Cardinal Invariants and Large Continuum

Mejía Guzmán Diego Alejandro

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論文内容の要旨

氏　名 Mejía Guzmán Diego Alejandro

専　攻 情報科学

論文題目（外国語の場合は，その和訳を併記すること。）

Cardinal Invariants and Large Continuum

大きな連続体の下での基数不変量

指導教員 Brendle Jörg

（注）2，000 字～4，000 字でまとめること。

We use and improve some forcing techniques of iterations with finite support to construct models of inequalities between classical (and some not classical) cardinal invariants with large continuum. Here, *small continuum* means that $\mathfrak{c}$, the *size of the set of real numbers*, is less than or equal to $\aleph_2$. Otherwise, we say that the continuum is *large*.

The first and more elaborated forcing technique that we look at is Shelah's theory of *template iterations*, which extends the method of construction of a finite support iteration to an iteration along an arbitrary linear order by using a well-founded class of subsets of the linear order. This technique has been developed originally to prove the consistency of $\mathfrak{d} < \mathfrak{a}$ with ZFC (the standard formal system where modern mathematics are formalized) where $\mathfrak{a}$ is the *almost disjointness number* and $\mathfrak{d}$ is the *dominating number*, thus solving a long-standing problem about cardinal invariants.

We present, in Chapter 2, a version of the theory of template iterations where non-definable forcing notions are allowed in the forcing construction. Hence, many important properties of finite support iterations with definable forcing notions become available for those template iterations and they have many consequences in the theory of cardinal invariants. As an application in this dissertation, a typical argument to force the *groupwise-density number* $\mathfrak{g}$ to be small is extended to get larger values of $\mathfrak{g}$ in models constructed from template iterations.

One problem of interest in this thesis is to obtain models where the cardinal invariants $\mathfrak{s}$, $\mathfrak{b}$ and $\mathfrak{a}$ assume pairwise different values, where $\mathfrak{s}$ is the splitting number and $\mathfrak{b}$ is the bounding number. Shelah's original models with template iterations satisfies $\mathfrak{s} = \aleph_1 < \mathfrak{b} < \mathfrak{a}$, but models of $\mathfrak{b} < \mathfrak{s} < \mathfrak{a}$ or $\mathfrak{s} < \mathfrak{b} < \mathfrak{a}$ are still unknown. By the theory of template iterations that we develop in this work, we extend Shelah's construction to get a model of $\mathfrak{s} < \kappa < \mathfrak{b} < \mathfrak{a}$ where κ is a measurable cardinal in the ground model and where the cardinal invariants $\mathfrak{s}$, $\mathfrak{b}$ and $\mathfrak{a}$ can take arbitrary regular uncountable values. For all our applications of this theory, presented in Chapter 4, it has been relevant to establish in Chapter 3 a theory of preservation properties of cardinal invariants in the context of template iterations.

The second forcing technique that we use is *matrix iterations of ccc posets*, created by Blass and Shelah to force $\mathfrak{u} < \mathfrak{d}$ with large continuum (where $\mathfrak{u}$ is the *ultrafilter number*). Many improvements of such a construction have been carried out to obtain important consistency results about cardinal invariants, in particular, the construction of ccc posets to obtain versions with large continuum of well known consistency results about cardinal invariants that had been originally obtained with small continuum.

We use matrix iterations to obtain consistency results about the cardinal invariants in *Cichon's diagram*. In the 90's, Judah, Shelah and Brendle developed finite support iteration techniques (and a theory of preservation properties of cardinal invariants as well) to construct models where many cardinal invariants in Cichon's diagram assume different values. Although these techniques work very well to separate the cardinal invariants of the left hand side of the diagram, they only succeed to force two different values for the cardinals of the right hand side. Very little is known about models for at least three such different values that can be obtained in the typical finite support iteration framework. It seems that more involved forcing constructions are needed to obtain more examples in this sense. In Chapter 5 we obtain, by including Suslin ccc posets in a matrix iteration construction, models of some instances of the possible cases where the cardinal invariants of the right hand side of Cichon's diagram take three different values. Moreover, the typical finite support iteration techniques can be included in such a construction to force, at the same time, different values for the cardinal invariants of the left hand side.

The results in Chapter 6 concern gaps in quotients by F_σ ideals on ω, whose were obtained in joint work with J. Brendle. The research on gaps in quotients by (definable) ideals on ω started approximately 100 years ago when Hausdorff constructed his (ω_1,ω_1)-gap in the quotient algebra $\mathcal{P}(\omega)/\mathrm{Fin}$ where Fin denotes the ideal of finite subsets of ω. Many years later, Rothberger constructed his $(\omega,\mathfrak{b})$-gap and also proved that there are no (ω,κ)-gaps for any cardinal $\kappa<\mathfrak{b}$.

Recently, the research about gaps has been extended to quotients by (definable) ideals in general. For example, many results have been proved in relation with gaps in $\mathcal{P}(\omega)/\mathrm{Fin}$, for example, Mazur discovered that Hausdorff's construction can be carried out to obtain a (ω_1,ω_1)-gap in any quotient by an F_σ-ideal, fact that was generalized years later by Todorcevic when he showed that, for any $\mathcal{I}$ that is either an F_σ ideal or an analytic P-ideal, there is an embedding from $\mathcal{P}(\omega)/\mathrm{Fin}$ into $\mathcal{P}(\omega)/\mathcal{I}$ that preserves gaps.

For an ideal $\mathcal{I}$, a gap in $\mathcal{I}$ of type (ω,κ) is known as a *Rothberger gap in* $\mathcal{P}(\omega)/\mathcal{I}$. The *Rothberger number of* $\mathcal{I}$, denoted by $\mathfrak{b}(\mathcal{I})$, is the least cardinal κ such that there is a (ω,κ)-gap in the quotient $\mathcal{P}(\omega)/\mathcal{I}$. Rothberger's result above implies that $\mathfrak{b}(\mathrm{Fin})=\mathfrak{b}$ and, by Todorčević's result, $\mathfrak{b}(\mathcal{I})\leq\mathfrak{b}$ whenever $\mathcal{I}$ is either an F_σ ideal or an analytic P-ideal. Moreover, it is known that $\mathfrak{b}(\mathcal{I})=\mathfrak{b}$ in the latter case, but this may not be true for F_σ-ideals in general, e.g., Brendle constructed, by an argument with eventually different functions, an (ω,ω_1)-gap in $\mathcal{P}(\omega)/\mathcal{ED}_{\mathrm{fin}}$ where $\mathcal{ED}_{\mathrm{fin}}$ is the ideal in $\omega\times\omega$ generated by the graph of finitely many functions from ω into ω. To get a better understanding about Rothberger gaps in quotients by F_σ-ideals, we investigate their Rothberger numbers.

We focus our research on the Rothberger number for *fragmented ideals*, which is a subclass of F_σ ideals that includes Fin and $\mathcal{ED}_{\mathrm{fin}}$ as well. An important subclass is considered, say, the class of *gradually fragmented ideals*. Quite recently, Hrušák conjectured, in view of Brendle's construction of an (ω,ω_1)-gap in $\mathcal{P}(\omega)/\mathcal{ED}_{\mathrm{fin}}$, that $\mathfrak{b}(\mathcal{I})=\aleph_1$ for any fragmented not gradually fragmented ideal $\mathcal{I}$. Although this conjecture has not been verified yet, we prove, by using an argument with independent functions, that it is true for a large class of fragmented not gradually fragmented ideals that includes $\mathcal{ED}_{\mathrm{fin}}$.

On the other hand, we construct a definable ccc forcing notion that is used to prove statements (and consistency results) about Rothberger gaps in quotients

by gradually fragmented ideals. We first show that, for any such ideal, there is a ccc forcing notion that destroys all the Rothberger gaps in its quotient that lies in the ground model. Moreover, the combinatorics of this forcing notion implies that $\mathfrak{b}(\mathcal{I})\geq\mathrm{add}(\mathcal{N})$ for any gradually fragmented ideal $\mathcal{I}$ where $\mathrm{add}(\mathcal{N})$ is the *additivity of the null ideal* (of sets of real numbers).

We prove many consistency results about the Rothberger numbers of quotients by fragmented ideals. The fragmented ideals that are nowhere tall have a very simple structure (they are gradually fragmented and Fin is one of them) and their Rothberger number is equal to $\mathfrak{b}$. To see that these may be the only fragmented ideals satisfying that statement, we construct a model where $\mathrm{add}(\mathcal{N})<\mathfrak{b}$ and $\mathfrak{b}(\mathcal{I})\leq\mathrm{add}(\mathcal{N})$ for any somewhere tall (i.e., not nowhere tall) fragmented ideal (so $\mathfrak{b}(\mathcal{I})=\mathrm{add}(\mathcal{N})$ for any somewhere tall gradually fragmented ideal in this model). Besides, we show that it is consistent that there are infinitely many (even continuum many by assuming the existence of a weakly inaccessible cardinal) gradually fragmented ideals with pairwise different Rothberger numbers, which is considered the main consistency result in Chapter 6. A theory of preservation of Rothberger gaps in quotients by fragmented ideals has been developed to obtain all these consistency results.

（別紙１）　　　　　　　　　　　　論文審査の結果の要旨

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論文題目	Cardinal Invariants and Large Continuum 大きな連続体の下での基数不変量

審査委員	区　分	職　名	氏　　　名
	主　査	教授	渕野　昌
	副　査	教授	南部　隆夫
	副　査	教授	ブレンドル　ヤーグ
	副　査		
	副　査		印

要　　旨

In his thesis, Diego Mejía makes a number of important contributions to iterated forcing theory in the context of large continuum. In particular, he refines and generalizes several sophisticated forcing techniques, like matrix iterations or template iterations, and proves new preservation results for such iterated forcing constructions. As applications, he obtains a number of models where several cardinal invariants of the continuum have different values.

From the set-theoretic point of view, the combinatorial structure of the set of real numbers can be described by *cardinal invariants of the continuum*, that is, cardinal numbers defined as the minimal size of a set of real numbers satisfying a certain "largeness" property. Such cardinal invariants typically take values between the first uncountable cardinal $\aleph_1$ and the cardinality of the continuum $\mathfrak{c}$. For example, consider the Baire space N^N of functions from the set of natural numbers N to itself, ordered by *eventual dominance*. The *unbounding number* $\mathfrak{b}$ is the least size of an unbounded family of functions in this ordering while the *dominating number* $\mathfrak{d}$ is the smallest cardinality of a dominating (=cofinal) family of functions. Other important cardinal invariants of the continuum are the *almost disjointness number* $\mathfrak{a}$ (the least size of an infinite maximal almost disjoint family), the *splitting number* $\mathfrak{s}$, etc. The order relationship between these cardinal invariants is known: $\mathfrak{b} \leqq \mathfrak{d}$ is easy to see, but also $\mathfrak{b} \leqq \mathfrak{a}$ and $\mathfrak{s} \leqq \mathfrak{d}$ hold.

To show that for two cardinal invariants $\mathfrak{x}$ and $\mathfrak{y}$, the relation $\mathfrak{x} \leqq \mathfrak{y}$ cannot be proved – or, put otherwise, that $\mathfrak{x} > \mathfrak{y}$ is consistent – a model for the latter inequality has to be constructed, and this is usually done by the *method of forcing*. Very roughly speaking, forcing is a technique for extending a given model of set theory (called ground model) to a larger model (called generic extension) using a partial order such that the combinatorial properties of this extension depend on the combinatorics of the partial order in the ground model. For obtaining consistency results about the values of cardinal invariants, the process of taking the generic extension usually has to be iterated; the resulting method is called *iterated forcing*. There are two classical iteration techniques for making the continuum large, *countable support iteration of proper forcing* (csi) and *finite support iteration of ccc forcing* (fsi). The former applies to a much larger class of partial orders and has been used to obtain a plethora of independence results on the order relationship of cardinal invariants. Its major drawback is that it only can be used to construct models in which the continuum has size $\aleph_2$, the second uncountable cardinal. Fsi can be used for models with large continuum, but it has other drawbacks; for example, it necessarily adds Cohen reals.

For this reason, the search for new iteration techniques with large continuum (that is, continuum larger than $\aleph_2$) has been one of the central problems in set theory in recent decades. Progress has been made through the solution of concrete problems, like the proof of the consistency of $\mathfrak{d} < \mathfrak{a}$ by Shelah for which he developed the technique of *iterations along templates*. However, many instances of this problem still remain open.

Mejía's thesis makes major contributions to this general problem, with particular emphasis on models obtained by various iteration techniques with finite support.

The first three chapters of this thesis are largely a review of known material: Chapter one contains preliminaries about forcing theory and cardinal invariants of the continuum, Chapter two outlines *iterations along templates* – roughly speaking, iterations with finite support along non-wellfounded

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linear orders L where the initial segments of the iteration are constructed by recursion on a wellfounded subfamily of subsets of L, and which are thus more complicated than standard fsi. Chapter three is about *preservation theorems* for forcing iterations: these are techniques for showing iteratively along the iteration that certain objects are not added by the forcing; typically they are used for showing that certain cardinal invariants remain small in the final generic extension. Unlike the first two chapters, this chapter already contains new results about preservation in matrix iterations (Section 3.3) and template iterations (Section 3.4), which play a crucial role for the models in the remaining chapters.

Chapter four contains several consistency results about the order relationship of cardinal invariants in the context of large continuum, which are obtained by template iterations. The main result says that it is consistent that $\aleph_1 < \mathfrak{s} < \kappa < \mathfrak{b} = \mathfrak{d} < \mathfrak{a}$ where κ is a measurable cardinal in the ground model. This consistency was known in the special case $\mathfrak{s} = \aleph_1$, by Shelah's template technique (originally this was also done from a measurable, but the measurable was later removed). For the generalization, Mejía has to insert small Mathias forcings into the template framework. This is highly non-trivial because the original template framework only works for definable (Suslin ccc) forcing and Mathias forcing is non-definable. The price one has to pay for that is that ultrapowers of partial orders become once again relevant and that therefore the measurable is needed for the construction. (The existence of a measurable cardinal is a large cardinal axiom, and the system ZFC + "there is a measurable cardinal" has strictly larger consistency strength than ZFC alone.)

In Chapter five, Mejía further develops the technique of *matrix iterations* originally invented by Blass and Shelah in the eighties. Roughly speaking, this is a method for building an iteration, also with finite support, using a two-dimensional system of partial orders, with complete embeddings between them. Traditionally it has been used mostly for models with $\mathfrak{b} < \mathfrak{s}$ and their relatives, but Mejía actually uses them for simultaneously separating many cardinal invariants on the right-hand side of Cichoń's diagram. *Cichoń's diagram* is a chart showing the order relationship between various cardinal invariants related to measure and category (e.g., the cofinalities cof($\mathbf{M}$) and cof($\mathbf{N}$) of the ideals $\mathbf{M}$ of meager sets and $\mathbf{N}$ of Lebesgue null sets, etc), as well as $\mathfrak{b}$ and $\mathfrak{d}$. Many consistency results about the order relationship of these cardinals have been proved in the past, and it has been known that the cardinals on the left-hand side of the diagram can simultaneously all assume distinct values, but for the right-hand side this is unclear. Mejía obtains models where three of these cardinals are distinct, with further different values for those on the left-hand side, more than in any model constructed in the past.

The last chapter deals with gaps in definable quotients $P(N)/I$ of the power set $P(N)$ of the natural numbers N modulo an ideal I on N. Two families A and B of subsets of N are I-orthogonal if the intersection $a \cap b$ belongs to I for any a in A and b in B. A and B are separated if there is a set c such that all members of A are contained in c modulo I, while all members of B are disjoint from c modulo I. (A,B) is an I-gap if A and B are I-orthogonal but not separated. Classical results say that the only gaps that provably exist in $P(N)$/Fin are of type $(\aleph_1,\aleph_1)$ or of type $(\aleph_0,\mathfrak{b})$. Todorčević proved that for a large class of definable ideals I, including the F_σ and the analytic P-ideals, any type of gap which exists in $P(N)$/Fin also exists in $P(N)/I$, and addressed the problem of determining the gap spectrum of such quotients. Mejía considers this problem for the class of fragmented ideals, a subclass of the F_σ ideals. On the one hand, for a large class of fragmented not gradually fragmented ideals, like the ideal of linear growth, there is a gap of type $(\aleph_0,\aleph_1)$ in the quotient. On the other hand, for gradually fragmented ideals, like the ideal of polynomial growth, the least κ such that the quotient has a $(\aleph_0,\kappa)$ gap is between the additivity number of the null ideal (one of the cardinals in Cichoń's diagram) and $\mathfrak{b}$. This number κ is called the Rothberger number $\mathfrak{b}(I)$ of the ideal I, and may be considered as a cardinal invariant. In one more important consistency result, obtained by finite support iteration, Mejía shows that there is a model in which many gradually fragmented ideals simultaneously have pairwise distinct Rothberger number.

本研究は、「行列による反復法」や「テンプレートに沿った反復法」などの強制法の洗練された有限台集合反復法をさらに発展させることによって、連続体の濃度の大きいモデルの構成へ重要な貢献しており、連続体の基数不変量の大小関係についていくつかの新しい無矛盾性の結果を得たものとして価値ある結果を集めたものとなっている。提出された論文はシステム情報学研究科科学位論文評価基準を満たしており、学位申請者の Diego Mejía は、博士（学術）の学位を得る資格があると認める。