Flat braidzel surfaces for links

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博　士　論　文

Flat braidzel surfaces for links

（絡み目の平坦組み紐状曲面）

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Preface

This thesis is written under the subject "Flat braidzel surfaces for links" submitted for the degree of Doctor at Kobe University.

A link is a disjoint union of oriented circles embedded in the 3-sphere, and a knot is a link with one component. Knot theory is an analysis of the situation of knots and links in the 3-sphere. When I was an undergraduate student in Saga University, I began to learn knot theory through Kawauchi's book [7]. Since I read this book, I have been interested in Seifert surfaces for links. A Seifert surface for a link is a compact, oriented, and connected surface which has the link as its boundary. The notion of a Seifert surface is very important in knot theory. Indeed, some of link invariants are defined by using Seifert surfaces. When I was a graduate student, I read Nakamura's paper [13]. This paper says that any link has a braidzel surface as a Seifert surface. A braidzel surface is a surface defined as a generalization of pretzel surfaces by Rudolph [15]. I was interested in a braidzel surface and have studied it. Then, I came up with the notion of a flat braidzel surface as a special kind of braidzel surfaces. I showed that any link has a flat braidzel surface as a Seifert surface, and studied relationships between flat braidzel surfaces and links. I was defined a new integral invariant of a link, named the flat braidzel genus, and have studied it. Then, I could give an upper and lower bound for the flat braidzel genus. After the entrance into the doctorial course of Kobe University, I have continued studying properties of the flat braidzel surfaces. I was defined a new integral invariant of a link, named the flat braidzel length, and have studied it. Then, I could give a lower bound for the flat braidzel length. Moreover, I could give the table of knots with the flat braidzel length five or less.

In this thesis, I present my work about flat braidzel surfaces. I hope that the readers of this thesis are interested in flat braidzel surfaces and knot theory.

Acknowledgments

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Contents

Abstract

This thesis consists of the following two topics:

Chapter 1: On flat braidzel surfaces for links

Rudolph introduced the notion of braidzel surfaces as a generalization of pretzel surfaces, and Nakamura showed that any oriented link has a braidzel surface as a Seifert surface. In this chapter, we introduce the notion of flat braidzel surfaces as a special kind of braidzel surfaces, and show that any oriented link has a flat braidzel surface. We also introduce and study a new integral invariant of a link, named the flat braidzel genus, with respect to their flat braidzel surfaces. Moreover, we give a way to calculate the number of components, the distance from proper links, the Arf invariant, and a Seifert matrix of a given link through the flat braidzel presentation. This chapter is essentially published in [9].

Chapter 2: On the flat braidzel length of links

We introduce a new integral invariant of a link, named the flat braidzel length, with respect to the flat braidzel presentation. We give a lower bound for the flat braidzel length, and determine the flat braidzel length of an infinite family of links. Moreover, we give the table of knots with the flat braidzel length five or less. This chapter is essentially published in [10].

1 On flat braidzel surfaces for links

1.1 Introduction

An *n-braidzel surface* is the surface in S^3 which consists of two disks joined by n bands $b_1, b_2, \ldots, b_n$ such that; (1) the cores of the bands form the n-string braid β, and (2) each b_i ($i = 1, 2, \ldots, n$) may be half-twisted. Rudolph introduced the notion of braidzel surfaces as a generalization of pretzel surfaces in [15] on his study of the quasipositivity for a pretzel surface.

Definition 1.1.1. The *n-flat braidzel surface* is a braidzel surface such that all b_i have no twists. The surface is denoted by $F(\beta)$.

A *link* is a disjoint union of oriented circles embedded in S^3, and a *knot* is a link with one component. We say that a link L has a *Seifert surface* S if S is a compact, orientable, and connected surface in S^3 such that $\partial S = L$. It is well-known that any link has a Seifert surface (see [2, 4, 6, 7, 8, 12, 16]). Nakamura [13] showed the following theorem.

Theorem 1.1.2. ([13]) *Any link has a braidzel surface.*

In Section 1.2, we prove the following theorem.

Theorem 1.1.3. *Any link has a flat braidzel surface.*

Example 1.1.4. The three surfaces as in Figure 1.1 are a non-orientable braidzel surface, an orientable braidzel surface, and a flat braidzel surface for the same 6_3, respectively.

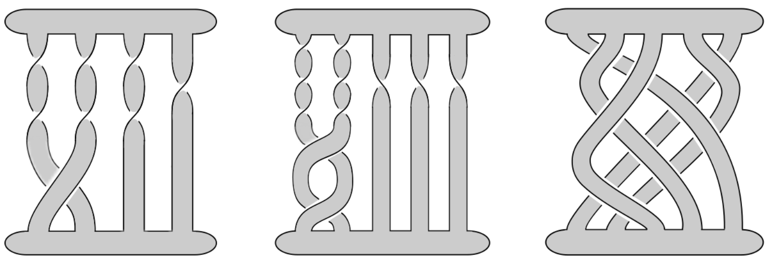


Figure 1.1: A non-orientable braidzel surface, an orientable braidzel surface, and a flat braidzel surface for 6_3.

By Theorem 1.1.3, any link can be represented by a braid β such that $L = \partial F(\beta)$. We call such a presentation of links the *flat braidzel presentation.* We remark that there exist infinitely many braids which give flat braidzel presentations of the same link.

Example 1.1.5. The knots with crossing number five or less have flat braidzel presentations as in Figure 1.2, respectively.

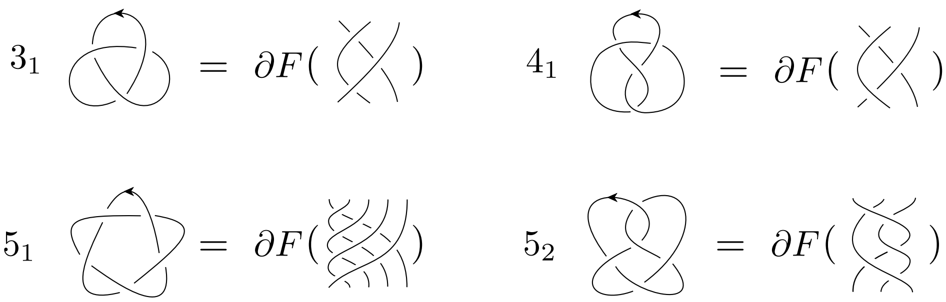


Figure 1.2: Examples of flat braidzel presentations of knots with crossing number five or less.

Nakamura [13] also introduced the *braidzel genus* which is defined as the minimal genus of all braidzel surfaces for L. We denote it by $g_b(L)$.

Definition 1.1.6. The *flat braidzel genus* of a link L is the minimal genus of all flat braidzel surfaces for L. We denote it by $g_{fb}(L)$.

By Definition 1.1.1, it follows that $g_{fb}(L) \geq g_b(L)$ for any link. We prove the following theorem in Section 1.3.

Theorem 1.1.7. *For any non-negative integer m, there exist infinitely many links L such that $g_{fb}(L) - g_b(L) = m$.*

The flat braidzel surface is useful to consider some elementary properties of links. In fact, from a braid β such that $L = \partial F(\beta)$, we can calculate the number of components (Theorem 1.4.2(1)), the distance from proper links (Theorem 1.4.2(2)), the Arf invariant (Theorem 1.4.8), and a Seifert matrix (Theorem 1.4.9) of L. We also give examples of calculation.

1.2 The flat braidzel surface

Throughout this thesis, we consider only orientable surfaces. The braidzel surface is orientable if and only if all b_i's have odd or even half-twists. Let B be a braidzel surface such that all b_i have odd half-twists. We turn the top disk of B right-handed π radians with fixing the other disk, around the straight line passing through centers of two disks as the axis. We call the operation repeating this $2k$ times the *k-wrenches* for the braidzel surface (see

Figure 1.3). Then, we obtain the braidzel surface such that all b_i have even half-twists. Therefore, we assume that each b_i of a braidzel surface has $|a_i|$ full-twists. If a_i is positive (resp. negative), then the full-twist of the band is right-handed (resp. left-handed). We denote it by $B(\beta; a_1, a_2, \ldots, a_n)$.

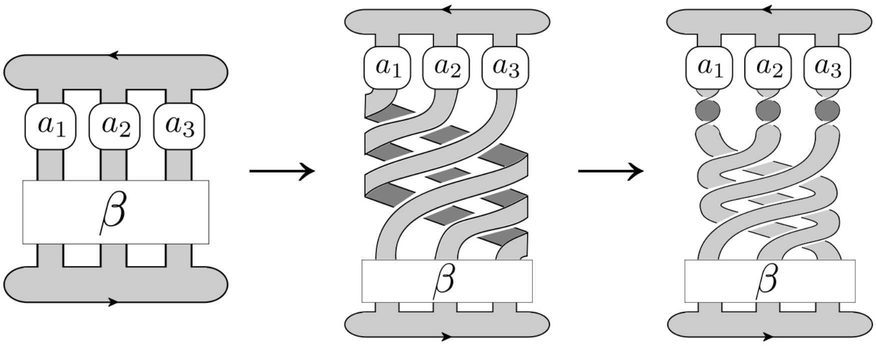


Figure 1.3: The 1-wrench for a 3-braidzel surface.

Let D and D' be the two disks for $B(\beta; a_1, a_2, \ldots, a_n)$, and $b_1, b_2, \ldots, b_n$ the bands attached to D from the left to the right (see Figure 1.4). Let α_i $(i = 1, 2, \ldots, n)$ be the arc of $\partial D \setminus (b_1 \cup b_2 \cup \cdots \cup b_n)$ between b_i and b_{i+1}. Similarly, let $b_{i(1)}, b_{i(2)}, \ldots, b_{i(n)}$ be the bands attached to D' from the left to the right, and α'_j $(j = 1, 2, \ldots, n)$ the arc of $\partial D' \setminus (b_{i(1)} \cup b_{i(2)} \cup \cdots \cup b_{i(n)})$ between $b_{i(j)}$ and $b_{i(j+1)}$. We regard the indices of b, α, and α' as elements of $\mathbb{Z}_n$. Let σ_β be the element of the symmetric group Σ_n of degree n associated with β. We remark that $i(\sigma_\beta(j)) = j$.

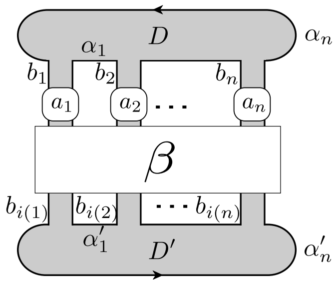


Figure 1.4: The name of each part of $B(\beta; a_1, a_2, \ldots, a_n)$.

Proof of Theorem 1.1.3. Let L be a link. By Theorem 1.1.2, L has a braidzel surface $B = B(\beta; a_1, a_2, \ldots, a_n)$. We perform $(-a_1)$-wrenches for B. Then, we get another braidzel surface $B(\Delta_n^{-2a_1}\beta; 0, a_2-a_1, \ldots, a_n-a_1)$ for L, where $\Delta_n = (\sigma_1\sigma_2\ldots\sigma_n)(\sigma_1\sigma_2\cdots\sigma_{n-1})\cdots(\sigma_1\sigma_2)\sigma_1$.

The proof is given by induction on $\varphi = |a_2 - a_1| + \cdots + |a_n - a_1|$. If $\varphi = 0$, then B is a flat braidzel surface. Assume that Theorem 1.1.3 holds when $\varphi < m$. If $\varphi = m$, then B has both a flat band and a twisted band. There exists i such that b_i is a flat band and b_{i+1} is a twisted band (see Figure 1.5(a)).

First, we replace the top full-twist of the band b_{i+1} with two bands b' and b'' as in Figure 1.5(b). As for a negative full-twist, we switch the crossing of b' and b''. We remark that this operation preserves the link type of its boundary. Second, we slide each root of b' and b'' connected in the boundary on the right side of b_{i+1} to α_{i+1}, and each root of b' and b'' connected in the boundary on the left side of b_{i+1} to α_i (see Figure 1.5(c)). Finally, we slide each root of b' and b'' connected on α_i to $\alpha'_{\sigma_\beta(i)}$ along b_i (see Figure 1.5(d)). Then, we get an $(n+2)$-braidzel surface which satisfies $\varphi = m - 1 < m$. Hence, from the assumption, the proof is complete. □

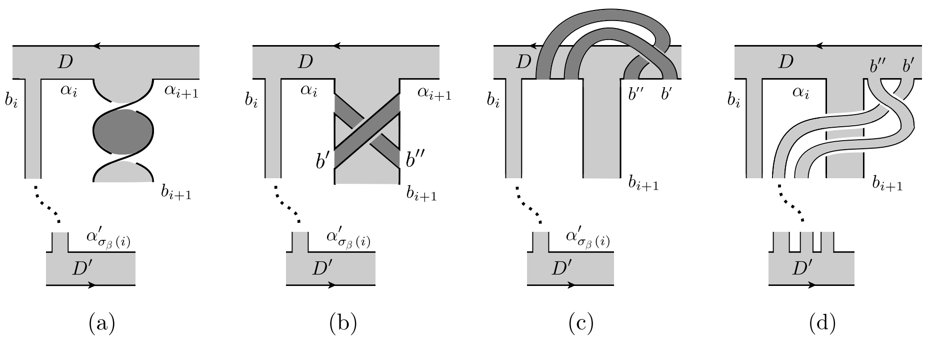


Figure 1.5: The deformation reducing the number of full-twists.

1.3 The flat braidzel genus and the braidzel genus

In this section, we study relationships between the flat braidzel genus $g_{fb}(L)$ and the braidzel genus $g_b(L)$. We easily see that if an r-component link L has an n-flat braidzel surface F, then we have

$$g_{fb}(L) \leq g(F) = \frac{n-r}{2},$$

where $g(F)$ is the genus of F.

For a braidzel surface $B(\beta; a_1, a_2, \dots, a_n)$, we define $i(B)$ as

$$i(B) = \min\{\#\{i|a_i \text{ is even}\}, \#\{i|a_i \text{ is odd}\}\},$$

where $\#A$ is the cardinality of a set A.

Theorem 1.3.1. *For any link L, we have*

$$g_{fb}(L) = \min\{g(B) + i(B)|B \text{ is a braidzel surface for } L\}.$$

Proof. Let F be a flat braidzel surface such that $g(F) = g_{fb}(L)$. Since $i(F) = 0$, we have

$$g_{fb}(L) = g(F) + i(F) \geq \min\{g(B) + i(B)\}.$$

To prove the reverse inequality, we use an isotopic deformation as in Figure 1.6. Let $B = B(\beta; a_1, a_2, \dots, a_n)$ be a braidzel surface for L. First, we deform two full-twists of a band b_i of B to the shape of two clasps. Second, we push one clasp out around the disk D, and slide the other clasp to the left side of D. Finally, we take down a part of b_i over D through the front of D. Then, we obtain a braidzel surface $B(\beta'; a_1, a_2, \dots, a_i - 2, \dots, a_n)$ for L. By repeating this deformation for all b_i, we obtain a braidzel surface $B(\beta''; \bar{a_1}, \bar{a_2}, \dots, \bar{a_n})$ from B. Here, $\bar{a_i}$ means 0 if a_i is even, 1 if a_i is odd.

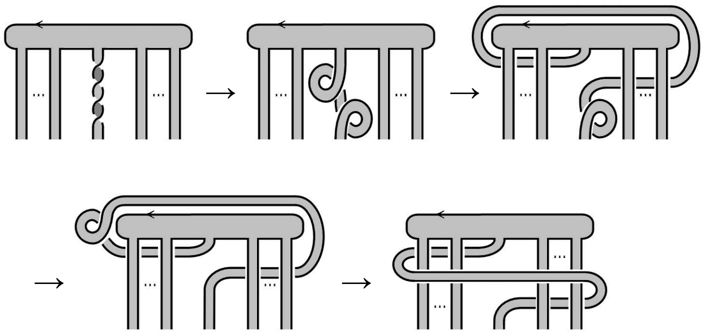


Figure 1.6: The deformation vanishing two full-twists of a band, preserving the genus of B.

This deformation changes a band which has even (resp. odd) full-twists to a flat band (resp. a band which has one full-twist). Then, the sum of the number of full-twists of all bands is equal to the number of odd a_i's, i.e., $\#\{i|a_i \text{ is odd}\}$.

Case 1. If $\#\{i|a_i \text{ is even}\} \geq \#\{i|a_i \text{ is odd}\}$, then we deform the braidzel surface to a flat braidzel surface as in the proof of Theorem 1.1.3. Then the number of bands increases by two for one full-twist, and the genus of B increases by one for one full-twist. Hence, we have

$$g_{fb}(L) \leq \min\{g(B) + i(B) | B \text{ is a braidzel surface for } L\}.$$

Case 2. If $\#\{i | a_i \text{ is even}\} \leq \#\{i | a_i \text{ is odd}\}$, then we perform a 1-wrench for B to obtain the braidzel surface $B(\Delta_n^2\beta; a_1+1, a_2+1, \ldots, a_n+1)$. Since the operation reverses the parity of each a_i, Case 2 is reduced to Case 1. □

From Theorem 1.3.1, we have the following corollary.

Corollary 1.3.2. *For any r-component link, we have*

$$g_{fb}(L) \leq 2g_b(L) + \left\lceil \frac{r}{2} - 1 \right\rceil.$$

Here, $\lceil x \rceil$ is the least integer not less than x.

Proof. Let B be an n-braidzel surface for L such that $g(B) = g_b(L)$. By definition, $i(B) \leq n/2$. If $i(B) = n/2$, then the deformation sliding the root of the most left band along the boundary of the disk D' of B makes $i(B) < n/2$ as in Figure 1.7. Therefore, by $g(B) = g_b(L) = (n-r)/2$ and Theorem 1.3.1, we have

$$g_{fb}(L) \leq g(B) + i(B) < g_b(L) + \frac{n}{2} = 2g_b(L) + \frac{r}{2}.$$

□

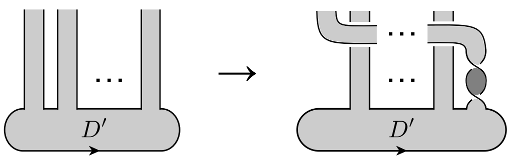


Figure 1.7: The deformation sliding the root of most left band along the boundary of the disk.

Definition 1.3.3. For an r-component link $L = K_1 \cup K_2 \cup \cdots \cup K_r$, we call the value

$$\#\{i | \text{lk}(K_i, L \setminus K_i) \equiv 1 \pmod 2\}/2$$

the *distance from proper links*, denoted by $d(L)$. If $d(L) = 0$, then L is called a *proper link*. Here, a knot is regarded as a proper link.

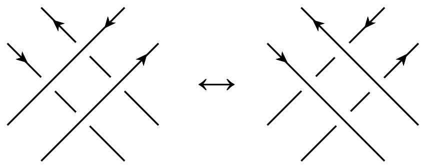


Figure 1.8: The pass move.

The local move of a link as in Figure 1.8 is called a *pass move*, and two links L and L' are *pass equivalent* if we can obtain L' from L by applying a finite sequence of pass moves [4].

For the pass move, we have the following lemma (refer to [11, Appendix]).

Lemma 1.3.4. *For any component K of a link L, the value* $\mathrm{lk}(K, L \backslash K) \pmod 2$ *does not change by a pass move.*

Theorem 1.3.5. *For any link, we have*

$$g_{fb}(L) \geq d(L) - 1.$$

Proof. Let $F(\beta)$ be an n-flat braidzel surface for L such that $g(F(\beta)) = g_{fb}(L)$. We denote each arc obtained by removing all bands of $F(\beta)$ from L by $\alpha_1, \alpha_2, \dots, \alpha_n$ and $\alpha'_1, \alpha'_2, \dots, \alpha'_n$ as defined at Section 1.2 as in Figure 1.3. Moreover, let K be a component of L such that K does not contain α_n and α'_n. Assume that K contains only α_i $(i = 1, 2, \dots, n-1)$ among $\alpha_1, \alpha_2, \dots, \alpha_{n-1}$. Then, considering a height function by vertical direction of Figure 1.3, we can observe the number of maximal points of K is one, hence the number of minimal points of K is also one. In other words, K contains only α'_j $(j = 1, 2, \dots, n-1)$ among $\alpha'_1, \alpha'_2, \dots, \alpha'_{n-1}$. Hence, K can be split from L by a finite sequence of pass moves. Therefore, by Lemma 1.3.4, we have $\mathrm{lk}(K, L \setminus K) \equiv 0 \pmod 2$. By the contraposition, if $\mathrm{lk}(K, L \setminus K) \equiv 1 \pmod 2$, then K contains more than one arc among $\alpha_1, \alpha_2, \dots, \alpha_{n-1}$. Let x be the number of components K of L such that K does not contain α_n, α'_n and $\mathrm{lk}(K, L \setminus K) \equiv 1 \pmod 2$. Let r be the number of components of L. Then, x components contain at least two arcs among $\alpha_1, \alpha_2, \dots, \alpha_n$, and the other $r - x$ components contain at least one arc. Hence, we have

$$n \geq 2x + (r - x) = x + r.$$

On the other hand, the number of all components of L which contain α_n or α'_n is one or two, and the number of all components K of L such that $\mathrm{lk}(K, L \setminus K) \equiv 1 \pmod 2$ is $2d(L)$ by the definition of $d(L)$. Hence, we have

$$2d(L) \geq x \geq 2d(L) - 2.$$

Consequently, we have

$$n \geq 2d(L) - 2 + r,$$

and hence

$$g_{fb}(L) = g(F(\beta)) = \frac{n-r}{2} \geq d(L) - 1.$$

□

From the above mentioned results, we obtain Theorem 1.1.7.

Proof of Theorem 1.1.7. We consider the $(2m+2)$-component link $L_{k,m}$ as in Figure 1.9. Here, the box in Figure 1.9 means $2k+1$ full-twists. Since $B(1; 2k{+}1, 0, 1, 0, \ldots, 1, 0)$ is a braidzel surface for $L_{k,m}$, we can see $g_b(L_{k,m}) = 0$. From Corollary 1.3.2, we have

$$\begin{aligned} g_{fb}(L_{k,m}) &\leq 2g_b(L_{k,m}) + \left\lceil \frac{r}{2} - 1 \right\rceil \\ &= 2 \cdot 0 + \frac{2m+2}{2} - 1 \\ &= m. \end{aligned}$$

On the other hand, since $d(L_{k,m}) = m+1$, Theorem 1.3.5 implies

$$g_{fb}(L_{k,m}) \geq d(L_{k,m}) - 1 = m.$$

Therefore, we have

$$g_{fb}(L_{k,m}) - g_b(L_{k,m}) = m - 0 = m.$$

□

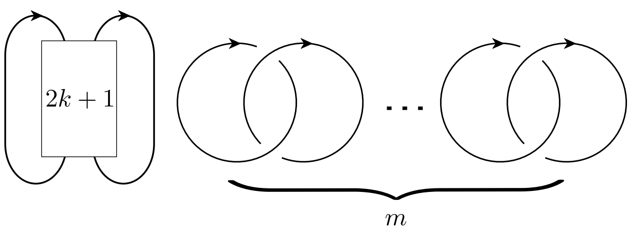


Figure 1.9: $L_{k,m}$.

1.4 Properties of a link and the flat braidzel presentation

1.4.1 The number of components and the distance from proper links through the flat braidzel presentation

Let $F(\beta_0)$ be an $(n-2)$-flat braidzel surface for an r-component link $L = K_1 \cup K_2 \cup \cdots \cup K_r$. Then, the surface as in Figure 1.10 is also a flat braidzel surface for L. Let $F(\beta)$ be this n-flat braidzel surface.

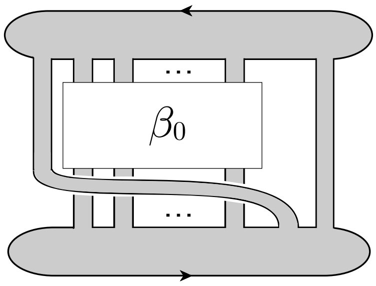


Figure 1.10: $F(\beta)$.

Definition 1.4.1. The bijective map $\rho_\beta : \mathbb{Z}_n \to \mathbb{Z}_n$ is defined as follows:

$$\rho_\beta(i) = \sigma_\beta^{-1}(\sigma_\beta(i+1) - 1).$$

We say that $i, j \in \mathbb{Z}_n$ are ρ_β *equivalent* if there exists an integer k such that $\rho_\beta^k(i) = j$. We denoted it by $i \sim_{\rho_\beta} j$.

Then the following theorem holds.

Theorem 1.4.2. *For an r-component link L which has an n-flat braidzel surface $F(\beta)$ as in* Figure 1.10*, we have*

$$\begin{aligned} &(1) \quad r = \#(\mathbb{Z}_n/\sim_{\rho_\beta}), \\ &(2) \quad d(L) = \#\{[i] \in \mathbb{Z}_n/\sim_{\rho_\beta} \mid \#[i] \equiv 0 \pmod 2\}/2, \end{aligned}$$

where $\#[i]$ is the number of elements in the ρ_β equivalence class $[i]$.

Proof of Theorem 1.4.2(1). The component of L which contains the arc α_i contains the left side boundary of band b_{i+1} and the arc $\alpha'_{\sigma_\beta(i+1)-1}$. Moreover, the component contains the right side boundary of band $b_{\sigma_\beta^{-1}(\sigma_\beta(i+1)-1)}$ and the arc $\alpha_{\sigma_\beta^{-1}(\sigma_\beta(i+1)-1)} = \alpha_{\rho_\beta(i)}$. □

To prove Theorem 1.4.2(2), we consider a projection image to a plane of L as in Figure 1.10. Let p_i be the projection of the component K_i of L which contains the arc α_i and $c(p_i)$ the number of self-intersection points of p_i. We remark that, by the proof of Theorem 1.4.2(1), each ρ_β equivalence class $[i] \in \mathbb{Z}_n/\sim_{\rho_\beta}$ is associated with K_i and p_i. Then, we have the following lemma.

Lemma 1.4.3. *For each $i = 1, 2, \dots, r$, it holds that*

$$\mathrm{lk}(K_i, L \setminus K_i) \equiv c(p_i) \pmod 2.$$

Proof. This lemma is proved by observing the contribution for $c(p_i)$ and $\mathrm{lk}(K_i, L \setminus K_i)$ of one crossing of bands. □

Moreover, we also have the following lemma.

Lemma 1.4.4. *For each ρ_β equivalence class $[i] \in \mathbb{Z}_n/\sim_{\rho_\beta}$, it holds that*

$$c(p_i) \equiv \#[i] + 1 \pmod 2.$$

Proof. We perform the smoothing for all the self-intersection points of p_i between two upward arcs and between two downward arcs. Then, we have $\#[i]$ closed curves. Since the number of the self-intersection points between an upward arc and a downward arc is even, the parity of $c(p_i)$ is equal to that of the number of the self-intersection points of p_i between two upward arcs and between two downward arcs. Moreover, since each smoothing changes the parity of the number of curves, the parity of the number of the self-intersection points of p_i between two upward arcs and between two downward arcs is equal to that of $\#[i] + 1$. □

Proof of Theorem 1.4.2(2). By Lemma 1.4.3 and Lemma 1.4.4, we have

$$\begin{aligned} 2d(L) &= \#\{[i] | \mathrm{lk}(K_i, L \setminus K_i) \equiv 1 \pmod 2\} \\ &= \#\{[i] | c(p_i) \equiv 1 \pmod 2\} \\ &= \#\{[i] | \#[i] \equiv 0 \pmod 2\}. \end{aligned}$$

□

Remark 1.4.5. By Theorem 1.4.2(2), we give an alternative proof of Theorem 1.3.5.

Let $F(\beta_0)$ be a flat braidzel surface for L such that $g(F(\beta_0)) = g_{fb}(L)$. Then by Theorem 1.4.2(2), we have

$$\begin{aligned} 2d(L) &= \#\{[i] | \#[i] \equiv 0 \pmod 2\}, \\ r - 2d(L) &= \#\{[i] | \#[i] \equiv 1 \pmod 2\}. \end{aligned}$$

Moreover, if $\#[i] \equiv 0 \pmod 2$, then $\#[i] \geq 2$, and if $\#[i] \equiv 1 \pmod 2$, then $\#[i] \geq 1$. Thus, we have

$$\begin{aligned} n &= \sum_{[i]\in\mathbb{Z}_n/\sim_{\rho_\beta}} \#[i] \\ &\geq 2 \cdot 2d(L) + (r - 2d(L)) \\ &= 2d(L) + r. \end{aligned}$$

Hence, we have

$$g_{fb}(L) = g(F(\beta_0)) = \frac{(n-2)-r}{2} \geq d(L) - 1.$$

1.4.2 The Arf invariant through the flat braidzel presentation.

Let L be a proper link, and S a Seifert surface for L. We consider the first homology group $H_1(S;\mathbb{Z}_2)$ of S with coefficients in $\mathbb{Z}_2$, and a function $q : H_1(S;\mathbb{Z}_2) \to \mathbb{Z}_2$ defined as $q(x) = \mathrm{lk}(l_x^+, l_x) \pmod 2$. Here, l_x is a loop on S which represents x in $H_1(S;\mathbb{Z}_2)$, and l_x^+ is a loop which is raised l_x from S to the positive direction. We define a function λ as follows:

$$\lambda(q) = \sum_{x\in H_1(S;\mathbb{Z}_2)} (-1)^{q(x)}.$$

The *Arf invariant* is an invariant of a proper link L, denoted by $\mathrm{Arf}(L)$. The Arf invariant takes a value in of $\mathbb{Z}_2$. It is known that the following proposition determines the value of $\mathrm{Arf}(L)$.

Proposition 1.4.6. ([6, Chapter 5]) *For a proper link L, it holds that $\lambda(q) > 0$ if and only if* $\mathrm{Arf}(L) = 0$, *and $\lambda(q) < 0$ if and only if* $\mathrm{Arf}(L) = 1$.

Definition 1.4.7. Let Σ_n be the symmetric group of degree n, and M a subset of $I_n = \{1, 2, \ldots, n\}$. Then, we define a map $\lambda : \Sigma_n \to \mathbb{Z}$ as

$$\lambda(\sigma) = \sum_{\substack{M\subset I_n \\ \#M:\text{even}}} (-1)^{\mathrm{sgn}(\sigma|_M)},$$

where

$$\mathrm{sgn}(\sigma|_M) = \#\{(i,j) \in M \times M | i < j,\ \sigma(i) > \sigma(j)\}.$$

Theorem 1.4.8. *For a proper link $L = \partial F(\beta)$, it holds that $\lambda(\sigma_\beta) > 0$ if and only if* $\mathrm{Arf}(L) = 0$, *and $\lambda(\sigma_\beta) < 0$ if and only if* $\mathrm{Arf}(L) = 1$.

Proof. This proof is inspired by Yasuhara's advice. We consider the Seifert surface S added two bands to $F(\beta)$ as in Figure 1.11. Let $\tilde{b}$ be the one of the bands which spans D and D'. We choose simple loops $l_{x_1}, l_{x_2}, \ldots, l_{x_n}, l_y$ presenting a basis $x_1, x_2, \ldots, x_n, y$ of $H_1(S;\mathbb{Z}_2)$ as in Figure 1.11. Here, we take loops l_{x_i} in band $\tilde{b}$ as the following. Let $\mathrm{p}_1, \mathrm{p}_2, \ldots, \mathrm{p}_n$ be points from the right to the left in $\tilde{b} \cap D$. Similarly, let $\mathrm{p}'_1, \mathrm{p}'_2, \ldots, \mathrm{p}'_n$ be points from the right to the left in $\tilde{b} \cap D'$. Then, we take l_{x_i} such that l_{x_i} passes on the segment $\mathrm{p}'_{\sigma_\beta(i)}$ to p_i.

For any element x of $H_1(S;\mathbb{Z}_2)$, we put $x = a_1x_1 + a_2x_2 + \cdots + a_nx_n + by$ $(a_i, b \in \mathbb{Z}_2)$ and $\tilde{x} = x - by$.

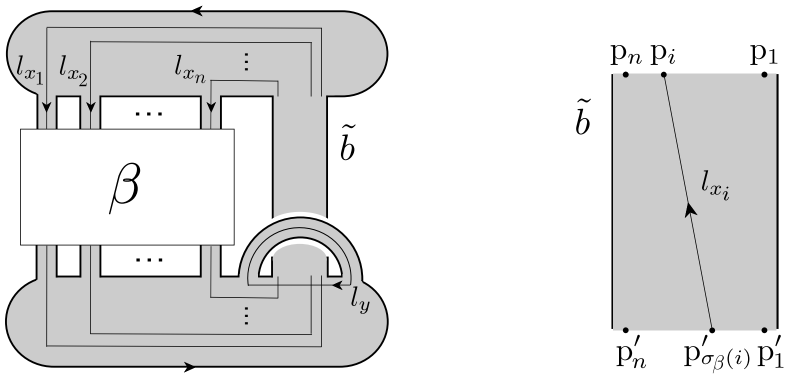


Figure 1.11: Simple loops $l_{x_1}, l_{x_2}, \ldots, l_{x_n}, l_y$ presenting a basis $x_1, x_2, \ldots, x_n, y$ of $H_1(S;\mathbb{Z}_2)$.

By $q(y) = 0$, we have $q(x) = q(\tilde{x}) + \tilde{x} \cdot (by)$. Moreover, by $q(x_i) = 0$ and $x_i \cdot y = 1$, we have

$$\begin{aligned} q(\tilde{x}) &= \sum_{j=2}^{n}\sum_{i=1}^{j-1} a_ia_j(x_i \cdot x_j) \pmod 2, \text{and} \\ \tilde{x} \cdot (by) &= (a_1 + a_2 + \cdots + a_n)b \pmod 2. \end{aligned}$$

Hence, we have

$$q(x) = \sum_{j=2}^{n}\sum_{i=1}^{j-1} a_ia_j(x_i \cdot x_j) + (a_1 + a_2 + \cdots + a_n)b \pmod 2.$$

Therefore, we have

$$\lambda(q) = 2 \sum_{a_1+\cdots+a_n=0} (-1)^{q(\tilde{x})}.$$

By $x_i \cdot x_j = \mathrm{sgn}(\sigma_\beta|_{\{i,j\}})$, we have

$$\begin{aligned} q(\tilde{x}) &= \sum_{j=2}^{n}\sum_{i=1}^{j-1} a_i a_j (x_i \cdot x_j) \pmod 2 \\ &= \sum_{j=2}^{n}\sum_{i=1}^{j-1} a_i a_j \mathrm{sgn}(\sigma_\beta|_{\{i,j\}}) \pmod 2 \\ &= \mathrm{sgn}(\sigma_\beta|_{M_x}) \pmod 2, \end{aligned}$$

where $M_x = \{i | a_i = 1\}$. Hence, we have

$$\begin{aligned} \frac{\lambda(q)}{2} &= \sum_{a_1+\cdots+a_n=0} (-1)^{q(\tilde{x})} \\ &= \sum_{a_1+\cdots+a_n=0} (-1)^{\mathrm{sgn}(\sigma_\beta|_{M_x})} \\ &= \sum_{\substack{M\subset I_n \\ \#M:\mathrm{even}}} (-1)^{\mathrm{sgn}(\sigma_\beta|_M)} \\ &= \lambda(\sigma_\beta). \end{aligned}$$

By Proposition 1.4.6, the proof is completed. □

1.4.3 The Seifert matrix through the flat braidzel presentation.

Let L be an r-component link, and S a Seifert surface for L. Let $l_1, l_2, \ldots, l_m$ ($m = 2g(S) + r - 1$) be loops which represent a basis of the first homology group of S. The *Seifert matrix* $V = (v_{ij})$ $(i, j = 1, 2, \ldots, m)$ of L is a square matrix of order m defined as $v_{ij} = \mathrm{lk}(l_i^+, l_j)$.

Let β be an n-string braid, and L a link such that $L = \partial F(\beta)$. We define a square matrix of order n, $\tilde{V} = (\tilde{v}_{ij})$, as follows:

$$\tilde{v}_{ij} = \begin{cases} \lceil \frac{w_{ij}}{2} \rceil & (i < j) \\ \lfloor \frac{w_{ij}}{2} \rfloor & (i > j) \\ 0 & (i = j) \end{cases},$$

where all strings s_i $(i = 1, 2, \ldots, n)$ of β are oriented from top to bottom, w_{ij} $(i, j = 1, 2, \ldots, n, i \neq j)$ is the sum of signs of all crossing points of s_i and s_j, and $\lceil x \rceil$ (resp. $\lfloor x \rfloor$) is the least (resp. greatest) integer not less (resp. greater) than x.

Theorem 1.4.9. *A Seifert matrix V of L is a square matrix of order $n + 1$ as follows:*

$$V = \left(\begin{array}{ccc|c} & & & 1 \\ & \tilde{V} & & \vdots \\ & & & 1 \\ \hline 0 & \cdots & 0 & 0 \end{array}\right).$$

Proof. We consider the Seifert surface S, and choose simple loops presenting a basis of the first homology of S as in Figure 1.11. For $i, j = 1, 2, \ldots, n$, we have $\mathrm{lk}(l^+_{x_i}, l_{x_j}) = \tilde{v}_{ij}, \mathrm{lk}(l^+_{x_i}, l_y) = 1, \mathrm{lk}(l^+_y, l_{x_i}) = 0$, and $\mathrm{lk}(l^+_y, l_y) = 0$. Therefore, we see that V is a Seifert matrix of L. □

1.4.4 Calculation examples.

Example 1 Let β_0 be the 3-string braid $\sigma_1\sigma_2\sigma_1$, and L the r-component link such that $L = \partial F(\beta_0)$. By Definition 1.4.1, we have

$$\rho_\beta(1) = \sigma_\beta^{-1}(\sigma_\beta(1+1)-1) = \sigma_\beta^{-1}(\sigma_\beta(2)-1) = \sigma_\beta^{-1}(3-1) = \sigma_\beta^{-1}(2) = 3,$$
$$\rho_\beta(3) = \sigma_\beta^{-1}(\sigma_\beta(3+1)-1) = \sigma_\beta^{-1}(\sigma_\beta(4)-1) = \sigma_\beta^{-1}(1-1) = \sigma_\beta^{-1}(5) = 5,$$
$$\rho_\beta(5) = \sigma_\beta^{-1}(\sigma_\beta(5+1)-1) = \sigma_\beta^{-1}(\sigma_\beta(1)-1) = \sigma_\beta^{-1}(4-1) = \sigma_\beta^{-1}(3) = 2,$$
$$\rho_\beta(2) = \sigma_\beta^{-1}(\sigma_\beta(2+1)-1) = \sigma_\beta^{-1}(\sigma_\beta(3)-1) = \sigma_\beta^{-1}(2-1) = \sigma_\beta^{-1}(1) = 4,$$
$$\rho_\beta(4) = \sigma_\beta^{-1}(\sigma_\beta(4+1)-1) = \sigma_\beta^{-1}(\sigma_\beta(5)-1) = \sigma_\beta^{-1}(5-1) = \sigma_\beta^{-1}(4) = 1.$$

Hence $1 \sim_{\rho_\beta} 3 \sim_{\rho_\beta} 5 \sim_{\rho_\beta} 2 \sim_{\rho_\beta} 4$, i.e. $[1] = \{1, 2, 3, 4, 5\}$. By Theorem 1.4.2, we have $r = 1$ and $d(L) = 0$.

Moreover, by Definition 1.4.7, we have

$$\begin{aligned} \lambda(\sigma_{\beta_0}) &= \mathrm{sgn}(\sigma_{\beta_0}|_{\{1,2\}}) + \mathrm{sgn}(\sigma_{\beta_0}|_{\{1,3\}}) + \mathrm{sgn}(\sigma_{\beta_0}|_{\{2,3\}}) \\ &= (-1) + (-1) + (-1) \\ &= -3. \end{aligned}$$

Therefore, by Theorem 1.4.8, we have $\mathrm{Arf}(L) = 1$.

Moreover, we have $w_{12} = w_{13} = w_{23} = 1$. Therefore, by Theorem 1.4.9, we have a Seifert matrix V of L as follows.

$$V = \begin{pmatrix} 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

Example 2 Let β_0 be the 4-string braid $\sigma_1\sigma_2^{-1}\sigma_3^{-1}\sigma_1^2\sigma_2$, and L the r-component link such that $L = \partial F(\beta_0)$. By Definition 1.4.1, we have

$$\begin{aligned}
\rho_\beta(1) &= \sigma_\beta^{-1}(\sigma_\beta(1+1)-1) = \sigma_\beta^{-1}(\sigma_\beta(2)-1) = \sigma_\beta^{-1}(4-1) = \sigma_\beta^{-1}(3) = 4,\\
\rho_\beta(4) &= \sigma_\beta^{-1}(\sigma_\beta(4+1)-1) = \sigma_\beta^{-1}(\sigma_\beta(5)-1) = \sigma_\beta^{-1}(2-1) = \sigma_\beta^{-1}(1) = 3,\\
\rho_\beta(3) &= \sigma_\beta^{-1}(\sigma_\beta(3+1)-1) = \sigma_\beta^{-1}(\sigma_\beta(4)-1) = \sigma_\beta^{-1}(3-1) = \sigma_\beta^{-1}(2) = 5,\\
\rho_\beta(5) &= \sigma_\beta^{-1}(\sigma_\beta(5+1)-1) = \sigma_\beta^{-1}(\sigma_\beta(6)-1) = \sigma_\beta^{-1}(6-1) = \sigma_\beta^{-1}(5) = 1,\\
\rho_\beta(2) &= \sigma_\beta^{-1}(\sigma_\beta(2+1)-1) = \sigma_\beta^{-1}(\sigma_\beta(3)-1) = \sigma_\beta^{-1}(1-1) = \sigma_\beta^{-1}(6) = 6,\\
\rho_\beta(6) &= \sigma_\beta^{-1}(\sigma_\beta(6+1)-1) = \sigma_\beta^{-1}(\sigma_\beta(1)-1) = \sigma_\beta^{-1}(5-1) = \sigma_\beta^{-1}(4) = 2.
\end{aligned}$$

Hence, we have $[1] = \{1,3,4,5\}, [2] = \{2,6\}$. By Theorem 1.4.2, we have $r = 2$ and $d(L) = 1$. Since L is not a proper link, $\mathrm{Arf}(L)$ is not defined.

Moreover, we have $w_{12} = 1, w_{13} = -1, w_{14} = -1, w_{23} = 2, w_{24} = 0, w_{34} = 1$. Therefore, by Theorem 1.4.9, we have a Seifert matrix V of L as follows.

$$V = \begin{pmatrix} 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 1 \\ -1 & 1 & 0 & 1 & 1 \\ -1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

2 On the flat braidzel length of links

2.1 Introduction

We define the *length* of an n-string braid $\beta = \sigma_{i_1}^{p_1}\sigma_{i_2}^{p_2}\cdots\sigma_{i_k}^{p_k}(i_j = 1, 2, \ldots, n-1,\ p_j \in \mathbb{Z})$ as $|p_1| + |p_2| + \cdots + |p_k|$. We denote it by $l(\beta)$.

Definition 2.1.1. The *flat braidzel length* of a link L is the minimal length of all braids which give flat braidzel presentations of L, and denoted by $l_{fb}(L)$, that is,

$$l_{fb}(L) = \min\{l(\beta)|\beta : \text{a braid such that } L = \partial F(\beta)\}.$$

The *canonical genus* of a link L is defined as the minimal genus of all Seifert surfaces obtained by Seifert's algorithm, denoted by $g_c(L)$ [5]. Nakamura [13] showed that, for any link, the canonical genus of the link is greater than or equal to the braidzel genus of the link. We can give a lower bound for the flat braidzel length by the following theorem.

Theorem 2.1.2. *Let L be a non-split r-component link, then we have*

$$l_{fb}(L) \geq g_c(L) + g_{fb}(L) + r - 1.$$

We denote by $\text{span}\Delta_L(t)$ the span of the Alexander polynomial $\Delta_L(t)$. Here, for convenience, we define $\text{span}0 = 0$. Then, we have the following corollary.

Corollary 2.1.3. *For any link L, we have $l_{fb}(L) \geq \text{span}\Delta_L(t)$*

In Section 2.2, we prove Theorem 2.1.2 and Corollary 2.1.3. In Section 2.3, we determine the flat braidzel length of an infinite family of links, including the connected sum of the $(2, q)$-torus knot and its mirror image. In Section 2.4, we prove the propositions given in Section 2.3. In Section 2.5, we give the table of knots with the flat braidzel length five or less.

2.2 Proofs of Theorem 2.1.2 and Corollary 2.1.3

We prove Theorem 2.1.2.

Proof of Theorem 2.1.2. Let β be an n-string braid such that $L = \partial F(\beta)$ and $l_{fb}(L) = l(\beta)$. Let $S(\beta)$ be the canonical Seifert surface obtained by applying Seifert's algorithm to the standard diagram of $\partial F(\beta)$. Here, the standard diagram of $\partial F(\beta)$ is the diagram with $4l(\beta)$ crossings. Around the four crossing points yielded by a crossing of bands, Seifert's algorithm is the operation as in Figure 2.1. Here, an arc which spans two Seifert circles represents a half-twist band. For example, if β is the 3-string braid $\sigma_1\sigma_2\sigma_1^{-1}$, then we have the standard diagram of $\partial F(\beta)$ as in Figure 2.2(a) and $S(\beta)$ as in Figure 2.2(b).

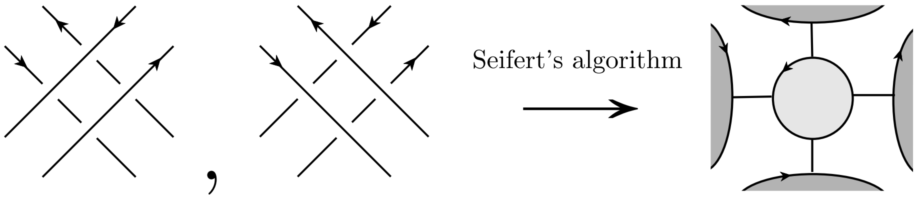
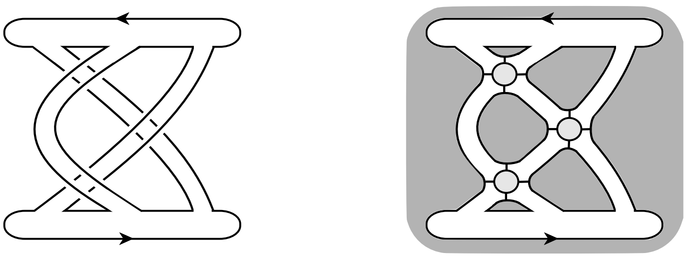


Figure 2.1: Seifert's algorithm around the four crossing points.



(a) (b)

Figure 2.2: The standard diagram of $\partial F(\sigma_1\sigma_2\sigma_1^{-1})$ and the canonical Seifert surface $S(\sigma_1\sigma_2\sigma_1^{-1})$.

Then, the number of Seifert disks of $S(\beta)$ is equal to $2l(\beta)+n$, and that of half-twist bands is equal to $4l(\beta)$. Therefore, we have

$$\begin{aligned} g(S(\beta)) &= 1 - \frac{\#\{\text{disk}\} - \#\{\text{band}\} + r}{2} \\ &= 1 + l(\beta) - \frac{n+r}{2}. \end{aligned}$$

Since $g(S(\beta))$ is equal to $(n-r)/2$, we have

$$l(\beta) = g(S(\beta)) + g(F(\beta)) + r - 1 \geq g_c(L) + g_{fb}(L) + r - 1.$$

□

By Theorem 2.1.2, we have Corollary 2.1.3.

Proof of Corollary 2.1.3. If L is a split link, then $\mathrm{span}\Delta_L(t) = 0$. Therefore, this inequality is trivial. We assume that L is a non-split link. It is known that, for any link L, it holds that $2g(L)+r-1 \geq \mathrm{span}\Delta_L(t)$ (see [2, 6, 7, 8]). By inequalities $g_c(L) \geq g_b(L)$, $g_{fb}(L) \geq g_b(L)$, and Theorem 2.1.2, we have

$$\begin{aligned} l_{fb}(L) &\geq g_c(L) + g_{fb}(L) + r - 1 \\ &\geq 2g_b(L) + r - 1 \\ &\geq 2g(L) + r - 1 \\ &\geq \mathrm{span}\Delta_L(t). \end{aligned}$$

□

2.3 The sufficient condition for the equality in Corollary 2.1.3

By the inequalities in the proof of Corollary 2.1.3, we have the following corollary.

Corollary 2.3.1. *If $L = \partial F(\beta)$ is a non-split link and $l(\beta) = \mathrm{span}\Delta_L(t)$, then we have the following equalities.*

$$\begin{aligned} &(1) \quad l_{fb}(L) = l(\beta) = n - 1. \\ &(2) \quad g(L) = g_c(L) = g_b(L) = g_{fb}(L) = \frac{n-r}{2}. \end{aligned}$$

Proof. By the assumption, the proofs of Theorem 2.1.2, and Corollary 2.1.3, we have

$$g(S(\beta)) + g(F(\beta)) = g_c(L) + g_{fb}(L) = 2g_b(L) = 2g(L).$$

Since $g(F(\beta))$ is equal to $(n-r)/2$, we have (1) and (2). □

If $l(\beta) = \mathrm{span}\Delta_L(t)$, then we can determine the flat braidzel length of L by Corollary 2.3.1(1). We consider which braids satisfy the equation $l(\beta) = \mathrm{span}\Delta_L(t)$. Theorem 2.3.2 is an answer to this question.

Let γ_i $(i = 0, 1, 2, \ldots), \delta_2, \delta_3$, and δ_4 be the $(i + 1)$-, 2-, 3-, and 4-string braids as follows (see Figure 2.3):

$$\begin{aligned}
\gamma_i &= \begin{cases} \text{1-string braid} & (i = 0), \\ (\sigma_1\sigma_3 \cdots \sigma_i)(\sigma_2^{-1}\sigma_4^{-1} \cdots \sigma_{i-1}^{-1}) & (i : \text{odd}), \\ (\sigma_1\sigma_3 \cdots \sigma_{i-1})(\sigma_2^{-1}\sigma_4^{-1} \cdots \sigma_i^{-1}) & (i : \text{even}, i \neq 0), \end{cases} \\
\delta_2 &= \sigma_1^2, \\
\delta_3 &= \sigma_1\sigma_2^{-1}\sigma_1, \\
\delta_4 &= \sigma_2\sigma_1^{-1}\sigma_3^{-1}\sigma_2.
\end{aligned}$$

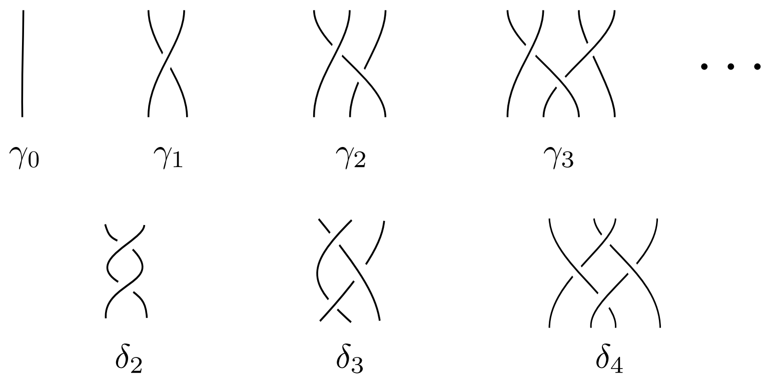


Figure 2.3: $\gamma_i, \delta_2, \delta_3$, and δ_4.

For a braid β, let $[\beta]$ be the set of all braids (in general, eight braids) obtained by performing π rotations to vertical or horizontal direction to β or the mirror image of β. For two braids β_1 and β_2, we define the split sum of β_1 and β_2 obtained by placing β_2 on the right of β_1, denoted by $\beta_1 \circ \beta_2$ (see Figure 2.4).

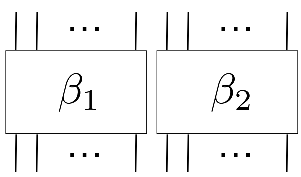


Figure 2.4: $\beta_1 \circ \beta_2$.

Theorem 2.3.2. *Let L be a non-split link, and β an n-string braid such that $L = \partial F(\beta)$. Then $l(\beta) = \mathrm{span}\Delta_L(t)$ if and only if β satisfies the following three conditions.*

$$\begin{array}{ll} (1) & \beta = \beta_1 \circ \beta_2 \circ \cdots \circ \beta_m. \\ (2) & \beta_k \in \bigcup_{i=0,1,\ldots} [\gamma_i] \quad \text{for some } k \ (1 \leq k \leq m). \\ (3) & \beta_h \in [\delta_2] \cup [\delta_3] \cup [\delta_4] \quad (1 \leq h \leq m, h \neq k). \end{array}$$

Example 2.3.3. Let $T(p, q)$ be the (p, q)-torus knot. The knot $\overline{K}$ means the mirror image of a knot K. Then, we have $\partial F(\gamma_{4q}) = T(2, q)\#\overline{T(2, q)}$. Since γ_{4q} satisfies the three conditions in Theorem 2.3.2, it follows by Corollary 2.3.1(1) that

$$l_{fb}(T(2, q)\#\overline{T(2, q)}) = 4q.$$

Corollary 2.3.4. *If $\Delta_L(t)$ is not monic, then we have*

$$l_{fb}(L) \geq \mathrm{span}\Delta_L(t) + 1.$$

We prove Theorem 2.3.2 and Corollary 2.3.4 in the next section.

2.4 Proofs of Theorem 2.3.2 and Corollary 2.3.4

Throughout this section, we assume that β is an n-string braid and $L = \partial F(\beta)$.

Let $s_1, s_2, \ldots, s_n$ be the strings of a braid β. We assume that all strings of β are oriented from the top to the bottom. We define w_{ij} $(i, j = 1, 2, \ldots, n, i \neq j)$ as the sum of the signs of all crossings of s_i and s_j. Let σ be a permutation of order n. For convenience, we arrange the order of the strings such that the starting points of the strings of β are $s_{\sigma(1)}, s_{\sigma(2)}, \ldots, s_{\sigma(n)}$ from the left to the right as in Figure 2.5.

Then, we define a square matrix $\tilde{V}_\beta$ of order n whose $(\sigma(i), \sigma(j))$ component is

$$\tilde{v}_{\sigma(i)\sigma(j)} = \begin{cases} \left\lceil \frac{w_{\sigma(i)\sigma(j)}}{2} \right\rceil & (i < j), \\ \left\lfloor \frac{w_{\sigma(i)\sigma(j)}}{2} \right\rfloor & (i > j), \\ 0 & (i = j), \end{cases}$$

where $\lceil x \rceil$ (resp. $\lfloor x \rfloor$) is the least (resp. greatest) integer not less (resp. greater) than x.

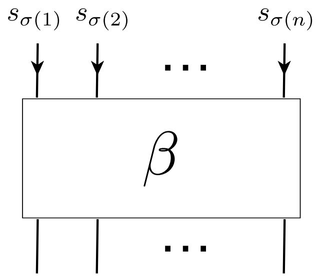


Figure 2.5: braid β.

Theorem 2.4.1. *The following matrix V_β is a Seifert matrix of L.*

$$V_\beta = \left(\begin{array}{ccc|c} & & & 1 \\ & \tilde{V}_\beta & & \vdots \\ & & & 1 \\ \hline 0 & \cdots & 0 & 0 \end{array}\right).$$

We remark that Theorem 2.4.1 is a generalization of Theorem 1.4.9.

Proof. Let F' be the Seifert surface for L obtained from attaching two bands to $F(\beta)$ as in Figure 2.6. We choose simple loops $x_1, x_2, \dots, x_{n+1}$ on F' presenting a basis of the first homology group $H_1(F')$ of F'. For $i, j = 1, 2, \dots, n$, we have

$$\begin{aligned} \mathrm{lk}(x^+_{\sigma(i)}, x_{\sigma(j)}) &= \tilde{v}_{\sigma(i)\sigma(j)}, \\ \mathrm{lk}(x^+_{\sigma(i)}, x_{n+1}) &= 1, \\ \mathrm{lk}(x^+_{n+1}, x_{\sigma(i)}) &= 0, \\ \mathrm{lk}(x^+_{n+1}, x_{n+1}) &= 0. \end{aligned}$$

Therefore, we see that V_β is a Seifert matrix of L. □

From now, we assume that β is the split sum of n_i-string braids β_i ($i = 1, 2, \dots, m$), that is, $\beta = \beta_1 \circ \beta_2 \circ \cdots \circ \beta_m$. Then, we have $n_1 + n_2 + \cdots + n_m = n$ and the Seifert matrix V_β of L is given by

$$V_\beta = \left(\begin{array}{cccc|c} \tilde{V}_{\beta_1} & & & O & 1 \\ & \tilde{V}_{\beta_2} & & & \vdots \\ & & \ddots & & \\ O & & & \tilde{V}_{\beta_m} & 1 \\ \hline 0 & \cdots & & 0 & 0 \end{array}\right).$$

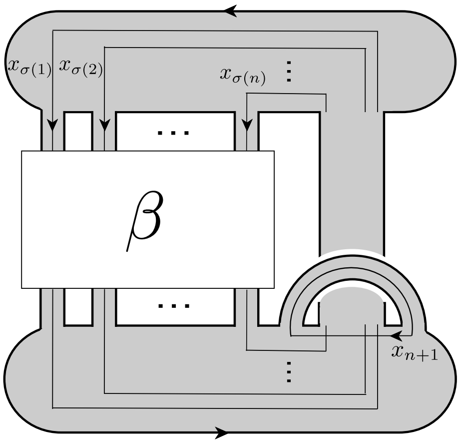


Figure 2.6: Simple loops $x_1, x_2, \ldots, x_{n+1}$ presenting a basis of $H_1(F')$.

Lemma 2.4.2. *If β satisfies the three conditions in* Theorem 2.3.2*, then we have $l(\beta) = \mathrm{span}\Delta_L(t)$.*

Proof. For γ_i, we choose σ as follows:

$$\sigma = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & \cdots \\ 2 & 1 & 4 & 3 & 6 & 5 & \cdots \end{pmatrix}.$$

Then, γ_i is as in Figure 2.7.

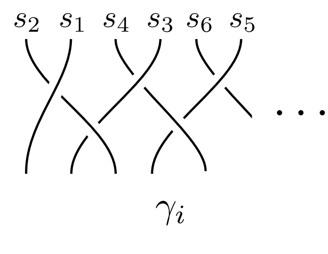


Figure 2.7: The order of the strings of γ_i.

We have the matrix $\tilde{V}_{\gamma_i}$ of order $i+1$ as follows:

$$\tilde{V}_{\gamma_i} = \begin{pmatrix} 0 & & & & O \\ 1 & 0 & & & \\ & -1 & \ddots & & \\ & & \ddots & 0 & \\ O & & & (-1)^{i-1} & 0 \end{pmatrix}.$$

Let γ_i' be a braid in $[\gamma_i]$. For γ_i', by choosing σ as similar to the case of γ_i, we have $\tilde{V}_{\gamma_i'} = \pm\tilde{V}_{\gamma_i}$.

Let δ_2', δ_3', and δ_4' be braids in $[\delta_2], [\delta_3]$, and $[\delta_4]$, respectively. By choosing suitable σ, we have matrices $\tilde{V}_{\delta_2'}, \tilde{V}_{\delta_3'}$, and $\tilde{V}_{\delta_4'}$ as follows.

$$\tilde{V}_{\delta_2'} = \pm\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \tilde{V}_{\delta_3'} = \pm\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -1 & 0 & 0 \end{pmatrix}, \tilde{V}_{\delta_4'} = \pm\begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix}.$$

Let A be the matrix $V_\beta - tV_\beta^T$ whose (i,j) component is a_{ij} $(i,j = 1,2,\ldots,n+1)$. Since V_β is a matrix of order $n+1$, we have

$$\det A = a_0 + a_1 t + a_2 t^2 + \cdots + a_{n+1}t^{n+1} \quad (a_0,\ldots,a_{n+1} \in \mathbb{Z}).$$

Since any component in the last row of A is $-t$ or 0, we have $a_0 = 0$. Since any component in the last column of A is 1 or 0, we have $a_{n+1} = 0$. We show $a_1 \neq 0$ and $a_n \neq 0$. By assumption, we have

$$\tilde{V}_{\beta_h} = \begin{cases} \tilde{V}_{\gamma_i'} & (h = k), \\ \tilde{V}_{\delta_2'} \text{ or } \tilde{V}_{\delta_3'} \text{ or } \tilde{V}_{\delta_4'} & (1 \leq h \leq m, h \neq k). \end{cases}$$

We define a permutation τ of order $n+1$ as follows. We remark that i $(1 \leq i \leq n+1)$ satisfies $n_1 + \cdots + n_{h-1} + 1 \leq i \leq n_1 + \cdots + n_{h-1} + n_h$ for some h.

If $h = k$, then

$$\tau(i) = \begin{cases} n+1 & (i = n_1 + \cdots + n_{k-1} + 1), \\ i-1 & (\text{otherwise}). \end{cases}$$

If $\beta_h \in [\delta_2]$, then

$$\tau(i) = \begin{cases} i+1 & (i = n_1 + \cdots + n_{h-1} + 1), \\ i-1 & (i = n_1 + \cdots + n_{h-1} + 2). \end{cases}$$

If $\beta_h \in [\delta_3]$, then

$$\tau(i) = \begin{cases} i+1 & (i = n_1 + \cdots + n_{h-1} + 1), \\ i+1 & (i = n_1 + \cdots + n_{h-1} + 2), \\ i-2 & (i = n_1 + \cdots + n_{h-1} + 3). \end{cases}$$

If $\beta_h \in [\delta_4]$, then

$$\tau(i) = \begin{cases} i+3 & (i = n_1 + \cdots + n_{h-1} + 1), \\ i+1 & (i = n_1 + \cdots + n_{h-1} + 2), \\ i-2 & (i = n_1 + \cdots + n_{h-1} + 3), \\ i-2 & (i = n_1 + \cdots + n_{h-1} + 4). \end{cases}$$

Finally, $\tau(n+1) = n_1 + \cdots + n_k$.

Then, we have $a_{1\tau(1)}a_{2\tau(2)}\cdots a_{(n+1)\tau(n+1)} = \pm t$. In the $(n_1+\cdots+n_{k-1}+1)$-st row of A, a component which has a non-zero constant term is only in the $(n+1)$-st column. Except for the $(n_1+\cdots+n_{k-1}+1)$-st row, the $(n+1)$-st row, and the $(n+1)$-st column of A, there exists a unique component which has a non-zero constant term in each row. Therefore, for a permutation τ' except τ, the smallest degree of $a_{1\tau'(1)}a_{2\tau'(2)}\cdots a_{(n+1)\tau'(n+1)}$ is not one. Therefore, we have $a_1 \neq 0$. By a similar argument, we see that $a_n \neq 0$. Therefore, we have $\mathrm{span}\Delta_L(t) = \mathrm{span}\ \det A = n-1 = l(\beta)$. □

We prove the converse of Lemma 2.4.2. We assume that $l(\beta) = \mathrm{span}\Delta_L(t)$. By Corollary 2.3.1, we remark that $l(\beta) = \mathrm{span}\Delta_L(t) = n-1$.

Let $\tilde{A}$ be the matrix $\tilde{V}_\beta - t\tilde{V}_\beta^T$ whose (i,j) component is $\tilde{a}_{ij}$ $(i,j = 1,2,\ldots,n)$. We define λ as the number of components of $\tilde{A}$ which have degree one term, that is,

$$\lambda = \#\{(i,j) | \mathrm{maxdeg}\ \tilde{a}_{ij} = 1\}.$$

Then, we have the following lemma.

Lemma 2.4.3. (1) $\lambda = n-1$.
(2) *If* maxdeg $\tilde{a}_{ij}$ = maxdeg $\tilde{a}_{i'j'} = 1$ *with* $(i,j) \neq (i',j')$, *then* $i \neq i'$ *and* $j \neq j'$.

Proof. (1) Since $\mathrm{span}\Delta_L(t) = \mathrm{span}\ \det A = n-1$, we have $\lambda \geq n-1$. We show $\lambda \leq n-1$. Let c_{ij} $(i<j)$ be the number of crossings of s_i and s_j. We remark that $|w_{ij}| \leq c_{ij}$. If $|w_{ij}| \leq 1$, then we have

$$(\tilde{a}_{ij}, \tilde{a}_{ji}) = \begin{cases} (1,-t) \text{ or } (\mathrm{t},-1) & (|w_{ij}| = 1), \\ (0,0) & (w_{ij} = 0). \end{cases}$$

Therefore, we have maxdeg $\tilde{a}_{ij}$ + maxdeg $\tilde{a}_{ji} \leq |w_{ij}| \leq c_{ij}$. If $|w_{ij}| \geq 2$, then we have $c_{ij} \geq 2$ and maxdeg $\tilde{a}_{ij}$ + maxdeg $\tilde{a}_{ji} \leq 2 \leq c_{ij}$. Therefore, we have maxdeg $\tilde{a}_{ij}$ + maxdeg $\tilde{a}_{ji} \leq c_{ij}$ for any (i,j). Since $\tilde{a}_{ii} = 0$ for any i, we have

$$\lambda = \sum_{i<j}(\mathrm{maxdeg}\ \tilde{a}_{ij} + \mathrm{maxdeg}\ \tilde{a}_{ji}) \leq \sum_{i<j} c_{ij} = l(\beta) = n-1.$$

(2) By (1) and the assumption that $\mathrm{span}\Delta_L(t) = n-1$, there exists at most one component which has degree one term in each low or column of A. □

Let c_i ($i = 1, 2, \ldots, n$) be the number of crossings of s_i and all other strings, that is,

$$c_i = \sum_{j=1, j\neq i}^{n} c_{ij}.$$

Lemma 2.4.4. *For any i, we have $c_i \leq 2$.*

Proof. We assume that there exists i such that $c_i \geq 3$.

Case 1. If there exists j such that $c_{ij} \geq 3$, then we have maxdeg $\tilde{a}_{ij}$ + maxdeg $\tilde{a}_{ji} \leq 2 < c_{ij}$. By the proof of Lemma 2.4.3 (1), we have

$$\lambda = \sum_{i<j} (\text{maxdeg } \tilde{a}_{ij} + \text{maxdeg } \tilde{a}_{ji}) < \sum_{i<j} c_{ij} = l(\beta) = n - 1.$$

It is a contradiction with Lemma 2.4.3 (1).

Case 2. If there exists j such that $c_{ij} = 2$, then $w_{ij} = 0$ or ± 2. If $w_{ij} = 0$, then we have maxdeg $\tilde{a}_{ij}$ + maxdeg $\tilde{a}_{ji} = 0 < c_{ij}$. Therefore, we have $\lambda < n - 1$. By a similar argument to Case 1, it is a contradiction with Lemma 2.4.3 (1). If $w_{ij} = \pm 2$, then maxdeg $\tilde{a}_{ij}$ = maxdeg $\tilde{a}_{ji} = 1$. By assumption, there exists $k (\neq j)$ such that $c_{ik} \geq 1$. By Case 1, we have $c_{ik} = 1$ or 2. If $c_{ik} = 1$, then maxdeg $\tilde{a}_{ik} = 1$ or maxdeg $\tilde{a}_{ki} = 1$. It is a contradiction with Lemma 2.4.3 (2). If $c_{ik} = 2$, then $w_{ik} = 0$ or ± 2. If $w_{ik} = 0$, then we have maxdeg $\tilde{a}_{ik}$ + maxdeg $\tilde{a}_{ki} = 0 < c_{ik}$. It is a contradiction with Lemma 2.4.3 (1). If $w_{ik} = \pm 2$, then maxdeg $\tilde{a}_{ik}$ = maxdeg $\tilde{a}_{ki} = 1$. It is a contradiction with Lemma 2.4.3 (2).

Case 3. If $c_{ij} \leq 1$, then there exist k and k' such that $c_{ij} = c_{ik} = c_{ik'} = 1$. It is a contradiction with Lemma 2.4.3 (2). □

By the following lemma, the proof of Theorem 2.3.2 is completed.

Lemma 2.4.5. *If $l(\beta) = \text{span}\Delta_L(t)$, then β satisfies the three conditions in* Theorem 2.3.2.

Proof. Let α_k be the number of i such that $c_i = k$. By Lemma 2.4.4, we have $\alpha_0 + \alpha_1 + \alpha_2 = n$, and $\alpha_1 + 2\alpha_2 = 2l(\beta)$. Since L is a non-split link, we have $\alpha_0 = 0$ or 1. Therefore, we have $(\alpha_0, \alpha_1, \alpha_2) = (0, 2, n-2)$ or $(1, 0, n-1)$. If $(\alpha_0, \alpha_1, \alpha_2) = (0, 2, n-2)$, then β has the projection which is a split sum of the projection of a braid in the $[\gamma_i]$ and the several copies of the projection δ_2, δ_3, and δ_4. If $(\alpha_0, \alpha_1, \alpha_2) = (1, 0, n-1)$, then β has the projection which is a split sum of one simple arc and the several copies of the projection of δ_2, δ_3, and δ_4. Moreover, if two crossings on the same string of β are over- (or under-)crossings, then it is a contradiction with Lemma 2.4.3 (2). Therefore, β satisfies the three conditions in Theorem 2.3.2. □

Finally, we prove Corollary 2.3.4.

Proof of Corollary 2.3.4. By the proof of Lemma 2.4.2, if $l(\beta) = \mathrm{span}\Delta_L(t)$, then the coefficient of the largest (smallest) degree of $\Delta_L(t)$ is ± 1. It is the contraposition of this corollary. $\square$

2.5 Table of knots with the flat braidzel length five of less

We have the following proposition.

Proposition 2.5.1.
(1) *L is a trivial link if and only if $l_{fb}(L) = 0$.*
(2) *For a non-trivial knot K, we have $l_{fb}(K) \geq 3$.*

Proof. (1) A trivial link has the flat braidzel presentation given by a trivial braid. A braid β such that $l(\beta) = 0$ is only a trivial braid.

(2) We assume that $l(\beta) = 1$. If β is a 2-string braid, then $\partial F(\beta)$ is the positive or negative Hopf link. Otherwise, $\partial F(\beta)$ is a trivial link. We assume that $l(\beta) = 2$. If β is the 2-string braid $\sigma_1^{\pm 2}$ or the 4-string braid $\sigma_1\sigma_3$ or $\sigma_1^{-1}\sigma_3^{-1}$, then $\partial F(\beta)$ is the $(2, \pm 4)$-torus link. If β is a 3-string braid in $[\sigma_1\sigma_2]$, then $\partial F(\beta)$ is the split sum of the Hopf link and the trivial knot. If β is a 3-string braid in $[\sigma_1\sigma_2^{-1}]$, then $\partial F(\beta)$ is the connected sum of two Hopf links. If β is the n-string braid $\sigma_i^{\pm 2}$ ($n \geq 3$, $i = 1, 2, \ldots, n-1$), then $\partial F(\beta)$ is a split sum of the $(3, \pm 3)$-torus link and a trivial link. Otherwise, $\partial F(\beta)$ is a trivial link. Therefore, if $l(\beta) \leq 2$, then $\partial F(\beta)$ is a trivial link or a link with two components or more. $\square$

We determine the knots with the flat braidzel length five or less. For the notation of knots, we refer to the knot tables in [1] and [16].

Theorem 2.5.2. *We have the table of knots with the flat braidzel length five or less as follows.*

$l_{fb}(K)$	K
3	$3_1, 4_1, 6_1, 9_{46}$
4	$5_2, 10_{140}, 11_{n49}, 3_1\#\overline{3_1}$
5	$8_1, 8_3, 8_8, 8_{20}, 10_3, 10_{137}, 11_{n139}, 11_{n141}, 12_{n523},$ $P(5,-5,-3), P(5,-5,-5), K_1, K_2, K_3, K_4$

Here, $P(p, q, r)$ is a 3-strand pretzel knot, K_1, K_2, K_3, and K_4 are $\partial F(\sigma_1^2\sigma_2^{-1}\sigma_4^{-1}\sigma_3)$, $\partial F(\sigma_1\sigma_2^{-1}\sigma_4^{-1}\sigma_3\sigma_2^{-1})$, $\partial F(\sigma_2\sigma_1^{-1}\sigma_3^{-1}\sigma_2^2)$, and $\partial F(\sigma_1\sigma_3\sigma_5^{-1}\sigma_6\sigma_5^{-1})$, respectively.

We give the proof of Theorem 2.5.2. Let K and β be a knot and an n-string braid such that $K = \partial F(\beta)$. We remark that n is odd, since $g(F(\beta)) = (n-1)/2 \in \mathbb{Z}$.

Lemma 2.5.3.
(1) *If β has the part as in* Figure 2.8(a1)*, then there exists β' such that $K = \partial F(\beta')$ and $l(\beta) > l(\beta')$.*
(2) *If β has the part as in* Figure 2.8(b1)*, then there exists β' such that $K = \partial F(\beta')$ and $l(\beta) > l(\beta')$.*

Proof. (1) If β has the part as in Figure 2.8(a1), then we can obtain β' by removing strings s_{i+1} and s_{i+2} from β (see Figure 2.8(a2)).

(2) If β has the part as in Figure 2.8(b1), then we can obtain β' by removing strings s_i, s_{i+1}, s_{i+2}, and s_{i+3} from β (see Figure 2.8(b2)). □

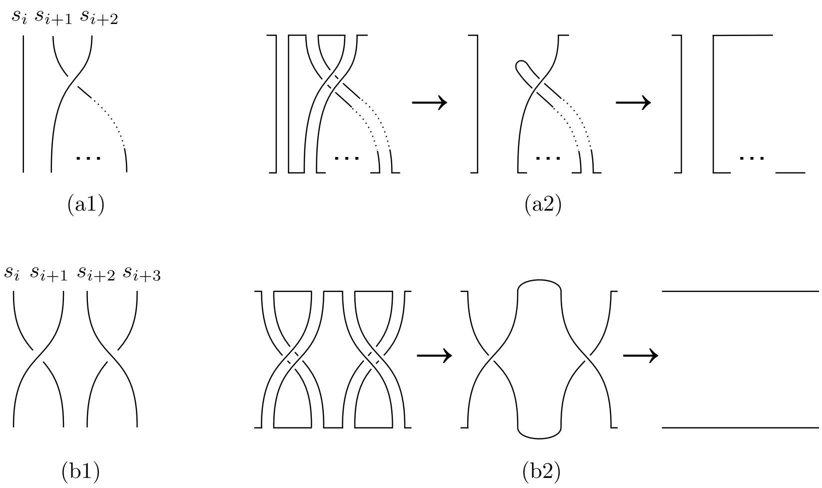


Figure 2.8: There exists β' such that $K = \partial F(\beta')$ and $l(\beta) > l(\beta')$.

By Lemma 2.5.3(1), we have the following lemma.

Lemma 2.5.4. *If $l_{fb}(K) = l(\beta)$, then we have $n \leq 2l(\beta) - 3$.*

Proof. Let α_k be the number as defined in the proof of Lemma 2.4.5. Then, we have the following two equalities.

$$\sum_{k=0,1,\dots} \alpha_k = n \quad \text{and} \quad \sum_{k=0,1,\dots} k\alpha_k = 2l(\beta).$$

Since K is a knot, we have $\alpha_0 = 0$ or 1. Therefore, we have

$$0 \leq \sum_{k=2,3\ldots} (k-1)\alpha_k = 2l(\beta) - n + \alpha_0 \leq 2l(\beta) - n + 1.$$

If $n = 2l(\beta) + 1$, then we have $\alpha_k = 0$ $(k \geq 2)$ and $(\alpha_0, \alpha_1) = (1, n-1)$. Therefore, β has a projection of the split sum of a single γ_0 and several copies of γ_1 (see Figure 1.5).

If $n = 2l(\beta)-1$, then we have $\alpha_k = 0$ $(k \geq 4)$, and $(\alpha_0, \alpha_1, \alpha_2, \alpha_3) = (0, n-1, 1, 0), (1, n-3, 2, 0)$, or $(1, n-2, 0, 1)$. If $(\alpha_0, \alpha_1, \alpha_2, \alpha_3) = (0, n-1, 1, 0)$, then β has a projection of the split sum of a braid in the $[\gamma_2]$ and several copies of γ_1. If $(\alpha_0, \alpha_1, \alpha_2, \alpha_3) = (1, n-3, 2, 0)$, then β has a projection which is one of the following three braids. (1) A split sum of a single γ_0, δ_2, and several copies of γ_1. (2) A split sum of a single γ_0, two braids in $[\gamma_2]$, and several copies of γ_1. (3) A split sum of a single γ_0, a braid in $[\gamma_3]$, and several copies of γ_1. If $(\alpha_0, \alpha_1, \alpha_2, \alpha_3) = (1, n-2, 0, 1)$, then β has a projection of the split sum of one simple arc, 3-string braid in $[\sigma_1\sigma_2]$, and several copies of a projection of γ_1.

In any case of the above, β satisfies the condition in Lemma 2.5.3 (1). It is a contradiction with the assumption. □

Proof of Theorem 2.5.2. From now, we denote σ_i (resp. σ_i^{-1}) by i (resp. $\bar{i}$) simply. For example, $12\bar{1}$ means $\sigma_1\sigma_2\sigma_1^{-1}$.

If $l_{fb}(K) = l(\beta) = 3$, then $n = 3$ by Lemma 2.5.4. A 3-string braid with the length three has a projection of a braid in [111], [112], or [121].

If $l_{fb}(K) = l(\beta) = 4$, then $n = 3$ or 5 by Lemma 2.5.4. By a simple observation, if $n = 3$, then $\partial F(\beta)$ is a three component link. Therefore, we have $n = 5$. By Lemma 2.5.3 (1), it is not difficult to see that it is sufficient to consider the braids which have a projection of a braid in [1124], [1214], or [1324].

If $l_{fb}(K) = l(\beta) = 5$, then $n = 3, 5$ or 7 by Lemma 2.5.4.

Case $n = 3$. A 3-string braid with the length five has a projection of a braid in [11111], [11112], [11122], [11121], [11221], [11211], [12221], [11212], [12112], or [12121].

Case $n = 5$. By Lemma 2.5.3 (1), it is not difficult to see that it is sufficient to consider braids which have a projection of a braid in [11243], [11324], [12432], or [21322].

Case $n = 7$. By Lemma 2.5.3 (1), it is not difficult to see that it is sufficient to consider braids which have a projection of a braid in [11246], [12146], or [13246].

We give the knots presented by the flat braidzel presentation for each braid in the following table. Here, O is the trivial knot. We omit braids β which have $i\bar{i}$ or the part as in Figure 2.8(a1) or (b1), since there exists β' such that $K = \partial F(\beta')$ and $l(\beta) > l(\beta')$ by Lemma 2.5.3. We assume that the initial of a braid is positive, since we disregard the difference between a knot and its mirror image.

β	$\partial F(\beta)$
111	9_{46}
112	O
$11\bar{2}$	6_1
121	3_1
$12\bar{1}$	4_1
$1\bar{2}1$	4_1
1124	O
$112\bar{4}$	O
$11\bar{2}4$	6_1
$11\bar{2}\bar{4}$	9_{46}
1214	5_2
$121\bar{4}$	O
$12\bar{1}4$	6_1
$12\bar{1}\bar{4}$	O
$1\bar{2}14$	10_{140}
$1\bar{2}1\bar{4}$	11_{n49}
1324	O
$132\bar{4}$	O
$13\bar{2}4$	O
$13\bar{2}\bar{4}$	$3_1\#\overline{3_1}$
$1\bar{3}2\bar{4}$	O
$1\bar{3}\bar{2}4$	6_1
11111	$P(5,-5,-5)$
11112	11_{n139}
$1111\bar{2}$	$P(5,-5,-3)$
11122	O
$111\bar{2}\bar{2}$	10_3
11121	3_1
$1112\bar{1}$	8_1
$111\bar{2}1$	8_1
$111\bar{2}\bar{1}$	11_{n141}
11221	O
$1122\bar{1}$	O
$11\bar{2}\bar{2}1$	10_3

β	$\partial F(\beta)$
$11\bar{2}\bar{2}\bar{1}$	O
11211	5_2
$112\bar{1}\bar{1}$	8_3
$11\bar{2}11$	6_1
$11\bar{2}\bar{1}\bar{1}$	8_3
12221	3_1
$1222\bar{1}$	8_1
$1\bar{2}\bar{2}\bar{2}1$	11_{n141}
11212	3_1
$1121\bar{2}$	3_1
$112\bar{1}2$	8_1
$112\bar{1}\bar{2}$	4_1
$11\bar{2}12$	8_1
$11\bar{2}1\bar{2}$	4_1
$11\bar{2}\bar{1}2$	11_{n141}
$11\bar{2}\bar{1}\bar{2}$	4_1
12112	5_2
$1211\bar{2}$	O
$12\bar{1}\bar{1}2$	8_3
$12\bar{1}\bar{1}\bar{2}$	6_1
$1\bar{2}112$	O
$1\bar{2}11\bar{2}$	8_3
$1\bar{2}\bar{1}\bar{1}2$	O
$1\bar{2}\bar{1}\bar{1}\bar{2}$	5_2
12121	5_2
$1212\bar{1}$	O
$121\bar{2}1$	O
$121\bar{2}\bar{1}$	O
$12\bar{1}21$	5_2
$12\bar{1}2\bar{1}$	O
$12\bar{1}\bar{2}1$	6_1
$1\bar{2}12\bar{1}$	8_3
$1\bar{2}1\bar{2}1$	O
$1\bar{2}\bar{1}\bar{2}1$	O

β	$\partial F(\beta)$
11243	O
$1124\bar{3}$	O
$112\bar{4}3$	O
$112\bar{4}\bar{3}$	O
$11\bar{2}43$	11_{n139}
$11\bar{2}4\bar{3}$	6_1
$11\bar{2}\bar{4}3$	K_1
$11\bar{2}\bar{4}\bar{3}$	6_1
11324	O
$1132\bar{4}$	8_{20}
$113\bar{2}4$	O
$113\bar{2}\bar{4}$	8_8
$11\bar{3}24$	9_{46}
$11\bar{3}2\bar{4}$	O
$11\bar{3}\bar{2}4$	10_{137}
$11\bar{3}\bar{2}\bar{4}$	O
12432	O
$1243\bar{2}$	O
$124\bar{3}2$	O
$124\bar{3}\bar{2}$	O
$12\bar{4}32$	O
$12\bar{4}3\bar{2}$	O
$12\bar{4}\bar{3}2$	O
$12\bar{4}\bar{3}\bar{2}$	O
$1\bar{2}432$	O
$1\bar{2}43\bar{2}$	8_{20}
$1\bar{2}4\bar{3}2$	O
$1\bar{2}4\bar{3}\bar{2}$	$3_1\#\overline{3_1}$
$1\bar{2}\bar{4}32$	6_1
$1\bar{2}\bar{4}3\bar{2}$	K_2
$1\bar{2}\bar{4}\bar{3}2$	8_{20}
$1\bar{2}\bar{4}\bar{3}\bar{2}$	O
21322	O
$213\bar{2}\bar{2}$	9_{46}

β	$\partial F(\beta)$
$21\bar{3}22$	9_{46}
$21\bar{3}\bar{2}\bar{2}$	O
$2\bar{1}322$	9_{46}
$2\bar{1}3\bar{2}\bar{2}$	O
$2\bar{1}\bar{3}22$	K_3
$2\bar{1}\bar{3}\bar{2}\bar{2}$	$3_1\#\overline{3_1}$
11246	O
$112\bar{4}\bar{6}$	O
$11\bar{2}46$	11_{n139}
$11\bar{2}\bar{4}\bar{6}$	9_{46}
12146	5_2
$121\bar{4}\bar{6}$	4_1
$12\bar{1}46$	6_1
$12\bar{1}\bar{4}\bar{6}$	3_1
$1\bar{2}146$	12_{n523}
$1\bar{2}1\bar{4}\bar{6}$	K_4
13246	O
$1324\bar{6}$	O
$132\bar{4}6$	O
$132\bar{4}\bar{6}$	O
$13\bar{2}46$	O
$13\bar{2}4\bar{6}$	O
$13\bar{2}\bar{4}6$	8_{20}
$13\bar{2}\bar{4}\bar{6}$	8_{20}
$1\bar{3}246$	O
$1\bar{3}24\bar{6}$	O
$1\bar{3}2\bar{4}6$	O
$1\bar{3}2\bar{4}\bar{6}$	O
$1\bar{3}\bar{2}46$	9_{46}
$1\bar{3}\bar{2}4\bar{6}$	6_1
$1\bar{3}\bar{2}\bar{4}6$	O
$1\bar{3}\bar{2}\bar{4}\bar{6}$	O

By observing the Conway polynomial and the Jones polynomial, and by comparing the table of knot invariants in [1], we see that K_1, K_2, K_3, and K_4 are knots with crossing number thirteen or more, and they have different knot types. $\square$

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