\subseteq\omega^\omega$ is a set of size $\leq h(|s|)$, ordered by $(t,F')\leq(s,F)$ iff $s\subseteq t$, $F\subseteq F'$ and $\forall_{i\in[|s|,|t|)}(\{f(i)\ /\ f\in F\}\subseteq t(i))$. If G is $\mathbb{LOC}^h$-generic, the *h-localization slalom* $\psi=\bigcup\mathrm{dom}G\in S(\omega,h)$ localizes all the reals in ω^ω of the ground model. Also, $V[G]=V[\psi]$.

By coding the members of $\mathbb{LOC}^h$ by reals in ω^ω, it is easy to see that this is a Suslin ccc pose; in fact, the domain, the order and the incompatibility relations are even Borel. Since the Borel sets $L^h_s=\{(t,F)\in\mathbb{LOC}^h\ /\ t=s,\ |F|\leq\frac{1}{2}h(|s|)\}$ for $s\in S_{<\omega}(\omega,h)$ are 2-linked, it is σ-linked.

Truss [Tr88] proved that $\mathbb{A}\lessdot\mathbb{LOC}^h$, so this forcing adds a G_δ null set that contains all the Borel null sets coded in the ground model. Also, $\mathbb{C}_\mathfrak{c}$, $\mathbb{D}$ and $\mathbb{E}$ are embeddable into $\mathbb{LOC}^h$. But $\mathbb{LOC}^h$ cannot be embedded into $\mathbb{A}$.

1.4 Some cardinal invariants of the continuum

We present some classical cardinal invariants of the continuum that are relevant for the main results of this thesis. For definitions, proofs of all the results of this section and further explanations, see [BaJ], [Ba10] and [Bl10].

For a set X and an ideal $\mathcal{I} \subseteq \mathcal{P}(X)$, define the following cardinal numbers.

$\mathrm{add}(\mathcal{I})$**,** the *additivity of* $\mathcal{I}$, which is the least size of a subfamily of $\mathcal{I}$ whose union is not in $\mathcal{I}$. Note that this cardinal is regular.

$\mathrm{cov}(\mathcal{I})$**,** the *covering of* $\mathcal{I}$, which is the least size of a subfamily of $\mathcal{I}$ whose union is all of X.

$\mathrm{non}(\mathcal{I})$**,** the *uniformity of* $\mathcal{I}$, which is the least size of a subset of X that is not in $\mathcal{I}$.

$\mathrm{cof}(\mathcal{I})$**,** the *cofinality of* $\mathcal{I}$, which is the least size of a cofinal subfamily of $\mathcal{I}$. Recall that $\mathcal{F} \subseteq \mathcal{I}$ is *cofinal* if, for any $A \in \mathcal{I}$, there exists a $C \in \mathcal{F}$ such that $A \subseteq C$.

We are particularly interested in the previous cardinal invariants for the ideals $\mathcal{M}$ and $\mathcal{N}$ for an uncountable Polish space. The following result states that these cardinals do not depend on the chosen uncountable Polish space.

1.4.1 Lemma. *Let X, Y be uncountable Polish spaces.*

(a) $\mathrm{add}(\mathcal{M}(X)) = \mathrm{add}(\mathcal{M}(Y))$ *and likewise for* cov, non *and* cof.

(b) If μ and ν are σ-finite continuous non-zero measures on X and Y respectively, then $\mathrm{add}(\mathcal{N}(X,\mu)) = \mathrm{add}(\mathcal{N}(Y,\nu))$ *and likewise for* cov, non *and* cof.

Proof. (a) is a direct consequence of Theorem 1.1.3. (b) follows from Theorem 1.1.4 and the fact that, for any σ-finite non-zero measure μ on a Polish space X there exists a probability measure μ' on X such that $\mathcal{N}(X,\mu) = \mathcal{N}(X,\mu')$. □

The cardinal invariants $\mathrm{add}(\mathcal{N})$ and $\mathrm{cof}(\mathcal{N})$ can be characterized as follows.

1.4.2 Theorem (Bartoszyński characterization [BaJ, Thm. 2.3.9])**.** *Let $h \in \omega^\omega$ that converges to infinity. Then,*

(a) $\mathrm{add}(\mathcal{N})$ *is the least size of a set $Y \subseteq \omega^\omega$ that cannot be localized by a single slalom in $S(\omega, h)$.*

(b) $\mathrm{cof}(\mathcal{N})$ *is the least size of a family $\mathcal{F} \subseteq S(\omega, h)$ with the property that every real in ω^ω is localized by some slalom in $\mathcal{F}$.*

The following is a well known characterization of covering and uniformity of category.

1.4.3 Theorem ([BaJ, Thm. 2.4.1 and 2.4.7])**.** *(a)* $\mathrm{non}(\mathcal{M})$ *is the least size of a family $\mathcal{F} \subseteq \omega^\omega$ such that, for any $x \in \omega^\omega$, there is an $f \in \mathcal{F}$ such that $\exists^\infty_{n<\omega}(f(n) = x(n))$.*

(b) $\mathrm{cov}(\mathcal{M})$ *is the least size size of a family $\mathcal{E} \subseteq \omega^\omega$ such that, for any $x \in \omega^\omega$, there is a $y \in \mathcal{E}$ such that $\forall^\infty_{n<\omega}(x(n) \neq y(n))$.*

Recall, from the Introduction, the *(un)bounding number* $\mathfrak{b}$ and the *dominating number* $\mathfrak{d}$ in the preorder $\langle \omega^\omega, \leq^* \rangle$. Recall also Cichon's diagram in Figure 1, where the vertical lines from bottom to top and the horizontal lines from left to right represent $\leq$ provable relations in ZFC. Also, the dotted lines represent $\mathrm{add}(\mathcal{M}) = \min\{\mathfrak{b}, \mathrm{cov}(\mathcal{M})\}$ and $\mathrm{cof}(\mathcal{M}) = \max\{\mathfrak{d}, \mathrm{non}(\mathcal{M})\}$.

Recall the splitting number $\mathfrak{s}$ from the Introduction. Given $\mathcal{F} \subseteq [\omega]^\omega$, say that A *reaps* $\mathcal{F}$ if A splits all the members of $\mathcal{F}$. $\mathcal{F}$ is said to be an *unreaping family* if it is not reaped by any infinite subset of ω. Define $\mathfrak{r}$, the *(un)reaping number*, as least size of an unreaping family. In relation with the cardinals in Cichon's diagram, we know that

1.4.4 Theorem. *(a)* $\mathfrak{s}$ *is less than or equal to* $\mathrm{non}(\mathcal{M})$*,* $\mathfrak{d}$ *and* $\mathrm{non}(\mathcal{N})$.

(b) $\mathfrak{r}$ *is greater than or equal to* $\mathrm{cov}(\mathcal{N})$*,* $\mathfrak{b}$ *and* $\mathrm{cov}(\mathcal{M})$.

A collection $\mathcal{F}$ of infinite subsets of ω is said to have the *finite intersection property (f.i.p.)* if the intersection of finitely many members of $\mathcal{F}$ is infinite. Note that, if $\mathcal{F}$ has the f.i.p., the set of all its finite intersections form a filter base. Define the following cardinal invariants.

$\mathfrak{p}$, the *pseudo-intersection number*, which is the least size of a filter base (or a family with the f.i.p.) that does not have a pseudo-intersection.

$\mathfrak{t}$, the *tower number*, which is the least order type of a $\subseteq^*$-decreasing sequence that has no pseudo-intersection.

It is clear that $\mathfrak{t}$ is an uncountable regular cardinal and that $\mathfrak{p} \leq \mathfrak{t}$. Moreover,

1.4.5 Theorem. *(a)* $\mathfrak{p}$ *is regular.*

(b) $\mathfrak{t}$ *is less than or equal to* $\mathrm{add}(\mathcal{M})$ *and* $\mathfrak{s}$.

Recently, it has been proved by Malliaris and Shelah [MS] that actually $\mathfrak{p} = \mathfrak{t}$, which was a long standing problem about cardinal invariants. Even though important, this result is not necessary for any of the conclusions of this thesis.

The *ultrafilter number* $\mathfrak{u}$ is defined as the least size of a filter base (or a family with the f.i.p.) that generates an ultrafilter. It is known that

1.4.6 Theorem. $\mathfrak{r} \leq \mathfrak{u}$.

Recall $\mathfrak{a}$, the *almost disjointness number*, from the Introduction.

1.4.7 Theorem. $\mathfrak{b} \leq \mathfrak{a}$.

A family $\mathcal{G} \subseteq [\omega]^\omega$ is *groupwise-dense* if it is downwards closed under $\subseteq^*$ and, for every interval partition $\langle I_n \rangle_{n<\omega}$ of ω, there is an $A \in [\omega]^\omega$ such that $\bigcup_{n \in A} I_n \in \mathcal{G}$. $\mathfrak{g}$, the *groupwise-density number*, is the least size of a collection of groupwise-dense families whose intersection is empty.

1.4.8 Theorem. $\mathfrak{t} \leq \mathfrak{g} \leq \mathfrak{d}$. *Also,* $\mathfrak{g}$ *is regular.*

Figure 1.1 describes the diagram of provable inequalities in ZFC of the cardinal invariants introduced in this section. Lines from bottom to top represent $\leq$. Dotted lines are as in Cichon's diagram (Figure 1) and the double line represents equality. The diagram is complete in the sense that the inequalities that are not stated, are consistent with ZFC, with the exception of $\mathfrak{r} < \mathfrak{a}$ and $\mathfrak{u} < \mathfrak{a}$ that are consistent modulo a measurable cardinal ([S04], see also [Br07]). Proofs and summaries about most of these consistency results can be found in [BaJ] and in [Bl10]. For example, with countable support iteration of proper forcing techniques, $\mathfrak{u} = \mathfrak{a} = \mathrm{non}(\mathcal{M}) = \mathrm{non}(\mathcal{N}) = \aleph_1 < \mathfrak{g} = \aleph_2$ holds in Miller's model and $\mathrm{cof}(\mathcal{N}) = \aleph_1 < \mathfrak{r} = \aleph_2$ is true in Silver's model. Also, the consistency of $\mathfrak{u} = \aleph_1 < \mathfrak{s} = \aleph_2$ is proved in [BlS87] and the consistency of $\mathfrak{r} < \mathfrak{u}$ is proved in [GS90].

Classical models with large continuum constructed with finite support iterations: $\mathfrak{a}, \mathrm{non}(\mathcal{M}) < \mathrm{cov}(\mathcal{M})$ is true in Cohen's model (an extension of a model of CH by $\mathbb{C}_\lambda$ with $\lambda > \aleph_1$ regular) and $\mathrm{non}(\mathcal{N}) = \mathfrak{a} = \mathfrak{d} = \aleph_1 < \mathrm{cov}(\mathcal{N})$ is true in the random model (an extension of a model of CH by $\mathbb{B}_\lambda$ with $\lambda > \aleph_1$ regular). Moreover, if we assume $\mathfrak{b} > \aleph_1$ in the ground model, $\mathrm{non}(\mathcal{N}) = \aleph_1 < \mathfrak{b}$ holds in an extension by $\mathbb{B}$. $\mathfrak{s} < \mathrm{add}(\mathcal{N})$ and $\mathfrak{g} < \mathrm{add}(\mathcal{N})$ hold in the amoeba model, that is, an extension of a model of CH by a fsi of amoeba forcing of length $\lambda > \aleph_1$ regular (see [JS88] for the first inequality and [Bl89] or Lemma 4.1.1 for the second) and $\mathrm{cov}(\mathcal{N}), \mathfrak{s}, \mathfrak{g} < \mathrm{add}(\mathcal{M})$ holds in Hechler's model (an extension of a model of CH by a fsi of Hechler forcing of length λ). The dual of Hechler's model (that is, assume that $\mathrm{add}(\mathcal{N})$ and $\mathfrak{p}$ are large in the ground model and perform a short iteration using Hechler forcing) satisfies $\mathrm{cof}(\mathcal{M}) < \mathfrak{r}, \mathrm{non}(\mathcal{N})$. A long fsi of Mathias forcing with ultrafilters yields a model of $\mathrm{cov}(\mathcal{N}) < \mathfrak{p}$. A model of $\mathfrak{g} < \mathfrak{s}$ is obtained by a matrix iteration construction as in [BlS84], e.g., define $\mathbb{P}_{\aleph_1,\nu} = \langle \langle \mathbb{P}_{\alpha,\xi}, \dot{\mathbb{Q}}_{\alpha,\xi} \rangle_{\xi<\nu} \rangle_{\alpha<\omega_1}$ as in Context 5.1.4 with $U = \nu$ and $\nu > \aleph_1$ regular (this is a typical matrix iteration construction for a model of $\mathfrak{b} < \mathfrak{s}$ where $\mathfrak{g} = \aleph_1$ because of Lemma 4.1.1).

Brendle [Br02] proved the consistency of $\aleph_1 < \mathrm{cof}(\mathcal{N}) < \mathfrak{a}$ by a template iteration construction.

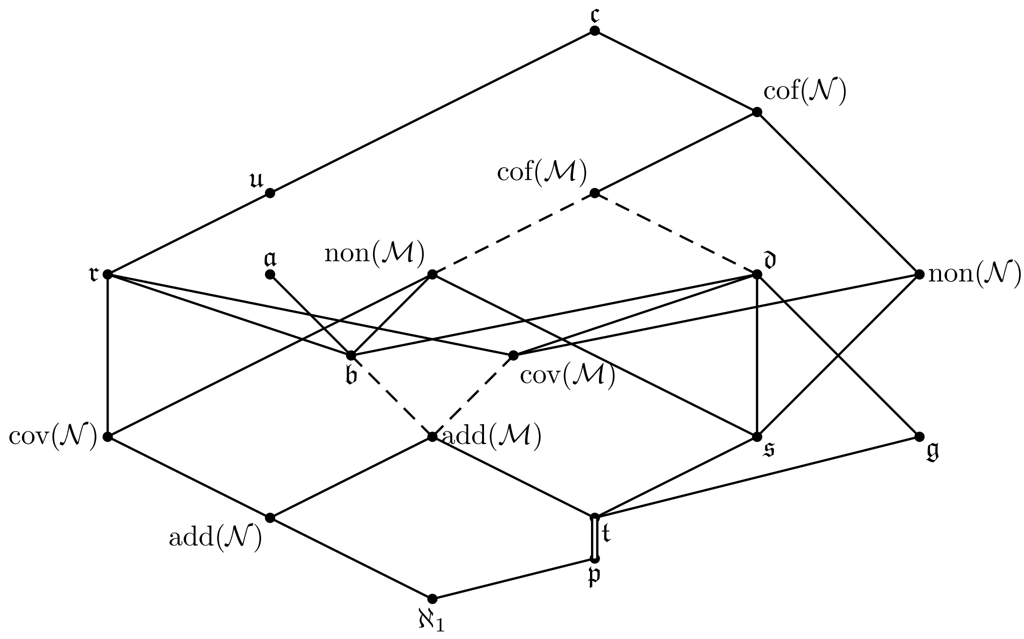

Figure 1.1: Diagram of some cardinal invariants

1.5 Gaps

Let $\mathbb{B}$ be a Boolean algebra and $\mathcal{A}, \mathcal{B} \subseteq \mathbb{B}$. $\mathcal{A}$ is *σ-directed* if, for any countable $\mathcal{C} \subseteq \mathcal{A}$ there is an $a \in \mathcal{A}$ such that $\forall_{c \in \mathcal{C}}(c \leq a)$. The pair $\langle \mathcal{A}, \mathcal{B} \rangle$ is *orthogonal* if $a \wedge b = 0$ for all $a \in \mathcal{A}$ and $b \in \mathcal{B}$. If, additionally, there is no $c \in \mathbb{B}$ such that $\forall_{a \in \mathcal{A}}(a \wedge c = 0)$ and $\forall_{b \in \mathcal{B}}(b \leq c)$, we say that the pair $\langle \mathcal{A}, \mathcal{B} \rangle$ is a *gap*. In the case that $\mathcal{A}$ and $\mathcal{B}$ are σ-directed, say that $\langle \mathcal{A}, \mathcal{B} \rangle$ is a *Hausdorff gap*. On the other hand, if one of $\mathcal{A}$ and $\mathcal{B}$ is countable, say that $\langle \mathcal{A}, \mathcal{B} \rangle$ is a *Rothberger gap*. For ordinals γ, δ, a *linear (γ, δ)-gap* is a gap of the form $\langle \{a_\alpha\}_{\alpha<\gamma}, \{b_\beta\}_{\beta<\delta} \rangle$ where both sequences $\{a_\alpha\}_{\alpha<\gamma}, \{b_\beta\}_{\beta<\delta}$ are strictly increasing. Equivalently, $\langle \{a_\alpha\}_{\alpha<\gamma}, \{b_\beta\}_{\beta<\delta} \rangle$ is a linear gap iff $\{a_\alpha\}_{\alpha<\gamma} \subseteq \mathbb{B}$ is strictly increasing, $\{-b_\beta\}_{\beta<\delta}$ is strictly decreasing, $a_\alpha \leq -b_\beta$ for all $\alpha < \gamma$, $\beta < \delta$ and there is no $c \in \mathbb{B}$ such that $a_\alpha \leq c \leq -b_\beta$ for all $\alpha < \gamma$, $\beta < \delta$. A linear (γ, δ)-gap where $\gamma = \omega$ (or $\delta = \omega$) is called *linear Rothberger gap.*

In this work, we are mainly interested in Rothberger gaps for quotients of the form $\mathcal{P}(\omega)/\mathcal{I}$ where $\mathcal{I}$ is an ideal on ω. Recall that Fin the ideal of finite subsets of ω. Given a pointclass Γ on the Cantor space, an ideal $\mathcal{I}$ is a *Γ-ideal* if the set of characteristic functions of elements of $\mathcal{I}$ belongs to Γ. We can extend the notions for gaps in $\mathbb{P}(\omega)/\mathcal{I}$ without looking at the equivalence classes modulo $\mathcal{I}$ but just at subsets of ω.

1.5.1 Definition. Let $\mathcal{I}$ be an ideal on ω, $\mathcal{A}, \mathcal{B} \subseteq \omega$.

(1) The pair $\langle \mathcal{A}, \mathcal{B} \rangle$ is *orthogonal (with respect to $\mathcal{I}$)* iff $\forall_{A \in \mathcal{A}} \forall_{B \in \mathcal{B}}(A \cap B \in \mathcal{I})$.

(2) $C \subseteq \omega$ *separates* $\langle \mathcal{A}, \mathcal{B} \rangle$ *(with respect to $\mathcal{I}$)* iff $\langle \mathcal{A}, \{C\} \rangle$ is orthogonal and $\forall_{B \in \mathcal{B}}(B \subseteq_{\mathcal{I}} C)$.

(3) The pair $\langle \mathcal{A}, \mathcal{B} \rangle$ is a *gap (on $\mathcal{I}$)* if it is an orthogonal pair and there is no $C \subseteq \omega$ separating it. When $\mathcal{A}$ (or $\mathcal{B}$) is countable, we say that $\langle \mathcal{A}, \mathcal{B} \rangle$ is a *Rothberger gap (on $\mathcal{I}$).*

Notions like "linear gap" can be extended likewise in this way.

1.5.2 Lemma (Hadamard [H84])**.** *If $\mathcal{I}$ is an ideal on ω and $\mathcal{A}, \mathcal{B} \subseteq \mathcal{P}(\omega)$ are countable, then $\langle \mathcal{A}, \mathcal{B} \rangle$ is not a gap on $\mathcal{I}$.*

The following results states also that a $(\omega, \mathfrak{b})$-gap on Fin can be constructed in ZFC.

1.5.3 Theorem (Rothberger [Ro41])**.** *There is a linear (ω, κ)-gap in $\mathcal{P}(\omega)/\mathrm{Fin}$ iff there is a well-ordered unbounded sequence in $\langle\omega, \leq^*\rangle$ of length κ.*

The following is an extension (in one direction) of the previous result to a large class of ideals.

1.5.4 Theorem (Todorčević [T98])**.** *If $\mathcal{I}$ is either an F_σ-ideal or an analytic P-ideal, then there exists an embedding $\psi : \mathcal{P}(\omega)/\mathrm{Fin} \to \mathcal{P}(\omega)/\mathcal{I}$ that preserves all gaps of $\mathcal{P}(\omega)/\mathrm{Fin}$.*

Theorems 1.5.3 and 1.5.4 directly imply

1.5.5 Corollary. *If $\mathcal{I}$ is either an F_σ-ideal or an analytic P-ideal, then $\mathcal{P}(\omega)/\mathcal{I}$ has a gap of type $(\omega, \mathfrak{b})$.*

Recall that, for a set Y, $\varphi : \mathcal{P}(Y) \to [0, +\infty]$ is a *submeasure on $\mathcal{P}(Y)$* if the following conditions hold.

- $\varphi(\varnothing) = 0$.
- $\varphi(x) < +\infty$ for any finite $x \subseteq Y$.
- *(Monotonicity)* $A \subseteq B \subseteq Y$ implies $\varphi(A) \leq \varphi(B)$.
- *(Finite subadditivity)* $\varphi(A \cup B) \leq \varphi(A) + \varphi(B)$ for all $A, B \subseteq Y$.

If $Y = \omega$, a submeasure φ on $\mathcal{P}(\omega)$ is *lower semicontinuous* if $\varphi(x) = \lim_{n\to+\infty} \varphi(x \cap n)$.

1.5.6 Theorem. *Let $\mathcal{I}$ be an ideal on ω.*

(1) (Mazur [Ma91]) *$\mathcal{I}$ is F_σ iff there is a lower semicontinuous submeasure φ on $\mathcal{P}(\omega)$ such that $\mathcal{I} = \mathrm{Fin}(\varphi) := \{x \subseteq \omega \;/\; \varphi(x) < +\infty\}$.*

(2) (Solecki [So96, So99]) *$\mathcal{I}$ is an analytic P-ideal iff there is a lower semicontinuous submeasure φ on $\mathcal{P}(\omega)$ such that $\mathcal{I} = \mathrm{Exh}(\varphi) := \{x \subseteq \omega \;/\; \lim_{n\to\infty} \varphi(x \smallsetminus n) = 0\}$. In particular, all analytic P-ideals are $F_{\sigma\delta}$.*

The following result is a well known fact about gaps on analytic P-ideals. We include a proof because we could not find a reference.

1.5.7 Corollary. *If $\mathcal{I}$ is an analytic P-ideal, then it has no (ω, κ)-gaps for $\kappa < \mathfrak{b}$.*

Proof. By Theorem 1.5.6(2), choose a lower semicontinuous submeasure φ such that $\mathcal{I} = \mathrm{Exh}(\varphi)$. Now, let $\langle\mathcal{A}, \mathcal{B}\rangle$ be an orthogonal pair such that $\mathcal{A} = \{A_n \;/\; n < \omega\}$ is a partition of ω (this can be assumed without loss of generality) and $|\mathcal{B}| < \mathfrak{b}$. Without loss of generality, we may assume that $\mathcal{I}$ is an ideal on $\omega \times \omega$ and $A_n = \{n\} \times \omega$. In this notation, for $x \subseteq \omega \times \omega$, $x \in \mathrm{Exh}(\varphi)$ iff, for all $\epsilon > 0$ there exists an $F \subseteq \omega \times \omega$ finite such that $\varphi(x \smallsetminus F) < \epsilon$. For $m < \omega$, we denote by $(A_n)_m := \{(n, k) \in A_n \;/\; k < m\}$. For each $0 < l < \omega$ and $B \in \mathcal{B}$, let $g_{B,l}(n)$ be the minimal m such that $\varphi((A_n \cap B) \smallsetminus (A_n)_m) < 1/(l \cdot 2^{n+1})$, which exists because $A_n \cap B \in \mathcal{I}$. As $\{g_{B,l} \;/\; B \in \mathcal{B},\ 0 < l < \omega\}$ has size $< \mathfrak{b}$, we can find $g \in \omega^\omega$ that dominates that set of functions. Put $C := \bigcup_{n<\omega} (A_n)_{g(n)}$. Clearly, $A_n \cap C = (A_n)_{g(n)} \in \mathcal{I}$ for every $n < \omega$, so it remains to show that $B \smallsetminus C \in \mathcal{I}$ for any $B \in \mathcal{B}$. Let $0 < l < \omega$ and choose $N < \omega$ such that $g_{B,l}(n) \leq g(n)$ for every $n \geq N$. Let $F := \bigcup_{n<N} (A_n)_{g_{B,l}(n)}$ and note that

$$\varphi((B \smallsetminus C) \smallsetminus F) \leq \varphi\Big(\bigcup_{n<\omega} (A_n \cap B \smallsetminus (A_n)_{g_{B,l}(n)})\Big) \leq \sum_{n<\omega} \frac{1}{l \cdot 2^{n+1}} = \frac{1}{l},$$

where the last inequality holds because of the lower semicontinuity of the submeasure. □

To finish this section, we recall the notion of tallness. An ideal $\mathcal{I}$ on ω is *tall* if, for any $Y \in [\omega]^\omega$, $\mathcal{I}{\restriction}Y$ contains an infinite subset of ω. Also, say that $\mathcal{I}$ is *somewhere tall* if there is some $\mathcal{I}$-positive Y such that $\mathcal{I}{\restriction}Y$ is tall (with respect to Y). $\mathcal{I}$ is *nowhere tall* if it is not somewhere tall.

CHAPTER 2

ITERATIONS ALONG A TEMPLATE

The theory of *iterations along templates* was introduced for the first time by Shelah [S04], originally developed to prove the consistency, with ZFC, of $\mathfrak{d} < \mathfrak{a}$, which was one long-standing open problem about inequalities between classical cardinal invariants. As this theory is an extension of the construction of iterations with finite support, it turns out to be useful to solve problems related to models with large continuum. A very interesting insight about template iterations has been done by Brendle in [Br02], [Br05] and [Br03], in particular, the latter contains the consistency, with ZFC, of $\aleph_1 < \mathfrak{d} < \mathfrak{a}$ where $\mathfrak{a}$ has countable cofinality.

Originally, the theory about templates was introduced for iterations with Suslin ccc posets. Later, the author found a way to expand this theory to iterations where non-definable posets are allowed [Me]. The main purpose in this chapter is to introduce the theory of iterations along templates with non-definable posets and also prove some general results about this type of iterations.

This chapter is organized as follows. In Section 2.1 we present *correctness*, which is a fundamental concept about forcing, introduced by Brendle, that is very useful to construct iterations that extend fsi, like shattered iterations [Br-1, Br-2] and template iterations [Br05]. Section 2.2 is about the relationship between Suslin ccc and correctness, which is essential to identify which definable ccc posets can be used in a template iteration. Templates, which are the supports where a template iteration grounds, are defined in Section 2.3 with some examples. Finally, we explain the construction of an iteration along a template for non-definable posets in Section 2.4 and also prove general results about ccc-ness and embeddability.

2.1 Correctness

The concept of correctness is originally developed for complete Boolean algebras [Br-1, Br-2, Br05], but notions and results can be translated in terms of posets in general. In this section, we present correctness for posets.

For this section, fix $M \subseteq N$ transitive models of ZFC. Recall the four-element lattice $I_4 := \{\wedge, 0, 1, \vee\}$ where $\vee$ is the largest element, $\wedge$ is the least element and $0, 1$ are in between.

2.1.1 Definition (Correct system of embeddings)**.** Let $\mathbb{P}_i$ be a poset for each $i \in I_4$ and let $e_{i,j} : \mathbb{P}_i \to \mathbb{P}_j$ be complete embeddings for $i < j$ in I_4 such that $e_{0,\vee} \circ e_{\wedge,0} = e_{1,\vee} \circ e_{\wedge,1}$. This system of embeddings is *correct* if, for each $p \in \mathbb{P}_0$ and $q \in \mathbb{P}_1$, if both have compatible reductions in $\mathbb{P}_\wedge$, then $e_{0,\vee}(p)$ and $e_{1,\vee}(q)$ are compatible in $\mathbb{P}_\vee$. An equivalent statement is that, for each $p \in \mathbb{P}_0$ and for every reduction $r \in \mathbb{P}_\wedge$ of p, $e_{\wedge,1}(r)$ is a reduction of $e_{0,\vee}(p)$. When the embeddings are not stated, we assume that they are the identity functions.

There is a restrictive version of this notion. For the model M, if $\mathbb{P}_\wedge, \mathbb{P}_0, e_{\wedge,0} \in M$ and $e_{\wedge,1}, e_{0,\vee}$ are just complete embeddings with respect to M, say that the system of embeddings is *correct with respect to*

M if, for any $p \in \mathbb{P}_\wedge$ and $q \in \mathbb{P}_0$, if p is a reduction of q with respect to $e_{\wedge,0}$, then $e_{\wedge,1}(p)$ is a reduction of $e_{0,\vee}(q)$ with respect to $e_{1,\vee}$. When the embeddings are understood, we say that $\langle \mathbb{P}_\wedge, \mathbb{P}_0, \mathbb{P}_1, \mathbb{P}_\vee \rangle$ is a correct system (with respect to M).

The results of this section are applications of this notion to two-step iterations, quotients and direct limits of posets.

2.1.2 Lemma. *Let $\mathbb{P} \in M$, $\mathbb{P}' \in N$ posets such that $\mathbb{P} \lessdot_M \mathbb{P}'$. If $\dot{\mathbb{Q}} \in M$ is a $\mathbb{P}$-name of a poset, $\dot{\mathbb{Q}}' \in N$ a $\mathbb{P}'$-name of a poset and $\mathbb{P}'$ forces (with respect to N) that $\dot{\mathbb{Q}} \lessdot_{M^{\mathbb{P}}} \dot{\mathbb{Q}}'$, then $\mathbb{P} * \dot{\mathbb{Q}} \lessdot_M \mathbb{P}' * \dot{\mathbb{Q}}'$. Also, $\langle \mathbb{P}, \mathbb{P} * \dot{\mathbb{Q}}, \mathbb{P}', \mathbb{P}' * \dot{\mathbb{Q}}' \rangle$ is a correct system with respect to M.*

Proof. First prove that, if $(p_0, \dot{q}_0), (p_1, \dot{q}_1) \in \mathbb{P} * \dot{\mathbb{Q}}$ are compatible in $\mathbb{P}' * \dot{\mathbb{Q}}'$, then they are also compatible in $\mathbb{P} * \dot{\mathbb{Q}}$. Let $(p', \dot{q}') \in \mathbb{P}' * \dot{\mathbb{Q}}'$ be a common extension. Find $A \in M$ a maximal antichain in $\mathbb{P}$ contained in $\{p \in \mathbb{P} \;/\; p \leq p_0, p_1 \text{ or } p_0 \perp p \text{ or } p_1 \perp p\}$. As A is also maximal antichain in $\mathbb{P}'$, there exists a $p_2 \in A$ compatible with p'. p_2 is a common extension of p_0, p_1 because p' is a common extension of p_0, p_1. Also, p_2 cannot force, with respect to $\mathbb{P}$ and M, that $\dot{q}_0 \perp \dot{q}_1$ because p' forces their compatibility with respect to $\mathbb{P}'$ and N. Therefore, there exists $p \leq p_2$ that forces $\dot{q}_0, \dot{q}_1$ compatible.

Now, let $\{(p_\alpha, q_\alpha) \;/\; \alpha < \delta\} \in M$ a maximal antichain of $\mathbb{P} * \dot{\mathbb{Q}}$. We claim first that $\mathbb{P}$ forces that $\{q_\alpha \;/\; p_\alpha \in \dot{G}, \alpha < \delta\}$ is a maximal antichain in $\dot{\mathbb{Q}}$, where $\dot{G}$ is a $\mathbb{P}$-name of the generic subset. Indeed, let $p \in \mathbb{P}$ be arbitrary and $\dot{q}$ be a $\mathbb{P}$-name for a condition in $\dot{\mathbb{Q}}$, For some $\alpha < \delta$, there exists a common extension $(r, \dot{s})$ of $(p, \dot{q}), (p_\alpha, \dot{q}_\alpha)$, so r forces that $p_\alpha \in \dot{G}$ and that $\dot{q}_\alpha, \dot{s}$ are compatible.

Let $(p', \dot{q}') \in \mathbb{P}' * \dot{\mathbb{Q}}'$. Clearly, p' forces (with respect to $\mathbb{Q}, N$) that $\{\dot{q}_\alpha \;/\; p_\alpha \in \dot{H}, \alpha < \delta\}$ is a maximal antichain in $\dot{\mathbb{Q}}'$, where $\dot{H}$ is the $\mathbb{P}'$-name of its generic subset. Hence, there are $\alpha < \delta$ and $p'' \leq p'$ in $\mathbb{P}'$ that forces $p_\alpha \in \dot{H}$ and $\dot{q}'$ compatible with $\dot{q}_\alpha$. Therefore, $(p', \dot{q}')$ is compatible with $(p_\alpha, \dot{q}_\alpha)$. □

2.1.3 Lemma. *Let $\langle \mathbb{P}, \mathbb{Q}, \mathbb{P}', \mathbb{Q}' \rangle$ be a correct system. Then, $\mathbb{P}'$ forces that $\mathbb{Q}/\mathbb{P} \lessdot_{V^{\mathbb{P}}} \mathbb{Q}'/\mathbb{P}'$.*

Proof. Correctness implies directly that $\Vdash_{\mathbb{P}'} \mathbb{Q}/\mathbb{P} \subseteq \mathbb{Q}'/\mathbb{P}'$. We prove first that $\mathbb{P}'$ forces that any pair of incompatible conditions in $\mathbb{Q}/\mathbb{P}$ are incompatible in $\mathbb{Q}'/\mathbb{P}'$. Let $p' \in \mathbb{P}'$, $q_0, q_1 \in \mathbb{Q}$ and $q' \in \mathbb{Q}'$ be such that $p' \Vdash_{\mathbb{P}'}$ "$q_0, q_1 \in \mathbb{Q}/\mathbb{P}$, $q' \in \mathbb{Q}'/\mathbb{P}'$ and $q' \leq q_0, q_1$". We need to find a $p'' \leq p'$ in $\mathbb{P}'$ which forces that q_0 and q_1 are compatible in $\mathbb{Q}/\mathbb{P}$. As $p' \Vdash_{\mathbb{P}'} q' \in \mathbb{Q}'/\mathbb{P}'$, p' is a reduction of q'. Find $p \in \mathbb{P}$ and $q \in \mathbb{Q}$ such that $q \leq q_0, q_1$, p is a reduction of q, p is a reduction of p' and q is a reduction of q'. Indeed, choose $p_0 \in \mathbb{P}$ a reduction of p'. Then, as p_0 is also a reduction of q', there exists a $q'' \in \mathbb{Q}'$ such that $q'' \leq q', p_0$. Then, we can find $q \in \mathbb{Q}$ a reduction of q'' such that $q \leq q_0, q_1, p_0$. Now, find $p \leq p_0$ in $\mathbb{P}$ such that it is a reduction of q. Clearly, p and q are as desired. Now, $p \Vdash_{\mathbb{P}} q \in \mathbb{Q}/\mathbb{P}$ and, as it is a reduction of p', find $p'' \in \mathbb{P}'$ such that $p'' \leq p, p'$. Thus, $p'' \Vdash_{\mathbb{P}'}$ "$q \in \mathbb{Q}/\mathbb{P}$" and $q \leq q_0, q_1$.

Now, let $\dot{A}$ be a $\mathbb{P}$-name for a maximal antichain in $\mathbb{Q}/\mathbb{P}$. Given $p' \in \mathbb{P}'$ and $q' \in \mathbb{Q}'$ such that $p' \Vdash_{\mathbb{P}'} q' \in \mathbb{Q}'/\mathbb{P}'$, we need to find $p'' \leq p'$ in $\mathbb{P}'$ and $q \in \mathbb{Q}$ such that p'' forces that $q \in \dot{A}$ and that it is compatible with q' in $\mathbb{Q}'/\mathbb{P}'$. Clearly, p' is a reduction of q', so there exists $q'' \in \mathbb{Q}'$ that extends both p' and q'. Now, let $q_2 \in \mathbb{Q}$ be a reduction of q''. Hence, as $\dot{A}$ is the $\mathbb{P}$-name of a maximal antichain in $\mathbb{Q}/\mathbb{P}$, there exist $q, q_3 \in \mathbb{Q}$ and $p \in \mathbb{P}$ such that $q_3 \leq q, q_2$ and p is a reduction of q_3 that forces $q \in \dot{A}$. Find $q_4 \in \mathbb{Q}$ such that $q_4 \leq p, q_3$. As $q_4 \leq q_2$, there exists $q''' \in \mathbb{Q}'$ extending q'' and q_4. Now, let $p'' \in \mathbb{P}'$ be a reduction of q''' such that $p'' \leq p, p'$. Thus, p'' forces that $q \in \dot{A}$, $q''' \in \mathbb{Q}'/\mathbb{P}'$ and $q''' \leq q, q'$. □

2.1.4 Corollary. *Let $\langle \mathbb{P}, \mathbb{Q}, \mathbb{P}', \mathbb{Q}' \rangle$ and $\langle \mathbb{Q}, \mathbb{R}, \mathbb{Q}', \mathbb{R}' \rangle$ be correct systems. Then, $\mathbb{P}'$ forces that the system $\langle \mathbb{Q}/\mathbb{P}, \mathbb{R}/\mathbb{P}, \mathbb{Q}'/\mathbb{P}', \mathbb{R}'/\mathbb{P}' \rangle$ is correct with respect to $V^{\mathbb{P}}$.*

Proof. By Lemma 2.1.3 we only need to prove correctness (to get, e.g., $\Vdash_{\mathbb{P}} \mathbb{Q}/\mathbb{P} \lessdot \mathbb{R}/\mathbb{P}$, note that $\langle \mathbb{P}, \mathbb{Q}, \mathbb{P}, \mathbb{R} \rangle$ is a correct system). In $V^{\mathbb{P}'}$, we know that $\mathbb{R}/\mathbb{P} \simeq (\mathbb{Q}/\mathbb{P}) * (\mathbb{R}/\mathbb{Q})$ and $\mathbb{R}'/\mathbb{P}' \simeq (\mathbb{Q}'/\mathbb{P}') * (\mathbb{R}'/\mathbb{Q}')$ by Corollary 1.2.5. As $\mathbb{Q}/\mathbb{P} \lessdot_{V^{\mathbb{P}}} \mathbb{Q}'/\mathbb{P}'$ and $\mathbb{Q}'/\mathbb{P}'$ forces that $\mathbb{R}/\mathbb{Q} \lessdot_{V^{\mathbb{Q}}} \mathbb{R}'/\mathbb{Q}'$ by Lemma 2.1.3, we get the correctness we are looking for from Lemma 2.1.2. □

2.1.5 Lemma (Embeddability of direct limits [Br-1], see also [Br05, Lemma 1.2])**.** *Let $I \in M$ be a directed set, $\langle \mathbb{P}_i \rangle_{i \in I} \in M$ and $\langle \mathbb{Q}_i \rangle_{i \in I} \in N$ directed systems of posets such that* *(i) for each $i \in I$, $\mathbb{P}_i \lessdot_M \mathbb{Q}_i$ and*

(ii) whenever $i \leq j$, $\langle \mathbb{P}_i, \mathbb{P}_j, \mathbb{Q}_i, \mathbb{Q}_j \rangle$ is a correct system with respect to M

Then, $\mathbb{P} := \text{limdir}_{i \in I} \mathbb{P}_i$ is a complete suborder of $\mathbb{Q} := \text{limdir}_{i \in I} \mathbb{Q}_i$ with respect to M and, for any $i \in I$, $\langle \mathbb{P}_i, \mathbb{P}, \mathbb{Q}_i, \mathbb{Q} \rangle$ is a correct system with respect to M.

Proof. Let $A \in M$ be a maximal antichain of $\mathbb{P}$. Let $q \in \mathbb{Q}$, so there is some $i \in I$ such that $q \in \mathbb{Q}_i$. Work within M. Enumerate $A := \{p_\alpha \mathrel{/} \alpha < \delta\}$ for some ordinal δ and, for each $\alpha < \delta$, choose $j_\alpha \geq i$ in I such that $p_\alpha \in \mathbb{P}_{j_\alpha}$. Now, if $p \in \mathbb{P}_i$, there is some $\alpha < \delta$ such that p is compatible with p_α in $\mathbb{P}_{j_\alpha}$, so there exists $p' \leq p$ which is a reduction of p_α with respect to $\mathbb{P}_i, \mathbb{P}_{j_\alpha}$.

The previous density argument implies, in N, that q is compatible with some $p \in \mathbb{P}_i$ which is a reduction of p_α for some $\alpha < \delta$. By (2), p is a reduction of p_α with respect to $\mathbb{Q}_i, \mathbb{Q}_{j_\alpha}$, which implies that q is compatible with p_α. □

2.1.6 Lemma. *Let $\langle \mathbb{P}_i \rangle_{i \in I}$ be a directed system of posets, $\mathbb{P}$ its direct limit. Assume that $\mathbb{Q}$ is a complete suborder of $\mathbb{P}_i$ for all $i \in I$. Then, $\mathbb{Q}$ forces that $\mathbb{P}/\mathbb{Q} = \text{limdir}_{i \in I} \mathbb{P}_i/\mathbb{Q}$.*

Proof. For $i \in I$, as $\langle \mathbb{Q}, \mathbb{P}_i, \mathbb{Q}, \mathbb{P} \rangle$ is a correct system, by Lemma 2.1.3 $\mathbb{Q}$ forces that $\mathbb{P}_i/\mathbb{Q}$ is a complete suborder of $\mathbb{P}/\mathbb{Q}$. It is easy to see that $\mathbb{Q}$ forces that $\mathbb{P}/\mathbb{Q} = \bigcup_{i \in I} \mathbb{P}_i/\mathbb{Q}$. □

2.2 More on Suslin ccc posets

All the material presented in this section is due to Brendle [Br-2, Br-1] (see also [Br05]).

2.2.1 Definition ([Br05])**.** Let $\mathbb{S}$ be a Suslin ccc poset.

(1) $\mathbb{S}$ is *correctness preserving* if, given a correct system $\langle \mathbb{P}_\wedge, \mathbb{P}_0, \mathbb{P}_1, \mathbb{P}_\vee \rangle$, the system $\langle \mathbb{P}_\wedge * \dot{\mathbb{S}}_\wedge, \mathbb{P}_0 * \dot{\mathbb{S}}_0, \mathbb{P}_1 * \dot{\mathbb{S}}_1, \mathbb{P}_\vee * \dot{\mathbb{S}}_\vee \rangle$ (with the obvious embeddings) is also correct, where each $\dot{\mathbb{S}}_i$ is a $\mathbb{P}_i$-name for $\mathbb{S}$.

(2) $\mathbb{S}$ is *Suslin σ-linked* if there exists a sequence $\{S_n\}_{n<\omega}$ of 2-linked subsets of $\mathbb{S}$ such that the statement "$x \in S_n$" is $\mathbf{\Sigma}^1_1$. Here, note that the statement "S_n is 2-linked" is $\mathbf{\Pi}^1_1$.

(3) $\mathbb{S}$ is *Suslin σ-centered* if there exists a sequence $\{S_n\}_{n<\omega}$ of centered subsets of $\mathbb{S}$ such that the statement "$x \in S_n$" is $\mathbf{\Sigma}^1_1$. Here, note that the statement "S_n is centered" is $\mathbf{\Pi}^1_2$, this because the statement "$p_0, \ldots, p_l$ have a common extension in $\mathbb{S}$" is $\mathbf{\Sigma}^1_1$.

According to the rules of the construction of template iterations, the Suslin ccc posets that are correctness preserving are the definable posets that can be used in such an iteration. The concepts of Suslin σ-linked and Suslin σ-centered are useful to get ccc forcings from template iterations (see Section 2.4). We prove that the examples of Suslin ccc posets presented in Section 1.3 are correctness preserving. Nevertheless, it is not known an example of a Suslin ccc notion that is not correctness preserving.

2.2.2 Conjecture (Brendle)**.** *Every Suslin ccc notion is correctness preserving.*

It is easy to note that $\mathbb{B}$, $\mathbb{A}$ and $\mathbb{LOC}^h$ are Suslin σ-linked, while $\mathbb{C}$, $\mathbb{D}$ and $\mathbb{E}$ are Suslin σ-centered. For these posets, the statement "$p_0, \ldots, p_l$ have a common extension" is Borel. Then, "S is centered" is $\mathbf{\Pi}^1_1$ for any $\mathbf{\Sigma}^1_1$ subset S of such a poset.

2.2.3 Lemma. $\mathbb{C}$ *is correctness preserving.*

Proof. Clear, because $\mathbb{P} * \dot{\mathbb{C}} \simeq \mathbb{P} \times \mathbb{C}$. □

2.2.4 Lemma. *Let $\mathbb{P}$ be a complete suborder of the poset $\mathbb{P}'$. Let $\dot{\mathbb{Q}}'$ be a $\mathbb{P}'$-name of a poset and $\dot{\mathbb{Q}}$ a $\mathbb{P}$-name of a poset such that $\Vdash_{\mathbb{P}'} \dot{\mathbb{Q}} \lessdot_{V^{\mathbb{P}}} \dot{\mathbb{Q}}'$. If $(p, \dot{q}) \in \mathbb{P} * \dot{\mathbb{Q}}$, $(p', \dot{q}') \in \mathbb{P}' * \dot{\mathbb{Q}}'$, p is a reduction of p' and $p' \Vdash \dot{q}' \parallel \dot{q}$, then there exists a $\mathbb{P}$-name $\dot{q}_0$ of a condition in $\dot{\mathbb{Q}}$ such that p forces, in $\mathbb{P}$, that $\dot{q}_0 \leq \dot{q}$ and that, for any $r \in \dot{\mathbb{Q}}$ extending $\dot{q}_0$ there exists a $p_1 \leq p'$ in $\mathbb{P}'/\mathbb{P}$ such that $p_1 \Vdash_{\mathbb{P}'/\mathbb{P}} r \parallel \dot{q}'$.*

Proof. Let G be $\mathbb{P}$-generic over V with $p \in \mathbb{P}$. Work in $V[G]$. Let A be an antichain contained in $D := \{r \in \mathbb{Q} \mathrel{/} r \leq q \text{ and } p' \Vdash_{\mathbb{P}'/\mathbb{P}} r \perp \dot{q}'\}$ which is maximal in D. Note that A is not a maximal antichain below q because, if so, $p' \Vdash_{\mathbb{P}'/\mathbb{P}} q \perp \dot{q}'$. Therefore, there exists a $q_0 \leq q$ in $\mathbb{Q}$ that is incompatible with all the members of A. Thus, any $r \leq q_0$ is not in D, so there is a $p_1 \leq p'$ in $\mathbb{P}'/\mathbb{P}$ such that $p_1 \Vdash_{\mathbb{P}'/\mathbb{P}} r \parallel \dot{q}'$. □

2.2.5 Lemma ([Br-2])**.** $\mathbb{B}$ *is correctness-preserving.*

Proof. Let $\langle \mathbb{P}_\wedge, \mathbb{P}_0, \mathbb{P}_1, \mathbb{P}_\vee \rangle$ be a correct system as in the notation of Definition 2.1.1. Without loss of generality, assume that the embeddings are the identity functions. For $i \in I_4$, let $\dot{\mathbb{B}}_i$ be a $\mathbb{P}_i$-name for random forcing. To prove that $\langle \mathbb{P}_\wedge * \dot{\mathbb{B}}_\wedge, \mathbb{P}_0 * \dot{\mathbb{B}}_0, \mathbb{P}_1 * \dot{\mathbb{B}}_1, \mathbb{P}_\vee * \dot{\mathbb{B}}_\vee \rangle$ is correct, it is enough to show that, if $(p_0, \dot{b}_0) \in \mathbb{P}_0 * \dot{\mathbb{B}}_0$ and $(p_1, \dot{b}_1) \in \mathbb{P}_1 * \dot{\mathbb{B}}_1$ have a common reduction in $\mathbb{P}_\wedge * \dot{\mathbb{B}}_\wedge$, then they can be extended to conditions $(q_0, \dot{c}_0)$ and $(q_1, \dot{c}_1)$, respectively, such that $q_i \Vdash \lambda(\dot{c}_i \cap \dot{c}_\wedge) > \frac{3}{4}\lambda(\dot{c}_\wedge)$ for $i = 0, 1$ where λ is the Lebesgue measure for 2^ω, $\dot{c}_\wedge$ is a $\mathbb{P}_\wedge$-name for a condition in $\dot{\mathbb{B}}_\wedge$ and q_0, q_1 have a common reduction in $\mathbb{P}_\wedge$. This is so because q_0, q_1 will be compatible in $\mathbb{P}_\vee$ by correctness and $\dot{c}_\vee = \dot{c}_0 \cap \dot{c}_1 \cap \dot{c}_\wedge$ is forced to be, by any common extension of q_0 and q_1, a condition in $\dot{\mathbb{B}}_\vee$ of measure $\geq \frac{1}{2}\lambda(\dot{c}_\wedge)$.

Let $(p, \dot{b})$ be a common reduction in $\mathbb{P}_\wedge * \dot{\mathbb{B}}_\wedge$ of $(p_0, \dot{b}_0)$ and $(p_1, \dot{b}_1)$. Let $(p_1', \dot{b}_1') \in \mathbb{P}_1 * \dot{\mathbb{B}}_1$ be a common extension of $(p, \dot{b})$ and $(p_1, \dot{b}_1)$, so p_1' forces that $\dot{b}_1' \subseteq \dot{b} \cap \dot{b}_1$. Choose $p' \leq p$ a reduction of p_1' in $\mathbb{P}_\wedge$.

2.2.6 Claim. *There is a $\mathbb{P}_\wedge$-name $\dot{b}_\wedge$ of a condition in $\dot{\mathbb{B}}_\wedge$ such that p' forces in $\mathbb{P}_\wedge$ that $\dot{b}_\wedge \subseteq \dot{b}$ and that, for any $c \in \dot{\mathbb{B}}_\wedge$ extending $\dot{b}_\wedge$, there is a condition $q \leq p_1'$ in $\mathbb{P}_1/\mathbb{P}_\wedge$ such that $q \Vdash_{\mathbb{P}_1/\mathbb{P}_\wedge} \lambda(\dot{b}_1' \cap c) > 0$.*

Proof. Direct consequence of Lemma 2.2.4. □

As $(p', \dot{b}_\wedge)$ is a reduction of $(p_0, \dot{b}_0)$, there is a common extension $(p_0', \dot{c}_0) \in \mathbb{P}_0 * \dot{\mathbb{B}}_0$. Then, $p_0' \Vdash \dot{c}_0 \subseteq \dot{b}_\wedge$. By the Lebesgue density Theorem 1.1.5, find $s \in 2^{<\omega}$ and $p_0'' \leq_{\mathbb{P}_0} p_0'$ that forces $\lambda(\dot{c}_0 \cap [s]) > \frac{3}{4}\lambda([s])$. Put $\dot{b}_\wedge'' = \dot{b}_\wedge \cap [s]$, so $p_0'' \Vdash \lambda(\dot{c}_0 \cap \dot{b}_\wedge'') > \frac{3}{4}\lambda(\dot{b}_\wedge'')$.

Let $p'' \leq p'$ be a reduction of p_0'' in $\mathbb{P}_\wedge$.

2.2.7 Claim. *There are $\mathbb{P}_\wedge$-names $\{\dot{c}^n\}_{n<\omega}$ for conditions in $\dot{\mathbb{B}}_\wedge$ and $\{\dot{p}_1^n\}_{n>\omega}$ of conditions in $\mathbb{P}_1/\mathbb{P}_\wedge$ such that p'' forces that $\{\dot{c}^n\}_{n<\omega}$ is a maximal antichain below $\dot{b}_\wedge''$ and, for each $n < \omega$, $\dot{p}_1^n \leq p_1'$ and forces, in $\mathbb{P}_1/\mathbb{P}_\wedge$, that $\lambda(\dot{b}_1' \cap \dot{c}^n) > \frac{3}{4}\lambda(\dot{c}^n)$.*

Proof. Let G be a $\mathbb{P}_\wedge$-generic over with $p'' \in G$. Work in $V[G]$. Let $c \subseteq b_\wedge''$ in $\mathbb{B}$ arbitrary. By Claim 2.2.6, there is a $q \leq p_1'$ that forces, in $\mathbb{P}_1/\mathbb{P}_\wedge$, $\lambda(\dot{b}_1' \cap c) > 0$. As in the paragraph preceding the claim, use the Lebesgue density theorem to get $c' \subseteq c$ and $q' \leq q$ that forces, in $\mathbb{P}_1/\mathbb{P}_\wedge$, $\lambda(\dot{b}_1' \cap c') > \frac{3}{4}\lambda(c')$. As c is arbitrary, there exists $\{c^n\}_{n<\omega}$ maximal antichain below $b_\wedge''$ such that, for any $n < \omega$, there exists a $p_1^n \leq p_1'$ that forces, in $\mathbb{P}_1/\mathbb{P}_\wedge$, $\lambda(\dot{b}_1' \cap c^n) > \frac{3}{4}\lambda(c^n)$. □

Note that there is an $n < \omega$ and a common extension $q_0 \in \mathbb{P}_0$ of p'' and p_0'' such that $q_0 \Vdash \lambda(\dot{c}_0 \cap \dot{c}^n) > \frac{3}{4}\lambda(\dot{c}^n)$. If it were not the case, then any common extension of p'' and p_0'' in $\mathbb{P}_0$ would force $\lambda(\dot{c}_0 \cap \dot{c}^n) \leq \frac{3}{4}\lambda(\dot{c}^n)$ for all $n < \omega$, but this implies that $\lambda(\dot{c}_0 \cap \dot{b}_\wedge'') \leq \frac{3}{4}\lambda(\dot{b}_\wedge'')$, which is false because p_0'' forces the contrary. Put $\dot{c}_\wedge = \dot{c}^n$.

Let $q \leq p''$ be a reduction of q_0 in $\mathbb{P}_\wedge$. As q forces that $\dot{p}_1^n \leq p_1'$ is in $\mathbb{P}_1/\mathbb{P}_\wedge$, there exists a $q_1' \leq p_1'$ in $\mathbb{P}_1$ and $q_\wedge \leq q$ in $\mathbb{P}_\wedge$ such that $q_\wedge \Vdash q_1' = \dot{p}_1^n \in \mathbb{P}_1/\mathbb{P}_\wedge$, so $q_\wedge$ is a reduction of q_1'. Let $q_1 \in \mathbb{P}_1$ be a common extension of $q_\wedge$ and q_1', so Claim 2.2.7 implies that any reduction of q_1 in $\mathbb{P}_\wedge$ forces that $q_1 \Vdash_{\mathbb{P}_1/\mathbb{P}_\wedge} \lambda(\dot{b}_1' \cap \dot{c}_\wedge) > \frac{3}{4}\lambda(\dot{c}_\wedge)$. Therefore, by Lemma 1.2.2, $q_1 \Vdash_{\mathbb{P}_1} \lambda(\dot{b}_1' \cap \dot{c}_\wedge) > \frac{3}{4}\lambda(\dot{c}_\wedge)$. Put $\dot{c}_1 = \dot{b}_1'$. Note also that any reduction of q_1 in $\mathbb{P}_\wedge$ is also a reduction of q_0, so the proof is complete. □

2.2.8 Lemma ([Br-1], see also [Br05, Lemma 1.3])**.** $\mathbb{D}$ *is correctness preserving.*

Proof. Let $\langle \mathbb{P}_\wedge, \mathbb{P}_0, \mathbb{P}_1, \mathbb{P}_\vee \rangle$ be a correct system. Assume $(p_0, (s_0, \dot{f}_0)) \in \mathbb{P}_0 * \dot{\mathbb{D}}_0$ and $(p_1, (s_1, \dot{f}_1)) \in \mathbb{P}_1 * \dot{\mathbb{D}}_1$ with a common reduction in $\mathbb{P}_\wedge * \dot{\mathbb{D}}_\wedge$. Show that they are compatible in $\mathbb{P}_\vee * \dot{\mathbb{D}}_\vee$. Consider $(p, (s, \dot{f})) \in \mathbb{P}_\wedge * \dot{\mathbb{D}}_\wedge$ such a common reduction with $|s| \geq |s_1|$.

2.2.9 Claim. *$s_1 \subseteq s$ and, for any $t \supseteq s$ in $\omega^{<\omega}$, if $q \leq p$ forces, in $\mathbb{P}_\wedge$, that $\dot{f}\restriction|t| \leq t$, then there is a $p'_1 \leq p_1, q$ in $\mathbb{P}_1$ that forces $\dot{f}_1\restriction|t| \leq t$.*

As $(p, (s, \dot{f}))$ is a reduction of $(p_0, (s_0, \dot{f}_0)) \in \mathbb{P}_0 * \dot{\mathbb{D}}_0$, let $(p'_0, (s'_0, \dot{f}'_0)) \in \mathbb{P}_0 * \dot{\mathbb{D}}_0$ be a common extension, so $s \subseteq s'_0$. Let $(p', (t, \dot{f}')) \leq (p, (s, \dot{f}))$ be a reduction of $(p'_0, (s'_0, \dot{f}'_0))$ in $\mathbb{P}_\wedge * \mathbb{D}_\wedge$ with $|s'_0| \leq |t|$, so $s'_0 \subseteq t$. By the claim, there is a $p'_1 \leq p_1, p'$ in $\mathbb{P}_1$ that forces $\dot{f}_1\restriction|t| \leq t$. As any reduction of p'_1 in $\mathbb{P}_\wedge$ is a reduction of p'_0, by correctness we get that p'_0, p'_1 are compatible in $\mathbb{P}_\vee$. Note that any common extension of p'_0 and p'_1 in $\mathbb{P}_\vee$ forces that $(s'_0, \dot{f}'_0)$ and $(s_1, \dot{f}_1)$ are compatible. □

2.2.10 Lemma. $\mathbb{E}$ *is correctness preserving.*

Proof. Imitate the proof of Lemma 2.2.8 and just replace $\dot{f}\restriction|t| \leq t$ by $\forall_{i\in[|s|,|t|)}\forall_{x\in\dot{F}}(x(i) \neq t(i))$ and $\dot{f}_1\restriction|t| \leq t$ by $\forall_{i\in[|s_1|,|t|)}\forall_{x\in\dot{F}_1}(x(i) \neq t(i))$ □

2.2.11 Lemma. *For $h \in \omega^\omega$ non-decreasing that converges to infinite, $\mathbb{LOC}^h$ is correctness preserving.*

Proof. Same idea of the proof of Lemma 2.2.8. □

2.3 Templates

We introduce Shelah's notion of a template (in a simpler way than in the original work [S04]), which represents the index set of a forcing iteration as defined in Section 2.4. Except for Lemmas 2.3.8 and 2.3.11, all definitions and results are, in essence, due to Shelah [S04], but for proofs we refer to [Br02].

For a linear order $L := \langle L, \leq_L \rangle$ and $x \in L$, denote $L_x := \{z \in L \;/\; z < x\}$.

2.3.1 Definition (Indexed template)**.** An *indexed template* is a pair $\langle L, \bar{\mathcal{I}} := \langle \mathcal{I}_x \rangle_{x\in L} \rangle$ such that L is a linear order, $\mathcal{I}_x \subseteq \wp(L_x)$ for all $x \in L$ and

(1) $\varnothing \in \mathcal{I}_x$,

(2) $\mathcal{I}_x$ is closed under finite unions and intersections,

(3) if $z < x$ then there is some $A \in \mathcal{I}_x$ such that $z \in A$.

(4) $\mathcal{I}_x \subseteq \mathcal{I}_y$ if $x < y$, and

(5) $\mathcal{I} := \bigcup_{x\in L} \mathcal{I}_x \cup \{L\}$ is well-founded with the subset relation. Let $\mathrm{Dp}_\mathcal{I} : \mathcal{I} \to \mathbf{On}$ be the rank function for this relation.

L is meant to be the index set of an iteration as in Section 2.4, which can be constructed thanks to the well-foundedness of $\mathcal{I}$. Note that properties (2) and (4) imply that $\mathcal{I}$ is closed under finite unions and intersections.

If $A \subseteq L$ and $x \in L$, define $\mathcal{I}_x\restriction A := \{A \cap X \;/\; X \in \mathcal{I}_x\}$ the *trace of $\mathcal{I}_x$ on A*. Put $\bar{\mathcal{I}}\restriction A := \langle \mathcal{I}_x\restriction A \rangle_{x\in A}$ and $\mathcal{I}\restriction A := \bigcup_{x\in A} \mathcal{I}_x\restriction A \cup \{A\}$. Although the notation $\mathcal{I}\restriction A$ has a different meaning when $\mathcal{I}$ is an ideal, we change its meaning only in the context of templates (note that it may be that $\mathcal{I}\restriction A \subsetneq \{A \cap X \;/\; X \in \mathcal{I}\}$). For $Z \subseteq A$ such that $Z = A \cap X$ for some $X \in \mathcal{I}$, let $X_Z = X_{\mathcal{I},Z}$ be a set in $\mathcal{I}$ of minimal rank such that $Z = A \cap X_Z$.

2.3.2 Lemma. *$\langle A, \bar{\mathcal{I}}\restriction A := \langle \mathcal{I}_x\restriction A \rangle_{x\in A} \rangle$ is an indexed template. Moreover, $\mathcal{J} := \{A \cap X \;/\; X \in \mathcal{I}\}$ is well founded and $\mathrm{Dp}_\mathcal{J}(A \cap X) \leq \mathrm{Dp}_\mathcal{I}(X)$ for all $X \in \mathcal{I}$.*

Proof. First note that, for $Z, Z' \in \mathcal{J}$, $Z \subseteq Z'$ iff $X_Z \subseteq X_{Z'}$. Indeed, if $Z \subseteq Z'$, then $A \cap (X_Z \cap X_{Z'}) = Z$ and $X_Z \cap X_{Z'} \in \mathcal{I}$. Therefore, by minimality of X_Z, $X_Z \cap X_{Z'} = X_Z$.

The previous argument implies that $\mathcal{J}$ is well-founded, moreover, $\mathrm{Dp}_{\mathcal{J}}(Z) \lneq \mathrm{Dp}_{\mathcal{I}}(X_Z)$ for all $Z \in \mathcal{J}$. If $Z = A \cap X$ for $X \in \mathcal{I}$, then $\mathrm{Dp}_{\mathcal{J}}(Z) \leq \mathrm{Dp}_{\mathcal{I}}(X_Z) \leq \mathrm{Dp}_{\mathcal{I}}(X)$ because of the minimality of X_Z. □

Also note that, if $X \subseteq A \subseteq L$, then $(\mathcal{I}_x{\restriction}A){\restriction}X = \mathcal{I}_x{\restriction}X$ for any $x \in L$, $(\bar{\mathcal{I}}{\restriction}A){\restriction}X = \bar{\mathcal{I}}{\restriction}X$ and $(\mathcal{I}{\restriction}A){\restriction}X = \mathcal{I}{\restriction}X$.

In view of the previous lemma, for a template $\langle L, \bar{\mathcal{I}}\rangle$ define $\Upsilon^{\bar{\mathcal{I}}} : \mathcal{P}(L) \to \mathbf{ON}$ such that $\Upsilon^{\bar{\mathcal{I}}}(X) = \mathrm{Dp}_{\mathcal{I}{\restriction}X}(X)$. Although this is not a rank function (that is, increasing with respect to $\subsetneq$), it can be used to define an iteration along the template $\langle L, \bar{\mathcal{I}}\rangle$ by recursion on $\Upsilon^{\bar{\mathcal{I}}}(X)$, this to have directly that such an iteration can be restricted to any subset of L. When the template is understood, we just denote $\Upsilon := \Upsilon^{\bar{\mathcal{I}}}$.

2.3.3 Lemma. *Fix $A \subseteq L$. $\Upsilon := \Upsilon^{\bar{\mathcal{I}}}$ has the following properties.*

(a) If $Y \in \mathcal{I}{\restriction}A$, then $\Upsilon(Y) \leq \mathrm{Dp}_{\mathcal{I}{\restriction}A}(Y)$.

(b) If $X \subseteq A$ then $\Upsilon(X) \leq \Upsilon(A)$.

(c) Let $x \in A$. If $Y \in \mathcal{I}_x{\restriction}A$ then $\Upsilon(Y) < \Upsilon(A)$.

Also, $\Upsilon^{\bar{\mathcal{I}}{\restriction}A} = \Upsilon{\restriction}\mathcal{P}(A)$.

Proof. For $X \subseteq A$, put $\mathcal{J}_X := \{X \cap Z \ / \ Z \in \mathcal{I}{\restriction}A\}$.

(a) $\Upsilon(Y) \leq \mathrm{Dp}_{\mathcal{J}_Y}(Y) = \mathrm{Dp}_{\mathcal{J}_Y}(Y \cap Y) \leq \mathrm{Dp}_{\mathcal{I}{\restriction}A}(Y)$ by Lemma 2.3.2.

(b) By Lemma 2.3.2, $\Upsilon(X) \leq \mathrm{Dp}_{\mathcal{J}_X}(A \cap X) \leq \mathrm{Dp}_{\mathcal{I}{\restriction}A}(A)$.

(c) By (a), $\Upsilon(Y) \leq \mathrm{Dp}_{\mathcal{I}{\restriction}A}(Y) < \mathrm{Dp}_{\mathcal{I}{\restriction}A}(A)$.

□

For $x \in L$, define $\hat{\mathcal{I}}_x := \{B \subseteq L_x \ / \ B \in \mathcal{I}_x{\restriction}(B \cup \{x\})\}$. This family is important at the time of the construction of an iteration because the generic object added at stage x is generic over all the intermediate extensions that come from any support in $\hat{\mathcal{I}}_x$ (see the comment after Theorem 2.4.1). Note that $B \in \hat{\mathcal{I}}_x$ if and only if $B \subseteq H$ for some $H \in \mathcal{I}_x$. Also, (1), (2) and (3) imply that any finite subset of L_x is in $\hat{\mathcal{I}}_x$.

2.3.4 Example. (1) Given a linear order L, $\mathcal{I}_x = [L_x]^{<\omega}$ for $x \in L$ form an indexed template on L. Note that $\hat{\mathcal{I}}_x = \mathcal{I}_x$ and, for $X \subseteq L$,

$$\Upsilon(X) = \begin{cases} |X| & \text{if } X \text{ is finite,} \\ \omega & \text{otherwise.} \end{cases}$$

(2) (Template for a fsi) Let δ be an ordinal number. Then, $\mathcal{I}_\alpha := \alpha + 1 = \{\xi \ / \ \xi \leq \alpha\}$ for $\alpha < \delta$ form an indexed template on δ. This is the template structure that corresponds to a fsi of length δ. Note that $\hat{\mathcal{I}}_\alpha = \mathcal{P}(\alpha)$ and, for $X \subseteq \delta$, $\Upsilon(X)$ is the order type of X.

2.3.5 Definition (Innocuous extension)**.** Let $\langle L, \bar{\mathcal{I}}\rangle$ be an indexed template and θ an uncountable cardinal.

(I) An indexed template $\langle L, \bar{\mathcal{J}}\rangle$ is a *θ-innocuous extension of* $\langle L, \bar{\mathcal{I}}\rangle$ if

(1) for every $x \in L$, $\mathcal{I}_x \subseteq \mathcal{J}_x$, and

(2) for any $x \in L$, $A \in \mathcal{J}_x$ and $X \subseteq A$ of size $< \theta$, there exists a $C \in \mathcal{I}_x$ containing X.

If in (2) we can even find $C \subseteq A$, say that $\langle L, \bar{\mathcal{J}}\rangle$ is a *strongly θ-innocuous extension of* $\langle L, \bar{\mathcal{I}}\rangle$.

(II) Let $\langle L', \bar{\mathcal{I}}'\rangle$ be an indexed template such that L' is a linear order extending L. $\langle L', \bar{\mathcal{I}}'\rangle$ is a *(strongly) θ-innocuous extension of* $\langle L, \bar{\mathcal{I}}\rangle$ if

(1) for every $x \in L$, $\mathcal{I}'_x \restriction L \subseteq \mathcal{I}'_x$ and

(2) $\langle L, \bar{\mathcal{I}}' \restriction L\rangle$ is a (strongly) θ-innocuous extension of $\langle L, \bar{\mathcal{I}}\rangle$.

The main point of this definition is that, when two iterations are defined along templates where one is an innocuous extension of the other and where some "coherence" is ensured in the construction of both iterations, we can get complete embeddability or equivalence between the resulting posets. The results that express this are Corollary 2.4.8 and Lemma 2.4.9.

2.3.6 Lemma ([Br02, Lemma 1.3] Adding small sets to a template)**.** *Let $\langle L, \bar{\mathcal{I}}\rangle$ be an indexed template, $L_0 \subseteq L$. For $x \in L$, define $\mathcal{J}_x := \{A \cup (B \cap L_0) \;/\; A, B \in \mathcal{I}_x\}$. Then, $\langle L, \bar{\mathcal{J}}\rangle$ is an indexed template which is a θ-innocuous extension of $\langle L, \bar{\mathcal{I}}\rangle$ and a strongly θ-innocuous extension of $\langle L_0, \bar{\mathcal{I}} \restriction L_0\rangle$ for any θ. Moreover, $\bar{\mathcal{J}} \restriction L_0 = \bar{\mathcal{I}} \restriction L_0$.*

Proof. According to the notation in Definition 2.3.1(5), $\mathcal{J} \subseteq \{A \cup (B \cap L_0) \;/\; A, B \in \mathcal{I}\}$, so we prove that the set in the right hand side is well-founded. Let $\{C_n\}_{n<\omega}$ be a $\subseteq$-decreasing sequence where, for any $n < \omega$, $C_n = A_n \cup (B_n \cap L_0)$ for some $A_n, B_n \in \mathcal{I}$. Without loss of generality, we may assume that $A_n \subseteq B_n$, so $\{B_n \cap L_0\}_{n<\omega}$ is a $\subseteq$-decreasing sequence. On the other hand, with $D_n = \bigcap_{i\leq n} A_i \in \mathcal{I}$, $\{D_n\}_{n<\omega}$ is $\subseteq$-decreasing. Then, by Lemma 2.3.2, there is an $N < \omega$ such that $B_n \cap L_0 = B_N \cap L_0$ and $D_N \subseteq A_n$ for any $n \geq N$. This implies $A_n \smallsetminus L_0 = A_N \smallsetminus L_0$, so $C_n = (A_n \smallsetminus L_0) \cup (B_n \cap L_0) = C_N$.

It is clear that $\mathcal{J}_x \restriction L_0 = \mathcal{I}_x \restriction L_0 \subseteq \mathcal{J}_x$ for any $x \in L$. Also note that any member of $\mathcal{J}_x$ is contained in a member of $\mathcal{I}_x$. This facts imply directly the claims about innocuity. □

Fix a measurable cardinal κ with a non-principal κ-complete ultrafilter $\mathcal{D}$ and let $\langle L, \bar{\mathcal{I}}\rangle$ be an indexed template. Put $L^* := L^\kappa/\mathcal{D}$, which is a linear order. For $\bar{x} = \langle x_\alpha\rangle_{\alpha<\kappa}/\mathcal{D} \in L^*$, let $\mathcal{I}^*_{\bar{x}}$ be the family of sets of the form $\bar{A} := [\{A_\alpha\}_{\alpha<\kappa}] = \prod_{\alpha<\kappa} A_\alpha/\mathcal{D}$ where $\{A_\alpha\}_{\alpha<\kappa}$ is a sequence of subsets of L such that $A_\alpha \in \mathcal{I}_{x_\alpha}$ for $\mathcal{D}$-many α. Identifying the members of L with constant functions in L^*, L^* extends the linear order L and $\mathcal{I}_x \subseteq \mathcal{I}'_x := \mathcal{I}^*_x \restriction L$ for all $x \in L$. For $\bar{x} \in L^*$, let $\mathcal{I}^\dagger_{\bar{x}} = \{\bar{A} \cup (\bar{B} \cap L) \;/\; \bar{A}, \bar{B} \in \mathcal{I}^*_{\bar{x}}\}$. Notice that $\mathcal{I}'_{\bar{x}} = \mathcal{I}^\dagger_{\bar{x}} \restriction L \subseteq \mathcal{I}^\dagger_{\bar{x}}$ for all $\bar{x} \in L^*$.

2.3.7 Lemma ([Br02, Lemma 2.1] Ultrapowers of templates)**.** *(a) $\langle L^*, \bar{\mathcal{I}}^*\rangle$ is an indexed template.*

(b) $\langle L, \bar{\mathcal{I}}'\rangle$ is an indexed template which is a strongly κ-innocuous extension of $\langle L, \bar{\mathcal{I}}\rangle$.

(c) $\langle L^*, \bar{\mathcal{I}}^\dagger\rangle$ is an indexed template which is a θ-innocuous extension of $\langle L^*, \bar{\mathcal{I}}^*\rangle$ and a strongly θ-innocuous extension of $\langle L, \bar{\mathcal{I}}'\rangle$ for any θ.

(d) $\langle L^*, \bar{\mathcal{I}}^\dagger\rangle$ is a strongly κ-innocuous extension of $\langle L, \bar{\mathcal{I}}\rangle$.

Proof. (a) Note that $\mathcal{I}^* = \{\bar{A} \;/\; A_\alpha \in \mathcal{I}$ for $\mathcal{D}$ many $\alpha\}$. We show that this family is well founded. Let $\{\bar{A}_n\}_{n<\omega}$ be a $\subseteq$-decreasing sequence of sets in that family. For $n < \omega$, let $D_n = \{\alpha < \kappa \;/\; A_{n+1,\alpha} \subseteq A_{n,\alpha}\}$, which is in $\mathcal{D}$. By κ-completeness of $\mathcal{D}$, $\bigcap_{n<\omega} D_n \in \mathcal{D}$, so $\{A_{n,\alpha}\}_{n<\omega}$ is $\subseteq$-decreasing for $\mathcal{D}$-many α. Again, by κ-completeness, there is an $N < \omega$ such that, for $\mathcal{D}$-many α, $\forall_{n\geq N}(A_{n,\alpha} = A_{N,\alpha})$. Therefore, $\bar{A}_n = \bar{A}_N$ for $n \geq N$.

(b) Lemma 2.3.2 and item (a) imply that $\langle L, \bar{\mathcal{I}}'\rangle$ is an indexed template. To see strong innocuity, let $x \in L$, $Z \in \mathcal{I}'_x$ and $X \subseteq Z$ of size $< \kappa$. Then, $Z = L \cap \bar{A}$ where $\bar{A} \in \mathcal{I}^*_x$. For $y \in X$, as $y \in Z$, the set $D_y = \{\alpha < \kappa \;/\; y \in A_\alpha\}$ is in $\mathcal{D}$. By κ-completeness, $D = \bigcap_{y\in X} D_y \in \mathcal{D}$. Now, consider the family $\mathcal{F} = \{\bigcap_{\alpha\in F} A_\alpha \;/\; F \in [D]^{<\omega}\} \subseteq \mathcal{I}_x$. As $\mathcal{I}_x$ is well-founded, there is a $\subseteq$-minimal $C \in \mathcal{F}$, that is, $C \subseteq A_\alpha$ for all $\alpha \in D$. Therefore, $\bigcap_{\alpha\in D} A_\alpha = C \in \mathcal{I}_x$. Clearly, $X \subseteq C \subseteq Z$.

(c) Consequence of Lemma 2.3.6.

(d) Immediate from (b).

□

Typically, given a poset $\mathbb{P}$ that comes from an iteration along the template $\langle L,\bar{\mathcal{I}}\rangle$, its ultrapower is (forcing equivalent to) an iteration along $\langle L^*,\bar{\mathcal{I}}^*\rangle$. Also, $\bar{\mathcal{I}}^\dagger$ is very close to $\bar{\mathcal{I}}^*$, so there is an iteration along $\langle L^*,\bar{\mathcal{I}}^\dagger\rangle$ that gives a poset which is forcing equivalent to the ultrapower of $\mathbb{P}$. This procedure is used for the inductive step of the construction of the chain of template iterations of the proof of Theorem 4.3.1. The use of $\bar{\mathcal{I}}^\dagger$, though it is used like $\bar{\mathcal{I}}^*$ to define the same iteration for $\mathbb{P}^\kappa/\mathcal{D}$, is preferred in order to ease the construction of the template in the limit step.

2.3.8 Lemma. *Fix $\theta<\kappa$ an infinite cardinal. Assume that $|\mathcal{I}{\restriction}X|<\theta$, for all $X\in[L]^{<\theta}$. Then, for every $\bar{X}\in[L^*]^{<\theta}$, $|\mathcal{I}^\dagger{\restriction}\bar{X}|<\theta$.*

Proof. Let $\bar{X}=\left\{\bar{x}^\xi\ /\ \xi<\nu\right\}$ for some $\nu<\theta$. For $\alpha<\kappa$ let $X_\alpha:=\left\{x^\xi_\alpha\ /\ \xi<\nu\right\}$. Then, $\bar{X}=[\{X_\alpha\}_{\alpha<\kappa}]$, so any $Z\in\mathcal{I}^\dagger{\restriction}\bar{X}$ comes from two objects of the form $\bar{Y}=[\{Y_\alpha\}_{\alpha<\kappa}]$ where $Y_\alpha\in\mathcal{I}{\restriction}X_\alpha$ for $\mathcal{D}$-many α. But, as $\theta<\kappa$ and each $|\mathcal{I}{\restriction}X_\alpha|<\theta$, there exists $\nu'<\theta$ such that $|\mathcal{I}{\restriction}X_\alpha|=\nu'$ for $\mathcal{D}$-many α. Therefore, $|\mathcal{I}^\dagger{\restriction}\bar{X}|\leq(\nu')^2<\theta$. □

Now we deal with the context of the construction of a "limit" of templates, which is relevant for the construction of the templates corresponding to the limit step in the proof of Theorem 4.3.1.

2.3.9 Lemma ([Br02, Lemma 1.4] Adding large sets to a template)**.** *Let $\langle L,\bar{\mathcal{I}}\rangle$ be an indexed template and μ an ordinal such that*

(i) $\mu\subseteq L$ is cofinal in L and

(ii) for $x\in L$ and $\alpha<\mu$ such that $\alpha\leq x$, if $A\in\mathcal{I}_x$ then $L_\alpha\cap A\in\mathcal{I}_x$.

For $x\in L$ put $\mathcal{J}_x=\mathcal{I}_x\cup\{A\cup L_\alpha\ /\ \alpha<\mu,\ \alpha\leq x$ and $A\in\mathcal{I}_x\}$. Then, $\langle L,\bar{\mathcal{J}}\rangle$ is an indexed template.

Proof. For notational purposes, consider $L_{-1}=\varnothing$. Note that $\mathcal{J}=\{A\cup L_\alpha\ /\ A\in\mathcal{I}$ and $\alpha\in[-1,\mu)\}$. Let $\{B_n\}_{n<\omega}$ be a $\subseteq$-decreasing sequence in $\mathcal{J}$, where $B_n=A_n\cup L_{\alpha_n}$ for some $A_n\in\mathcal{I}$ and $\alpha_n\in[-1,\mu)$. Let α_N be the minimal of $\{\alpha_n\ /\ n<\omega\}$, so we may assume $N=0$ and also that $B_0\neq L$. Note that $B_n=\bigcap_{k\leq n}B_k=L_{\alpha_0}\cup C_n$ where $C_n=\bigcup\{\bigcap_{k\leq n}X_k^{f(k)}\ /\ f\in 2^{n+1},f(0)=0\}$ with $X_k^0=A_k$ and $X_k^1=L_{\alpha_k}$. Note that $\{C_n\}_{n<\omega}$ is a decreasing sequence in $\mathcal{I}$, so there is an $n_0<\omega$ such that $C_n=C_{n_0}$ for all $n\geq n_0$, so also $B_n=B_{n_0}$. □

Fix an uncountable cardinal θ and consider a chain of indexed templates $\left\{\langle L^\alpha,\bar{\mathcal{I}}^\alpha\rangle\right\}_{\alpha<\delta}$ such that, for $\alpha<\beta<\delta$, $\langle L^\beta,\bar{\mathcal{I}}^\beta\rangle$ is a strongly θ-innocuous extension of $\langle L^\alpha,\bar{\mathcal{I}}^\alpha\rangle$. Moreover, assume that there is an ordinal $\mu\subseteq L^0$ such that, for all $\alpha<\delta$,

(i) μ is cofinal in L^α and

(ii) $L^\alpha_\xi\in\mathcal{I}^\alpha_\xi$ for all $\xi\in\mu$.

Define $L^\delta:=\bigcup_{\alpha<\delta}L^\alpha$ and, for $x\in L^\delta$, let $\mathcal{I}_x:=\bigcup_{\alpha\in[\alpha_x,\delta)}\mathcal{I}^\alpha_x$ where α_x is the least α such that $x\in L^\alpha$. Also, put $\mathcal{J}_x:=\mathcal{I}_x\cup\left\{L^\delta_\xi\cup A\ /\ \xi\in\mu,\xi\leq x\text{ and }A\in\mathcal{I}_x\right\}$.

2.3.10 Lemma ([Br02, Lemma 1.8] Chains of templates)**.** *(a) $\langle L^\delta,\bar{\mathcal{I}}\rangle$ is an indexed template which is a strongly θ-innocuous extension of $\langle L^\alpha,\bar{\mathcal{I}}^\alpha\rangle$ for all $\alpha<\delta$.*

(b) $\langle L^\delta,\bar{\mathcal{J}}\rangle$ is an indexed template which is a strongly θ-innocuous extension of $\langle L^\alpha,\bar{\mathcal{I}}^\alpha\rangle$ for all $\alpha<\delta$. Moreover, if cf$(\delta)\geq\theta$*, then $\langle L^\delta,\bar{\mathcal{J}}\rangle$ is a strongly θ-innocuous extension of $\langle L^\delta,\bar{\mathcal{I}}\rangle$.*

Proof. (a) Assume, by the contrary, that $\{B_n\}_{n<\omega}$ is an strictly $\subseteq$-decreasing sequence in $\mathcal{I}^\delta$. Choose $\{\alpha_n\}_{n<\omega}\subseteq\delta$ increasing such that $B_n\in\mathcal{I}^{\alpha_n}$, so $B_n\in\mathcal{I}^{\alpha_n}{\restriction}L^{\alpha_0}$. Choose $x_n\in B_n\smallsetminus B_{n+1}$ and put $X_n=\{x_m\ /\ m\geq n\}$. As $X_n\subseteq B_n$ is countable, by strong innocuity find $C_n\in\mathcal{I}^{\alpha_0}$ such that $X_n\subseteq C_n\subseteq B_n$. Put $D_n=\bigcap_{i\leq n}C_i\in\mathcal{I}^{\alpha_0}$. Note that, $x_n\in D_n\smallsetminus D_{n+1}$, so $\{D_n\}_{n<\omega}$ is strictly $\subseteq$-decreasing in the well-founded $\mathcal{I}^{\alpha_0}$, a contradiction.

(b) Consequence of Lemma 2.3.9. Strong innocuity is easy to prove.

□

Note that properties (i) and (ii) also hold for the template $\langle L^\delta, \bar{\mathcal{J}}\rangle$, but (ii) may not hold for $\langle L^\delta, \bar{\mathcal{I}}\rangle$. Although, in many cases, both templates lead to equivalent template iteration constructions when $\mathrm{cf}(\delta) \geq \theta$, $\bar{\mathcal{J}}$ is preferred over $\bar{\mathcal{I}}$ because of property (ii).

As Lemma 2.3.8, the following result states that, in the resulting template, it is preserved the property of having small templates when restricting to a small set.

2.3.11 Lemma. *Assume that $\nu \leq \theta$ is a regular cardinal and that, for each $\alpha < \delta$ and $X \in [L^\alpha]^{<\nu}$, $|\mathcal{I}^\alpha{\restriction}X| < \nu$. Then, $|\mathcal{I}{\restriction}X| < \nu$ and $|\mathcal{J}{\restriction}X| < \nu$ for any $X \in [L^\delta]^{<\nu}$.*

Proof. If $\mathrm{cf}(\delta) < \nu$, choose an increasing cofinal sequence $\{\alpha_\eta\}_{\eta<\mathrm{cf}(\delta)}$ for δ and note that $(\mathcal{I}{\restriction}X) \smallsetminus \{X\} \subseteq \bigcup_{\eta<\mathrm{cf}(\delta)} \mathcal{I}^{\alpha_\eta}{\restriction}(X \cap L^{\alpha_\eta})$ for any $X \subseteq L^\delta$, so it has size $< \nu$ when X does. In the case that $\mathrm{cf}(\delta) \geq \nu$, if $X \in [L^\delta]^{<\nu}$, there exists an $\alpha < \delta$ such that $X \subseteq L^\alpha$. We claim that $\mathcal{I}{\restriction}X = \mathcal{I}^\alpha{\restriction}X$. If $Z \in \mathcal{I}{\restriction}X$, then $Z = X \in \mathcal{I}^\alpha{\restriction}X$ or $Z = X \cap H$ for some $H \in \mathcal{I}^\beta_\xi$ with $\xi \in \mu$ and $\alpha < \beta < \delta$. As $|Z| < \nu$, by strong θ-innocuity, we can find a $C \in \mathcal{I}^\alpha_\xi$ such that $Z \subseteq C \subseteq H$, so $Z = C \cap X \in \mathcal{I}^\alpha{\restriction}X$.

For the case of $\mathcal{J}$, note that $\left\{L^\delta_\xi \cap X \ /\ \xi \leq \mu\right\}$ has size $\leq |X|$. As, for any $X \subseteq L^\delta$, $\mathcal{J}{\restriction}X = \mathcal{I}{\restriction} X \cup \left\{(L^\delta_\xi \cap X) \cup Z \ /\ \xi < \mu \text{ and } Z \in \mathcal{I}{\restriction}X\right\}$, then it has size $< \nu$ when X does. □

2.4 Template iterations

We present the theory of template iterations for non-definable posets. Although this approach is general, the proofs of the criteria of construction of the iterations and ccc-ness are not different from those in [Br05], actually, our presentation is based on this reference. With some generality, we can say that the original version of template iterations with definable forcing (in [S04]) corresponds to Example 2.4.3 with $L_C = \varnothing$.

The original idea of a construction of an iteration along a template $\langle L, \bar{\mathcal{I}}\rangle$ is to define, by recursion on $A \in \mathcal{I}$, the poset $\mathbb{P}{\restriction}A$ that represents the iteration defined on the support A. Later, as $\langle Z, \bar{\mathcal{I}}{\restriction}Z\rangle$ is also a template for any $Z \subseteq L$, $\mathbb{P}{\restriction}Z$, the iteration defined on the support Z, can also be defined. However, in this dissertation, we use the function $\Upsilon^{\bar{\mathcal{I}}}$ (defined in Section 2.3) to prove that we can directly define $\mathbb{P}{\restriction}Z$ for all $Z \subseteq L$.

Like a finite support iteration, we consider names of posets with which we force at a stage $x \in L$, but for this we also need a support, say, if $B \in \hat{\mathcal{I}}_x$, $\dot{\mathbb{Q}}^B_x$ is a $\mathbb{P}{\restriction}B$-name of a poset (given by reals) that we use to get $\mathbb{P}{\restriction}(B \cup \{x\}) \simeq \mathbb{P}{\restriction}B * \dot{\mathbb{Q}}^B_x$. As these names for posets depend on some support, we need some additional properties to get a well defined iteration. All these conditions and properties for this construction are explained in the following result.

2.4.1 Theorem (Iteration along a template)**.** *Given a template $\langle L, \bar{\mathcal{I}}\rangle$, a partial order $\mathbb{P}{\restriction}A$ can be defined by recursion on $\alpha = \Upsilon(A)$ for all $A \subseteq L$, this with the following conditions and properties.*

(1) For $x \in L$ and $B \in \hat{\mathcal{I}}_x$, $\dot{\mathbb{Q}}^B_x$ is a $\mathbb{P}{\restriction}B$-name of a poset given by reals. The following conditions should hold.

- *(i) If $E \subseteq B$ and $\mathbb{P}{\restriction}E$ is a complete suborder of $\mathbb{P}{\restriction}B$, then $\Vdash_{\mathbb{P}{\restriction}B} \dot{\mathbb{Q}}^E_x \lessdot_{V^{\mathbb{P}{\restriction}E}} \dot{\mathbb{Q}}^B_x$.*
- *(ii) If $E \in \hat{\mathcal{I}}_x$, $\mathbb{P}{\restriction}(B\cap E)$ is a complete suborder of both $\mathbb{P}{\restriction}B$ and $\mathbb{P}{\restriction}E$, and $\dot{q}$ is a $\mathbb{P}{\restriction}(B\cap E)$-name for a real such that $\Vdash_{\mathbb{P}{\restriction}E} \dot{q} \in \dot{\mathbb{Q}}^E_x$ and $\Vdash_{\mathbb{P}{\restriction}B} \dot{q} \in \dot{\mathbb{Q}}^B_x$, then $\Vdash_{\mathbb{P}{\restriction}(B\cap E)} \dot{q} \in \dot{\mathbb{Q}}^{B\cap E}_x$.*
- *(iii) If $B', D \subseteq B$ and $\langle \mathbb{P}{\restriction}(B' \cap D), \mathbb{P}{\restriction}B', \mathbb{P}{\restriction}D, \mathbb{P}{\restriction}B\rangle$ is a correct system, then the system $\langle \mathbb{P}{\restriction}(B' \cap D) * \dot{\mathbb{Q}}^{B'\cap D}_x, \mathbb{P}{\restriction}B' * \dot{\mathbb{Q}}^{B'}_x, \mathbb{P}{\restriction}D * \dot{\mathbb{Q}}^D_x, \mathbb{P}{\restriction}B * \dot{\mathbb{Q}}^B_x\rangle$ is correct.*

(2) The partial order $\mathbb{P}{\restriction}A$ is defined as:

(i) $\mathbb{P}{\restriction}A$ *consists of all finite partial functions* p *with domain contained in* A *such that* $p = \varnothing$ *or, if* $|p| > 0$ *and* $x = \max(\mathrm{dom}p)$*, then there exists a* $B \in \mathcal{I}_x{\restriction}A$ *such that* $p{\restriction}L_x \in \mathbb{P}{\restriction}B$ *and* $p(x)$ *is a* $\mathbb{P}{\restriction}B$*-name for a condition in* $\dot{\mathbb{Q}}_x^B$*.*

(ii) *The ordering on* $\mathbb{P}{\restriction}A$ *is given by:* $q \leq_A p$ *if* $\mathrm{dom}p \subseteq \mathrm{dom}q$ *and either* $p = \varnothing$ *or, when* $p \neq 0$ *and* $x = \max(\mathrm{dom}q)$*, there is a* $B \in \mathcal{I}_x{\restriction}A$ *such that* $q{\restriction}L_x \in \mathbb{P}{\restriction}B$ *and, either* $x \notin \mathrm{dom}p$*,* $p \in \mathbb{P}{\restriction}B$ *and* $q{\restriction}L_x \leq_B p$*, or* $x \in \mathrm{dom}p$*,* $p{\restriction}L_x \in \mathbb{P}{\restriction}B$*,* $q{\restriction}L_x \leq_B p{\restriction}L_x$ *and* $p(x), q(x)$ *are* $\mathbb{P}{\restriction}B$*-names for conditions in* $\dot{\mathbb{Q}}_x^B$ *such that* $q{\restriction}L_x \Vdash_{\mathbb{P}{\restriction}B} q(x) \leq p(x)$*.*

Within this recursion, the following properties are proved.

(a) *If* $p \in \mathbb{P}{\restriction}A$*,* $x \in A$ *and* $\max(\mathrm{dom}p) < x$*, then there exists a* $B \in \mathcal{I}_x{\restriction}A$ *such that* $p \in \mathbb{P}{\restriction}B$*.*

(b) *For* $D \subseteq A$*,* $\mathbb{P}{\restriction}D \subseteq \mathbb{P}{\restriction}A$ *and, for* $p, q \in \mathbb{P}{\restriction}D$*,* $q \leq_D p$ *iff* $q \leq_A p$*.*

(c) $\mathbb{P}{\restriction}A$ *is a poset.*

(d) $\mathbb{P}{\restriction}A$ *is obtained from some posets of the form* $\mathbb{P}{\restriction}B$ *with* $B \subsetneq A$ *in the following way:*

(i) *If* $x = \max(A)$ *exists and* $A_x := A \cap L_x \in \hat{\mathcal{I}}_x$*, then* $\mathbb{P}{\restriction}A = \mathbb{P}{\restriction}A_x * \dot{\mathbb{Q}}_x^{A_x}$*.*

(ii) *If* $x = \max(A)$ *but* $A_x \notin \hat{\mathcal{I}}_x$*, then* $\mathbb{P}{\restriction}A$ *is the direct limit of the* $\mathbb{P}{\restriction}B$ *where* $B \subseteq A$ *and* $B \cap L_x \in \mathcal{I}_x{\restriction}A$*.*

(iii) *If* A *does not have a maximum element, then* $\mathbb{P}{\restriction}A$ *is the direct limit of the* $\mathbb{P}{\restriction}B$ *where* $B \in \mathcal{I}_x{\restriction}A$ *for some* $x \in A$ *(in the case* $A = \varnothing$*, it is clear that* $\mathbb{P}{\restriction}A = \mathbb{1}$*).*

(e) *If* $D \subseteq A$*, then* $\mathbb{P}{\restriction}D$ *is a complete suborder of* $\mathbb{P}{\restriction}A$*.*

(f) *If* $D \subseteq L$ *then* $\mathbb{P}{\restriction}(A \cap D) = \mathbb{P}{\restriction}A \cap \mathbb{P}{\restriction}D$*.*

(g) *If* $D, A' \subseteq A$ *then* $\langle \mathbb{P}{\restriction}(A' \cap D), \mathbb{P}{\restriction}A', \mathbb{P}{\restriction}D, \mathbb{P}{\restriction}A \rangle$ *is a correct system.*

Proof. By just changing certain notation, the proof follows the same lines as [Br05, Thm. 2.2]. Note that Lemma 2.3.3 guaranties that the definition in (2) can be done recursively by the function Υ.

(a) Denote $z := \max(\mathrm{dom}p)$. By (2)(i), there is an $E \in \mathcal{I}_z{\restriction}A$ such that $p{\restriction}L_z \in \mathbb{P}{\restriction}E$ and $p(z)$ is a $\mathbb{P}{\restriction}E$-name for a condition in $\dot{\mathbb{Q}}_z^E$. By Definition 2.3.1 and Lemma 2.3.2, there is some $B \in \mathcal{I}_x A$ such that $z \in B$. Without loss of generality, we may assume that $E \in \mathcal{I}_z B$ (as $E = A \cap H$ and $B = A \cap H'$ for some $H \in \mathcal{I}_z$ and $H' \in \mathcal{I}_x$, just redefine B as $A \cap (H \cup H')$). Thus, $p \in \mathbb{P}{\restriction}B$.

(b) Let $p \in \mathbb{P}{\restriction}D$ and assume that $p \neq \varnothing$, so let $x = \max(\mathrm{dom}p)$. By (2), there is an $E \in \mathcal{I}_x{\restriction}D$ such that $p{\restriction}L_x \in \mathbb{P}{\restriction}E$ and $p(x)$ is a $\mathbb{P}{\restriction}E$-name for a condition in $\dot{\mathbb{Q}}_x^E$. Also, there exists an $H \in \mathcal{I}_x$ such that $E = D \cap H$. Put $B := A \cap H \in \mathcal{I}_x{\restriction}A$. As $E \subseteq B$ and $\Upsilon(B) < \Upsilon(A)$ (see Lemma 2.3.3), by induction hypothesis and (e), we have that $\mathbb{P}{\restriction}E$ is a complete suborder of $\mathbb{P}{\restriction}B$, so $p{\restriction}L_x \in \mathbb{P}{\restriction}B$. Moreover, by (1)(i), $p(x)$ is a $\mathbb{P}{\restriction}B$-name for a condition in $\dot{\mathbb{Q}}_x^B$, so $p \in \mathbb{P}{\restriction}A$.

Now, fix $p, q \in \mathbb{P}{\restriction}D$. Assume that $q \leq_D p$ and $x = \max(\mathrm{dom}p) = \max(\mathrm{dom}q)$. By (2)(ii), there exists an $E \in \mathcal{I}_x{\restriction}D$ such that $p{\restriction}L_x, q{\restriction}L_x \in \mathbb{P}{\restriction}E$, $q{\restriction}L_x \leq_E p{\restriction}L_x$, $p(x)$ and $q(x)$ are $\mathbb{P}{\restriction}E$-names for conditions in $\dot{\mathbb{Q}}_x^E$ such that $q{\restriction}L_x \Vdash_{\mathbb{P}{\restriction}E} q(x) \leq_{\dot{\mathbb{Q}}_x^E} p(x)$. Also, there is an $H \in \mathcal{I}_x$ such that $E = D \cap H$. Put $B = A \cap H$ so, by induction hypothesis, $q{\restriction}L_x \leq_B p{\restriction}L_x$, $p(x)$ and $q(x)$ are $\mathbb{P}{\restriction}B$-names for conditions in $\dot{\mathbb{Q}}_x^B$ and $q{\restriction}L_x \Vdash_{\mathbb{P}{\restriction}B} q(x) \leq_{\dot{\mathbb{Q}}_x^B} p(x)$. Clearly, $q \leq_A p$. The case $\max(\mathrm{dom}p) < \max(\mathrm{dom}q)$ is treated similarly.

To prove the converse, assume $q \leq_A p$ and $x = \max(\mathrm{dom}p) = \max(\mathrm{dom}q)$. $p, q \in \mathbb{P}{\restriction}D$ implies that there is an $E \in \mathcal{I}_x{\restriction}D$ such that $p{\restriction}L_x, q{\restriction}L_x \in \mathbb{P}{\restriction}E$ and $p(x)$ and $q(x)$ are $\mathbb{P}{\restriction}E$-names for conditions in $\dot{\mathbb{Q}}_x^E$ (here, induction hypothesis and (e) are used). On the other hand, $q \leq_A p$ implies that there is a $B \in \mathcal{I}_x{\restriction}A$ such that the statement in (2)(ii) holds. Without loss of generality, we may assume that $E \subseteq B$ (there are $H, H' \in \mathcal{I}_x$ such that $E = D \cap H$ and $B = A \cap H'$, so just redefine B as $A \cap (H \cup H')$). By induction hypothesis, $q{\restriction}L_x \leq_E p{\restriction}L_x$ and $q{\restriction}L_x \Vdash_{\mathbb{P}{\restriction}E} q(x) \leq_{\dot{\mathbb{Q}}_x^E} p(x)$, so $q \leq_E p$. The case $\max(\mathrm{dom}p) < \max(\mathrm{dom}q)$ is treated similarly, but it requires (a).

(c) Reflexivity of $\leq_A$ is easy by the induction hypothesis, so we prove transitivity. Let $p, q, r \in \mathbb{P}{\restriction}A$ be such that $r \leq_A q$ and $q \leq_A p$. Assume that $x = \max(\mathrm{dom}p) = \max(\mathrm{dom}q) = \max(\mathrm{dom}r)$ (the other cases are treated similarly). We can find a $B \in \mathcal{I}_x{\restriction}A$ such that $p{\restriction}L_x, q{\restriction}L_x, r{\restriction}L_x \in \mathbb{P}{\restriction}B$, $r{\restriction}L_x \leq_B q{\restriction}L_x$, $q{\restriction}L_x \leq_B p{\restriction}L_x$ and $p(x), q(x), r(x)$ are $\mathbb{P}{\restriction}B$-names for conditions in $\dot{\mathbb{Q}}_x^B$ such that $r{\restriction}L_x \Vdash_{\mathbb{P}{\restriction}B} r(x) \leq q(x)$ and $q{\restriction}L_x \Vdash_{\mathbb{P}{\restriction}B} q(x) \leq p(x)$. By induction hypothesis, it is clear that $r{\restriction}L_x \leq_B p{\restriction}L_x$ and $r{\restriction}L_x \Vdash_{\mathbb{P}{\restriction}B} r(x) \leq p(x)$.

(d) (i) It is enough to show that the set $\{p \in \mathbb{P}{\restriction}A \ / \ x = \max(\mathrm{dom}p)$ and $p{\restriction}L_x \in \mathbb{P}{\restriction}A_x\}$ is dense in $\mathbb{P}{\restriction}A$. Let $p \in \mathbb{P}{\restriction}A$. If either $p = \varnothing$ or $\max(\mathrm{dom}p) < x$ then, by (a), $p \in \mathbb{P}{\restriction}B$ for some $B \in \mathcal{I}_x A$, so $p \in \mathbb{P}{\restriction}A_x$ and $p^\frown\langle \mathbb{1}_{\dot{\mathbb{Q}}_x^{A_x}}\rangle \leq_A p$. On the other hand, if $\max(\mathrm{dom}p) = x$, then it is clear that $p{\restriction}L_x \in \mathbb{P}{\restriction}A_x$.

(ii) Let $p \in \mathbb{P}{\restriction}A$. If either $p = \varnothing$ or $\max(\mathrm{dom}p) < x$ then there is a $B \in \mathcal{I}_x{\restriction}A$ such that $p \in \mathbb{P}{\restriction}B$ (by (a)), so assume that $\max(\mathrm{dom}p) = x$. There is an $E \in \mathcal{I}_x{\restriction}A$ such that $p{\restriction}L_x \in \mathbb{P}{\restriction}E$ and $p(x)$ is a $\mathbb{P}{\restriction}E$-name for a condition in $\dot{\mathbb{Q}}_x^E$. Put $B := E \cup \{x\}$. It is clear that $B \cap L_x = E \in \mathcal{I}_x{\restriction}B$ and that $p \in \mathbb{P}{\restriction}B$. On the other hand, by the previous case, $\{\mathbb{P}{\restriction}B \ / \ B \subseteq A$ and $B \cap L_x \in \mathcal{I}_x{\restriction}A\}$ is a directed system of posets, so $\mathbb{P}{\restriction}A$ is its direct limit.

(iii) Let $p \in \mathbb{P}{\restriction}A$ and $y = \max(\mathrm{dom}p)$. As there is some $x \in A$ above y, there is some $B \in \mathcal{I}_x{\restriction}A$ such that $p \in \mathbb{P}{\restriction}B$ by (a). On the other hand, by induction hypothesis, $\{\mathbb{P}{\restriction}B \ / \ \exists_{x\in A}(B \in \mathcal{I}_x{\restriction}A)\}$ is a directed system of posets, so $\mathbb{P}{\restriction}A$ is its direct limit.

(e) We argue by cases from (d) for A.

(i) If $x \notin D$ then $D \subseteq A_x$. By induction hypothesis (as $\Upsilon(A_x) < \Upsilon(A)$), $\mathbb{P}{\restriction}D$ is a complete suborder of $\mathbb{P}{\restriction}A_x$. It is clear that $\mathbb{P}{\restriction}A_x$ is a complete suborder of $\mathbb{P}{\restriction}A$. Assume $x \in D$ otherwise. Note that $D_x := D \cap L_x \in \hat{\mathcal{I}}_x$, so $\mathbb{P}{\restriction}D_x$ is a complete suborder of $\mathbb{P}{\restriction}A_x$. Then, by (1)(i), Lemma 2.1.2 and (d)(i), it is clear that $\mathbb{P}{\restriction}D$ is a complete suborder of $\mathbb{P}{\restriction}A$.

(ii) We proceed by the following cases.

- $D_x := D \cap L_x \in \hat{\mathcal{I}}_x$. Then, there is some $B_x \in \mathcal{I}_x{\restriction}A$ such that $D_x = D \cap B_x$. Put $B := B_x \cup \{x\}$. As in the previous case, $\mathbb{P}{\restriction}D$ is a complete suborder of $\mathbb{P}{\restriction}B$ and the latter is a complete suborder of $\mathbb{P}{\restriction}A$ by (d)(ii).
- $D_x \notin \hat{\mathcal{I}}_x$. We first assume that $x \in D$. Then, $\mathbb{P}{\restriction}D = \mathrm{limdir}_{E\in\mathcal{D}}\mathbb{P}{\restriction}E$ where $\mathcal{D} := \{E \subseteq D \ / \ E \cap L_x \in \mathcal{I}_x{\restriction}D\}$. Clearly, $\mathcal{D} = \{B \cap D \ / \ B \in \mathcal{A}\}$ where $\mathcal{A} := \{B \subseteq A \ / \ B \cap L_x \in \mathcal{I}_x{\restriction}A\}$ and, for each $B, B' \in \mathcal{A}$, if $B \subseteq B'$, $\langle \mathbb{P}{\restriction}(B \cap D \cap L_x), \mathbb{P}{\restriction}(B' \cap D \cap L_x), \mathbb{P}{\restriction}(B \cap L_x), \mathbb{P}{\restriction}(B' \cap L_x))\rangle$ is a correct system by induction hypothesis and (g), so $\langle \mathbb{P}{\restriction}(B \cap D, \mathbb{P}{\restriction}(B' \cap D), \mathbb{P}{\restriction}B, \mathbb{P}{\restriction}B')\rangle$ is a correct system by (1)(iii). Therefore, by Lemma 2.1.5, $\mathbb{P}{\restriction}D$ is a complete suborder of $\mathbb{P}{\restriction}A$.
  Now, we assume that $x \notin D$. $D_x \notin \hat{\mathcal{I}}_x$ implies that, whenever D has a maximum z, $D_z := D \cap L_z \notin \hat{\mathcal{I}}_z$, so $\mathbb{P}{\restriction}D$ is described as a direct limit from (d)(ii) or (iii). In either case, $\mathbb{P}{\restriction}D = \mathrm{limdir}_{B\in\mathcal{A}}\mathbb{P}{\restriction}B \cap D$ (the system of posets is directed because of inductive hypothesis and case (i)), so, as in the previous argument, $\mathbb{P}{\restriction}D$ is a complete suborder of $\mathbb{P}{\restriction}A$.

(iii) If $D \in \hat{\mathcal{I}}_x$ for some $x \in A$, we can find some $B \in \mathcal{I}_x{\restriction}A$ such that $D \subseteq B$, so $\mathbb{P}{\restriction}D$ is a complete suborder of $\mathbb{P}{\restriction}B$ (by induction hypothesis) and, by (d)(iii), it is clear that the latter is a complete suborder of $\mathbb{P}{\restriction}A$. So assume that $D \notin \hat{\mathcal{I}}_x$ for any $x \in A$. Again, as in the previous paragraph, $\mathbb{P}{\restriction}D = \mathrm{limdir}_{B\in\mathcal{A}}\mathbb{P}{\restriction}B \cap D$ with $\mathcal{A} := \{B \subseteq A \ / \ \exists_{x\in A}(B \in \mathcal{I}_x{\restriction}A)\}$, so Lemma 2.1.5 implies that $\mathbb{P}{\restriction}D$ is a complete suborder of $\mathbb{P}{\restriction}A$.

(f) We prove the statement for all $D \subseteq L$ with $\Upsilon(D) \leq \alpha$. By (b), it is clear that $\mathbb{P}{\restriction}(A \cap D) \subseteq \mathbb{P}{\restriction}A \cap \mathbb{P}{\restriction}D$. To prove the converse contention, assume $p \in \mathbb{P}{\restriction}A \cap \mathbb{P}{\restriction}D$ with $x = \max(\mathrm{dom}p)$. Then, there are $B \in \mathcal{I}_x{\restriction}A$ and $E \in \mathcal{I}_x{\restriction}D$ such that $p{\restriction}L_x \in \mathbb{P}{\restriction}B \cap \mathbb{P}{\restriction}E$ and $p(x)$ is a $\mathbb{P}{\restriction}B$-name for a condition in $\dot{\mathbb{Q}}_x^B$ as well as a $\mathbb{P}{\restriction}E$-name for a condition in $\dot{\mathbb{Q}}_x^E$. Without loss of generality, we may assume that $B \cap E \in \mathcal{I}_x{\restriction}(A \cap D)$. By induction hypothesis, as $\Upsilon(B), \Upsilon(E) < \alpha$, $\mathbb{P}{\restriction}(B \cap E) = \mathbb{P}{\restriction}B \cap \mathbb{P}{\restriction}E$, so $p{\restriction}L_x \in \mathbb{P}{\restriction}(B \cap E)$. Clearly, $p(x)$ is a $\mathbb{P}{\restriction}(B \cap E)$-name for a real. Thus, by (1)(ii), $p(x)$ is a $\mathbb{P}{\restriction}(B \cap E)$-name for a condition in $\dot{\mathbb{Q}}_x^{B \cap E}$.

(g) We split into cases according to (d) for A.

(i) Here, $A'_x := A' \cap L_x$ and $D_x := D \cap L_x$ are subsets of A_x, so they are in $\hat{\mathcal{I}}$. By induction hypothesis, $\langle \mathbb{P}{\restriction}(A' \cap D \cap L_x), \mathbb{P}{\restriction}A'_x, \mathbb{P}{\restriction}D_x, \mathbb{P}{\restriction}A_x \rangle$ is a correct system, so the result follows (do cases for x being in A' or in D and use (1)(iii) for the case $x \in A' \cap D$).

(ii) Let $p \in \mathbb{P}{\restriction}A'$ and $r \in \mathbb{P}{\restriction}(A' \cap D)$ a reduction of p. We first assume that $D_x \in \hat{\mathcal{I}}$. Find $B \in \mathcal{A} := \{B \subseteq A \mathbin{/} B \cap L_x \in \mathcal{I}_x{\restriction}A\}$ such that $D \subseteq B$ and $p \in \mathbb{P}{\restriction}B$ (by (d)(ii)). Put $B' := A' \cap B$, so $p \in \mathbb{P}{\restriction}B'$ by (f) and, as $A' \cap D = B' \cap D$, r is a reduction of p with respect to $\mathbb{P}{\restriction}(A' \cap D), \mathbb{P}{\restriction}B'$. On the other hand, by the previous case, $\langle \mathbb{P}{\restriction}(A' \cap D), \mathbb{P}{\restriction}B', \mathbb{P}{\restriction}D, \mathbb{P}{\restriction}B \rangle$ is a correct system, so r is a reduction of p with respect to $\mathbb{P}{\restriction}D, \mathbb{P}{\restriction}B$. Thus, this is also with respect to $\mathbb{P}{\restriction}D, \mathbb{P}{\restriction}A$.

Now, assume that $D_x \notin \hat{\mathcal{I}}$, so $\mathbb{P}{\restriction}D = \mathrm{limdir}_{B \in \mathcal{A}} \mathbb{P}{\restriction}(B \cap D)$ (like in the proof of (e)). Choose $B \in \mathcal{A}$ such that $p \in \mathbb{P}{\restriction}B$ and $r \in \mathbb{P}{\restriction}(B \cap D)$. Put $B' = A' \cap B$. As before, we have that $p \in \mathbb{P}{\restriction}B'$, $r \in \mathbb{P}{\restriction}(B' \cap D)$ and $\langle \mathbb{P}{\restriction}(B' \cap D), \mathbb{P}{\restriction}B', \mathbb{P}{\restriction}(B \cap D), \mathbb{P}{\restriction}B \rangle$ is a correct system. Clearly, r is a reduction of p with respect to $\mathbb{P}{\restriction}(B' \cap D), \mathbb{P}{\restriction}B'$ and, by correctness, the same with respect to $\mathbb{P}{\restriction}(B \cap D), \mathbb{P}{\restriction}B$. We claim that r is a reduction of p with respect to $\mathbb{P}{\restriction}D, \mathbb{P}{\restriction}A$. Indeed, if $q \leq_D r$, find $B_1 \in \mathcal{A}$ containing B such that $q \in \mathbb{P}{\restriction}(B_1 \cap D)$. The system $\langle \mathbb{P}{\restriction}(B \cap D), \mathbb{P}{\restriction}B, \mathbb{P}{\restriction}(B_1 \cap D), \mathbb{P}{\restriction}B_1 \rangle$ is also correct, which implies that r is a reduction of p with respect to $\mathbb{P}{\restriction}(B_1 \cap D), \mathbb{P}{\restriction}B_1$, so q is compatible with p in $\mathbb{P}{\restriction}B_1$ (and so in $\mathbb{P}{\restriction}A$).

(iii) By doing cases on whether $\exists_{x \in A}(D \in \mathcal{I}_x{\restriction}A)$ or not, a similar argument as before (using facts from the proof of (e) as well) works.

□

Condition (1), particularly item (ii), implies that, when we step into the generic extension of $\mathbb{P}{\restriction}L$, the generic object added at stage x is generic over the intermediate extension by $\mathbb{P}{\restriction}B$ for any $B \in \hat{\mathcal{I}}_x$. In general, as L_x may not belong to $\hat{\mathcal{I}}_x$ (that is, to $\mathcal{I}_x$), this object added at stage x need not be generic over the intermediate extension by $\mathbb{P}{\restriction}L_x$ or over the extension for any subset of L_x that is not in $\hat{\mathcal{I}}_x$.

The following examples present the types of template iterations that are used in our applications in Chapter 4.

2.4.2 Example (Fsi in terms of a template iteration). Let δ be an ordinal and consider the template $\bar{\mathcal{I}}$ defined in Example 2.3.4(2). An iteration along $\langle \delta, \bar{\mathcal{I}} \rangle$ defined as in Theorem 2.4.1 is equivalent to the fsi $\langle \mathbb{P}{\restriction}\alpha, \dot{\mathbb{Q}}_\alpha^\alpha \rangle_{\alpha < \delta}$. Unlike a generic fsi, this iteration has the feature that it can be restricted to any subset of δ. To be more precise, if $X \subseteq \delta$, then $\mathbb{P}{\restriction}X$ is equivalent to the fsi $\langle \mathbb{P}{\restriction}(X \cap \alpha), \dot{\mathbb{Q}}_\alpha^{X \cap \alpha} \rangle_{\alpha \in X}$ that is a complete suborder of $\mathbb{P}{\restriction}\delta$. Recall that, for any $\alpha < \delta$, $\hat{\mathcal{I}}_\alpha = \mathcal{P}(\alpha)$, so the generic object added at stage α is generic over the intermediate extension by $\mathbb{P}{\restriction}X$ for any $X \subseteq \alpha$.

Of course, the proof of Theorem 2.4.1 is much simpler for this template, for it is enough to have the conditions in (1) and prove, by induction on $\alpha \leq \delta$, that $\mathbb{P}{\restriction}X$ is defined for any $X \subseteq \alpha$ and that properties (a)-(g) hold (notice that, in the case of the proof, by recursion on $\alpha \leq \delta$, $\mathbb{P}{\restriction}X$ is defined for all X of order type α).

2.4.3 Example. Let $L = L_S \cup L_C$ be a disjoint union. For $x \in L$ define the orders $\dot{\mathbb{Q}}_x^B$ for $B \in \hat{\mathcal{I}}_x$ according to one of the following cases.

(i) If $x \in L_S$, $\dot{\mathbb{Q}}_x^B$ is a $\mathbb{P}{\restriction}B$-name for $\mathbb{S}_x^{V^{\mathbb{P}{\restriction}B}}$, where $\mathbb{S}_x$ is a fixed Suslin correctness-preserving ccc poset coded in the ground model.

(ii) If $x \in L_C$, for a fixed $C_x \in \hat{\mathcal{I}}_x$ and a $\mathbb{P}{\restriction}C_x$-name $\dot{\mathbb{Q}}_x$ for a poset given by reals,

$$\dot{\mathbb{Q}}_x^B = \begin{cases} \dot{\mathbb{Q}}_x & \text{if } C_x \subseteq B \\ \mathbb{1} & \text{otherwise.} \end{cases}$$

It is a straightforward calculation to see that the properties stated in (1) of Theorem 2.4.1 hold, so the template iteration can be defined.

The following result is about complete embeddability between two template iterations. Although it is stated in a general way, Corollary 2.4.8 presents a particular case corresponding to what we need for our applications.

2.4.4 Theorem (Complete embeddability of template iterations)**.** *Let L be a linear order, $\bar{\mathcal{I}}$ and $\bar{\mathcal{J}}$ templates on L such that $\mathcal{I}_x \subseteq \mathcal{J}_x$ for all $x \in L$. Consider two template iterations $\mathbb{P}{\restriction}\langle L, \bar{\mathcal{I}}\rangle$ and $\check{\mathbb{P}}{\restriction}\langle L, \bar{\mathcal{J}}\rangle$ such that the following conditions hold.*

(1) For $x \in L$ and $B \in \hat{\mathcal{I}}_x$, if $\mathbb{P}{\restriction}B$ is a complete suborder of $\check{\mathbb{P}}{\restriction}B$, then $\Vdash_{\check{\mathbb{P}}{\restriction}B} \dot{\mathbb{Q}}_x^B \lessdot_{V^{\mathbb{P}{\restriction}B}} \dot{\check{\mathbb{Q}}}_x^B$.

(2) Whenever $B \in \hat{\mathcal{I}}_x$, $A \subseteq B$ and $\langle \mathbb{P}{\restriction}A, \check{\mathbb{P}}{\restriction}A, \mathbb{P}{\restriction}B, \check{\mathbb{P}}{\restriction}B\rangle$ is a correct system, then the system $\langle \mathbb{P}{\restriction}A * \dot{\mathbb{Q}}_x^A, \check{\mathbb{P}}{\restriction}A * \dot{\check{\mathbb{Q}}}_x^A, \mathbb{P}{\restriction}B * \dot{\mathbb{Q}}_x^B, \check{\mathbb{P}}{\restriction}B * \dot{\check{\mathbb{Q}}}_x^B\rangle$ is correct.

(3) For $B \subseteq L$, $x \in B$, if $C \in \mathcal{J}_x{\restriction}B$ and $p \in \check{\mathbb{P}}{\restriction}C$, then there exists an $A \in \mathcal{I}_x{\restriction}B$ such that $p \in \check{\mathbb{P}}{\restriction}A$.

(4) For $B \subseteq L$, $x \in B$, if $C \in \mathcal{J}_x{\restriction}B$ and $\dot{q}$ is a $\check{\mathbb{P}}{\restriction}C$-name for a condition in $\dot{\check{\mathbb{Q}}}_x^C$, then there exists an $A \in \mathcal{I}_x{\restriction}B$ such that $\dot{q}$ is a $\check{\mathbb{P}}{\restriction}A$-name for a condition in $\dot{\check{\mathbb{Q}}}_x^A$.

Then, the following hold for each $B \subseteq L$.

(a) $\mathbb{P}{\restriction}B$ is a complete suborder of $\check{\mathbb{P}}{\restriction}B$.

(b) If $A \subseteq B$, then $\langle \mathbb{P}{\restriction}A, \check{\mathbb{P}}{\restriction}A, \mathbb{P}{\restriction}B, \check{\mathbb{P}}{\restriction}B\rangle$ is a correct system.

Proof. We proceed by induction on $\Upsilon^{\bar{\mathcal{I}}}(B)$. The non-trivial case is when $B \neq \varnothing$. According to Theorem 2.4.1, consider the following cases:

(i) *Case $x = \max(B)$ and $B_x = B \cap L_x \in \hat{\mathcal{I}}_x$.* Then, $\mathbb{P}{\restriction}B = \mathbb{P}{\restriction}B_x * \dot{\mathbb{Q}}_x^{B_x}$ and $\check{\mathbb{P}}{\restriction}B = \check{\mathbb{P}}{\restriction}B_x * \dot{\check{\mathbb{Q}}}_x^{B_x}$. Then, by induction hypothesis, (1) and Lemma 2.1.2, $\mathbb{P}{\restriction}B$ is a complete suborder of $\check{\mathbb{P}}{\restriction}B$. This gives (a).
For (b), if $x \in A$, note that $A_x = A \cap L_x \in \hat{\mathcal{I}}_x$. By inductive hypothesis, $\langle \mathbb{P}{\restriction}A_x, \mathbb{P}{\restriction}B_x, \check{\mathbb{P}}{\restriction}A_x, \check{\mathbb{P}}{\restriction}B_x\rangle$ is a correct system, so $\langle \mathbb{P}{\restriction}A, \mathbb{P}{\restriction}B, \check{\mathbb{P}}{\restriction}A, \check{\mathbb{P}}{\restriction}B\rangle$ is a correct system by (2). The conclusion is simpler when $x \notin A$.

(ii) *Case $x = \max(B)$ and $B_x \notin \hat{\mathcal{I}}_x$.* Then, with $\mathcal{B} := \{B' \subseteq B \;/\; B' \cap L_x \in \mathcal{I}_x{\restriction}B\}$, $\mathbb{P}{\restriction}B = \text{limdir}_{B' \in \mathcal{B}} \mathbb{P}{\restriction}B'$. By induction hypothesis and Lemma 2.1.5, it is enough to prove that $\check{\mathbb{P}}{\restriction}B = \text{limdir}_{B' \in \mathcal{B}} \check{\mathbb{P}}{\restriction}B'$ to see that $\mathbb{P}{\restriction}B$ is a complete suborder of $\check{\mathbb{P}}{\restriction}B$. If $p \in \check{\mathbb{P}}{\restriction}B$, then, in the case that $x = \max(\text{dom}(p))$, there exists an $A' \in \mathcal{J}_x{\restriction}B$ such that $p{\restriction}L_x \in \check{\mathbb{P}}{\restriction}A'$ and $p(x)$ is a $\check{\mathbb{P}}{\restriction}A'$-name for a condition in $\dot{\check{\mathbb{Q}}}_x^{A'}$. By (3) and (4), we can find $C \in \mathcal{I}_x{\restriction}B$ such that $p{\restriction}L_x \in \check{\mathbb{P}}{\restriction}C$ and $p(x)$ is a $\check{\mathbb{P}}{\restriction}C$-name for a condition in $\dot{\check{\mathbb{Q}}}_x^C$, so $p \in \check{\mathbb{P}}{\restriction}(C \cup \{x\})$ with $C \cup \{x\} \in \mathcal{B}$. The case $\max(\text{dom}(p)) < x$ is treated in a similar way.
For (b), let $A \subseteq B$ and $p \in \mathbb{P}{\restriction}A$ which is a reduction of $q \in \mathbb{P}{\restriction}B$ and prove that p is a reduction of q with respect to the posets $\check{\mathbb{P}}{\restriction}A$ and $\check{\mathbb{P}}{\restriction}B$. Find $B' \in \mathcal{B}$ such that $p, q \in \mathbb{P}{\restriction}B'$. Put $A' = A \cap B'$, so $p \in \mathbb{P}{\restriction}A'$. It is easy to notice that p is a reduction of q with respect to the posets $\mathbb{P}{\restriction}A'$ and $\mathbb{P}{\restriction}B'$ so, by induction hypothesis, p is a reduction of q with respect to the posets $\check{\mathbb{P}}{\restriction}A'$ and $\check{\mathbb{P}}{\restriction}B'$. As $\langle \check{\mathbb{P}}{\restriction}A', \check{\mathbb{P}}{\restriction}A, \check{\mathbb{P}}{\restriction}B', \check{\mathbb{P}}{\restriction}B\rangle$ is a correct system, our claim is proved.

(iii) *Case B does not have a maximum element.* Then, $\mathbb{P}\restriction B = \mathrm{limdir}_{B'\in\mathcal{B}'}\mathbb{P}\restriction B'$ where $\mathcal{B}' := \{B'\subseteq B \,/\, \exists_{x\in B}(B'\in\mathcal{I}_x\restriction B)\}$. Like in the previous case, (3) and (4) imply $\check{\mathbb{P}}\restriction B = \mathrm{limdir}_{B'\in\mathcal{B}}\check{\mathbb{P}}\restriction B'$. Then, by Lemma 2.1.5, $\mathbb{P}\restriction B$ is a complete suborder of $\check{\mathbb{P}}\restriction B$. The argument for (b) is very similar to the one of the previous case.

□

For our applications, we are interested in template iterations that produce ccc posets. The following result presents some conditions for this.

2.4.5 Lemma (Ccc-ness of template iterations)**.** *Consider an indexed template $\langle L,\overline{\mathcal{I}}\rangle$ and $\mathbb{P}\restriction L$ a corresponding template iteration such that the following conditions hold.*

(i) For any $x\in L$ and $B\in\hat{\mathcal{I}}_x$ there are $\mathbb{P}\restriction B$-names $\langle\dot{Q}^B_{n,x}\rangle_{n<\omega}$ witnessing that $\dot{\mathbb{Q}}^B_x$ is σ-linked and

(ii) if $D\subseteq B$ then $\Vdash_{\mathbb{P}\restriction B}\dot{Q}^D_{n,x}\subseteq\dot{Q}^B_{n,x}$ for all $n<\omega$.

Then, $\mathbb{P}\restriction L$ has the Knaster condition.

Proof. The idea is the same as the proof of [Br05, Lemma 2.3]. By induction on $\Upsilon(A)$ it is easy to prove that any $p\in\mathbb{P}\restriction A$ can be extended to a $q\in\mathbb{P}\restriction A$ such that there is a function $f_q:\mathrm{dom}q\to\omega$ and, for any $x\in\mathrm{dom}q$, there is a $B\in\mathcal{I}_x\restriction A$ such that $q\restriction L_x\in\mathbb{P}\restriction B$ and $q\restriction L_x\Vdash q(x)\in\dot{Q}^B_{f_q(x),x}$.

We prove that, whenever $p,q\in\mathbb{P}\restriction A$ are as above and f_p and f_q are compatible functions, then p and q are compatible conditions. Enumerate $\mathrm{dom}p\cup\mathrm{dom}q=\{x_k \,/\, k<m\}$ in increasing order. Construct conditions r_k and sets B_k for $k\leq m$ such that

- $B_k\in\mathcal{I}_{x_{k+1}}\restriction B_{k+1}$ for $k<m$, $B_m=A$,
- $\mathrm{dom}r_k=\{x_j \,/\, j<k\}$ and $p\restriction L_{x_k},q\restriction L_{x_k},r_k\in\mathbb{P}\restriction B_k$, (for $k=m$, $p\restriction L_{x_m}=p$, likewise for q),
- $r_k\leq p\restriction L_{x_k},q\restriction L_{x_k}$ and
- for all $k<m$, $r_{k+1}\restriction L_{x_k}=r_k$, r_k forces, in $\mathbb{P}\restriction B_k$, that $r_{k+1}(x_k)$ extends both $p(x_k)$ and $q(x_k)$ (when they exist). Also, $p\restriction L_{x_k}$ forces $p(x_k)\in\dot{Q}^{B_k}_{f_p(x_k),x_k}$ and $q\restriction L_{x_k}$ forces $q(x_k)\in\dot{Q}^{B_k}_{f_q(x_k),x_k}$

$\langle B_k\rangle_{k\leq m}$ is constructed by regressive recursion on $k\leq m$ such that $p\restriction L_{x_k}$ forces $p(x_k)\in\dot{Q}^{B_k}_{f_p(x_k),x_k}$ and $q\restriction L_{x_k}$ forces $q(x_k)\in\dot{Q}^{B_k}_{f_q(x_k),x_k}$. Construct r_k by recursion on $k\leq m$. Put $r_0=\varnothing$. Assume we have constructed r_k ($k<m$). If $x_k\in\mathrm{dom}p\smallsetminus\mathrm{dom}q$, put $r_{k+1}=r_k{}^\frown\langle p(x_k)\rangle_{x_k}$; if $x_k\in\mathrm{dom}q\smallsetminus\mathrm{dom}p$, put $r_{k+1}=r_k{}^\frown\langle q(x_k)\rangle_{x_k}$; otherwise, if $x_k\in\mathrm{dom}p\cap\mathrm{dom}q$, $p\restriction L_{x_k},q\restriction L_{x_k},r_k\in\mathbb{P}\restriction B_k$, $p\restriction L_{x_k}$ forces $p(x)\in\dot{Q}^{B_k}_{n_k,x_k}$ and $q\restriction L_{x_k}$ forces $q(x)\in\dot{Q}^{B_k}_{n_k,x_k}$ where $n=f_p(x_k)=f_q(x_k)$. As r_k extends both $p\restriction L_{x_k}$ and $q\restriction L_{x_k}$, it forces that $p(x_k)$ and $q(x_k)$ are compatible in $\dot{\mathbb{Q}}^B_{x_k}$, so let $r_{k+1}(x_k)$ be a $\mathbb{P}\restriction B_k$-name of a common extension.

A typical delta-system argument and the previous facts imply that $\mathbb{P}\restriction A$ has the Knaster condition. □

For this lemma, if the template $\langle L,\overline{\mathcal{I}}\rangle$ is as in Example 2.3.4(2), to obtain that $\mathbb{P}\restriction L$ has the ccc conditions (i) and (ii) can be replaced by $\Vdash_{\mathbb{P}\restriction B}$ "$\dot{\mathbb{Q}}^B_x$ has the ccc" for any $x\in L$ and $B\in\hat{\mathcal{I}}_x$. The reason of this, as explained in Example 2.4.2, is that $\mathbb{P}\restriction X$ is a fsi for any $X\subseteq L$.

2.4.6 Corollary ([Br05, Lemmas 2.3, 2.4])**.** *Any template iteration defined as in Example 2.4.3 where $L_C=\varnothing$ and where only Suslin σ-linked correctness-preserving posets are involved satisfies the Knaster condition. Moreover, any condition and any name of a real for this template iteration poset has a support of countable size, that is, if $A\subseteq L$, $p\in\mathbb{P}\restriction A$ and $\dot{x}$ is a $\mathbb{P}\restriction A$-name for a real, then there exists a $C\in[A]^{\leq\omega}$ such that $p\in\mathbb{P}\restriction C$ and $\dot{x}$ is a $\mathbb{P}\restriction C$-name.*

The last assertion of the preceding corollary follows from the next result. In contrast with this, when we consider iterations as in Example 2.4.3 with $L_C\neq\varnothing$, it is not possible to guarantee that the supports of a condition or of a name for a real have countable size.

2.4.7 Lemma (Small support for ccc template iterations)**.** *Fix θ a cardinal with uncountable cofinality. Consider a template iteration defined as in Example 2.4.3 where*

- *for $x \in L_S$, $\mathbb{S}_x$ is a Suslin σ-linked correctness-preserving forcing notion and*
- *for $x \in L_C$, $\dot{\mathbb{Q}}_x$ is a $\mathbb{P}{\restriction}C_x$-name for a σ-linked poset of reals such that each linked component contains the trivial condition, and $|C_x| < \theta$.*

Then, for each $A \subseteq L$, $\mathbb{P}{\restriction}A$ has the Knaster condition and each condition and name of a real for this poset has a support of size $< \theta$ contained in A.

Proof. The Knaster condition follows from Lemma 2.4.5. This proof is based in [Br05, Lemma 2.4]. Proceed by induction on $\Upsilon(A)$, let $p \in \mathbb{P}{\restriction}A$ and $x = \max(\mathrm{dom}p)$. There exists a $B \in \mathcal{I}_x{\restriction}A$ such that $p{\restriction}L_x \in \mathbb{P}{\restriction}B$ and $p(x)$ is a $\mathbb{P}{\restriction}B$-name for a condition in $\dot{\mathbb{Q}}_x^B$. By induction hypothesis, there exists $D \subseteq B$ of size $< \theta$ such that $p \in \mathbb{P}{\restriction}D$ and $p(x)$ is a $\mathbb{P}{\restriction}D$-name for a real. If $x \in L_S$, then clearly $p(x)$ is a name for a condition in $\dot{\mathbb{Q}}_x^D = \dot{\mathbb{S}}_x$, so $p \in \mathbb{P}{\restriction}(D \cup \{x\})$. When $x \in L_C$, if $C_x \not\subseteq B$ then $p(x)$ will be the trivial condition, so that $p \in \mathbb{P}{\restriction}(D \cup \{x\})$. Else, if $C_x \subseteq B$, we may assume $C_x \subseteq D$, so $p(x)$ is a $\mathbb{P}{\restriction}D$-name for a condition in $\dot{\mathbb{Q}}_x^B = \dot{\mathbb{Q}}_x$. Then, $p \in \mathbb{P}{\restriction}(D \cup \{x\})$.

Now, if $\dot{x}$ is a $\mathbb{P}{\restriction}A$-name for a real, note that it can be determined by countably many conditions $\langle r_n \rangle_{n<\omega}$ in $\mathbb{P}{\restriction}A$. As each r_n has a support of size $< \theta$ and θ has uncountable cofinality, we can find $X \subseteq A$ of size $< \theta$ such that $r_n \in \mathbb{P}{\restriction}X$ for all $n < \omega$. This implies that $\dot{x}$ is a $\mathbb{P}{\restriction}X$-name. □

The following is a consequence of Theorem 2.4.4 that fits for the purposes of our applications. Although this type of results was considered originally to get only forcing equivalence, we need to extend to cases where we can get complete embeddability, fact that is needed in order to deal with the limit steps of small cofinality in the proof of Theorem 4.3.1.

2.4.8 Corollary (Complete embeddability of template iterations, particular case)**.** *Let θ be a cardinal with uncountable cofinality, L a linear order, $\bar{\mathcal{I}}$ and $\bar{\mathcal{J}}$ templates on L such that $\langle L, \bar{\mathcal{J}} \rangle$ is a θ-innocuous extension of $\langle L, \bar{\mathcal{I}} \rangle$. Consider two template iterations $\mathbb{P}{\restriction}\langle L, \bar{\mathcal{I}} \rangle$ and $\check{\mathbb{P}}{\restriction}\langle L, \bar{\mathcal{J}} \rangle$ defined with the conditions of Lemma 2.4.7, such that*

(0') The same L_S and L_C are considered for both iterations.

(1') For $x \in L_S$, the same Suslin forcing $\mathbb{S}_x$ is considered for both template iterations.

(2') For $x \in L_C$ either $\check{C}_x = C_x$ and $\dot{\check{\mathbb{Q}}}_x = \dot{\mathbb{Q}}_x$, or $C_x = \varnothing$ and $\dot{\mathbb{Q}}_x$ is the trivial forcing.

Then, the following hold for each $B \subseteq L$.

(a) $\mathbb{P}{\restriction}B$ is a complete suborder of $\check{\mathbb{P}}{\restriction}B$.

(b) If $A \subseteq B$, then $\langle \mathbb{P}{\restriction}A, \check{\mathbb{P}}{\restriction}A, \mathbb{P}{\restriction}B, \check{\mathbb{P}}{\restriction}B \rangle$ is a correct system.

Proof. It is enough to prove conditions (1)-(4) of Theorem 2.4.4.

(1) Straightforward from (0'), (1') and (2').

(2) For $x \in L_S$, the result follows because $\mathbb{S}_x$ is a correctness-preserving Suslin ccc notion. For $x \in L_C$, it is straightforward from (2').

(3) Let $B \subseteq L$, $x \in B$, $C \in \mathcal{J}_x{\restriction}B$ and $p \in \check{\mathbb{P}}{\restriction}C$. By Lemma 2.4.7, there exists $K \subseteq C$ such that $p \in \check{\mathbb{P}}{\restriction}K$ and $|K| < \theta$. Then, by θ-innocuity, there exists $H \in \mathcal{I}_x$ such that $K \subseteq H$, so $K \subseteq A$ and $p \in \check{\mathbb{P}}{\restriction}A$, where $A := B \cap H \in \mathcal{I}_x{\restriction}B$.

(4) Let $B \subseteq L$, $x \in B$, $C \in \mathcal{J}_x{\restriction}B$ and $\dot{q}$ a $\check{\mathbb{P}}{\restriction}C$-name for a condition in $\dot{\check{\mathbb{Q}}}_x^C$. A similar argument as before works with Lemma 2.4.7. It is clear for $x \in L_S$, so assume $x \in L_C$. If $\check{C}_x \subseteq C$, find $K \subseteq C$ such that $\dot{q}$ is a $\check{\mathbb{P}}{\restriction}K$-name for a real, $|K| < \theta$ and $\check{C}_x \subseteq K$. Then, $\dot{q}$ is a $\check{\mathbb{P}}{\restriction}K$-name for a condition in $\dot{\check{\mathbb{Q}}}_x$ so, by θ-innocuity, find an $A \in \mathcal{I}_x{\restriction}B$ containing K, so that $\dot{q}$ is a $\check{\mathbb{P}}{\restriction}A$-name for a condition in $\dot{\check{\mathbb{Q}}}_x$. The case $\check{C}_x \nsubseteq C$ is simpler because $\dot{q}$ is a $\check{\mathbb{P}}{\restriction}C$-name for the trivial condition.

□

We conclude this section with a version of a known result of forcing equivalence for the template iterations of Lemma 2.4.7.

2.4.9 Lemma (Forcing equivalence between template iterations, analog to [Br02, Lemma 1.7])**.** *Assume that $\langle L, \bar{\mathcal{J}}\rangle$ is a θ-innocuous extension of $\langle L, \bar{\mathcal{I}}\rangle$. Consider $\mathbb{P}{\restriction}\langle L, \bar{\mathcal{I}}\rangle$ and $\check{\mathbb{P}}{\restriction}\langle L, \bar{\mathcal{J}}\rangle$ template iterations satisfying the hypothesis in Corollary 2.4.8, but in (2') always assume that $\check{C}_x = C_x$ and $\dot{\check{\mathbb{Q}}}_x = \dot{\mathbb{Q}}_x$. Then, there exists a dense embedding $F : \check{\mathbb{P}}{\restriction}L \to \mathbb{P}{\restriction}L$.*

Proof. Proceed like in the proof of [Br02, Lemma 1.7]. By recursion on $\Upsilon^{\bar{\mathcal{J}}}(B)$, construct $F_B : \check{\mathbb{P}}{\restriction}B \to \mathbb{P}{\restriction}B$ such that

(1) F_B is a dense embedding and

(2) $F_B \subseteq F_{B'}$ whenever $B, B' \in \mathcal{J}$ and $B \subseteq B'$.

Let $p \in \check{\mathbb{P}}{\restriction}B$. If $p = \varnothing$, put $F_B(\varnothing) = \varnothing$, so assume that $p \neq \varnothing$. Let $x := \max(\text{dom}p)$ and find $\bar{B} \in \mathcal{J}_x{\restriction}B$ such that $p{\restriction}L_x \in \check{\mathbb{P}}{\restriction}\bar{B}$ and $p(x)$ is a $\check{\mathbb{P}}{\restriction}\bar{B}$-name for a condition in $\dot{\check{\mathbb{Q}}}_x^{\bar{B}}$. Consider the following cases.

(i) $x \in L_S$. By hypothesis, there exists an $\bar{A} \subseteq \bar{B}$ of size $< \theta$ such that $p{\restriction}L_x \in \check{\mathbb{P}}{\restriction}\bar{A}$ and $p(x)$ is a $\check{\mathbb{P}}{\restriction}\bar{A}$-name for a condition in $\mathbb{S}_x^{V^{\check{\mathbb{P}}{\restriction}\bar{A}}}$. By innocuity, there exists a $\bar{C} \in \mathcal{I}_x{\restriction}B \subseteq \mathcal{J}_x{\restriction}B$ containing $\bar{A}$, so $p{\restriction}L_x \in \check{\mathbb{P}}{\restriction}\bar{C}$ and $p(x)$ is a $\check{\mathbb{P}}{\restriction}\bar{C}$-name for a condition in $\mathbb{S}_x^{V^{\check{\mathbb{P}}{\restriction}\bar{C}}}$. As $\Upsilon^{\bar{\mathcal{J}}}(\bar{C}) < \Upsilon^{\bar{\mathcal{J}}}(B)$, the embedding $F_{\bar{C}}$ has already been defined. So let $F_B(p) := F_{\bar{C}}(p{\restriction}L_x)^\frown\langle p_0(x)\rangle_x$ where $p_0(x)$ is the $\mathbb{P}{\restriction}\bar{C}$-name associated to $p(x)$ with respect to the embedding $F_{\bar{C}}$. Notice that, because of (2), $F_B(p)$ does not depend on the choice of $\bar{C}$.

(ii) $x \in L_C$ *and* $C_x \subseteq \bar{B}$, so $\dot{\check{\mathbb{Q}}}_x^{\bar{B}} = \dot{\mathbb{Q}}_x$. Proceed like before, but take $\bar{A}$ such that $C_x \subseteq \bar{A}$.

(iii) $x \in L_C$ *but* $C_x \nsubseteq \bar{B}$, so $\dot{\check{\mathbb{Q}}}_x^{\bar{B}} = \mathbb{1}$, that is, $p(x)$ is forced to be the trivial condition. Proceed as in[1] (i).

□

[1]Here, $F_B(p) = F_{\bar{B}}(p{\restriction}L_x)$ would be ok, but proceeding as in (i) guarantees that $\text{dom}F_B(p) = \text{dom}p$.

CHAPTER 3

PRESERVATION PROPERTIES

Judah and Shelah [JS90] and Brendle [Br91] developed techniques to get models of cardinal invariants with large continuum from fsi of ccc forcing. They did not only use Suslin ccc notions as in the examples in Section 1.3 to increase the value of certain cardinal invariant in a fsi iteration of ccc posets, but also created some preservation properties to ensure that a certain invariant does not become too big (or too small) in the final extension of such an iteration. These preservation properties can be contextualized in a very general theory, for example, properties in [JS90] and [Br91] are summarized and generalized in [BaJ, Sect. 6.4 and 6.5] and in [G92].

In this chapter, we present this general theory of preservation properties and extend its use to template iterations and matrix iterations. We introduce this theory in Section 3.1 and generalize many known results, specially the preservation on directs limits which is fundamental to extend the applications of preservation properties to template iterations. We present many well-known and less-known examples of such properties in Section 3.2. In Section 3.3, we generalize a property studied in [BlS84] and in [BrF11] to preserve unbounded reals along iterations, which is useful for matrix iterations. Section 3.4 contains the preservation results for template iterations and an interesting preservation result about "adding new reals", proved by the author as a generalization of a similar result for fsi of Suslin ccc posets.

3.1 Preservation of $\sqsubset$-unbounded families

Fix, for this section, an uncountable regular cardinal θ and a cardinal $\lambda \geq \theta$.

3.1.1 Context ([G92],[BaJ, Sect. 6.4]). Fix $\langle \sqsubset_n \rangle_{n<\omega}$ an increasing sequence of 2-place closed relations in ω^ω such that, for any $n<\omega$ and $g \in \omega^\omega$, $(\sqsubset_n)^g = \{f \in \omega^\omega \ / \ f \sqsubset_n g\}$ is (closed) nwd.

For $f, g \in \omega^\omega$, say that g $\sqsubset$-*dominates* f if $f \sqsubset g$. $F \subseteq \omega^\omega$ is a $\sqsubset$-*unbounded family* if no function in ω^ω dominates all the members of F. Associate with this notion the cardinal $\mathfrak{b}_\sqsubset$, which is the least size of a $\sqsubset$-unbounded family. Dually, say that $C \subseteq \omega^\omega$ is a $\sqsubset$-*dominating family* if any real in ω^ω is dominated by some member of C. The cardinal $\mathfrak{d}_\sqsubset$ is the least size of a $\sqsubset$-dominating family. For a set Y and a real $f \in \omega^\omega$, say that f *is* $\sqsubset$-*unbounded over* Y if $\forall_{g \in \omega^\omega \cap Y}(f \not\sqsubset g)$, which we denote by $f \not\sqsubset Y$.

Although this context is defined for ω^ω, the domain and codomain of $\sqsubset$ can be any uncountable Polish space coded by reals in ω^ω.

3.1.2 Lemma. $\mathfrak{b}_\sqsubset \leq \mathrm{non}(\mathcal{M})$ *and* $\mathrm{cov}(\mathcal{M}) \leq \mathfrak{d}_\sqsubset$.

Proof. Immediate from the fact that $(\sqsubset)^g$ is meager for any $g \in \omega^\omega$. □

3.1.3 Definition. Let $F \subseteq \omega^\omega$. Say that F is θ-$\sqsubset$-*unbounded* if, for any $X \subseteq \omega^\omega$ of size $< \theta$, there is an $f \in F$ such that $f \not\sqsubset X$.

Clearly, any θ-$\sqsubset$-unbounded family is $\sqsubset$-unbounded, so

3.1.4 Lemma. *If $F \subseteq \omega^\omega$ is θ-$\sqsubset$-unbounded, then $\mathfrak{b}_\sqsubset \leq |F|$ and $\theta \leq \mathfrak{d}_\sqsubset$.*

The following is a property that expresses when a forcing notion preserves θ-$\sqsubset$-unbounded families of the ground model.

3.1.5 Definition (Judah and Shelah [JS90], [BaJ, Def. 6.6.4]). A forcing notion $\mathbb{P}$ is *θ-$\sqsubset$-good* if the following property holds[1]: For any $\mathbb{P}$-name $\dot{h}$ for a real in ω^ω, there exists a nonempty $Y \subseteq \omega^\omega$ (in the ground model) of size $< \theta$ such that, for any $f \in \omega^\omega$, if $f \not\sqsubset Y$ then $\Vdash f \not\sqsubset \dot{h}$.

Say that $\mathbb{P}$ is *$\sqsubset$-good* if it is $\aleph_1$-$\sqsubset$-good.

3.1.6 Lemma ([BaJ, Lemma 6.4.8]). *Assume that $\mathbb{P}$ is θ-$\sqsubset$-good*[2].

(a) If $F \subseteq \omega^\omega$ is θ-$\sqsubset$-unbounded, then $\mathbb{P}$ forces that F is still θ-$\sqsubset$-unbounded.

(b) If $\mathfrak{d}_\sqsubset \geq \lambda$, then $\mathbb{P}$ forces that $\mathfrak{d}_\sqsubset \geq \lambda$.

Proof. (a) Let $\{\dot{h}_\alpha\}_{\alpha<\delta}$ be a sequence of $\mathbb{P}$-names for reals in ω^ω with $\delta < \theta$ ordinal. For $\alpha < \delta$, let $Y_\alpha \subseteq \omega^\omega$ of size $< \theta$ that witnesses goodness of $\mathbb{P}$ for $\dot{h}_\alpha$. Put $Y = \bigcup_{\alpha<\delta} Y_\alpha$, which has size $< \theta$ because of the regularity of θ. It is clear that $f \in \omega^\omega$ and $f \not\sqsubset Y$ imply $\Vdash f \not\sqsubset \dot{h}_\alpha$ for any $\alpha < \delta$.

(b) Let $\{\dot{g}_\alpha\}_{\alpha<\gamma}$ be a sequence of $\mathbb{P}$-names for reals in ω^ω with $\gamma < \lambda$ ordinal. For $\alpha < \gamma$, let $Z_\alpha \subseteq \omega^\omega$ of size $< \theta$ that witnesses goodness of $\mathbb{P}$ for $\dot{g}_\alpha$. Put $Z = \bigcup_{\alpha<\gamma} Z_\alpha$, which has size $< \lambda$. Therefore, Y is not $\sqsubset$-dominating, so there is some $f \in \omega^\omega$ such that $f \not\sqsubset Y$. Then, $\Vdash f \not\sqsubset \dot{g}_\alpha$ for any $\alpha < \gamma$, that is, $\mathbb{P}$ forces that $\{\dot{g}_\alpha\}_{\alpha<\gamma}$ is not $\sqsubset$-dominating.

□

Note that $\theta < \theta'$ implies that any θ-$\sqsubset$-good poset is θ'-$\sqsubset$-good. Also, if $\mathbb{P} \lessdot \mathbb{Q}$ and $\mathbb{Q}$ is θ-$\sqsubset$-good, then $\mathbb{P}$ is θ-$\sqsubset$-good. The following result shows that small forcing notions are good.

3.1.7 Lemma. *Any poset of size $< \theta$ is θ-$\sqsubset$-good. In particular, $\mathbb{C}$ is $\sqsubset$-good.*

Proof. Put $\mathbb{P} = \{p_\alpha \mathbin{/} \alpha < \mu\}$ where $\mu := |\mathbb{P}| < \kappa$. Let $\dot{h}$ be a $\mathbb{P}$-name for a real in ω^ω. For each $\alpha < \mu$, choose $\langle q^\alpha_n\rangle_{n<\omega}$ a decreasing sequence in $\mathbb{P}$ and $h_\alpha \in \omega^\omega$ such that $q^\alpha_0 = p_\alpha$ and, for every $n < \omega$, $q^\alpha_n \Vdash \dot{h}{\restriction}n = h_\alpha{\restriction}n$. It suffices to prove that, if $f \in \omega^\omega$ and $\forall_{\alpha<\mu}(f \not\sqsubset h_\alpha)$ then $\Vdash f \not\sqsubset \dot{h}$, that is, $\forall_{p\in\mathbb{P}}\forall_{m<\omega}\exists_{q\leq p}(q \Vdash f \not\sqsubset_m \dot{h})$. Fix $p \in \mathbb{P}$ and $m < \omega$, so there exists an $\alpha < \mu$ such that $p = p_\alpha$. As $f \not\sqsubset h_\alpha$ and $(\sqsubset_m)_f := \{g \in \omega^\omega \mathbin{/} f \sqsubset_m g\}$ is closed, there exists $n < \omega$ such that $[h_\alpha{\restriction}n] \cap (\sqsubset_m)_f = \varnothing$, so $q^\alpha_n \Vdash [\dot{h}{\restriction}n] \cap (\sqsubset_m)_f = \varnothing$. Therefore, $q^\alpha_n \Vdash f \not\sqsubset_m \dot{h}$ with $q^\alpha_n \leq p_\alpha = p$. □

Judah and Shelah [JS90] proved that θ-$\sqsubset$-goodness is preserved in fsi of θ-$\sqsubset$-good posets. We generalize the preservation in the limits steps in Theorem 3.1.9.

3.1.8 Lemma ([BaJ, Lemma 6.4.11]). *Let $\mathbb{P}$ be a poset and $\dot{\mathbb{Q}}$ a $\mathbb{P}$-name for a poset. If $\mathbb{P}$ is θ-cc, θ-$\sqsubset$-good and $\mathbb{P}$ forces that $\dot{\mathbb{Q}}$ is θ-$\sqsubset$-good, then $\mathbb{P} * \dot{\mathbb{Q}}$ is θ-$\sqsubset$-good.*

Proof. Let $\dot{h}$ be a $\mathbb{P} * \dot{\mathbb{Q}}$-name for a real in ω^ω. As $\mathbb{P}$ forces that $\dot{h}$ is a $\dot{\mathbb{Q}}$-name for a real, by θ-cc and the regularity of θ, there is a $\mathbb{P}$-name $\dot{Z} = \{\dot{g}_\alpha\}_{\alpha<\nu}$ for a set of reals and some $\nu < \theta$ such that $\mathbb{P}$ forces that $\dot{Z}$ witnesses goodness of $\dot{\mathbb{Q}}$ for $\dot{h}$. Now, let Y_α be a witness of goodness of $\mathbb{P}$ for $\dot{g}_\alpha$, so $Y = \bigcup_{\alpha<\nu} Y_\alpha$ has size $< \theta$ and witnesses goodness of $\mathbb{P} * \dot{\mathbb{Q}}$ for $\dot{h}$. □

3.1.9 Theorem (Preservation of goodness in short direct limits). *Let I be a directed partial order, $\langle\mathbb{P}_i\rangle_{i\in I}$ a directed system and $\mathbb{P} = \mathrm{limdir}_{i\in I}\mathbb{P}_i$. If $|I| < \theta$ and $\mathbb{P}_i$ is θ-$\sqsubset$-good for any $i \in I$, then $\mathbb{P}$ is θ-$\sqsubset$-good.*

[1] According to [BaJ, Def. 6.6.4], our property is called *really θ-$\sqsubset$-good* while θ-$\sqsubset$-good stands for another property. However, [BaJ, Lemma 6.6.5] states that really θ-$\sqsubset$-good implies θ-$\sqsubset$-good, and it is also easy to see that the converse is true for θ-cc posets, see details in [Me13a, Lemma 2].

[2] Note that θ or λ may not be cardinals in some $\mathbb{P}$-extension.

Proof. Let $\dot{h}$ be a $\mathbb{P}$-name for a real in ω^ω. For $i \in I$, find a $\mathbb{P}_i$-name for a real $\dot{h}_i$ and a sequence $\{\dot{p}^i_m\}_{m<\omega}$ of $\mathbb{P}_i$-names that represents a decreasing sequence of conditions in $\mathbb{P}/\mathbb{P}_i$ such that $\mathbb{P}_i$ forces that $\dot{p}^i_m \Vdash_{\mathbb{P}/\mathbb{P}_i} \dot{h}{\restriction}m = \dot{h}_i{\restriction}m$. For each $i \in I$ choose $Y_i \subseteq \omega^\omega$ of size $< \theta$ that witnesses goodness of $\mathbb{P}_i$ for $\dot{h}_i$. As $|I| < \theta$, $Y = \bigcup_{i\in I} Y_i$ has size $< \theta$ by regularity of θ.

We prove that Y witnesses goodness of $\mathbb{P}$ for $\dot{h}$. Assume, towards a contradiction, that $f \in \omega$, $f \not\sqsubset Y$ and that there are $p \in \mathbb{P}$ and $n < \omega$ such that $p \Vdash_{\mathbb{P}} f \sqsubset_n \dot{h}$. Choose $i \in I$ such that $p \in \mathbb{P}_i$. Let G be $\mathbb{P}_i$-generic over the ground model V with $p \in G$. Then, by the choice of Y_i, $f \not\sqsubset h_i$, in particular, $f \not\sqsubset_n h_i$. As $C := (\sqsubset_n)_f = \{g \in \omega^\omega \,/\, f \sqsubset_n g\}$ is closed, there is an $m < \omega$ such that $[h_i{\restriction}m] \cap C = \varnothing$. Thus, $p^i_m \Vdash_{\mathbb{P}/\mathbb{P}_i} [\dot{h}\restriction m] \cap C = \varnothing$, that is, $p^i_m \Vdash_{\mathbb{P}/\mathbb{P}_i} f \not\sqsubset_n \dot{h}$. On the other hand, by hypothesis, $\Vdash_{\mathbb{P}/\mathbb{P}_i} f \sqsubset_n \dot{h}$, a contradiction. □

3.1.10 Corollary (Preservation of goodness in well ordered direct limits)**.** *Let δ be a limit ordinal and $\{\mathbb{P}_\alpha\}_{\alpha<\delta}$ be a sequence of posets such that, for $\alpha < \beta < \delta$, $\mathbb{P}_\alpha$ is a complete suborder of $\mathbb{P}_\beta$. If $\mathbb{P}_\delta = \mathrm{limdir}_{\alpha<\delta}\mathbb{P}_\alpha$ is θ-cc and $\mathbb{P}_\alpha$ is θ-$\sqsubset$-good for any $\alpha < \delta$, then $\mathbb{P}_\delta$ is θ-$\sqsubset$-good.*

Proof. First assume that $\mathrm{cf}(\delta) < \theta$, so there is an increasing sequence $\{\alpha_\xi\}_{\xi<\mathrm{cf}(\delta)}$ that converges to δ. Then, $\mathbb{P}_\delta = \mathrm{limdir}_{\xi<\mathrm{cf}(\delta)}\mathbb{P}_{\alpha_\xi}$, which implies that $\mathbb{P}_\delta$ is θ-$\sqsubset$-good by Theorem 3.1.9.

Now, assume that $\mathrm{cf}(\delta) \geq \theta$. Let $\dot{h}$ be a $\mathbb{P}_\delta$-name for a real. By θ-cc, there is an $\alpha < \theta$ such that $\dot{h}$ is a $\mathbb{P}_\alpha$-name. Then, by hypothesis, there is $Y \subseteq \omega^\omega$ of size $< \theta$ that witnesses goodness of $\mathbb{P}_\alpha$ for $\dot{h}$. It is clear that Y also witnesses goodness of $\mathbb{P}_\delta$. □

3.1.11 Corollary (Preservation of goodness in fsi [BaJ, Lemma 6.4.12])**.** *Let $\mathbb{P}_\delta = \langle \mathbb{P}_\alpha, \dot{\mathbb{Q}}_\alpha \rangle_{\alpha<\delta}$ be a fsi of θ-cc forcing notions. If, for each $\alpha < \delta$, $\mathbb{P}_\alpha$ forces that $\dot{\mathbb{Q}}_\alpha$ is θ-$\sqsubset$-good, then $\mathbb{P}_\delta$ is θ-$\sqsubset$-good.*

Proof. Prove by induction on $\alpha \leq \delta$ that $\mathbb{P}_\alpha$ is θ-$\sqsubset$-good. Step $\alpha = 0$ follows from Lemma 3.1.7, successor step from Lemma 3.1.8 and the limit step is a direct consequence of Corollary 3.1.10. □

The following results show how to add $\sqsubset$-unbounded families with Cohen reals, in order to get values for $\mathfrak{b}_\sqsubset$ and $\mathfrak{d}_\sqsubset$. These are essential to prove the main results of this thesis.

3.1.12 Lemma. *Let ν be an uncountable regular cardinal, $\langle \mathbb{P}_\alpha \rangle_{\alpha<\nu}$ a $\lessdot$-increasing sequence of forcing notions and $\mathbb{P}_\nu = \mathrm{limdir}_{\alpha<\nu}\mathbb{P}_\alpha$. If*

(i) for each $\alpha < \nu$, $\mathbb{P}_{\alpha+1}$ adds a Cohen real over $V^{\mathbb{P}_\alpha}$, and

(ii) $\mathbb{P}_\nu$ is ccc,

then, $\mathbb{P}_\nu$ adds a ν-$\sqsubset$-unbounded family (of Cohen reals) of size ν. Moreover, it forces $\mathfrak{b}_\sqsubset \leq \nu$ and $\nu \leq \mathfrak{d}_\sqsubset$.

Proof. Let $\dot{c}_\alpha$ be a $\mathbb{P}_{\alpha+1}$-name of a Cohen real over $V^{\mathbb{P}_\alpha}$. Then, $\mathbb{P}_\nu$ forces that $\{\dot{c}_\alpha \,/\, \alpha < \nu\}$ is a ν-$\sqsubset$-unbounded family. Indeed, if $\{\dot{x}_\xi\}_{\xi<\mu}$ is a sequence of $\mathbb{P}_\nu$-names for reals with $\mu < \nu$, by (ii) there is an $\alpha < \nu$ such that $\{\dot{x}_\xi\}_{\xi<\mu}$ is a sequence of $\mathbb{P}_\alpha$-names, so $\mathbb{P}_{\alpha+1}$ forces that $\dot{c}_\alpha \not\sqsubset \dot{x}_\xi$ for all $\xi < \mu$. This last assertion holds because $(\sqsubset)^g$ is an F_σ meager set for any $g \in \omega^\omega$ (see Context 3.1.1).

The second statement is a consequence of Lemma 3.1.4. □

3.1.13 Lemma. *Let $\delta \geq \theta$ be an ordinal and $\mathbb{P}_\delta = \langle \mathbb{P}_\alpha, \dot{\mathbb{Q}}_\alpha \rangle_{\alpha<\delta}$ be a fsi such that,*

i) for $\alpha < \theta$, $\dot{\mathbb{Q}}_\alpha$ is forced (by $\mathbb{P}_\alpha$) to be ccc and to have two incompatible conditions, and

ii) for $\theta \leq \alpha < \delta$, $\dot{\mathbb{Q}}_\alpha$ is forced to be θ-cc and θ-$\sqsubset$-good.

Then,

(a) $\mathbb{P}_\theta$ adds a θ-$\sqsubset$-unbounded family (of Cohen reals) of size θ.

(b) *The family added in (a) is forced to be a θ-$\sqsubset$-unbounded family by $\mathbb{P}_\delta$. In particular, it forces that* $\mathfrak{b}_\sqsubset \leq \theta \leq \mathfrak{d}_\sqsubset$.

Proof. (a) This is a direct consequence of Lemmas 1.3.5 and 3.1.12.

(b) Let $\dot{C}$ be a $\mathbb{P}_\theta$-name for a family of reals as in (a). Step in V_θ. Note that $\mathbb{P}_\delta/\mathbb{P}_\theta$ is equivalent to the fsi $\langle \mathbb{P}_\alpha/\mathbb{P}_\theta, \dot{\mathbb{Q}}_\alpha \rangle_{\theta \leq \alpha < \delta}$. Thus, by Corollary 3.1.11, $\mathbb{P}_\delta/\mathbb{P}_\theta$ is θ-$\sqsubset$-good. Hence, by Lemma 3.1.6, it forces that C is θ-$\sqsubset$-unbounded.

□

3.2 Examples of preservation properties

We present examples of relations of Context 3.1.1 that are useful for the proofs of the main result. Also, we discuss the cardinal invariants given by those relations and the type of forcing notions that are good for them.

3.2.1 Example (Preserving non-meager sets)**.** Let $n < \omega$. For $f, g \in \omega^\omega$, define $f \neq_n g \Leftrightarrow \forall_{k \geq n}(f(k) \neq g(k))$, so $f \neq g \Leftrightarrow \forall^\infty_{k<\omega}(f(k) \neq g(k))$. From Theorem 1.4.3 it is clear that $\mathfrak{b}_{\neq} = \text{non}(\mathcal{M})$ and $\mathfrak{d}_{\neq} = \text{cov}(\mathcal{M})$.

Every forcing notion that adds an eventually different real is not $\text{cov}(\mathcal{M})$-$\neq$-good, so this applies to $\mathbb{LOC}^h$, $\mathbb{A}$ $\mathbb{B}$, $\mathbb{D}$, $\mathbb{E}$ and $\mathbb{L}_\mathcal{F}$ with a filter base $\mathcal{F}$.

3.2.2 Example (Preserving unbounded families)**.** For $n < \omega$ and $f, g \in \omega^\omega$, $f \leq^*_n g$ denotes $\forall_{k \geq n}(f(k) \leq g(k))$, so $\leq^* = \bigcup_{n<\omega} \leq^*_n$. Clearly, $\mathfrak{b}_{\leq^*} = \mathfrak{b}$ and $\mathfrak{d}_{\leq^*} = \mathfrak{d}$.

Any $\mathfrak{b}$-$\leq^*$-good poset is $\leq^*$-good, moreover, $\mathbb{P}$ is $\mathfrak{b}$-$\leq^*$-good iff for any $\dot{h}$ $\mathbb{P}$-name for a real, there is a $g \in \omega^\omega$ such that, for any real $f \not\leq^* g$, $\Vdash f \not\leq^* \dot{h}$.

3.2.3 Lemma. *If $\mathbb{P}$ is θ-cc and ω^ω-bounding, then $\mathbb{P}$ is θ-$\leq^*$-good. In particular, random forcing $\mathbb{B}$ is $\leq^*$-good.*

Proof. Let $\dot{h}$ be a $\mathbb{P}$-name for a real in ω^ω. By Definition 1.3.8, there are a maximal antichain $A \subseteq \mathbb{P}$ and $Y = \{g_q \,/\, q \in A\}$ such that $q \Vdash \dot{h} \leq g_q$ for each $q \in A$. Assume $f \in \omega^\omega$ and $f \not\leq^* Y$. If $p \in \mathbb{P}$, find $q \in A$ compatible with p, so $f \not\leq^* g_q$. This implies $q \Vdash f \not\leq^* \dot{h}$, so any common extension of p and q forces $f \not\leq^* \dot{h}$. □

3.2.4 Lemma (Miller [Mi81])**.** $\mathbb{E}$ *is $\leq^*$-good.*

Proof. Let $\dot{g}$ be a $\mathbb{E}$-name for a real in ω^ω. We want find $h \in \omega^\omega$ such that, for any $f \in \omega^\omega$ that is not dominated by h, $\Vdash f \not\leq^* \dot{g}$. For $s \in \omega^{<\omega}$ and $n < \omega$, define $h_{s,n} : \omega \to \omega + 1$ such that $h_{s,n}(i)$ is the minimal $j \leq \omega$ such that, for any $F \subseteq \omega^\omega$ of size n, $(s, F) \nVdash \dot{g}(i) > j$.

3.2.5 Claim. $h_{s,n} \in \omega^\omega$.

Proof. Suppose that there is an $i < \omega$ such that $h_{s,n}(i) = \omega$, that is, for any $j < \omega$ there is an $F_j \subseteq \omega^\omega$ of size n such that $(s, F_j) \Vdash \dot{g}(i) > j$. Put $F_j = \{f^l_j \,/\, l < n\}$. Find $D \subseteq \omega$ infinite with $D = \{m_j \,/\, j < \omega\}$ increasing enumeration and, for $l < n$ construct $A^l \subseteq D$ infinite and $f^l : A^l \to \omega$ such that, for any $k < \omega$,

- if $m_k \in A^l$, then $f^l_{m_j}(k) = f^l(m_k)$ for any $j > k$;
- if $m_k \in D \smallsetminus A^l$, then $f^l_{m_j}(k) > j$ for any $j > k$.

The construction is done by recursion. Put $D^{-1}=\omega$. Let $k<\omega$ and assume that we have $D^{k-1}\subseteq\omega$ infinite and that $d_k:=\{m_j\,/\,j<k\}$, $A^l\cap d_k$ and $f^l{\restriction}(A^l\cap d_k)$ have been defined in such a way that $D^{k-1}\cap\sup(d_k+1)=\varnothing$. Put $m_k=\min D^{k-1}$ and $I^{-1}=D^{k-1}\smallsetminus\{m_k\}$. For $l<n$ assume that we have $I^{l-1}\subseteq D^{k-1}\smallsetminus\{m_k\}$ infinite. If the set $\{f^l_j(k)\,/\,j\in I^{l-1}\}$ is finite, put $A^l\cap(m_k+1)=(A^l\cap d_k)\cup\{m_k\}$ and choose $f^l(m_k)\in\omega$ and $I^l\subseteq I^{l-1}$ infinite such that $f^l_j(k)=f^l(m_k)$ for any $j\in I^l$; otherwise, put $A^l\cap(m_k+1)=A^l\cap d_k$ and choose $I^l\subseteq I^{l-1}$ infinite such that the sequence $\{f^l_j(k)\}_{j\in I^l}$ is strictly increasing and above $k+1$. At the end, Put $D^k=I^{n-1}$.

For each $l<n$ choose some $g^l\in\omega^\omega$ such that $g^l(k)=f^l(m_k)$ for any $k<\omega$ such that $m_k\in A^l$. Put $F':=\{g^l\,/\,l<n\}$ and find $(t,F'')\leq(s,F')$ in $\mathbb{E}$ and $j_0<\omega$ such that $(t,F'')\Vdash\dot{g}(i)=j_0$. Choose $j\geq j_0$ such that j is above $\{t(k)\,/\,k<|t|\}\cup\{|t|\}$, so $(t,F_{m_j}\cup F'')$ is a common extension of (t,F'') and (s,F_{m_j}). Indeed, assume $k\in|t|\smallsetminus|s|$, $l<k$ and argue from these two cases: if $m_k\in A^l$ then $f^l_{m_j}(k)=g^l(k)\neq t(k)$ because $(t,F'')\leq(s,F')$; if $m_k\notin A^l$, then $f^l_{m_j}(k)>j>t(k)$.

Therefore, $(t,F_{m_j}\cup F'')$ forces that $j_0=\dot{g}(i)>m_j\geq j$, a contradiction. □

Now, find $h\in\omega^\omega$ that dominates $\{h_{s,n}\,/\,s\in\omega^{<\omega},\ n<\omega\}$. Assume that $f\in\omega^\omega$ is not dominated by h. Fix $(s,F)\in\mathbb{E}$ and $i_0<\omega$. Then, there is an $i>i_0$ such that $f(i)>h_{s,n}(i)$ where $n:=|F|$. As $(s,F)\not\Vdash\dot{g}(i)>h_{s,n}(i)$, there is $(t,F'')\leq(s,F)$ that forces $\dot{g}(i)\leq h_{s,n}(i)<f(i)$. □

Note that any forcing that adds a dominating real is not $\mathfrak{d}$-$\leq^*$-good, so this applies to $\mathbb{D}$, $\mathbb{LOC}^h$ and $\mathbb{L}_{\mathcal{F}}$ for any filter base $\mathcal{F}$.

3.2.6 Example (Preserving splitting families)**.** For $n<\omega$ and $A,B\in[\omega]^\omega$, define $A\propto_n B\Leftrightarrow(B\smallsetminus n\subseteq A$ or $B\smallsetminus n\subseteq\omega\smallsetminus A)$, so $A\propto B$ iff either $B\subseteq^* A$ or $B\subseteq^*\omega\smallsetminus A$. Note that $A\not\propto B$ iff A splits B. Clearly, $\mathfrak{b}_\propto=\mathfrak{s}$ and $\mathfrak{d}_\propto=\mathfrak{r}$.

3.2.7 Lemma (Baumgartner and Dordal [BD85], see also [Br09, Lemma 3.8])**.** $\mathbb{D}$ *is* $\propto$*-good.*

3.2.8 Lemma. $\mathbb{B}$ *is not* $\propto$*-good. Moreover, any forcing that adds a random real is not* $\propto$*-good.*

Proof. Consider $\mathbb{B}=\mathcal{B}(2^\omega)\smallsetminus\mathcal{N}$ and let $\langle I_n\rangle_{n<\omega}$ be an interval partition of ω such that $|I_n|=2^n$ for each $n<\omega$. Enumerate $I_n=\{k_t\,/\,t\in 2^n\}$. Construct a $\mathbb{B}$-name of a subset of ω such that, for each $n<\omega$ and $t\in 2^n$, $[t]\Vdash k_t\in\dot{x}$ and $[r]\Vdash k_t\notin\dot{x}$ for any $r\in 2^n\smallsetminus\{t\}$. Note that $\mathbb{B}$ forces that $\dot{x}$ is an infinite set.

Now, if $\{z_n\}_{n<\omega}$ is a sequence of infinite subsets of ω, then we can construct an infinite set $a\subseteq\omega$ such that, for any $n<\omega$, a splits z_n and $|a\cap I_n|\leq 1$, moreover, $a\cap I_0=a\cap I_1=\varnothing$. Then, for any $B\in\mathbb{B}$ that forces $a\cap I_n\cap\dot{x}\neq\varnothing$ there is a (unique) $t=t^n_B\in 2^n$ such that $k_t\in a$ and $B\subseteq_{\mathcal{N}}[t]$. Indeed, choose $B'\subseteq B$ in $\mathbb{B}$ and $t\in 2^n$ such that $B'\Vdash k_t\in a\cap\dot{x}$, so $k_t\in a$ and $B'\subseteq_{\mathcal{N}}[t]$. As $|a\cap I_n|\leq 1$, B is incompatible with $[r]$ for any $r\in 2^n\smallsetminus\{t\}$, so $B\subseteq_{\mathcal{N}}[t]$.

We claim that any B that forces $a\cap\dot{x}\neq\varnothing$ has Lebesgue measure $\leq\frac{1}{2}$. Let $\mathcal{A}$ be a maximal antichain below B such that, for any $A\in\mathcal{A}$ there is an $n\in[2,\omega)$ such that $A\Vdash a\cap I_n\cap\dot{x}\neq\varnothing$ an let n_A be the minimal such n. For $2\leq m<\omega$, let $\mathcal{A}_m=\{A\in\mathcal{A}\,/\,n_A=m\}$ (this may be empty) and $B_m=\bigcup\mathcal{A}_m$. $\{B_m\,/\,m<\omega\}\smallsetminus\{\varnothing\}$ is still a maximal antichain below B and, for $B_m\neq\varnothing$ there is a $t^m\in 2^m$ such that $B_m\subseteq_{\mathcal{N}}[t^m]$. This last statement holds because, for any $A,A'\in\mathcal{A}_m$, as $t^m_A,t^m_{A'}\in a\cap I_m$, $t^m_A=t^m_{A'}$. Therefore, $\lambda(B)=\sum_{m=2}^\infty\lambda(B_m)\leq\sum_{m=2}^\infty\lambda([t^m])\leq\sum_{m=2}^\infty 2^{-m}=\frac{1}{2}$.

Moreover, $\Vdash|a\cap\dot{x}|<\aleph_0$. This is because, by a similar argument as before, if $T\subseteq 2^{<\omega}$ is a tree such that $\lambda([T]\cap[t])>0$ for any $t\in T$, any $B\subseteq[T]$ in $\mathbb{B}$ that forces $\exists_{n\geq|\mathrm{stem}(T)|+2}(a\cap I_n\cap\dot{x}\neq\varnothing)$ has measure $\leq\frac{1}{2}\lambda([T])$. □

If $\mathcal{U}$ is an ultrafilter on ω, then both $\mathbb{M}_{\mathcal{U}}$ and $\mathbb{L}_{\mathcal{U}}$ are not $\propto$-good. This is because both posets add an infinite subset of ω that cannot be splitted by any infinite subset of ω of the ground model. We do not know whether $\mathbb{E}$ is $\propto$-good.

3.2.9 Example (Preserving finitely splitting families)**.** For $a\in[\omega]^\omega$ and an interval partition $\bar{J}=\langle J_n\rangle_{n<\omega}$ of ω, define $a\rhd_n\bar{J}\Leftrightarrow(\forall_{k\geq n}(J_k\not\subseteq a)$ or $\forall_{k\geq n}(J_k\not\subseteq\omega\smallsetminus a))$, so $a\rhd\bar{J}\Leftrightarrow(\forall^\infty_{k<\omega}(J_k\not\subseteq$

a) or $\forall^\infty_{k<\omega}(J_k \not\subseteq \omega \smallsetminus a)$). $a \not\triangleright \bar{J}$ is read a *splits* $\bar{J}$. It is proved in [KWe96] that $\mathfrak{b}_\triangleright = \max\{\mathfrak{b}, \mathfrak{s}\}$ and $\mathfrak{d}_\triangleright = \min\{\mathfrak{d}, \mathfrak{r}\}$.

3.2.10 Lemma. *Any $\leq^*$-good poset is $\triangleright$-good. In particular, $\mathbb{B}$ and $\mathbb{E}$ are $\triangleright$-good.*

Proof. Let $\dot{\bar{J}}$ be a $\mathbb{P}$-name of an interval partition of ω. By $\leq^*$-goodness, choose $h \in \omega^\omega$ such that $\Vdash \exists^\infty_n(\max(\dot{J}_n) < f(n))$ for any $f \in \omega^\omega$ with $f \not\leq^* h$. Without loss of generality, assume that h is strictly increasing and $h(0) > 0$. Define $\hat{h} \in \omega^\omega$ recursively, where $\hat{h}(0) = 0$ and $\hat{h}(n+1) = h(\hat{h}(n))$. Put $J'_n := [\hat{h}(2n), \hat{h}(2n+2))$ and $\bar{J}' := \langle J'_n \rangle_{n\in\omega}$. It is enough to prove $\Vdash a \not\triangleright \dot{\bar{J}}$ for any $a \in [\omega]^\omega$ such that $a \not\triangleright \bar{J}'$. Indeed, define $f \in \omega^\omega$ such that

$$f(n) = \begin{cases} h(n)+1 & \text{if } n \in [\hat{h}(2k), \hat{h}(2k+1)) \text{ and } J'_k \subseteq a \text{ for some } k < \omega, \\ 0 & \text{otherwise.} \end{cases}$$

It is clear that $f \not\leq^* h$, so $\Vdash \exists^\infty_n(\max(\dot{J}_n) < f(n))$. Now, let G be a $\mathbb{P}$-generic set over the ground model. In $V[G]$: fix $m < \omega$ and choose $n, k' \in \omega$ such that $\hat{h}(k') > m$, $n \in [\hat{h}(k'), \hat{h}(k'+1))$ and $\max(J_n) < f(n)$. As $f(n)$ cannot be 0, then $k' = 2k$ for some $k \in \omega$, $J'_k \subseteq a$ and $f(n) = h(n)+1$. It is easy to check that $J_n \subseteq [n, f(n)) \subseteq J'_k \subseteq a$. This gives us $\exists^\infty_n(J_n \subseteq a)$. To get $\exists^\infty_n(J_n \subseteq \omega \smallsetminus a)$, do the same argument but change a by $\omega \smallsetminus a$ in the definition of f. □

No ccc poset that adds dominating reals is $\triangleright$-good. If there were such a poset and $\kappa = \mathfrak{c}$, a fsi of length κ^+ of this poset would force $\kappa^+ \leq \mathfrak{b} \leq \mathfrak{b}_\triangleright \leq \aleph_1 \leq \kappa$ (the last inequality by Lemma 3.1.13), which is false.

3.2.11 Example (Preserving null covering families)**.** Fix $\bar{I} = \langle I_n \rangle_{n<\omega}$ an interval partition of ω such that $|I_n| \geq n$ for all $n < \omega$. For $n < \omega$ and $f, g \in 2^\omega$, define $f \pitchfork^{\bar{I}}_n g \Leftrightarrow \forall_{k\geq n}(f{\restriction}I_k \neq g{\restriction}I_k)$, so $f \pitchfork^{\bar{I}} g \Leftrightarrow \forall^\infty_{k<\omega}(f{\restriction}I_k \neq g{\restriction}I_k)$. Note that $(\pitchfork^{\bar{I}})^g$ is a co-null F_σ meager set for any $g \in 2^\omega$.

This preservation property is related to cov($\mathcal{N}$) and non($\mathcal{N}$) in the following way.

3.2.12 Lemma. cov($\mathcal{N}$) $\leq \mathfrak{b}_{\pitchfork^{\bar{I}}} \leq$ non($\mathcal{M}$) *and* cov($\mathcal{M}$) $\leq \mathfrak{d}_{\pitchfork^{\bar{I}}} \leq$ non($\mathcal{N}$)

Proof. $\mathfrak{b}_{\pitchfork^{\bar{I}}} \leq$ non($\mathcal{M}$) and cov($\mathcal{M}$) $\leq \mathfrak{d}_{\pitchfork^{\bar{I}}}$ are immediate from Lemma 3.1.2. If $F \subseteq 2^\omega$ is a $\pitchfork^{\bar{I}}$-unbounded family, then $\{2^\omega \smallsetminus (\pitchfork^{\bar{I}})^g \,/\, g \in F\} \subseteq \mathcal{N}$ covers all the reals. Dually, any not null subset of 2^ω is a $\pitchfork^{\bar{I}}$-dominating family. □

3.2.13 Lemma ([Br91, Lemma 1*])**.** *Let $\mu < \theta$ be an infinite cardinal. Then, every μ-centered poset is θ-$\pitchfork^{\bar{I}}$-good.*

Proof. Let $\mathbb{P}$ be a poset such that $\mathbb{P} = \bigcup_{\alpha<\mu} P_\alpha$ where each P_α is centered. Let $\dot{g}$ be a $\mathbb{P}$-name for a real in 2^ω. Given $\alpha < \mu$ and $n < \omega$, by Lemma 1.2.7 find a $s_{\alpha,n} \in 2^{I_n}$ such that no $p \in P_\alpha$ forces that $s_{\alpha,n} \neq \dot{g}{\restriction}I_n$. Put $h_\alpha = \bigcup_{n<\omega} s_{\alpha,n}$.

Assume that $f \in 2^\omega$ and $f \not\pitchfork h_\alpha$ for all $\alpha < \mu$. Let $p \in \mathbb{P}$, $n < \omega$ and find $\alpha < \mu$ such that $p \in P_\alpha$. Find $k \geq n$ such that $f{\restriction}I_k = h_\alpha{\restriction}I_k$. Then, $p \not\Vdash h_\alpha{\restriction}I_k \neq \dot{g}{\restriction}I_k$, so there is a $q \leq p$ that forces $f{\restriction}I_k = h_\alpha{\restriction}I_k = \dot{g}{\restriction}I_k$. □

Any $\mathfrak{d}_{\pitchfork^{\bar{I}}}$-$\pitchfork^{\bar{I}}$-good poset $\mathbb{P}$ does not add random reals. Indeed, let $\dot{x}$ be a $\mathbb{P}$-name for a real in 2^ω, so there is $Y \subseteq 2^\omega$ of size $< \mathfrak{d}_{\pitchfork^{\bar{I}}}$ that witnesses goodness for $\dot{x}$. Then, we can find $f \in 2^\omega$ such that $f \not\pitchfork^{\bar{I}} Y$, so $\Vdash \dot{x} \in 2^\omega \smallsetminus (\pitchfork^{\bar{I}})^f$.

3.2.14 Example (Preserving "union of null sets is not null")**.** Fix $h \in \omega^\omega$ that converges to infinity. Given $n < \omega$, for $x \in \omega^\omega$ and $\psi \in S(\omega, h)$, define the relation $x \in^*_{h,n} \psi$ iff $\forall_{k\geq n}(x(k) \in \psi(k))$, so $x \in^*_h \in \psi \Leftrightarrow x \in^* \psi$ (see Subsection 1.3.6). By Bartoszyński characterization (Theorem 1.4.2), $\mathfrak{b}_{\in^*_h} =$ add($\mathcal{N}$) and $\mathfrak{d}_{\in^*_h} =$ cof($\mathcal{N}$).

3.2.15 Lemma (Judah and Shelah [JS90]). *If $\mu < \theta$ is an infinite cardinal, then every μ-centered poset is θ-$\in^*_h$-good.*

Proof. Let $\mathbb{P}$ be a poset such that $\mathbb{P} = \bigcup_{\alpha<\mu} P_\alpha$. Let $\dot\psi$ be a $\mathbb{P}$-name for an slalom in $S(\omega, h)$. For $\alpha < \mu$ and $k < \omega$, let $\psi_\alpha(k) = \{j < \omega \ / \ \exists_{p\in P_\alpha}(p \Vdash j \in \dot\psi(k))\}$. Centeredness of P_α implies that $\psi_\alpha \in S(\omega, h)$.

Assume that $x \in \omega^\omega$ is such that $x \notin^* \psi_\alpha$ for any $\alpha < \mu$. Let $p \in \mathbb{P}$ and $n < \omega$ be arbitrary. There exists an $\alpha < \mu$ such that $p \in P_\alpha$. Then, choose $k \geq n$ such that $x(k) \notin \psi_\alpha(k)$, so $p \not\Vdash x(k) \in \dot\psi(k)$, i.e., there is a $q \leq p$ that forces $x(k) \notin \dot\psi(k)$. $\square$

Judah and Shelah [JS90, Def. 3.3] defined a similar property for the preservation of "union of null sets is not null". We generalize their notion in the following way.

3.2.16 Definition. Let $\bar H \subseteq \omega^\omega$ countable such that each of its members converges to infinity. Put $S(\omega, \bar H) = \bigcup_{h\in\bar H} S(\omega, h)$. For $n < \omega$, $x \in \omega^\omega$ and $\psi \in S(\omega, \bar H)$, define the relation $x \in^*_{\bar H,n} \psi \Leftrightarrow \forall_{k\geq n}(x(k) \in \psi(k))$, so $x \in^*_{\bar H} \psi$ is equivalent to $x \in^* \psi$. Note that $\text{add}(\mathcal{N}) = \mathfrak{b}_{\in^*_h} \leq \mathfrak{b}_{\in^*_{\bar H}}$ and $\mathfrak{d}_{\in^*_{\bar H}} \leq \mathfrak{d}_{\in^*_h} = \text{cof}(\mathcal{N})$ for any $h \in \bar H$.

3.2.17 Lemma. $\mathfrak{b}_{\in^*_{\bar H}} = \text{add}(\mathcal{N})$ *and* $\mathfrak{d}_{\in^*_{\bar H}} = \text{cof}(\mathcal{N})$.

Proof. If $h' \in \omega^\omega$ dominates all the functions in $\bar H$, then $\mathfrak{b}_{\in^*_{\bar H}} \leq \mathfrak{b}_{\in^*_{h'}} = \text{add}(\mathcal{N})$ and $\text{cof}(\mathcal{N}) = \mathfrak{d}_{\in^*_{h'}} \leq \mathfrak{d}_{\in^*_{\bar H}}$. $\square$

As Lemma 3.2.15 is valid for any h that converges to infinity, it follows directly that

3.2.18 Corollary. *If $\mu < \theta$ is an infinite cardinal, then every μ-centered poset is θ-$\in^*_{\bar H}$-good.*

For a Boolean algebra $\mathbb{B}'$, say that $\mu : \mathbb{B}' \to [0, 1]$ is a *strictly positive finitely additive (s.p.f.a.) measure* if $\mu(\mathbb{1}) = 1$, $\mu(a \vee b) = \mu(a) + \mu(b)$ for all $a, b \in \mathbb{B}'$ such that $a \wedge b = 0$ and $\mu(a) = 0$ iff $a = 0$. It is clear that any Boolean algebra with a s.p.f.a. measure is ccc. Examples of such Boolean algebras are random forcing $\mathbb{B}$ and the completion of any σ-centered poset. To see the last statement, let $\mathbb{A}$ be a complete Boolean algebra that is σ-centered, so $\mathbb{A} \setminus \{0\} = \bigcup_{n<\omega} U_n$ where each U_n is an ultrafilter in $\mathbb{A}$ (in the sense that $\mathbb{1} \in U_n$, $0 \notin U_n$, U_n is closed under intersections, it is upwards $\leq$-closed and, for any $a \in \mathbb{A}$, either $a \in U_n$ or $-a \in U_n$). Given $a \in \mathbb{A}$, define $\mu(a) = \sum\{1/2^{n+1} \ / \ a \in U_n,\ n < \omega\}$, which is clearly a s.p.f.a. measure on $\mathbb{A}$.

3.2.19 Lemma (Kamburelis [K89]). *Let $\bar H$ as in Definition 3.2.16 such that, for any $h \in \bar H$ there exists an $h' \in \bar H$ such that $\frac{h}{h'}$ converges to 0. Then, any Boolean algebra with a s.p.f.a measure is $\in^*_{\bar H}$-good. In particular, random forcing is $\in^*_{\bar H}$-good.*

Proof. Let $\mathbb{A}$ be a boolean algebra and $\mu : \mathbb{A} \to [0, 1]$ a s.p.f.a. measure. As $\mathbb{A}$ is isomorphic to the class $\mathcal{C}$ of clopen subsets of its Stone space, $\mathcal{C}$ inherits the measure from $\mathbb{A}$ and, by compactness of the Stone space, the measure on $\mathcal{C}$ is even σ-additive. Thus, the (forcing) completion of $\mathcal{C}$ admits a strictly positive σ-additive measure, so, without loss of generality, we may assume that $\mathbb{A}$ is a complete Boolean algebra and that μ is σ-additive.

Recall that, for a formula φ in the forcing language of $\mathbb{A}$, $\|\varphi\|$ denotes the maximum condition in $\mathbb{A}$ that forces φ. To prove that $\mathbb{A}$ is $\in^*_{\bar H}$-good, since $\mathbb{A}$ is ccc, we can assume that $\dot\psi$ is an $\mathbb{A}$-name for a slalom in $S(\omega, h)$ for some $h \in \bar H$. Let $h' \in \bar H$ be such that $\frac{h}{h'}$ converges to 0. For $n < \omega$, define $\psi'(n) = \{j < \omega \ / \ \mu(\|j \in \dot\psi(n)\|) \geq \frac{h(n)+1}{h'(n)+1}\}$.

3.2.20 Claim. $|\psi'(n)| \leq h'(n)$.

Proof. Assume the contrary, that is, there is some $c \subseteq \psi'(n)$ of size $h'(n)+1$. For any $y \subseteq \psi'(n)$ consider $i(y) = \max\{|x| \,/\, x \subseteq y,\ \bigwedge_{j\in x} \|j \in \dot\psi(n)\| \neq 0\}$. By [Kelley59, Prop. 1],

$$\inf_{j\in\psi'(n)}\{\mu(\|j \in \dot\psi(n)\|)\} \leq \inf\left\{\frac{i(y)}{|y|} \,/\, \varnothing \subsetneq y \subseteq \psi'(n)\right\},$$

in particular, $\frac{h(n)+1}{h'(n)+1} \leq \frac{i(c)}{h'(n)+1}$, so $h(n) < i(c)$, which is a contradiction to the fact that $\Vdash |\dot\psi(n)| \leq h(n)$. □

Assume that $x \in \omega^\omega$ is not localized by ψ' and show that $\Vdash x \not\in^* \dot\psi$. $x \not\in^* \psi'$ means that $\exists^\infty_{k<\omega}(\mu(\|x(k) \in \dot\psi(k)\|) < \frac{h(k)+1}{h'(k)+1})$ so, as $\frac{h(k)+1}{h'(k)+1}$ converges to 0, $\mu(\|\forall_{k\geq n}(x(k) \in \dot\psi(k))\|) = 0$ for any $n < \omega$. Thus $\mu(\|x \in^* \dot\psi\|) = 0$, i.e., $\Vdash x \not\in^* \dot\psi$. □

3.2.21 Example (Preserving new reals)**.** Consider, for $f, g \in \omega^\omega$ and $n < \omega$, $f =^*_n g$ defined as $\forall_{k\geq n}(f(n) = g(n))$. Then, $f =^* g$ iff $\forall^\infty_{k<\omega}(f(k) = g(k))$. Note that $\mathfrak{b}_{=^*} = 2$ and $\mathfrak{d}_{=^*} = \mathfrak{c}$.

In this case, the associated cardinal invariants are not that important. We are interested in the meaning of "$f \in \omega^\omega$ is $=^*$-unbounded over M", which is equivalent to $f \notin M$ when M is a model of some finite subset of axioms of ZFC.

3.2.22 Lemma. *If $\theta \leq \mathfrak{c}$, any θ-cc poset is θ-$=^*$-good. In particular, any ccc forcing is $=^*$-good.*

Proof. Let $\mathbb{P}$ be a θ-cc poset. If $\dot h$ is a $\mathbb{P}$-name for a real, let $A \subseteq \mathbb{P}$ be a maximal antichain such that, for any $p \in A$, either $p \Vdash f \notin V$ or there is an $f_p \in V$ such that $p \Vdash \dot h = f_p$. If $f \neq^* f_p$ for all $p \in A$ for which f_p exists, then $\Vdash \dot h \neq^* f$. □

The preservation property of Example 3.2.21 will be important to prove that, in a template iteration of certain type, any real added at some stage of the iteration cannot be added at other different stages (see Corollary 3.4.7).

3.3 Preservation of $\sqsubset$-unbounded reals

For this section, fix $M \subseteq N$ transitive models of ZFC. We discuss a property of preserving unbounded reals over M along parallel iterations from M and N. After defining the property, we present some examples and, at the end, prove some general results about the preservation of this property through iterations. Except of Lemmas 3.3.6, 3.3.7 and 3.3.8, the material of this section is based on results from [BlS84] and [BrF11].

Consider $\sqsubset$ from Context 3.1.1 with parameters in M and fix $c \in N$ a $\sqsubset$-unbounded real over M. As Cohen reals over M that belong to N are $\sqsubset$-unbounded over M, typically c is such a real.

3.3.1 Definition. Let $\mathbb{P} \in M$ and $\mathbb{Q} \in N$ be posets such that $\mathbb{P} \lessdot_M \mathbb{Q}$. Define

$$(\star, \mathbb{P}, \mathbb{Q}, M, N, \sqsubset, c): \quad \text{for every } \dot h \in M\ \mathbb{P}\text{-name for a real, } \Vdash_{\mathbb{Q},N} c \not\sqsubset \dot h.$$

This means that c is forced by $\mathbb{Q}$ (in N) to be $\sqsubset$-unbounded over $M^{\mathbb{P}}$.

This notion is fundamental to get consistency results about cardinal invariants using matrix iterations. We apply this property to get an interesting result in Section 3.4 and for the construction of models with matrix iterations in Chapter 5.

The first known example of preservation of unbounded reals is the following.

3.3.2 Lemma (Blass and Shelah [BlS84, Main Lemma])**.** *In M, let $\mathcal{U}$ be an ultrafilter on ω. If $c \in N$ is $\leq^*$-unbounded over M, then there exists an ultrafilter $\mathcal{V} \in N$ such that $\mathcal{U} \subseteq \mathcal{V}$, $\mathbb{M}_{\mathcal{U}} \lessdot_M \mathbb{M}_{\mathcal{V}}$ and $(\star, \mathbb{M}_{\mathcal{U}}, \mathbb{M}_{\mathcal{V}}, M, N, \leq^*, c)$.*

Proof. For $(s,F)\in[\omega]^{<\omega}\times[\omega]^{\omega}$, say that $t\in[\omega]^{<\omega}$ *is permitted by* (s,F) if $s\subseteq t\subseteq s\cup F$. If $A\in N$ is a subset of $\mathbb{M}_{\mathcal{U}}$, t *is permitted by* A means that t is permitted by some condition in A. $C\in[\omega]^{\omega}\cap N$ is *forbidden by* s, A if there is no finite subset of ω permitted by (s,C) and A.

If $\dot{z}\in M$ is a $\mathbb{M}_{\mathcal{U}}$-name for a real in ω^{ω}, it is coded by a sequence of maximal antichains $\langle B^{\dot{z}}_n\rangle_{n<\omega}\in M$ in $\mathbb{M}_{\mathcal{U}}$ and a sequence of functions $\{g^{\dot{z}}_n\}_{n<\omega}\in M\cap\prod_{n<\omega}\omega^{B^{\dot{z}}_n}$ such that, for any $n<\omega$ and $p\in B^{\dot{z}}_n$, $p\Vdash_{\mathbb{M}_{\mathcal{U}},M}\dot{z}(n)=g^{\dot{z}}_n(p)$. For $s\in[\omega]^{<\omega}$, say that $C\in[\omega]^{\omega}\cap N$ *is forbidden by* $s,\dot{z}$ if C is forbidden by $s,\bigcup_{n<\omega}\{p\in B^{\dot{z}}_n\ /\ g^{\dot{z}}_n(p)<c(n)\}$.

In N, consider $\mathcal{I}$ the ideal on ω generated by the finite subsets of ω and the sets $C\in[\omega]^{\omega}$ such that, for some $s\in[\omega]^{<\omega}$, C is either

(i) forbidden by s, A for some $A\in M$ maximal antichain in $\mathbb{M}_{\mathcal{U}}$, or

(ii) forbidden by $s,\dot{z}$ for some $\mathbb{M}_{\mathcal{U}}$-name $\dot{z}\in M$ for a real in ω^{ω}.

3.3.3 Claim. $\mathcal{I}\cap\mathcal{U}=\varnothing$.

Proof. Assume the contrary, that is, there are sets $\{C_k\}_{k<l}$ and $\{D_{k'}\}_{k'<l'}$ such that each C_k is forbidden by some s_k, A_k as in (i), each $D_{k'}$ is forbidden by some $t_{k'},\dot{z}_{k'}$ as in (ii) and the union of all these sets is in $\mathcal{U}$. Without loss of generality, as being forbidden is downwards closed under $\subseteq$, we may assume that $\{C_k\}_{k<l}\cup\{D_{k'}\}_{k'<l'}$ is a pairwise disjoint family, that any member of this family has empty intersection with any s_k and $t_{k'}$, and that $l=l'$ when both are not 0. This is because, in the process of making them pairwise disjoint, those sets that become finite can be ignored, so it may happen that some of the two sequences disappear. When both do not disappear, make their length equal by splitting some sets. Let $1\leq e\leq 2$ be the number of sequences that did not disappear and put $J=\bigcup_{k<l}(C_k\cup D_k)$ (ignore in this union the sets that disappeared). Note that $J\in M$ because it is in $\mathcal{U}$.

3.3.4 Claim. *There exists an $f\in\omega^{\omega}\cap M$ such that, for any $n<\omega$, $f(n)\geq n$ and, whenever we partition $J\cap[n,f(n))$ into $e\cdot l$ pieces, at least one of the pieces a satisfies*

- $\forall_{k<l}\exists_{q\subseteq a}(s_k\cup q$ *is permitted by* $A_k)$ *and*
- $\forall_{k<l}\exists_{q\subseteq a}(t_k\cup q$ *is permitted by* $\{p\in B^{\dot{z}_k}_n\ /\ g^{\dot{z}_k}_n(p)<f(n)\})$.

Proof. Work in M. Fix $n<\omega$ and assume that there is no such $f(n)$. Construct a tree of height ω such that level n' is formed by the sequences of length $e\cdot l$ that partitions $J\cap[n,n+n')$ into sets that do not satisfy the claim, which order is $\{q_k\}_{k<e\cdot l}\leq\{a_k\}_{k<e\cdot l}$ iff $q_k\subseteq a_k$ for all $k<e\cdot l$. As the levels are finite and non-empty, by König lemma we can find a partition of $J\cap[n,\omega)$ into $e\cdot l$ non-empty pieces such that none of them satisfy the claim. Then, there is one piece K that belongs to $\mathcal{U}$. For each $k<l$, as $(s_k,K),(t_k,K)\in\mathbb{M}_{\mathcal{U}}$ and A_k and $B^{\dot{z}_k}_n$ are maximal antichains, we can find $q^1_k,q^2_k\subseteq K$ such that $s_k\cup q^1_k$ is permitted by A_k and $t_k\cup q^2_k$ is permitted by $B^{\dot{z}_k}_n$. Thus, we can easily find $f(n)$ such that $f(n)\geq\sup(\{n\}\cup\bigcup_{k<l}(q^1_k\cup q^2_k)+1)$ and satisfies the claim, a contradiction. □

Then, as $\{C_k\cap[n,f(n))\ /\ k<l\}\cup\{D_k\cap[n,f(n))\ /\ k<l\}$ is a partition of $J\cap[n,f(n))$ into $e\cdot l$ pieces, one of these pieces a satisfies the previous claim. If a is of the form $C_k\cap[n,f(n))$, then there is a $q\subseteq a$ that is permitted by (s_k,C_k) and A_k, contradicting that C_k is forbidden by s_k,A_k. Hence, a is of the form $D_k\cap[n,f(n))$. Then, there is a $q\subseteq a$ that is permitted by (t_k,D_k) and $\{p\in B^{\dot{x}_k}_n\ /\ g^{\dot{x}_k}_n(p)<f(n)\}$), so $c(n)\leq f(n)$ because D_k is forbidden. As this is true for any $n<\omega$, we get that $c\leq f$ with $f\in M$, but c is $\leq^*$-unbounded over M, a contradiction. □

In N, as $\mathcal{U}\cup\{\omega\smallsetminus Y\ /\ Y\in\mathcal{I}\}$ has the finite intersection property, there exists an ultrafilter $\mathcal{V}$ containing it. It is clear that $\mathcal{V}\cap\mathcal{I}=\varnothing$. We prove:

- $\mathbb{M}_{\mathcal{U}}\lessdot_M\mathbb{M}_{\mathcal{V}}$. Let $A\in M$ be a maximal antichain in $\mathbb{M}_{\mathcal{U}}$ and $(s,F)\in\mathbb{M}_{\mathcal{V}}$. As $F\notin\mathcal{I}$, F is not forbidden by s,A, so there is a $t\in[\omega]^{<\omega}$ permitted by (s,F) and by some $p\in A$, so it is clear that (s,F) is compatible with p.

- $\Vdash_{\mathbb{M}_{\mathcal{V}},N} c \not\leq^* \dot{z}$ *for any* $\dot{z} \in M$ $\mathbb{M}_{\mathcal{U}}$*-name for a real in* ω^ω. Towards a contradiction, assume that there are $\dot{z} \in M$, $m < \omega$ and a $(s,F) \in \mathbb{M}_{\mathcal{V}}$ that forces $\forall_{k\geq m}(c(k) \leq \dot{z}(k))$. By making finitely many changes to $\dot{z}$, we may assume that (s,F) forces $c \leq \dot{z}$. As F is not forbidden by $s, \dot{z}$, there is a $t \in [\omega]^{<\omega}$ permitted by (s,F) and by some $p \in B_n^{\dot{z}}$ with $g_n^{\dot{z}}(p) < c(n)$ for some $n < \omega$. Then, (s,F) is compatible with p, so there is a common extension $q \in \mathbb{M}_{\mathcal{V}}$. As $p \Vdash_{\mathbb{M}_{\mathcal{U}},M} \dot{h}(n) = g_n^{\dot{z}}(p) < c(n)$, we get that q forces $\dot{h}(n) < c(n)$ and $c \leq \dot{h}$, a contradiction.

□

3.3.5 Lemma. *If* $c \in \omega^\omega \cap N$ *is* $\leq^*$*-unbounded over* M*, then it is* $\in^*_{\bar{H}}$*-unbounded over* M*. In particular, in Lemma 3.3.2 we can also conclude* $(\star, \mathbb{M}_{\mathcal{U}}, \mathbb{M}_{\mathcal{V}}, M, N, \in^*_{\bar{H}}, c)$*.*

Proof. Let $\psi \in S(\omega, \bar{H}) \cap M$. Let $h \in \omega^\omega \cap M$ defined as $h(n) = \sup(\psi(n)+1)$ for any $n < \omega$. Then, $\exists^\infty_{n<\omega}(h(n) < c(n))$, which implies $\exists^\infty_{n<\omega}(c(n) \notin \psi(n))$. □

Lemma 3.3.2 does not hold for $\propto$ because the Mathias real added cannot be spitted by any infinite subset of ω in the ground model and also because $\Vdash_{\mathbb{M}_{\mathcal{V}},N} \dot{m}_{\mathcal{V}} = \dot{m}_{\mathcal{U}}$. We do not know whether the lemma is valid for $\pitchfork^{\bar{I}}$ or for $\in^*_{\bar{H}}$ (this last without using a $\leq^*$-unbounded real, as it happens in Lemma 3.3.5). However, this can be done for Laver forcing with an ultrafilter.

3.3.6 Lemma. *In* M*, let* $\mathcal{U}$ *be an ultrafilter on* ω *and, in* N*, let* $\mathcal{V}$ *be an ultrafilter on* ω *such that* $\mathcal{U} \subseteq \mathcal{V}$*. Then,*

(a) (Shelah [S04], see also [Br06, Lemma 2.1] and [Br07, Lemma 8]) $\mathbb{L}_{\mathcal{U}} \lessdot_M \mathbb{L}_{\mathcal{V}}$*.*

(b) Let φ *be a* $\mathbf{\Sigma}^1_1$*-statement in the forcing language of* $\mathbb{P}$ *with parameters in* M*. Fix* $s \in \omega^{<\omega}$ *and assume that no* $T \in \mathbb{L}_{\mathcal{U}}$ *with stem* s *forces, in* M*, that* $\neg\varphi$*. Then, no* $T \in \mathbb{L}_{\mathcal{V}}$ *with stem* s *forces, in* N*, that* $\neg\varphi$*.*

(c) $(\star, \mathbb{L}_{\mathcal{U}}, \mathbb{L}_{\mathcal{V}}, M, N, \in^*_{\bar{H}}, c)$ *holds for any* $c \in N$ *that is* $\pitchfork^{\bar{I}}$*-unbounded over* M*.*

(d) $(\star, \mathbb{L}_{\mathcal{U}}, \mathbb{L}_{\mathcal{V}}, M, N, \pitchfork^{\bar{I}}, c)$ *holds for any* $c \in N$ *that is* $\pitchfork^{\bar{I}}$*-unbounded over* M*.*

Proof. (a) It is clear that $\mathbb{L}_{\mathcal{U}} \subseteq \mathbb{L}_{\mathcal{V}}$ and that incompatibilities are preserved. Let $D \in M$ be a dense subset of $\mathbb{L}_{\mathcal{U}}$. Consider the rank function ρ_D defined for the proof of pure decision of $\mathbb{L}_{\mathcal{U}}$ (Theorem 1.2.9). We prove, by induction on this rank function, that, for all $s \in \omega^{<\omega}$, any $T \in \mathbb{L}_{\mathcal{V}}$ with stem s is compatible with a member of D. If $\rho_D(s) = 0$ then there is $S \in D$ with stem s, so clearly any $T \in \mathbb{L}_{\mathcal{V}}$ with stem s is compatible with it. Now, if $\rho_D(s) = \alpha > 0$ and $T \in \mathbb{L}_{\mathcal{V}}$ with stem s, there exists a $j < \omega$ such that $t := s^\frown\langle j\rangle \in T$ and $\rho_D(t) < \alpha$. By induction hypothesis, $T_t = \{r \in T \,/\, r \subseteq t \text{ or } t \subseteq r\}$ is compatible with some $S \in D$, so T is compatible with the same S.

(b) By pure decision (Theorem 1.2.9), choose $S \in \mathbb{L}_{\mathcal{U}}$ with stem s such that either $S \Vdash_{\mathbb{L}_{\mathcal{U}},M} \varphi$ or $S \Vdash_{\mathbb{L}_{\mathcal{U}},M} \neg\varphi$. By hypothesis, the second option is not possible, so $S \Vdash_{\mathbb{L}_{\mathcal{U}},M} \varphi$. By (a) and Lemma 1.2.1, in N, $S \Vdash_{\mathbb{L}_{\mathcal{V}}} \varphi$. Now, if $T \in \mathbb{L}_{\mathcal{V}}$ has stem s, then it is compatible with S, so it cannot force $\neg\varphi$.

(c) It is enough to prove the statement for $\in^*_h$ for any $h \in \omega^\omega$ that converges to infinity. Let $c \in \omega^\omega \cap N$ be a real such that no slalom in $S(\omega, h) \cap M$ localizes it. In M, fix $\dot{\psi}$ a $\mathbb{L}_{\mathcal{U}}$-name for a slalom in $S(\omega, h)$. For $s \in \omega^{<\omega}$ let $L_{\mathcal{U},s} := \{T \in \mathbb{L}_{\mathcal{U}} \,/\, \mathrm{stem}(T) = s\}$, which is centered. For $n < \omega$, let $\psi'_s(n) = \{j < \omega \,/\, \exists_{T\in L_{\mathcal{U},s}}(T \Vdash j \in \dot{\psi}(n))\}$. By the centeredness of $L_{\mathcal{U},s}$, $\psi'_s \in S(\omega, h)$.

In N, fix $T \in \mathbb{L}_{\mathcal{V}}$ and $n < \omega$. Put $s = \mathrm{stem}(T)$. As $c \notin^* \psi'_s$, there is a $k > n$ such that $c(k) \notin \psi'_s(k)$, that is, in M, no condition in $L_{\mathcal{U}}, s$ forces that $c(k) \in \dot{\psi}(k)$. By (b), no condition in $\mathbb{L}_{\mathcal{V}}$ with stem s forces that $c(k) \in \dot{\psi}(k)$, so there is $S \subseteq T$ in $\mathbb{L}_{\mathcal{V}}$ that forces $c(k) \notin \dot{\psi}(k)$.

(d) In M, let $\dot{x}$ be a $\mathbb{L}_{\mathcal{U}}$-name of a real in 2^ω. For $s \in \omega^{<\omega}$, as $L_{\mathcal{U},s}$ is centered, by Lemma 1.2.7, for each $n < \omega$ we can find $z_s{\restriction}I_n \in 2^{I_n}$ such that no condition in $L_{\mathcal{U},s}$ forces that $z_s{\restriction}I_n \neq \dot{x}{\restriction}I_n$. Put $z_s = \bigcup_{n<\omega} z_s{\restriction}I_n \in 2^\omega \cap M$.

In N, fix $T \in \mathbb{L}_{\mathcal{V}}$ and $n < \omega$ any natural number. Put $s = \mathrm{stem}(T)$. Choose $k > n$ such that $c{\restriction}I_k = z_s{\restriction}I_k$. By (b), no condition in $\mathbb{L}_{\mathcal{V}}$ with stem s forces that $z_s{\restriction}I_k \neq \dot{x}{\restriction}I_k$, so there is $S \in \mathbb{L}_{\mathcal{V}}$ extending T that forces $c{\restriction}I_k = z_s{\restriction}I_k = \dot{x}{\restriction}I_k$.

□

Like in the case of Mathias forcing with an ultrafilter, the previous result is not valid for $\propto$. Moreover, as $\Vdash_{\mathbb{L}_{\mathcal{V}},N} \dot{l}_{\mathcal{V}} = \dot{l}_{\mathcal{U}}$, the result is not valid for $\leq^*$.

We prove a general preservation result of unbounded reals for Suslin ccc notions. Before, consider the following preliminary result.

3.3.7 Lemma. *Let $\varphi(x_0, \dots, x_{n-1})$ be a G_δ statement. Then, $p \Vdash_{\mathbb{S}} \varphi(\dot{x}_0, \dots \dot{x}_{n-1})$ is a $\mathbf{\Pi}^1_1$-statement.*

Proof. Let $\{T_l\}_{l<\omega}$ be a sequence of subtrees of $(\omega^n)^{<\omega}$ such that $\bigcup_{l<\omega}[T_l] = \{(x_0, \dots, x_{n-1}) \in (\omega^\omega)^n \,/\, \neg\varphi(x_0, \dots, x_{n-1})\}$. We can code $\dot{x}_0, \dots \dot{x}_{n-1}$ by a real given by maximal antichains $\{A^k\}_{k<\omega}$ where $A^k = \{q^k_j \,/\, j < \omega\}$ and functions $\{g^k\}_{k<\omega}$ where $g^k : \omega \to (\omega^{<\omega})^n$ such that $q^k_j \Vdash g^k(j) = (\dot{x}_0{\restriction}k, \dots, \dot{x}_{n-1}{\restriction}k)$. Observe that $p \Vdash (\dot{x}_0, \dots \dot{x}_{n-1}) \notin \bigcup_{l<\omega}[T_l]$ iff

$$\forall_{q\leq p}\forall_{l<\omega}\exists_{k<\omega}\exists_{j<\omega}(q \parallel q^k_j \text{ and } g^k(j) \notin T_l).$$

The latter is clearly a $\mathbf{\Pi}^1_1$-statement. □

3.3.8 Lemma. *Let $\mathbb{S}$ be a Suslin ccc poset with parameters in M. If $\mathbb{S}$ is $\sqsubset$-good in M, then the property $(\star, \mathbb{S}^M, \mathbb{S}^N, M, N, \sqsubset, c)$ holds.*

Proof. In M, fix $\dot{h}$ a $\mathbb{S}$-name for a real and choose $Y \subseteq \omega^\omega$ a witness of the goodness of $\mathbb{S}$ for $\dot{h}$. By Lemma 3.3.7, the statement $\forall_{f\in\omega^\omega}(f \not\sqsubset Y \Rightarrow \Vdash f \not\sqsubset \dot{h})$ is $\mathbf{\Pi}^1_1$ so, as it is true in M, then it is also true in N. But, as $c \not\sqsubset Y$, $\Vdash_N c \not\sqsubset \dot{h}$. □

When the same poset is used in M and N, the preservation of an unbounded real is guaranteed.

3.3.9 Lemma (Brendle and Fischer [BrF11, Lemma 11])**.** *Let $\mathbb{P} \in M$ be a poset. Then, $(\star, \mathbb{P}, \mathbb{P}, M, N, \sqsubset, c)$ holds.*

Proof. In M, let $\dot{h}$ be a $\mathbb{P}$-name for a real in ω^ω. Fix $p \in \mathbb{P}$ and $n < \omega$. Choose $\{p_k\}_{k<\omega}$ a decreasing sequence in $\mathbb{P}$ and $g \in \omega^\omega$ such that $p_0 = p$ and $p_k \Vdash \dot{h}{\restriction}k = g{\restriction}k$. In N, $c \not\sqsubset g$, so $c \not\sqsubset_n g$, which implies that there is a $k < \omega$ such that $[g{\restriction}k] \cap (\sqsubset_n)_c = \varnothing$, this because $(\sqsubset_n)_c$ is a closed set. As $p_k \Vdash_N \dot{h}{\restriction}k = g{\restriction}k$, we get that $p_k \Vdash_N c \not\sqsubset_n \dot{h}$. □

To finish this section, we prove that preservation of unbounded reals is preserved in fsi. We even generalize this for direct limits.

3.3.10 Lemma. *Let $\mathbb{P} \in M$, $\mathbb{P}' \in N$ posets such that $(\star, \mathbb{P}, \mathbb{P}', M, N, \sqsubset, c)$ holds. Also, let $\dot{\mathbb{Q}} \in M$ be a $\mathbb{P}$-name of a poset and $\dot{\mathbb{Q}}' \in N$ a $\mathbb{P}'$-name of a poset such that $\mathbb{P}'$ forces (with respect to N) that $(\star, \dot{\mathbb{Q}}, \dot{\mathbb{Q}}', M^{\mathbb{P}}, N^{\mathbb{P}'}, \sqsubset, c)$. Then $(\star, \mathbb{P} * \dot{\mathbb{Q}}, \mathbb{P}' * \dot{\mathbb{Q}}', M, N, \sqsubset, c)$ holds.*

Proof. From Lemma 2.1.2 it is clear that $\mathbb{P} * \dot{\mathbb{Q}} \lessdot_M \mathbb{P}' * \dot{\mathbb{Q}}'$. $(\star, \mathbb{P}, \mathbb{P}', M, N, \sqsubset, c)$ indicates that $\Vdash_{\mathbb{P}',N} c \not\sqsubset M^{\mathbb{P}}$ and, as it forces $(\star, \dot{\mathbb{Q}}, \dot{\mathbb{Q}}', M^{\mathbb{P}}, N^{\mathbb{P}'}, \sqsubset, c)$, then $\Vdash_{\mathbb{P}',N}$"$\Vdash_{\dot{\mathbb{Q}}',N^{\mathbb{P}'}} c \not\sqsubset M^{\mathbb{P}*\dot{\mathbb{Q}}'}$". □

3.3.11 Lemma. *Let $I \in M$ be a directed set, $\langle\mathbb{P}_i\rangle_{i\in I} \in M$ and $\langle\mathbb{Q}_i\rangle_{i\in I} \in N$ directed systems of posets such that*

(i) for each $i \in I$, $(\star, \mathbb{P}_i, \mathbb{Q}_i, M, N, \sqsubset, c)$ holds and

(ii) whenever $i \leq j$, $\langle\mathbb{P}_i, \mathbb{P}_j, \mathbb{Q}_i, \mathbb{Q}_j\rangle$ is a correct system with respect to M

Then, $(\star, \mathbb{P}, \mathbb{Q}, M, N, \sqsubset, c)$ *where* $\mathbb{P} := \text{limdir}_{i\in I}\mathbb{P}_i$ *and* $\mathbb{Q} := \text{limdir}_{i\in I}\mathbb{Q}_i$*. Moreover, for any* $i \in I$*,* $\langle \mathbb{P}_i, \mathbb{P}, \mathbb{Q}_i, \mathbb{Q}\rangle$ *is a correct system with respect to* M*.*

Proof. By Lemma 2.1.5, it is enough to prove that, if $\dot{h}$ is a $\mathbb{P}$-name for a real in ω^ω, then $\Vdash_{\mathbb{Q},N} c \not\sqsubset \dot{h}$. Assume, towards a contradiction, that there are $q \in \mathbb{Q}$ and $n < \omega$ such that $q \Vdash_{\mathbb{Q},N} c \sqsubset_n \dot{h}$. Choose $i \in I$ such that $q \in \mathbb{Q}_i$.

Let G be $\mathbb{Q}_i$-generic over N with $q \in G$. By assumption, $\Vdash_{\mathbb{Q}/\mathbb{Q}_i,N[G]} c \sqsubset_n \dot{h}$. In $M[G\cap\mathbb{P}]$, find $g \in \omega^\omega$ and a decreasing chain $\{p_k\}_{k<\omega}$ in $\mathbb{P}/\mathbb{P}_i$ such that $p_k \Vdash_{\mathbb{P}/\mathbb{P}_i,M[G\cap\mathbb{P}]} \dot{h}{\restriction}k = g{\restriction}k$. In $N[G]$, by hypothesis, $c \not\sqsubset g$ so there is a $k < \omega$ such that $[g{\restriction}k] \cap (\sqsubset_n)_c = \varnothing$. Then, as $\mathbb{P}/\mathbb{P}_i \lessdot_{M[G\cap\mathbb{P}]} \mathbb{Q}/\mathbb{Q}_i$ by Lemma 2.1.3, $p_k \Vdash_{\mathbb{Q}/\mathbb{Q}_i,N[G]} [\dot{h}\restriction k] \cap (\sqsubset_n)_c = \varnothing$, that is, $p_k \Vdash_{\mathbb{Q}/\mathbb{Q}_i,N[G]} c \not\sqsubset_n \dot{h}$, which is a contradiction. □

3.3.12 Corollary. *Let* $\mathbb{P}_\delta = \langle \mathbb{P}_\alpha, \dot{\mathbb{Q}}_\alpha\rangle$ *be a fsi in* M *and* $\mathbb{P}'_\delta = \langle \mathbb{P}'_\alpha, \dot{\mathbb{Q}}'_\alpha\rangle$ *a fsi in* N*. Assume that, for any* $\alpha < \delta$*, if* $\mathbb{P}_\alpha \lessdot_M \mathbb{P}'_\alpha$ *and* $\mathbb{P}'_\alpha$ *forces (in* N*)* $(\star, \dot{\mathbb{Q}}_\alpha, \dot{\mathbb{Q}}'_\alpha, M^{\mathbb{P}_\alpha}, N^{\mathbb{P}'_\alpha}, \sqsubset, c)$*. Then,* $(\star, \mathbb{P}_\alpha, \mathbb{P}'_\alpha, M, N, \sqsubset, c)$ *holds for any* $\alpha \le \delta$*.*

3.4 Theorems of preservation for template iterations

We show how can we get preservation results as in Section 3.1 for iterations along a template. Consider Context 3.1.1 and fix θ an uncountable regular cardinal.

3.4.1 Theorem. *Consider* $\langle L, \bar{\mathcal{I}}\rangle$ *an indexed template and* $\mathbb{P}{\restriction}L$ *a corresponding template iteration such that it is* θ*-cc and* $\nu \le \theta$ *is an uncountable cardinal such that*

(i) for all $B \in [L]^{<\nu}$*,* $\mathcal{I}{\restriction}B$ *has size* $< \nu$*,*

(ii) for all $A \in \mathcal{I}$*, every condition and name for a real in* $\mathbb{P}{\restriction}A$ *has a support of size* $< \nu$ *and*

(iii) for all $x \in L$ *and* $B \in \mathcal{I}_x$*,* $\Vdash_{\mathbb{P}\restriction B} \dot{\mathbb{Q}}^B_x$ *is* θ*-*$\sqsubset$*-good.*

Then, $\mathbb{P}{\restriction}L$ *is* θ*-*$\sqsubset$*-good. Moreover, if* L' *is an initial segment of* L *such that* $\forall_x \in L \smallsetminus L'(L' \in \hat{\mathcal{I}}_x)$*, then* $\mathbb{P}{\restriction}L'$ *forces that* $\mathbb{P}{\restriction}L/\mathbb{P}{\restriction}L'$ *is* θ*-*$\sqsubset$*-good.*

Proof. We prove, by induction on $\Upsilon(A)$ with $L' \subseteq A$, that $\mathbb{P}{\restriction}L'$ forces that $\mathbb{P}{\restriction}A/\mathbb{P}{\restriction}L'$ is θ-$\sqsubset$-good. We may assume that $L' \subsetneq A$. Proceed by cases.

(1) *A has a maximum x and* $A_x = A \cap L_x \in \hat{\mathcal{I}}_x$*.* By Lemma 1.2.3, $\mathbb{P}\restriction A/\mathbb{P}\restriction L'$ is equivalent to $(\mathbb{P}{\restriction}A_x/\mathbb{P}{\restriction}L') * \dot{\mathbb{Q}}^{A_x}_x$, so it is θ-$\sqsubset$-good by Lemma 3.1.8 and induction hypothesis.

(2) *A has a maximum x but* $A_x \notin \hat{\mathcal{I}}_x$*.* Then, $\mathbb{P}{\restriction}A = \text{limdir}_{B\in\mathcal{A}}\mathbb{P}{\restriction}B$ where $\mathcal{A} := \{B \subseteq A \mathbin{/} B \cap L_x \in \mathcal{I}_x{\restriction}A \text{ and } L' \subseteq B\}$. Let $\dot{h} \in V$ be a $\mathbb{P}{\restriction}A$-name for a real. If there exists a $B \in \mathcal{A}$ such that $\dot{h}$ is a $\mathbb{P}\restriction B$-name then, in V', any witness of goodness of $\mathbb{P}\restriction B/\mathbb{P}\restriction L'$ for $\dot{h}$ is also a witness of goodness of $\mathbb{P}\restriction A/\mathbb{P}\restriction L'$, this because $\mathbb{P}\restriction B/\mathbb{P}\restriction L'$ is a complete suborder of $\mathbb{P}\restriction A/\mathbb{P}\restriction L'$ by Lemma 2.1.3. So assume that $\dot{h}$ is not a $\mathbb{P}{\restriction}B$-name for any $B \in \mathcal{A}$. By (ii), there is $C \in [A \smallsetminus L']^{<\nu}$ such that $\dot{h}$ is a $\mathbb{P}{\restriction}(L' \cup C)$-name with $x \in C$. As $L' \in \hat{\mathcal{I}}_x$, note that

$$\mathcal{C} := \big\{D \subseteq L' \cup C \mathbin{/} L' \subseteq D \text{ and } D \cap L_x \in \mathcal{I}_x{\restriction}(L' \cup C)\big\} \subseteq \big\{B \cap (L' \cup C) \mathbin{/} B \in \mathcal{A}\big\} \subseteq \big\{L' \cup E \mathbin{/} E \subseteq C \text{ and } E \cap L_x \in \mathcal{I}_x{\restriction}C\big\}.$$

and $\mathcal{C}$ is cofinal in the latter. As $\mu := |\mathcal{I}\restriction C| < \nu$ by (i), this equation implies that $|\mathcal{C}| \le \mu$, so enumerate $\mathcal{C} := \{D_\alpha \mathbin{/} \alpha < \mu\}$ where each $D_\alpha = B_\alpha \cap (L' \cup C)$ with some $B_\alpha \in \mathcal{A}$. Note also that $(L' \cup C) \cap L_x \notin \mathcal{C}$ (if so, there exists a $B \in \mathcal{A}$ such that $L' \cup C \subseteq B$, so $\dot{h}$ would be a $\mathbb{P}\restriction B$-name, which is false), so $\mathbb{P}\restriction(L' \cup C) = \text{limdir}\,\{\mathbb{P}{\restriction}D_\alpha \mathbin{/} \alpha < \mu\}$ and, by Lemma 2.1.6, $\mathbb{P}{\restriction}(L' \cup C)/\mathbb{P}{\restriction}L' = \text{limdir}\,\{\mathbb{P}{\restriction}D_\alpha/\mathbb{P}{\restriction}L' \mathbin{/} \alpha < \mu\}$. By induction hypothesis, as $\mathbb{P}{\restriction}D_\alpha/\mathbb{P}{\restriction}L'$ is a complete suborder of $\mathbb{P}{\restriction}B_\alpha/\mathbb{P}{\restriction}L'$, then both are θ-$\sqsubset$-good for any $\alpha < \mu$. Therefore, by Theorem 3.1.9, $\mathbb{P}{\restriction}(L' \cup C)/\mathbb{P}{\restriction}L'$ is θ-$\sqsubset$-good. Any family of reals that witnesses this goodness for $\dot{h}$ works for the goodness of $\mathbb{P}{\restriction}A/\mathbb{P}{\restriction}L'$ for $\dot{h}$.

(3) *A does not have a maximum element.* So $\mathbb{P}{\restriction}A = \mathrm{limdir}_{B\in\mathcal{B}}\mathbb{P}{\restriction}B$ where $\mathcal{B} := \{B \in \mathcal{I}_x{\restriction}A \;/\; x \in A \text{ and } L' \subseteq B\}$. Let $\dot{h}$ a $\mathbb{P}{\restriction}A$-name for a real. If there is no $B \in \mathcal{B}$ such that $\dot{h}$ is a $\mathbb{P}{\restriction}B$-name, proceed to find $C \subseteq A \smallsetminus L'$ of size $< \nu$ such that $\dot{h}$ is a $\mathbb{P}{\restriction}(L' \cup C)$-name and, without loss of generality, assume that C doesn't have a maximum. Proceed exactly as in the previous case.

□

3.4.2 Remark. Shelah's models ([S04], see also [Br02]) for the consistency of $\mathfrak{d} < \mathfrak{a}$ with ZFC use template iterations like in Example 2.4.3 where $L_C = \varnothing$ and $\mathbb{S}_x = \mathbb{D}$ for every $x \in L_S = L$. By Lemma 3.2.7, the conditions of Theorem 3.4.1 with $\theta = \aleph_1$ and $\sqsubset = \propto$ hold for those template iterations and, thus, $\mathfrak{s} = \aleph_1$ in the generic extension. Therefore, if $\aleph_1 < \mu < \lambda$ are regular cardinals and $\lambda^\omega = \lambda$, there is a model of ZFC that satisfies $\mathfrak{s} = \aleph_1 < \mathfrak{b} = \mathfrak{d} = \mu < \mathfrak{a} = \mathfrak{c} = \lambda$.

3.4.3 Theorem. *Consider $\langle L, \bar{\mathcal{I}}\rangle$ an indexed template and $\mathbb{P}{\restriction}L$ a corresponding template iteration such that it is ccc and, for any $A \subseteq L$.*

(i) whenever A has a maximum x and $A \cap L_x \notin \hat{\mathcal{I}}_x$, if $\dot{h}$ is a $\mathbb{P}{\restriction}A$-name for a real, then there exists an increasing sequence $\langle B_n\rangle_{n<\omega}$ in $\mathcal{B}_A := \{B \subseteq A \;/\; B \cap L_x \in \mathcal{I}_x{\restriction}A\}$ such that $\dot{h}$ is a $\mathbb{P}{\restriction}C$-name for a real, where $C := \bigcup_{n<\omega} B_n$, and $\mathbb{P}{\restriction}C = \mathrm{limdir}_{n<\omega}\mathbb{P}{\restriction}B_n$,

(ii) whenever A does not have a maximum and $\dot{h}$ is a $\mathbb{P}{\restriction}A$-name for a real, then there exists an increasing sequence $\langle B_n\rangle_{n<\omega}$ in $\mathcal{B}_A := \{B \subseteq A \;/\; \exists_{x\in A}(B \in \mathcal{I}_x{\restriction}A)\}$ like in (i), and

(iii) for all $x \in L$ and $B \in \hat{\mathcal{I}}_x$, $\Vdash_{\mathbb{P}{\restriction}B} \dot{\mathbb{Q}}_x^B$ is θ-$\sqsubset$-good.

Then, $\mathbb{P}{\restriction}L$ is θ-$\sqsubset$-good.

Proof. By induction on $\Upsilon(A)$. Proceed by cases.

(1) *A has a maximum x and $A_x = A \cap L_x \in \hat{\mathcal{I}}_x$.* Use Lemma 3.1.8.

(2) *A has a maximum x but $A_x \notin \hat{\mathcal{I}}_x$.* If $\dot{h}$ is a $\mathbb{P}{\restriction}A$-name for a real, use (i) to find $\langle B_n\rangle_{n<\omega}$ and C. Then, by induction hypothesis and Theorem 3.1.9, $\mathbb{P}{\restriction}C$ is θ-$\sqsubset$-good. Any family of reals that witnesses this goodness for $\dot{h}$ also works for $\mathbb{P}{\restriction}A$.

(3) *A does not have a maximum.* Proceed like in case (2) and use (ii).

□

3.4.4 Corollary. *Consider a template iteration given by the template $\mathcal{I}$ for a fsi (see Example 2.4.2) where all the names for posets involved correspond to ccc forcing notions. Then, conditions (i) and (ii) of Theorem 3.4.3 hold, moreover, (i) is irrelevant because, whenever $A \subseteq L$ have a maximum x, $A \cap L_x \in \hat{\mathcal{I}}_x$.*

To finish this section, we prove a result that states that reals added at certain stage of an iteration cannot be added at other stages. To see this, we give a more general result.

3.4.5 Theorem. *In a (ground) model V of* ZFC, *let $\mathbb{P}{\restriction}\langle L, \bar{\mathcal{I}}\rangle$ be a template iteration. Fix $x \in L$ such that $L_x \in \mathcal{I}_x$, let $\dot{c}$ be a $\mathbb{P}{\restriction}(L_x \cup \{x\})$-name for a real and assume that $\mathbb{P}{\restriction}(L_x \cup \{x\})$ forces that $\dot{c}$ is $\sqsubset$-unbounded over $V^{\mathbb{P}{\restriction}L_x}$. If*

(i) for any $y \in L$, if $B \in \hat{\mathcal{I}}_y$ and $L_x \cup \{x\} \subseteq B$, then $\Vdash_{\mathbb{P}{\restriction}B} (\star, \dot{\mathbb{Q}}_y^{B\smallsetminus\{x\}}, \dot{\mathbb{Q}}_y^B, V^{\mathbb{P}{\restriction}(B\smallsetminus\{x\})}, V^{\mathbb{P}{\restriction}B}, \sqsubset, \dot{c})$,

then, $\mathbb{P}{\restriction}L$ forces that $\dot{c}$ is $\sqsubset$-unbounded over $V^{\mathbb{P}{\restriction}(L\smallsetminus\{x\})}$.

Proof. Put $\bar{L}_x = L_x \cup \{x\}$. Let G be $\mathbb{P}{\restriction}\bar{L}_x$-generic over V. Put $M = V[G \cap \mathbb{P}{\restriction}L_x]$ and $N = V[G]$. We prove, by induction on $\Upsilon(A)$ with $\bar{L}_x \subseteq A$ and $A \in V$, that $(\star, \mathbb{P}{\restriction}(A \smallsetminus \{x\})/\mathbb{P}{\restriction}L_x, \mathbb{P}{\restriction}A/\mathbb{P}{\restriction}\bar{L}_x, M, N, \sqsubset, c)$. Proceed by cases.

(1) *A has a maximum y and $A_y = A \cap L_y \in \hat{\mathcal{I}}_y$.* We may assume that $x < y$. Clearly, $A_y \smallsetminus \{x\} \in \hat{\mathcal{I}}_y$. By (i), $\mathbb{P}{\restriction}A_y$ forces $(\star, \dot{\mathbb{Q}}_y^{A_y \smallsetminus \{x\}}, \dot{\mathbb{Q}}_y^{A_y}, V^{\mathbb{P}\restriction(A_y \smallsetminus \{x\})}, V^{\mathbb{P}\restriction A_y}, \sqsubset, \dot{c})$, so inductive hypothesis and Lemma 3.3.10 implies $(\star, \mathbb{P}{\restriction}(A \smallsetminus \{x\})/\mathbb{P}{\restriction}L_x, \mathbb{P}{\restriction}A/\mathbb{P}{\restriction}\bar{L}_x, M, N, \sqsubset, c)$.

(2) *A has a maximum y but $A_y \notin \hat{\mathcal{I}}_y$.* Clearly, $x < y$. With $\mathcal{B} = \{B \subseteq A \mid B \cap L_x \in \mathcal{I}_x{\restriction}A$ and $\bar{L}_x \subseteq B\}$, $\mathbb{P}{\restriction}A = \text{limdir}_{B\in\mathcal{B}}\mathbb{P}{\restriction}B$ and $\mathbb{P}{\restriction}A \smallsetminus \{x\} = \text{limdir}_{B\in\mathcal{B}}\mathbb{P}{\restriction}(B \smallsetminus \{x\})$, so the same applies with quotients with $\mathbb{P}{\restriction}\bar{L}_x$ and $\mathbb{P}{\restriction}L_x$, respectively. Moreover, if $B \subseteq B'$ are in $\mathcal{B}$, by Lemma 2.1.4 $\langle \mathbb{P}{\restriction}B \smallsetminus \{x\}/\mathbb{P}{\restriction}L_x, \mathbb{P}{\restriction}B' \smallsetminus \{x\}/\mathbb{P}{\restriction}L_x, \mathbb{P}{\restriction}B/\mathbb{P}{\restriction}\bar{L}_x, \mathbb{P}{\restriction}B'/\mathbb{P}{\restriction}\bar{L}_x\rangle$ is a correct system with respect to M. Thus, by induction hypothesis and Theorem 3.3.11, $(\star, \mathbb{P}{\restriction}(A \smallsetminus \{x\})/\mathbb{P}{\restriction}L_x, \mathbb{P}{\restriction}A/\mathbb{P}{\restriction}\bar{L}_x, M, N, \sqsubset, c)$ holds.

(3) *A does not have a maximum.* Proceed exactly as in the previous case.

□

3.4.6 Corollary. *In a (ground) model V of* ZFC*, let $\mathbb{P}{\restriction}\langle L, \bar{\mathcal{I}}\rangle$ be a template iteration as in Example 2.4.3 such that, for every $x \in L_S$ and $B \in \hat{\mathcal{I}}_x$, $\mathbb{P}{\restriction}B$ forces that $\dot{\mathbb{Q}}_x^B$ is $\sqsubset$-good. Let $x \in L$ such that $L_x \in \mathcal{I}_x$, $\dot{c}$ a $\mathbb{P}{\restriction}(L_x \cup \{x\})$-name for a real and assume that $\mathbb{P}{\restriction}(L_x \cup \{x\})$ forces that $\dot{c}$ is $\sqsubset$-unbounded over $V^{\mathbb{P}\restriction L_x}$. Then, $\mathbb{P}{\restriction}L$ forces that $\dot{c}$ is $\sqsubset$-unbounded over $V^{\mathbb{P}\restriction(L \smallsetminus \{x\})}$.*

Proof. By Lemmas 3.3.8 and 3.3.9, this iteration satisfies condition (i) of Theorem 3.4.5. □

3.4.7 Corollary. *In a (ground) model V of* ZFC*, let $\mathbb{P}{\restriction}\langle L, \bar{\mathcal{I}}\rangle$ be a template iteration as in Example 2.4.3. Let $x \in L$ such that $L_x \in \mathcal{I}_x$, $\dot{c}$ a $\mathbb{P}{\restriction}(L_x \cup \{x\})$-name for a real and assume that $\mathbb{P}{\restriction}(L_x \cup \{x\})$ forces that $\dot{c} \notin V^{\mathbb{P}\restriction L_x}$. Then, $\mathbb{P}{\restriction}L$ forces that $\dot{c} \notin \mathbb{P}{\restriction}(L \smallsetminus \{x\})$.*

Proof. Use Corollary 3.4.6 with $\sqsubset$ as $=^*$ (see Example 3.2.21) and Lemma 3.2.22. □

CHAPTER 4

APPLICATIONS OF TEMPLATE ITERATIONS

This section includes the proof of some main results of this dissertation. We first show how to extend Blass's argument [Bl89] to get larger values of $\mathfrak{g}$ in models constructed by template iterations. The other result is to force $\mathfrak{s} < \kappa < \mathfrak{b} < \mathfrak{a}$ with a poset constructed by ultrapowers of template iterations, where κ is a measurable cardinal in the ground model and $\mathfrak{s}$ is allowed to take any arbitrary uncountable regular value below κ.

In Section 4.1, we show how to extend Blass's result about $\mathfrak{g}$ and how to use it in template iterations for fsi (see Example 2.4.2). Here, we use fsi techniques from [Br91] to force values for some cardinal invariants of Section 1.4. In Section 4.2, we explain Shelah's approach [S04] of forcing with ultrapowers of a poset by a measurable cardinal. We use this for the construction with ultrapowers of the model of $\mathfrak{s} < \kappa < \mathfrak{b} < \mathfrak{a}$ with κ measurable in Section 4.3.

4.1 Models with finite support iterations

Recall the following known result about the construction of models with small $\mathfrak{g}$.

4.1.1 Lemma ([Bl89], see also [Br10, Lemma 1.17])**.** *Let θ be an uncountable regular cardinal, $\langle V_\alpha\rangle_{\alpha\leq\theta}$ an increasing sequence of transitive models of* ZFC *such that*

(i) $[\omega]^\omega \cap (V_{\alpha+1} \smallsetminus V_\alpha) \neq \varnothing$,

(ii) $\langle [\omega]^\omega \cap V_\alpha\rangle_{\alpha<\theta} \in V_\theta$ *and*

(iii) $[\omega]^\omega \cap V_\theta = \bigcup_{\alpha<\theta} [\omega]^\omega \cap V_\alpha$.

Then, in V_θ, $\mathfrak{g} \leq \theta$.

Proof. In V_θ, for $\alpha < \theta$ let $C_\alpha = \{x \in [\omega]^\omega \ / \ \neg\exists_{a\in[\omega]^\omega\cap V_\alpha}(a \subseteq^* x)\}$. By (iii), $\bigcap_{\alpha<\theta} C_\alpha = \varnothing$, so it is enough to prove that each C_α is groupwise-dense ((ii) implies that $\langle C_\alpha\rangle_{\alpha<\theta} \in V_\theta$). $\subseteq^*$-closed is clear. Let $\langle I_n\rangle_{n<\omega}$ be an interval partition of ω. By (iii), we can find $\beta \in [\alpha, \theta)$ such that $\langle I_n\rangle_{n<\omega} \in V_\beta$.

In V_β, consider F a perfect a.d. family, e.g., by identifying ω with $2^{<\omega}$, F is the family of subsets of the form $a_f = \{f{\restriction}k \ / \ k < \omega\}$. Let M be a mad family extending F. Now, in $V_{\beta+1}$, any set of the form a_f with $f \in 2^\omega \cap V_{\beta+1} \smallsetminus V_\beta$ has finite intersection with any member of M because, for any $x \in M \smallsetminus F$, the statement "$\forall_{f\in 2^\omega}(|a_f \cap x| < \aleph_0)$" is $\mathbf{\Pi}^1_1$. By (i), choose such an f and put $z = \bigcup_{n\in a_f} I_n$. We show that $z \in C_\beta$, which implies that $z \in C_\alpha$. Towards a contradiction, assume that there is an $a \in [\omega]^\omega \cap V_\beta$ that is almost contained in z. Put $d = \{n < \omega \ / \ a \cap I_n \neq \varnothing\}$. Clearly, $d \in V_\beta$ and, as $a \subseteq^* z$, $d \subseteq^* a_f$, so d is almost disjoint from M where M is mad in V_β, a contradiction. □

For the following results, fix uncountable regular cardinals $\mu_1 \leq \mu_2 \leq \mu_3 \leq \nu$ and a cardinal $\lambda \geq \nu$. Also, products of the form $\alpha\gamma$ denote ordinal product.

4.1.2 Theorem. *If* $\lambda^{<\mu_3} = \lambda$, *then it is consistent with* ZFC *that* $\mathrm{add}(\mathcal{N}) = \mu_1$, $\mathrm{cov}(\mathcal{N}) = \mu_2$, $\mathfrak{p} = \mathfrak{s} = \mathfrak{g} = \mu_3$, $\mathrm{non}(\mathcal{N}) = \mathfrak{r} = \mathfrak{c} = \lambda$ *and that one of the following statements hold.*

(a) $\mathrm{non}(\mathcal{M}) = \mu_3$ *and* $\mathrm{cov}(\mathcal{M}) = \lambda$.

(b) $\mathrm{add}(\mathcal{M}) = \mathrm{cof}(\mathcal{M}) = \nu$.

(c) $\mathfrak{b} = \mu_3$, $\mathrm{non}(\mathcal{M}) = \mathrm{cov}(\mathcal{M}) = \nu$ *and* $\mathfrak{d} = \lambda$.

Proof. (a) Consider $\langle L = \lambda, \bar{\mathcal{I}}\rangle$ the template corresponding to a fsi of length λ (see Example 2.3.4). For each $\alpha < \lambda$ enumerate $[\alpha]^{<\mu_3} := \{C_{\alpha,\beta}\}_{\beta<\lambda}$. Fix a bijection $g : \lambda \to \lambda^3$ such that $g^{-1}(\alpha, \beta, \gamma) \geq \alpha, \beta, \gamma$ for any $\alpha, \beta, \gamma < \lambda$. Consider a template iteration $\mathbb{P}{\restriction}\langle\lambda, \bar{\mathcal{I}}\rangle$ as in Example 2.4.3 such that $L_S = \{\xi < \lambda \,/\, \exists_\delta(\xi = 4\delta)\}$, $\mathbb{S}_\xi = \mathbb{C}$ for $\xi \in L_S$ and, for each $\xi \in L_C$, if $\xi = 4\delta_\xi + r_\xi$ with $0 < r_\xi < 4$ and $g(\delta_\xi) = (\alpha, \beta, \gamma)$, then

- $C_\xi := C_{\alpha,\beta}$.
- $\{\dot{\mathbb{A}}_{\alpha,\beta,\eta}\}_{\eta<\lambda}$ is an enumeration of the $\mathbb{P}{\restriction}C_{\alpha,\beta}$-names for *all* the suborders of $\mathbb{A}$ of size $< \mu_1$.
- $\{\dot{\mathbb{B}}_{\alpha,\beta,\eta}\}_{\eta<\lambda}$ is an enumeration of the $\mathbb{P}{\restriction}C_{\alpha,\beta}$-names for *all* the suborders of $\mathbb{B}$ of size $< \mu_2$.
- $\{\dot{\mathcal{F}}_{\alpha,\beta,\eta}\}_{\eta<\lambda}$ is an enumeration of the $\mathbb{P}{\restriction}C_{\alpha,\beta}$-names for *all* the filter bases of size $< \mu_3$.
- If $r_\xi = 1$, then $\dot{\mathbb{Q}}_\xi = \dot{\mathbb{A}}_{\alpha,\beta,\gamma}$.
- If $r_\xi = 2$, then $\dot{\mathbb{Q}}_\xi = \dot{\mathbb{B}}_{\alpha,\beta,\gamma}$.
- If $r_\xi = 3$, then $\dot{\mathbb{Q}}_\xi = \mathbb{M}_{\dot{\mathcal{F}}_{\alpha,\beta,\gamma}}$.

By Lemma 2.4.7, $\mathbb{P}{\restriction}L$ is ccc and every name of a real has a support of size $< \mu_3$. Also, by Theorem 3.4.3, Lemmas 3.1.7, 3.2.13, 3.2.19 and Corollary 3.2.18, $\mathbb{P}{\restriction}L$ is μ_1-$\in^*_{\bar{H}}$-good, μ_2-$\pitchfork^{\bar{I}}$-good and μ_3-$\eqcirc$-good. We prove that $\mathbb{P}{\restriction}L$ forces the following.

- $\mathrm{add}(\mathcal{N}) = \mu_1$. To force $\geq$, let $\{\dot{N}_\eta\}_{\eta<\mu}$ be a sequence of $\mathbb{P}{\restriction}L$-names of Borel null sets with $\mu < \mu_1$. Then, there is an $\alpha < \lambda$ such that all the $\dot{N}_\eta$ ($\eta < \mu$) are $\mathbb{P}{\restriction}\alpha$-names (i.e., their Borel codes), so we can find a $\beta < \lambda$ such that these are $\mathbb{P}{\restriction}C_{\alpha,\beta}$-names. Step into $V^{\mathbb{P}{\restriction}C_{\alpha,\beta}}$. Find a model M of a large finite amount of ZFC such that $\{N_\eta \,/\, \eta < \mu\} \subseteq M$ and $|M| \leq \mu$. Now, back in V, find $\gamma < \lambda$ such that $\dot{\mathbb{A}}_{\alpha,\beta,\gamma}$ is a $\mathbb{P}{\restriction}C_{\alpha,\beta}$-name for $\mathbb{A}^{\dot{M}}$. Thus, with $\xi = 4g^{-1}(\alpha, \beta, \gamma) + 1$, $\mathbb{P}{\restriction}(C_{\alpha,\beta} \cup \{\xi\})$ adds a Borel null set that covers all the null sets in $\dot{M}$, in particular, it covers $\{\dot{N}_\eta \,/\, \eta < \mu\}$.
  To force $\mathrm{add}(\mathcal{N}) \leq \mu_1$, note that $\mathbb{P}{\restriction}\mu_1$ adds a μ_1-$\in^*_h$-unbounded family of Cohen reals of size μ_1 by Lemma 3.1.12. Also, this unbounded family is preserved in the $\mathbb{P}{\restriction}L$ extension because $\mathbb{P}{\restriction}L/\mathbb{P}{\restriction}\mu_1$ is forced by $\mathbb{P}{\restriction}\mu_1$ to be μ_1-$\in^*_{\bar{H}}$-good by Theorem 3.4.3 and the fact that $\mathbb{P}{\restriction}L/\mathbb{P}{\restriction}\mu_1$ is equivalent to a template iteration corresponding to a fsi on $[\mu_1, \lambda)$. Therefore, $\mathbb{P}{\restriction}L$ forces $\mathrm{add}(\mathcal{N}) = \mathfrak{b}_{\in^*_{\bar{H}}} \leq \mu_1$.
- $\mathrm{cov}(\mathcal{N}) = \mu_2$. Similar argument as before, but use the small suborders of $\mathbb{B}$ of the iteration and use μ_2-$\pitchfork^{\bar{I}}$-goodness to force $\mathrm{cov}(\mathcal{N}) \leq \mathfrak{b}_{\pitchfork^{\bar{I}}} \leq \mu_2$ (see Lemma 3.2.12).
- $\mathfrak{p} = \mathrm{non}(\mathcal{M}) = \mu_3$. To force $\mu_3 \leq \mathfrak{p}$ use the Mathias posets with small filter bases of the iteration like before. To force $\mathrm{non}(\mathcal{M}) \leq \mu_3$, use μ_3-$\eqcirc$-goodness.
- $\mathrm{cov}(\mathcal{M}) = \mathfrak{c} = \lambda$. As $|\mathbb{P}{\restriction}L| \leq \lambda$ and $\lambda^\omega = \lambda$, then $\mathfrak{c} \leq \lambda$ is clearly forced. To force $\lambda \leq \mathrm{cov}(\mathcal{M})$ note that, for any regular cardinal κ such that $\mu_3 \leq \kappa \leq \lambda$, $\mathbb{P}{\restriction}\kappa$ adds a κ-$\eqcirc$-unbounded family (by Lemma 3.1.12) that is preserved until the $\mathbb{P}{\restriction}L$-extension. Then, by Lemma 3.1.4, $\mathbb{P}{\restriction}L$ forces $\kappa \leq \mathfrak{d}_{\eqcirc} = \mathrm{cov}(\mathcal{M})$. As this is true for any such regular κ, this implies $\lambda \leq \mathrm{cov}(\mathcal{M})$ in the $\mathbb{P}{\restriction}L$-extension.

- $\mathfrak{g} = \mu_3$. As $\mathfrak{p} \leq \mathfrak{g}$, we only need to force $\mathfrak{g} \leq \mu_3$. Consider a partition $\{A_\eta\}_{\eta<\mu_3}$ of λ such that each A_η has size λ an intersects L_S. For each $\eta < \mu_3$, put $E_\eta = \bigcup_{\xi<\eta} A_\xi$. Now, if G is $\mathbb{P}{\restriction}L$-generic over V, in $W_{\mu_3} = V[G]$ let $W_\eta = V[G \cap \mathbb{P}{\restriction}E_\eta]$ for each $\eta < \mu_3$. It is enough to see that the conditions of Lemma 4.1.1 hold for the sequence $\{W_\eta\}_{\eta\leq\mu_3}$. Conditions (ii) and (iii) follow from the fact that, in V, $\mathbb{P}{\restriction}L = \text{limdir}_{\eta<\mu_3}\mathbb{P}{\restriction}E_\eta$, this because any condition in $\mathbb{P}{\restriction}L$ has a support of size $< \mu_3$. To see (i), in V, let $\xi \in A_\eta \cap L_S = (E_{\eta+1} \smallsetminus E_\eta) \cap L_S$, so $\mathbb{P}{\restriction}((E_\eta \cap \xi) \cup \{\xi\})$ adds a Cohen real over $V^{\mathbb{P}\restriction(E_\eta\cap\xi)}$ and, by Corollary 3.4.7, this new real does not belong to $V^{\mathbb{P}\restriction E_\eta} = W_\eta$.

(b) Let $\langle L = \lambda\nu, \bar{\mathcal{I}}\rangle$ be the template corresponding to a fsi of length $\lambda\nu$. Fix a bijection $h : \lambda \to \lambda\times\lambda\times 3$ and, for each $\alpha < \nu$, enumerate $[\lambda\alpha]^{<\mu_3} := \{C_{\alpha,\beta}\}_{\beta<\lambda}$. Perform a template iteration $\mathbb{P}{\restriction}\langle L, \bar{\mathcal{I}}\rangle$ such that $L_S = \{\lambda\alpha + 2\eta \;/\; \alpha < \nu,\ \eta < \lambda\}$, $\mathbb{S}_\xi = \mathbb{D}$ for each $\xi \in L_S$ and, for each $\xi \in L_C$, if $\xi = \lambda\alpha + 2\eta + 1$ for some $\alpha < \nu$, $\eta < \lambda$ and $h(\eta) = (\beta, \gamma, r)$, then

- $C_\xi := C_{\alpha,\beta}$
- $\{\dot{\mathbb{A}}_{\alpha,\beta,\eta}\}_{\eta<\lambda}$ is an enumeration of the $\mathbb{P}{\restriction}C_{\alpha,\beta}$-names for *all* the suborders of $\mathbb{A}$ of size $< \mu_1$.
- $\{\dot{\mathbb{B}}_{\alpha,\beta,\eta}\}_{\eta<\lambda}$ is an enumeration of the $\mathbb{P}{\restriction}C_{\alpha,\beta}$-names for *all* the suborders of $\mathbb{B}$ of size $< \mu_2$.
- $\{\dot{\mathcal{F}}_{\alpha,\beta,\eta}\}_{\eta<\lambda}$ is an enumeration of the $\mathbb{P}{\restriction}C_{\alpha,\beta}$-names for *all* the filter bases of size $< \mu_3$.
- If $r = 0$, then $\dot{\mathbb{Q}}_\xi = \dot{\mathbb{A}}_{\alpha,\beta,\gamma}$.
- If $r = 1$, then $\dot{\mathbb{Q}}_\xi = \dot{\mathbb{B}}_{\alpha,\beta,\gamma}$.
- if $r = 2$, then $\dot{\mathbb{Q}}_\xi = \mathbb{M}_{\dot{\mathcal{F}}_{\alpha,\beta,\gamma}}$.

As in (a), $\mathbb{P}{\restriction}L$ is ccc and every name of a real has a support of size $< \mu_3$. Also, by Theorem 3.4.3 and Lemmas 3.1.7, 3.2.13, 3.2.18 and 3.2.7, $\mathbb{P}{\restriction}L$ is μ_1-$\in^*_{\bar{H}}$-good, μ_2-$\pitchfork^{\bar{I}}$-good and μ_3-$\propto$-good. Similar arguments from (a) show that $\mathbb{P}{\restriction}L$ forces $\text{add}(\mathcal{N}) = \mu_1$, $\text{cov}(\mathcal{N}) = \mu_2$, $\mathfrak{p} = \mathfrak{g} = \mu_3$ and $\mathfrak{c} \leq \lambda$. We show that $\mathbb{P}{\restriction}L$ forces the following.

- $\mathfrak{s} \leq \mu_3$. $\mathbb{P}{\restriction}\mu_3$ adds a μ_3-$\propto$-unbounded family that is preserved in $\mathbb{P}{\restriction}L$ by μ_3-$\propto$-goodness. So, in the final extension, $\mathfrak{s} = \mathfrak{b}_\propto \leq \mu_3$.
- $\lambda \leq \text{non}(\mathcal{N})$. By Lemma 3.1.12, if κ is regular and $\mu_3 \leq \kappa \leq \lambda$, $\mathbb{P}{\restriction}\kappa$ adds a κ-$\pitchfork^{\bar{I}}$-unbounded family that is preserved in the $\mathbb{P}{\restriction}L$-extension. Therefore, $\kappa \leq \mathfrak{d}_{\pitchfork^{\bar{I}}} \leq \text{non}(\mathcal{N})$ is true in that extension for any such regular κ. Therefore, $\lambda \leq \text{non}(\mathcal{N})$.
- $\lambda \leq \mathfrak{r}$. Same argument as before, but use μ_3-$\propto$-goodness and κ-$\propto$-unbounded families.
- $\text{add}(\mathcal{M}) = \text{cof}(\mathcal{M}) = \nu$. As $\mathbb{P}{\restriction}L = \text{limdir}_{\alpha<\nu}\mathbb{P}{\restriction}\lambda\alpha$, by Lemma 3.1.12, it adds a ν-$\eqcirc$-unbounded family of Cohen reals, which make $\text{non}(\mathcal{M}) \leq \nu \leq \text{cov}(\mathcal{M})$. We are left to prove $\nu \leq \mathfrak{b}$ and $\mathfrak{d} \leq \nu$. Indeed, $\mathbb{P}{\restriction}\lambda(\alpha+1)$ adds a dominating real $\dot{d}_\alpha$ over $V^{\mathbb{P}\restriction\alpha}$ for any $\alpha < \nu$, so it is easy to see, in the $\mathbb{P}{\restriction}L$-extension, that $\{\dot{d}_\alpha \;/\; \alpha < \nu\}$ is a dominating family and that any family of reals of size $< \nu$ can be dominated by some $\dot{d}_\alpha$.

(c) Use the same template, bijection and enumerations of subsets of size $< \mu_3$ as in (b). Perform a template iteration $\mathbb{P}{\restriction}\langle L, \bar{\mathcal{I}}\rangle$ such that $L_S = \{\lambda\alpha + 2\eta \;/\; \alpha < \nu,\ \eta < \lambda\}$, $\mathbb{S}_\xi = \mathbb{E}$ for each $\xi \in L_S$ and, for each $\xi \in L_C$, if $\xi = \lambda\alpha + 2\eta + 1$ for some $\alpha < \nu$, $\eta < \lambda$ and $h(\eta) = (\beta, \gamma, r)$, then

- $C_\xi := C_{\alpha,\beta}$
- $\{\dot{\mathbb{A}}_{\alpha,\beta,\eta}\}_{\eta<\lambda}$ is an enumeration of the $\mathbb{P}{\restriction}C_{\alpha,\beta}$-names for *all* the suborders of $\mathbb{A}$ of size $< \mu_1$.
- $\{\dot{\mathbb{B}}_{\alpha,\beta,\eta}\}_{\eta<\lambda}$ is an enumeration of the $\mathbb{P}{\restriction}C_{\alpha,\beta}$-names for *all* the suborders of $\mathbb{B}$ of size $< \mu_2$.
- $\{\dot{\mathcal{F}}_{\alpha,\beta,\eta}\}_{\eta<\lambda}$ is an enumeration of the $\mathbb{P}{\restriction}C_{\alpha,\beta}$-names for *all* the filter bases of size $< \mu_3$.
- If $r = 0$, then $\dot{\mathbb{Q}}_\xi = \dot{\mathbb{A}}_{\alpha,\beta,\gamma}$.
- If $r = 1$, then $\dot{\mathbb{Q}}_\xi = \dot{\mathbb{B}}_{\alpha,\beta,\gamma}$.
- if $r = 2$, then $\dot{\mathbb{Q}}_\xi = \mathbb{M}_{\dot{\mathcal{F}}_{\alpha,\beta,\gamma}}$.

With the same techniques as before, $\mathbb{P}{\restriction}L$ forces the desired statements. Just notice that, to force $\mathfrak{b},\mathfrak{s}\leq\mu_3$ and $\lambda\leq\mathfrak{d},\mathfrak{r}$, we should use that $\mathbb{P}{\restriction}L$ is μ_3-$\triangleright$-good.

□

The same type of argument as in the previous proof leads to the following results.

4.1.3 Theorem. *Assume $\lambda^{<\mu_2}=\lambda$. It is consistent with* ZFC *that* $\mathrm{add}(\mathcal{N})=\mu_1$, $\mathfrak{p}=\mathfrak{b}=\mathfrak{s}=\mathfrak{g}=\mu_2$, $\mathrm{cov}(\mathcal{N})=\mathrm{non}(\mathcal{M})=\mathrm{cov}(\mathcal{M})=\mathrm{non}(\mathcal{N})=\nu$ *and* $\mathfrak{d}=\mathfrak{r}=\mathfrak{c}=\lambda$.

Proof. As in the proof of Theorem 4.1.2, perform a fsi (as a template iteration) of length $\lambda\nu$ alternating between $\mathbb{B}$, $\mathbb{C}$, suborders of $\mathbb{A}$ of size $<\mu_1$ and $\mathbb{M}_{\mathcal{F}}$ with filter base $\mathcal{F}$ of size $<\mu_2$ such that we use a book-keeping for the small posets. □

4.1.4 Theorem. *Assume $\lambda^{<\mu_1}=\lambda$. It is consistent that* $\mathfrak{p}=\mathfrak{g}=\mu_1$, $\mathrm{cov}(\mathcal{N})=\mathrm{add}(\mathcal{M})=\mathrm{cof}(\mathcal{M})=\mathrm{non}(\mathcal{N})=\nu$ *and* $\mathfrak{c}=\lambda$ *and that one of the following statements hold.*

(a) $\mathrm{add}(\mathcal{N})=\mu_1$ *and* $\mathrm{cof}(\mathcal{N})=\lambda$.

(b) $\mathrm{add}(\mathcal{N})=\mathrm{cof}(\mathcal{N})=\nu$.

Proof. (a) Perform a fsi (as a template iteration) of length $\lambda\nu$ alternating between $\mathbb{B}$, $\mathbb{D}$, suborders of $\mathbb{A}$ of size $<\mu_1$ and $\mathbb{M}_{\mathcal{F}}$ with filter base $\mathcal{F}$ of size $<\mu_1$ with a book-keeping to track all the small posets.

(b) Perform a fsi (as a template iteration) of length $\lambda\nu$ alternating between $\mathbb{LOC}^h$ and $\mathbb{M}_{\mathcal{F}}$ with filter base $\mathcal{F}$ of size $<\mu_1$ with a book-keeping to track all the small posets.

□

In this last result we do not know how to obtain values for $\mathfrak{s}$, $\mathfrak{r}$ and $\mathfrak{u}$. For example, in (a) we would like to obtain $\mathfrak{s}\leq\mu_1$ and $\lambda\leq\mathfrak{r}$, but the preservation properties related to $\mathfrak{s}$ (and $\mathfrak{r}$) that we know so far do not work for $\mathbb{B}$ and $\mathbb{D}$ at the same time, i.e., $\mathbb{D}$ is $\propto$-good but $\mathbb{B}$ is not and, although $\mathbb{B}$ is $\triangleright$-good, $\mathbb{D}$ is not because it adds dominating reals.

4.2 Forcing with ultrapowers

We present some facts, introduced by Shelah [S04] (see also [Br02] and [Br07]) about forcing with the ultrapower of a ccc poset by a measurable cardinal. Fix a measurable cardinal κ and a κ-complete ultrafilter $\mathcal{D}$ on κ.

Fix a poset $\mathbb{P}$. For notation, if $p\in\mathbb{P}^\kappa$, denote $p_\alpha=p(\alpha)$. For $p,q\in\mathbb{P}^\kappa$ say that $p\leq_{\mathcal{D}}q$ iff $p_\alpha\leq q_\alpha$ for $\mathcal{D}$-many α. The poset $\mathbb{P}^\kappa/\mathcal{D}$, ordered by $\bar{p}\leq\bar{q}$ iff $p\leq_{\mathcal{D}}q$, is the *$\mathcal{D}$-ultrapower of* $\mathbb{P}$.

4.2.1 Lemma (Shelah [S04], see also [Br02, Lemma 0.1])**.** *Consider $i:\mathbb{P}\to\mathbb{P}^\kappa/\mathcal{D}$ defined as $i(r)=\bar{r}$ where $r_\alpha=r$ for all $\alpha<\kappa$. Then, i is a complete embedding iff $\mathbb{P}$ is κ-cc.*

Proof. It is clear that i is increasing and that preserves incompatibilities. If $\{r_\alpha\ /\ \alpha<\lambda\}$ is a maximal antichain in $\mathbb{P}$ with $\lambda\geq\kappa$, put $p:=\langle r_\alpha\rangle_{\alpha<\kappa}$ and note that $\bar{p}\perp i(r_\alpha)$ for all $\alpha<\lambda$. This proves ($\Rightarrow$).

To see the converse, let $A\subseteq\mathbb{P}$ be a maximal antichain and $\bar{p}\in\mathbb{P}^\kappa/\mathcal{D}$. For each $\alpha<\kappa$ choose an $r_\alpha\in A$ such that $p_\alpha\parallel r_\alpha$. As $|A|<\kappa$, by κ-completeness of $\mathcal{D}$ we can find an $r\in A$ such that $r_\alpha=r$ for $\mathcal{D}$-many α. Therefore, $\bar{p}\parallel i(r)$. □

4.2.2 Lemma (Shelah [S04], see also [Br02, Lemma 0.2])**.** *If $\mu<\kappa$ and $\mathbb{P}$ is μ-cc, then $\mathbb{P}^\kappa/\mathcal{D}$ is also μ-cc. The same holds for μ-Knaster, μ-centered and μ-linked in place of μ-cc.*

Proof. We prove the lemma for the case of μ-cc. Let $\{\bar{p}^\xi\ /\ \xi<\mu\}$ be an antichain in $\mathbb{P}^\kappa/\mathcal{D}$. For distinct $\xi,\xi'<\mu$ let $A_{\xi,\xi'}:=\{\alpha<\kappa\ /\ p^\xi_\alpha\perp p^{\xi'}_\alpha\}\in\mathcal{D}$. As there are $<\kappa$-many such $A_{\xi,\xi'}$, their intersection is in $\mathcal{D}$. So, if $\alpha<\kappa$ is in such intersection, then $\{p^\xi_\alpha\ /\ \xi<\mu\}$ is an antichain in $\mathbb{P}$ of size μ. □

Fix a ccc poset $\mathbb{P}$. We analyze how $\mathbb{P}^\kappa/\mathcal{D}$-names for reals looks like in terms of $\mathbb{P}$-names of reals. For reference, consider ω^ω. First we show how to construct a $\mathbb{P}^\kappa/\mathcal{D}$-name from a sequence $\langle\dot{f}_\alpha\rangle_{\alpha<\kappa}$ of $\mathbb{P}$-names of reals. For each $\alpha<\omega$ and $n<\omega$, let $\{p^{n,j}_\alpha\ /\ j<\omega\}$ be a maximal antichain in $\mathbb{P}$ and $k^n_\alpha:\omega\to\omega$ a function such that $p^{n,j}_\alpha\Vdash\dot{f}_\alpha(n)=k^n_\alpha(j)$ for all $j<\omega$. Put $p^{n,j}=\langle p^{n,j}_\alpha\rangle_{\alpha<\kappa}$ and note that, for $n<\omega$, $\{\bar{p}^{n,j}\ /\ j<\omega\}$ is a maximal antichain in $\mathbb{P}^\kappa/\mathcal{D}$ by ω_1-completeness of $\mathcal{D}$. Also, as $\mathfrak{c}<\kappa$, there exists a $D\in\mathcal{D}$ and, for each $n<\omega$, a function $k^n:\omega\to\omega$ such that $k^n_\alpha=k^n$ for all $\alpha\in D$. Define $\dot{f}=\langle\dot{f}_\alpha\rangle_{\alpha<\kappa}/\mathcal{D}$ the $\mathbb{P}^\kappa/\mathcal{D}$-name for a real such that, for any $n,j<\omega$, $\bar{p}^{n,j}\Vdash\dot{f}(n)=k^n(j)$. Note that, if $\langle\dot{g}_\alpha\rangle_{\alpha<\kappa}$ is a sequence of $\mathbb{P}$-names of reals and $\Vdash_{\mathbb{P}}\dot{f}_\alpha=\dot{g}_\alpha$ for $\mathcal{D}$-many α, then $\Vdash_{\mathbb{P}^\kappa/\mathcal{D}}\dot{f}=\dot{g}$ where $\dot{g}=\langle\dot{g}_\alpha\rangle_{\alpha<\kappa}/\mathcal{D}$.

We show that any $\mathbb{P}^\kappa/\mathcal{D}$-name $\dot{f}$ for a real can be described in this way. For each $n<\omega$, let $A^n:=\{\bar{p}^{n,j}\ /\ j<\omega\}$ be a maximal antichain in $\mathbb{P}^\kappa/\mathcal{D}$ and $k^n:\omega\to\omega$ such that $\bar{p}^{n,j}\Vdash\dot{f}(n)=k^n(j)$. By κ-completeness of $\mathcal{D}$, we can find $D\in\mathcal{D}$ such that, for all $\alpha\in D$, $\{p^{n,j}_\alpha\ /\ j<\omega\}$ is a maximal antichain in $\mathbb{P}$ for any $n<\omega$. Let $\dot{f}_\alpha$ be the $\mathbb{P}$-name of a real such that $p^{n,j}_\alpha\Vdash_{\mathbb{P}}\dot{f}_\alpha=k^n(j)$. For $\alpha\in\kappa\smallsetminus D$ just choose any $\mathbb{P}$-name $\dot{f}_\alpha$ for a real, so we get that $\Vdash_{\mathbb{P}^\kappa/\mathcal{D}}\dot{f}=\langle\dot{f}_\alpha\rangle_{\alpha<\kappa}/\mathcal{D}$. This characterization of $\mathbb{P}^\kappa/\mathcal{D}$-names of reals is very useful to get consistency results about reals by using ultrapowers of posets. The next two results are consequences of this characterization that are very useful for the applications in Section 4.3.

4.2.3 Lemma. *Fix $m<\omega$ and a $\mathbf{\Sigma}^1_m$ property $\varphi(x)$ of reals. Let $\langle\dot{f}_\alpha\rangle_{\alpha<\kappa}$ be a sequence of $\mathbb{P}$-names of reals and put $\dot{f}=\langle\dot{f}_\alpha\rangle_{\alpha<\kappa}/\mathcal{D}$. Then, for $\bar{p}\in\mathbb{P}^\kappa/\mathcal{D}$, $\bar{p}\Vdash\varphi(\dot{f})$ iff $p_\alpha\Vdash_{\mathbb{P}}\varphi(\dot{f}_\alpha)$ for $\mathcal{D}$-many α.*

Proof. This is proved by induction on $m<\omega$. Recall that $\mathbf{\Sigma}^1_0=\mathbf{\Pi}^1_0$ corresponds to the pointclass of closed sets. Thus, if $\varphi(x)$ is a $\mathbf{\Sigma}^1_0$-property of reals, there exists a tree $T\subseteq\omega^\omega$ such that, for $x\in\omega^\omega$, $\varphi(x)$ iff $x\in[T]:=\{z\in\omega^\omega\ /\ \forall_{k<\omega}(z{\restriction}k\in T)\}$.

As in the previous discussion choose, for each $n<\omega$, a maximal antichain $\{\bar{p}^{n,j}\ /\ j<\omega\}$ on $\mathbb{P}^\kappa/\mathbb{D}$ and a function $k^n:\omega\to\omega$ such that $\bar{p}^{n,j}\Vdash\dot{f}(n)=k^n(j)$ and $p^{n,j}_\alpha\Vdash\dot{f}_\alpha(n)=k^n(j)$ for $\mathcal{D}$-many α. First, assume that $p_\alpha\Vdash f_\alpha\in[T]$ for $\mathcal{D}$-many α and fix $k<\omega$. If $\bar{q}\leq\bar{p}$, we can find a decreasing sequence $\{\bar{q}^i\}_{i\leq k}$ and a $t\in\omega^k$ such that $\bar{q}^0=\bar{q}$ and $\bar{q}^{i+1}\leq\bar{p}^{i,t(i)}$ for any $i<k$. Therefore, $\bar{q}^k\Vdash\dot{f}{\restriction}k=k^n\circ t$ and, for $\mathcal{D}$-many α, $q^k_\alpha\Vdash\dot{f}_\alpha{\restriction}k=k^n\circ t$, so $k^n\circ t\in T$.

Now, assume that $p_\alpha\not\Vdash f_\alpha\in[T]$ for $\mathcal{D}$-many α. Without loss of generality, we may assume that there is a $k<\omega$ such that $p_\alpha\Vdash\dot{f}_\alpha{\restriction}k\notin T$ for $\mathcal{D}$-many α. To prove $\bar{p}\Vdash\dot{f}{\restriction}k\notin T$ repeat the same argument as before, but note that this time we get $k^n\circ t\notin T$.

For the inductive step, assume that $\varphi(x)$ is $\mathbf{\Sigma}^1_{m+1}$, so $\varphi(x)\Leftrightarrow\exists_{y\in\omega^\omega}\psi(x,y)$ where $\psi(x,y)$ is some $\mathbf{\Pi}^1_m(\omega^\omega\times\omega^\omega)$-statement (notice that, if this theorem is valid for all $\mathbf{\Sigma}^1_m$-statements, then it is also valid for $\mathbf{\Pi}^1_m$). First assume that $p_\alpha\Vdash\exists_{z\in\omega^\omega}\psi(\dot{f}_\alpha,z)$ for $\mathcal{D}$-many α and, for those α, choose a $\mathbb{P}$-name $\dot{g}_\alpha$ such that $p_\alpha\Vdash\psi(\dot{f}_\alpha,\dot{g}_\alpha)$. By induction hypothesis, $\bar{p}\Vdash\psi(\dot{f},\dot{g})$ where $\dot{g}=\langle\dot{g}_\alpha\rangle_{\alpha<\kappa}/\mathcal{D}$. The converse is also easy. □

To finish this section, we prove that forcing with the ultrapower $\mathbb{P}^\kappa/\mathcal{D}$ of a ccc poset $\mathbb{P}$ destroys the maximality of big mad families in the $\mathbb{P}$-extension.

4.2.4 Lemma (Shelah [S04], see also [Br02, Lemma 0.3])**.** *Let $\dot{A}$ be a $\mathbb{P}$-name of a ad family such that $\Vdash_{\mathbb{P}}|\dot{\mathcal{A}}|\geq\kappa$. Then, $\Vdash_{\mathbb{P}^\kappa/\mathcal{D}}\dot{\mathcal{A}}$ is not maximal.*

Proof. Let $r\in\mathbb{P}$ and $\lambda\geq\kappa$ be a cardinal such that $r\Vdash_{\mathbb{P}}\dot{\mathcal{A}}=\{\dot{A}_\xi\ /\ \xi<\lambda\}$. Put $\dot{A}=\langle\dot{A}_\alpha\rangle_{\alpha<\kappa}/\mathcal{D}$ (this can be defined in a similar way by associating the characteristic function to each set), and show that it is a $\mathbb{P}^\kappa/\mathcal{D}$-name of an infinite subset of ω and $i(r)\Vdash\forall_{\xi<\lambda}(|\dot{A}_\xi\cap\dot{A}|<\aleph_0)$. But this is straightforward from Lemma 4.2.3. □

4.3 A model with $\mathfrak{a}$ large

This section is devoted to the proof of the following result.

4.3.1 Theorem. *Let κ be a measurable cardinal, $\theta < \kappa < \mu < \lambda$ uncountable regular cardinals such that $\theta^{<\theta} = \theta$ and $\lambda^\kappa = \lambda$. Then, there exists a ccc poset that forces $\mathfrak{s} = \theta < \kappa < \mathfrak{b} = \mathfrak{d} = \mu < \mathfrak{a} = \mathfrak{c} = \lambda$. Moreover, there is such a poset that also forces* $\mathrm{add}(\mathcal{N}) = \mathrm{cov}(\mathcal{N}) = \mathfrak{p} = \mathfrak{g} = \theta$*,* $\mathrm{add}(\mathcal{M}) = \mathrm{cof}(\mathcal{M}) = \mu$ *and* $\mathrm{non}(\mathcal{N}) = \mathfrak{r} = \lambda$.

Fix $\mathcal{D}$ a non-principal κ-complete ultrafilter on κ.

4.3.2 Definition (Appropriate template iteration)**.** A template iteration $\mathbb{P}{\restriction}\langle L, \mathcal{I}\rangle$ is *appropriate (for the proof of Theorem 4.3.1)* if the following conditions hold.

(I) $\lambda\mu \subseteq L$ is cofinal in L, $|L| = \lambda$ and $0 = \min(L)$.

(II) Every $x \in L$ has an immediate successor and, for $\alpha \in \lambda\mu$, $\alpha + 1$ is the immediate successor of α.

(III) If $\gamma \in \lambda\mu$ is a limit ordinal of cofinality $\neq \kappa$, then $\gamma = \sup_L \{\alpha \in \lambda\mu \,/\, \alpha < \gamma\}$.

(IV) L is partitioned into four disjoint sets L_H, L_A, L_F and L_T.

(V) $|L_H \cap [\lambda\xi, \lambda(\xi + 1)) \cap \mathbf{Ord}| = \lambda$ for any $\xi < \mu$.

(VI) For each $\alpha \in \lambda\mu$, $L_\alpha \in \mathcal{I}_\alpha$.

(VII) If $X \in [L]^{<\theta}$, then $|\mathcal{I}{\restriction}X| < \theta$.

(VIII) For $x \in L_H$ and $B \in \hat{\mathcal{I}}_x$, $\dot{\mathbb{Q}}_x^B$ is a $\mathbb{P}{\restriction}B$-name for $\mathbb{D}^{V^{\mathbb{P}{\restriction}B}}$.

(IX) For $x \in L_A$ there is a fixed $C_x \in \hat{\mathcal{I}}_x$ of size $< \theta$ and a $\mathbb{P}{\restriction}C_x$-name $\dot{\mathbb{A}}_x$ of a suborder of $\mathbb{A}$ of size $< \theta$ such that, for every $B \in \hat{\mathcal{I}}_x$,

$$\dot{\mathbb{Q}}_x^B = \begin{cases} \dot{\mathbb{A}}_x & \text{if } C_x \subseteq B, \\ \dot{\mathbb{1}} & \text{otherwise.} \end{cases}$$

(X) For $x \in L_F$ there is a fixed $C_x \in \hat{\mathcal{I}}_x$ of size $< \theta$ and $\dot{\mathcal{F}}_x$ a $\mathbb{P}{\restriction}C_x$-name for a filter base on ω of size $< \theta$ such that, for every $B \in \hat{\mathcal{I}}_x$,

$$\dot{\mathbb{Q}}_x^B = \begin{cases} \mathbb{M}_{\dot{\mathcal{F}}_x} & \text{if } C_x \subseteq B, \\ \dot{\mathbb{1}} & \text{otherwise.} \end{cases}$$

(XI) For $x \in L_T$ and $B \in \hat{\mathcal{I}}$, $\dot{\mathbb{Q}}_x^B$ is the trivial forcing.

(XII) Given $\dot{\mathbb{R}}$ a $\mathbb{P}{\restriction}L$-name for a subalgebra of $\mathbb{A}$ of size $< \theta$, there exists an $x \in L_A$ such that $\Vdash_{\mathbb{P}{\restriction}L} \dot{\mathbb{R}} = \dot{\mathbb{A}}_x$.

(XIII) Given $\dot{\mathcal{F}}$ a $\mathbb{P}{\restriction}L$-name for a filter base on ω of size $< \theta$, there exists an $x \in L_F$ such that $\Vdash_{\mathbb{P}{\restriction}L} \dot{\mathcal{F}} = \dot{\mathcal{F}}_x$.

Notice that an appropriate template iteration $\mathbb{P}{\restriction}\langle L, \mathcal{I}\rangle$ satisfies the hypothesis of Lemma 2.4.7, so it has the Knaster condition and the support of each condition and of each name of a real has size $< \theta$. Therefore, by Theorem 3.4.1 and Lemmas 3.1.7, 3.2.13 and 3.2.7, $\mathbb{P}{\restriction}L$ is θ-$\pitchfork^{\bar{I}}$-good and θ-$\propto$-good. Similar as in the proof of Theorem 4.1.2(b), we show that $\mathbb{P}{\restriction}L$ forces $\mathrm{add}(\mathcal{N}) = \mathrm{cov}(\mathcal{N}) = \mathfrak{p} = \mathfrak{g} = \mathfrak{s} = \theta$, $\mathrm{add}(\mathcal{M}) = \mathrm{cof}(\mathcal{M}) = \mu$ and $\mathrm{non}(\mathcal{N}) = \mathfrak{r} = \mathfrak{c} = \lambda$.

- $\theta \leq \mathrm{add}(\mathcal{N})$. Let G be $\mathbb{P}{\restriction}L$-generic over V. In $V[G]$, let $\{N_\eta \,/\, \eta < \nu\}$ be a family of Borel null sets with $\nu < \theta$. Let M be a model of a finite large amount of ZFC such that it contains the codes of all the N_η ($\eta < \nu$) and $|M| \leq \nu$. Back in V, by (XII), $\mathbb{P}{\restriction}L$ forces that $\mathbb{A}^{\dot{M}} = \dot{\mathbb{A}}_x$ for some $x \in L_A$, so $\mathbb{P}{\restriction}C_x * \dot{\mathbb{A}}_x$ adds a Borel null set that covers all the Borel null sets in $\dot{M}$, in particular, it covers $\{\dot{N}_\eta \,/\, \eta < \nu\}$.

- $\theta \leq \mathfrak{p}$. Similar argument as before, but use (XIII).

- $\text{cov}(\mathcal{N}) \leq \theta$. By (V), find $\alpha < \lambda$ minimal such that $|L_H \cap \alpha| = \theta$. Notice that $\text{cf}(\alpha) = \theta$ and, by (III), $\mathbb{P}{\restriction}L_\alpha = \text{limdir}_{\varepsilon<\alpha}\mathbb{P}{\restriction}L_\varepsilon$. Then, by Lemma 3.1.12, $\mathbb{P}{\restriction}L_\alpha$ adds a θ-$\pitchfork^{\bar{I}}$-unbounded family of size θ that is preserved in $\mathbb{P}{\restriction}L$ because $\mathbb{P}{\restriction}L/\mathbb{P}{\restriction}L_\alpha$ is θ-$\pitchfork^{\bar{I}}$-good by Theorem 3.4.1. Therefore, $\text{cov}(\mathcal{N}) \leq \mathfrak{b}_{\pitchfork^{\bar{I}}} \leq \theta$ is true in the $\mathbb{P}{\restriction}L$-extension.

- $\mathfrak{s} \leq \theta$. Same argument as before, but adding a θ-$\propto$-unbounded family.

- $\mathfrak{g} \leq \theta$. By (V), find an increasing sequence $\{E_\alpha\}_{\alpha<\theta}$ of subsets of L such that its union is L and, for each $\alpha < \theta$, $L_H \cap \lambda\mu \cap E_{\alpha+1} \smallsetminus E_\alpha \neq \varnothing$. Let G be $\mathbb{P}{\restriction}L$-generic over V and put $W_\theta = V[G]$ and $W_\alpha = V[G \cap \mathbb{P}{\restriction}E_\alpha]$ for any $\alpha < \theta$. It is enough to show that $\{W_\alpha\}_{\alpha\leq\theta}$ satisfies the conditions of Lemma 4.1.1. Conditions (ii) and (iii) hold because, in V, $\mathbb{P}{\restriction}L = \text{limdir}_{\alpha<\theta}\mathbb{P}{\restriction}E_\alpha$. To see (i), choose a $\beta \in L_H \cap \lambda\mu \cap E_{\alpha+1} \smallsetminus E_\alpha$ and note that $\mathbb{P}{\restriction}((E_\alpha \cap L_\beta) \cup \{\beta\})$ adds a Cohen real over $V^{\mathbb{P}{\restriction}(E_\alpha\cap L_\beta)}$ so, by Corollary 3.4.7 and (VI), that real does not belong to $V^{\mathbb{P}{\restriction}E_\alpha} = W_\alpha$.

- $\lambda \leq \mathfrak{r}$. By (V), $\mathbb{P}{\restriction}L_\lambda = \text{limdir}_{\alpha<\lambda}\mathbb{P}{\restriction}L_\alpha$, so it adds a λ-$\propto$-unbounded family of size λ that is preserved in the $\mathbb{P}{\restriction}L$-extension. Therefore, $\lambda \leq \mathfrak{d}_\propto = \mathfrak{r}$ in that extension.

- $\lambda \leq \text{non}(\mathcal{N})$. A similar argument like before, but adding a λ-$\pitchfork^{\bar{I}}$-unbounded family.

- $\mathfrak{c} \leq \lambda$. As $\theta^\omega = \theta$, by (VII), for any $X \in [L]^{<\theta}$ it is easy to see, by induction on $Z \in \mathcal{I}{\restriction}X$, that $|\mathbb{P}{\restriction}Z| \leq \theta$, so $|\mathbb{P}{\restriction}X| \leq \theta$. Then, as $\mathbb{P}{\restriction}L = \text{limdir}_{X\in[L]^{<\theta}}\mathbb{P}{\restriction}X$, its cardinality is $\leq \lambda$ because $\lambda^\kappa = \lambda$. Thus, it forces $\mathfrak{c} \leq \lambda$.

- $\text{add}(\mathcal{M}) = \text{cof}(\mathcal{M}) = \mu$. $\mathbb{P}{\restriction}L = \text{limdir}_{\xi<\mu}\mathbb{P}{\restriction}L_{\lambda\xi}$ and, by (V), $\mathbb{P}{\restriction}L_{\lambda\xi+1}$ add a dominating real $\dot{d}_\xi$ over $V^{\mathbb{P}{\restriction}L_{\lambda\xi}}$. $\mathbb{P}{\restriction}L$ forces that $\{\dot{d}_\xi \,/\, \xi < \mu\}$ is a dominating family and that any family of reals of size $< \mu$ is bounded by some $\dot{d}_\xi$, which implies $\mu \leq \mathfrak{b} \leq \mathfrak{d} \leq \mu$. On the other hand, by Lemma 3.1.12, $\mathbb{P}{\restriction}L$ adds a μ-$\eqcirc$-unbounded family of size μ, so it forces $\text{non}(\mathcal{M}) \leq \mu \leq \text{cov}(\mathcal{M})$.

Therefore, to prove Theorem 4.3.1, it is enough to construct an appropriate template iteration that forces $\mathfrak{a} \geq \lambda$. This will be done by constructing a chain of appropriate template iterations of length λ such that the inductive step is done by taking ultrapowers (so we can use Lemma 4.2.4 to force $\mathfrak{a}$ to be large). Before proceeding with this construction, we explain how we deal with the inductive and limit steps for the construction of that chain.

Fix an appropriate template iteration $\mathbb{P}{\restriction}\langle L, \mathcal{I}\rangle$. Recall from the context of Lemma 2.3.7 the templates $\mathcal{I}^*$ and $\mathcal{I}^\dagger$ associated to the ultrapower L^* of the linear order L. We show how to construct, in a canonical way, an appropriate template iteration $\mathbb{P}^\dagger{\restriction}\langle L^*, \mathcal{I}^\dagger\rangle$ that is forcing equivalent to the ultrapower of $\mathbb{P}{\restriction}L$.

As $\text{cf}(\lambda\mu) = \mu > \kappa$, it is easy to note that $\lambda\mu$ is still cofinal in L^*. By standard arguments with ultrapowers, as $\lambda^\kappa = \lambda$, conditions (I)-(III) of Definition 4.3.2 are satisfied by L^*. Let $L^*_H := L^\kappa_H/\mathcal{D}$, L^*_A, L^*_F and L^*_T defined likewise. (IV)-(VII) for $\langle L^*, \mathcal{I}^*\rangle$ and $\langle L^*, \mathcal{I}^\dagger\rangle$ are clear, the last one by Lemma 2.3.8. Notice that $L^*_H \cap L = L_H$, $L^*_A \cap L = L_A$, $L^*_F \cap L = L_F$ and $L^*_T \cap L = L_T$.

4.3.3 Lemma (Ultrapower of a template iteration). *There is a template iteration $\mathbb{P}^*{\restriction}\langle L^*, \mathcal{I}^*\rangle$ such that (VIII)-(XI) hold and, for any $\bar{A} = [\{A_\alpha\}_{\alpha<\kappa}] \subseteq L^*$, there is an onto embedding $F_{\bar{A}} : \prod_{\alpha<\kappa}\mathbb{P}{\restriction}A_\alpha/\mathcal{D} \to \mathbb{P}^*{\restriction}\bar{A}$ such that, for any $\bar{D} = [\{D_\alpha\}_\alpha] \subseteq \bar{A}$, $F_{\bar{D}} \subseteq F_{\bar{A}}$.*

Proof. To define the desired template iteration $\mathbb{P}^*{\restriction}\langle L^*, \mathcal{I}^*\rangle$, it is enough to show how C_x, $\dot{A}_x$ and $\dot{\mathcal{F}}_x$ are defined for (IX) and (X). This is done in parallel with the construction, by recursion on $\Upsilon^{\bar{\mathcal{I}}^*}(\bar{A})$, of the desired onto embeddings.

(IX) For $\bar{x} \in L^*_A$, let $\bar{C}_{\bar{x}} = [\{C_{x_\alpha}\}_{\alpha<\kappa}] \in \hat{\mathcal{I}}^*_{\bar{x}}$. Define the $\mathbb{P}^*{\restriction}\bar{C}_{\bar{x}}$-name $\dot{\mathbb{A}}^*_{\bar{x}} = \langle \dot{\mathbb{A}}_{x_\alpha}\rangle_{\alpha<\kappa}$ in the following way. By ccc-ness, for $\mathcal{D}$-many α there is a cardinal $\nu_\alpha < \theta$ such that $\mathbb{P}{\restriction}C_{x_\alpha}$ forces $|\dot{\mathbb{A}}_{x_\alpha}| \leq \nu_\alpha$. As $\theta < \kappa$, there is a cardinal $\nu < \theta$ such that $\nu_\alpha = \nu$ for $\mathcal{D}$-many α. For those α, let $\dot{\mathbb{A}}_{x_\alpha} = \{\dot{a}_{\alpha,\xi} \,/\, \xi < \nu\}$ and $\dot{a}^*_\xi = \langle \dot{a}_{\alpha,\xi}\rangle_{\alpha<\kappa}/\mathcal{D}$, which is a $\prod_{\alpha<\kappa}\mathbb{P}{\restriction}C_{x_\alpha}/\mathcal{D}$-name for a real. But, by induction hypothesis, this ultraproduct is equivalent to $\mathbb{P}^*{\restriction}\bar{C}_{\bar{x}}$ by the function $F_{\bar{C}_{\bar{x}}}$, so let $\dot{\mathbb{A}}^*_{\bar{x}}$ be a $\mathbb{P}^*{\restriction}\bar{C}_{\bar{x}}$-name for $\{\dot{a}^*_\xi \mathrel{/} \xi < \nu\}$. By Lemma 4.2.3, $\mathbb{P}^*{\restriction}\bar{C}_{\bar{x}}$ forces that $\dot{\mathbb{A}}^*_{\bar{x}}$ is a suborder of $\mathbb{A}$.

Note that, if $x \in L_A$, then $\bar{C}_x = C_x$ because $|C_x| < \theta$, so $\dot{\mathbb{A}}^*_x = \dot{\mathbb{A}}_x$.

(X) For $\bar{x} \in L^*_F$, let $\bar{C}_{\bar{x}} = [\{C_{x_\alpha}\}_{\alpha<\kappa}] \in \hat{\mathcal{I}}^*_{\bar{x}}$. By a similar argument as before and Lemma 4.2.3, we can define a $\mathbb{P}^*{\restriction}\bar{C}_{\bar{x}}$-name $\dot{\mathcal{F}}^*_{\bar{x}} := \langle \dot{\mathcal{F}}_{x_\alpha}\rangle_{\alpha<\kappa}/\mathcal{D}$ of a filter base on ω of size $< \theta$.

Like before, if $x \in L_F$, then $C^*_x = C_x$ and $\dot{\mathcal{F}}^*_x = \dot{\mathcal{F}}_x$.

Now, we construct $F_{\bar{A}}$. Let $\bar{p} \in \prod_{\alpha<\kappa} \mathbb{P}{\restriction}A_\alpha/\mathcal{D}$, that is, $p_\alpha \in \mathbb{P}{\restriction}A_\alpha$ for $\mathcal{D}$-many α. Let $x_\alpha := \max(\mathrm{dom}(p_\alpha))$, so there exists a $B_\alpha \in \mathcal{I}_{x_\alpha}{\restriction}A$ such that $p_\alpha{\restriction}L_{x_\alpha} \in \mathbb{P}{\restriction}B_\alpha$ and $p_\alpha(x_\alpha)$ is a $\mathbb{P}{\restriction}B_\alpha$-name for a condition in $\dot{\mathbb{Q}}^{B_\alpha}_{x_\alpha}$. Let $\bar{r} := \langle p_\alpha{\restriction}L_{x_\alpha}\rangle_{\alpha<\kappa}/\mathcal{D}$ and $p(\bar{x}) := \langle p_\alpha(x_\alpha)\rangle_{\alpha<\kappa}/\mathcal{D}$ which is a $\mathbb{P}^*{\restriction}\bar{B}$-name for a real (by induction hypothesis), where $\bar{B} := [\{B_\alpha\}_{\alpha<\kappa}] \in \mathcal{I}^*_{\bar{x}}{\restriction}\bar{A}$. By considering cases on (VIII), (IX), (X) and (XI), by Lemma 4.2.3, $p(\bar{x})$ is actually a $\mathbb{P}^*{\restriction}\bar{B}$-name for a condition in $\dot{\mathbb{Q}}^{*\bar{B}}_{\bar{x}}$, so define $F_{\bar{A}}(\bar{p}) = F_{\bar{B}}(\bar{r})^\frown\langle p(\bar{x})\rangle_{\bar{x}}$. Note that this definition does not depend on $\bar{B}$. □

A template iteration $\mathbb{P}^\dagger{\restriction}\langle L^*, \bar{\mathcal{I}}^\dagger\rangle$ can be defined in a similar way as in the previous proof, so that $\mathbb{P}^\dagger{\restriction}\bar{A}$ is forcing equivalent to $\prod_{\alpha<\kappa}\mathbb{P}{\restriction}A_\alpha/\mathcal{D}$ for any $\bar{A} = [\{A_\alpha\}_{\alpha<\kappa}] \subseteq L^*$. Notice that $\langle L^*, \bar{\mathcal{I}}^\dagger\rangle$ is a θ-innocuous extension of $\langle L^*, \bar{\mathcal{I}}^*\rangle$ (Lemma 2.3.7), so, by Lemma 2.4.9, $\mathbb{P}^\dagger{\restriction}\bar{A}$ *is forcing equivalent to* $\mathbb{P}^*{\restriction}\bar{A}$.

4.3.4 Lemma. *$\mathbb{P}^*{\restriction}\langle L^*, \bar{\mathcal{I}}^*\rangle$ and $\mathbb{P}^\dagger{\restriction}\langle L^*, \bar{\mathcal{I}}^\dagger\rangle$ are appropriate template iterations. Moreover, $\mathbb{P}{\restriction}A$ is forcing equivalent to $\mathbb{P}^*{\restriction}A$ and $\mathbb{P}^\dagger{\restriction}A$ for any $A \subseteq L$.*

Proof. It remains to prove conditions (XII) and (XIII) for both iterations. As every set in $\mathcal{I}^\dagger_{\bar{x}}$ is contained in some set in $\mathcal{I}^*_{\bar{x}}$ for any $\bar{x} \in L^*$, it is enough to consider only the case for $\bar{\mathcal{I}}^*$. As the proof of both conditions are similar, we only show (XIII). Let $\dot{\bar{\mathcal{F}}}$ be a $\mathbb{P}^*{\restriction}L^*$-name for a filter base on ω of size $< \theta$. By ccc-ness, find $\nu < \theta$ such that $\dot{\bar{\mathcal{F}}}$ is forced to have size $\leq \nu$ and let $\dot{\bar{\mathcal{F}}} = \left\{\dot{\bar{U}}_\epsilon \mathrel{/} \epsilon < \nu\right\}$. Each $\dot{\bar{U}}_\epsilon$ is of the form $\langle \dot{U}_{\alpha,\epsilon}\rangle_{\alpha<\kappa}/\mathcal{D}$ where each $\dot{U}_{\alpha,\epsilon}$ is a $\mathbb{P}{\restriction}L$-name for an infinite subset of ω. As $\nu < \theta$, $\dot{\mathcal{F}}_\alpha := \left\{\dot{U}_{\alpha,\epsilon} \mathrel{/} \epsilon < \nu\right\}$ is a $\mathbb{P}{\restriction}L$-name for a filter base for $\mathcal{D}$-many α, so, by (XI), there exists an $x_\alpha \in L_F$ such that $\Vdash_{\mathbb{P}{\restriction}L} \dot{\mathcal{F}}_\alpha = \dot{\mathcal{F}}_{x_\alpha}$. Then, $\Vdash_{\mathbb{P}^*{\restriction}L^*} \dot{\bar{\mathcal{F}}} = \dot{\mathcal{F}}^*_{\bar{x}}$.

The second part of the proof follows from Lemma 2.4.9 because, for $x \in L$, $\mathcal{I}^*_x{\restriction}L = \mathcal{I}^\dagger_x{\restriction}L$ and $\langle L, \bar{\mathcal{I}}^*{\restriction}L\rangle$ is a strongly θ-innocuous extension of $\langle L, \bar{\mathcal{I}}\rangle$. □

Now, we explain how we deal, in general, with the limit step. Let $\delta \leq \lambda$ be a limit ordinal and consider a chain $\{\langle L^\alpha, \bar{\mathcal{I}}^\alpha\rangle\}_{\alpha<\delta}$ of templates and appropriate template iterations $\mathbb{P}^\alpha{\restriction}\langle L^\alpha, \bar{\mathcal{I}}^\alpha\rangle$ with the following properties for all $\alpha < \beta < \delta$.

(1) $\langle L^\beta, \bar{\mathcal{I}}^\beta\rangle$ is a strongly θ-innocuous extension of $\langle L^\alpha, \bar{\mathcal{I}}^\alpha\rangle$.

(2) For $x \in L^\alpha$, its immediate successor in L^α is the same as in L^β.

(3) $L^\alpha_H = L^\beta_H \cap L^\alpha$, $L^\alpha_A \subseteq L^\beta_A$ and $L^\alpha_F \subseteq L^\beta_F$.

(4) If $x \in L^\alpha_A$, then $C^\alpha_x = C^\beta_x$ and $\Vdash_{\mathbb{P}^\beta{\restriction}C^\beta_x} \dot{\mathbb{A}}^\alpha_x = \dot{\mathbb{A}}^\beta_x$.

(5) If $x \in L^\alpha_F$, then $C^\alpha_x = C^\beta_x$ and $\Vdash_{\mathbb{P}^\beta{\restriction}C^\beta_x} \dot{\mathcal{F}}^\alpha_x = \dot{\mathcal{F}}^\beta_x$.

Corollary 2.4.8 implies that $\mathbb{P}^\alpha{\restriction}X$ is a complete suborder of $\mathbb{P}^\beta{\restriction}X$ for any $X \subseteq L_\alpha$.

Consider L^δ and the templates $\bar{\mathcal{I}}$ and $\bar{\mathcal{J}}$ as in the context of Lemma 2.3.10. Let $L^\delta_H = \bigcup_{\alpha<\delta} L^\alpha_H$, L^δ_A a set disjoint from L^δ_H that contains $\bigcup_{\alpha<\delta} L^\alpha_A$, L^δ_F a set disjoint from $L^\delta_H \cup L^\delta_A$ that contains $\bigcup_{\alpha<\delta} L^\alpha_F$ and $L^\delta_T = L^\delta \setminus (L^\delta_H \cup L^\delta_A \cup L_F)$. Properties (I)-(V) are straightforward for L^δ, moreover, properties (1)-(3) hold for any $\alpha < \delta$ by replacing β by δ and for both templates $\bar{\mathcal{I}}$ and $\bar{\mathcal{J}}$. (VII) also holds for both templates because of Lemma 2.3.11. Nevertheless, (VI) holds for $\bar{\mathcal{J}}$ but it need not hold for $\bar{\mathcal{I}}$.

We show how to define template iterations $\mathbb{P}^\delta_0{\restriction}\langle L^\delta,\bar{\mathcal{I}}\rangle$ and $\mathbb{P}^\delta_1{\restriction}\langle L^\delta,\bar{\mathcal{J}}\rangle$ such that they are close to be appropriate and have nice agreement with the template iterations $\mathbb{P}^\alpha{\restriction}\langle L^\alpha,\bar{\mathcal{I}}^\alpha\rangle$ for $\alpha<\delta$. We just need to be specific about (IX) and (X) in order to define the iterations. In the case of $\bar{\mathcal{I}}$, for $x\in L^\delta_A\cup L^\delta_F$, if there is some $\alpha<\delta$ such that $x\in L^\alpha_A\cup L^\alpha_F$, let $C^\delta_x:=C^\alpha_x$ and $\dot{\mathbb{A}}^\delta_x=\dot{\mathbb{A}}^\alpha_x$ or $\dot{\mathcal{F}}^\delta_x=\dot{\mathcal{F}}^\alpha_x$, depending on the case. Otherwise, choose C^δ_x and $\dot{\mathbb{A}}^\delta_x$ or $\dot{\mathcal{F}}^\delta_x$ freely.

For this to be defined, it is necessary to proceed inductively and guarantee that, for each $\alpha<\delta$, $\mathbb{P}^\alpha{\restriction}X$ is a complete suborder of $\mathbb{P}^\delta_0{\restriction}X$ for any $X\subseteq L^\alpha$, but this can be done along the way using Corollary 2.4.8. Notice that (4) and (5) hold in this case by replacing β by δ (for $\mathbb{P}^\delta_0{\restriction}C^\delta_x$).

$\mathbb{P}^\delta_1{\restriction}\langle L^\delta,\bar{\mathcal{J}}\rangle$ is defined in the same way by just ensuring to make the same choices of C^δ_x, $\dot{\mathbb{A}}^\delta_x$ and $\dot{\mathcal{F}}^\delta_x$ as for $\bar{\mathcal{I}}$. The same conclusions as in the previous case hold in the same way. However, it is not always the case that properties (XII) and (XIII) hold, moreover, they will depend on the particular "free" choices of C^δ_x, $\dot{\mathbb{A}}^\delta_x$ and $\dot{\mathcal{F}}^\delta_x$. There is one case in which both properties hold for both template iterations.

4.3.5 Lemma (Direct limit of a chain of template iterations)**.** *Assume that* $\mathrm{cf}(\delta)\geq\theta$, $L^\delta_A=\bigcup_{\alpha<\delta}L^\alpha_A$ *and* $L^\delta_F=\bigcup_{\alpha<\delta}L^\alpha_F$*. Then, both template iterations* $\mathbb{P}^\delta_0{\restriction}\langle L^\delta,\bar{\mathcal{I}}\rangle$ *and* $\mathbb{P}^\delta_1{\restriction}\langle L^\delta,\bar{\mathcal{J}}\rangle$ *are forcing equivalent and satisfy (XII) and (XIII). Moreover,* $\mathbb{P}^\delta_0{\restriction}L^\delta=\mathrm{limdir}_{\alpha<\delta}\mathbb{P}^\alpha{\restriction}L^\alpha$ *and* $\mathbb{P}^\delta_1{\restriction}\langle L^\delta,\bar{\mathcal{J}}\rangle$ *is appropriate.*

Proof. By Lemmas 2.3.10 and 2.4.9, both template iterations are equivalent, so it is enough to prove (XII) and (XIII) for the iteration along $\bar{\mathcal{I}}$.

We claim that $\mathbb{P}^\delta_0{\restriction}A=\mathrm{limdir}_{\alpha<\delta}\mathbb{P}^\alpha{\restriction}(L^\alpha\cap A)$ for any $A\subseteq L^\delta$. Proceed by induction on $\Upsilon^{\bar{\mathcal{I}}}(A)$. Let $p\in\mathbb{P}^\delta_0{\restriction}A$ and $x=\max(\mathrm{dom}(p))$, so there exists a $B\in\mathcal{I}_x{\restriction}A$ such that $p{\restriction}L^\delta_x\in\mathbb{P}^\delta_0{\restriction}B$ and $p(x)$ is a $\mathbb{P}^\delta_0{\restriction}B$-name for a real in $\dot{\mathbb{Q}}^{\delta,B}_{0,x}$. By induction hypothesis and ccc-ness, find $\alpha<\delta$ such that $x\in L^\alpha$, $B\in\mathcal{I}^\alpha_x{\restriction}(L^\alpha\cap A)$, $p{\restriction}L^\delta_x=p{\restriction}L^\alpha_x\in\mathbb{P}^\alpha{\restriction}B$ and $p(x)$ is a $\mathbb{P}^\alpha{\restriction}B$-name for a real. In the case that $x\in L^\delta_H$, then $x\in L^\alpha_H$ by (3), so $p(x)$ is a $\mathbb{P}^\alpha{\restriction}B$-name for a condition in Hechler forcing; in the case that $x\in L^\delta_A$ and $C_x\subseteq B$, increasing α if necessary, $x\in L^\alpha_A$, so clearly $p(x)$ is a $\mathbb{P}^\alpha{\restriction}B$-name for a condition in $\dot{\mathbb{A}}^\alpha_x$ (the case $C_x\nsubseteq B$ is easier); if $x\in L^\delta_F$ and $C_x\subseteq B$, like before, $p(x)$ is a $\mathbb{P}^\alpha{\restriction}B$-name for a condition in $\mathbb{M}_{\dot{\mathcal{F}}^\alpha_x}$; and if $x\in L^\delta_T$, then $x\in L^\alpha_T$ and so $p(x)$ is clearly a name for the trivial condition. Then, in any case, $p\in\mathbb{P}^\alpha{\restriction}(L^\alpha\cap A)$.

We show (XIII) ((XII) is proved in a similar way). Let $\dot{\mathcal{F}}$ be a $\mathbb{P}^\delta_0{\restriction}L^\delta$-name for a filter base on ω of size $<\theta$. Then, as $\mathrm{cf}(\delta)\geq\theta$ and the previous claim, find $\alpha<\delta$ such that $\dot{\mathcal{F}}$ is a $\mathbb{P}^\alpha{\restriction}L^\alpha$-name, so there exists an $x\in L^\alpha_F\subseteq L^\delta_F$ such that $\dot{\mathcal{F}}$ is forced to be equal to $\dot{\mathcal{F}}^\alpha_x=\dot{\mathcal{F}}^\delta_x$. □

The use of the indexed template $\bar{\mathcal{I}}$ is to prove the preceding result. But, for the construction of the model we want, $\bar{\mathcal{J}}$ is the one used for the limit step.

Proof of Theorem 4.3.1. Fix a bijective enumeration $\{2\alpha+1\ /\ \alpha<\lambda\}:=\{\tau_{\alpha,\beta}\ /\ \alpha,\beta<\lambda\}$ (the odd ordinals below λ), a bijection $g:\lambda\to\lambda\times\theta$ and an increasing enumeration $\langle\delta_\alpha\rangle_{\alpha<\lambda}$ of 0 and all the limit ordinals below λ that have cofinality $<\theta$. For an ordered pair $z=(x,y)$, denote $(z)_0:=x$ and $(z)_1:=y$.

By recursion on $\gamma\leq\lambda$, define a chain of templates $\{\langle L^\gamma,\bar{\mathcal{I}}^\gamma\rangle\}_{\gamma\leq\lambda}$ such that they satisfy conditions (I)-(VII) and (1)-(3). It is also required that, for $\gamma<\delta_\alpha$, $(L^\gamma_F\cup L^\gamma_A)\cap\{\lambda\xi+\tau_{\alpha,\beta}\ /\ \xi<\mu\text{ and }\beta<\lambda\}=\varnothing$ and, if $\delta_\alpha\leq\gamma$, then $\{\lambda\xi+\tau_{\alpha,2\beta}\ /\ \xi<\mu\text{ and }\beta<\lambda\}\subseteq L^\gamma_A$ and $\{\lambda\xi+\tau_{\alpha,2\beta+1}\ /\ \xi<\mu\text{ and }\beta<\lambda\}\subseteq L^\gamma_F$.

Let $L^0:=\lambda\mu$ and $\bar{\mathcal{I}}^0$ be the template corresponding to a fsi of length $\lambda\mu$ (Example 2.3.4(2)). Put $L^0_H:=\{2\alpha\ /\ \alpha<\lambda\mu\}=\{\lambda\xi+2\alpha\ /\ \xi<\mu\text{ and }\alpha<\lambda\}$, $L^0_A:=\{\lambda\xi+\tau_{0,2\beta}\ /\ \xi<\mu\text{ and }\beta<\lambda\}$, $L^0_F:=\{\lambda\xi+\tau_{0,2\beta+1}\ /\ \xi<\mu\text{ and }\beta<\lambda\}$ and $L^0_T:=L^0\smallsetminus(L^0_H\cup L^0_F)=\{\lambda\xi+\tau_{\alpha,\beta}\ /\ \xi<\mu,\alpha,\beta<\lambda\text{ and }\alpha\neq0\}$. Clearly, conditions (I)-(VII) hold for $\langle L^0,\bar{\mathcal{I}}^0\rangle$.

Given $\langle L^\gamma,\bar{\mathcal{I}}^\gamma\rangle$, let $\langle L^{\gamma+1},\bar{\mathcal{I}}^{\gamma+1}\rangle:=\langle(L^\gamma)^*,(\bar{\mathcal{I}}^\gamma)^\dagger\rangle$ as in the previous discussion of ultrapowers. Clearly, (I)-(VII) hold and, moreover, $L^{\gamma+1}_H\cap L^\gamma=L^\gamma_H$, the same for the other type of sets, so (2) and (3) hold. For (1), recall that $\langle L^{\gamma+1},\bar{\mathcal{I}}^{\gamma+1}\rangle$ is a strongly θ-innocuous extension of $\langle L^\gamma,\bar{\mathcal{I}}^\gamma\rangle$, so it is needed to prove that it is also a strongly θ-innocuous extension of $\langle L^\beta,\bar{\mathcal{I}}^\beta\rangle$ for each $\beta<\gamma$. Indeed, the non-trivial part is to see that, for $x\in L^\beta$, $\mathcal{I}^{\gamma+1}_x{\restriction}L^\beta\subseteq\mathcal{I}^{\gamma+1}_x$. If $A\in\mathcal{I}^{\gamma+1}_x{\restriction}L^\beta$, then, as $L^\beta\subseteq L^\gamma$, there exists $\bar{H}\in(\mathcal{I}^\gamma_x)^*$ such that $A=\bar{H}\cap L^\beta$. Then, $A=[\{H_\alpha\cap L^\beta\}_{\alpha<\kappa}]\in(\mathcal{I}^\gamma_x)^*$ because $\mathcal{I}^\gamma_x{\restriction}L^\beta\subseteq\mathcal{I}^\gamma_x$.

If δ is a limit ordinal, define $L^\delta := \bigcup_{\beta<\delta} L^\beta$ and $\bar{\mathcal{I}}^\delta := \bar{\mathcal{J}}$ according to the previous discussion about chains of templates. Hence, it is only needed to be specific about how to define L^δ_A and L^δ_F. If $\mathrm{cf}(\delta) \geq \theta$, put $L^\delta_A := \bigcup_{\beta<\delta} L^\beta_A$ and $L^\delta_F := \bigcup_{\beta<\delta} L^\beta_F$. Otherwise, let $L^\delta_A := \bigcup_{\beta<\delta} L^\beta_A \cup \{\lambda\xi + \tau_{\alpha,2\beta} \ / \ \xi < \mu \text{ and } \beta < \lambda\}$ and $L^\delta_F := \bigcup_{\beta<\delta} L^\beta_F \cup \{\lambda\xi + \tau_{\alpha,2\beta+1} \ / \ \xi < \mu \text{ and } \beta < \lambda\}$ where $\alpha < \lambda$ is such that $\delta = \delta_\alpha$. Clearly, (I)-(VII) and (1)-(3) are satisfied.

Again, by recursion on $\gamma \leq \lambda$, define appropriate template iterations $\mathbb{P}^\gamma{\restriction}\langle L^\gamma, \bar{\mathcal{I}}^\gamma\rangle$ such that (4) and (5) are satisfied for them.

Looking at the template $\langle L^0, \bar{\mathcal{I}}^0\rangle$, for each $\xi < \mu$ enumerate $[\lambda\xi]^{<\theta} := \left\{C^0_{\xi,\alpha} \ / \ \alpha < \lambda\right\}$. In an inductive way, define the iteration $\mathbb{P}^0{\restriction}\langle L^0, \bar{\mathcal{I}}^0\rangle$ in the following way.

- For each $\xi < \mu$ and $\alpha < \lambda$, let $\langle \dot{\mathbb{A}}^0_{\xi,\alpha,\eta}\rangle_{\eta<\theta}$ and $\langle \dot{\mathcal{F}}^0_{\xi,\alpha,\eta}\rangle_{\eta<\theta}$ be enumerations of all the $\mathbb{P}^0{\restriction}C^0_{\xi,\alpha}$-names of suborders of $\mathbb{A}$ of size $< \theta$ and all the names of filter bases on ω of size $< \theta$, respectively. This can be done because $|\mathbb{P}^0{\restriction}X| \leq \theta$ when $|X| < \theta$ (see the explanation of this above in the justification of $\mathfrak{c} \leq \lambda$ in the extension of an appropriate iteration).

- For $x \in L^0$ and $B \in \hat{\mathcal{I}}_x$, $\dot{\mathbb{Q}}^{0,B}_x$ is defined as indicated in (VIII)-(XI). For (IX), if $x \in L^0_A$ then $x = \lambda\xi + \tau_{0,2\beta}$ for some $\beta < \lambda$, so put $C^0_x := C^0_{\xi,(g(\beta))_0}$ and $\dot{\mathbb{A}}^0_x := \dot{\mathbb{A}}^0_{\xi,g(\beta)}$; for (X), if $x \in L^0_F$ then $x = \lambda\xi + \tau_{0,2\beta+1}$ for some $\beta < \lambda$, so put $C^0_x := C^0_{\xi,(g(\beta))_0}$ and $\dot{\mathcal{F}}^0_x := \dot{\mathcal{F}}^0_{\xi,g(\beta)}$.

To see that $\mathbb{P}^0{\restriction}\langle L^0, \bar{\mathcal{I}}^0\rangle$ is appropriate, it remains to prove (XII) and (XIII). We show (XIII). If $\dot{\mathcal{F}}$ is a $\mathbb{P}^0{\restriction}L^0$-name for a filter base of size $< \theta$ then, as the support of every name of a real has size $< \theta$, find $C \in [L^0]^{<\theta}$ such that $\dot{\mathcal{F}}$ is a $\mathbb{P}^0{\restriction}C$-name. Clearly, there exist $\xi < \mu$ and $\alpha < \lambda$ such that $C = C^0_{\xi,\alpha}$, so $\dot{\mathcal{F}}$ is forced to be equal to $\dot{\mathcal{F}}^0_{\xi,\alpha,\eta} = \dot{\mathcal{F}}^0_{\lambda\xi+\tau_{0,2g^{-1}(\alpha,\eta)+1}}$ for some $\eta < \theta$.

The iteration $\mathbb{P}^{\gamma+1}{\restriction}\langle L^{\gamma+1}, \bar{\mathcal{I}}^{\gamma+1}\rangle$ is defined from $\mathbb{P}^\gamma{\restriction}\langle L^\gamma, \bar{\mathcal{I}}^\gamma\rangle$ as explained in the previous discussion of ultrapowers. From the proof of Lemma 4.3.3, (4) and (5) are satisfied.

For the limit step for δ limit consider two cases. When $\mathrm{cf}(\delta) \geq \theta$, define $\mathbb{P}^\delta{\restriction}\langle L^\delta, \bar{\mathcal{I}}^\delta\rangle$ as in Lemma 4.3.5. So assume that $\mathrm{cf}(\delta) < \theta$, that is, $\delta = \delta_\epsilon$ for some $\epsilon < \lambda$. For each $\xi < \mu$, enumerate $[L^\delta_{\lambda\xi}]^{<\theta} := \{C^\delta_{\xi,\alpha} \ / \ \alpha < \lambda\}$. As it was done for $\mathbb{P}^0{\restriction}L^0$, define the iteration corresponding to δ inductively in the following way.

- For each $\xi < \mu$ and $\alpha < \lambda$, let $\langle \dot{\mathbb{A}}^\delta_{\xi,\alpha,\eta}\rangle_{\eta<\theta}$ and $\langle \dot{\mathcal{F}}^\delta_{\xi,\alpha,\eta}\rangle_{\eta<\theta}$ be an enumeration of all the $\mathbb{P}^\delta{\restriction}C^\delta_{\xi,\alpha}$-names of suborders of $\mathbb{A}$ of size $< \theta$ and of filter bases on ω of size $< \theta$, respectively.

- For $x \in L^0$ and $B \in \hat{\mathcal{I}}_x$, $\dot{\mathbb{Q}}^{0,B}_x$ is defined as indicated in (VIII)-(XI). For (X), if $x \in \bigcup_{\beta<\delta} L^\beta_A$, let $C^\gamma_x := C^\beta_x$ and $\dot{\mathbb{A}}^\gamma_x := \dot{\mathbb{A}}^\beta_x$ for some $\beta < \delta$ such that $x \in L^\beta_A$ (this does not depend on the chosen β by (4)); if $x \in L^\delta_A \smallsetminus \bigcup_{\beta<\delta} L^\beta_A$, then $x = \lambda\xi + \tau_{\epsilon,2\beta}$ for some (unique) $\beta < \lambda$, so put $C^\delta_x := C^\delta_{\xi,(g(\beta))_0}$ and $\dot{\mathbb{A}}^\delta_x := \dot{\mathbb{A}}^\delta_{\xi,g(\beta)}$. For (XI), if $x \in \bigcup_{\beta<\delta} L^\beta_F$, let $C^\gamma_x := C^\beta_x$ and $\dot{\mathcal{F}}^\gamma_x := \dot{\mathcal{F}}^\beta_x$ for some $\beta < \delta$ such that $x \in L^\beta_F$; if $x \in L^\delta_F \smallsetminus \bigcup_{\beta<\delta} L^\beta_F$, then $x = \lambda\xi + \tau_{\epsilon,2\beta+1}$ for some (unique) $\beta < \lambda$, so put $C^\delta_x := C^\delta_{\xi,(g(\beta))_0}$ and $\dot{\mathcal{F}}^\delta_x := \dot{\mathcal{F}}^\delta_{\xi,g(\beta)}$.

According to the previous discussion with chains of templates, we only need to prove (XII) and (XIII) for $\mathbb{P}^\delta{\restriction}L^\delta$, but its proof follows the same lines as in the case of $\mathbb{P}^0{\restriction}L^0$.

It only remains to prove that $\mathbb{P}^\lambda{\restriction}L^\lambda$ forces that $\mathfrak{a} = \lambda$. Let $\dot{\mathcal{A}}$ be a $\mathbb{P}^\lambda{\restriction}L^\lambda$-name for an a.d. family of size $\nu < \lambda$ with $\nu \geq \kappa$ (we don't need to consider a.d. families of size $< \kappa$ because $\mathfrak{b}$ is forced to be equal to $\mu > \kappa$). By Lemma 4.3.5, $\mathbb{P}^\lambda{\restriction}L^\lambda = \mathrm{limdir}_{\alpha<\lambda}\mathbb{P}^\alpha{\restriction}L^\alpha$, so there exists an $\alpha < \lambda$ such that $\dot{\mathcal{A}}$ is a $\mathbb{P}^\alpha{\restriction}L^\alpha$-name. As $\mathbb{P}^{\alpha+1}{\restriction}L^{\alpha+1}$ is forcing equivalent to the ultrapower of $\mathbb{P}^\alpha{\restriction}L^\alpha$ (Lemma 4.3.3), by Lemma 4.2.4 this poset forces that $\dot{\mathcal{A}}$ is not mad, and so does $\mathbb{P}^\lambda{\restriction}L^\lambda$. □

4.3.6 Remark. Shelah's models discussed in Remark 3.4.2 satisfy $\mathrm{cov}(\mathcal{N}) = \mathfrak{s} = \mathfrak{g} = \aleph_1 < \mathrm{add}(\mathcal{M}) = \mathrm{cof}(\mathcal{M}) = \mu < \mathrm{non}(\mathcal{N}) = \mathfrak{a} = \mathfrak{r} = \mathfrak{c} = \lambda$ by the same arguments as in this section for the values of $\mathfrak{g}$, $\mathrm{cov}(\mathcal{N})$, $\mathrm{non}(\mathcal{N})$ and $\mathfrak{r}$.

CHAPTER 5

MATRIX ITERATIONS

The first example of a matrix iteration was constructed by Blass and Shelah [BlS84] using Mathias forcing with ultrapowers to force $\mathfrak{u} < \mathfrak{d}$ with large continuum. Year later, Brendle and Fischer [BrF11] improved this construction by showing how to add dominating reals up to some intermediate extensions in the matrix. Also, they constructed a matrix iteration with ultrapowers to get a model where $\mathfrak{a}$ is large.

In Section 5.1 we define matrix iterations in general and fix the type of matrix iteration that we use for our applications. Here, we use posets like in the previous paragraph, but we also show how to use Suslin ccc posets into the matrix. In Section 5.2 we prove our main (consistency) results about Cichon's diagram with models that come from matrix iterations, this by also using results of Chapter 3.

5.1 Matrix iterations of ccc posets

Fix two ordinals δ and γ.

5.1.1 Definition (Blass and Shelah [BlS84] (see also [BrF11])). For an increasing sequence $\{V_{\alpha,0}\}_{\alpha\leq\delta}$ of transitive models of ZFC, a *matrix iteration of ccc posets* is given by $\mathbb{P}_{\delta,\gamma} = \langle\langle\mathbb{P}_{\alpha,\xi}, \dot{\mathbb{Q}}_{\alpha,\xi}\rangle_{\xi<\gamma}\rangle_{\alpha\leq\delta}$ with the following conditions.

(1) For all $\alpha \leq \delta$, $\mathbb{P}_{\alpha,\gamma} = \langle\mathbb{P}_{\alpha,\xi}, \dot{\mathbb{Q}}_{\alpha,\xi}\rangle_{\xi<\gamma}$ is a fsi of ccc posets in $V_{\alpha,0}$.

(2) For all $\xi < \gamma$ and $\alpha < \beta \leq \delta$, if $\mathbb{P}_{\alpha,\xi} \lessdot_{V_{\alpha,\xi}} \mathbb{P}_{\beta,\xi}$, then $\Vdash_{\mathbb{P}_{\beta,\xi},V_{\beta,0}} \dot{\mathbb{Q}}_{\alpha,\xi} \lessdot_{V_{\alpha,0}^{\mathbb{P}_{\alpha,\xi}}} \dot{\mathbb{Q}}_{\beta,\xi}$.

By Lemmas 2.1.2 and 2.1.5, (2) implies that $\mathbb{P}_{\alpha,\xi} \lessdot_{V_{\alpha,0}} \mathbb{P}_{\beta,\xi}$ for any $\alpha \leq \beta \leq \delta$ and $\xi \leq \gamma$. Moreover, if $\xi \leq \eta \leq \gamma$, $\langle\mathbb{P}_{\alpha,\xi}, \mathbb{P}_{\alpha,\eta}, \mathbb{P}_{\beta,\xi}, \mathbb{P}_{\beta,\eta}\rangle$ is a correct system with respect to $V_{\alpha,0}$. Denote by $V_{\alpha,\xi}$ the $\mathbb{P}_{\alpha,\xi}$-generic extension. Figure 5.1 shows how a matrix iteration can be pictured as a system of parallel fsi.

Before introducing the type of matrix iteration that we will use for our applications in Section 5.2, we need the following preliminary results about ultrafilters on ω.

5.1.2 Lemma. *Let δ be a limit ordinal of uncountable cofinality and $\{V_\alpha\}_{\alpha\leq\delta}$ a increasing sequence of transitive models of* ZFC *such that $[\omega]^\omega \cap V_\delta = \bigcup_{\alpha<\delta}[\omega]^\omega \cap V_\alpha$. Let $\langle\mathcal{U}_\alpha\rangle_{\alpha<\delta} \in V_\delta$ be a sequence such that, for any $\alpha \leq \beta < \delta$, $\mathcal{U}_\alpha$ is an ultrafilter on ω in V_α and $\mathcal{U}_\alpha \subseteq \mathcal{U}_\beta$. Then, $\mathcal{U}_\delta := \bigcup_{\alpha<\delta}\mathcal{U}_\alpha$ is an ultrafilter in V_δ. Moreover, if $\mathbb{M}_{\mathcal{U}_\alpha} \lessdot_{V_\alpha} \mathbb{M}_{\mathcal{U}_\beta}$ for all $\alpha \leq \beta < \delta$, then $\mathbb{M}_{\mathcal{U}_\alpha} \lessdot_{V_\alpha} \mathbb{M}_{\mathcal{U}_\delta}$ for any $\alpha < \delta$.*

Proof. In V_δ, it is clear that $\mathcal{U}_\delta \subseteq [\omega]^\omega$ contains the cofinite subsets on ω and it is closed under intersections. Also, $[\omega]^\omega \cap V_\delta = \bigcup_{\alpha<\delta}[\omega]^\omega \cap V_\alpha$ implies that $\mathcal{U}_\delta$ is upwards $\subseteq$-closed and that, for any $X \subseteq \omega$, there is an $\alpha < \delta$ such that either $X \in \mathcal{U}_\alpha$ or $\omega \smallsetminus X \in \mathcal{U}_\alpha$, so $\mathcal{U}_\delta$ is an ultrafilter on ω.

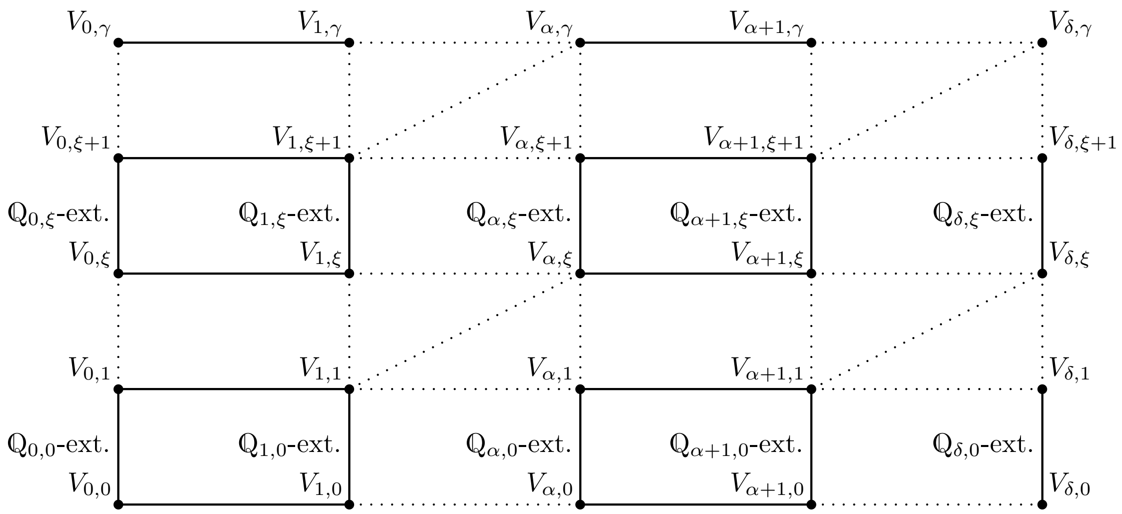

Figure 5.1: Matrix iteration

For the second statement, let $\alpha < \delta$ and $A \in V_\alpha$ a maximal antichain in $\mathbb{M}_{\mathcal{U}_\alpha}$. If $(s,F) \in \mathbb{M}_{\mathcal{U}_\delta}$, then there is a $\beta \in [\alpha,\delta)$ such that $F \in \mathcal{U}_\beta$, i.e., $(s,F) \in \mathbb{M}_{\mathcal{U}_\beta}$. As $\mathbb{M}_{\mathcal{U}_\alpha} \lessdot_{V_\alpha} \mathbb{M}_{\mathcal{U}_\beta}$, then A is a maximal antichain in $\mathbb{M}_{\mathcal{U}_\beta}$, so (s,F) is compatible with some member of A with respect to $\mathbb{M}_{\mathcal{U}_\beta}$. □

5.1.3 Lemma (Blass and Shelah [BlS84]). *Let δ be a limit ordinal of countable cofinality and $\{V_\alpha\}_{\alpha\leq\delta}$ a increasing sequence of transitive models of* ZFC. *Assume that $\langle\mathcal{U}_\alpha\rangle_{\alpha<\delta} \in V_\delta$ is a sequence such that, for any $\alpha \leq \beta < \delta$, $\mathcal{U}_\alpha$ is an ultrafilter on ω in V_α, $\mathcal{U}_\alpha \subseteq \mathcal{U}_\beta$ and $\mathbb{M}_{\mathcal{U}_\alpha} \lessdot_{V_\alpha} \mathbb{M}_{\mathcal{U}_\beta}$. Then, there is an ultrafilter $\mathcal{U}_\delta$ in V_δ such that $\mathcal{U}_\alpha \subseteq \mathcal{U}_\delta$ and $\mathbb{M}_{\mathcal{U}_\alpha} \lessdot_{V_\alpha} \mathbb{M}_{\mathcal{U}_\delta}$ for any $\alpha < \delta$.*

Proof. Proceed like in the proof of Lemma 3.3.2. Fix $\{\alpha_n\}_{n<\omega}$ a strictly increasing sequence of ordinals with limit δ. To ease the notation, we may assume that $\alpha_n = n$ and $\delta = \omega$. For $(s,F) \in [\omega]^{<\omega} \times [\omega]^\omega$ with $\sup(s+1) \leq \min(F)$, recall that $t \in [\omega]^{<\omega}$ *is permitted by* (s,F) if $s \subseteq t \subseteq s \cup F$. If $A \in V_n$ is a maximal antichain in $\mathbb{M}_{\mathcal{U}_n}$, t *is permitted by* A means that t is permitted by some condition in A. $C \in [\omega]^\omega \cap V_\omega$ is *forbidden by* s, A if there is no finite subset of ω permitted by (s,C) and A.

In V_ω, let $\mathcal{I}$ be the ideal generated by the finite subsets of ω and the sets forbidden by some s, A where $s \in [\omega]^{<\omega}$ and $A \in V_n$ is a maximal antichain in $\mathbb{M}_{\mathcal{U}_n}$ for some $n < \omega$. The same argument as in the proof of Lemma 3.3.2 works to get $\bigcup_{n<\omega}\mathcal{U}_n \cap \mathcal{I} = \varnothing$. Therefore, there exists an ultrafilter $\mathcal{U}_\omega$ that contains $\bigcup_{n<\omega}\mathcal{U}_n$ and such that $\mathcal{U}_\omega \cap \mathcal{I} = \varnothing$. Like in Lemma 3.3.2, $\mathbb{M}_{\mathcal{U}_n} \lessdot_{V_n} \mathbb{M}_{\mathcal{U}_\omega}$ for all $n < \omega$. □

5.1.4 Context. Let δ, γ be ordinals. Put $\gamma = S \cup U \cup L \cup T$ as a disjoint union and fix a function $\Delta : T \to \delta$. For $\xi \in S$ fix $\mathbb{S}_\xi$ a Suslin ccc poset with parameters in V. Define the matrix iteration $\mathbb{P}_{\delta,\gamma} = \langle\langle\mathbb{P}_{\alpha,\xi}, \dot{\mathbb{Q}}_{\alpha,\xi}\rangle_{\xi<\gamma}\rangle_{\alpha\leq\delta}$ as follows.

(1) $V_{0,0} = V$ (the ground model) and, by a fsi of Cohen forcing of length δ, let $V_{\alpha,0}$ be the intermediate extension on $\alpha \leq \delta$. For $\alpha < \delta$, denote by $c_\alpha \in V_{\alpha+1,0}$ the Cohen real over $V_{\alpha,0}$ added by the iteration.

(2) For a fixed $\xi < \gamma$, $\dot{\mathbb{Q}}_{\alpha,\xi}$ is defined, recursively, for all $\alpha \leq \delta$ according to one of the following cases.

  (i) For $\xi \in S$, $\dot{\mathbb{Q}}_{\alpha,\xi} = \dot{\mathbb{S}}_\xi$ as a $\mathbb{P}_{\alpha,\xi}$-name of $\mathbb{S}_\xi$.

  (ii) For $\xi \in U$ define, by recursion on $\alpha \leq \delta$, a $\mathbb{P}_{\alpha,\xi}$-name $\dot{\mathcal{U}}_{\alpha,\xi}$ of an ultrafilter on ω such that

- For $\alpha < \beta \leq \delta$, $\Vdash_{\mathbb{P}_{\beta,\xi},V_{\beta,0}}$"$\dot{\mathcal{U}}_{\alpha,\xi} \subseteq \dot{\mathcal{U}}_{\beta,\xi}$ and $\mathbb{M}_{\dot{\mathcal{U}}_{\alpha,\xi}} \lessdot_{V_{\alpha,\xi}} \mathbb{M}_{\dot{\mathcal{U}}_{\beta,\xi}}$".
- It is forced by $\mathbb{P}_{\alpha,\xi}$ that $\dot{\mathcal{U}}_{\alpha,\xi}$ contains the Mathias reals and Laver reals (of the form $\mathrm{ran}(\dot{l}_{\dot{\mathcal{U}}_{\alpha,\xi}})$) added by $\mathbb{P}_{\alpha,\xi'+1}$ for each $\xi' \in U \cup L$, $\xi' < \xi$ (see also (iii)).
- If $(\star, \mathbb{P}_{\alpha,\xi}, \mathbb{P}_{\alpha+1,\xi}, V_{\alpha,0}, V_{\alpha+1,0}, \leq^*, c_\alpha)$ holds, $(\star, \mathbb{M}_{\dot{\mathcal{U}}_{\alpha,\xi}}, \mathbb{M}_{\dot{\mathcal{U}}_{\alpha+1,\xi}}, V_{\alpha,\xi+1}, V_{\alpha+1,\xi+1}, \leq^*, c_\alpha)$ is forced by $\mathbb{P}_{\alpha+1,\xi}$.
- If $\beta \leq \delta$ has uncountable cofinality, then $\Vdash_{\mathbb{P}_{\beta,\xi}} \dot{\mathcal{U}}_{\beta,\xi} = \bigcup_{\alpha<\beta} \dot{\mathcal{U}}_{\alpha,\xi}$.

This construction is done by recursion on $\alpha \leq \delta$. For $\alpha = 0$, consider $\dot{\mathcal{U}}_{0,\xi}$ containing all the Mathias reals and Laver reals added by $\mathbb{P}_{0,\xi'+1}$ for each $\xi' \in U \cup L$, $\xi' < \xi$. In the successor step, use Lemma 3.3.2 to produce $\dot{\mathcal{U}}_{\alpha+1}$ (if c_α is not $\leq^*$-unbounded over $V_{\alpha,\xi}$, just make sure that $\mathbb{M}_{\mathcal{U}_{\alpha,\xi}} \lessdot_{V_{\alpha,\xi}} \mathbb{M}_{\mathcal{U}_{\alpha+1,\xi}}$). In the limit step, for limit α use Lemma 5.1.3 or 5.1.2, according to the cofinality of α (Lemma 5.1.5 is also relevant here). Note that, for any $\alpha \leq \delta$, $\Vdash_{\mathbb{P}_{\alpha,\xi+1},V_{\alpha,0}} \dot{m}_{\alpha,\xi} = \dot{m}_{0,\xi}$ (the corresponding Mathias reals).

Put $\dot{\mathbb{Q}}_{\alpha,\xi} = \mathbb{M}_{\dot{\mathcal{U}}_{\alpha,\xi}}$.

(iii) For $\xi \in L$ define, by recursion on $\alpha \leq \delta$, a $\mathbb{P}_{\alpha,\xi}$-name $\dot{\mathcal{U}}_{\alpha,\xi}$ of an ultrafilter on ω such that

- For $\alpha < \beta \leq \delta$, $\Vdash_{\mathbb{P}_{\beta,\xi}} \dot{\mathcal{U}}_{\alpha,\xi} \subseteq \dot{\mathcal{U}}_{\beta,\xi}$.
- It is forced by $\mathbb{P}_{\alpha,\xi}$ that $\dot{\mathcal{U}}_{\alpha,\xi}$ contains the Mathias reals and the Laver reals added by $\mathbb{P}_{\alpha,\xi'+1}$ for each $\xi' \in U \cup L$, $\xi' < \xi$.
- If $\beta \leq \delta$ has uncountable cofinality, then $\Vdash_{\mathbb{P}_{\beta,\xi}} \dot{\mathcal{U}}_{\beta,\xi} = \bigcup_{\alpha<\beta} \dot{\mathcal{U}}_{\alpha,\xi}$.

For $\alpha = 0$ define $\dot{\mathcal{U}}_{0,\xi}$ as in (ii). For $\beta \leq \delta$, if it is successor or has countable cofinality, let $\dot{\mathcal{U}}_{\beta,\xi}$ be a $\mathbb{P}_{\beta,\xi}$-name of an ultrafilter containing $\bigcup_{\alpha<\beta} \dot{\mathcal{U}}_{\alpha,\xi}$; if β has uncountable cofinality, put $\dot{\mathcal{U}}_{\beta} = \bigcup_{\alpha<\beta} \dot{\mathcal{U}}_{\alpha,\xi}$.

Put $\dot{\mathbb{Q}}_{\alpha,\xi} = \mathbb{L}_{\dot{\mathcal{U}}_{\alpha,\xi}}$. Lemmas 3.3.6, 5.1.2 and 5.1.5 are relevant to ensure that the mentioned properties hold.

(iv) For $\xi \in T$ fix a $\mathbb{P}_{\Delta(\xi),\xi}$-name $\dot{\mathbb{T}}_\xi$ of a ccc poset whose conditions are reals. Put

$$\dot{\mathbb{Q}}_{\alpha,\xi} = \begin{cases} \mathbb{1} & \text{if } \alpha \leq \Delta(\xi), \\ \dot{\mathbb{T}}_\xi & \text{if } \alpha > \Delta(\xi). \end{cases}$$

It is clear that this satisfies the conditions of Definition 5.1.1.

To ensure that a matrix iteration as in the previous context can be constructed and satisfies the conditions of Definition 5.1.1, the following lemma has to be proved along with the recursive construction.

5.1.5 Lemma (Blass and Shelah [BlS84] (see also [BrF11, Lemma 15])). *Assume that δ has uncountable cofinality and $\xi \leq \gamma$.*

(a) If $p \in \mathbb{P}_{\delta,\xi}$ then there exists an $\alpha < \delta$ such that $p \in \mathbb{P}_{\alpha,\xi}$.

(b) If $\dot{h} \in V_{\delta,0}$ is a $\mathbb{P}_{\delta,\xi}$-name for a real, then there exists an $\alpha < \delta$ such that $\dot{h} \in V_{\alpha,0}$ is a $\mathbb{P}_{\alpha,\xi}$-name.

Proof. We prove the lemma by induction on $\xi \leq \gamma$. As $\{V_{\alpha,0}\}_{\alpha\leq\delta}$ is constructed from a fsi of Cohen forcing, conditions (i) and (ii) of Lemma 5.1.2 are satisfied, so the step $\xi = 0$ is clear. For the successor step, let $r = (p, \dot{q}) \in \mathbb{P}_{\delta,\xi+1} = \mathbb{P}_{\delta,\xi} * \dot{\mathbb{Q}}_{\delta,\xi}$ so, by induction hypothesis, there is an $\alpha < \delta$ such that $p \in \mathbb{P}_{\alpha,\xi}$ and $\dot{q} \in V_{\alpha,0}$ is a $\mathbb{P}_{\alpha,\xi}$-name for a real. Do cases from (2) of Context 5.1.4.

(i) $\xi \in S$. It is clear that $\dot{q}$ is a $\mathbb{P}_{\alpha,\xi}$-name of a condition in $\mathbb{S}_\xi$.

(ii) $\xi \in U$. By ccc-ness, there is a $\beta \in [\alpha, \delta)$ such that $\mathbb{P}_{\delta,\xi}$ forces that $\dot{q} \in \mathbb{M}_{\dot{\mathcal{U}}_{\beta,\xi}}$, so $(p, \dot{q}) \in \mathbb{P}_{\beta,\xi+1}$.

(iii) $\xi \in L$. Same argument as before.

(iv) $\xi \in T$. Increase α so that $\Delta(\xi) \leq \alpha$, so $\mathbb{P}_{\alpha,\xi}$ forces that $\dot{q} \in \dot{\mathbb{T}}_\xi$.

Let $\dot{h} \in V_{\delta,0}$ be a $\mathbb{P}_{\delta,\xi+1}$-name for a real. By ccc-ness, $\dot{h}$ is coded by a sequence $\langle (A_n, g_n) \rangle_{n<\omega}$ of pairs of a maximal antichains in $\mathbb{P}_{\delta,\xi+1}$ and functions $g_n : A_n \to \omega$ that decides the values of $\dot{h}(n)$. Then, there is some $\alpha < \delta$ such that this sequence is in $V_{\alpha,0}$ and that the maximal antichains are contained in $\mathbb{P}_{\alpha,\xi+1}$, so $\dot{h} \in V_{\alpha,0}$ and it is actually a $\mathbb{P}_{\alpha,\xi+1}$-name for a real.

For the limit step, let $\eta \leq \gamma$ be limit, so $\mathbb{P}_{\alpha,\eta} = \text{limdir}_{\xi<\eta}\mathbb{P}_{\alpha,\xi}$ for all $\alpha \leq \delta$. (a) is clear. (b) follows for the same argument of the successor step. □

Given a relation $\sqsubset$ coded in V as in Context 3.1.1, in Context 5.1.4 it is clear, by (1), that the Cohen real c_α is $\sqsubset$-unbounded over $V_{\alpha,0}$. We are interested in preserving c_α to be $\sqsubset$-unbounded over $V_{\alpha,\gamma}$.

5.1.6 Theorem. *Fix $\alpha < \delta$ and assume:*

(i) For all $\xi \in S$, $\mathbb{P}_{\alpha,\xi}$ forces that $\mathbb{S}_\xi$ is $\sqsubset$-good.

(ii) If $U \neq \varnothing$, then $L = \varnothing$ and $\sqsubset$ is $\leq^*$.

(iii) If $L \neq \varnothing$, then $\sqsubset$ is $\in^*_{\bar{H}}$ or $\pitchfork^{\bar{I}}$.

Then, $(\star, \mathbb{P}_{\alpha,\xi}, \mathbb{P}_{\alpha+1,\xi}, V_{\alpha,0}, V_{\alpha+1,0}, \sqsubset, c_\alpha)$ holds for all $\xi \leq \gamma$. Moreover, if δ has uncountable cofinality and (i)-(iii) hold for any $\alpha < \delta$, then $\mathbb{P}_{\delta,\gamma}$ forces $\mathfrak{d}_\sqsubset \geq \text{cf}(\delta)$.

Proof. The first part is a direct consequence of Corollary 3.3.12 and Lemmas 3.3.8, 3.3.9 and 3.3.6. Now, assume that δ has uncountable cofinality and that (i)-(iii) hold for any $\alpha < \delta$. Step in $V_{\delta,\gamma}$. If Y is a set of reals of size $< \text{cf}(\delta)$, by Lemma 5.1.5, there is an $\alpha < \delta$ such that $Y \subseteq V_{\alpha,\gamma}$. Then, as c_α is $\sqsubset$-unbounded over $V_{\alpha,\gamma}$, $c_\alpha \not\sqsubset Y$, so Y is not a $\sqsubset$-dominating family. □

5.2 Applications

We present the main results of this text concerning applications of matrix iterations. Fix $\mu_1 \leq \mu_2 \leq \nu \leq \kappa$ uncountable regular cardinals and a cardinal $\lambda \geq \kappa$ such that $\lambda^{<\nu} = \lambda$. Consider $t : \kappa\nu \to \kappa$ such that $t^{-1}[\{\alpha\}]$ is cofinal in $\kappa\nu$ for every $\alpha < \kappa$ (e.g., t defined as $t(\kappa\delta + \alpha) = \alpha$ works). Fix a bijection $g : \lambda \to \kappa \times \lambda$ and, given an ordered pair $z = (x, y)$, denote $(z)_0 = x$.

5.2.1 Theorem. *It is consistent with* ZFC *that* $\text{add}(\mathcal{N}) = \mathfrak{p} = \text{non}(\mathcal{M}) = \mathfrak{g} = \nu$, $\text{cof}(\mathcal{N}) = \kappa$, $\mathfrak{c} = \lambda$, *and that one of the following statements hold.*

(a) $\text{cov}(\mathcal{M}) = \mathfrak{r} = \kappa$.

(b) $\text{cov}(\mathcal{M}) = \nu$ *and* $\mathfrak{d} = \mathfrak{r} = \text{non}(\mathcal{N}) = \kappa$.

(c) $\text{non}(\mathcal{N}) = \mathfrak{u} = \nu$, $\mathfrak{d} = \kappa$.

(d) $\text{non}(\mathcal{N}) = \nu$, $\mathfrak{d} = \mathfrak{r} = \kappa$.

(e) $\text{cof}(\mathcal{M}) = \nu$ *and* $\text{non}(\mathcal{N}) = \mathfrak{r} = \kappa$.

(f) $\text{cof}(\mathcal{M}) = \mathfrak{u} = \nu$ *and* $\text{non}(\mathcal{N}) = \kappa$.

(g) $\text{cof}(\mathcal{M}) = \text{non}(\mathcal{N}) = \mathfrak{u} = \nu$.

Proof. (a) According to Context 5.1.4, construct a matrix iteration $\mathbb{P}_{\kappa,\lambda\kappa\nu} = \langle\langle \mathbb{P}_{\alpha,\xi}, \dot{\mathbb{Q}}_{\alpha,\xi} \rangle_{\xi<\lambda\kappa\nu} \rangle_{\alpha\leq\kappa}$ where $U = L = \varnothing$, $S = \{\lambda\rho + 2\gamma \;/\; \rho < \kappa\nu, \gamma < \lambda\}$ and the following.

(i) $S_\xi = \mathbb{C}$ for any $\xi \in S$.

(ii) If $\xi = \lambda\rho + 1$ for some (unique) $\rho < \kappa\nu$, put $\Delta(\xi) = t(\rho)$ and $\dot{\mathbb{T}}_\xi = \dot{\mathbb{A}}_\rho$ is a $\mathbb{P}_{t(\rho),\xi}$-name for $\mathbb{A}^{V_{t(\rho),\xi}}$.

(iii) If $\xi = \lambda\rho + 3$ for some $\rho < \kappa\nu$, put $\Delta(\xi) = t(\rho)$ and $\dot{\mathbb{T}}_\xi = \mathbb{M}_{\dot{\mathcal{U}}_\rho}$ where $\dot{\mathcal{U}}_\rho$ is a $\mathbb{P}_{t(\rho),\xi}$-name for an ultrafilter on ω.

For $\alpha < \kappa$ and $\rho < \kappa\nu$, fix a sequence of $\mathbb{P}_{\alpha,\lambda\rho}$-names $\langle \dot{\mathcal{F}}^\rho_{\alpha,\gamma} \rangle_{\gamma<\lambda}$ of <u>all</u> the filter bases of size $< \nu$.

(iv) If $\xi = \lambda\rho + 2(2+\epsilon) + 1$ for some $\rho < \kappa\mu$ and $\epsilon < \lambda$, put $\Delta(\xi) = (g(\epsilon))_0$ and $\dot{\mathbb{T}}_\xi = \mathbb{M}_{\dot{\mathcal{F}}^\rho_{g(\epsilon)}}$.

We prove the following statements in $V_{\kappa,\lambda\kappa\nu}$.

- $\mathfrak{c} = \lambda$. It is clear that $\lambda^\omega = \lambda$ is true in $V_{\kappa,0}$. In this model, it is clear that $|\mathbb{P}_{\kappa,\lambda\kappa\nu}| \leq \lambda$, so it forces that $\mathfrak{c} \leq \lambda$. The other inequality follows because $\mathbb{P}_{\kappa,\lambda}$ adds λ-many Cohen reals.
- $\mathrm{cov}(\mathcal{M}) = \mathrm{cof}(\mathcal{N}) = \mathfrak{r} = \kappa$. For each $\rho < \kappa\nu$ let N_ρ be a Borel null set coded in $V_{t(\rho)+1,\lambda\rho+2}$ that covers all the Borel null sets coded in $V_{t(\rho),\lambda\rho+1}$ (this is added by $\mathbb{A}_\rho$). $\mathrm{cof}(\mathcal{N}) \leq \kappa$ is a consequence from the following

  5.2.2 Claim. *Any family of size $< \nu$ of Borel null sets is covered by some N_ρ.*

  Proof. Let $\mathcal{E}$ be such a family. By Lemma 5.1.5, there are $\alpha < \kappa$ and $\delta < \kappa\nu$ such that all the sets in $\mathcal{E}$ are coded in $V_{\alpha,\lambda\delta}$. Find $\rho \in [\delta, \kappa\nu)$ such that $t(\rho) = \alpha$. Clearly, all the sets in $\mathcal{E}$ are coded in $V_{\alpha,\xi}$ with $\xi = \lambda\rho + 1$, so they are all covered by N_ρ. □

  For each $\rho < \kappa\nu$ let $m_\rho \in V_{t(\rho)+1,\lambda\rho+4}$ be a Mathias real added by $\mathbb{M}_{\mathcal{U}_\rho}$, which $\propto$-dominates all the infinite subsets of ω that belong to $V_{t(\rho),\lambda\rho+3}$. $\mathfrak{r} \leq \kappa$ is straightforward from the following

  5.2.3 Claim. *Any family of size $< \nu$ of infinite subsets of ω is $\propto$-dominated by some m_ρ.*

  Proof. Let $\mathcal{C}$ be such a family. By a similar argument as in Lemma 5.2.2, find $\rho < \kappa\nu$ such that $\mathcal{C} \subseteq V_{t(\rho),\xi}$ with $\xi = \lambda\rho + 3$. Clearly, m_ρ $\propto$-dominates all the sets in $\mathcal{C}$. □

  $\mathrm{cov}(\mathcal{M}) = \mathfrak{d}_{\neq} \geq \kappa$ follows from Theorem 5.1.6.
- $\mathrm{add}(\mathcal{N}) = \mathfrak{p} = \mathrm{non}(\mathcal{M}) = \mathfrak{g} = \nu$. $\nu \leq \mathrm{add}(\mathcal{N})$ is a direct consequence of Claim 5.2.2.
  As a fsi in $V_{\kappa,0}$, $\mathbb{P}_{\kappa,\lambda\kappa\nu}$ adds a ν-$\neq$-unbounded family of Cohen reals (Lemma 3.1.12), so $\mathrm{non}(\mathcal{M}) = \mathfrak{b}_{\neq} \leq \nu$ is true in $V_{\kappa,\lambda\kappa\nu}$.
  $\mathfrak{g} \leq \nu$ because $V_{\kappa,\lambda\kappa\nu} = \bigcup_{\epsilon<\nu} V_{\kappa,\lambda\kappa\epsilon}$ and the conditions of Lemma 4.1.1 are clearly satisfied.
  We are left with $\nu \leq \mathfrak{p}$. If $\mathcal{F}$ is a filter base of size $< \nu$, then there are $\alpha < \delta$ and $\rho < \kappa\nu$ such that $\mathcal{F} \in V_{\alpha,\lambda\rho}$. Then, there is a $\gamma < \lambda$ such that $\mathcal{F} = \mathcal{F}^\rho_{\alpha,\gamma}$, so the Mathias real added by $\mathbb{M}_{\mathcal{F}^\rho_{g(\epsilon)}}$, with $\epsilon < \lambda$ such that $g(\epsilon) = (\alpha, \gamma)$, is a pseudo-intersection of $\mathcal{F}$.

(b) Use an iteration as in (a), but only change (i) to $\mathbb{S}_\xi = \mathbb{E}$ for all $\xi \in S$. The same proofs in (a) works in this case, but except that we do not get $\mathrm{cov}(\mathcal{M}) \geq \kappa$ in $V_{\kappa,\lambda\kappa\nu}$. Instead, we get $\min\{\mathfrak{d}, \mathfrak{r}\} = \mathfrak{d}_{\triangleright} \geq \kappa$ and $\mathrm{non}(\mathcal{N}) \geq \mathfrak{d}_{\pitchfork I} \geq \kappa$ by Theorem 5.1.6.

We only have to prove $\mathrm{cov}(\mathcal{M}) = \mathfrak{d}_{\neq} \leq \nu$ in $V_{\kappa,\lambda\kappa\nu}$, but note that $\mathbb{P}_{\kappa,\lambda\kappa\nu}$, as a fsi in $V_{\kappa,0}$, adds ν cofinally many eventually different reals, whose form a $\neq$-dominating family of size ν.

(c) Use an iteration as in (a) but with the following changes. Let $S = \{\lambda\rho + 4\gamma \mathbin{/} \rho < \kappa\nu, \gamma < \lambda\}$, $U = \{\lambda\rho + 4\gamma + 2 \mathbin{/} \rho < \kappa\nu, \gamma < \lambda\}$ and put

$$S_\xi = \begin{cases} \mathbb{C} & \text{if } \xi = \lambda\rho + 8\gamma, \\ \mathbb{B} & \text{if } \xi = \lambda\rho + 8\gamma + 4. \end{cases}$$

(iii) can be ignored as well. By Theorem 5.1.6, $\mathfrak{d} = \mathfrak{d}_{\leq^*} \geq \kappa$ holds in $V_{\kappa,\lambda\kappa\nu}$. $\mathrm{non}(\mathcal{N}) \leq \nu$ because of the ν cofinally many random reals added by the iteration. We show $\mathfrak{u} \leq \nu$. For $\delta < \nu$, consider $M_\delta = m_{\kappa,\xi} \in V_{\kappa,\xi+1}$, with $\xi = \lambda\kappa\delta + 2$, the Mathias real added by $\mathbb{M}_{\mathcal{U}_{\kappa,\xi}}$ in the iteration. By the construction of the matrix iteration, $\{M_\delta\}_{\delta<\nu}$ is a $\subseteq^*$-decreasing family. Moreover, for any $X \subseteq \omega$, there is some $\nu < \delta$ such that $X \in V_{\kappa,\lambda\kappa\delta}$, so either $M_\delta \subseteq^* X$ or $M_\delta \subseteq^* \omega \smallsetminus X$. This means that $\{M_\delta\}_{\delta<\nu}$ generates an ultrafilter.

(d) Use an iteration as in (a) but put

$$S_\xi = \begin{cases} \mathbb{C} & \text{if } \xi = \lambda\rho + 4\gamma, \\ \mathbb{B} & \text{if } \xi = \lambda\rho + 4\gamma + 2. \end{cases}$$

(e) As in (a), but put $S_\xi = \mathbb{D}$ for all $\xi \in S$.

(f) Do the following changes in (a). Let $S = \{\lambda\rho + 4\gamma \ / \ \rho < \kappa\nu, \gamma < \lambda\}$, $L = \{\lambda\rho + 4\gamma + 2 \ / \ \rho < \kappa\nu, \gamma < \lambda\}$ and put $\mathbb{S}_\xi = \mathbb{D}$ for all $\xi \in S$. Here, (iii) can be ignored.

(g) Use an iteration as in (f), but put

$$\mathbb{S}_\xi = \begin{cases} \mathbb{D} & \text{if } \xi = \lambda\rho + 8\gamma, \\ \mathbb{B} & \text{if } \xi = \lambda\rho + 8\gamma + 4. \end{cases}$$

□

5.2.4 Theorem. *It is consistent with* ZFC *that* $\text{add}(\mathcal{N}) = \mu_1$, $\text{cov}(\mathcal{N}) = \mathfrak{p} = \text{non}(\mathcal{M}) = \mathfrak{g} = \nu$, $\text{cof}(\mathcal{N}) = \mathfrak{c} = \lambda$ *and that one of the following statements hold.*

(a) $\text{cov}(\mathcal{M}) = \text{cof}(\mathcal{M}) = \mathfrak{r} = \text{non}(\mathcal{N}) = \kappa$.

(b) $\text{cov}(\mathcal{M}) = \nu$ *and* $\text{non}(\mathcal{N}) = \mathfrak{d} = \mathfrak{r} = \text{cof}(\mathcal{M}) = \kappa$.

(c) $\text{cof}(\mathcal{M}) = \nu$ *and* $\text{non}(\mathcal{N}) = \mathfrak{r} = \kappa$.

(d) $\text{cof}(\mathcal{M}) = \mathfrak{u} = \nu$ *and* $\text{non}(\mathcal{N}) = \kappa$.

(e) $\text{non}(\mathcal{N}) = \mathfrak{u} = \nu$ *and* $\mathfrak{d} = \text{cof}(\mathcal{M}) = \kappa$.

(f) $\text{non}(\mathcal{N}) = \nu$ *and* $\mathfrak{d} = \mathfrak{r} = \text{cof}(\mathcal{M}) = \kappa$.

Proof. (a) According to Context 5.1.4, construct a matrix iteration $\mathbb{P}_{\kappa,\lambda\kappa\nu} = \langle\langle\mathbb{P}_{\alpha,\xi}, \dot{\mathbb{Q}}_{\alpha,\xi}\rangle_{\xi<\lambda\kappa\nu}\rangle_{\alpha\leq\kappa}$ where $U = L = \varnothing$, $S = \{\lambda\rho + 2\gamma \ / \ \rho < \kappa\nu, \gamma < \lambda\}$ and the following.

(i) $S_\xi = \mathbb{C}$ for any $\xi \in S$.

(ii) If $\xi = \lambda\rho + 1$ for some (unique) $\rho < \kappa\nu$, put $\Delta(\xi) = t(\rho)$ and $\dot{\mathbb{T}}_\xi = \dot{\mathbb{B}}_\rho$ is a $\mathbb{P}_{t(p),\xi}$-name for $\mathbb{B}^{V_{t(\rho),\xi}}$.

(iii) If $\xi = \lambda\rho + 3$ for some $\rho < \kappa\nu$, put $\Delta(\xi) = t(\rho)$ and $\dot{\mathbb{T}}_\xi = \dot{\mathbb{D}}_\rho$ is a $\mathbb{P}_{t(p),\xi}$-name for $\mathbb{D}^{V_{t(\rho),\xi}}$.

(iv) If $\xi = \lambda\rho + 5$ for some $\rho < \kappa\nu$, put $\Delta(\xi) = t(\rho)$ and $\dot{\mathbb{T}}_\xi = \mathbb{M}_{\dot{\mathcal{U}}_\rho}$ where $\dot{\mathcal{U}}_\rho$ is a $\mathbb{P}_{t(\rho),\xi}$-name for an ultrafilter on ω.

For $\alpha < \kappa$ and $\rho < \kappa\nu$, fix sequences of $\mathbb{P}_{\alpha,\lambda\rho}$-names $\langle\dot{\mathbb{A}}^\rho_{\alpha,\gamma}\rangle_{\gamma<\lambda}$ and $\langle\dot{\mathcal{F}}^\rho_{\alpha,\gamma}\rangle_{\gamma<\lambda}$ for <u>all</u> the suborders of $\mathbb{A}$ of size $< \mu_1$ and <u>all</u> the filter bases of size $< \nu$, respectively.

(v) If $\xi = \lambda\rho + 2(3 + 2\epsilon) + 1$ for some $\rho < \kappa\mu$ and $\epsilon < \lambda$, put $\Delta(\xi) = (g(\epsilon))_0$ and $\dot{\mathbb{T}}_\xi = \mathbb{M}_{\dot{\mathcal{F}}^\rho_{g(\epsilon)}}$.

(vi) If $\xi = \lambda\rho + 2(3 + 2\epsilon + 1) + 1$ for some $\rho < \kappa\mu$ and $\epsilon < \lambda$, put $\Delta(\xi) = (g(\epsilon))_0$ and $\dot{\mathbb{T}}_\xi = \dot{\mathbb{A}}^\rho_{g(\epsilon)}$.

From the proof of Theorem 5.2.1(a), it is clear that $\mathfrak{c} = \lambda$, $\text{cov}(\mathcal{M}) = \mathfrak{r} = \kappa$ and $\mathfrak{p} = \text{non}(\mathcal{M}) = \mathfrak{g} = \nu$ are true in $V_{\kappa,\kappa\lambda\nu}$. For this iteration, we get $\nu \leq \text{cov}(\mathcal{N})$ and $\text{non}(\mathcal{N}) \leq \kappa$ by the following

5.2.5 Claim. *There is a set of reals* $\{r_\rho \ / \ \rho < \kappa\nu\}$ *such that, for any family of size* $< \nu$ *of Borel null sets, some* r_ρ *does not belong to its union.*

Proof. Let $r_\rho \in V_{t(\rho)+1,\lambda\rho+2}$ be a random real over $V_{t(\rho),\lambda\rho+1}$ added by $\mathbb{B}_\rho$. The proof is similar to Claims 5.2.2 and 5.2.3. □

To get $\mathfrak{d} \leq \kappa$, note that

5.2.6 Claim. *There is a set of reals $\{d_\rho \,/\, \rho < \kappa\nu\} \subseteq \omega^\omega$ such that, for any subset of ω^ω of size $< \nu$, some d_ρ dominates it.*

Proof. Consider $d_\rho \in V_{t(\rho)+1,\lambda\rho+4}$ be a dominating real over $V_{t(\rho),\lambda\rho+3}$ added by $\mathbb{D}_\rho$. Conclude as in Claims 5.2.2, 5.2.3 and 5.2.5. □

Note that $\mathbb{P}_{\kappa,\lambda\kappa\nu}$ is μ_1-$\in^*_{\bar{H}}$-good. Therefore, for any regular $\theta \in [\mu_1, \lambda)$, $\mathbb{P}_{\kappa,\theta}$, as a fsi, adds a θ-$\in^*_{\bar{H}}$-unbounded family that is preserved in $V_{\kappa,\lambda\kappa\nu}$, so $\text{add}(\mathcal{N}) = \mathfrak{b}_{\in^*_{\bar{H}}} \leq \mu_1$ and $\theta \leq \text{cof}(\mathcal{N})$ for any such θ, that is, $\lambda \leq \text{cof}(\mathcal{N})$.

It remains to prove $\mu_1 \leq \text{add}(\mathcal{N})$ in $V_{\kappa,\lambda\kappa\nu}$ but the argument is similar to the proof of $\nu \leq \mathfrak{p}$ (see also Theorem 4.1.2).

(b) Use the an iteration as in (a), but put $\mathbb{S}_\xi = \mathbb{E}$ for all $\xi \in S$.

(c) In (a), just put $S_\xi = \mathbb{D}$ for all $\xi \in S$. (iii) can be ignored.

(d) Do the following changes in (a). Let $S = \{\lambda\rho + 4\gamma \,/\, \rho < \kappa\nu, \gamma < \lambda\}$, $L = \{\lambda\rho + 4\gamma + 2 \,/\, \rho < \kappa\nu, \gamma < \lambda\}$ and put $\mathbb{S}_\xi = \mathbb{D}$ for all $\xi \in S$. Here, (iii) and (iv) can be ignored.

(e) Use an iteration as in (a) but with the following changes. Let $S = \{\lambda\rho + 4\gamma \,/\, \rho < \kappa\nu, \gamma < \lambda\}$, $U = \{\lambda\rho + 4\gamma + 2 \,/\, \rho < \kappa\nu, \gamma < \lambda\}$ and put

$$S_\xi = \begin{cases} \mathbb{C} & \text{if } \xi = \lambda\rho + 8\gamma, \\ \mathbb{B} & \text{if } \xi = \lambda\rho + 8\gamma + 4. \end{cases}$$

(ii) and (iv) can be ignored.

(f) Use an iteration as in (a) but put

$$S_\xi = \begin{cases} \mathbb{C} & \text{if } \xi = \lambda\rho + 4\gamma, \\ \mathbb{B} & \text{if } \xi = \lambda\rho + 4\gamma + 2. \end{cases}$$

(ii) can be ignored.

□

5.2.7 Theorem. *It is consistent with* ZFC *that* $\text{add}(\mathcal{N}) = \mu_1$, $\text{cov}(\mathcal{N}) = \mu_2$, $\mathfrak{p} = \text{non}(\mathcal{M}) = \nu$, $\mathfrak{d} = \text{cof}(\mathcal{M}) = \kappa$, $\text{non}(\mathcal{N}) = \mathfrak{c} = \lambda$ *and that one of the following statements hold.*

(a) $\text{cov}(\mathcal{M}) = \mathfrak{r} = \kappa$.

(b) $\mathfrak{u} = \nu$.

(c) $\text{cov}(\mathcal{M}) = \nu$ *and* $\mathfrak{r} = \kappa$.

Proof. (a) Perform a matrix iteration as in the proof of Theorem 5.2.4(a), but change the following. Ignore (ii) and, additionally, for $\alpha < \kappa$ and $\rho < \kappa\nu$ consider a sequence $\langle \dot{\mathbb{B}}^\rho_{\alpha,\gamma} \rangle_{\gamma<\lambda}$ of $\mathbb{P}_{\alpha,\lambda\rho}$-names for <u>all</u> the suborders of $\mathbb{B}$ of size $< \mu_2$.

(iv) If $\xi = \lambda\rho + 2(2 + 3\epsilon) + 1$ for some $\rho < \kappa\mu$ and $\epsilon < \lambda$, put $\Delta(\xi) = (g(\epsilon))_0$ and $\dot{\mathbb{T}}_\xi = \mathbb{M}_{\dot{\mathcal{F}}^\rho_{g(\epsilon)}}$.

(v) If $\xi = \lambda\rho + 2(2 + 3\epsilon + 1) + 1$ for some $\rho < \kappa\mu$ and $\epsilon < \lambda$, put $\Delta(\xi) = (g(\epsilon))_0$ and $\dot{\mathbb{T}}_\xi = \dot{\mathbb{A}}^\rho_{g(\epsilon)}$.

(vi) If $\xi = \lambda\rho + 2(2 + 3\epsilon + 2) + 1$ for some $\rho < \kappa\mu$ and $\epsilon < \lambda$, put $\Delta(\xi) = (g(\epsilon))_0$ and $\dot{\mathbb{T}}_\xi = \dot{\mathbb{B}}^\rho_{g(\epsilon)}$.

From the proof of 5.2.1(a), it is clear that $\mathfrak{c} = \lambda$, $\mathrm{cov}(\mathcal{M}) = \mathfrak{r} = \kappa$ and $\mathfrak{p} = \mathrm{non}(\mathcal{M}) = \mathfrak{g} = \nu$ are true in $V_{\kappa,\kappa\lambda\nu}$. For this iteration, we get $\mathfrak{d} \leq \kappa$ by Claim 5.2.6.

Note that $\mathbb{P}_{\kappa,\lambda\kappa\nu}$ is μ_1-$\in^*_{\bar{H}}$-good and μ_2-$\pitchfork^{\bar{I}}$-good. Therefore, it is easy to conclude, in $V_{\kappa,\lambda\kappa\nu}$, that $\mathrm{add}(\mathcal{N}) \leq \mu_1$, $\mathrm{cov}(\mathcal{N}) \leq \mathfrak{b}_{\pitchfork^{\bar{I}}} \leq \mu_2$ and $\lambda \leq \mathfrak{d}_{\pitchfork^{\bar{I}}} \leq \mathrm{non}(\mathcal{N})$.

The same technique used in Theorem 4.1.2 and the idea of the proof of $\nu \leq \mathfrak{p}$ work to get $\mu_1 \leq \mathrm{add}(\mathcal{N})$ and $\mu_2 \leq \mathrm{cov}(\mathcal{N})$ in $V_{\kappa,\lambda\kappa\nu}$.

(b) Do the following changes in (a). Let $S = \{\lambda\rho + 4\gamma \ / \ \rho < \kappa\nu, \gamma < \lambda\}$, $U = \{\lambda\rho + 4\gamma + 2 \ / \ \rho < \kappa\nu, \gamma < \lambda\}$ and put $\mathbb{S}_\xi = \mathbb{E}$ for all $\xi \in S$. (iii) can be ignored.

(c) In (a), put $\mathbb{S}_\xi = \mathbb{E}$ for all $\xi \in S$.

□

CHAPTER 6

ROTHBERGER GAPS ON FRAGMENTED IDEALS

Recall the notions of gaps of Section 1.5. For an ideal $\mathcal{I}$ on ω, $\mathfrak{b}(\mathcal{I})$, *Rothberger number of* $\mathcal{I}$, is defined as the least ordinal δ such that there exists a (ω, δ)-gap in $\mathcal{P}(\omega)/\mathcal{I}$ (which is, in fact, an uncountable regular cardinal when it exists). Our main results about Rothberger gaps are described with this number. Also, we focus on a particular subclass of F_σ ideals that is called *fragmented*.

This chapter is structured as follows. In Section 6.1 we introduce, in detail, the Rothberger number of an ideal and the notion of fragmented ideal, plus some preliminary results that can be interpreted in terms of the Rothberger number. We prove in Section 6.2 that, for a large subclass of fragmented ideals, any ideal in that subclass has Rothberger number equal to ω_1, which is one of our main results about Rothberger gaps.

There is a subclass of the fragmented ideals, known as *gradually fragmented*, whose Rothberger gaps can be destroyed by a two-step iteration of Suslin ccc posets. We show how to do this in Section 6.3. On the other hand, we explain how to preserve Rothberger gaps for fragmented ideals in Section 6.4 in a similar way as the preservation results in Section 3.1. At the end, we use the results of the previous two sections to prove the main consistency results concerning Rothberger gaps on fragmented ideals.

6.1 Rothberger number and fragmented ideals

For an ideal $\mathcal{I}$ on ω, we can define the *Rothberger number* $\mathfrak{b}(\mathcal{I})$ *of* $\mathcal{I}$ in terms of Definition 1.5.1, where it is not necessary to look at linear Rothberger gaps. We show in the next result that many restrictions on the definition of $\mathfrak{b}(\mathcal{I})$ can be considered and the value of the cardinal invariant remains unchanged.

6.1.1 Lemma. *Let* $\mathfrak{b}(\mathcal{I})$ *be the least cardinal number* λ *such that there exists a Rothberger gap* $\langle\mathcal{A},\mathcal{B}\rangle$ *with* $|\mathcal{B}| = \lambda$*. The following restrictions on* $\mathcal{A}$ *and* $\mathcal{B}$ *can be applied without affecting the value of* $\mathfrak{b}(\mathcal{I})$.

(I) $\mathcal{A}$ *can either be*

- *(i) a disjoint family, even a partition of* ω,
- *(ii) a* $\subseteq$*-increasing sequence of length* ω*, even with union equal to* ω*, or*
- *(iii) a* $\subseteq$*-increasing,* $\subsetneq_{\mathcal{I}}$*-increasing sequence of length* ω*, even with union equal to* ω.

Moreover, it can be assumed that all the members of $\mathcal{A}$ *are* $\mathcal{I}$*-positive.*

(II) $\mathcal{B}$ *can either be*

- *(i) a* $\subseteq_{\mathcal{I}}$*-well-ordered sequence or* *(ii) a $\subsetneq_{\mathcal{I}}$-well-ordered sequence.*

Moreover, it can be assumed that all the members of $\mathcal{B}$ are $\mathcal{I}$-positive. For (i) or (ii), $\mathfrak{b}(\mathcal{I})$ is the least order type of such $\mathcal{B}$.

Proof. In each of these cases, objects in $\mathcal{I}$ can be ignored, so this explain why we can assume that $\mathcal{A}$ and $\mathcal{B}$ contain only $\mathcal{I}$-positive sets.

(I) Let $\langle \mathcal{A}, \mathcal{B} \rangle$ be a Rothberger gap with $\mathcal{A} = \{A_n \mathbin{/} n < \omega\}$.

(i) Put $A'_n = A_n \setminus \bigcup_{k<n} A_k$, so $\mathcal{A}' = \{A'_n \mathbin{/} n < \omega\} \setminus \mathcal{I}$ is a disjoint family and $\langle \mathcal{A}', \mathcal{B} \rangle$ is still a gap. Moreover, if $\omega \setminus \bigcup \mathcal{A}' = \{l_n \mathbin{/} n \leq \eta\}$ with some $\eta \leq \omega$ and $\mathcal{A}' = \{A''_n \mathbin{/} n < \omega\}$ (it has to be infinite because it is a half of a gap), put $A^3_n = A''_n \cup \{l_n\}$ for each $n \leq \eta$ and $A^3_n = A''_n$ for $n > \eta$. Clearly, $\mathcal{A}^3 = \{A^3_n \mathbin{/} n < \omega\}$ is a partition of ω into $\mathcal{I}$-positive sets and $\langle \mathcal{A}^3, \mathcal{B} \rangle$ is a gap.

(ii) It follows from the proof of (I)(iii).

(iii) By (i), we may assume that $\mathcal{A}$ is a partition of ω into $\mathcal{I}$-positive sets. Put $A'_n = \bigcup_{n<\omega} A_n$, so $\mathcal{A}' = \{A'_n \mathbin{/} n < \omega\}$ is $\subseteq$-increasing, $\subsetneq_{\mathcal{I}}$-increasing and its union is ω. It is clear that $\langle \mathcal{A}', \mathcal{B} \rangle$ is a gap.

(II) Let $\langle \mathcal{A}, \mathcal{B} \rangle$ be a Rothberger gap with $\mathcal{A} = \{A_n \mathbin{/} n < \omega\}$ and $\mathcal{B}$ of minimal size λ such that there is such an $\mathcal{A}$ (even we can assume that $\mathcal{A}$ is restricted as in (I)). Without loss of generality, we may assume that all members of $\mathcal{B}$ are $\mathcal{I}$-positive.

(i) See the proof of (II)(ii).

(ii) Enumerate $\mathcal{B} = \{B_\alpha \mathbin{/} \alpha < \lambda\}$. Construct, by recursion, a $\subsetneq_{\mathcal{I}}$-increasing sequence $\langle B'_\alpha \rangle_{\alpha<\lambda}$ of $\mathcal{I}$-positive sets orthogonal with $\mathcal{A}$ such that $\forall_{\xi\leq\alpha}(B_\xi \subseteq_{\mathcal{I}} B'_\alpha)$ for any $\alpha < \lambda$. Put $B'_0 = B_0$. For the successor step, find $\gamma < \lambda$ minimal such that $B_\gamma \nsubseteq_{\mathcal{I}} B'_\alpha$ (if such a γ does not exists, then $\langle \mathcal{A}, \mathcal{B} \rangle$ would be separated by B'_α), so put $B'_{\alpha+1} = B'_\alpha \cup B_\gamma$. For the limit step, by minimality of λ, $\langle \mathcal{A}, \{B'_\xi \mathbin{/} \xi < \alpha\} \rangle$ can be separated by a set $C \subseteq \omega$. As $\langle \mathcal{A}, \mathcal{B} \rangle$ is a gap, there exists a minimal $\gamma < \lambda$ such that $B_\gamma \nsubseteq_{\mathcal{I}} C$, so it is clear that $\gamma \geq \alpha$. Put $B'_\alpha = B_\gamma \cup C$.
It is clear that $\langle \mathcal{A}, \{B'_\alpha \mathbin{/} \alpha < \lambda\} \rangle$ is a gap.

□

Note that Lemma 6.1.1 implies that $\mathfrak{b}(\mathcal{I})$ is a regular cardinal, if it exists. Note that there are no gaps on $\mathcal{I}$ when $\mathcal{I}$ is a maximal ideal.

The following result becomes very practical in this chapter to find $\mathfrak{b}(\mathcal{I})$ in our applications.

6.1.2 Lemma. *Let $\mathcal{I}$ be an ideal on ω, $X \subseteq \omega$ infinite. Then, every gap on $\mathcal{I}{\restriction}X$ (as an ideal on X) is a gap on $\mathcal{I}$. In particular, $\mathfrak{b}(\mathcal{I}) \leq \mathfrak{b}(\mathcal{I}{\restriction}X)$.*

Many results of Section 1.5 can be stated in terms of the Rothberger number of an ideal.

6.1.3 Theorem. *Let $\mathcal{I}$ be an ideal on ω.*

(a) $\mathfrak{b}(\mathcal{I})$ is uncountable (Hadamard [H84]).

(b) $\mathfrak{b}(\mathrm{Fin}) = \mathfrak{b}$ (Rothberger [Ro41]).

(c) If $\mathcal{I}$ is either F_σ or an analytic P-ideal, then $\mathfrak{b}(\mathcal{I}) \leq \mathfrak{b}$. (Todorčević [T98]).

(d) If $\mathcal{I}$ is an analytic P-ideal, then $\mathfrak{b}(\mathcal{I}) = \mathfrak{b}$.

Proof. (a) is Lemma 1.5.2, (b) follows from Theorem 1.5.3, (c) is Corollary 1.5.5 and (d) is Corollary 1.5.7. □

In view of the previous theorem, we are interested in investigating $\mathfrak{b}(\mathcal{I})$ for F_σ ideals $\mathcal{I}$. Because of their combinatorial simplicity, we focus our research in the following subclass of F_σ ideals.

6.1.4 Definition (Fragmented ideals (Hrušák, Rojas-Rebolledo and Zapletal [HrRZ])**.** (1) An ideal $\mathcal{I}$ is *fragmented* if there exists a partition $\{a_i\}_{i<\omega}$ of ω into non-empty finite sets and, for each $i<\omega$, a submeasure $\varphi_i : \mathcal{P}(a_i) \to [0,+\infty)$ such that $x \in \mathcal{I}$ iff $\{\varphi_i(x \cap a_i)\}_{i<\omega}$ is bounded (in $[0,+\infty)$). In this case, we say that $\mathcal{I} = \mathcal{I}\langle a_i, \varphi_i\rangle_{i<\omega}$.

(2) A fragmented ideal $\mathcal{I} = \mathcal{I}\langle a_i, \varphi_i\rangle_{i<\omega}$ is *gradually fragmented* if, for any $k<\omega$, there exists an $m \in \omega$ such that

$$\forall_{l<\omega}\forall^\infty_{i<\omega}\forall_{B\subseteq\mathcal{P}(a_i)}\big[\big(|B|\leq l \text{ and } \forall_{b\in B}(\varphi_i(b)\leq k)\big) \Rightarrow \varphi_i\big(\bigcup B\big)\leq m\big].$$

In this case, a function $f:\omega\to\omega$ *witnesses the gradual fragmentation of* $\mathcal{I}$ if, for any $k<\omega$, $f(k)$ satisfies the same property as m above.

Denote $\bar{\varphi}(x) = \sup_{i<\omega}\{\varphi_i(x)\}$ for any $x\subseteq\omega$. $\bar{\varphi}$ turns out to be a lower semicontinuous submeasure on $\mathcal{P}(\omega)$ with $\mathcal{I} = \mathrm{Fin}(\bar{\varphi})$. Thus, any fragmented ideal is F_σ.

The following important dichotomy proved in [HrRZ] implies that the gradual fragmentation of a fragmented ideal does not depend on the choice of the partition and the submeasures. Recall that, if $\mathcal{I}$ is an ideal on ω, $P\subseteq\mathcal{I}$ is *strongly unbounded* if it is infinite and the union of every infinite subset of $\mathcal{I}$ is $\mathcal{I}$-positive.

6.1.5 Theorem (Hrušák, Rojas-Rebolledo, Zapletal [HrRZ, Thm. 2.6])**.** *If $\mathcal{I} = \mathcal{I}\langle a_i, \varphi_i\rangle_{i<\omega}$ is a fragmented not gradually fragmented ideal for some $\langle a_i, \varphi_i\rangle_{i<\omega}$, then $\mathcal{I}$ contains a perfect strongly unbounded subset.*

This result becomes a dichotomy because no gradually fragmented ideal contains a perfect strongly unbounded subset. To see this, let $\mathcal{I} = \mathcal{I}\langle a_i, \varphi_i\rangle_{i<\omega}$ be a gradually fragmented ideal and $P\subseteq\mathcal{I}$ perfect. As $\mathcal{I} = \bigcup_{k<\omega} C_k$ where $C_k = \{x\subseteq\omega \,/\, \forall_{i<\omega}(\varphi_i(x\cap a_i)\leq k)\}$ is closed, there is one $k<\omega$ such that $P\cap C_k$ has size $\mathfrak{c}$. By gradual fragmentation, find $m<\omega$ and a strictly increasing sequence $\{N_l\}_{l<\omega}$ such that

$$\forall_{l<\omega}\forall_{i\geq N_l}\forall_{B\subseteq\mathcal{P}(a_i)}\big[\big(|B|\leq l+2 \text{ and } \forall_{b\in B}(\varphi_i(b)\leq k)\big) \Rightarrow \varphi_i\big(\bigcup B\big)\leq m\big].$$

By recursion, as $P\cap C_k$ is perfect, find $\{x_l\}_{l<\omega}$ a 1-1 sequence in $P\cap C_k$ such that $x_{l+1}\cap\bigcup_{i<N_l} a_i = x_l\cap\bigcup_{i<N_l} a_i$ for all $l<\omega$. Hence, $\varphi_i(a_i\cap\bigcup_{n<\omega} x_n) = \varphi_i(a_i\cap(\bigcup_{n\leq l+1} x_n))\leq m$ for all $i\in[N_l, N_{l+1})$ and $l<\omega$, that is, $\bigcup_{n<\omega} x_n\in\mathcal{I}$. Therefore, P is not strongly unbounded.

We need the following notation with functions for the rest of this chapter. id_X is the identity function from X to X. For $f,g\in\omega^\omega$ and $c<\omega$, we extend the use of the notation for operations with natural numbers to functions, that is, $f\cdot g = fg$ is the function such that $(fg)(i) = f(i)\cdot g(i) = f(i)g(i)$, $(f^g)(i) = f(i)^{g(i)}$, $(cf)(i) = c\cdot f(i)$, etc. We may use this notation for real valued functions as well. Also, natural numbers may represent constant functions, that is, a natural number n may represent the constant function from ω to $\{n\}$. This will be clear from the context.

6.1.6 Remark. (1) A fragmented ideal may be trivial, e.g., choose any partition of ω into non-empty finite sets and use the zero-measure at each piece of the partition. The trivial ideal clearly is gradually fragmented. A fragmented ideal $\mathcal{I}\langle a_i, \varphi_i\rangle_{i<\omega}$ is not trivial iff $\forall_{k<\omega}\exists^\infty_{i<\omega}(\varphi_i(a_i)>k)$.

(2) If $\varphi:\mathcal{P}(Y)\to[0,+\infty]$ is a submeasure, then $\varphi'(x) = \lceil\varphi(x)\rceil$ (least integer above $\varphi(x)$ if it is $<+\infty$, or $+\infty$ otherwise) is also a submeasure. Therefore, if $\mathcal{I} = \mathcal{I}\langle a_i, \varphi_i\rangle_{i<\omega}$ is a fragmented ideal, then $\mathcal{I} = \mathcal{I}\langle a_i, \varphi'_i\rangle_{i<\omega}$ where $\varphi'_i:\mathcal{P}(a_i)\to\omega$ such that $\varphi'_i(x) = \lceil\varphi_i(x)\rceil$.

(3) If c is a positive real, then $\mathcal{I}\langle a_i, \varphi_i\rangle_{i<\omega} = \mathcal{I}\langle a_i, c\varphi_i\rangle_{i<\omega}$.

Before introducing some examples of fragmented ideals, we prove some generalities related to tallness.

6.1.7 Lemma. *Let $\mathcal{I} = \mathcal{I}\langle a_i, \varphi_i\rangle_{i<\omega}$ be a fragmented ideal. The following are equivalent.*

(i) $\mathcal{I}$ is tall.

(ii) There exists a $k \leq \omega$ such that, for every $i < \omega$ and $j \in a_i$, $\varphi_i(\{j\}) \leq k$.

(iii) $\forall_{m<\omega}\exists_{l>m}\forall_{i<\omega}\forall_{x\subseteq a_i}(\varphi_i(x) > m \Rightarrow \exists_{x'\subseteq x}(m < \varphi_i(x') \leq l))$.

(iv) The previous formula but with $m = 0$.

(v) The formula of (iii) with $\exists_{m<\omega}$ instead of the universal quantifier.

Proof. To see (i) implies (ii), assume the negation of (ii). Therefore, we can find $W := \{j_k \;/\; k < \omega\} \subseteq \omega$ such that $\bar{\varphi}(\{j_k\}) > k$ for any $k < \omega$. Then, it is clear that $\mathcal{I}{\restriction}W$ does not contain infinite sets.

Assume (ii) to prove (iii). Let $k > 0$ be as in (ii). Now, for $m < \omega$, $l = m + k$ works. By contradiction, assume that there are $i < \omega$ and $x \subseteq a_i$ such that $\varphi_i(x) > m$ and all its subsets have submeasure not in $(m, m + k]$. In particular, $\varphi_i(x) > m + k$. When extracting one point of x, its submeasure is still bigger than m and, then, bigger than $m + k$. By repeating this process, we get $\varphi_i(\varnothing) > m + k$ at the end, which is a contradiction.

To finish, we prove (v) implies (i). Let m and $l > m$ be as in (v) and assume that $W \subseteq \omega$ is infinite. Now, for each $i < \omega$, if $\varphi_i(W \cap a_i) > m$ then there exists a $y_i \subseteq W \cap a_i$ with submeasure in $(m, l]$. If $\varphi_i(W \cap a_i) \leq m$, put $y_i = W \cap a_i$. Then, $y := \bigcup_{i<\omega} y_i \subseteq W$ is infinite and $\bar{\varphi}(y) \leq l$, so $y \in \mathcal{I}{\restriction}W$. □

6.1.8 Corollary. *Any somewhere tall fragmented ideal is not a P-ideal.*

Proof. By restricting the ideal to some infinite subset of ω, we may assume that $\mathcal{I} = \mathcal{I}\langle a_i, \varphi_i\rangle_{i<\omega}$ is a non-trivial tall fragmented ideal. Find $L \subseteq \omega$ infinite such that $\{\varphi_i(a_i)\}_{i\in L}$ is strictly increasing. Choose $\{l_k\}_{k<\omega}$ strictly increasing by applying, recursively, Lemma 6.1.7(iii) and starting with $l_0 = 0$. Also, construct a strictly increasing sequence $\{N_k\}_{k<\omega}$ of natural numbers such that $\varphi_i(a_i) > \sum_{j\leq k} l_j + l_k$ for all $i \geq N_k$, $i \in L$. By recursion on k, choose $x_i^k \subseteq a_i$ for $i \geq N_k$, $i \in L$ such that $l_k < \varphi_i(x_i^k) \leq l_{k+1}$ and $x_i^k \cap x_i^j = \varnothing$ for each $j < k$. Indeed, for $i \geq N_k$, $\varphi_i(\bigcup_{j<k} x_i^j) \leq \sum_{j\leq k} l_j$, so its complement with respect to a_i has submeasure bigger than l_k. Thus, by Lemma 6.1.7(iii), there exists an $x_i^k \subseteq a_i \smallsetminus \bigcup_{j<k} x_i^j$ as required.

Put $x^k := \bigcup\{x_i^k \;/\; i \geq N_k \text{ and } i \in L\}$, which is clearly in $\mathcal{I}$. Now, if $y \subseteq \omega$ is such that $x^k \subseteq^* y$ for all $k < \omega$, we get that $l_k < \varphi_{i_k}(x_{i_k}^k)$ for some $i_k \in L$ such that $x_{i_k}^k \subseteq y$. Therefore, $y \notin \mathcal{I}$. □

On the other hand, nowhere tall fragmented ideals can be simply characterized. For example, a non-trivial fragmented ideal $\mathcal{I} = \mathcal{I}\langle a_i, \varphi_i\rangle_{i<\omega}$ where $\{|a_i|\}_{i<\omega}$ is bounded is nowhere tall. Indeed, if $X \subseteq \omega$ is $\mathcal{I}$-positive then $\{\bar{\varphi}(\{j\})\}_{j\in X}$ is unbounded and, by Lemma 6.1.7, $\mathcal{I}{\restriction}X$ is not tall. A converse of this and the mentioned characterization is stated as follows.

6.1.9 Lemma. *If $\mathcal{I}$ is a nowhere tall fragmented ideal on ω, then $\mathcal{I} = \mathcal{I}\langle a_i, \varphi_i\rangle_{i<\omega}$ where $a_i = \{i\}$. Moreover, $\mathcal{I}$ is gradually fragmented and can only be one of the following ideals:*

(i) $\mathcal{I} = \{x \subseteq \omega \;/\; x \subseteq^* A\}$ for some $A \subseteq \omega$, or

(ii) $\mathcal{I}$ is the ideal generated by some infinite partition of ω into infinite sets.

Proof. Let $\mathcal{I} = \mathcal{I}\langle a_i', \varphi_i'\rangle_{i<\omega}$. By Lemma 6.1.7(ii), $\mathcal{I}$ nowhere tall means that, for any $\mathcal{I}$-positive $X \subseteq \omega$, $\{\bar{\varphi}'(\{j\})\}_{j\in X}$ is unbounded. Therefore, for any $x \subseteq \omega$, $x \in \mathcal{I}$ iff $\{\bar{\varphi}'(\{j\})\}_{j\in x}$ is bounded, so $\mathcal{I} = \langle a_i, \varphi_i\rangle_{i<\omega}$ where $a_i = \{i\}$ and $\varphi_i(\{i\}) = \lceil\bar{\varphi}'(\{i\})\rceil$. Here, the identity function witnesses the gradual fragmentation of $\mathcal{I}$. Now, for $m < \omega$, put $I_m := \{i < \omega \;/\; \varphi_i(\{i\}) = m\}$. Note that $\{I_m\}_{m<\omega}$ is a partition of ω and that $\mathcal{I}$ is generated by this partition. If I_m is infinite for infinitely many $m < \omega$, then we easily get (ii). Otherwise, if there is some $N < \omega$ such that I_m is finite for all $m \geq N$, then we get (i) with $A := \bigcup_{m<N} I_m$. □

Note that the case (i) gives us a P-ideal, so $\mathfrak{b}(\mathcal{I})=\mathfrak{b}$ for such non-trivial $\mathcal{I}$ by Theorem 6.1.3. In case (ii), $\mathcal{I}$ is not a P-ideal but we are going to prove in Corollary 6.3.6 that still $\mathfrak{b}(\mathcal{I})=\mathfrak{b}$. In Section 6.5 we prove that every somewhere tall ideal has, consistently, Rothberger number strictly less than $\mathfrak{b}$.

As a first general example, we prove that any somewhere tall fragmented ideal that is characterized by measures (finitely additive submeasures) is not gradually fragmented.

6.1.10 Lemma. *Let $\mathcal{I}=\mathcal{I}\langle a_j,\varphi_j\rangle_{j<\omega}$ be a somewhere tall fragmented ideal such that all the φ_j are measures. Then, $\mathcal{I}$ is not gradually fragmented.*

Proof. Without loss of generality, by restricting the ideal to an $\mathcal{I}$-positive set, we may assume that $\mathcal{I}$ is tall and non-trivial. Moreover, by Lemma 6.1.7(ii) and Remark 6.1.6(3), we may assume that, for all $j<\omega$ and $k\in a_j$, $0<\varphi_j(\{k\})\leq 1$ (we restrict the ideal again to those points that have non-zero measure). Let $i<\omega$ and $L_i:=\{j<\omega\ /\ \varphi_j(a_j)\geq i+1\}$, which is infinite. To see that $\mathcal{I}$ is not gradually fragmented we show that, for each $j\in L_i$, there exists a pairwise disjoint family $B_j\subseteq\mathcal{P}(a_j)$ of size $\leq 2\cdot i$ such that $\forall_{b\in B_j}(\varphi_j(b)\leq 1)$ and $\varphi_j(\bigcup B_j)>i$.

6.1.11 Claim. *Let $x\subseteq a_j$ such that $\varphi_j(x)\geq 1$. Then, there exists a $y\subseteq x$ such that $\frac{1}{2}<\varphi_j(y)\leq 1$.*

Proof. Let y be a subset of x of maximal measure ≤ 1. If $\varphi_j(y)\leq\frac{1}{2}$ there exists a $k\in x\smallsetminus y$ so, as $\varphi_j(y\cup\{k\})>1$, we get that $\frac{1}{2}<\varphi(\{k\})\leq 1$, which contradicts the maximality of y. □

Construct $B_j=\{b_{j,k}\ /\ k<l\}$ by recursion on k, where $\frac{1}{2}<\varphi_j(b_{j,k})\leq 1$ (l is defined at the end). Assume we have got $b_{k'}$ for $k'<k$. If $\varphi_j(\bigcup_{k'<k}b_{j,k'})>i$ put $l=k$ and stop the recursion. Otherwise, $\varphi_j(a_j\smallsetminus\bigcup_{k'<k}b_{j,k'})\geq 1$, so we get $b_{j,k}\subseteq a_j\smallsetminus\bigcup_{k'<k}b_{j,k'}$ by application of the claim. If this recursion reaches $2\cdot i$ steps, put $l=2\cdot i$. Note that $\varphi_j(\bigcup B_j)=\sum_{k<2\cdot i}\varphi_j(b_{j,k})>i$. □

6.1.12 Example. (1) Given $c\in\omega^\omega$, $c\geq 2$ that converges to infinity and any partition $P=\{a_i\}_{i<\omega}$ of ω into non-empty finite sets, denote by $\mathcal{I}_c(P):=\mathcal{I}\langle a_i,\varphi_i\rangle_{i<\omega}$ where $\varphi_i(x)=\log_{c(i)}(|x|+1)$ for $x\subseteq a_i$. In view of Remark 6.1.6(2), $\varphi_i(x)$ can also be defined as the least $k<\omega$ such that $|x|<c(i)^k$.

This ideal is tall and gradually fragmented. Indeed, $f:\omega\to\omega$, $f(k)=k+1$ witnesses the gradual fragmentation of the ideal, as

$$\forall_{l<\omega}\forall_{i,c(i)\geq l}\forall_{B\subseteq\mathcal{P}(a_i)}\big[\big(|B|\leq l\text{ and }\forall_{x\in B}(|x|<c(i)^k)\big)\Rightarrow|\bigcup B|<c(i)^{k+1}\big].$$

This ideal is not trivial iff $\forall_{k<\omega}\exists^\infty_{i<\omega}(|a_i|\geq c(i)^k)$.

This is a generalization of the polynomial growth ideal $\mathcal{I}_P$, which is $\mathcal{I}_c(\langle a_i\rangle_{i<\omega})$ where $\{a_i\}_{i<\omega}$ is the interval partition of ω such that $|a_i|=2^i$ and $c=\max\{id_\omega,2\}$.

(2) An equivalent definition of the ideal $\mathcal{ED}_{\text{fin}}$ mentioned in the introduction is given by $\mathcal{ED}_{\text{fin}}=\mathcal{I}\langle a_i,\varphi_i\rangle_{i<\omega}$ where $\{a_i\}_{i<\omega}$ is the interval partition such that $|a_i|=i+1$ and $\varphi_i(x)=|x|$ for $x\subseteq a_i$. As this ideal is tall and characterized by measures, Lemma 6.1.10 implies that $\mathcal{ED}_{\text{fin}}$ is not gradually fragmented.

(3) Let $g:\omega\to\omega\smallsetminus\{0\}$ and $\{a_i\}_{i<\omega}$ a partition of ω into non-empty finite sets. Define $\mathcal{I}=\mathcal{I}\langle a_i,\varphi_i\rangle_{i<\omega}$ where $\varphi_i(x)=|x|/g(i)$. This ideal is tall and, if it is non-trivial, that is, the sequence of reals $\{|a_i|/g(i)\}_{i<\omega}$ is not bounded, then $\mathcal{I}$ is not gradually fragmented by Lemma 6.1.10. It is clear that $\mathcal{ED}_{\text{fin}}$ is a particular case of this ideal. Also, the linear growth ideal $\mathcal{I}_L$ is a particular case with $\{a_i\}_{i<\omega}$ the interval partition of ω such that $|a_i|=2^i$ and $g(i)=i+1$.

(4) Let $g:\omega\to\omega\smallsetminus\{0\}$ and $\{a_i\}_{i<\omega}$ a partition of ω into non-empty finite sets. Define $\mathcal{I}=\mathcal{I}\langle a_i,\varphi_i\rangle_{i<\omega}$ where $\varphi_i(x)=|x|^{1/g(i)}$. $\mathcal{I}$ is tall and it is gradually fragmented iff $\exists_{m<\omega}\forall_{l<\omega}\forall^\infty_{i<\omega}(\min\{l,|a_i|\}\leq m^{g(i)})$. If such m exists, $f(k)=m\cdot k$ witnesses the gradual fragmentation of $\mathcal{I}$. Indeed, for $l<\omega$, let $N<\omega$ be such that $\forall_{i\geq N}(\min\{l,|a_i|\}\leq m^{g(i)})$ so, for $i\geq N$ and $B\subseteq\mathcal{P}(a_i)$ of size $\leq l$ such that all its members have size $\leq k^{g(i)}$, $|\bigcup B|\leq(m\cdot k)^{g(i)}$.

For the other direction, assume that $\forall_{m<\omega}\exists_{l<\omega}\exists^{\infty}_{i<\omega}(m^{g(i)} < \min\{l,|a_i|\})$. For $m<\omega$ choose $l<\omega$ and $W\subseteq\omega$ infinite such that $m^{g(i)} < \min\{l,|a_i|\}$ for all $i\in W$. Then, for any $B\subseteq\mathcal{P}(a_i)$ of size $m^{g(i)}+1$ whose members are singletons (such a family exists), $|\bigcup B| > m^{g(i)}$.

The following result is a characterization of fragmented not gradually fragmented ideals. This was taken from the proof of Theorem 6.1.5 in [HrRZ].

6.1.13 Lemma. *Let $\mathcal{I}=\mathcal{I}\langle a_j,\varphi_j\rangle_{j<\omega}$ be a fragmented not gradually fragmented ideal. Then, there exist $k<\omega$, a sequence $\langle C_i\rangle_{i<\omega}$ of pairwise disjoint infinite subsets of ω and a sequence $\{l_i\}_{i<\omega}$ of natural numbers such that, for any $i<\omega$ and $j\in C_i$, there exists a pairwise disjoint family B_j of subsets of a_j such that $|B_j|=l_i$, $\forall_{b\in B_j}(0<\varphi_j(b)\leq k)$ and $i<\varphi_j(\bigcup B_j)\leq i+k$.*

Proof. As $\mathcal{I}$ is not gradually fragmented, there exists a $k<\omega$ such that, for any $i<\omega$, there are $l'_i<\omega$ and $W'_i\subseteq\omega$ infinite such that, for any $j\in W'_i$, there exists a $B'_j\subseteq\mathcal{P}(a_j)$ of size $\leq l'_i$ such that $\forall_{b\in B'_j}(\varphi_j(b)\leq k)$ and $\varphi_j(\bigcup B'_j)>i$. By taking complements between the members of B'_j, it is easy to find a pairwise disjoint family $B''_j\subseteq\mathcal{P}(a_j)$ of size $\leq l'_i$ such that $\bigcup B''_j=\bigcup B'_j$ and $\forall_{b\in B'_j}(0<\varphi_j(b)\leq k)$. A similar argument as in the proof of (ii) implies (iii) of Lemma 6.1.7 shows that there is a $B_j\subseteq B''_j$ such that $i<\varphi_j(\bigcup B_j)\leq i+k$.

It is clear that, for each $i<\omega$, there exist $l_i\leq l'_i$ and $W''_i\subseteq W'_i$ infinite such that $|B_j|=l_i$ for all $j\in W''_i$. Finally, find $C_i\subseteq W''_i$ infinite and pairwise disjoint. □

As a final remark, note that a fragmented ideal $\mathcal{I}=\mathcal{I}\langle a_i,\varphi_i\rangle_{i<\omega}$ is coded by the real $\langle a_i,\varphi_i\rangle_{i<\omega}$, so the formula $x\in\mathcal{I}$ is clearly F_σ and expressions like "$\mathcal{I}$ is gradually fragmented" and "$\mathcal{I}$ is tall" (see Lemma 6.1.7) are Borel statements and, therefore, absolute notions.

6.2 Fragmented ideals with short gaps

In this section, we present a wide class of fragmented not gradually fragmented ideals that have, provably in ZFC, Rothberger number equal to $\aleph_1$. In fact, we present two different arguments for this. The first (Theorem 6.2.1), discovered by Brendle in 2009, is based on eventually different functions and was used originally to show $\mathfrak{b}(\mathcal{ED}_{\text{fin}})=\aleph_1$; in fact it can be used for the ideals in Example 6.1.12(3) as well. The second method (Theorems 6.2.3 and 6.2.5), based on independent functions, seems to apply to a larger class of ideals (including those of Examples 6.1.12(3) and (4)).

6.2.1 Theorem (Brendle (2009)). $\mathfrak{b}(\mathcal{ED}_{\mathit{fin}})=\aleph_1$.

Proof. Construct a disjoint family $\mathcal{A}=\{A_n\ /\ n<\omega\}$ of subsets of ω such that, for each $n<\omega$, $\lim_{i\to+\infty}|A_n\cap a_i|=+\infty$. To see that this can be done, construct, by induction on $n<\omega$, a $\leq$-increasing sequence $\langle e_n\rangle_{n<\omega}$ of functions in ω^ω such that $e_n\leq id_\omega$, e_n converges to infinity and $e_{n+1}-e_n$ converges to infinity. For each $i<\omega$, consider a bijection $g_i:i+1\to a_i$ and put $A_n:=\bigcup_{n<\omega}g_i[[e_n(i),e_{n+1}(i))]$ and $\bar{A}_n:=\bigcup_{k\leq n}A_k$.

For each $n<\omega$ let N_n be such that $A_n\cap a_i\neq\varnothing$ for every $i\geq N_n$. As $\lim_{i\to+\infty}|A_n\cap a_i|=+\infty$, there exists a pairwise eventually different family of functions $\{f_{n,\alpha}\}_{\alpha<\omega_1}$ in $\prod_{i\geq N_n}(A_n\cap a_i)$, that is, if $\alpha\neq\beta$ then $\forall^\infty_{i<\omega}(f_{n,\alpha}(i)\neq f_{n,\beta}(i))$.

Construct, by induction, a $\subseteq_{\mathcal{ED}_{\text{fin}}}$-increasing sequence $\mathcal{B}=\{B_\alpha\}_{\alpha<\omega_1}$ that is $\mathcal{ED}_{\text{fin}}$-orthogonal with $\mathcal{A}$ and such that $\forall_{\beta<\alpha}\forall^\infty_{n<\omega}(\text{ran}f_{n,\beta}\subseteq B_\alpha)$. Indeed, let $B_0=\varnothing$ and $B_{\alpha+1}=B_\alpha\cup\bigcup_{n<\omega}\text{ran}f_{n,\alpha}$. For the limit step, if $\alpha<\omega_1$ limit, let $B_\alpha=\bigcup_{n<\omega}\left[(B_{\alpha_n}\smallsetminus\bar{A}_n)\cup\bigcup_{k<n}\text{ran}f_{n,\beta_k}\right]$ where $\{\alpha_n\}_{n<\omega}$ is a strictly increasing sequence converging to α and $\alpha=\{\beta_k\ /\ k<\omega\}$ is an enumeration. Note that $B_{\alpha_n}\smallsetminus B_\alpha\subseteq B_{\alpha_n}\cap\bar{A}_n\in\mathcal{ED}_{\text{fin}}$, $B_\alpha\cap A_n\subseteq\bigcup_{k<n}\left((B_{\alpha_k}\cap A_n)\cup\text{ran}f_{n,\beta_k}\right)\in\mathcal{ED}_{\text{fin}}$ and $\forall_{n>k}(\text{ran}f_{n,\beta_k}\subseteq B_\alpha)$.

We claim that $\langle\mathcal{A},\mathcal{B}\rangle$ is an $\mathcal{ED}_{\text{fin}}$-gap. Assume the contrary, so there exists a C that separates $\langle\mathcal{A},\mathcal{B}\rangle$. By recursion on $n<\omega$, construct a decreasing chain $\{X_n\}_{n<\omega}$ of infinite subsets of ω and $F_n\subseteq\omega_1$ finite such that $\forall_{\alpha\in\omega_1\smallsetminus F_n}\forall^\infty_{i\in X_n}(f_{n,\alpha}(i)\notin C)$. Start with $X_{-1}=\omega$. Suppose that X_n has been constructed

$(n \geq -1)$. As $C \cap A_{n+1} \in \mathcal{ED}_{\text{fin}}$, there exists an $l < \omega$ such that $\forall_{i<\omega}(|C \cap A_{n+1} \cap a_i| \leq l)$. By recursion in $j < \omega$ choose, if possible, $Y_j \subseteq \omega$ infinite and $\alpha_j \in \omega_1 \setminus \{\alpha_k \ / \ k < j\}$ such that $Y_0 = X_n$, $Y_{j+1} \subseteq Y_j$ and $\forall_{i \in Y_{j+1}}(f_{n+1,\alpha_j}(i) \in C)$. Note that this construction must stop at l, at the latest, that is, Y_{l+1} and α_l cannot exist. For otherwise, as $\{f_{n+1,\alpha}\}_{\alpha<\omega_1}$ is a sequence of pairwise eventually different functions, there exists an $i \in Y_{l+1}$ such that all $f_{n+1,\alpha_j}(i)$ are different for $j \leq l$ and then, as $\{f_{n+1,\alpha_j}(i) \ / \ j \leq l\} \subseteq C \cap A_{n+1} \cap a_i$, $|C \cap A_{n+1} \cap a_i| > l$, which is impossible. Now, once the construction stops at $l_0 \leq l$, $F_{n+1} := \{\alpha_j \ / \ j < l_0\}$ and $X_{n+1} := Y_{l_0}$ are as required.

Let X be a pseudo-intersection of $\{X_n\}_{n<\omega}$. Choose $\alpha < \omega_1$ strictly above all the ordinals in $\bigcup_{n<\omega} F_n$. Note that, for any $n < \omega$, $\text{ran} f_{n,\alpha} \subseteq B_{\alpha+1}$ and $\forall^\infty_{i \in X}(f_{n,\alpha}(i) \notin C)$. On the other hand, as $B_{\alpha+1} \subseteq_{\mathcal{ED}_{\text{fin}}} C$, there exists a $k < \omega$ such that $\forall_{i<\omega}(|a_i \cap B_{\alpha+1} \setminus C| \leq k)$. We can find an $i \in X$ such that $i \geq N_n$ and $f_{n,\alpha}(i) \notin C$ for all $n \leq k$. Then, $\{f_{n,\alpha}(i) \ / \ n \leq k\} \subseteq a_i \cap B_{\alpha+1} \setminus C$ so, as $f_{n,\alpha}(i) \in A_n$, it is clear that $|a_i \cap B_{\alpha+1} \setminus C| > k$, a contradiction. □

6.2.2 Corollary. *If $\mathcal{I}$ is a non-trivial ideal as defined in Example 6.1.12(3), then $\mathfrak{b}(\mathcal{I}) = \aleph_1$.*

Proof. By Lemma 6.1.2, we may assume that $|a_i| = (i+1)g(i)$ for any $i < \omega$. Let $\{a_{i,j}\}_{j<i+1}$ be a partition of a_i into sets of size $g(i)$. Let $\{b_i\}_{i<\omega}$ be the interval partition of ω such that $|b_i| = i+1$ and let $b_i = \{k_{i,j} \ / \ j < i+1\}$ be an enumeration. Define the finite-one function $h : \omega \to \omega$ such that $h^{-1}[\{k_{i,j}\}] = a_{i,j}$. Note that, for $x \subseteq \omega$, $x \in \mathcal{ED}_{\text{fin}}$ iff $h^{-1}[x] \in \mathcal{I}$, so $F : \mathcal{P}(\omega)/\mathcal{ED}_{\text{fin}} \to \mathcal{P}(\omega)/\mathcal{I}$, $F([x]) = [h^{-1}[x]]$, is an embedding (of Boolean algebras).

It suffices to show that F preserves gaps. Let $\langle \mathcal{A}, \mathcal{B} \rangle$ be $\mathcal{ED}_{\text{fin}}$-orthogonal. If the $\mathcal{I}$-orthogonal pair $\langle \{h^{-1}[A] \ / \ A \in \mathcal{A}\}, \{h^{-1}[B] \ / \ B \in \mathcal{A}\} \rangle$ is separated by a subset C of ω, then $H(C) := \bigcup_{i<\omega}\{k_{i,j} \ / \ |C \cap a_{i,j}| \geq \frac{1}{2}g(i)\}$ separates $\langle \mathcal{A}, \mathcal{B} \rangle$. Indeed, if $A \in \mathcal{A}$ then there exists an $l < \omega$ such that $|C \cap h^{-1}[A \cap b_i]| = |C \cap h^{-1}[A] \cap a_i| \leq l \cdot g(i)$ for all $i < \omega$. Therefore, $|H(C) \cap A \cap b_i| \leq 2 \cdot l$ for every $i < \omega$, so $H(C) \cap A \in \mathcal{ED}_{\text{fin}}$. Likewise, as $\omega \setminus H(C) = \bigcup_{i<\omega}\{k_{i,j} \ / \ |a_{i,j} \setminus C| > \frac{1}{2}g(i)\}$, $B \setminus H(C) \in \mathcal{ED}_{\text{fin}}$ for any $B \in \mathcal{B}$. □

We will obtain a generalization of the previous two results in two ways. The first part for our generalization focuses on the class of fragmented not gradually fragmented ideals that can be characterized by uniform submeasures. For a finite set a, we say that a submeasure $\varphi : \mathcal{P}(a) \to [0, +\infty)$ is *uniform* if it only depends on the size of the sets.

6.2.3 Theorem. *Let $\mathcal{I} = \mathcal{I}\langle a_j, \varphi_j \rangle_{j<\omega}$ be a fragmented not gradually fragmented ideal such that all the φ_j are uniform submeasures. Then, $\mathfrak{b}(\mathcal{I}) = \aleph_1$.*

Proof. Let k be as in Lemma 6.1.13. By multiplying all the submeasures by $\frac{1}{k}$ (see Remark 6.1.6(3)), we may assume that $k = 1$. Therefore, we can write $\mathcal{I} = \mathcal{I}\langle a_{i,j,k}, \varphi_{i,j,k} \rangle_{i,j,k<\omega}$, where the submeasures are uniform, such that there is a sequence $\{l_i\}_{i<\omega}$ of natural numbers such that, for any $i < \omega$, there is $W_i \subseteq \omega \times \omega$ infinite and, for any $(j,k) \in W_i$, there exists $B_{i,j,k}$ a pairwise disjoint family of subsets of $a_{i,j,k}$ such that $|B_{i,j,k}| = l_i$, $\forall_{b \in B_{i,j,k}}(0 < \varphi_{i,j,k}(b) \leq 1)$ and $i < \varphi_{i,j,k}(\bigcup B_{i,j,k}) \leq i+1$. By Lemma 6.1.2, we may assume that $W_i = \omega \times \omega$ and $a_{i,j,k} = \bigcup B_{i,j,k}$ for all $i, j, k < \omega$. Also, without loss of generality, $\varphi_{i,j,k} = \lceil \varphi_{i,j,k} \rceil$ (Remark 6.1.6(2)), so $\forall_{b \in B_{i,j,k}}(\varphi_{i,j,k}(b) = 1)$ and $\varphi(a_{i,j,k}) = i+1$.

Fix $i, j, k < \omega$. For each $m \leq i+1$ let $s_{i,j,k}(m)$ be the maximal $n \leq |a_{i,j,k}|$ such that all the subsets of $a_{i,j,k}$ of size n have submeasure equal to m. By uniformity, it is clear that $s_{i,j,k}(m)$ exists and $s_{i,j,k}(m) < s_{i,j,k}(m+1)$ for $m \leq i$. Also, note that $s_{i,j,k}(0) = 0$. By induction on $m \leq i$, it is easy to prove that $m \cdot s_{i,j,k}(1) \leq s_{i,j,k}(m)$.

For each $0 < m < \omega$, $i \geq m$ and $j, k < \omega$, let $n_{i,j,k}(m)$ and $r_{i,j,k}(m) < s_{i,j,k}(m)$ be such that $|a_{i,j,k}| = s_{i,j,k}(m) \cdot n_{i,j,k}(m) + r_{i,j,k}(m)$. Note that $m \cdot s_{i,j,k}(1) \cdot n_{i,j,k}(m) \leq s_{i,j,k}(m) \cdot n_{i,j,k}(m) \leq |a_{i,j,k}| \leq l_i \cdot s_{i,j,k}(1)$, where the last inequality holds because $|B_{i,j,k}| = l_i$. Therefore, $n_{i,j,k}(m) \leq l_i$. Thus, for a fixed $i < \omega$, there is an infinite $V_i \subseteq \omega \times \omega$ such that, for all $0 < m \leq i$, there is an $n_i(m) \leq l_i$ such that $n_{i,j,k}(m) = n_i(m)$ for all $(j,k) \in V_i$. Again, by Lemma 6.1.2, we may assume that $V_i = \omega \times \omega$.

Fix $k<\omega$ and put

$$P_k=\prod\big\{\{x\subseteq a_{i,j,k}\ /\ |x|=s_{i,j,k}(k)\}\ /\ j<\omega, i\geq k\big\}.$$

Say that a family $\mathcal{F}\subseteq P_k$ is *independent* if, for any finite $F\subseteq\mathcal{F}$ and for all $i\geq k$, there are infinitely many j's such that $\{f(i,j)\ /\ f\in F\}$ is either pairwise disjoint or its union is $a_{i,j,k}$. It is easy to see that adding a Cohen real adds a real $c\in P_k$ such that, whenever $\mathcal{F}$ is an independent family in the ground model, $\mathcal{F}\cup\{c\}$ is independent in the extension. Therefore, there exists an independent family $\mathcal{F}_k\subseteq P_k$ of size $\aleph_1$. Say $\mathcal{F}_k=\{f_{k,\alpha}\ /\ \alpha<\omega_1\}$.

For $k<\omega$ let $A_k:=\bigcup_{i,j<\omega}a_{i,j,k}$ and, for $\alpha<\omega_1$, let $B_\alpha:=\bigcup\{f_{k,\alpha}(i,j)\ /\ j,k<\omega, i\geq k\}$. As $A_k\cap B_\alpha=\bigcup\{f_{k,\alpha}(i,j)\ /\ j<\omega, i\geq k\}$ and $\varphi_{i,j,k}(f_{k,\alpha}(i,j))=k$, we get that $A_k\cap B_\alpha\in\mathcal{I}$, that is, $\langle\{A_k\}_{k<\omega},\{B_\alpha\}_{\alpha<\omega_1}\rangle$ is $\mathcal{I}$-orthogonal. We want to show that it is an $\mathcal{I}$-gap.

Assume that B separates $\langle\{A_k\}_{k<\omega},\{B_\alpha\}_{\alpha<\omega_1}\rangle$. Find $\Gamma\subseteq\omega_1$ uncountable and $m<\omega$ such that, for all $\alpha\in\Gamma$, $\bar\varphi(B_\alpha\smallsetminus B)\leq m$ (here, $\bar\varphi(X)=\sup_{i,j,k<\omega}\{\varphi_{i,j,k}(X\cap a_{i,j,k})\}$). Let $k>2m$ and find $i\geq k$ such that $i+1-k>2\cdot\bar\varphi(A_k\cap B)$. Choose $H\subseteq\Gamma$ of size $n_i(k)$. By independence, there are infinitely many j's such that $\{f_{k,\alpha}(i,j)\ /\ \alpha\in H\}$ is a disjoint family because, in the case that its union is $a_{i,j,k}$, we have $r_{i,j,k}(k)=0$ and the family will be disjoint anyway. Work with one of these j's. For any $\alpha\in H$, as $\varphi_{i,j,k}(f_{k,\alpha}(i,j)\smallsetminus B)\leq m$ and $\varphi_{i,j,k}(f_{k,\alpha}(i,j))=k$, we obtain $|f_{k,\alpha}(i,j)\smallsetminus B|<\frac{1}{2}|f_{k,\alpha}(i,j)|=\frac{1}{2}s_{i,j,k}(k)$. Thus, $|\bigcup_{\alpha\in H}f_{k,\alpha}(i,j)\smallsetminus B|<\frac{1}{2}n_i(k)s_{i,j,k}(k)=\frac{1}{2}|\bigcup_{\alpha\in H}f_{k,\alpha}(i,j)|$, which implies that $\varphi_{i,j,k}(\bigcup_{\alpha\in H}f_{k,\alpha}(i,j)\cap B)\geq\frac{1}{2}\varphi_{i,j,k}(\bigcup_{\alpha\in H}f_{k,\alpha}(i,j))$. But, because $r_{i,j,k}(k)<s_{i,j,k}(k)$, we have $\varphi_{i,j,k}(\bigcup_{\alpha\in H}f_{k,\alpha}(i,j))\geq i+1-k>2\cdot\varphi_{i,j,k}(a_{i,j,k}\cap B)$, a contradiction. □

6.2.4 Corollary. *Let $\mathcal{I}$ be a fragmented not gradually fragmented ideal as in Example 6.1.12(4). Then,* $\mathfrak{b}(\mathcal{I})=\aleph_1$.

The second part corresponds to fragmented not gradually fragmented ideals that can be characterized by measures.

6.2.5 Theorem. *Let $\mathcal{I}=\mathcal{I}\langle a_j,\varphi_j\rangle_{j<\omega}$ be a somewhere tall fragmented ideal such that all the φ_j are measures. Then,* $\mathfrak{b}(\mathcal{I})=\aleph_1$.

Proof. $\mathcal{I}$ is not gradually fragmented by Lemma 6.1.10. Like in the first part of the proof of Theorem 6.2.3, we may assume that $\mathcal{I}=\mathcal{I}\langle a_{i,j,k},\varphi_{i,j,k}\rangle_{i,j,k<\omega}$ is given by measures and that there is a sequence $\{l_i\}_{i<\omega}$ of natural numbers such that, for any $i,j,k<\omega$, there exists $B_{i,j,k}$ a pairwise disjoint family of subsets of $a_{i,j,k}$ such that $|B_{i,j,k}|=l_i$, $\forall_{b\in B_{i,j,k}}(0<\varphi_{i,j,k}(b)\leq 1)$, $\bigcup B_{i,j,k}=a_{i,j,k}$ and $i<\varphi_{i,j,k}(a_{i,j,k})\leq i+1$.

For $i,j,k<\omega$, there exist $n_{i.j,k}<\omega$ and $r_{i,j,k}\leq k$ such that $i+1=(k+1)\cdot n_{i,j,k}+r_{i,j,k}$. As $(k+1)\cdot n_{i,j,k}\leq i+1$, we may assume that there is an $n_{i,k}<\omega$ such that $n_{i,j,k}=n_{i,k}$ for all but finitely many $j<\omega$. To see this, construct a decreasing family $\{W_{i,k}\}_{i,k<\omega}$ (with respect to a well order of $\omega\times\omega$) of infinite subsets of ω such that, for each $i,k<\omega$, there is an $n_{i,k}<\omega$ such that $n_{i,j,k}=n_{i,k}$ for all $j\in W_{i,k}$. Let W be a pseudo-intersection of $\{W_{i,k}\}_{i,k<\omega}$. By restricting the ideal, we may assume that $W=\omega$ (the set corresponding to the j coordinates). Also note that, for fixed k, the sequence $\{n_{i,k}\}_{i<\omega}$ converges to infinity because $i+1<(k+1)\cdot(n_{i,k}+1)$.

Start as in the proof of Theorem 6.2.3 but change "$|x|=s_{i,j,k}(k)$" to "$\varphi_{i,j,k}(x)\in(k,k+1]$" in the definition of P_k. After choosing Γ and m, proceed as follows. Choose $k>m$ and find $i\geq k$ such that $\bar\varphi(A_k\cap B)<n_{i,k}-1$. Now, for $H\subseteq\Gamma$ of size $n_{i,k}-1$, by independence there are infinitely many j's such that $n_{i,j,k}=n_{i,k}$ and $\{f_{k,\alpha}(i,j)\ /\ \alpha\in H\}$ is a disjoint family. Work with one of these j's. As $\varphi_{i,j,k}(\bigcup_{\alpha\in H}f_{k,\alpha}(i,j)\smallsetminus B)\leq m\cdot(n_{i,k}-1)$, $\varphi_{i,j,k}(\bigcup_{\alpha\in H}f_{k,\alpha}(i,j)\cap B)>(n_{i,k}-1)\cdot(k-m)\geq n_{i,k}-1$. Thus, $\varphi_{i,j,k}(B\cap a_{i,j,k})>n_{i,k}-1$, a contradiction. □

Note that if $\mathcal{I}$ is a fragmented not gradually fragmented ideal and $\omega=X\cup Y$ is a disjoint union, then $\mathcal{I}|X$ or $\mathcal{I}|Y$ is not gradually fragmented. Because of this, we can mix Theorems 6.2.3 and 6.2.5 to obtain

6.2.6 Corollary. *Let $\mathcal{I} = \mathcal{I}\langle a_j, \varphi_j\rangle_{j<\omega}$ be a fragmented not gradually fragmented ideal such that, for all but finitely many $j < \omega$, either φ_j is a measure or a uniform submeasure. Then, $\mathfrak{b}(\mathcal{I}) = \aleph_1$.*

To finish this section, we explain a way of how to obtain a fragmented not gradually fragmented ideal from a fragmented ideal. Let $\mathcal{I} = \mathcal{I}\langle a_i, \varphi_i\rangle_{i<\omega}$ be a fragmented ideal. Now, let $\langle a_{i,j}\rangle_{i,j<\omega}$ be a partition of ω such that, for a fixed $i < \omega$ and all $j < \omega$, $|a_{i,j}| = |a_i|$ and $\varphi_{i,j} : \mathcal{P}(a_{i,j}) \to [0, +\infty)$ is the *submeasure associated with* $\langle a_i, \varphi_i\rangle$, that is, if $h_{i,j} : a_{i,j} \to a_i$ is the (unique) strictly increasing bijection, then $\varphi_{i,j}(x) = \varphi_i(h_{i,j}[x])$ for any $x \subseteq a_{i,j}$. Let $\hat{\mathcal{I}}$ be the fragmented ideal associated to $\langle a_{i,j}, \varphi_{i,j}\rangle_{i,j<\omega}$. Roughly speaking, $\hat{\mathcal{I}}$ is the ideal obtained by taking countably many copies of the ideal $\mathcal{I}$.

6.2.7 Lemma. *With the notation of the previous paragraph,*

(a) if $\mathcal{I}$ is nowhere tall, then $\hat{\mathcal{I}}$ is also nowhere tall;

(b) if $\mathcal{I}$ is somewhere tall, then $\hat{\mathcal{I}}$ is not gradually fragmented.

Proof. (a) Let $X \subseteq \omega$ be $\hat{\mathcal{I}}$-positive, that is, $\{\varphi_{i,j}(X \cap a_{i,j})\}_{i,j<\omega}$ is an unbounded set of non-negative reals. Then, there exist $W \subseteq \omega$ infinite and a function $g : W \to \omega$ such that $\{\varphi_{i,g(i)}(X \cap a_{i,g(i)})\}_{i\in W}$ converges to infinity. Put $X_1 = \bigcup_{i\in W} X \cap a_{i,g(i)} \subseteq X$ and $X_1' = \bigcup_{i\in W}(h_{i,g(i)}[X \cap a_{i,g(i)}])$. Clearly, $\langle X_1, \hat{\mathcal{I}}\restriction X_1\rangle$ and $\langle X_1', \mathcal{I}\restriction X_1'\rangle$ are isomorphic and, as the second ideal is not tall, neither is the first ideal.

(b) Without loss of generality, we may assume that $\mathcal{I}$ is tall and non-trivial. By Lemma 6.1.7, let k be such that $\varphi_i(\{c\}) \leq k$ for all $c \in a_i$ and $i < \omega$. Now let $m < \omega$ be arbitrary. Choose an $i < \omega$ such that $\varphi_i(a_i) > m$ and let $l := |a_i|$. Note that, for any $j < \omega$, the family $B_{i,j} = \{\{c\} \,/\, c \in a_{i,j}\}$ has size l and satisfies $\forall_{b\in B_{i,j}}(\varphi_{i,j}(b) \leq k)$ and $\varphi_{i,j}(\bigcup B_{i,j}) > m$.

□

6.2.8 Corollary. *Let $\mathcal{I} = \mathcal{I}\langle a_i, \varphi_i\rangle_{i<\omega}$ be a somewhere tall fragmented ideal such that, for any $i < \omega$, either φ_i is uniform or is a measure. Then, $\mathfrak{b}(\hat{\mathcal{I}}) = \aleph_1$.*

6.3 Destroying gaps of gradually fragmented ideals

We present in this section a way to destroy Rothberger gaps for gradually fragmented ideals by a ccc poset. Moreover, for the case of an ideal like in Example 6.1.12(1), we can even find a natural cardinal invariant that is less than or equal to its Rothberger number. As a consequence of our discussion we obtain two basic ZFC-results: the Rothberger number of any gradually fragmented ideal is larger or equal to add($\mathcal{N}$) (Corollary 6.3.5) and the Rothberger number of any fragmented nowhere tall ideal is $\mathfrak{b}$ (Corollary 6.3.6).

To fix some notation, for $b, h \in \omega^\omega$ let $\mathbb{R}_b := \prod_{i<\omega} b(i)$ and $S(b,h) := \prod_{i<\omega}[b(i)]^{\leq h(i)}$. Also, for $n < \omega$ put $S_n(b,h) := \prod_{i<n}[b(i)]^{\leq h(i)}$ and $S_{<\omega}(b,h) := \bigcup_{n<\omega} S_n(b,h)$. The forcing notions and cardinal invariants involved in the destruction of Rothberger gaps of gradually fragmented ideals are, respectively, parameterized versions of the localization forcings and of the cardinal invariant add($\mathcal{N}$).

6.3.1 Definition (Localization posets and cardinal invariants)**.** Let $b, h \in \omega^\omega$ such that $b > 0$.

(1) Define $\mathfrak{b}_{\mathrm{Loc}}(b,h)$ as the minimal size of a subset of $\mathbb{R}_b$ that cannot be localized by any slalom in $S(b,h)$ (if it exists).

(2) If $h \in \omega^\omega$ is a non-decreasing function, for $\mathcal{F} \subseteq \mathbb{R}_b$, define the poset

$$\mathbb{LOC}^h_{b,\mathcal{F}} := \{(s,F) \,/\, s \in S_{<\omega}(b,h),\, F \subseteq \mathcal{F} \text{ and } |F| \leq h(|s|)\}$$

ordered by $(s', F') \leq (s, F)$ iff $s \subseteq s'$, $F \subseteq F'$ and $\forall_{i\in[|s|,|s'|)}(\{x(i) \,/\, x \in F\} \subseteq s'(i))$. Put $\mathbb{LOC}^h_b := \mathbb{LOC}^h_{b,\mathbb{R}_b}$.

6.3.2 Remark. (1) If $X \subseteq \omega$ is infinite, then $\mathfrak{b}_{\text{Loc}}(b,h) \leq \mathfrak{b}_{\text{Loc}}(b{\restriction}X, h{\restriction}X)$.

(2) Denote $[h<b] := \{i<\omega \ / \ h(i)<b(i)\}$. Then, $\mathfrak{b}_{\text{Loc}}(b,h)$ exists iff $[h<b]$ is infinite. In this case, $\mathfrak{b}_{\text{Loc}}(b,h) \leq \text{non}(\mathcal{M})$ and, moreover, $\mathfrak{b}_{\text{Loc}}(b,h) = \mathfrak{b}_{\text{Loc}}(b{\restriction}[h<b], h{\restriction}[h<b])$.

(3) If $[h<b]$ is infinite and $h{\restriction}[h<b]$ does not converge to infinity, then $\mathfrak{b}_{\text{Loc}}(b,h)$ is finite. To see this, without loss of generality, assume $h<b$. As h does not converge to infinity, there is an $m<\omega$ and $X \subseteq \omega$ infinite such that $h{\restriction}X = m$, so, by (1), $\mathfrak{b}_{\text{Loc}}(b,h) \leq \mathfrak{b}_{\text{Loc}}(b{\restriction}X, h{\restriction}X) = m+1$.

(4) If $h{\restriction}[h<b]$ converges to infinity, then $\text{add}(\mathcal{N}) \leq \mathfrak{b}_{\text{Loc}}(b,h)$ by the Bartoszyński's characterization of $\text{add}(\mathcal{N})$ (Theorem 1.4.2)

6.3.3 Lemma. *In the notation of Definition 6.3.1, if h converges to infinity, then $\mathbb{LOC}^h_{b,\mathcal{F}}$ is σ-linked and generically adds a slalom in $S(b,h)$ that localizes all the reals in $\mathcal{F}$. In particular, $\mathbb{LOC}^h_b$ generically adds a slalom in $S(b,h)$ that localizes all the ground model reals in $\mathbb{R}_b$.*

Proof. σ-linked is witnessed by $Q_s := \{(t,F) \in \mathbb{LOC}^h_{b,\mathcal{F}} \ / \ t = s \text{ and } |F| \leq h(|s|)/2\}$ for $s \in S_{<\omega}(b,h)$. Convergence of h to infinity is needed for the density of $\bigcup_{s\in S_{<\omega}(b,h)} Q_s$. If $\dot{G}$ is the $\mathbb{LOC}^h_{b,\mathcal{F}}$-name of the generic subset, then $\bigcup \text{dom}\dot{G}$ is the name of the slalom that localizes all the reals in $\mathcal{F}$. □

It is clear that $\mathbb{LOC}^h_b$ is a Suslin σ-linked poset. Moreover, as in Lemma 2.2.11, $\mathbb{LOC}^h_b$ is correctness-preserving.

6.3.4 Theorem. *Let $\mathcal{I}$ be a gradually fragmented ideal. Then, there exists a function $b \in \omega^\omega$ and a $\mathbb{D}$-name $\dot{h}$ of a non-decreasing function in ω^ω that converges to infinity such that the forcing $\mathbb{D} * \mathbb{LOC}^{\dot{h}}_b$ destroys all the $\mathcal{I}$-Rothberger gaps of the ground model.*

Proof. In V (the ground model), let $\mathcal{I} = \mathcal{I}\langle a_i, \varphi_i\rangle_{i<\omega}$ be a gradually fragmented ideal and $f \in \omega^\omega$ a function that witnesses its gradual fragmentation. Let $d \in \omega^\omega$ be a dominating real added generically over V by $\mathbb{D}$. In $V[d]$, by the gradual fragmentation of $\mathcal{I}$, find a non-decreasing sequence $\{N_l\}_{l<\omega}$ of natural numbers that converges to infinity and such that

$$(+)\ \forall_{n\leq l}\forall_{i\geq N_l}\forall_{B\subseteq\mathcal{P}(a_i)}\big[\big(|B|\leq l \text{ and } \forall_{x\in B}(\varphi_i(x)\leq d(n))\big) \Rightarrow \varphi_i(\bigcup B) \leq f(d(n))\big].$$

$N_0 = 0$ can be assumed. Let $h:\omega\to\omega$ be defined as $h(i) = l$ when $i \in [N_l, N_{l+1})$.

Back in V, let $\dot{h}$ be the $\mathbb{D}$-name of h and put $b(i) = \mathcal{P}(a_i)$. Now, let $\langle\mathcal{A},\mathcal{B}\rangle$ be $\mathcal{I}$-orthogonal with $|\mathcal{A}| = \aleph_0$ and we show that $\mathbb{D} * \mathbb{LOC}^{\dot{h}}_b$ adds a subset of ω that separates $\langle\mathcal{A},\mathcal{B}\rangle$, moreover, we can even find a $\mathbb{D}$-name $\dot{\mathcal{F}}$ of a subset of $\mathbb{R}_b$ of size $\leq |\mathcal{B}|$ such that $\mathbb{D} * \mathbb{LOC}^{\dot{h}}_{b,\dot{\mathcal{F}}}$ adds such a subset of ω. Put $\mathcal{A} = \{A_n \ / \ n<\omega\}$. Without loss of generality, we may assume that $\mathcal{A}$ is a partition of ω and that, for each $i<\omega$, $\forall_{n>i}(A_n \cap a_i = \varnothing)$, so $\{A_n\cap a_i\}_{n\leq i}$ becomes a partition of a_i. For each $B\in\mathcal{B}$, let $g_B\in\omega^\omega$ be such that $g_B(n) = \lceil\bar{\varphi}(B\cap A_n)\rceil$. Step into $V[d]$ and, for each $B\in\mathcal{B}$, define $x_B \in \mathbb{R}_b$ such that $x_B(i) = \bigcup_{n\leq i} x_B(i,n)$ where $x_B(i,n) = B\cap A_n\cap a_i$ if $\varphi_i(B\cap A_n\cap a_i)\leq d(n)$ and, otherwise, $x_B(i,n) = \varnothing$. Put $\mathcal{F} := \{x_B \ / \ B\in\mathcal{B}\}$.

Let ψ be a slalom in $S(b,h)$ added generically over $V[d]$ by $\mathbb{LOC}^h_{b,\mathcal{F}}$. Now, work in $V[d][\psi]$. Without loss of generality, we may assume that, for every $i<\omega$, $\forall_{x\in\psi(i)}\forall_{n\leq i}(\varphi_i(x\cap A_n)\leq d(n))$ (just take out those x of $\psi(i)$ that do not satisfy that property). Put $C := \bigcup_{i<\omega}\bigcup_{x\in\psi(i)} x$. This C separates $\langle\mathcal{A},\mathcal{B}\rangle$.

- *$C\cap A_n \in \mathcal{I}$ for all $n<\omega$, moreover, $\varphi_i(C\cap A_n\cap a_i)\leq f(d(n))$ for all $i\geq N_n$.* It is enough to consider $i\geq n$ (because $A_n\cap a_i = \varnothing$ for all $i<n$). Let $l<\omega$ be such that $i\in[N_l,N_{l+1})$, so $|\psi(i)|\leq h(i) = l$ and, by (+), as $n\leq l$, $C\cap A_n\cap a_i = \bigcup_{x\in\psi(i)}(x\cap A_n)$ and $\varphi_i(x\cap A_n)\leq d(n)$ for all $x\in\psi(i)$, we have that $\varphi_i(C\cap A_n\cap a_i)\leq f(d(n))$.

- *$B\smallsetminus C\in\mathcal{I}$ for all $B\in\mathcal{B}$.* Note that $g_B\leq^* d$, so there exists an $m<\omega$ such that, for every $n\geq m$ and $i\geq n$, $\varphi_i(B\cap A_n\cap a_i)\leq d(n)$. Also, as $x_B\in^*\psi$, we may assume (by enlarging m) that $x_B(i)\in\psi(i)$ for all $i\geq m$. Then, $B\cap A_n\cap a_i\subseteq C\cap A_n\cap a_i$ for all $i\geq n\geq m$, so $B\cap(\bigcup_{n\geq m}A_n)\subseteq^* C\cap(\bigcup_{n\geq m}A_n)$. As $A_n\cap B\in\mathcal{I}$ for any $n<\omega$, it follows that $B\subseteq_{\mathcal{I}} C$.

□

The previous proof also indicates that the forcing $\mathbb{D} * \mathbb{LOC}^h$ destroys the Rothberger gaps of the ground model for any gradually fragmented ideal $\mathcal{I}$. But, as any localization forcing $\mathbb{LOC}^{h'}$ adds a dominating real, the following result comes as a consequence.

6.3.5 Corollary. *If $\mathcal{I}$ is a gradually fragmented ideal, then* $\mathrm{add}(\mathcal{N}) \leq \mathfrak{b}(\mathcal{I})$.

Proof. Let $\mathcal{I} = \mathcal{I}\langle a_i, \varphi_i\rangle_{i<\omega}$ and $\langle \mathcal{A}, \mathcal{B}\rangle$ be $\mathcal{I}$-orthogonal where $\mathcal{A}$ is a partition of ω and $|\mathcal{B}| < \mathrm{add}(\mathcal{N})$. Let χ be a large enough regular cardinal and $M \preceq H_\chi$ such that $\mathcal{B} \cup \{\mathcal{A}, \mathcal{B}, \langle a_i, \varphi_i\rangle_{i<\omega}\} \subseteq M$ and $|M| < \mathrm{add}(\mathcal{N})$. As $\mathrm{add}(\mathcal{N}) \leq \mathfrak{b}$, there exists a real $d \in \omega^\omega$ dominating $M \cap \omega^\omega$. Define b, h and $\mathcal{F} = \{g_B \,/\, B \in \mathcal{B}\}$ as in the proof of Theorem 6.3.4. Let $N \preceq H_\chi$ be such that $M \cup \{d\} \subseteq N$ and $|N| < \mathrm{add}(\mathcal{N})$. As $\mathrm{add}(\mathcal{N}) \leq \mathfrak{b}_{\mathrm{Loc}}(b, h)$, we can find $\psi \in S(b, h)$ that localizes all the reals in $\mathbb{R}_b \cap N$. Like in the proof of Theorem 6.3.4, we can construct C that separates $\langle \mathcal{A}, \mathcal{B}\rangle$. □

If in the proof of Theorem 6.3.4 we consider a partition $\langle a_i\rangle_{i<\omega}$ such that $\langle |a_i|\rangle_{i<\omega}$ is bounded, then the resulting forcing $\mathbb{LOC}^h_b$ does not add anything new, which means that we can destroy $\mathcal{I}$-Rothberger gaps in this case by just adding dominating reals. Therefore, as a consequence of Lemma 6.1.9, it follows that

6.3.6 Corollary. *If $\mathcal{I}$ is a nowhere tall fragmented ideal on ω, then* $\mathfrak{b}(\mathcal{I}) = \mathfrak{b}$.

In the particular case of the gradually fragmented ideals in 6.1.12(1), we even get a nice lower bound for the Rothberger number for each of these ideals.

6.3.7 Lemma. *Let $b, h \in \omega^\omega$ be functions converging to infinity such that $b \geq 2$ and h is non-decreasing. If $c \in \omega^\omega$ is such that $2 \leq c$ and $h \leq^* c$ and $P = \{a_i\}_{i<\omega}$ is a partition of ω into non-empty finite sets such that $|a_i| \leq \log_2 b(i)$ for all but finitely many $i < \omega$, then* $\min\{\mathfrak{b}_{Loc}(b, h), \mathfrak{b}\} \leq \mathfrak{b}(\mathcal{I}_c(P))$. *Also, the forcing $\mathbb{D} * \dot{\mathbb{LOC}}^h_b$ destroys the Rothberger gaps of $\mathcal{I}_c(P)$.*

Proof. As $b' \leq^* b$ implies $\mathfrak{b}_{\mathrm{Loc}}(b, h) \leq \mathfrak{b}_{\mathrm{Loc}}(b', h)$, it is enough to assume that $b(i) = 2^{|a_i|}$ for all $i < \omega$. Note that, in the proof of Theorem 6.3.4, any sequence $\{N_l\}_{l<\omega}$ such that $c(i) \geq l$ for all $i \geq N_l$ serves for the purposes of that proof, so it can be defined in the ground model. In particular, choose such a sequence with the property $\forall_{i\in[N_l,N_{l+1})}(h(i) = l \leq c(i))$ for all but finitely many $l < \omega$. Define $h'(i) = l$ when $i \in [N_l, N_{l+1})$. By the argument of the same proof, $\min\{\mathfrak{b}_{\mathrm{Loc}}(b, h'), \mathfrak{b}\} \leq \mathfrak{b}(\mathcal{I}_c(P))$ and, as $h' =^* h$, it is clear that $\mathfrak{b}_{\mathrm{Loc}}(b, h') = \mathfrak{b}_{\mathrm{Loc}}(b, h)$. □

To finish this section, we prove that it is consistent that $\mathfrak{b} < \mathfrak{b}_{\mathrm{Loc}}(b, h)$ for all the pair of functions $b, h \in \omega^\omega$ for which $\mathfrak{b}_{\mathrm{Loc}}(b, h)$ is infinite.

6.3.8 Lemma. $\mathbb{LOC}^h_b$ *is $\leq^*$-good.*

Proof. The proof is very similar to (even simpler than) the proof of Lemma 3.2.4. Let $\dot{g}$ be a $\mathbb{LOC}^h_b$-name for a real in ω^ω and find an $h \in \omega^\omega$ such that, for any $f \in \omega^\omega$ that is not dominated by h, $\Vdash f \not\leq^* \dot{g}$. For $s \in S_{<\omega}(b, h)$ and $n \leq h(|s|)$, define $h_{s,n} : \omega \to \omega + 1$ such that $h_{s,n}(i)$ is the minimal $j \leq \omega$ such that, for any $F \subseteq \mathbb{R}_b$ of size n, $(s, F) \not\Vdash \dot{g}(i) > j$.

6.3.9 Claim. $h_{s,n} \in \omega^\omega$.

Proof. Towards a contradiction, assume that there is an $i < \omega$ such that, for any $j < \omega$ there is an $F_j \subseteq \mathbb{R}_b$ of size n such that $(s, F_j) \Vdash \dot{g}(i) > j$. Put $F_j = \{f^l_j \,/\, l < n\}$. Find a strictly increasing sequence $\{m_j\}_{j<\omega}$ of natural numbers and, for $l < n$, construct $f^l \in \mathbb{R}_b$ such that, for any $k < \omega$, $f^l_{m_j}(k) = f^l(k)$ for any $j > k$. This construction is carried out as in the proof of Claim 3.2.5 and it is simpler because, for fixed $l < n$ and $k < \omega$, the set $\{f^l_j(k) \,/ j < \omega\}$ is always finite.

Put $F' = \{f^l \,/\, l < n\}$ and find $(t, F'') \leq (s, F')$ in $\mathbb{LOC}^h_b$ with $h(|t|) \geq |F''| + n$ and $j_0 < \omega$ such that $(t, F'') \Vdash \dot{g}(i) = j_0$. Choose $j \geq j_0$ above $|t|$, so $(t, F_{m_j} \cup F'')$ is a common extension of (t, F'') and (s, F_{m_j}). Indeed, if $k \in |t| \smallsetminus |s|$ and $l < k$, then $f^l_{m_j}(k) = f^l(k) \in t(k)$ because $(t, F'') \leq (s, F')$.

Therefore, $(t, F_{m_j} \cup F'')$ forces that $j_0 = \dot{g}(i) > m_j \geq j$, a contradiction. □

Let $h \in \omega^\omega$ be a function dominating $\{h_{s,n} \,/\, s \in S_{<\omega}(b,h),\ n \leq h(|s|)\}$. Exactly as in the proof of Lemma 3.2.4, h is as required. $\square$

6.3.10 Lemma. *Let $b, h \in \omega^\omega$ such that $b > 0$ and h converges to infinity. Then, there is an non-decreasing function h' that converges to infinity and $b' \in \omega^\omega$, $b' > 0$, such that $b' > h'$ and $\mathfrak{b}_{Loc}(b', h') \leq \mathfrak{b}_{Loc}(b, h)$.*

Proof. Without loss of generality, assume that $0 < h < b$. Define $\{N_l\}_{l,\omega} \subseteq \omega$ strictly increasing by recursion: $N_0 = 0$ and $N_{l+1} > N_l$ is such that $h(i) > \max_{j \leq N_l}\{h(j)\} + 1$ for all $i \geq N_{l+1}$. Denote $I_l = [N_l, N_{l+1})$ and put $h'(0) = 0$ and $h'(l) = h(N_{l-1})$ for all $0 < l < \omega$. Clearly, h' is strictly increasing. Put $b'(l) = \prod_{i \in I_l} b(i)$.

To see $\mathfrak{b}_{\mathrm{Loc}}(b', h') \leq \mathfrak{b}_{\mathrm{Loc}}(b, h)$, find $F : \mathbb{R}_b \to \mathbb{R}_{b'}$ and $F' : S(b', h') \to S(b, h)$ such that, for any $x \in \mathbb{R}_b$ and $\psi' \in S(b', h')$, $F(x) \in^* \psi'$ implies $x \in^* F'(\psi')$. F is the bijection given by $F(x) = \{x{\restriction}I_l\}_{l<\omega}$ and, for $\psi' \in S(b', h')$, let $F'(\psi')(i) = \{\sigma(i) \,/\, \sigma \in \psi'(l)\}$ for $i \in I_l$ and $l < \omega$. Here, $|F'(\psi')(i)| \leq |\psi'(l)| \leq h'(l) < h(i)$, so $F'(\psi') \in S(b, h)$. Now, if $F(x) \in^* \psi'$, then there is an $m < \omega$ such that $x{\restriction}I_l \in \psi'(l)$ for all $l \geq m$. Thus, for $i \geq N_m$, $x(i) \in F'(\psi')(i)$. $\square$

6.3.11 Theorem. *Let $\kappa \leq \lambda$ be regular uncountable cardinals such that $\lambda^{<\kappa} = \lambda$. Then, there is a ccc poset that forces $\mathfrak{b} = \kappa$ and $\mathfrak{b}_{Loc}(b, h) = \lambda$ for all $b, h \in \omega^\omega$ with $b > 0$ for which $\mathfrak{b}_{Loc}(b, h)$ exists and is infinite. Moreover, this poset forces $\mathfrak{b}(\mathcal{I}) = \mathfrak{b} = \kappa$ for any gradually fragmented ideal $\mathcal{I}$.*

Proof. By a book-keeping argument (as used in the results of Section 4.1), if $\kappa \leq \lambda$ are uncountable regular cardinals and $\lambda^{<\kappa} = \lambda$, it is possible to perform a fsi $\mathbb{P}_\lambda = \langle \mathbb{P}_\alpha, \dot{\mathbb{Q}}_\alpha \rangle_{\alpha<\lambda}$ alternating between suborders of $\mathbb{D}$ of size $< \kappa$ and posets of the form $\mathbb{LOC}^h_b$, with $b > 0$ and h non-decreasing and converging to infinity, such that, for any $\alpha < \lambda$ and $\mathbb{P}_\alpha$-names $\dot{b}$ and $\dot{h}$ of such reals, there is a $\beta \in [\alpha, \lambda)$ such that $\dot{\mathbb{Q}}_\beta$ is $\mathbb{LOC}^{\dot{h}}_{\dot{b}}$, likewise for any $\mathbb{P}_\alpha$-name for a suborder of $\mathbb{D}$ of size $< \kappa$. In a $\mathbb{P}_\lambda$-extension V_λ, it is clear that $\mathfrak{c} \leq \lambda$ and that $\lambda \leq \mathfrak{b}_{\mathrm{Loc}}(b, h)$ for any $b, h \in \omega^\omega \cap V_\lambda$ with $b > 0$ and h non-decreasing that converges to infinity. Therefore, $\mathfrak{b}_{\mathrm{Loc}}(b, h) = \lambda$ when such a $\mathfrak{b}_{\mathrm{Loc}}(b, h)$ exists. By Remark 6.3.2 and Lemma 6.3.10, $\mathfrak{b}_{\mathrm{Loc}}(b, h) = \lambda$ for any $b, h \in \omega^\omega \cap V_\lambda$ with $b > 0$ and $h{\restriction}[h < b]$ convergent to infinity (i.e., when $\mathfrak{b}_{\mathrm{Loc}}(b, h)$ exists and is infinite).

On the other hand, the iteration is κ-$\leq^*$-good and, by the use of small suborders of $\mathbb{D}$, $\mathbb{P}_\lambda$ forces that $\mathfrak{b} = \kappa$ (see results in Section 4.1 for details of this type of argument).

The second statement is a direct consequence of the first. Let $\mathcal{I}$ be a fragmented ideal and $\langle \mathcal{A}, \mathcal{B} \rangle$ an $\mathcal{I}$-orthogonal family with $\mathcal{A}$ a partition of ω and $|\mathcal{B}| < \kappa = \mathfrak{b}$. Define g_B for $B \in \mathcal{B}$ as in the proof of Theorem 6.3.4, so, as $\mathfrak{b} = \kappa$, there is a $d \in \omega^\omega$ that dominates $\{g_B \,/\, B \in \mathcal{B}\}$. Now, if b and h are defined as in the proof of Theorem 6.3.4, as $\mathfrak{b}_{\mathrm{Loc}}(b, h) \geq \kappa$, we can find a subset of ω that separates $\langle \mathcal{A}, \mathcal{B} \rangle$. $\square$

6.4 Preservation properties

We present some properties that help us to preserve the Rothberger number of a tall fragmented ideal small under certain forcing extensions. Actually, we present a new cardinal invariant that serves as upper bound for some of these Rothberger numbers and study a property for preserving this invariant small under generic extensions. Many ideas involved for this are taken from [KaO14].

For this section, fix $\mathcal{I} = \mathcal{I}\langle a_i, \varphi_i \rangle_{i<\omega}$ a tall fragmented ideal, $2^{\bar{a}} := \{\mathcal{P}(a_i)\}_{i<\omega}$, $\bar{L} := \{L_n\}_{n<\omega}$ a partition of ω into infinite sets, $A_n := \bigcup_{i \in L_n} a_i$ and $\mathcal{A} := \{A_n \,/ n < \omega\}$, which is also a partition of ω into infinite sets. Let $\mathcal{O}(\mathcal{I}, \bar{L})$ be the collection of all the subsets of ω that are $\mathcal{I}$-orthogonal with $\mathcal{A}$. In our applications, we will have that $\lim_{i\to+\infty} \varphi_i(a_i) = +\infty$ (a useful assumption for applying Theorem 6.4.2 and for saying something about the Rothberger number of $\mathcal{I}$), but this is not a general requirement for the results in this section.

6.4.1 Definition. Let $\rho \in \omega^\omega$, $\rho > 0$.

(1) For $\psi \in \prod_{i<\omega} \mathcal{P}(\mathcal{P}(a_i))$ and $Y \in \mathcal{O}(\mathcal{I}, \bar{L})$, define $Y \in' \psi$ iff $\forall_{n<\omega} \exists^\infty_{i \in L_n} (Y \cap a_i \in \psi(i))$.

(2) $\mathfrak{b}^\rho(\mathcal{I},\bar{L})$ is the least size of a subset Ψ of $S(2^{\bar{a}},\rho)$ that *covers* $\mathcal{O}(\mathcal{I},\bar{L})$, that is, for any $Y\in\mathcal{O}(\mathcal{I},\bar{L})$, there exists a $\psi\in\Psi$ such that $Y\in'\psi$.

From now on in this chapter, fix $\mathcal{E}\subseteq\omega^\omega$ such that

(i) For any $e\in\mathcal{E}$, e is non-decreasing, converges to infinity, $e\leq id_\omega$ and $id_\omega-e$ converges to infinity.

(ii) If $e\in\mathcal{E}$ then there exists an $e'\in\mathcal{E}$ such that $e+1\leq^* e'$.

(iii) If $\mathcal{C}\subseteq\mathcal{E}$ is countable, there exists an $e\in\mathcal{E}$ that $\leq^*$-dominates all the reals in $\mathcal{C}$.

This family can be constructed by recursion on $\alpha<\omega_1$ in such a way that $\mathcal{E}=\{e_\alpha\ /\ \alpha<\omega_1\}$ is $\leq^*$-increasing and $e_{\alpha+1}=^* e_\alpha+1$ for all $\alpha<\omega$. Lemma 6.5.5 can also be used (with $g=id_\omega$ and $H=id_\omega+1$).

For $b,\rho\in\omega^\omega$, put $\widetilde{S}(b,\rho):=\bigcup_{e\in\mathcal{E}}S(b,\rho^e)$. For $L\subseteq\omega$, denote by $S(b,\rho)\restriction L:=\{\psi\restriction L\ /\ \psi\in S(b,\rho)\}$, likewise for $\widetilde{S}(b,\rho)\restriction L$. For $m<\omega$, put $\mathcal{P}_{m,i}(\mathcal{I}):=\{x\subseteq a_i\ /\ \varphi_i(x)\leq m\}$, $S(\mathcal{I},L,m,\rho):=S(\{\mathcal{P}_{m,i}(\mathcal{I})\}_{i<\omega},\rho)\restriction L$ and $\widetilde{S}(\mathcal{I},L,m,\rho):=\widetilde{S}(\{\mathcal{P}_{m,i}(\mathcal{I})\}_{i<\omega},\rho)\restriction L$. Note that $\widetilde{S}(b,1)=S(b,1)$ and $\widetilde{S}(\mathcal{I},L,m,1)=S(\mathcal{I},L,m,1)$.

6.4.2 Theorem. *If* $\lim_{i\to+\infty}\varphi_i(a_i)/\rho(i)=+\infty$ *then* $\mathfrak{b}(\mathcal{I})\leq\mathfrak{b}^\rho(\mathcal{I},\bar{L})$.

Proof.

6.4.3 Claim. *Let $L\subseteq\omega$ be infinite, $A:=\bigcup_{i\in L}a_i$, $n<\omega$, $f\in\omega^\omega$ and $\{\psi_k\}_{k<\omega}$ a sequence of slaloms such that $\psi_k\in S(\mathcal{I},L,f(k),\rho)$. Then, there exists a $Z\in\mathcal{I}\restriction A$ such that*

(i) $\forall^\infty_{i\in L}(\varphi_i(Z\cap a_i)>n)$ *and*

(ii) $\forall_{k<\omega}\forall^\infty_{i\in L}\forall_{x\in\psi_k(i)}(x\cap Z=\varnothing)$.

Proof. For $k<\omega$, put $m_k:=(k+1)\cdot\max_{j\leq k}\{f(j)\}$. Let $\{N_k\}_{k<\omega}$ be a strictly increasing sequence of natural numbers such that $\varphi_i(a_i)>n+m_k\cdot\rho(i)$ for all $i\geq N_k$. Now, by tallness, find $l<\omega$ as in Lemma 6.1.7(iii) applied to $m=n$. For $i\in L\cap[N_k,N_{k+1})$, as $\varphi_i(\bigcup_{j\leq k}\bigcup_{x\in\psi_j(i)}x)\leq m_k\cdot\rho(i)$, $a_i\smallsetminus\bigcup_{j\leq k}\bigcup_{x\in\psi_j(i)}x$ has submeasure bigger than n, so, by tallness, it contains a z_i with submeasure in $(n,l]$. Therefore, $Z=\bigcup_{i\in L\cap[N_0,\omega)}z_i$ is as required. □

6.4.4 Claim. *Let $L\subseteq\omega$ infinite, $A:=\bigcup_{i\in L}a_i$, $\psi\in S(2^{\bar{a}},\rho)\restriction L$ and $n<\omega$. Then, there exists a $Z_\psi\in\mathcal{I}\restriction A$ such that*

(i) $\forall^\infty_{i\in L}(\varphi_i(Z_\psi\cap a_i)>n)$ *and*

(ii) $\forall_{k<\omega}\forall^\infty_{i\in L}\forall_{x\in\psi(i)}(\varphi_i(x)\leq k\Rightarrow x\cap Z_\psi=\varnothing)$.

Proof. For each $k<\omega$ put $\psi_k(i)=\{x\in\psi(i)\ /\varphi_i(x)\leq k\}$ and apply the previous claim with $f=id_\omega$. □

Now, let $\Psi\subseteq S(2^{\bar{a}},\rho)$ be a witness of $\mathfrak{b}^\rho(\mathcal{I},\bar{L})$. For each $\psi\in\Psi$ and $n<\omega$, let $Z_{\psi,n}\in\mathcal{I}\restriction A_n$ be as in Claim 6.4.4 applied to L_n, A_n, $\psi\restriction L_n$ and n. Put $Z_\psi:=\bigcup_{n<\omega}Z_{\psi,n}$, which is clearly in $\mathcal{O}(\mathcal{I},\bar{L})$. It is enough to prove that the orthogonal pair $\langle\mathcal{A},\{Z_\psi\ /\ \psi\in\Psi\}\rangle$ is an $\mathcal{I}$-gap. Let $X\in\mathcal{O}(\mathcal{I},\bar{L})$ and choose $\psi\in\Psi$ such that $X\in'\psi$. We show that, for any $n<\omega$ there is some $i<\omega$ such that $\varphi_i(a_i\cap Z_\psi\smallsetminus X)>n$. Choose $m<\omega$ such that $\bar{\varphi}(X\cap A_n)\leq m$, that is, $\varphi_i(X\cap a_i)\leq m$ for all $i\in L_n$. By Claim 6.4.4, choose a large enough $i\in L_n$ such that $X\cap a_i\in\psi(i)$, $\varphi_i(Z_\psi\cap a_i)>n$ and $X\cap a_i\cap Z_\psi=\varnothing$, so $\varphi_i(a_i\cap Z_\psi\smallsetminus X)>n$. □

6.4.5 Definition. Let $\rho\in\omega^\omega$ with $\rho>0$ and θ a cardinal number.

(1) $\Psi''\subseteq S(2^{\bar{a}},\rho^{id_\omega})$ is said to be a *θ-ρ-strong covering family (with respect to $\mathcal{I}$ and $\bar{L}$)* if, for any $\Psi\subseteq\bigcup_{m,n<\omega}\widetilde{S}(\mathcal{I},L_n,m,\rho)$ of size $<\theta$, there exists a $\psi''\in\Psi''$ such that, for all $n<\omega$ and $\psi\in\Psi$ such that $\mathrm{dom}\psi=L_n$, $\exists^\infty_{i\in L_n}(\psi''(i)\supseteq\psi(i))$.

(2) A poset $\mathbb{P}$ is θ-$\langle\bar{L},\mathcal{I},\rho\rangle$-*good* if, for every $m,n<\omega$ and $\dot{\psi}$ a $\mathbb{P}$-name for a slalom in $\widetilde{S}(\mathcal{I},L_n,m,\rho)$, there exists a nonempty $\Psi\subseteq\widetilde{S}(\mathcal{I},L_n,m,\rho)$ of size $<\theta$ such that, for any $\psi''\in S(2^{\bar{a}},\rho^{id_\omega})$, if $\forall_{\psi'\in\Psi}\exists^\infty_{i\in L_n}(\psi''(i)\supseteq\psi'(i))$, then $\Vdash\exists^\infty_{i\in L_n}(\psi''(i)\supseteq\dot{\psi}(i))$.

Note that (2) is simpler for the case $\rho=1$.

The property in (2) is, actually, an intersection of countably many versions of a simpler property that is defined in [KaO14, Def. 6]. Although θ-$\langle\bar{L},\mathcal{I},\rho\rangle$-goodness is not expressed in terms of a binary relation as in Context 3.1.1, it works in a similar way as goodness in Definition 3.1.5, say, it is a key property for preserving θ-ρ-strong covering families when θ is regular uncountable and it is preserved in finite support iterations. We are interested in using this property to preserve the cardinal $\mathfrak{b}^{\rho^{id_\omega}}(\mathcal{I},\bar{L})$ small in generic extensions.

6.4.6 Lemma. *Let θ be an uncountable regular cardinal.*

(a) If $\Psi''\subseteq S(2^{\bar{a}},\rho^{id_\omega})$ is a θ-ρ-strong covering family, then $\mathfrak{b}^{\rho^{id_\omega}}(\mathcal{I},\bar{L})\leq|\Psi''|$.

(b) If $\mathbb{P}$ is a θ-$\langle\bar{L},\mathcal{I},\rho\rangle$-good poset, then $\mathbb{P}$ preserves θ-ρ-strong covering families.

(c) Let $\langle\mathbb{P}_\alpha\rangle_{\alpha<\theta}$ be a $\lessdot$-increasing sequence of posets and $\mathbb{P}=\mathrm{limdir}_{\alpha<\theta}\mathbb{P}_\alpha$ such that

(i) for each $\alpha<\theta$, $\mathbb{P}_{\alpha+1}$ adds a Cohen real over $V^{\mathbb{P}_\alpha}$ and

(ii) $\mathbb{P}$ is ccc.

Then, $\mathbb{P}$ adds a θ-ρ-strong covering family of size θ (of Cohen reals).

(d) If $\mathbb{P}$ is a θ-$\langle\bar{L},\mathcal{I},\rho\rangle$-good θ-cc poset and $\dot{\mathbb{Q}}$ is a $\mathbb{P}$-name for a θ-$\langle\bar{L},\mathcal{I},\rho\rangle$-good poset, then $\mathbb{P}*\dot{\mathbb{Q}}$ is θ-$\langle\bar{L},\mathcal{I},\rho\rangle$-good.

(e) Let $\langle\mathbb{P}_i\rangle_{i\in I}$ be a directed system of θ-$\langle\bar{L},\mathcal{I},\rho\rangle$-good posets with $|I|<\theta$. If $\mathbb{P}=\mathrm{limdir}_{i\in I}\mathbb{P}_i$ is ccc, then it is θ-$\langle\bar{L},\mathcal{I},\rho\rangle$-good.

(f) Any fsi of θ-$\langle\bar{L},\mathcal{I},\rho\rangle$-good ccc posets is θ-$\langle\bar{L},\mathcal{I},\rho\rangle$-good.

Proof. (a) We prove that Ψ'' covers $\mathcal{O}(\mathcal{I},\bar{L})$. Let $Y\in\mathcal{O}(\mathcal{I},\bar{L})$ and, for $n<\omega$, put $\psi_n\in\widetilde{S}(\mathcal{I},L_n,\bar{\varphi}(Y\cap A_n),\rho)$ where $\psi_n(i)=\{Y\cap a_i\}$ for all $i\in L_n$. As Ψ'' is a θ-ρ-strong covering family, there exists a $\psi''\in\Psi''$ such that, for all $n<\omega$, $\exists^\infty_{i\in L_n}(\psi''(i)\supseteq\psi_n(i))$. Therefore, $Y\in'\psi''$.

(b) Let Ψ'' be a θ-ρ-strong covering family. Let $\nu<\theta$ and, for $n,m<\omega$, let $\dot{\Psi}_{n,m}=\{\dot{\psi}_{n,m,\alpha}\ /\ \alpha<\nu\}$ be $\mathbb{P}$-names for slaloms in $\widetilde{S}(\mathcal{I},L_n,m,\rho)$. For each $n,m<\omega$ and $\alpha<\nu$, let $\Psi'_{m,n,\alpha}\subseteq\widetilde{S}(\mathcal{I},L_n,m,\rho)$ be a witness of goodness for n, m and $\dot{\psi}_{n,m,\alpha}$, so it has size $<\theta$. As Ψ'' is a θ-ρ-strong covering family, there exists a $\psi''\in\Psi''$ such that $\exists^\infty_{i\in L_n}(\psi''(i)\supseteq\psi'(i))$ for all $\psi'\in\Psi'_{n,m,\alpha}$, $n,m<\omega$ and all $\alpha<\nu$. Thus, $\mathbb{P}$ forces that $\exists^\infty_{i\in L_n}(\psi''(i)\supseteq\dot{\psi}_{n,m,\alpha})$.

(c) Consider Cohen forcing $\mathbb{C}=S_{<\omega}(2^{\bar{a}},\rho^{id_\omega})$ ordered by end extension. If $\dot{\psi}''$ is a $\mathbb{C}$-name for the Cohen generic real then, for any $n<\omega$ and $\psi\in S(2^{\bar{a}},\rho^{id_\omega})\restriction L_n$, $\mathbb{C}$ forces that $\exists^\infty_{i\in L_n}(\dot{\psi}''(i)=\psi(i))$.

For $\alpha<\theta$, let $\dot{\psi}''_\alpha$ be a $\mathbb{P}_{\alpha+1}$-name of a $\mathbb{C}$-generic real over $V^{\mathbb{P}_\alpha}$. By the fact in the previous paragraph, it is clear that $\mathbb{P}$ forces that $\{\dot{\psi}''_\alpha\ /\ \alpha<\theta\}$ is a θ-ρ-strong covering family.

(d) Same proof as Lemma 3.1.8.

(e) Similar to Theorem 3.1.9. Fix $m,n<\omega$ and let $\dot{\psi}$ be a $\mathbb{P}$-name for a slalom in $\widetilde{S}(\mathcal{I},L_n,m,\rho)$. By ccc-ness, find $e\in\mathcal{E}$ such that $\dot{\psi}$ is a $\mathbb{P}$-name for a slalom in $S(\mathcal{I},L_n,m,\rho^e)$. For each $i\in I$, find a $\mathbb{P}_i$-name $\dot{\psi}_i$ for a slalom in $S(\mathcal{I},L_n,m,\rho^e)$ and $\{\dot{p}^i_l\}_{l<\omega}$ a sequence of $\mathbb{P}_i$-names for a decreasing sequence in $\mathbb{P}/\mathbb{P}_i$ such that $\mathbb{P}_i$ forces $\dot{p}^i_l\Vdash_{\mathbb{P}/\mathbb{P}_i}\dot{\psi}\restriction l=\dot{\psi}_i\restriction l$. By goodness, there is a nonempty

$\Psi_i \subseteq \widetilde{S}(\mathcal{I}, L_n, m, \rho)$ of size $< \theta$ that witnesses goodness of $\mathbb{P}_i$ for $\dot{\psi}_i$. Let $\Psi = \bigcup_{i\in I} \Psi_i$, which has size $< \theta$.

Assume that $\psi'' \in S(2^{\bar{a}}, \rho^{id_\omega})$ and $\forall_{\psi'\in\Psi}\exists^\infty_{j\in L_n}(\psi''(j) \supseteq \psi'(j))$. Towards a contradiction, assume that there are $p \in \mathbb{P}$ and $N < \omega$ such that $p \Vdash \forall_{j\in L_n}(j \geq N \Rightarrow \psi''(j) \not\supseteq \dot{\psi}(j))$. Choose $i \in I$ such that $p \in \mathbb{P}_i$ and let G be $\mathbb{P}_i$-generic over V with $p \in G$. In $V[G]$, $\Vdash_{\mathbb{P}/\mathbb{P}_i} \forall_{j\in L_n}(j \geq N \Rightarrow \psi''(j) \not\supseteq \dot{\psi}(j))$. On the other hand, by goodness of $\mathbb{P}_i$ (in V), $\exists^\infty_{j\in L_n}(\psi''(j) \supseteq \psi_i(j))$, so choose $j \in L_n$ such that $j > N$ and $\psi''(j) \supseteq \psi_i(j)$. Also, $\Vdash_{\mathbb{P}/\mathbb{P}_i} \psi''(j) \not\supseteq \dot{\psi}(j)$. But, as $p^i_{j+1} \Vdash_{\mathbb{P}/\mathbb{P}_i} \dot{\psi}(j) = \psi_i(j)$, then p^i_{j+1} forces $\psi''(j) \supseteq \dot{\psi}(j)$, a contradiction.

(f) Direct from (d) and (e).

□

Fix θ an uncountable regular cardinal. We explore some conditions for a poset to be θ-$\langle \bar{L}, \mathcal{I}, \rho\rangle$-good.

6.4.7 Lemma. *Let $\nu < \theta$ be an infinite cardinal. Any ccc ν-centered poset is θ-$\langle \bar{L}, \mathcal{I}, \rho\rangle$-good.*

Proof. Let $\mathbb{P} = \bigcup_{\alpha<\nu} P_\alpha$ be a poset where each P_α is centered. Fix $n, m < \omega$ and let $\dot{\psi}$ be a $\mathbb{P}$-name for a real in $\widetilde{S}(\mathcal{I}, L_n, m, \rho)$. By ccc-ness, we can find $e \in \mathcal{E}$ such that $\dot{\psi}$ is forced to be in $S(\mathcal{I}, L_n, m, \rho^e)$. For each $\alpha < \nu$ and $i \in L_n$, choose a $\psi'_\alpha(i) \subseteq \mathcal{P}_{m,i}(\mathcal{I})$ of size $\leq \rho(i)^{e(i)}$ such that $\forall_{p\in P_\alpha}(p \not\Vdash \psi'_\alpha(i) \neq \dot{\psi}(i))$ (Lemma 1.2.7). Then, $\psi'_\alpha \in S(\mathcal{I}, L_n, m, \rho^e)$. Put $\Psi := \{\psi'_\alpha \,/\, \alpha < \nu\}$.

Let $\psi'' \in S(2^{\bar{a}}, \rho^{id_\omega})$ and assume that $\exists^\infty_{i\in L_n}(\psi''(i) \supseteq \psi'_\alpha(i))$ for any $\alpha < \nu$. We show that $\Vdash \exists^\infty_{i\in L_n}(\psi''(i) \supseteq \dot{\psi}(i))$. Let $p \in \mathbb{P}$ and $i_0 \in \omega$ be arbitrary, choose $\alpha < \nu$ such that $p \in P_\alpha$ and also find $i > i_0$ in L_n such that $\psi''(i) \supseteq \psi'_\alpha(i)$. By definition of $\psi'_\alpha(i)$, there exists a $q \leq p$ such that $q \Vdash \psi'_\alpha(i) = \dot{\psi}(i)$ so, clearly, $q \Vdash \psi''(i) \supseteq \dot{\psi}(i)$. □

We also want conditions implying that a poset like in Definition 6.3.1 satisfies a preservation property like in Definition 6.4.5(2). The following generalization of σ-linkedness is useful for this.

6.4.8 Definition ([KaO14])**.** Let $\pi, \rho \in \omega^\omega$. A poset $\mathbb{Q}$ is *$\langle\pi, \rho\rangle$-linked* if there exists a sequence $\langle Q_{i,j}\rangle_{i<\omega, j<\rho(i)}$ of subsets of $\mathbb{Q}$ such that

(i) $Q_{i,j}$ is $\pi(i)$-linked and

(ii) for any $q \in \mathbb{Q}$, $\forall^\infty_{i<\omega}\exists_{j<\rho(i)}(q \in Q_{i,j})$.

Note that, if $\pi \geq^* 2$ and $\rho >^* 0$, then $\langle\pi, \rho\rangle$-linked implies σ-linked.

6.4.9 Lemma. *Let $\pi \in \omega^\omega$ be such that $\left|[\mathcal{P}(a_i)]^{\leq\rho(i)^i}\right| \leq \pi(i)$ for all but finitely many $i < \omega$. Then, any $\langle\pi, \rho\rangle$-linked poset is 2-$\langle \bar{L}, \mathcal{I}, \rho\rangle$-good.*

Proof. Assume that $\langle Q_{i,j}\rangle_{i<\omega, j<\rho(i)}$ witnesses $\langle\pi, \rho\rangle$-linkedness of a poset $\mathbb{Q}$. Fix $n, m < \omega$ and let $\dot{\psi}$ be a $\mathbb{Q}$-name for a real in $\widetilde{S}(\mathcal{I}, L_n, m, \rho)$. By ccc-ness, find $e \in \mathcal{E}$ such that $\dot{\psi}$ is a $\mathbb{Q}$-name for a real in $S(\mathcal{I}, L_n, m, \rho^e)$. For all but finitely many $i \in L_n$, for every $j < \rho(i)$, as $\pi(i) \geq \left|[\mathcal{P}(a_i)]^{\leq\rho(i)^i}\right|$ and $Q_{i,j}$ is $\pi(i)$-linked, by Lemma 1.2.7 there is a $Y_{i,j} \subseteq \mathcal{P}_{m,i}(\mathcal{I})$ of size $\leq \rho(i)^{e(i)}$ such that $\forall_{p\in Q_{i,j}}(p \not\Vdash \dot{\psi}(i) \neq Y_{i,j})$. Put $\psi'(i) := \bigcup_{j<\rho(i)} Y_{i,j}$.

There exists an $e' \in \mathcal{E}$ such that $e+1 \leq^* e'$, so we may assume, by changing $\psi'(i)$ at finitely many i if necessary, that $\psi' \in S(\mathcal{I}, L_n, m, \rho^{e'})$. $\{\psi'\}$ witnesses goodness of $\mathbb{Q}$ for $\dot{\psi}$. Indeed, let $\psi'' \in S(2^{\bar{a}}, \rho^{id_\omega})$ such that $\exists^\infty_{i\in L_n}(\psi''(i) \supseteq \psi'(i))$. For $p \in \mathbb{Q}$ and $i_0 \in \omega$, choose an $i > i_0$ in L_n and a $j < \rho(i)$ such that $\psi''(i) \supseteq \psi'(i)$ and $p \in Q_{i,j}$. Then, there exists a $q \leq p$ such that $q \Vdash Y_{i,j} = \dot{\psi}(i)$ so, clearly, $q \Vdash \psi''(i) \supseteq \psi'(i) \supseteq Y_{i,j} = \dot{\psi}(i)$. □

Moreover, as in the proof of Lemma 3.2.15, we can prove that, when π converges to infinity, $\langle\pi, \rho\rangle$-linked posets has a preservation property like in Definition 3.2.16.

6.4.10 Lemma. *Let $\pi, \rho \in \omega^\omega$ be such that $\lim_{k\to+\infty} \pi(k) = +\infty$ and assume $g \in \omega^\omega$ converges to infinity. Then, there is a $\leq^*$-increasing definable sequence $\bar{H} = \{g_k\}_{k<\omega}$ with $g_0 = g$ and such that any $\langle\pi,\rho\rangle$-linked poset is $\in^*_{\bar{H}}$-good.*

Proof. This is a direct consequence of the following fact.

6.4.11 Claim. *Let $\{m_k\}_{k<\omega}$ be a strictly increasing sequence of natural numbers such that $g(k) < \pi(m_k) < \pi(m_{k+1})$ and let $g' \in \omega^\omega$ such that $\forall^\infty_{k<\omega}(g'(k) \geq g(k)\cdot\rho(m_k))$. Then, if $\mathbb{Q}$ is $\langle\pi,\rho\rangle$-linked and $\dot{\psi}$ is a $\mathbb{Q}$-name for a slalom in $S(\omega, g)$, there exists a $\psi' \in S(\omega, g')$ such that, for any $f \in \omega^\omega$ such that $f \notin^* \psi'$, $\Vdash f \notin^* \dot{\psi}$.*

Proof. Let $\langle Q_{k,j}\rangle_{k<\omega, j<\rho(k)}$ be a witness of the linkedness of $\mathbb{Q}$. For any $k < \omega$ and $j < \rho(m_k)$, put $z_{k,j} := \{l < \omega \ / \ \exists_{q\in Q_{m_k,j}}(q \Vdash l \in \dot{\psi}(k))\}$. As $Q_{m_k,j}$ is $\pi(m_k)$-linked and $g(k) < \pi(m_k)$, by linkedness it is clear that $|z_{k,j}| \leq g(k)$. Put $\psi'(k) := \bigcup_{j<\rho(m_k)} z_{k,j}$, so it is clear that $\psi' \in S(\omega, g')$. Let $f \in \omega^\omega$ such that $\exists^\infty_{k<\omega}(f(k) \notin \psi'(k))$ and we show that $\Vdash \exists^\infty_{k<\omega}(f(k) \notin \dot{\psi}(k))$. For $p \in \mathbb{Q}$ and $k_0 < \omega$, find $k > k_0$ and $j < m_k$ such that $f(k) \notin \psi'(k)$ and $p \in Q_{m_k,j}$. In particular, $f(k) \notin z_{k,j}$. By definition of $z_{k,j}$, $p \nVdash f(k) \in \dot{\psi}(k)$, so there is a $q \leq p$ such that $q \Vdash f(k) \notin \dot{\psi}(k)$. □

□

6.4.12 Lemma. *Let $b, h \in \omega^\omega$ be non-decreasing functions with $b > 0$ and h converging to infinity. Let $\pi, \rho \in \omega^\omega$. If $\{m_k\}_{k<\omega}$ is a non-decreasing sequence of natural numbers that converges to infinity and, for all but finitely many $k < \omega$, $k\cdot\pi(k) \leq h(m_k)$ and $k\cdot|[b(m_k-1)]^{\leq k}|^{m_k} \leq \rho(k)$, then $\mathbb{LOC}^h_{b,\mathcal{F}}$ is $\langle\pi,\rho\rangle$-linked for any $\mathcal{F} \subseteq \mathbb{R}_b$.*

Proof. Choose $1 < M < \omega$ such that, for any $k \geq M$, $k\cdot\pi(k) \leq h(m_k)$ and $k\cdot|[b(m_k-1)]^{\leq k}|^{m_k} \leq \rho(k)$. Find a non-decreasing sequence $\{n_k\}_{k<\omega}$ of natural numbers that converges to infinity such that, for all $k \geq M$, $n_k \leq k, m_k$ and $|S_{n_k}(b,h)| \leq k$. Let $S_k := \{s \in S_{m_k}(b,h) \ / \ \forall_{i\in[n_k,m_k)}(|s(i)| \leq k)\}$ when $k \geq M$. Note that

$$|S_k| = |S_{n_k}(b,h)| \cdot \left|\prod_{i\in[n_k,m_k)} [b(i)]^{\leq k}\right| \leq k\cdot|[b(m_k-1)]^{\leq k}|^{m_k} \leq \rho(k).$$

For each $k \geq M$ and $s \in S_k$, put $Q_{k,s} := \{(t,F) \in \mathbb{LOC}^h_{b,\mathcal{F}} \ / \ t = s \text{ and } |F|\cdot\pi(k) \leq h(m_k)\}$. It is clear that $Q_{k,s}$ is $\pi(k)$-linked for all $s \in S_k$. To conclude that $\mathbb{LOC}^h_{b,\mathcal{F}}$ is $\langle\pi,\rho\rangle$-linked, we show that, given $(t,F) \in \mathbb{LOC}^h_{b,\mathcal{F}}$, for all but finitely many k we can extend (t,F) to some condition in $Q_{k,s}$ for some $s \in S_k$. Choose $N < \omega$ such that $M, |F| \leq N$ and $|t| \leq n_N$. Extend (t,F) to $(t',F) \in \mathbb{LOC}^h_{b,\mathcal{F}}$ such that $|t'| = n_N$. Now, for all $k \geq N$, we can extend (t',F) to $(s,F) \in \mathbb{LOC}^h_{b,\mathcal{F}}$ such that $s \in S_k$ because $|F| \leq k$. For the same reason, we get $|F|\cdot\pi(k) \leq k\cdot\pi(k) \leq h(m_k)$ and, thus, $(s,F) \in Q_{k,s}$. □

6.5 Many different Rothberger numbers

In this section, we prove all our main consistency results for fragmented ideals. Fix, from now on, $\mu \leq \nu \leq \kappa$ uncountable regular cardinals and a cardinal λ such that $\lambda^{<\kappa} = \lambda$. The first result says that it is consistent that the Rothberger numbers for all somewhere tall fragmented ideals are strictly less than $\mathfrak{b}$.

6.5.1 Theorem. *There exists a ccc poset that forces $\mathfrak{b}(\mathcal{I}) \leq \mu$ for any somewhere tall fragmented ideal $\mathcal{I}$, $\mathrm{add}(\mathcal{N}) = \mu$, $\mathfrak{b} = \kappa$ and $\mathfrak{c} = \lambda$. In particular, this poset forces $\mathfrak{b}(\mathcal{I}) = \mathrm{add}(\mathcal{N}) = \mu$ for any somewhere tall gradually fragmented ideal $\mathcal{I}$.*

Proof. In a similar way as in the results of Section 4.1, perform a fsi $\mathbb{P}_\lambda = \langle \mathbb{P}_\alpha, \dot{\mathbb{Q}}_\alpha \rangle_{\alpha<\lambda}$ alternating between Cohen forcing $\mathbb{C}$, suborders of $\mathbb{A}$ of size $< \mu$ and suborders of $\mathbb{D}$ of size $< \kappa$ and, by a book-keeping argument, we make sure that all those suborders of the extension are used in the iteration. By the techniques of the same results, $\mathbb{P}_\lambda$ forces $\text{add}(\mathcal{N}) = \mu$, $\mathfrak{b} = \kappa$ and $\mathfrak{c} = \lambda$.

In V, fix $\bar{L} = \{L_n\}_{n<\omega}$ a partition of ω into infinite sets. Now, in V_λ, let $\mathcal{I} = \mathcal{I}\langle a_i, \varphi_i \rangle_{i<\omega}$ be a somewhere tall fragmented ideal and, by Lemma 6.1.2, without loss of generality, assume that it is tall and $\lim_{n\to+\infty} \varphi_i(a_i) = +\infty$. As $\mathcal{I}$ is represented by a real number, there exists $\alpha < \lambda$ such that $\langle a_i, \varphi_i \rangle_{i<\omega} \in V_\alpha$. By Lemma 6.4.6(c), there is a μ-1-strong covering family Ψ'' of size μ in V_β with $\beta := \alpha + \mu$ (ordinal sum). By Lemmas 6.4.7 and 6.4.6(f), $\mathbb{P}_{[\beta,\lambda)} = \mathbb{P}_\lambda / \mathbb{P}_\beta$ (the remaining part of the iteration from β) is μ-$\langle \bar{L}, \mathcal{I}, 1 \rangle$-good, so this μ-1-strong covering family Ψ'' is preserved in V_λ. By Theorem 6.4.2, $\mathfrak{b}(\mathcal{I}) \leq \mathfrak{b}^1(\mathcal{I}, \bar{L}) \leq |\Psi''| = \mu$.

The last statement follows from Corollary 6.3.5. □

The following shows that, for an ideal as in Example 6.1.12(1), we can find a poset that puts its Rothberger number strictly between $\text{add}(\mathcal{N})$ and $\mathfrak{b}$. In particular, this holds for the polynomial growth ideal $\mathcal{I}_P$ as well.

6.5.2 Theorem. *Let $\mathcal{I} = \mathcal{I}_c(P)$ be a gradually fragmented ideal as in Example 6.1.12(1) and assume it is non-trivial. Then, there exists a ccc poset that forces* $\text{add}(\mathcal{N}) = \mu$, $\mathfrak{b}(\mathcal{I}) = \nu$, $\mathfrak{b} = \kappa$ *and* $\mathfrak{c} = \lambda$.

Proof. By Lemma 6.1.2, it is enough to assume that $|a_i| \geq c(i)^i$ for every $i < \omega$ (this assumption is only used to prove $\mathfrak{b}(\mathcal{I}) \leq \nu$ in the forcing extension defined below). Let $h = c$ and $b \in \omega^\omega$ any function such that $b(i) \geq 2^{|a_i|}$ for any $i < \omega$. By Lemma 6.4.12, we can find $\pi, \rho \in \omega^\omega$ such that π converges to infinity and $\mathbb{LOC}^h_{b,\mathcal{F}}$ is $\langle \pi, \rho \rangle$-linked for any $\mathcal{F} \subseteq \mathbb{R}_b$. Also, by Lemma 6.4.10, find $\bar{H}$ such that any $\langle \pi, \rho \rangle$-linked poset is $\in^*_{\bar{H}}$-good. Fix $\bar{L}$ as in Section 6.4.

As in the results of Section 4.1, perform a fsi $\mathbb{P}_\lambda = \langle \mathbb{P}_\alpha, \dot{\mathbb{Q}}_\alpha \rangle_{\alpha<\lambda}$ alternating between Cohen forcing $\mathbb{C}$, suborders of $\mathbb{A}$ of size $< \mu$, $\mathbb{LOC}^h_{b,\mathcal{F}}$ with $|\mathcal{F}| < \nu$ and suborders of $\mathbb{D}$ of size $< \kappa$. By a book-keeping argument, we make sure that all such possible suborders of the extension are used in the iteration. As $\mathbb{P}_\lambda$ is μ-$\in^*_{\bar{H}}$-good, ν-$\langle \bar{L}, \mathcal{I}, 1 \rangle$-good and κ-$\leq^*$-good, it follows that $\text{add}(\mathcal{N}) = \mu$, $\mathfrak{b} = \kappa$ and $\mathfrak{c} = \lambda$ in any $\mathbb{P}_\lambda$-extension V_λ.

Now, in V_λ, $\nu \leq \mathfrak{b}_{\text{Loc}}(b, h)$ (this implies $\nu \leq \mathfrak{b}(\mathcal{I})$ by Lemma 6.3.7). Indeed, let $\mathcal{F} \subseteq \mathbb{R}_b$ of size $< \nu$, so there is some $\alpha < \lambda$ such that $\mathcal{F} \in V_\alpha$. Now, at some point of the remaining part of the iteration, the poset $\mathbb{LOC}^h_{b,\mathcal{F}}$ is used to add a slalom $\psi \in S(b, h)$ that localizes $\mathcal{F}$.

Finally, we prove that $\mathfrak{b}^1(\mathcal{I}, \bar{L}) \leq \nu$ is true in V_λ (so $\mathfrak{b}(\mathcal{I}) \leq \nu$ by Theorem 6.4.2). This is because, by Lemma 6.4.6, the iteration adds a ν-1-strong covering family of size ν in V_ν that is preserved in V_λ. □

The following result states that, no matter which (uncountable regular) values one wants to force for $\text{add}(\mathcal{N})$ and $\mathfrak{b}$, it is consistent to find as many as possible gradually fragmented ideals that have pairwise different Rothberger numbers between $\text{add}(\mathcal{N})$ and $\mathfrak{b}$. The same happens with the cardinal invariants of the type of Definition 6.3.1(1).

6.5.3 Theorem. *Let $\delta \leq \kappa$ be an ordinal and $\{\nu_\xi\}_{\xi<\delta}$ a non-decreasing sequence of regular cardinals in $[\mu, \kappa]$. Then, there are a sequence $\{\mathcal{I}_\xi\}_{\xi<\delta}$ of tall gradually fragmented ideals, a sequence of pair of functions $\{(b_\xi, h_\xi)\}_{\xi<\delta}$ and a ccc poset that forces* $\text{add}(\mathcal{N}) = \mu$, $\mathfrak{b} = \kappa$, $\mathfrak{c} = \lambda$ *and* $\mathfrak{b}(\mathcal{I}_\xi) = \mathfrak{b}_{Loc}(b_\xi, h_\xi) = \nu_\xi$ *for all $\xi < \delta$.*

For the proof of this theorem, we use another characterization of the unbounding number $\mathfrak{b}$. To fix some notation, define an elementary exponentiation operation $\sigma : \omega \times \omega \to \omega$ by $\sigma(n, 0) = 1$ and $\sigma(n, m+1) = n^{\sigma(n,m)}$. Put $\rho : \omega \to \omega$ such that $\rho(0) = 2$ and $\rho(i+1) = \sigma(\rho(i), i+3)$. For a function $x \in \omega^\omega$ define, by recursion on $k < \omega$, $x^{[0]} = x$ and $x^{[k+1]} = 2^{\rho^2 \cdot x^{[k]}}$. Now, let

$$\mathbb{R}^\rho := \{x \in \omega^\omega \,/\, \forall_{k<\omega} \forall^\infty_{i<\omega} (x^{[k]}(i) \leq \rho(i+1))\}.$$

6.5.4 Lemma. *Let $x \in \mathbb{R}^\rho$. Then,*

(a) $id_\omega \in \mathbb{R}^\rho$.

(b) The functions 2^x, $x^{id_\omega \cdot \rho^{id_\omega}}$, $id_\omega \cdot x$ *and* y *defined as* $y(i) = |[x(i)]^{\leq \rho(i)^i}|$ *are in* $\mathbb{R}^\rho$.

(c) $z \in \mathbb{R}^\rho$ *where* $z(i) = \max\{x(j)\}_{j \leq i}$.

(d) $\forall^\infty_{i<\omega}(i \cdot |[x(i-1)]^{\leq i}|^i \leq \rho(i))$.

Proof. (a) It is enough to show that $id_\omega^{[k]}(i) \leq \sigma(\rho(i), 2k+1)$ for all $i < \omega$ and $k < \omega$ by induction on k. The case $k = 0$ is clear. For the induction step,

$$id_\omega^{[k+1]}(i) = 2^{id_\omega^{[k]}(i) \cdot \rho(i)^2} \leq \rho(i)^{\sigma(\rho(i),2k+1) \cdot \rho(i)^{\rho(i)}} \leq \rho(i)^{\sigma(\rho(i),2k+2)} = \sigma(\rho(i), 2k+3).$$

(b) Clear because $2^x \leq^* x^{[1]}$, $x^{id_\omega \cdot \rho^{id_\omega}} \leq^* x^{[2]}$, $id_\omega \cdot x \leq^* x^{[1]}$ and $y \leq^* x^{[1]}$.

(c) Let N_k be minimal such that $\forall_{i \geq N_k}(x^{[k]}(i) \leq \rho(i+1))$. As the sequence $\{x^{[k]}(i)\}_{k<\omega}$ is strictly increasing for each $i < \omega$, $\{N_k\}_{k<\omega}$ is non-decreasing and converges to infinity. Let $\{k_j\}_{j<\omega}$ be a strictly increasing sequence of natural numbers such that $\rho(N_{k_j}+1)$ is bigger than $x^{[j]}(i)$ for all $i < N_j$. Then, $z^{[j]}(i) \leq \rho(i+1)$ for all $i \geq N_{k_j}$.

(d) Note that $(i+1)|[x(i)]^{\leq (i+1)}|^{i+1} \leq 2^{x(i) \cdot (i+1) \cdot 2} \leq x^{[1]}(i)$ for all but finitely many i such that $x(i) \neq 0$. The case $x(i) = 0$ is straightforward.

□

To proceed with the proof of Theorem 6.5.3, we first need to see that $\mathfrak{b}$ is the least size of a $\leq^*$-unbounded family in $\mathbb{R}^\rho$. But we can prove a more general result instead. Fix $g, H \in \omega^\omega$ such that H is strictly increasing and $id_\omega < H$. Define $\mathbb{R}^g_H := \{x \in \omega^\omega \ / \ \forall_{k<\omega}(x^{[k]} \leq^* g)\}$ where $x^{[0]} = x$ and $x^{[k+1]} = H \circ x^{[k]}$. With the particular case $g(i) = \rho(i+1)$ and $H = 2^{\rho^2 \cdot id_\omega}$, what we need is just a consequence of the following

6.5.5 Lemma. *Assume that $\mathbb{R}^g_H \neq \varnothing$. Then, $\mathfrak{b}$ is the least size of a $\leq^*$-unbounded family in $\mathbb{R}^g_H$.*

Proof. For $x \in \mathbb{R}^g_H$ and $k < \omega$, let N^x_k be the minimal $N < \omega$ such that $\forall_{i \geq N}(x^{[k]}(i) \leq g(i))$. $\{N^x_k\}_{k<\omega}$ is non-decreasing and converges to infinity because $H > id_\omega$. Consider the function H' of natural numbers such that $H'(m)$ is the maximal $n < \omega$ such that $H(n) \leq m$. Note that the domain of H' is $[H(0), \omega)$ and that $H(n) \leq m$ iff $H'(m)$ is defined and $n \leq H'(m)$. Also, $H'(m) < m$ for all $m \geq H(0)$. Define, for $k < \omega$, the function C_k on a subset of ω by $C_0(i) = g(i)$ and $C_{k+1}(i) = H'(C_k(i))$.

6.5.6 Claim. *Let $x \in \omega^\omega$, $i, k < \omega$. Then, $x^{[k]}(i) \leq g(i)$ iff $C_k(i)$ exists and $x(i) \leq C_k(i)$.*

Proof. Fix $i < \omega$ and $0 < M < \omega$. Define, when possible, C^M_k for $k < \omega$ such that $C^M_0 = M$ and $C^M_{k+1} = H'(C^M_k)$. It is enough to prove, by induction on k that, for all $0 < M < \omega$, $x^{[k]}(i) \leq M$ iff C^M_k exists and $x(i) \leq C^M_k$ (our claim is the particular case $M = g(i)$). The case $k = 0$ is trivial, so we proceed to prove the inductive step. $x^{[k+1]}(i) \leq M$ is equivalent to $x^{[k]}(i) \leq C^M_1$ which is equivalent, by induction hypothesis, to the existence of $C^{C^M_1}_k$ and $x(i) \leq C^{C^M_1}_k = C^M_{k+1}$. □

Note that, as there exists a $c \in \mathbb{R}^g_H$, the functions C_k are defined for all but finitely many natural numbers because, by the previous claim, $\forall_{i \geq N^c_k}(c(i) \leq C_k(i))$. Now, consider W as the set of non-decreasing functions $z \in \omega^\omega$ such that $\forall_{k<\omega}(z(k) \geq N^c_k)$. It is clear that $\mathfrak{b}$ is the least size of a $\leq^*$-unbounded family in W.

Define the function $F : W \to \mathbb{R}^g_H$ such that, $F(z) = F_z : \omega \to \omega$, $F_z(i) = C_k(i)$ when $i \in [z(k), z(k+1))$ (we do not care about the values below $z(0)$). Claim 6.5.6 guarantees that $F_z \in \mathbb{R}^g_H$. Also, let $F' : \mathbb{R}^g_H \to \omega^\omega$, $F'(x) = F'_x$ such that $F'_x(i) = N^x_i$. The lemma follows from the fact that, for any $x \in \mathbb{R}^g_H$ and $z \in W$,

(i) $F_z \leq^* x$ implies $z \leq^* F'_x$, and

(ii) $F'_x \leq^* z$ implies $x \leq^* F_z$.

To prove (i), assume that there is a $\bar{k} < \omega$ such that $\forall_{i \geq z(\bar{k})}(F_z(i) \leq x(i))$. Let $k' < \omega$ be minimal such that $z(\bar{k}) < z(k')$ and prove that $z(k) \leq N^x_k$ for all $k \geq k'$. By contradiction, assume that there is a minimal $k \geq k'$ such that $N^x_k < z(k)$, so there exists an $i \in [z(k-1), z(k))$ with $i \geq N^x_k$. Then, $C_{k-1}(i) = F_z(i) \leq x(i)$ and $x(i) \leq C_k(i)$ (by Claim 6.5.6). But $C_k(i) = H'(C_{k-1}(i)) < C_{k-1}(i)$, a contradiction.

For (ii), assume that there is a $\bar{k} < \omega$ such that $N^x_k \leq z(k)$ for all $k \geq \bar{k}$. If $i \geq z(\bar{k})$, we can find a $k \geq \bar{k}$ such that $i \in [z(k), z(k+1))$, so $x(i) \leq C_k(i) = F_z(i)$ because $i \geq z(k) \geq N^x_k$. □

Proof of Theorem 6.5.3. Without loss of generality, we may assume that $\mathfrak{b} = \kappa$ in the ground model V. Construct, for $\xi < \delta$, functions $h_\xi, a_\xi, b_\xi, \pi_\xi \in \omega^\omega$ such that

(a) $h_0 = id^2_\omega$ and, for $\xi > 0$, h_ξ is non-decreasing, converges to infinity and, for $\eta < \xi$, $id_\omega \cdot \pi_\eta \leq^* h_\xi$

(b) $a_\xi > 0$ is non-decreasing, converges to infinity and $h_\xi^{id_\omega \cdot \rho^{id_\omega}} \leq^* a_\xi$,

(c) b_ξ and π_ξ are defined as $b_\xi = 2^{a_\xi}$ and $\pi_\xi(i) = |[b_\xi(i)]^{\leq \rho(i)^i}|$, and

(d) $\forall^\infty_{i<\omega}(i \cdot |[b_\xi(i-1)^{\leq i}]|^i \leq \rho(i))$.

We can construct all those functions in $\mathbb{R}^\rho$. To see this, fix $\xi < \delta$ and assume that we have all these functions for $\eta < \xi$. By Lemma 6.5.4, $id_\omega \cdot \pi_\eta \in \mathbb{R}^\rho$ for all $\eta < \xi$, so there exists a non-decreasing function $h_\xi \in \mathbb{R}^\rho$ bounding them by Lemma 6.5.5. Put $a_\xi := \max\{h_\xi^{id_\omega \cdot \rho^{id_\omega}}, 1\}$, which is in $\mathbb{R}^\rho$. Clearly, $b_\xi, \pi_\xi \in \mathbb{R}^\rho$ and (d) is true.

For each $\xi < \delta$, let $P_\xi = \{a_{\xi,i}\}_{i<\omega}$ be the interval partition of ω such that $|a_{\xi,i}| = a_\xi(i)$, define $c_\xi(i) = \max\{h_\xi(i), 2\}$ and let $\mathcal{I}_\xi := \mathcal{I}_{c_\xi}(P_\xi)$ (see Example 6.1.12(1)). Perform a fsi $\mathbb{P}_{(3+\delta)\cdot\lambda} := \langle \mathbb{P}_\alpha, \dot{\mathbb{Q}}_\alpha \rangle_{\alpha < (3+\delta)\cdot\lambda}$ such that, for $\gamma < \lambda$,

(i) If $\alpha = (3+\delta)\cdot\gamma$, let $\dot{\mathbb{Q}}_\alpha$ be a $\mathbb{P}_\alpha$-name for Cohen forcing,

(ii) if $\alpha = (3+\delta)\cdot\gamma + 1$, let $\dot{\mathbb{Q}}_\alpha$ be a $\mathbb{P}_\alpha$-name for a suborder of $\mathbb{A}$ of size $< \mu$,

(iii) if $\alpha = (3+\delta)\cdot\gamma + 2$, let $\dot{\mathbb{Q}}_\alpha$ be a $\mathbb{P}_\alpha$-name for a suborder of $\mathbb{D}$ of size $< \kappa$,

(iv) if $\alpha = (3+\delta)\cdot\gamma + 3 + \xi$ for $\xi < \delta$, let $\dot{\mathbb{Q}}_\alpha = \mathbb{LOC}^{h_\xi}_{b_\xi, \dot{\mathcal{F}}_\alpha}$ where $\dot{\mathcal{F}}_\alpha$ is a $\mathbb{P}_\alpha$-name of a subset of $\mathbb{R}_{b_\xi}$ of size $< \nu_\xi$.

By a book-keeping argument, as in the results of Section 4.1, we make sure to use all such posets of the extension in the iteration. Choose $\bar{L} = \{L_n\}_{n<\omega}$ any partition of ω into infinite sets.

6.5.7 Claim. *For every $\xi < \delta$ and $\alpha < (3+\delta)\cdot\lambda$, $\mathbb{P}_\alpha$ forces that $\dot{\mathbb{Q}}_\alpha$ is ν_ξ-$\langle \bar{L}, \mathcal{I}_\xi, \rho \rangle$-good.*

Proof. Let $\alpha = (3+\delta)\cdot\gamma + \xi''$ for some $\gamma < \lambda$ and $\xi'' < 3+\delta$. Step in V_α. If $\xi'' \leq 2$ then $\dot{\mathbb{Q}}_\alpha$ is μ-$\langle \bar{L}, \mathcal{I}_\xi, \rho \rangle$-good by Lemma 6.4.7; if $\xi'' = 3 + \xi'$ for some $\xi' < \delta$ then, if $\xi' \leq \xi$, as $|\mathbb{Q}_\alpha| < \nu_{\xi'} \leq \nu_\xi$, $\dot{\mathbb{Q}}_\alpha$ is ν_ξ-$\langle \bar{L}, \mathcal{I}_\xi, \rho \rangle$-good by Lemma 6.4.7; if $\xi < \xi'$, by (a),(d) and Lemma 6.4.12 (with $m_k = k$), $\mathbb{Q}_\alpha = \mathbb{LOC}^{h_{\xi'}}_{b_{\xi'}, \mathcal{F}_\alpha}$ is $\langle \pi_\xi, \rho \rangle$-linked, thus, by (c) and Lemma 6.4.9, $\dot{\mathbb{Q}}_\alpha$ is ν_ξ-$\langle \bar{L}, \mathcal{I}_\xi, \rho \rangle$-good. □

6.5.8 Claim. *In V, there is a sequence $\bar{H} = \{g_k\}_{k<\omega}$ of reals in ω^ω that converges to infinity such that $\mathbb{P}_\alpha$ forces that $\dot{\mathbb{Q}}_\alpha$ is μ-$\in^*_{\bar{H}}$-good.*

Proof. By Lemma 6.4.10, find $\bar{H} = \{g_k\}_{k<\omega}$ such that any $\langle id_\omega, \rho \rangle$-linked poset is $\in^*_{\bar{H}}$-good. Now, step in V_α. If $\alpha = (3+\delta)\cdot\gamma + \xi'$ for some $\xi' < 3+\delta$, when $\xi' \leq 2$ then $\dot{\mathbb{Q}}_\alpha$ is μ-$\in^*_{\bar{H}}$-good by Corollary 3.2.18; else, if $\xi' = 3 + \xi$ for some $\xi < \delta$, the claim holds because $\mathbb{Q}_\alpha$ is $\langle id_\omega, \rho \rangle$-linked (see the proof of the previous claim). □

It is known that, in $V_{(3+\delta)\cdot\lambda}$, $\mathrm{add}(\mathcal{N}) = \mu$, $\mathfrak{b} = \kappa$ and $\mathfrak{c} = \lambda$. By the same argument as in Theorem 6.5.2, we get, for $\xi < \delta$, $\nu_\xi \leq \mathfrak{b}_{\mathrm{Loc}}(b_\xi, h_\xi) \leq \mathfrak{b}(\mathcal{I}_\xi)$ (the last inequality by Lemma 6.3.7). On the other hand, by Claim 6.5.7 and Lemma 6.4.6, for $\xi < \delta$, we add in V_{ν_ξ} a ν_ξ-ρ-strong covering family (with respect to $\mathcal{I}_\xi$ and $\bar{L}$) of size ν_ξ that is preserved in $V_{(3+\delta)\cdot\lambda}$, so $\mathfrak{b}^{\rho^{id_\omega}}(\mathcal{I}_\xi, \bar{L}) \leq \nu_\xi$. But, by Theorem 6.4.2, as $\lim_{i\to+\infty} \varphi_i(a_{\xi,i})/(\rho(i)^i) = +\infty$ by (b), we get $\mathfrak{b}(\mathcal{I}_\xi) \leq \mathfrak{b}^{\rho^{id_\omega}}(\mathcal{I}_\xi, \bar{L}) \leq \nu_\xi$. □

To obtain (consistently) continuum many pairwise different Rothberger numbers, it is necessary that the continuum is a weakly inaccessible cardinal. Indeed, let $\{\mathcal{I}_\xi\}_{\xi<\mathfrak{c}}$ be a sequence of ideals such that the numbers $\mathfrak{b}(\mathcal{I}_\xi)$ are pairwise different. As there are continuum many and all of them are $\leq \mathfrak{b}$, we obtain $\mathfrak{c} = \mathfrak{b}$, so $\mathfrak{c}$ is regular. Also, as there are $\mathfrak{c}$-many different cardinals below $\mathfrak{c}$, $\mathfrak{c}$ has to be a limit cardinal. Likewise, the existence of $\mathfrak{b}$-many different Rothberger numbers implies that $\mathfrak{b}$ is weakly inaccessible.

6.5.9 Corollary. *Assume that λ is a weakly inaccessible cardinal such that $\lambda^{<\lambda} = \lambda$, and let $\mu < \lambda$ be a regular cardinal. For any collection of pairwise different regular cardinals $\{\nu_\xi\}_{\xi<\lambda} \subseteq [\mu, \lambda]$, there exist tall gradually fragmented ideals $\mathcal{I}_\xi$, $(b_\xi, h_\xi) \in \omega^\omega \times \omega^\omega$ for $\xi < \lambda$ and a ccc poset that forces* $\mathrm{add}(\mathcal{N}) = \mu$, $\mathfrak{b} = \mathfrak{c} = \lambda$ *and* $\mathfrak{b}(\mathcal{I}_\xi) = \mathfrak{b}(b_\xi, h_\xi) = \nu_\xi$ *for any* $\xi < \lambda$.

QUESTIONS AND DISCUSSIONS

Theorem 4.3.1, where we forced $\mathfrak{s} < \kappa < \mathfrak{b} = \mathfrak{d} < \mathfrak{a}$ modulo the existence of a measurable cardinal κ in the ground model and where $\mathfrak{s}$ can take any arbitrary regular uncountable value below κ, is an extension of Shelah's argument to force $\kappa < \mathfrak{d} < \mathfrak{a}$. As Shelah could modify the template iteration construction in order to obtain the consistency of $\mathfrak{s} = \aleph_1 < \mathfrak{b} = \mathfrak{d} < \mathfrak{a}$ with ZFC alone by replacing the ultrapower argument with an isomorphism-of-names argument, it is natural to think that our argument for Theorem 4.3.1 could be modified in order to get the consistency of $\aleph_1 < \mathfrak{s} < \mathfrak{b} < \mathfrak{a}$ with ZFC alone. In Shelah's proof it is used a template iteration where Hechler forcing is the only iterand at each stage, so it is possible to localize by restricting the template to countable sets to get many of them pairwise isomorphic, so an isomorphism-of-names argument becomes possible to force $\mathfrak{a}$ to be large. However, as in our argument we also use Mathias forcing with small filter bases, the construction is not that uniform and it is uncertain how to localize and get many pairwise isomorphic restrictions of the template iteration, this to allow an isomorphism-of-names argument.

Problem K. *Is it consistent with* ZFC *(alone) that* $\aleph_1 < \mathfrak{s} < \mathfrak{b} < \mathfrak{a}$*?*

Our construction in Theorem 4.3.1 also forces values for other cardinal invariants. In fact, the starting point was to construct a fsi as in Theorem 4.1.2(b) with $\mu_1 = \mu_2 = \mu_3 = \theta$ and $\nu = \mu$, which forces $\text{add}(\mathcal{N}) = \text{cov}(\mathcal{N}) = \mathfrak{p} = \mathfrak{g} = \mathfrak{s} = \theta < \text{add}(\mathcal{M}) = \text{cof}(\mathcal{M}) = \mu < \text{non}(\mathcal{N}) = \mathfrak{r} = \mathfrak{c} = \lambda$. Later, we proceeded with ultrapowers through the measurable κ in order to get a model of the same statements and, additionally, $\mathfrak{a} = \lambda$. It is natural to ask whether a similar construction, starting with an iteration as in Theorem 4.1.2(b) that forces $\text{add}(\mathcal{N}) < \text{cov}(\mathcal{N}) < \mathfrak{p} = \mathfrak{g} = \mathfrak{s} = \theta < \cdots$, can be done to get the same consistency result plus $\mathfrak{a} = \lambda$. However, Theorem 3.4.1, the preservation result for template iterations that we use in our proof, is not enough to guarantee that $\text{add}(\mathcal{N})$ and $\text{cov}(\mathcal{N})$ can be preserved small with different values, this because there may be names of reals that have a support on the template of size larger than the value we want to force for $\text{add}(\mathcal{N})$. An alternative is to guarantee that the hypothesis of the preservation Theorem 3.4.3 holds, but it is unclear how to deal with the resulting template iterations that result after taking δ many ultrapowers with δ of cofinality $< \theta$.

Problem L. *Is it consistent that* $\text{add}(\mathcal{N}) < \text{cov}(\mathcal{N}) < \mathfrak{s} < \mathfrak{b} < \mathfrak{a}$*, even with the existence of a measurable cardinal?*

A simpler forcing construction, like for the consistency of $\mathfrak{d} < \mathfrak{a}$ and $\mathfrak{u} < \mathfrak{a}$ with a measurable cardinal in [S04] (see also [Br07]) where it is not necessary to look at the template structure, can be considered to attack Problem L (and even for a simpler proof of Theorem 4.3.1), but we do not know how to prove a preservation theorem (in the sense of the results of Section 3.1) for such a construction.

One interesting discovering in this project is that we can use matrix iterations to get models where cardinal invariants of the right hand side of Cichon's diagram (or, in general, cardinal invariants above $\text{cov}(\mathcal{M})$) can assume at most three different values. However, this technique does not seems to work to get models where four or more values can be assumed. One idea to get such models is to extend the construction of a matrix iteration to more dimensions, for instance, a three dimensional iteration. The main problem here is that we cannot guarantee anymore the embeddability between the intermediate stages when we do such a construction towards an interesting model. A solution to this problem may work to give a positive answer to Problem B(1) and (2), possibly assuming the existence of a measurable cardinal.

As discussed in the Introduction, in an ongoing project by M. Goldstern, J. Kellner, S. Shelah and A. Fischer, it is proven that, with a large product construction, one way in which the cardinal invariants of the right hand side of Cichon's diagram assume 5 different values is consistent. However, such a forcing is, typically, ω^ω-bounding, so it does not work to get models where $\mathfrak{d} > \aleph_1$, i.e., where $\mathrm{cov}(\mathcal{M}) < \mathfrak{d}$.

Problem M. *Is it consistent that* $\mathrm{cov}(\mathcal{M}) < \mathfrak{d} < \mathrm{non}(\mathcal{N}) < \mathrm{cof}(\mathcal{N}) < \mathfrak{c}$*?*

In our models with fsi and matrix iterations, discussed in Sections 4.1 and 5.2 respectively, we could get models where the continuum can be singular. As it is provable in ZFC that $\mathrm{add}(\mathcal{I}) \leq \mathrm{cf}(\mathrm{non}(\mathcal{I}))$, $\mathrm{add}(\mathcal{I}) \leq \mathrm{cf}(\mathrm{cof}(\mathcal{I}))$ (see [BaJS89]) and that $\mathrm{cov}(\mathcal{I}) = \mathrm{cof}(\mathcal{I}) \Rightarrow \mathrm{non}(\mathcal{I}) \leq \mathrm{cf}(\mathrm{cof}(\mathcal{I}))$ (a result from Fremlin, see [Br91, Prop. 1]), many assumptions about the regularity of the cardinals is optimal. All the questions about this are summarized as follows.

Problem N. *(a) To what extend can* μ_2 *be assumed to be singular in Theorems 4.1.2 and 5.2.7?*

(b) Can we get the consistency result in Theorem 4.1.3 by only assuming that $\mathrm{cf}(\nu) \geq \mu_2$*?*

(c) Can we get the consistency results in Section 5.2 by only assuming that $\mathrm{cf}(\kappa) \geq \nu$*?*

(d) Can we get the consistency results in Theorem 5.2.4(c) and (d) with $\mu_1 \leq \mathrm{cf}(\kappa) < \nu$*?*

In our results in Section 5.2 we need κ regular because of the application of Theorem 5.1.6 in a matrix iteration construction. It seems that a similar result to force $\mathfrak{d}_{\sqsubset} \geq \kappa$ with κ singular cannot be proved in such generality, that is, it may depend on the way iterands are arranged into the matrix construction.

Note that most of the statements in Section 5.2 can be grouped in pairs in such a way that, when one consistency result contains $\mathfrak{r} = \kappa$, there is other consistency result with the same statements but changing $\mathfrak{r} = \kappa$ to $\mathfrak{u} = \nu$. Of course, for those results that contain $\mathrm{cov}(\mathcal{M}) = \kappa$ it is not possible to get a corresponding consistency result with $\mathfrak{u} = \nu$ because $\mathrm{cov}(\mathcal{M}) \leq \mathfrak{r} \leq \mathfrak{u}$. However, there are statements that could have a corresponding pair.

Problem O. *Can we get similar consistency results as Theorems 5.2.1(b), (g) and 5.2.4(b) by interchanging* $\mathfrak{u} = \nu$ *with* $\mathfrak{r} = \kappa$*?*

In the case of Theorems 5.2.1(b) and 5.2.4(b), we can think of adding cofinally many steps in the matrix construction where we use Mathias forcing with an ultrafilter as it is explained in Context 5.1.4(2)(i). However, it is not clear that the resulting poset forces $\kappa \leq \mathrm{non}(\mathcal{N})$ because we do not know whether the Mathias posets used preserve the $\pitchfork^{\bar{I}}$-unbounded reals added from the beginning of the construction. In the case of Theorem 5.2.1(g), as in the corresponding matrix iteration it is used random and Hechler forcing (as explained in Context 5.1.4(2)(i)) in cofinally many steps, it is not clear how they preserve the splitting reals added at the beginning of the iteration. This problem is related to the reason why we do not how to force $\mathfrak{s} \leq \mu_1$ and $\lambda \leq \mathfrak{r}$ in Theorem 4.1.4.

Recall that, in Section 5.2, whenever a consistency result contains $\mathfrak{r} = \kappa$ it is not stated a value for $\mathfrak{u}$.

Problem P. *Can we force* $\mathfrak{u} = \kappa$ *or* $\mathfrak{u} = \lambda$ *in the consistency results of Section 5.2 where* $\mathfrak{r} = \kappa$ *is forced?*

In Chapter 6, we proved in ZFC that every gradually fragmented ideal has Rothberger number above $\mathrm{add}(\mathcal{N})$, while there is a large class of fragmented not gradually fragmented ideals where each ideal there has Rothberger number $\aleph_1$. However, it is not known whether a positive answer to Problem H can be proved. Even if this is not true, it may happen that there is such a dichotomy for a large class of definable ideals and that there is a natural combinatorial characterization that suggests which way it goes.

Problem Q. *(a) For every fragmented ideal* $\mathcal{I}$*, either* $\mathrm{add}(\mathcal{N}) \leq \mathfrak{b}(\mathcal{I})$ *or* $\mathfrak{b}(\mathcal{I}) = \aleph_1$.

(b) *For every F_σ ideal $\mathcal{I}$, either* $\mathrm{add}(\mathcal{N}) \leq \mathfrak{b}(\mathcal{I})$ *or* $\mathfrak{b}(\mathcal{I}) = \aleph_1$.

(c) *For every analytic ideal $\mathcal{I}$, either* $\mathrm{add}(\mathcal{N}) \leq \mathfrak{b}(\mathcal{I})$ *or* $\mathfrak{b}(\mathcal{I}) = \aleph_1$.

Also, in (a), (b) or (c), give a combinatorial characterization of the ideals satisfying either case of the dichotomy.

Given an ideal as in Example 6.1.12(1), it is consistent that its Rothberger number is strictly bigger than $\mathrm{add}(\mathcal{N})$ (Theorem 6.5.2). However,

Problem R. *(a) Is there a (gradually) fragmented ideal $\mathcal{I}$ with* $\mathfrak{b}(\mathcal{I}) = \mathrm{add}(\mathcal{N})$*? Or is it consistent that* $\mathfrak{b}(\mathcal{I}) > \mathrm{add}(\mathcal{N})$ *for every gradually fragmented ideal $\mathcal{I}$?*

(b) Is there an analytic ideal $\mathcal{I}$ such that $\mathfrak{b}(\mathcal{I}) = \mathrm{add}(\mathcal{N})$*?*

Perhaps the iteration constructed in the proof of Theorem 6.3.11 would preserve some witness of $\mathrm{add}(\mathcal{N})$ of the ground model, but this is not clear because, as seen in Lemma 6.4.10, such a preservation property is not uniform for each iterand that comes from the forcing of Definition 6.3.1.

It is also interesting, as proposed in Problem G, to look at the gap spectrum, in the sense of *linear gaps*, of quotients by analytic ideals. For non-trivial fragmented ideals we have, so far, two type of linear Rothberger gaps, say, of type $(\omega, \mathfrak{b})$ and of type $(\omega, \mathfrak{b}(\mathcal{I}))$. In the case of $\mathcal{P}(\omega)/\mathrm{Fin}$ there is a characterization of its gap spectrum by Rothberger's result in Theorem 1.5.3, that is, there is a linear (ω, κ)-gap in $\mathcal{P}(\omega)/\mathrm{Fin}$ iff there is a well-ordered unbounded sequence in $\langle \omega, \leq^* \rangle$ of length κ. This spectrum is inherited in every quotient by an F_σ and by an analytic P-ideal (Theorem 1.5.4), but it could be larger. By Theorem 6.5.1, $(\omega, \mathfrak{b}(\mathcal{I}))$ is a different type for quotients by somewhere tall fragmented ideals. In particular, (ω, ω_1) is in the spectrum of $\mathbb{P}(\omega)/\mathcal{ED}_{\mathrm{fin}}$ and $(\omega, \mathfrak{b}(\mathcal{I}))$ is in the spectrum of $\mathbb{P}(\omega)/\mathcal{I}_P$ where $\mathrm{add}(\mathcal{N}) < \mathfrak{b}(\mathcal{I}_P) < \mathfrak{b}$ is consistent.

Problem S. *Characterize those κ for which there is a linear (ω, κ)-gap in the quotient of a known ideal, like $\mathcal{ED}_{\mathrm{fin}}$, $\mathcal{I}_L$ and $\mathcal{I}_P$.*

By the consistency result proved in Theorem 6.5.3, we can construct (consistently) F_σ ideals with many types of gaps of the form (ω, ν) with $\mathrm{add}(\mathcal{N}) < \nu < \mathfrak{b}$. For example, let $\{\nu_n\}_{n<\omega}$ be a sequence of regular cardinals in $(\mathrm{add}(\mathcal{N}), \mathfrak{b})$ and assume that, for each $n < \omega$, there is a gradually fragmented ideal $\mathcal{I}_n = \mathcal{I}\langle a_{n,i}, \varphi_{n,i}\rangle_{i<\omega}$ with $\mathfrak{b}(\mathcal{I}_n) = \nu_n$. Given a bijection $g : \omega \to \omega \times \omega$, construct a fragmented ideal $\mathcal{I} = \langle b_j, \varphi_j \rangle_{j<\omega}$ where $\langle b_j \rangle_{j<\omega}$ is a partition of ω with $|b_j| = |a_{g(j)}|$ and φ_j is the submeasure associated with $\langle a_{g(j)}, \varphi_{g(j)} \rangle$ for any $j < \omega$ (see the paragraph before Lemma 6.2.7). Lemma 6.1.2 implies that $\mathbb{P}(\omega)/\mathcal{I}$ has a linear gap of type (ω, ν_n) for each $n < \omega$. Moreover, the gap spectrum of $\mathbb{P}(\omega)/\mathcal{I}$ is contained in $\mathbb{P}(\omega)/\hat{\mathcal{I}}$ and $\mathbb{P}(\omega)/\hat{\mathcal{I}}$ has a gap of type (ω, ω_1) (Corollary 6.2.8).

It is consistent with the existence of a weakly inaccessible cardinal that there is an F_σ-ideal with continuum many types of gaps of the form (ω, ν) with $\mathrm{add}(\mathcal{N}) < \nu < \mathfrak{b}$. Indeed, by Corollary 6.5.9, given a sequence $\{\nu_f\}_{f \in 2^\omega} \subseteq (\mathrm{add}(\mathcal{N}), \mathfrak{b})$ of regular cardinals, we may assume that there is a sequence $\langle \mathcal{I}_f \rangle_{f \in 2^\omega}$ of fragmented ideals on ω such that $\mathfrak{b}(\mathcal{I}_f) = \nu_f$ for all $f \in 2^\omega$. Consider the perfect a.d. family $\mathcal{A} = \{a_f \,/\, f \in 2^\omega\}$ on $2^{<\omega}$ where $a_f = \{f{\restriction}k \,/\, k < \omega\}$ and let $\mathcal{I}$ be the ideal on $2^{<\omega}$ generated by $\bigcup_{f \in 2^\omega} \mathcal{I}'_f$ where $x' \in \mathcal{I}'_f$ iff there is an $x \in \mathcal{I}_f$ such that $x' = \{f{\restriction}k \,/\, k \in x\}$. It is clear that $\mathcal{I}$ is an analytic ideal and, by Lemma 6.1.2, it contains a linear (ω, ν_f)-gap for each $f \in 2^\omega$.

Along with our consistency results with gaps, we also proved, assuming a weakly inaccessible cardinal, that there are continuum many different cardinal invariants of the type $\mathfrak{b}(b, h)$ (Corollary 6.5.9). Although the existence of continuum many different Rothberger numbers for F_σ ideals implies that the continuum is weakly inaccessible, the same may not be true for the previous type of cardinals.

Problem T. *Is it consistent, with* ZFC *alone, that there are continuum many different cardinal invariants of the form* $\mathfrak{b}(b, h)$*?*

Kellner [Kell08] proved the consistency, with ZFC alone, of the existence of continuum many different cardinals of the form $c^\forall(b, h)$, where this cardinal invariant is the dual of $\mathfrak{b}(b, h)$, that is, the least size of a set $C \subseteq S(b,h)$ such that every real in $\mathbb{R}_b$ is localized by some slalom in C. This has been done by a large product construction, so it may be possible to use this technique to give a positive answer to Problem T.

BIBLIOGRAPHY

[BD85] J. Baumgartner, P. Dordal: *Adjoining dominating functions.* J. Symb. Logic 50, no. 1 (1985) 94-101.

[BaJS89] T. Bartoszyński, J. Ihoda (H. Judah), S. Shelah. *The cofinality of cardinal invariants related to measure and category.* J. Symb. Logic 54 (1989) 719-726.

[BaJ] T. Bartoszyński, H. Judah: *Set Theory. On the Structure of the Real Line.* A. K. Peters, Massachusetts, 1995.

[Ba10] T. Bartoszyński: *Invariants of measure and category.* In: A. Kanamori, M. Foreman (eds.), Handbook of Set-Theory, Springer, Heidelberg, 2010, pp. 491-555.

[BlS84] A. Blass, S. Shelah: *Ultrafilters with small generating sets.* Israel J. Math. 65 (1984) 259-271.

[BlS87] A. Blass, S. Shelah: *There may be simple $P_{\aleph_1}$ and $P_{\aleph_2}$ points and the Rudin-Keisler ordering may be downward directed.* Ann. Pure Appl. Logic 33 (1987) 213-243.

[Bl88] A. Blass: *Selective ultrafilters and homogeneity.* Ann. Pure Appl. Logic 38, no. 3 (1988) 215-255.

[Bl89] A. Blass: *Applications of superperfect forcing and its relatives.* In: J. Steprans, S. Watson (eds.), Set Theory and Its Applications, in: Lecture Notes in Math. 1401, Springer-Verlag, Berlin, 1989, pp. 18-40.

[Bl10] A. Blass: *Combinatorial cardinal characteristics of the continuum.* In: A. Kanamori, M. Foreman (eds.), Handbook of Set-Theory, Springer, Heidelberg, 2010, pp. 395-490.

[Br91] J. Brendle: *Larger cardinals in Cichoń's diagram,* J. Symb. Logic 56, no. 3 (1991) 795-810.

[Br98] J. Brendle: *Mob families and mad families.* Arch. Math. Logic 37 (1998) 183-197.

[Br02] J. Brendle: *Mad families and iteration theory.* In: Logic and Algebra, Y. Zhang (ed.), Contemp. Math. 302, Amer. Math. Soc., Providence, RI, 2002, pp. 1-31.

[Br03] J. Brendle: *The almost disjointness number may have countable cofinality.* Trans. Amer. Math. Soc. 355 (2003) 2633-2649.

[Br05] J. Brendle: *Templates and iterations, Luminy 2002 lecture notes.* Kyōto daigaku sūrikaiseki kenkyūsho kōkyūroku (2005) 1-12.

[Br06] J. Brendle: *Van Douwen's diagram for dense sets of rationals.* Ann. Pure Appl. Logic 143 (2006) 54-69.

[Br07] J. Brendle: *Mad families and ultrafilters.* AUC 49 (2007) 19-35.

[Br09] J. Brendle: *Forcing and the structure of the real line: the Bogotá lectures.* Lecture notes, 2009.

[Br10] J. Brendle: *Aspects of iterated forcing: the Hejnice lectures.* Lecture notes, 2010.

[BrF11] J. Brendle, V. Fischer: *Mad families, splitting families and large continuum.* J. Symb. Logic 76, no. 1 (2011) 198-208.

[Br-1] J. Brendle: *Measure, category and forcing theory.* Preprint.

[Br-2] J. Brendle: *Shattered iterations.* In preparation.

[BrMe] J. Brendle, D.A. Mejía: *Rothberger gaps in fragmented ideals.* Submitted.

[BrRa14] J. Brendle, D. Raghavan: *Bounding, splitting and almost disjointness.* Ann. Pure Appl. Logic 165 (2014) 631-651.

[C] P. J. Cohen: *Set theory and the continuum hypothesis.* W. A. Benjamin, 1966.

[Fa] I. Farah: *Analytic quotients: theory of liftings for quotients over analytic ideals on the integers.* Mem. Amer. Math. Soc. 148, no. 702, 2000, pp. 177.

[Go] K. Gödel: *The consistency of the continuum hypothesis.* Princeton University Press, 1940.

[GS90] M. Goldstern, S. Shelah: *Ramsey ultrafilters and the reaping number* - $\mathrm{con}(\mathfrak{r} < \mathfrak{u})$. Ann. Pure Appl. Logic 49 (1990) 121-142.

[G92] M. Goldstern: *Tools for your forcing construction.* In: Judah H. (ed.) Set theory of the reals, Israel Math. Conf. Proc., Bar Ilan University, 1992, pp. 305-360.

[H84] J. Hadamard: *Sur les caractères de convergence des séries a termes positifs et sur les fonctions indéfiniment croissantes.* Acta Math. 18 (1894) 319-336.

[HarW] G. H. Hardy, E. M. Wright: *An introduction to the theory of numbers* 6th edition, Oxford University Press, 2008.

[Ha1909] F. Hausdorff: *Die Graduierung nach dem Endverlauf.* Abh. König. Sächs. Ges. Wiss. Math.-Phys. Kl. 31 (1909) 296-334.

[Ha36] F. Hausdorff: *Summen von* $\aleph_1$ *Mengen.* Fund. Math. 26 (1936) 241-255.

[Hr11] M. Hrušák: *Combinatorics of filters and ideals*. Contemp. Math. 533 (2011) 29-69.

[HrRZ] M. Hrušák, D. Rojas-Rebolledo, J. Zapletal: *Cofinalities of Borel ideals.* Math. Logic Quart., to appear.

[Je] T. Jech: *Set theory.* 3rd millenium edition, Springer, Heidelberg, 2002.

[JS88] J. Ihoda (H. Judah), S. Shelah: *Souslin forcing.* J. Symb. Logic 53, no. 4 (1988) 1188-1207.

[JS90] H. Judah, S. Shelah: *The Kunen-Miller chart (Lebesgue measure, the Baire property, Laver reals and preservation theorems for forcing).* J. Symb. Logic 55, no. 3 (1990) 909-927.

[K89] A. Kamburelis: *Iterations of boolean algebras with measure.* Arch. Math. Logic 29 (1989) 21-28.

[KWe96] A. Kamburelis, B. Weglorz: *Splittings.* Arch. Math. Logic 35 (1996) 263-277.

[KaO14] S. Kamo, N. Osuga: *Many different covering numbers of Yorioka's ideals.* Arch. Math. Logic 53 (2014) 43-56.

[Kan] A. Kanamori: *The higher infinite, large cardinals in set theory from their beginnings.* 2nd edition, Springer, Berlin-Heidelberg, 2009.

[Kank13] T. Kankaanpää: *Remarks on gaps in* $\mathrm{Dense}(\mathbb{Q})/\mathrm{nwd}$. Math. Logic Quart. 59 (2013) 51-61.

[Ke] A. S. Kechris: *Classical descriptive set theory.* Springer-Verlag, New York, 1995.

[Kelley59] J. Kelley: *Measures on Boolean algebras.* Pac. J. Math. 9 (1959) 1165-1177.

[Kell08] J. Kellner: *Even more simple cardinal invariants.* Arch. Math. Logic 47 (2008) 503-515.

[KellS09] J. Kellner, S. Shelah: *Decisive creatures and large continuum.* J. Symb. Logic 74 (2009) 73-104.

[KellS12] J. Kellner, S. Shelah: *Creature forcing and large continuum: the joy of halving.* Arch. Math. Logic 51 (2012) 49-70.

[Ku] K. Kunen: *Set theory: an introduction to independence proofs.* North-Holland, Amsterdam, 1980.

[Ku84] K. Kunen: *Random and Cohen reals.* In: K.Kunen and J. Vaughan (eds.), Handbook of set-theoretic topology, North-Holland, Amsterdam, 1980, pp. 887-911.

[MS] M. Malliaris, S. Shelah: *Cofinality spectrum theorems in model theory, set theory and general topology.* `http://arxiv.org/abs/1208.5424` (2012) Accessed 9 November 2012.

[Ma91] K. Mazur: *F_σ-ideals and $\omega_1\omega_1^*$-gaps in the Boolean algebra $\mathcal{P}(\omega)/\mathcal{I}$.* Fund. Math. 138 (1991) 103-111.

[Me13a] D. A. Mejía: *Matrix iterations and Cichon's diagram,* Arch. Math. Logic 52 (2013) 261-278.

[Me13b] D. A. Mejía: *Models of some cardinal invariants with large continuum.* Kyōto daigaku sūrikaiseki kenkyūsho kōkyūroku (2013) 36-48.

[Me] D. A. Mejía: *Template iterations with non-definable ccc forcing notions.* Submitted.

[Mi81] A. Miller: *Some properties of measure and category.* Trans. Amer. Math. Soc. 266 (1981) 93-114.

[RosS] A. Roslanowski, S. Shelah: *Norms on possibilities I. Forcing with trees and creatures.* Mem. Amer. Math. Soc. 141, no. 671, 1999, xii+167 pp.

[Ro41] F. Rothberger: *Sur les familles indénombrables de suites de nombres naturels et les problèmes concernant la propríeté C.* Math. Proc. Cambridge Philos. Soc. 37 (1941) 109-126.

[S84] S. Shelah: *On cardinal invariants of the continuum.* Contemp. Math. 31 (1984) 184-207.

[S04] S. Shelah: *Two cardinal invariants of the continuum ($\mathfrak{d} < \mathfrak{a}$) and FS linearly ordered iterated forcing.* Acta Math. 192 (2004) 187-223 (publication number 700).

[So96] S. Solecki: *Analytic ideals.* Bull. Symb. Logic 2, no. 3 (1996) 339-348.

[So99] S. Solecki: *Analytic ideals and their applications.* Ann. Pure Appl. Logic 99 (1999) 51-72.

[T] S. Todorčević: *Partition Problems in Topology.* Amer. Math. Soc., Providence, 1989.

[T98] S. Todorčević: *Gaps in analytic quotients.* Fund. Math. 156 (1998) 85-97.

[Tr88] J.K. Truss: *Connections between different amoeba algebras.* Fund. Math. 130 (1988) 137-155.