Mixed Hodge structures of the moduli spaces of parabolic connections

光明, 新

(Degree)
博士 (理学)
(Date of Degree)
2015-03-25
(Date of Publication)
2016-03-01
(Resource Type)
doctoral thesis
(Report Number)
甲第6318号
(URL)
https://hdl.handle.net/20.500.14094/D1006318

博　士　論　文

Mixed Hodge structures of the moduli spaces of parabolic connections

(放物接続のモジュライ空間の混合ホッジ構造)

平成 27 年 1 月

神戸大学大学院理学研究科

氏　　名　　光明 新

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Abstract

This thesis deals with several topics on the mixed Hodge structures of the moduli spaces of parabolic connections, the moduli space of parabolic Higgs bundles and character varieties. Since these moduli spaces are quasi-projective varieties, but never projective, it is interesting to investigate the weights of these mixed Hodge structures and the invariance of weights under smooth deformations.

By the nonabelian Hodge theory, the moduli spaces of parabolic connections and the moduli space of parabolic Higgs bundles are diffeomorphic, and, by the Riemann–Hilbert correspondence, the moduli spaces of parabolic connections and character varieties are analytic isomorphic. Since those isomorphisms are not algebraic, relationships between the mixed Hodge structures of those moduli spaces are not clear.

First, we investigate the mixed Hodge structures of the moduli spaces of parabolic connections and the moduli space of parabolic Higgs bundles.

Second, we investigate the mixed Hodge structures of character varieties. We compute a few examples. Moreover, we study the boundary components of compactifications of character varieties. This study is motivated by a conjecture on the configurations of the boundary components, due to C. Simpson.

This thesis presents the following two consequences:

- We show that the mixed Hodge polynomials (and the Poincaré polynomials) of the moduli spaces of parabolic connections are independent of the choice of generic eigenvalues. As a result, we have
  (1) the mixed Hodge structures of the moduli spaces of parabolic connections are pure;
  (2) the Poincaré polynomials of character varieties of generic spectral types are independent of the choice of generic eigenvalues by the Riemann–Hilbert correspondence.
- We investigate a conjecture of C. Simpson on the boundary components of compactifications of character varieties. We verify the conjecture for a few examples.

Acknowledgments

The author expresses gratitude to adviser Professor Masa-Hiko Saito for his warm encouragement and his valuable guidance.

He also would like to thank Michi-aki Inaba, Kentaro Mitsui, Takuro Mochizuki, Hiraku Nakajima, and Carlos Simpson for their valuable comments and discussions.

Introduction

We fix integers $n > 0, d$ and $g \geq 0$. Let Σ be a smooth complex projective curve of genus g. The nonabelian Hodge theory of Σ gives the equivalence of categories related to the following three moduli spaces: the moduli space of semistable Higgs bundles of rank n and of degree 0 on Σ (denoted by $\mathcal{M}_{Dol}(\Sigma)$); the moduli space of (semistable) holomorphic connections of rank n and of degree 0 on Σ (denoted by $\mathcal{M}_{DR}(\Sigma)$); and the character variety $\mathrm{Hom}(\pi_1(\Sigma), \mathrm{GL}(n, \mathbb{C}))/\mathrm{GL}(n, \mathbb{C})$, whose points parametrize representations of the fundamental group $\pi_1(\Sigma)$ into $\mathrm{GL}(n, \mathbb{C})$ (denoted by $\mathcal{M}_B(\Sigma)$). These moduli spaces are related to each other in the following way. First, the moduli space of semistable λ-*connections* $\mathcal{M}_{Hod}(\Sigma)$ gives the relationship between $\mathcal{M}_{Dol}(\Sigma)$ and $\mathcal{M}_{DR}(\Sigma)$. Here, we call (E, ∇) a λ-connection if E is a vector bundle on Σ and $\nabla\colon E \to E \otimes \Omega^1_\Sigma$ is a homomorphism of sheaves satisfying $\nabla(ae) = a\nabla(e) + \lambda d(a) \otimes e$ where $\lambda \in \mathbb{C}, a \in \mathcal{O}_\Sigma$ and $e \in E$. Then, we have the natural map $\lambda\colon \mathcal{M}_{Hod}(\Sigma) \to \mathbb{C}^1$ such that $\lambda^{-1}(0) = \mathcal{M}_{Dol}(\Sigma)$ and $\lambda^{-1}(1) = \mathcal{M}_{DR}(\Sigma)$. Finally, the Riemann–Hilbert correspondence gives an analytic isomorphism between $\mathcal{M}_{DR}(\Sigma)$ and $\mathcal{M}_B(\Sigma)$.

In this thesis, we consider variants of those moduli spaces in the case of punctured curves. We fix an integer $k \geq 0$ and a k-tuple $\boldsymbol{\mu} = (\mu^1, \dots, \mu^k)$ of partitions of n, that is, $\mu^i = (\mu^i_1, \dots, \mu^i_{r_i})$ satisfies $\mu^i_1 \geq \mu^i_2 \geq \cdots$ and $\mu^i_1 + \cdots + \mu^i_{r_i} = n$ for $i = 1, \dots, k$. We take k-distinct points $p_1, \dots, p_k$ on Σ, and define a divisor by $D := p_1 + \cdots + p_k$.

Definition (Parabolic Higgs bundles). We call $(E, \Phi, \{l^{(i)}_*\}_{1\leq i\leq k})$ a *parabolic Higgs bundle of rank n, of degree d, and of type* $\boldsymbol{\mu}$ if

(1) E is an algebraic vector bundle on Σ of rank n and of degree d,
(2) $\Phi\colon E \to E \otimes \Omega^1_\Sigma(D)$ is an $\mathcal{O}_\Sigma$-homomorphism, and
(3) for each p_i, $l^{(i)}_*$ is a filtration $E|_{p_i} = l^{(i)}_1 \supset l^{(i)}_2 \supset \cdots \supset l^{(i)}_{r_i} \supset l^{(i)}_{r_i+1} = 0$ such that $\dim(l^{(i)}_j / l^{(i)}_{j+1}) = \mu^i_j$ and $\Phi|_{p_i}(l^{(i)}_j) \subset l^{(i)}_{j+1} \otimes \Omega^1_\Sigma(D)|_{p_i}$ for $j = 1, \dots, r_i$.

The $\mathcal{O}_\Sigma$-homomorphism Φ is called a *Higgs field*.

Definition (Parabolic connections). We call $(E, \nabla, \{l^{(i)}_*\}_{1\leq i\leq k})$ a (*regular singular*) $\boldsymbol{\xi}$-*parabolic connection of rank n, of degree d, and of type* $\boldsymbol{\mu}$ if

(1) E is an algebraic vector bundle on Σ of rank n and of degree d,
(2) $\nabla\colon E \to E \otimes \Omega^1_\Sigma(D)$ is a connection, and
(3) for each p_i, $l^{(i)}_*$ is a filtration $E|_{p_i} = l^{(i)}_1 \supset l^{(i)}_2 \supset \cdots \supset l^{(i)}_{r_i} \supset l^{(i)}_{r_i+1} = 0$ such that $\dim(l^{(i)}_j / l^{(i)}_{j+1}) = \mu^i_j$ and $(\mathrm{Res}_{p_i}(\nabla) - \xi^i_j \mathrm{id}_{E|_{p_i}})(l^{(i)}_j) \subset l^{(i)}_{j+1}$ for $j = 1, \dots, r_i$.

Here, we put $r := \sum r_i$ and $\boldsymbol{\xi} := (\xi^i_j)^{1\leq i\leq k}_{1\leq j\leq r_i} \in \mathbb{C}^r$ satisfying $d + \sum_{i,j} \mu^i_j \xi^i_j = 0$ (see Remark 1.1.2).

We consider the following three moduli spaces: the moduli space of semistable parabolic Higgs bundles on Σ of rank n, of degree d, and of type $\boldsymbol{\mu}$; the moduli space of semistable $\boldsymbol{\xi}$-parabolic connections on Σ of rank n, of degree d, and of type $\boldsymbol{\mu}$; and the (*generic*) $\mathrm{GL}(n,\mathbb{C})$-*character variety*, whose points parametrize representations of the fundamental group of $\Sigma \setminus D$ into $\mathrm{GL}(n,\mathbb{C})$ with prescribed images in $\mathcal{C}_1, \ldots, \mathcal{C}_k$ at the punctures. Here, $(\mathcal{C}_1, \ldots, \mathcal{C}_k)$ is a generic k-tuple of semisimple conjugacy classes of $\mathrm{GL}(n,\mathbb{C})$ such that, for each $i = 1, \ldots, k$, $\{\mu^i_1, \mu^i_2, \ldots\}$ is the set of the multiplicities of the eigenvalues of any matrix in $\mathcal{C}_i$. These moduli spaces are connected non-singular algebraic varieties of dimension

$$n^2(2g - 2 + k) - \sum_{i,j} (\mu^i_j)^2 + 2$$

(see [**15**], [**16**], [**21**], [**22**] and [**29**]). Note that, for *any* $\boldsymbol{\xi}$, the moduli space of semistable $\boldsymbol{\xi}$-parabolic connections on Σ is non-singular by the parabolic structures and the stability. On the other hand, *only* for generic $(\mathcal{C}_1, \ldots, \mathcal{C}_k)$, the character variety is non-singular. We denote the three moduli spaces by $\mathcal{M}^{\boldsymbol{\mu}}_{Dol}(\mathbf{0})$, $\mathcal{M}^{\boldsymbol{\mu}}_{DR}(\boldsymbol{\xi})$, and $\mathcal{M}^{\boldsymbol{\mu}}_{B}(\boldsymbol{\nu})$, respectively. Here, $\boldsymbol{\nu}$ denotes the eigenvalues of the any matrix of each conjugacy class in $(\mathcal{C}_1, \ldots, \mathcal{C}_k)$ and $\mathbf{0}$ means that Higgs fields have nilpotent residues at each puncture.

For the case of the punctured curve $\Sigma \setminus D$, we study relationships between those moduli spaces. We put

$$\Xi^{\boldsymbol{\mu},d}_n := \left\{ \left(\lambda, (\xi^i_j)^{1\leq i\leq k}_{1\leq j\leq r_i}\right) \in \mathbb{C} \times \mathbb{C}^r \;\middle|\; \lambda d + \sum_{i,j} \mu^i_j \xi^i_j = 0 \right\}.$$

Definition (Parabolic λ-connections). For $(\lambda, \boldsymbol{\xi}) \in \Xi^{\boldsymbol{\mu},d}_n$, we call $(\lambda, E, \nabla, \{l^{(i)}_*\}_{1\leq i\leq k})$ a $\boldsymbol{\xi}$*-parabolic λ-connection of rank n, of degree d, and of type $\boldsymbol{\mu}$* if

1. (1) E is an algebraic vector bundle on Σ of rank n and of degree d,
2. (2) $\nabla\colon E \to E \otimes \Omega^1_\Sigma(D)$ is a λ-connection, that is, ∇ is a homomorphism of sheaves satisfying $\nabla(fa) = \lambda a \otimes df + f\nabla(a)$ for $f \in \mathcal{O}_\Sigma$ and $a \in E$, and
3. (3) for each p_i, $l^{(i)}_*$ is a filtration $E|_{p_i} = l^{(i)}_1 \supset l^{(i)}_2 \supset \cdots \supset l^{(i)}_{r_i} \supset l^{(i)}_{r_i+1} = 0$ such that $\dim(l^{(i)}_j / l^{(i)}_{j+1}) = \mu^i_j$ and $(\mathrm{Res}_{p_i}(\nabla) - \xi^i_j \mathrm{id}_{E|_{p_i}})(l^{(i)}_j) \subset l^{(i)}_{j+1}$ for $j = 1, \ldots, r_i$.

We construct the moduli space of semistable parabolic λ-connections over $\Xi^{\boldsymbol{\mu},d}_n$ as a subscheme of the coarse moduli scheme of semistable parabolic Λ^1_D-tuples constructed in [**23**], denoted by

$$\pi : \mathcal{M}^{\boldsymbol{\mu}}_{Hod} \longrightarrow \Xi^{\boldsymbol{\mu},d}_n.$$

We have $\pi^{-1}(1, \boldsymbol{\xi}) = \mathcal{M}^{\boldsymbol{\mu}}_{DR}(\boldsymbol{\xi})$ and $\pi^{-1}(0, \mathbf{0}) = \mathcal{M}^{\boldsymbol{\mu}}_{Dol}(\mathbf{0})$. On the other hand, by the moduli theoretic description of the Riemann-Hilbert correspondence (see [**23**], [**21**] and [**22**]), we obtain the analytic isomorphism $\mathcal{M}^{\boldsymbol{\mu}}_{DR}(\boldsymbol{\xi}) \cong \mathcal{M}^{\boldsymbol{\mu}}_{B}(\boldsymbol{\nu})$ where $\boldsymbol{\nu} = rh_d(\boldsymbol{\xi})$. Here, rh_d is the map defined by $\xi^i_j \mapsto \exp(-2\pi\sqrt{-1}\xi^i_j)$ for $i = 1, \ldots, k$ and $j = 1, \ldots, r_i$.

The mixed Hodge structures of the moduli space of parabolic connections and character varieties. For smooth projective varieties, one can define the Hodge structure on the cohomology groups of the smooth projective varieties. Deligne generalized the Hodge structure to any complex algebraic varieties, not necessarily smooth or projective, i.e., one can define the *mixed Hodge structure* on the cohomology groups of the varieties ([**6**], [**7**]). The moduli spaces $\mathcal{M}^{\boldsymbol{\mu}}_{Dol}(\mathbf{0})$, $\mathcal{M}^{\boldsymbol{\mu}}_{DR}(\boldsymbol{\xi})$, and $\mathcal{M}^{\boldsymbol{\mu}}_{B}(\boldsymbol{\nu})$ are smooth. However, these moduli spaces are *not* projective. The purpose of this thesis is to study the mixed Hodge structures of these moduli spaces.

The first theorem of this thesis is the following

Theorem A ((Ch.V, Theorem.1.2.1), (Ch.V, Theorem.1.2.2), and (Ch.V, Corollary 1.2.3)).

(1) *The ordinary rational cohomology groups of the fibers of* $\pi : \mathcal{M}^{\boldsymbol{\mu}}_{Hod} \to \Xi^{\boldsymbol{\mu},d}_n$ *are isomorphic. Moreover, the isomorphism preserves the mixed Hodge structures on the cohomology groups of the fibers.*
(2) *In particular, we have the isomorphism*

$$H^k(\mathcal{M}^{\boldsymbol{\mu}}_{DR}(\boldsymbol{\xi}), \mathbb{Q}) \cong H^k(\mathcal{M}^{\boldsymbol{\mu}}_{Dol}(\mathbf{0}), \mathbb{Q})$$

which preserves the mixed Hodge structures.
(3) *Since the mixed Hodge structure of* $H^k(\mathcal{M}^{\boldsymbol{\mu}}_{Dol}(\mathbf{0}), \mathbb{Q})$ *is pure of weight* k*, the mixed Hodge structure on* $H^k(\mathcal{M}^{\boldsymbol{\mu}}_{DR}(\boldsymbol{\xi}), \mathbb{Q})$ *is pure of weight* k*.*
(4) *For* $\boldsymbol{\mu} = (\mu^i_j)$ *such that* g.c.d.$(\mu^i_j) = 1$*, the Poincaré polynomials of character varieties* $\mathcal{M}^{\boldsymbol{\mu}}_{B}(\boldsymbol{\nu})$ *are independent of the choice of generic eigenvalues.*

The proof of this theorem is done by constructing a relative compactification of $\pi : \mathcal{M}^{\boldsymbol{\mu}}_{Hod} \to \Xi^{\boldsymbol{\mu},d}_n$ via a Simpson's idea [**48**].

On the other hand, there are some interesting conjecture on the mixed Hodge structures of character varieties, for example the *Hausel-Letellier-Rodriguez-Villegas formula* [**15**, Conjecture 1.2.1]. (This is a conjecture on a combinatorial formula for the compactly supported mixed Hodge polynomial of a character variety). In this thesis, we compute the mixed Hodge structures of a few example of character varieties of genus 0. Then we may verify this conjecture for these examples. The computation of the examples are done by the explicit description of character varieties via the classical invariant theory. For simplicity of descriptions of the invariant rings, we consider $\mathrm{SL}(n,\mathbb{C})$-character varieties of genus 0, that is,

$$\{(M_1, \dots, M_k) \in \mathcal{C}_1 \times \cdots \times \mathcal{C}_k \mid M_1 M_2 \cdots M_k = I_n\} /\!/ \mathrm{SL}(n, \mathbb{C})$$

which is the categorical quotient of the simultaneous $\mathrm{SL}(n,\mathbb{C})$-action. Here, $(\mathcal{C}_1, \dots, \mathcal{C}_k)$ is a tuple of semisimple conjugacy classes of $\mathrm{SL}(n,\mathbb{C})$ of type $\boldsymbol{\mu}$ and I_n is the identity matrix. We denote by $\hat{\mathcal{M}}^{\boldsymbol{\mu}}_{B}(\boldsymbol{\nu})$ the $\mathrm{SL}(n,\mathbb{C})$-character varieties of $g = 0$.

Remark. We can define character varieties whose the structure groups are $\mathrm{GL}(n,\mathbb{C})$, $\mathrm{PGL}(n,\mathbb{C})$ and $\mathrm{SL}(n,\mathbb{C})$. However, for $g = 0$, those character varieties are the same.

In the case of $g = 0$ and $\dim \hat{\mathcal{M}}_B^{\boldsymbol{\mu}}(\boldsymbol{\nu}) = 2$, $\mathrm{SL}(n, \mathbb{C})$-character varieties can be classified into four cases, which can be listed as follows:

$$\begin{aligned}
\boldsymbol{\mu} &= ((1,1),(1,1),(1,1),(1,1)),\\
\boldsymbol{\mu} &= ((1,1,1),(1,1,1),(1,1,1)),\\
\boldsymbol{\mu} &= ((2,2),(1,1,1,1),(1,1,1,1)),\\
\boldsymbol{\mu} &= ((3,3),(2,2,2),(1,1,1,1,1,1)).
\end{aligned}$$

In the first and second types, the $\mathrm{SL}(n, \mathbb{C})$-character varieties are known to be an affine cubic surface by the classical invariant theory. ([**10**], [**25**], [**26**], [**30**]). Then we can compute the mixed Hodge structures of the first and second types.

Compactifications of character varieties and the configurations of the boundary divisors. The next purpose is to study the configuration of boundary divisor of compactifications of character varieties. This study is motivated by a conjecture due to Simpson [**50**], which is explained as follows. We choose a smooth compactification $\overline{\mathcal{M}_B^{\boldsymbol{\mu}}(\boldsymbol{\nu})}$ of $\mathcal{M}_B^{\boldsymbol{\mu}}(\boldsymbol{\nu})$ such that $D_{g,\boldsymbol{\mu}}^B = \overline{\mathcal{M}_B^{\boldsymbol{\mu}}(\boldsymbol{\nu})} \setminus \mathcal{M}_B^{\boldsymbol{\mu}}(\boldsymbol{\nu})$ is a divisor with normal crossings. We call the divisor $D_{g,\boldsymbol{\mu}}^B$ a *boundary divisor* of the compactification $\overline{\mathcal{M}_B^{\boldsymbol{\mu}}(\boldsymbol{\nu})}$. Let $\overline{N}_{g,\boldsymbol{\mu}}^B$ be a small neighborhood of $D_{g,\boldsymbol{\mu}}^B$ in $\overline{\mathcal{M}_B^{\boldsymbol{\mu}}(\boldsymbol{\nu})}$, and let $N_{g,\boldsymbol{\mu}}^B = \overline{N}_{g,\boldsymbol{\mu}}^B \cap \mathcal{M}_B^{\boldsymbol{\mu}}(\boldsymbol{\nu}) = \overline{N}_{g,\boldsymbol{\mu}}^B \setminus D_{g,\boldsymbol{\mu}}^B$. Let $\Delta(D_{g,\boldsymbol{\mu}}^B)$ be a simplicial complex whose n-dimensional simplices correspond to the irreducible components of intersections of $k+1$ distinct components of $D_{g,\boldsymbol{\mu}}^B$. This is called the *boundary complex* or *Stepanov complex* of a compactification of $\mathcal{M}_B^{\boldsymbol{\mu}}(\boldsymbol{\nu})$ (see [**52**], [**53**], and [**37**]).

Theorem ([**52**], [**53**], and [**37**]). *The homotopy type of boundary complex $\Delta(D_{g,\boldsymbol{\mu}}^B)$ is independent of the choice of compactifications.*

We have a continuous map, well-defined up to homotopy,

$$N_{g,\boldsymbol{\mu}}^B \longrightarrow \Delta(D_{g,\boldsymbol{\mu}}^B).$$

On the other hand, let $\mathcal{M}_{Dol}^{\boldsymbol{\mu}}(\mathbf{0}) \to \mathbb{A}^{\frac{d_{g,\boldsymbol{\mu}}}{2}}$ be the Hitchin fibration. We have a canonical orbifold compactification of $\mathcal{M}_{Dol}^{\boldsymbol{\mu}}(\mathbf{0})$ via the relative compactification of the moduli space of parabolic λ-connections (see Section 1 of Chapter V). The divisor at infinity is the quotient

$$D_{g,\boldsymbol{\mu}}^{Dol} := \mathcal{M}_{Dol}^{\boldsymbol{\mu}}(\mathbf{0})^*/\mathbb{C}^*.$$

Here, $\mathcal{M}_{Dol}^{\boldsymbol{\mu}}(\mathbf{0})^*$ is the complement of the nilpotent cone. Let $\overline{N}_{g,\boldsymbol{\mu}}^{Dol}$ be a small neighborhood of $D_{g,\boldsymbol{\mu}}^{Dol}$, and let $N_{g,\boldsymbol{\mu}}^{Dol} = \overline{N}_{g,\boldsymbol{\mu}}^{Dol} \cap \mathcal{M}_{Dol}^{\boldsymbol{\mu}}(\mathbf{0}) = \overline{N}_{g,\boldsymbol{\mu}}^{Dol} \setminus D_{g,\boldsymbol{\mu}}^{Dol}$. The Hitchin fibration gives us a continuous map to the sphere at infinity in the Hitchin base

$$N_{g,\boldsymbol{\mu}}^{Dol} \longrightarrow S^{d_{g,\boldsymbol{\mu}}-1}.$$

Conjecture (C. Simpson [**51**]). (1) *There exists a homotopy-commutative diagram*

$$\begin{array}{ccc}
N_{g,\boldsymbol{\mu}}^{Dol} & \xrightarrow{\cong} & N_{g,\boldsymbol{\mu}}^{B} \\
\big\downarrow & & \big\downarrow \\
S^{d_{g,\boldsymbol{\mu}}-1} & \xrightarrow{\cong} & \Delta(D_{g,\boldsymbol{\mu}}^B).
\end{array}$$

(2) *In particular, there exists a non-singular compactification of* $\mathcal{M}_B^{\boldsymbol{\mu}}(\boldsymbol{\nu})$ *such that the boundary complex is a simplicial decomposition of sphere* $S^{d_{g,\boldsymbol{\mu}}-1}$.

Remark (See [**51**]). The assertion (1) of this conjecture due to Simpson is true in the case where $\boldsymbol{\mu} = ((1,1),(1,1),(1,1),(1,1))$.

The second theorem of this thesis is the following

Theorem B ((Ch.VI, Corollary 1.1.4 and Theorem 2.3.2)). *The assertion* (2) *of this conjecture due to Simpson is true in the following cases*:

(1) $g = 0, n = 3, k = 3, \boldsymbol{\mu} = ((1,1,1),(1,1,1),(1,1,1)),\ d_{g,\boldsymbol{\mu}} = 2$;
(2) $g = 0, n = 2, k = 5, \boldsymbol{\mu} = ((1,1),(1,1),(1,1),(1,1),(1,1)),\ d_{g,\boldsymbol{\mu}} = 4$.

We consider $\mathrm{SL}(n,\mathbb{C})$-character varieties $\hat{\mathcal{M}}_B^{\boldsymbol{\mu}}(\boldsymbol{\nu})$. The case (1) of Theorem B is verified by the classical invariant theory [**30**]. However, it seems that the application of the classical invariant theory is difficult for general cases. Then, we construct compactifications of $\mathrm{SL}(n,\mathbb{C})$-character varieties as follows. Following [**3**], we can construct a compactification of the *representation variety* [**3**]

$$\hat{\mathcal{U}}^{\boldsymbol{\mu}}(\boldsymbol{\nu}) := \{(A_1, B_1, \dots, A_g, B_g; M_1, \dots, M_k) \in SL_n(\mathbb{C})^{2g} \times \mathcal{C}_1 \times \cdots \times \mathcal{C}_k \mid (A_1,B_1)\cdots(A_g,B_g)M_1\cdots M_k = I_n\}.$$

Then, we take the GIT quotient of this compactification of $\hat{\mathcal{U}}^{\boldsymbol{\mu}}(\boldsymbol{\nu})$, which gives a compactification $\overline{\hat{\mathcal{M}}_B^{\boldsymbol{\mu}}(\boldsymbol{\nu})}$ of $\hat{\mathcal{M}}_B^{\boldsymbol{\mu}}(\boldsymbol{\nu})$. As special cases, we consider the cases where $g = 0, n = 2, k \geq 4, \boldsymbol{\mu} = ((1,1),\dots,(1,1))$. For $k = 4$, we obtain the same result as the classical invariant theory [**10**]. For $k = 5$ (i.e., the case (2) of Theorem B), $\overline{\hat{\mathcal{M}}_B^{\boldsymbol{\mu}}(\boldsymbol{\nu})}$ has singular points. We show that a suitable blowing up of $\overline{\hat{\mathcal{M}}_B^{\boldsymbol{\mu}}(\boldsymbol{\nu})}$ has a boundary whose boundary complex is a simplicial decomposition of S^3. It seems that the configuration of the boundary divisor $D_{0,\boldsymbol{\mu}}^B$ is rather complicated for $k \geq 6$.

The conjecture due to Simpson is related to the P=W conjecture due to Hausel et al ([**4**]). First, we consider compact curve cases. The non-abelian Hodge theory for compact curves states that character varieties $\mathcal{M}_B$ are diffeomorphic to moduli spaces $\mathcal{M}_{Dol}$ of semi-stable Higgs bundles. Then, we have the induced isomorphism between the rational cohomology groups of $\mathcal{M}_B$ and $\mathcal{M}_{Dol}$. The P=W conjecture assert that the isomorphism of the rational cohomology groups exchanges the weight filtration on the cohomology groups of $\mathcal{M}_B$ with the perverse Leray filtration associated with the Hitchin fibration on the cohomology groups of $\mathcal{M}_{Dol}$. The P=W conjecture is verified in the case where $n = 2$ ([**4**]). We may extend the conjecture to punctured curve cases. On the other hand, there exists a natural isomorphism from the reduced homology of the boundary complex $\Delta(D_{g,\boldsymbol{\mu}}^B)$ to the $2l$-th graded piece of the weight filtration on the cohomology of $\mathcal{M}_B^{\boldsymbol{\mu}}(\boldsymbol{\nu})$:

$$\widetilde{H}_{i-1}(\Delta(D_{g,\boldsymbol{\mu}}^B),\mathbb{Q}) \cong Gr_{2l}^W H^{2l-i}(\mathcal{M}_B^{\boldsymbol{\mu}}(\boldsymbol{\nu})).$$

(For example, see [**37**, Theorem 4.4]). By the isomorphism, the assertion (2) of the conjecture due to Simpson above implies that there exists only 1-dimensional weight $2d_{g,\boldsymbol{\mu}}$ part in the middle degree $d_{g,\boldsymbol{\mu}}$ cohomology of the character variety, which is also a consequence of the P=W conjecture.

The organization of this thesis is as follows. From Chapter 1 to Chapter 4, we recall the basic notations and basic results. In Chapter 1, we recall the basic facts on connections, Higgs fields. The main reference is Sabbah's book [**39**]. In Chapter 2, after recalling the moduli problems and geometric invariant theory, we review the moduli problem of Λ^1_D-triples due to Inaba-Iwasaki-Saito [**23**]. Moreover the character varieties can be introduced in the GIT setting. In Chapter 3, we recall basic results on Hodge theory and mixed Hodge theory of open algebraic varieties or singular algebraic varieties due to Deligne [**6**], [**7**]. In Chapter 4, nonabelian Hodge theory and Riemann-HIlbert correspndence are reviewed. In last two chapters (Chapter 5 and 6), we will prove our main results.

CHAPTER 1

Connections

In this chapter, we recall several well-known facts on holomorphic connections, meromorphic connections, and connections which have regular singularities. Our main reference is the book of Sabbah [**39**].

1. Holomorphic integrable connections and Higgs fields

Let X be a complex analytic manifold, and let $\pi\colon E \to X$ be a holomorphic vector bundle on X. We use the same notation E for the locally free sheaf associated to the vector bundle E. Let Ω_X^i be the sheaf of holomorphic differential i-forms on X.

1.1. Holomorphic connections.

Definition 1.1.1. A *holomorphic connection* ∇ on a holomorphic vector bundle $\pi\colon E \to X$ is a $\mathbb{C}$-linear holomorphism of sheaves

$$\nabla\colon E \longrightarrow \Omega_X^1 \otimes E$$

satisfying the *Leibniz rule*: for any open set U of X, any section $s \in \Gamma(U, E)$ and any holomorphic function $f \in \mathcal{O}(U)$

$$\nabla(f \cdot s) = f\nabla(s) + df \otimes s \in \Gamma(U, \Omega_X^1 \otimes E).$$

In general, one can extend the holomrphic connection ∇ on the section of E as on the section of $\Omega_X^1 \otimes E$ (with values in $\Omega_X^2 \otimes E$):

$$\nabla(\omega \otimes s) = d\omega \otimes s - \omega \wedge \nabla(s).$$

Then, we define the *curvature* R_∇ of the connection ∇ as the following

$$R_\nabla := \nabla \circ \nabla\colon E \longrightarrow \Omega_X^2 \otimes E,$$

which is an $\mathcal{O}_X$-linear homomorphism.

Definition 1.1.2. The connection $\nabla\colon E \to \Omega_X^1 \otimes E$ is said to be *integral*, or also *flat*, if its curvature R_∇ vanishes identically.

Remark 1.1.3. When $\dim X = 1$, the sheaf Ω_X^2 is zero and therefore the integrability condition is fulfilled by any connection.

Definition 1.1.4. A ∇-*horizontal holomorphic section* of E on an open set U of X is a section $s \in \Gamma(U, E)$ satisfying $\nabla(s) = 0$. We denote by E^∇ the sheaf of ∇-horizontal holomorphic sections of E.

Theorem 1.1.5 (Cauchy–Kowalevski theorem). *Let $\nabla\colon E \to \Omega^1_X \otimes E$ be a holomorphic integral connection on some bundle E of rank d. Then,*

(1) *The sheaf E^∇ is a locally constant sheaf of $\mathbb{C}$-vector space of rank d;*
(2) *The sheaf $\mathcal{O}_X \otimes_{\mathbb{C}_X} E^\nabla$ is locally free sheaf of $\mathcal{O}_X$-modules, the connection on this sheaf defined by $\nabla(f \otimes s) = df \otimes s$ is flat and $(\mathcal{O}_X \otimes_{\mathbb{C}_X} E^\nabla)^\nabla = E^\nabla$;*
(3) *The natural homomorphism $\mathcal{O}_X \otimes_{\mathbb{C}_X} E^\nabla \to \mathcal{E}$ is an isomorphism of bundles with connection.*

Proof. For example, see [**39**, Ch.0, Theorem 12.8]. □

1.2. Higgs fields.

Definition 1.2.1. A *Higgs field* on a holomorphic bundle E is an $\mathcal{O}_X$-linear homomorphism

$$\Phi\colon E \longrightarrow \Omega^1_X \otimes E$$

which fulfills the condition $\Phi \wedge \Phi = 0$. We also say that (E, Φ) is a *Higgs bundle.*

Remark 1.2.2. When $\dim X = 1$, the condition $\Phi \wedge \Phi = 0$ is fulfilled by any $\mathcal{O}_X$-linear homomorphism $\Phi\colon E \to \Omega^1_X \otimes E$.

2. Meromorphic connections

2.1. Meromorphic bundles, lattice. Let Z be a smooth hypersurface in a complex anaytic manifold X. For the open set U of X, let f be a holomorphic function on $U \setminus Z \cap U$ and, for any chart V of X contained in U, in which $Z \cap V$ is defined by the vanishing if some coordinate, z_1 for instance, there exists an integer m such that $z_1^m f(z_1, \ldots, z_n)$ is locally bounded in the neighborhood of any point of $Z \cap U$, that is, can be extended as a holomorphic function on V. We call the function f *meromorphic function along* Z.

Definition 2.1.1. Let $\mathcal{O}_X(*Z)$ be a sheaf of meromorphic functions along Z. A *meromorphic bundle on M with pole along Z* is a locally free sheaf of $\mathcal{O}_X(*Z)$-module of finite rank. A *lattice* of this meromorphic bundle is a locally free $\mathcal{O}_X$-submodules of this meromorphic bundle, which has the same rank.

A meromorphic bundle M can contain *no* lattice at all, and can also contains some lattice which are non-isomorphic each other. If there exists a lattice E for a meromorphic bundle M with pole along Z, then E coincides with M on $M \setminus Z$ and we have

$$M = \mathcal{O}_X(*Z) \otimes_{\mathcal{O}_X} E.$$

For the existence of lattices, we have the following

Proposition 2.1.2. *Let X be a Riemann surface and let $Z \subset X$ be a discrete set of points. Then any meromorphic bundle on X with poles at the points of Z contains at least a lattice.*

Proof. For example, see [**39**, Ch.0, Propsition 8.4]. □

2.2. Meromorphic connections. Let M be a meromorphic bundle with pole along a hypersurface Z. We define a connection on M as a $\mathbb{C}$-linear homomorphism

$$\nabla\colon M \longrightarrow \Omega^1_X \otimes_{\mathcal{O}_X} M$$

satisfying the Leibniz rule. Note that, in local basis of M over $\mathcal{O}_X(*Z)$, the matrix of the connection has entries in $\mathcal{O}_X(*Z) \otimes_{\mathcal{O}_X} \Omega^1_X = \Omega^1_X(*Z)$.

We say that a connection on a meromorphic bundle is *integral* (or *flat*) if its restriction to $X \setminus Z$ is an integrable connection on the holomorphic bundle $M_{X\setminus Z}$. The horizontal sections $M^{\nabla}_{X\setminus Z}$ is then a locally constant sheaf of finite dimensional vector space. The locally constant sheaf determine the linear representation of the fundamental group $\pi_1(X\setminus Z, *)$. The linear representation is called the *monodromy representation* attached to the meromorphic bundle with connection (M, ∇).

Definition 2.2.1. Let M be a meromorphic bundle with pole along a hypersurface Z. Assume that there exists a lattice E of M. For a connection ∇ on M, we define the *meromorphic connection* ∇ *on* E

$$\nabla\colon E \longrightarrow \Omega^1_X(*Z) \otimes_{\mathcal{O}_X} E$$

by $\nabla(E) \subset \nabla(M) \subset \Omega^1_X \otimes M = \Omega^1_X(*Z) \otimes_{\mathcal{O}_X} E$.

We say that E is a *logarithmic lattice* of the meromorphic bundle with connection (M, ∇) if $\nabla(E) \subset \Omega^1_X\langle \log Z\rangle \otimes_{\mathcal{O}_X} E$, i.e. the meromorphic connection on E is the following

$$\nabla\colon E \longrightarrow \Omega^1_X\langle \log Z\rangle \otimes_{\mathcal{O}_X} E.$$

3. Connections which have regular singularities

In this section, we treat the following topics

- the normal form of a connection on D which has regular singularity at 0 (Proposition 3.2.5);
- the construction of the canonical logarithmic lattice for a connection on D which has regular singularity at 0 (Proposition 3.2.6);
- the equivalence of the category of connections on D which have regular singularity at 0 with some condition and the category of local systems on the disc D (Theorem 3.2.8).

Here, we put $D := \{t \in \mathbb{C} \mid |t| < r\}$ where r is arbitrarily small.

3.1. Definition.

Definition 3.1.1. Let M be a meromorphic bundle with pole along a hypersurface Z and ∇ be a connection on M. We say that (M, ∇) has *regular singular along* Z if ∇ is a flat connection and, in the neighborhood U of any point $z \in Z$, there exists a logarithmic lattice E_U of $M_{|U}$.

The condition of regular singularity is local on Z and one does not ask for the existence of a logarithmic lattice globally on M. Nevertheless, one can show this global existence.

3.2. Local study for Riemann surfaces. In this section, we consider connections which have regular singularities on the disc $D := \{t \in \mathbb{C} \mid |t| < r\}$ where r is arbitrarily small, so that 0 is the only possible pole of the meromorphic function that we consider. More precisely, we consider the germ $(\mathbb{C}, 0)$ of a Riemann surface.

We describe the germ of meromorphic bundle of rank d on the germ $(\mathbb{C}, 0)$ equipped with the connection. The ring of germ of meromorphic functions with poles at 0 is the field $\mathbb{C}\{t\}[1/t]$ of converging meromorphic Laurent series. For simplicity, we denote by $\boldsymbol{k}$ this field. The natural action of the derivation makes it a *differential field*. Therefore, a germ of meromorphic bundle with connection is nothing but a $\boldsymbol{k}$-vector space M of rank d equipped with a derivation ∇ compatible to the derivation of $\boldsymbol{k}$, denoted by (M, ∇). The derivation ∇ is described as $d + \Omega$ where $\Omega = A(t)dt$ and the entries of $A(t)$ are germs of meromorphic functions with pole at 0. We call a germ of meromorphic bundle with connection a $(\boldsymbol{k}, \nabla)$*-space*.

Two $(\boldsymbol{k}, \nabla)$-vector space with connection (M_1, ∇_1) and (M_2, ∇_2) are isomorphic if and only if the corresponding matrices Ω_1 and Ω_2 are related by a meromorphic base change $P \in \mathrm{GL}(d, \boldsymbol{k})$ via the relation

$$\Omega_2 = P^{-1}\Omega_1 P + P^{-1}dP.$$

and the matrices A_1 and A_2 by the relation

$$A_2 = P^{-1}A_1 P + P^{-1}P' \tag{3.2.1}$$

where $\Omega_i = A_i(t)dt$ for $i = 1, 2$.

A $(\boldsymbol{k}, \nabla)$-vector space (M, ∇) of rank d defined by a matrix $A_1(t)$ has a *regular singular at* 0, if there exists a matrix $P \in \mathrm{GL}(d, \boldsymbol{k})$ such that the matrix $A_2(t)$ obtained after the base change (3.2.1) has at most a *simple pole at* $t = 0$. Otherwise, the singularity is called *irregular*.

A *lattice* of the $(\boldsymbol{k}, \nabla)$-vector space (M, ∇) is a free $\mathbb{C}\{t\}$-module $E \subset M$ such that

$$\boldsymbol{k} \otimes_{\mathbb{C}\{t\}} E = M.$$

Example 3.2.1. We define an *elementary regular model* as a $(\boldsymbol{k}, \nabla)$-vector space equipped with a basis in which the connection matrix is written as

$$\Omega(t) = (\alpha \mathrm{Id} + N)\frac{dt}{t} \tag{3.2.2}$$

where $\alpha \in \mathbb{C}$ and N is a nilpotent matrix. Suppose that N has a single Jordan block of size $d \geq 0$. Then, we have the following

(1) the horizontal sections s on any simply connected open set U of $D \setminus \{0\}$ is obtain by taking the linear combinations with coefficients in $\mathbb{C}$ of the columns of the matrix

$$t^{-(\alpha \mathrm{Id}+N)} := t^{-\alpha}\left(\mathrm{Id} - N\log t + N^2\frac{(\log t)^2}{2!} + \cdots + (-N)^d\frac{(\log t)^d}{d!}\right);$$

(2) the monodromy representation defined by the horizontal sections is given by

$$T = \exp(-2\pi\sqrt{-1}(\alpha \mathrm{Id} + N));$$

(3) the horizontal sections have *moderate growth near the origin*, that is, for any horizontal section s on the neighborhood of a closed angular sector with angle $< 2\pi$, there exists an integer $n \geq 0$ and a constant $C > 0$ such that

$$\|s(t)\| \leq C|t|^{-n}.$$

on this closed sector.

We show that any $(\boldsymbol{k}, \nabla)$-vector space with regular singularity is isomorphic to a direct sum of elementary regular models as follows.

Lemma 3.2.2. (1) *Let $\mathbb{C}[[t]]$ be a field of formal power series, and let $\widehat{\Omega} = \widehat{A}(t)$ has entries in $\mathbb{C}[[t]]$. We assume that any two eigenvalues of the matrix $\widehat{A}(0)$ do not differ by a nonzero integer. Then, there exists a matrix $\widehat{P} \in \mathrm{GL}(d, \mathbb{C}[[t]])$ such that*

$$\widehat{P}^{-1}\widehat{A}\widehat{P} + t\widehat{P}^{-1}\widehat{P}' = \widehat{A}(0).$$

(2) *Moreover, the matrix of the base change $\widehat{P}$ has converging entries.*

(3) *Therefore, let (M, ∇) be $(\boldsymbol{k}, \nabla)$-vector space with regular singularity. Let E be a logarithmic lattice of M. If the eigenvalues of the residue $\mathrm{Res}_E(\nabla)$ of the connection on E do not differ by a nonzero integer, then there exists a basis of E over $\mathbb{C}\{t\}$ such that the connection matrix takes the form $A_0 dt/t$ where A_0 is constant.*

Proof. We show the first assertion. Put $\widehat{A}(t) = A_0 + A_1 t + \cdots$. We search for a matrix $\widehat{P} \in \mathrm{GL}(d, \mathbb{C}[[t]])$ satisfying

$$\widehat{P}^{-1}\widehat{A}\widehat{P} + t\widehat{P}^{-1}\widehat{P}' = A_0.$$

One can regard this relation as a differential equation satisfying by $\widehat{P}$:

$$t\widehat{P}' = \widehat{P}A_0 - \widehat{A}P. \tag{3.2.3}$$

We set a priori $\widehat{P} = \mathrm{Id} + tP_1 + t^2P_2 + \cdots$ where P_l are constant matrices. The term of degree l is the following

$$lP_l = P_l A_0 - A_0 P_l + \Phi_l(P_1, \ldots, P_{l-1}; A_0, \ldots, A_l)$$

where Φ_l is a matrix depending in a polynomial way on its variables. Hence, we have

$$P_l(l\mathrm{Id} - A_0) - (-A_0)P_l = \Phi_l(P_1, \ldots, P_{l-1}; A_0, \ldots, A_l). \tag{3.2.4}$$

We determine the matrices P_l by the inductive way. We take an integer $l \geq 1$. Let us assume that we have determined the matrices P_k, $(k \geq l-1)$. We apply the following lemma to the equation (3.2.4).

Lemma 3.2.3 (For example, see [**39**, Ch.II, Lemma 2.16]). *Let $U \in \mathrm{M}(p, \mathbb{C})$ and $V \in \mathrm{M}(q, \mathbb{C})$ be two matrices. Then, the following properties are equivalent*

(1) *for any matrix Y of size $q \times p$ with entries in $\mathbb{C}$, there exists a unique matrix X of the same kind satisfying $XU - VX = Y$;*
(2) *the square matrices U and V have no common eigenvalue.*

Then we can determine a matrix P_l.

We show the second assertion. Note that equation (3.2.3) defines $\widehat{P}$ as a horizontal section of a differential linear system of rank d^2 and that matrix of this system has at most a simple pole at 0. Then the convergence follows from the following lemma.

Lemma 3.2.4 (For example, see [**39**, Ch.II, Lemma 2.18]). *Let $\mathbb{C}\{t\}$ be the ring of convergent power series, and let $A(t)$ be a matrix in $\mathrm{M}(d, \mathbb{C}\{t\})$. Any vector $u(t)$ with entries in $\mathbb{C}[[t]]$ which is solution of the system $tu'(t)+A(t)u(t) = 0$ has converging entries.*

By the first and second assertions, we obtain the third assertion. □

Then we have the normal form of a connection on D which has regular singularity at 0.

Proposition 3.2.5. *Any $(\boldsymbol{k}, \nabla)$-vector space with regular singularity is isomorphic to a direct sum of elementary regular model.*

PROOF. Let $\Omega := A(t)dt/t$ which is a connection matrix of ∇. Here, $A(t)$ has entries in $\mathbb{C}[[t]]$ which is the field of formal power series $\sum_{n\geq 0} a_n t^n$. Let $A(t) := A_0 + \sum_{j\geq 1} A_j t^j$.

We can change the eigenvalues by the following transformation: First, by a base change with constant matrix, we reduce to the case where the matrix A_0 is blockdiagonal where each block corresponds to some eigenvalue λ_i, $(i = 1, \ldots, p)$ of A_0. Then, let us write

$$A(t) = \begin{pmatrix} A_0^{(1)} & 0 \\ 0 & A_0^{(2)} \end{pmatrix} + \sum_{j\geq 1} A_j t^j$$

where $A_0^{(1)}$ is the block corresponding to the eigenvalue λ_1 and $A_0^{(2)}$ to the eigenvalues λ_j, $(j = 2, \ldots, p)$. Secondary, we set

$$Q(t) := \begin{pmatrix} t\mathrm{Id} & 0 \\ 0 & \mathrm{Id} \end{pmatrix}.$$

Then, the eigenvalues of the constant part B_0 of the matrix $B := Q^{-1}AQ + tQ^{-1}Q'$ are $\lambda_1+1, \lambda_2, \ldots, \lambda_p$. By the finite sequence of such changes, we obtain a matrix B_0 such that the eigenvalues of the residue $\mathrm{Res}_E(\nabla)$ of the connection on E do not differ by a nonzero integer.

By Lemma 3.2.2 and normalization of a constant matrix, we have the assertion. □

For any $(\boldsymbol{k}, \nabla)$-vector space with regular singularity, we construct the *canonical* logarithmic lattice. We call the lattice the *local Deligne lattice*.

Proposition 3.2.6 (Local Deligne lattice). *Let (M, ∇) be $(\boldsymbol{k}, \nabla)$-vector space with regular singularity. Let σ be a section of the natural projection $\mathbb{C} \to \mathbb{C}/\mathbb{Z}$ (so that two complex numbers in the image $\mathrm{Im}(\sigma)$ of σ do not differ by a nonzero integer). Then, there exists a unique logarithmic lattice ${}^{\sigma}V_{(M,\nabla)}$ of (M, ∇) such that the eigenvalues of $\mathrm{Res}_{{}^{\sigma}V_{(M,\nabla)}}(\nabla)$ are contained in the region $\mathrm{Im}(\sigma)$. Moreover, for any homomorphism $\phi\colon (M, \nabla) \to (M', \nabla')$, we have*

$$\phi({}^{\sigma}V_{(M,\nabla)}) = {}^{\sigma}V_{(\mathrm{Im}\phi,\nabla)} = {}^{\sigma}V_{(M',\nabla')} \cap \mathrm{Im}(\phi).$$

Proof. There exists a logarithmic lattice E of M by definition of regular singularity. We can change the eigenvalues by the transformation as in the the proof of Proposition 3.2.5. By the finite sequence of the transformations, we obtain the logarithmic lattice E such that the eigenvalues of $\mathrm{Res}_{{}^\sigma V_{(M,\nabla)}}(\nabla)$ are contained in $\mathrm{Im}(\sigma)$.

We show the uniqueness of the logarithmic lattices. If E and E' are two such lattice, then there exists a basis $\boldsymbol{e}$ of E (resp. $\boldsymbol{e}'$ of E') in which the matrix $A_0 dt/t$ (resp. $A'_0 dt/t$) of ∇ has eigenvalues in $\mathrm{Im}(\sigma)$ by Proposition 3.2.5. Here, the matrices A_0 and A'_0 are constant. We search for a matrix $P \in \mathrm{GL}(d, \boldsymbol{k})$ satisfying

$$P^{-1}A_0P + tP^{-1}P' = A'_0.$$

One can regard this relation as a differential equation satisfying by P:

$$tP' = PA'_0 - A_0P.$$

We set a priori $P = t^{-n}P_{-n} + \cdots + P_0 + tP_1 + \cdots$ where P_l are constant matrices to be determined and $P_0^{-1}A'_0P_0 = A_0$. The term of degree l is the following

$$lP_l = P_lA'_0 - A_0P_l \quad \text{where } l \neq 0.$$

Hence,

$$P_l(l\mathrm{Id} - A'_0) - (-A_0)P_l = 0.$$

We have $P_l = 0$ $(l \neq 0)$ by the Lemma 3.2.3. Then, the matrix of the base change $P \in \mathrm{GL}(d, \boldsymbol{k})$ from $\boldsymbol{e}$ to $\boldsymbol{e}'$ is constant and conjugates A_0 and A'_0. We have the uniqueness.

We consider a homomorphism $\phi\colon (M, \nabla) \to (M', \nabla')$. Since the image by ϕ of a lattice is a lattice of the image of ϕ, the second assertion follows from uniqueness. □

Now, we consider the functor between the category of $(\boldsymbol{k}, \nabla)$-vector spaces with regular singularity and the category of vector spaces equipped with the automorphism $T = \exp(-2\pi\sqrt{-1}\mathrm{Res}_{{}^\sigma V_{(M,\nabla)}}\nabla)$:

$$(M, \nabla) \longmapsto ({}^\sigma H_{(M,\nabla)}, T) \tag{3.2.5}$$

where ${}^\sigma H_{(M,\nabla)} := {}^\sigma V_{(M,\nabla)}/t^\sigma V_{(M,\nabla)}$ and T is the monodoromy representation.

We show the functor (3.2.5) is an equivalence of categories. By Proposition 4.2.2 as below, we show the functor is fully faithful and essentially surjective. First, we show that the functor is essentially surjective:

Lemma 3.2.7. *Let T be a linear automorphism of $\mathbb{C}^d$ (defining a local system of rank d on some disc punctured D^*). Then, there exists a unique (up to constant) bundle with meromorphic connection on D with logarithmic pole at 0 for which*

(1) *the local system it defines on D^* is that associated to T;*
(2) *the residue at 0 of the connection has eigenvalues in the region $\mathrm{Im}(\sigma)$.*

Proof. We consider a $\mathbb{C}$-vector space $V_d = \mathbb{C}^d$ with the linear automorphism T. One can decompose in a unique way $V_d = \bigoplus_\lambda F_\lambda$ (Jordan decomposition of T) where F_λ is a direct sum of spaces $\mathbb{C}[T, T^{-1}]/(T - \lambda)^k$. We put

$$\mathcal{E} := V_d \otimes_{\mathbb{C}} \mathcal{O}_D.$$

We endow the vector bundle associated to $\mathcal{E}$ with the meromorphic connection ∇ which has a connection matrix having the $k \times k$-matrix

$$(\alpha \mathrm{Id} + N)\frac{dt}{t}.$$

Here, α is the unique logarithm of λ which belongs to $\mathrm{Im}(\sigma)$, N is a Jordan block of size k, and t is a coordinate on D. The bundle with meromorphic connection $(\mathcal{E}, \nabla)$ is desired bundle (see Example 3.2.1 (2)).

We show the uniqueness. If we have two such bundles with connections (E, ∇) and (E', ∇'), then the bundle with meromorphic connection $\mathcal{H}om_{\mathcal{O}_D}(\mathcal{E}, \mathcal{E}')$ also has a logarithmic pole at 0 and eigenvalues of the residue of its connection are obtained as the differences of the eigenvalues of $\mathrm{Res}\nabla$ and $\mathrm{Res}\nabla'$. Therefore, the only integral difference is 0, by assumption. Since the bundles E and E' have same monodromy on D^*, there exists an isomorphism of the associated locally constant sheaves on D^*. Hence, we get an invertible horizontal section of $\mathcal{H}om_{\mathcal{O}_D}(\mathcal{E}, \mathcal{E}')$ on D^*. We extend the section on D^* to the section on D as follows. Since the singularity of $\mathcal{H}om_{\boldsymbol{k}}(M, M')$ is regular, this section has moderate growth. Hence, this section is meromorphic at 0. We regard the the connection on $\mathcal{H}om_{\mathcal{O}_D}(\mathcal{E}, \mathcal{E}')$ as a following differential linear system of rank d^2:

$$tu'(t) + A(t)u(t) = 0$$

where $u(t) \in \boldsymbol{k}^{d^2}$ and $A(t) \in \mathrm{M}(d^2, \mathbb{C}[[t]])$. Since the unique integral eigenvalue of the residue of the connection on $\mathcal{H}om_{\mathcal{O}_D}(\mathcal{E}, \mathcal{E}')$ is more than 0, this section is holomorphic. In fact, if u_l denotes the coefficient of t^l in $u(t)$, then the term $(l\mathrm{Id} - A(0))u_l$ can be expressed in the term of the u_j for $j < l$. Since $l\mathrm{Id} - A(0)$ is invertible for $l < 0$, we deduce that $u_l = 0$ for $l < 0$ by induction. Therefore, we have the extended holomorphic section of $\mathcal{H}om_{\mathcal{O}_D}(\mathcal{E}, \mathcal{E}')$ on D. The section determine the isomorphism between (E, ∇) and (E', ∇'). □

Lastly, we show the functor (3.2.5) is fully faithful. We consider the map

$$\mathrm{Hom}((M, \nabla), (M', \nabla')) \to \mathrm{Hom}(({}^{\sigma}V_{(M,\nabla)}, T), ({}^{\sigma}V_{(M',\nabla')}, T')). \tag{3.2.6}$$

First, we show that the map (3.2.6) is injective. If $\varphi \in \mathrm{Hom}((M, \nabla), (M', \nabla'))$ satisfying $\varphi({}^{\sigma}V_{(M,\nabla)}) \subset t \cdot {}^{\sigma}V_{(M',\nabla)}$, then we have ${}^{\sigma}V_{\varphi((M,\nabla))} = t \cdot {}^{\sigma}V_{\varphi((M,\nabla))}$ by Proposition 3.2.6. Therefore, we have $\varphi = 0$. Second, we show that the map (3.2.6) is surjective. By choosing suitable bases, we are reduced to showing that, if A_0 and A_0' are two square matrices of size d and d' respectively having eigenvalues on $\mathrm{Im}(\sigma)$, then any matrix $P(t)$ of size $d' \times d$ satisfying $PA_0 - A_0'P = tP'$ is constant. This follows from Lemma 3.2.3, as the only integral eigenvalue of the linear operator $P \mapsto PA_0 - A_0'P$ is 0. Then we have the following

Theorem 3.2.8. *The functor*

$$(M, \nabla) \longmapsto {}^{\sigma}H_{(M,\nabla)} := {}^{\sigma}V_{(M,\nabla)}/t^{\sigma}V_{(M,\nabla)}$$

is an equivalence between the category of $(\boldsymbol{k}, \nabla)$-vector spaces with regular singularity and the category of the vector spaces equipped with the automorphism $T = \exp(-2\pi\sqrt{-1}\mathrm{Res}_{{}^{\sigma}V_{(M,\nabla)}}\nabla)$.

4. Appendix: the language of categories

4.1. Definitions. Now, we recall the language of categories briefly. Let $\mathcal{C}$ and $\mathcal{C}'$ be categories. A (*covariant*) *functor* from $\mathcal{C}$ to $\mathcal{C}'$ consists in given a mapping $F\colon \mathrm{Ob}(\mathcal{C}) \to \mathrm{Ob}(\mathcal{C}')$ and, for any pair (X, Y) of objects of $\mathcal{C}$, of a mapping $F\colon \mathrm{Hom}(X, Y) \to \mathrm{Hom}(F(X), F(Y))$ compatible with composition and preserving the identity morphism. A covariant functor is called simply a functor.

Definition 4.1.1. Let $\mathcal{C}$ and $\mathcal{C}'$ be categories, and F and G be functors from $\mathcal{C}$ to $\mathcal{C}'$. A *natural transformation* (or *functorial morphism*) $\rho\colon F \to G$ is a family

$$\{\rho(X)\colon F(X) \to G(X)\}_{X \in \mathrm{Ob}(\mathcal{C})}$$

such that, for a morphism $f\colon X \to Y$ of $\mathcal{C}$, the following diagram commutes:

$$\begin{array}{ccc} F(Y) & \xrightarrow{\rho(Y)} & G(Y) \\ \downarrow{\scriptstyle F(f)} & & \downarrow{\scriptstyle G(f)} \\ F(X) & \xrightarrow{\rho(X)} & G(X) \end{array}$$

If there exist natural transforms $\rho\colon F \to G$ and $\tau\colon G \to F$ satisfying $\tau\rho = \mathrm{id}_F$ and $\rho\tau = \mathrm{id}_G$ (where id_F means that the natural transformation $F \to F$ which is the identity mapping on any set), then the functor F, G are said to be *isomorphic*.

Definition 4.1.2. Let $\mathcal{C}$ and $\mathcal{C}'$ be categories, and F be a functor from $\mathcal{C}$ to $\mathcal{C}'$. For the functor F, the functor F' is said to be *inverse functor* (resp. *quasi-inverse functor*), if $F \circ F'$ and $F' \circ F$ are equal (resp. isomorphic) to the identity functors of each categories.

Let $\mathcal{C}$ and $\mathcal{C}'$ be categories, and F be a functor from $\mathcal{C}$ to $\mathcal{C}'$. The functor F is said to be an *isomorphism of category*, if there exists an inverse functor F'. On the other hand, The functor F is said to be an *equivalence of categories*, if there exists a quasi-inverse functor F'.

4.2. criterion of equivalence. We explain the criterion of equivalence of categories.

Definition 4.2.1. We say that a functor $F\colon \mathcal{C} \to \mathcal{C}'$ is *fully faithful* if, for any pair (X, Y) of objects on $\mathcal{C}$, the map $F\colon \mathrm{Hom}(X, Y) \to \mathrm{Hom}(F(X), F(Y))$ is a bijection. We say that a functor $F\colon \mathcal{C} \to \mathcal{C}'$ is *essentially surjective* if, for any object X' of $\mathcal{C}'$, there exists an object X of $\mathcal{C}$ such that $F(X)$ is isomorphic to X'.

We have the criterion:

Proposition 4.2.2. *Let $\mathcal{C}$ and $\mathcal{C}'$ be categories, and F be a functor from $\mathcal{C}$ to $\mathcal{C}'$. The functor F is an equivalence of categories if and only if it is fully faithful and essentially surjective.*

Proof. For example, see [**33**, IV, §4, Theorem 1]. □

CHAPTER 2

Moduli Problems and Geometric Invariant Theory

The purpose of this chapter is the construction of the moduli space of *semistable parabolic Λ^1_D-triples* and *character varieties* via the geometric invariant theory (GIT). The construction of the moduli space of semistable parabolic Λ^1_D-triples is due to Inaba–Iwasaki–Saito [**23**]. We construct the moduli space of semistable parabolic connections and the moduli space of semistable parabolic Higgs bundles as the subvarieties of the moduli space of semistable parabolic Λ^1_D-triples in Chapter IV.

1. Moduli Problem

Let k be a field, let S be k-scheme of finite type. Let $\mathcal{C}$ be a category, $\mathcal{C}^{op}$ the opposite category, i.e. the category with same objects and reversed arrows.

A *moduli functor* or *moduli problem* is a functor from the (opposite) category of S-schemes to the category of sets which associates to an S-scheme X the equivalence classes of families of geometric objects parametrized by X:

$$\begin{aligned} (Sch/S)^{op} &\longrightarrow (Sets) \\ X &\longmapsto \{\text{families of geometric objects over } X\}/\cong . \end{aligned}$$

On the other hand, for a S-scheme X, we define the *functor of points*

$$\underline{X}\colon (Sch/S)^{op} \longrightarrow (Sets)$$

as follows. For a S-scheme Y, we put $\underline{X}(Y) = \mathrm{Hom}_S(Y,X)$ and, for a S-morphism $f\colon Y \to Z$, we give the map of set $\underline{X}(f)\colon \mathrm{Hom}_S(Z,X) \to \mathrm{Hom}_S(Y,X)$ by the composition $g \circ f$ where $g \in \mathrm{Hom}_S(Z,X)$.

Now, any S-morphism $f\colon X \to Y$ induces a natural transformation $\underline{f}\colon \underline{X} \to \underline{Y}$. By Yoneda's Lemma, the functor of points $\underline{X}$ determines the S-scheme X. More precisely, if two functor $\underline{X}$ and $\underline{Y}$ are isomorphic as functor, then X and Y are isomorphic as S-scheme.

1.1. Fine Moduli.

For a moduli problem F, a *solution* of F is an isomorphism of the functor F with the functor $\underline{X}$ of some S-scheme X:

Definition 1.1.1. A functor $F\colon (Sch/S)^{op} \to (Sets)$ is said to be *representable*, and to be *represented by a S-scheme X*, if it is isomorphic to the functor $\underline{X}$. The S-scheme X is called a *fine moduli space* for the functor F.

Example 1.1.2. Let k be a field, let V be a finite dimensional vector space and let r be an integer where $0 \leq r \leq \dim V$. Let

$$\underline{\mathrm{Grass}}(V, r) \colon (Sch/k)^{op} \longrightarrow (Sets)$$

be the functor which associates to any k-scheme T of finite type the set of all subsheaves $K \subset \mathcal{O}_T \otimes_k V$ with the quotient $F := \mathcal{O}_T \otimes_k V/K$ which is locally free of rank r. *Then the functor* $\underline{\mathrm{Grass}}(V, r)$ *is represented by a smooth projective variety* $\mathrm{Grass}(V, r)$ (for example, see [**20**, Example 2.2.2]).

Example 1.1.3. The previous example can be generalized to the case where V is replaced by a coherent sheaf $\mathcal{V}$ on a k-scheme S of finite type. We define a functor

$$\underline{\mathrm{Grass}}_S(\mathcal{V}, r) \colon (Sch/S)^{op} \longrightarrow (Sets)$$

as follows:

$$\underline{\mathrm{Grass}}_S(\mathcal{V}, r)(T) := \left\{ [\mathcal{O}_T \otimes \mathcal{V} \twoheadrightarrow F] \middle| F\colon \text{locally free on } T \text{ of rank } r \right\},$$

which is the set of equivalence classes. Here, the equivalence relation is defined by the isomorphism of the quotient including the surjective:

$$\begin{array}{ccc} \mathcal{O}_T \otimes \mathcal{V} & \twoheadrightarrow & F \\ & \searrow & \downarrow \cong \\ & & F'. \end{array}$$

Then the functor $\underline{\mathrm{Grass}}_S(V, r)$ *is represented by a projective* S*-scheme* $\pi\colon \mathrm{Grass}_S(V, r) \to S$ (for example, see [**20**, Example 2.2.3]).

Example 1.1.4 (Quot-functor). Let k be a field, let S be k-scheme of finite type. Let $f\colon X \to S$ be a projective morphism and $\mathcal{O}_X(1)$ an f-ample line bundle on X. Let $\mathcal{H}$ be a coherent $\mathcal{O}_X$-module and $P \in \mathbb{Q}[t]$ a polynomial. We define a functor

$$\mathcal{Q} := \underline{\mathrm{Quot}}_{X/S}(\mathcal{H}, P) \colon (Sch/S)^{op} \longrightarrow (Sets)$$

as follows:

$$\mathcal{Q}(T) := \left\{ [\mathcal{O}_T \otimes \mathcal{H} \twoheadrightarrow F] \,\middle|\, \begin{array}{l} F\colon \text{coherent sheaf on } X_T \text{ which is flat over } T \text{ and} \\ \chi(F(t)) = P \text{ for any } t \in T \end{array} \right\},$$

which is the set of equivalence classes. Here, X_T is the pull-back of X and the equivalence relation is defined by the isomorphism of the quotient including the surjective:

$$\begin{array}{ccc} \mathcal{O}_T \otimes \mathcal{H} & \twoheadrightarrow & F \\ & \searrow & \downarrow \cong \\ & & F'. \end{array}$$

If $g\colon T \to S$ is an S-morphism, let $\mathcal{Q}(g)\colon \mathcal{Q}(T) \to \mathcal{Q}(T')$ be the map that sends $\mathcal{O}_T \otimes \mathcal{H} \twoheadrightarrow F$ to $\mathcal{O}_{T'} \otimes \mathcal{H} \twoheadrightarrow g_X^* F$.

Theorem 1.1.5 (For example, see [**20**, Theorem 2.2.4]). *The functor* $\underline{\mathrm{Quot}}_{X/S}(\mathcal{H}, P)$ *is represented by a projective* S*-scheme* $\pi\colon \mathrm{Quot}_{X/S}(\mathcal{H}, P) \to S$.

Then we have a *universal* or *tautological quotient*

$$[\widetilde{\rho}\colon \mathcal{O}_Q \otimes \mathcal{H} \twoheadrightarrow \widetilde{F}] \in \mathcal{Q}(Q).$$

Any quotient $[\rho\colon \mathcal{O}_T \otimes \mathcal{H} \twoheadrightarrow F]$ is equivalent to the pull-back of $\widetilde{\rho}$ under a uniquely determined S-morphism $\phi_\rho\colon T \to Q$, the *classifying map* associated to ρ.

In the case $X = S$, the polynomial P reduces to a number and $\mathrm{Quot}_{X/S}(\mathcal{H}, P)$ is $\mathrm{Grass}_S(\mathcal{H}, P)$.

If $S = \mathrm{Spec}(k)$ and $\mathcal{H} = \mathcal{O}_X$, then a quotient of $\mathcal{H}$ corresponds to a closed subscheme of X. In this context, the Quot-scheme is usually called the *Hilbert scheme* of closed subschemes of X of given Hilbert polynomial P and is denote by $\mathrm{Hilb}^P(X) = \mathrm{Quot}_X(\mathcal{O}_X, P)$.

1.2. Coarse Moduli. Unfortunately, in many moduli problem one cannot expect to have representable functors. For this reason, Mumford introduce the following notation of *coarse moduli*.

Definition 1.2.1. Given a functor

$$F\colon (\mathit{Sch}/S)^{op} \longrightarrow (\mathit{Sets}),$$

a S-scheme X is said to be a *best approximation* to F if it satisfies the following condition.

- There exists a natural transformation $\rho\colon F \to \underline{X}$ which is universal among natural transformation from F to functors of points. In other wards, given any $\tau\colon F \to \underline{Y}$ there exists a unique S-morphism $f\colon X \to Y$ making the following commute:

$$\begin{array}{ccc} F & \xrightarrow{\ \rho\ } & \underline{X} \\ & \searrow^{\tau} & \downarrow{\scriptstyle \underline{f}} \\ & & \underline{Y}. \end{array}$$

A best approximation X is called a *coarse moduli space* for the functor F if it satisfies, in addition:

- for every algebraically closed field $k' \supset k$, the set map
$$\rho(\mathrm{Spec}(k'))\colon F(\mathrm{Spec}(k')) \longrightarrow \underline{X/S}(\mathrm{Spec}(k'))$$
is bijective.

Note that fine moduli implies coarse, and the coarse (or best approximation) implies unique up to isomorphism by the universal property.

2. Geometric Invariant Theory

The *geometric invariant theory* (*GIT*) is a method of constructing quotients by a group action in algebraic geometry. In this thesis, we use this method to construct the moduli space of semistable parabolic Λ^1_D-triples (Section 3) and character varieties (Section 4).

Let k be an algebraically closed field of characteristic zero.

2.1. Group action and the linearization. An *algebraic group* over k is a k-scheme G of finite type together with morphism

$$\mu\colon G\times G\longrightarrow G,\quad \epsilon\colon \mathrm{Spec}(k)\longrightarrow G\quad\text{and}\quad \iota\colon G\longrightarrow G$$

defining the group multiplication, the unit element and taking the inverse, and satisfying the usual axioms for groups. Since we assume that the characteristic of k is zero, any algebraic group is smooth. An algebraic group is affine if and only if it is isomorphic to a closed subgroup of some $\mathrm{GL}(N,\mathbb{C})$.

A (left) *action* of an algebraic group G on a k-scheme X is a morphism

$$\sigma\colon G\times X\longrightarrow X$$

which satisfies the usual associativity rules. A morphism $\varphi\colon X\to Y$ of k-schemes with G-actions σ_X and σ_Y, respectively, is *G-equivariant*, if $\sigma_Y\circ(\mathrm{id_G}\times\varphi)=\varphi\circ\sigma_X$.

Let $\sigma\colon G\times X\to X$ be a group action as above, and let $x\in X$ be a closed point. The *orbit* of x is the image of the composite

$$\sigma_x\colon G\cong\{x\}\times G\hookrightarrow X\times G\xrightarrow{\sigma} X.$$

It is a locally closed smooth subscheme of X, since G act transitively on its closed points. We put $G_x:=\sigma_x^{-1}(x)\subset G$, which is a subgroup of G. We call G_x the *isotropy subgroup* or the *stabilizer* of x in G. If V is a G-representation space, let V^G denote the linear subspace of invariant elements.

Definition 2.1.1. Let $\sigma\colon G\times X\to X$ be a group action of an algebraic group G on a k-scheme X. A *categorical quotient* for σ is a k-scheme that is a best approximation of the functor

$$\underline{X}/\underline{G}\colon (\mathit{Sch}/k)^{\mathrm{op}}\longrightarrow(\mathit{Sets}),\quad T\longmapsto \underline{X}(T)/\underline{G}(T).$$

Moreover, let Y be a categorical quotient. The object Y is said to be *universal categorical quotient* if for any morphism $\underline{g}\colon\underline{U}\to\underline{Y}$ where $U\in\mathrm{Ob}(\mathit{Sch}/k)^{\mathrm{op}}$, the object U is a best approximation of the fiber product $\mathcal{U}=\underline{U}\times_{\underline{Y}}(\underline{X}/\underline{G})$.

Even if a categorical quotient exists, it can be far from being an "orbit space". For example, let the multiplicative group $\mathbb{G}\cong\mathrm{Spec}(k[T,T^{-1}])$ act on the affine space $\mathbb{A}^n$ by homothetic. Then, the projection $\mathbb{A}^n\to\mathrm{Spec}(k)$ is a categorical quotient. However, it is not really an an orbit space. We need notions which are closer to the intuitive idea of a quotient.

Definition 2.1.2. Let G an affine algebraic group over k acting on a k-scheme X. A morphism $\varphi\colon X\to Y$ is a *good quotient*, if

(1) φ is affine and invariant;
(2) φ is surjective, and $U\subset Y$ is open if and only if $\varphi^{-1}(U)\subset X$ is open;
(3) the natural morphism $\mathcal{O}_Y\to(\varphi_*\mathcal{O}_X)^G$ is an isomorphism;
(4) if W is an invariant closed subset of X, then $\varphi(W)$ is a closed subset of Y. If W_1 and W_2 are disjoint invariant closed subsets of X, then $\varphi(W_1)\cap\varphi(W_2)\neq\emptyset$.

If a good quotient exists, we denote Y by $X/\!/G$. The morphism φ is said to be a *geometric quotient* if the geometric fibers of φ are the orbits of geometric points of X. Finally, φ is a *universal good* (*geometric*) *quotient* if $Y' \times_Y X \to Y'$ is a good (geometric) quotient for any morphism $Y' \to Y$ of k-schemes.

Remark 2.1.3.
- Any (universal) good quotient is a (universal) categorical quotient.
- If $\varphi\colon X \to X/\!/G$ is a good quotient and if X is irreducible, reductive, integral, or normal, then the same holds for $X/\!/G$.

When we consider a quotient of a projective scheme by an action of an algebraic group, we need linearize the action. Then we give the precise definition for a group action on invertible sheaf that is compatible with a given group action on the supporting scheme.

Definition 2.1.4. Let X a k-scheme of finite type, G an algebraic k-group and $\sigma\colon G \times X \to X$ a group action. A *G-linearization* of a invertible sheaf $\mathcal{L}$ is an isomorphism of $\mathcal{O}_{G\times X}$-sheaves

$$\Phi\colon \sigma^*\mathcal{L} \longrightarrow p_2^*\mathcal{L},$$

where $p_2\colon G \times X \to X$ is the projection such that

$$(\mu \times \mathrm{id}_X)^*\Phi = p_{23}^*\Phi \circ (\mathrm{id}_G \times \sigma)^*\Phi$$

where μ is the group multiplication and $p_{23}\colon G \times G \times X \to G \times X$ is the projection onto the last two factors.

Remark 2.1.5. The tensor product of two G-linearized invertible sheaves, and the inverse of a G-linearized invertible sheaf both carry canonical G-linearization.

There are three ways to consider this definition. First, if g and x are k-rational points in G and X, respectively, and if we write gx for $\sigma(g,x)$, then the G-linearization Φ provide an isomorphism of fibers of $\mathcal{L}$

$$\Phi_{g,x}\colon \mathcal{L}_{gx} \longrightarrow \mathcal{L}_x.$$

And the cycle condition translates into

$$\Phi_{g,x} \circ \Phi_{h,gx} = \Phi_{hg,x}\colon \mathcal{L}_{hgx} \longrightarrow \mathcal{L}_x.$$

Second, we consider the G-linearization Φ by the line bundle L corresponding to the invertible sheaf $\mathcal{L}$. More precisely, we lift the action of G on X to a *bundle action* of G on L as follows. The G-linearrization Φ induces a morphism

$$\sigma_L\colon L \times_k G = L \times_{X,p_2} (G \times X) \to L \times_{X,\sigma} (G \times X) \to L$$

such that the diagram

$$\begin{array}{ccc} G \times L & \xrightarrow{\sigma_L} & L \\ \downarrow & & \downarrow \\ G \times X & \xrightarrow{\sigma} & X \end{array}$$

commutes. The cocycle condition for Φ implies that σ_L is group action of G on L, and the commutativity of the diagram says that the projection $\pi\colon L \to X$ is equaivariant.

Third, we consider a dual action of G on $H^0(X,\mathcal{L})$ induced by a G-linearization, which is the most important. This action is given by the composition:

$$\begin{array}{ccccc} H^0(X,\mathcal{L}) & \xrightarrow{\sigma^*} & H^0(G\times X,\sigma^*\mathcal{L}) & \xrightarrow{\Phi} & H^0(G\times X,{p_1}^*\mathcal{L}) \\ & & & & \downarrow\cong \\ & & & & H^0(G,\mathcal{O}_G)\otimes H^0(X,\mathcal{L}) \end{array}$$

where the last isomorphism follows from Künneth formula. The conditions for a dual action result from the cocycle condition on Φ. We denote by $H^0(X,\mathcal{L})^G$ the set of invariant global sections of $\mathcal{L}$.

Example 2.1.6 ($\mathrm{PGL}(n,k)$-action on $\mathbb{P}^n$). Let $\mathbb{P}^n := \mathrm{Proj}\ k[X_1,\dots,X_n]$. The algebraic group $\mathrm{PGL}(n,k)$ is the open subset of

$$\mathbb{P}^{n^2+2n}\cong \mathrm{Proj}\ k[a_{00},\dots,a_{0,n};a_{10},\dots,a_{1n};\dots;a_{n0},\dots,a_{nn}]$$

where $\det(a_{ij})\neq 0$. Then, the morphism

$$\sigma\colon \mathrm{PGL}(n,k)\times\mathbb{P}^n\longrightarrow\mathbb{P}^n \tag{2.1.1}$$

can be define by the condition

$$\sigma^*(\mathcal{O}_{\mathbb{P}^n}(1))\cong p_1^*[\mathcal{O}_{\mathbb{P}^{n^2+2n}}(1)]\otimes p_2^*[\mathcal{O}_{\mathbb{P}^n}(1)]$$
$$\sigma^*(X_i)=\sum_{j=0}^{n}p_1^*(a_{ij})\otimes p_2^*(X_j).$$

Unfortunately, $\mathcal{O}_{\mathbb{P}^{n^2+2n}}(1)$ is not trivial on $\mathrm{PGL}(n,k)$, since its order on $\mathrm{Pic}[\mathrm{PGL}(n,k)]$ is $n+1$. Therefore, $\mathcal{O}_{\mathbb{P}^n}(1)$ admits *no* $\mathrm{PGL}(n,k)$-linearization.

On the other hand, $\mathcal{O}_{\mathbb{P}^n}(n+1)$ admits a $\mathrm{PGL}(n)$-linearization. In fact, the line bundle corresponding to $\mathcal{O}_{\mathbb{P}^n}(n+1)$ is the n-th exterior power of the cotangent bundle on $\mathbb{P}^n$. Any action on $\mathbb{P}^n$ lifts to an an action on the cotangent bundle, hence to this n-th exterior power.

Example 2.1.7 ($\mathrm{SL}(n+1,k)$-action on $\mathbb{P}^n$). Let $\overline{\omega}\colon \mathrm{SL}(n+1,k)\to\mathrm{PGL}(n,k)$ be the canonical isogeny and consider the induced action $\tau=\sigma\circ(\overline{\omega}\times 1_{\mathbb{P}^n})$ of $\mathrm{SL}(n+1,k)$ on $\mathbb{P}^n$ where σ is the $\mathrm{PGL}(n,k)$-action (2.1.1). Then $\mathcal{O}_{\mathbb{P}^n}(1)$ admits a $\mathrm{SL}(n+1,k)$-linearization. It is well-known that $\mathrm{SL}(n+1,k)$ acts on the affine cone $\mathbb{A}^{n+1}$ over $\mathbb{P}^n$, so that the projection

$$\pi\colon \mathbb{A}^{n+1}\setminus\{0\}\longrightarrow\mathbb{P}^n$$

is $\mathrm{SL}(n+1,k)$-linear. On the other hand, let L be the line bundle corresponding to $\mathcal{O}_{\mathbb{P}^n}(1)$. The line bundle L is obtained from $\mathbb{A}^{n+1}$ by blowing up at 0, and the projection $L\to\mathbb{P}^n$ is obtained by π. Therefore, $\mathrm{SL}(n+1,k)$ acts on L, compatibly with the projection $L\to\mathbb{P}^n$. The invertible sheaf $\mathcal{O}_{\mathbb{P}^n}(1)$ is the dual of the invertible sheaf corresponding to L.

2.2. GIT quotient.

Definition 2.2.1. An algebraic group G is called *reductive*, if its unipotent radical (i.e. its maximal connected unipotent subgroup) is trivial.

For example, all tori $\mathbb{G}_m^N$ and groups $\mathrm{GL}(n,\mathbb{C})$, $\mathrm{SL}(n,\mathbb{C})$, $\mathrm{PGL}(n,\mathbb{C})$ are reductive.

Theorem 2.2.2 (Affine GIT quotient). *Let G be a reductive group acting on an affine k-scheme X of finite type. Let $A(X)$ be the affine coordinate ring of X and we put $Y = \mathrm{Spec}(A(X)^G)$. Then, $A(X)^G$ is finite generated over k, so that Y is of finite type, and the natural map $\pi\colon X \to Y$ is a universal good quotient for the action G.*

Proof. See [**35**, Theorem 1.1] or [**36**, Theorem 3.4 and Theorem 3.5]. □

Assume that X is a projective scheme with an action of a reductive group G and that $\mathcal{L}$ is G-linearized ample line bundle on X. For example, the reductive group G acts on X through $\mathrm{SL}(n+1,k)$-transformation of the projective space:

$$\begin{array}{ccc} G \times X & \longrightarrow & X \\ \downarrow & & \downarrow \cap \\ \mathrm{SL}(n+1,k) \times \mathbb{P}^n & \longrightarrow & \mathbb{P}^n. \end{array}$$

Then $\mathcal{O}_X(1)$ is a G-linearized ample line bundle on X.

Let $R := \bigoplus_{n\geq 0} H^0(X, \mathcal{L}^{\otimes n})$ be the associated homogeneous coordinate ring. Then $R^G := \bigoplus_{n\geq 0} H^0(X, \mathcal{L}^{\otimes n})^G$ is a finitely generated $\mathbb{Z}$-graded k-algebra (see [**36**, Proposition 3.4]). For $r \gg 0$, the variety Proj (R^G) is just the image of X under the linear system $H^0(X, \mathcal{L}^{\otimes r})^G$:

$$\begin{aligned} X &\dashrightarrow \mathbb{P}((H^0(X, \mathcal{L}^{\otimes r})^G)^*), \\ x &\longmapsto ev_x \end{aligned} \tag{2.2.1}$$

where $ev_x(s) := s(x)$, $s \in H^0(X, \mathcal{L}^{\otimes r})$. We fix a basis for $H^0(X, \mathcal{L}^{\otimes r})$, denoted by $s_0, \ldots, s_k$. Then we can consider the rational map (2.2.1) as

$$x \longmapsto [s_0(x) : \ldots : s_k(x)] \in \mathbb{P}^k.$$

The rational map (2.2.1) is defined on points for which $ev_x \not\equiv 0$ (equivalently the $s_i(x)$ are not all zero).

Definition 2.2.3. A point $x \in X$ is *semistable* with respect to a G-linearized ample line bundle $\mathcal{L}$ if there is an integer r and an invariant global section $s \in H^0(X, \mathcal{L}^{\otimes r})^G$ with $s(x) \neq 0$.

A point which is not semistable is called *unstable.*

We denote by $X^{ss}(\mathcal{L})$ the set of all semistable points on X. Then we can define the map

$$X^{ss}(\mathcal{L}) \longrightarrow \mathbb{P}((H^0(X, \mathcal{L}^{\otimes r})^G)^*). \tag{2.2.2}$$

Theorem 2.2.4 (GIT quotient). *Let G be a reductive group acting on a projective scheme X with G-linearized ample line bundle $\mathcal{L}$. We put $Y =$ Proj (R^G). Then the natural morphism $\pi\colon X^{ss}(\mathcal{L}) \to Y$ is a universal good quotient for the action.*

Proof. See [**35**, Theorem 1.10] or [**36**, Theorem 3.21]. □

The universal good quotient

$$\pi\colon X^{ss}(\mathcal{L}) \longrightarrow \text{Proj }(R^G)$$

is clearly constant on G-orbits, i.e. it factors through the set-theoretic quotient $X^{ss}(\mathcal{L})/G$. However, the universal good quotient π may constant on more than just G-orbits. Then we need another definition.

Definition 2.2.5. Let $x \in X$ be a semistable point with respect to a G-linearized ample line bundle $\mathcal{L}$. The point $x \in X$ is *stable* if in addition the stabilizer G_x is finite and G-orbit of x is closed in the open set of all semistable points in X.

A point is called *properly semistable* if it is semistable which is not stable.

We denote by $X^s(\mathcal{L})$ the set of all stable points on X. Let $x \in X$ be a stable point. Then, for any $y \in X \setminus Gx$, there exists a invariant global section $s \in H^0(X, \mathcal{L}^r)^G$ such that $s(x) \neq s(y)$. In fact, if $s(x) = s(y)$ for any $s \in H^0(X, \mathcal{L}^{\otimes r})^G$, then the intersection of the closures of the orbits Gx and Gy is nonempty, i.e. $\overline{Gx} \cap \overline{Gy} \neq \emptyset$ (see [**36**, Proposition 3.11]). Since the orbit Gx is closed set, we have $Gx \subset \overline{Gy}$. On the other hand, since the stabilizer G_x is finite, we have $\dim Gx = \dim G$. Then, we have $Gx = Gy$. We consider the restriction map of (2.2.3)

$$X^s(\mathcal{L}) \longrightarrow \text{Proj }(R^G) \subset \mathbb{P}((H^0(X, \mathcal{L}^{\otimes r})^G)^*). \tag{2.2.3}$$

The map only contracts single orbit. More properly, we have the following

Theorem 2.2.6. *Let G be a reductive group acting on a projective scheme X with G-linearized ample line bundle $\mathcal{L}$. Let $\pi\colon X^{ss}(\mathcal{L}) \to X//G$ be a universal good quotient for the action. We denote the image $\pi(X^s(\mathcal{L}))$ by $(X//G)^s$. Then, the restriction map $\pi\colon X^s(\mathcal{L}) \to (X//G)^s$ is a universal geometric quotient.*

Proof. See [**35**, Theorem 1.10] or [**36**, Theorem 3.21]. □

2.3. Mumford–Hilbert criterion. Let $\lambda\colon \mathbb{G}_m \to G$ be a non-trivial one-parameter subgroup of G. Then the action of G on X induces an action of $\mathbb{G}_m$ on X. Since X is projective, the orbit map $\mathbb{G}_m \to X$; $t \mapsto \sigma(x, \lambda(t))$ extends in an unique way to a morphism $f\colon \mathbb{A}^1 \to X$ such that the diagram

$$\begin{array}{ccc} \mathbb{G}_m & \xrightarrow{\lambda} & G, \\ \downarrow & & \downarrow \\ \mathbb{A}^1 & \xrightarrow{f} & X, \end{array}$$

commutes, where $\mathbb{G}_m = \mathbb{A}^1 \setminus \{0\} \to \mathbb{A}^1$ is the inclusion, and $G \to X$ is the orbit map, that is, $g \mapsto \sigma(x, g)$. We put

$$\lim_{t\to 0} \sigma(x, \lambda(t)) := f(0).$$

The point $f(0)$ is a fixed point of the action of $\mathbb{G}_m$ on X via λ. In particular, $\mathbb{G}_m$ acts on the fiber of $L(f(0))$ with a certain weight r, that is, Φ is the linearization of L, then $\Phi(f(0), \lambda(t)) = t^r \cdot \mathrm{id}_{\mathrm{L(f(0))}}$. Define the number

$$\mu^L(x, \lambda) := -r.$$

Theorem 2.3.1 (Mumford–Hilbert criterion). *A point $x \in X$ is semistable (resp. stable) if and only if for all non-trivial one-parameter subgroups $\lambda\colon \mathbb{G}_m \to G$, one has*

$$\mu^L(x,\lambda) \geq 0 \quad (\textit{resp. } >).$$

Proof. See [**35**, Theorem 2.1] or [**36**, Theorem 4.9]. □

3. Moduli problem for parabolic Λ^1_D-triples

3.1. Definitions. We recall the definition of a *parabolic Λ^1_D-triple* defined in [**23**]. Let D be an effective divisor on a nonsingular curve Σ. We define Λ^1_D as $\mathcal{O}_\Sigma \otimes \Omega^1_\Sigma(D)^\vee$ with the bimodule structure given by

$$f(a,v) = (fa, fv) \quad (f, a \in \mathcal{O}_\Sigma,\ v \in \Omega^1_\Sigma(D)^\vee),$$
$$(a,v)f = (fa + v(f), fv) \quad (f, a \in \mathcal{O}_\Sigma,\ v \in \Omega^1_\Sigma(D)^\vee).$$

Definition 3.1.1. We say $(E_1, E_2, \Phi, F_*(E_1))$ a *parabolic Λ^1_D-triple on Σ of rank n and of degree d* if

(1) E_1 and E_2 are vector bundles on Σ of rank n and of degree d,
(2) $\Phi\colon \Lambda^1_D \otimes E_1 \to E_2$ is a left $\mathcal{O}_\Sigma$-homomorphism, and
(3) $E_1 = F_1(E_1) \supset F_2(E_1) \supset \cdots \supset F_l(E_1) \supset F_{l+1}(E_1) = E_1(-D)$ is a filtration by coherent subsheaves.

Note that to give a left $\mathcal{O}_\Sigma$-homomorphism $\Phi : \Lambda^1_D \otimes E_1 \to E_2$ is equivalent to give an $\mathcal{O}_\Sigma$-homomorphism $\phi : E_1 \to E_2$ and a morphism $\nabla : E_1 \to E_2 \otimes \Omega^1_\Sigma(D)$ such that $\nabla(fa) = \phi(a) \otimes df + f\nabla(a)$ for $f \in \mathcal{O}_\Sigma$ and $a \in E_1$. We also denote the parabolic Λ^1_D-triple $(E_1, E_2, \Phi, F_*(E_1))$ by $(E_1, E_2, \phi, \nabla, F_*(E_1))$.

We take positive integers β_1, β_2, γ and rational numbers $0 < \alpha'_1 < \cdots < \alpha'_l < 1$. We assume $\gamma \gg 0$. Set $\boldsymbol{\alpha}' := (\alpha'_1, \ldots, \alpha'_l)$ and $\boldsymbol{\beta} = (\beta_1, \beta_2)$.

Definition 3.1.2. A parabolic Λ^1_D-triple $(E_1, E_2, \phi, \nabla, F_*(E_1))$ is $(\boldsymbol{\alpha}', \boldsymbol{\beta})$-*stable* (resp. $(\boldsymbol{\alpha}', \boldsymbol{\beta})$-*semistable*) if for any subbundle $(F_1, F_2) \subset (E_1, E_2)$ satisfying $(0,0) \neq (F_1, F_2) \neq (E_1, E_2)$ and $\Phi(\Lambda^1_D \otimes F_1) \subset F_2$, the inequality

$$\frac{\beta_1 \deg F_1(-D) + \beta_2(\deg F_2 - \gamma \operatorname{rank} F_2) + \beta_1 \sum_{j=1}^l \alpha'_j \operatorname{length}(F_j(E_1) \cap F_1)/(F_{j+1}(E_1) \cap F_1))}{\beta_1 \operatorname{rank} F_1 + \beta_2 \operatorname{rank} F_2}$$
$$\underset{(\text{resp. } \leq)}{<} \frac{\beta_1 \deg E_1(-D) + \beta_2(\deg E_2 - \gamma \operatorname{rank} E_2) + \beta_1 \sum_{j=1}^l \alpha'_j \operatorname{length}((F_j(E_1))/(F_{j+1}(E_1)))}{\beta_1 \operatorname{rank} E_1 + \beta_2 \operatorname{rank} E_2}$$

holds.

Let S be a connected noetherian scheme and $\pi_S\colon \mathcal{X} \to S$ be a smooth projective morphism whose geometric fibers are curves of genus g. Let $\mathcal{D} \subset \mathcal{X}$ be an effective Cartier divisor which is flat over S. We can consider the $\mathcal{O}_\mathcal{X}$-bimodule structure on $\Lambda^1_{\mathcal{D}/S} := \mathcal{O}_\mathcal{X} \otimes (\Omega^1_{\mathcal{X}/S(\mathcal{D})})^\vee$.

Fix a positive integers n, d, and $\{d_i\}_{1 \leq i \leq l}$.

Definition 3.1.3. We define the moduli functor $\overline{\mathcal{MF}}^{\mathcal{D},\boldsymbol{\alpha}',\boldsymbol{\beta},\gamma}_{\mathcal{X}/S}(n,d,\{d_i\})$ of the category of locally noetherian scheme over S to the category of sets by

$$\overline{\mathcal{MF}}^{\mathcal{D},\boldsymbol{\alpha}',\boldsymbol{\beta},\gamma}_{\mathcal{X}/S}(n,d,\{d_i\})(T) := \{(E_1,E_2,\Phi,F_*(E_1))\}/\sim$$

where T is a locally noetherian scheme over S and

(1) E_1, E_2 are vector bundle on $\mathcal{X}\times_S T$ such that for any generic point s of T, $\operatorname{rank}(E_1)_s = \operatorname{rank}(E_2)_s = n$, $\deg(E_1)_s = \deg(E_2)_s = d$,
(2) $\Phi\colon \Lambda^1_{\mathcal{D}/S}\otimes_{\mathcal{O}_\mathcal{X}} E_1 \to E_2$ is a homomorphism of left $\mathcal{O}_{\mathcal{X}\times_S T}$-modules,
(3) $E_1 = F_1(E_1)\supset F_2(E_1)\supset\cdots\supset F_l(E_1)\supset F_{l+1}(E_1) = E_1(-\mathcal{D}_T)$ is a filtration of E_1 by coherent subsheaves such that $E_1/F_{i+1}(E_1)$ is flat over T and for any geometric point s of T, $\operatorname{length}((E_1/F_{i+1}(E_1))_s) = d_i$,
(4) for any geometric point s on S, the parabolic $\Lambda^1_{\mathcal{D}_s}$-tuple $((E_1)_s,(E_2)_s,\Phi_s,F_*(E_1)_s)$ is $(\boldsymbol{\alpha}',\boldsymbol{\beta})$-stable.

Here, $(E_1,E_2,\Phi,F_*(E_1))\sim(E_1',E_2',\Phi',F_*(E_1'))$ if there exist a line bundle $\mathcal{L}$ on T and isomorphisms $\sigma_i\colon E_j \xrightarrow{\cong} E_i'\otimes\mathcal{L}$ for $i=1,2$ such that $\sigma_1(F_{i+1}(E_1)) = F_{i+1}(E_1')\otimes\mathcal{L}$ for any i and the diagram

$$\begin{array}{ccc}
\Lambda^1_{\mathcal{D}/S}\otimes_{\mathcal{O}_\mathcal{X}} E_1 & \xrightarrow{\ \Phi\ } & E_2 \\
\cong\downarrow \mathrm{id}\otimes\sigma_1 & & \cong\downarrow \sigma_2 \\
\Lambda^1_{\mathcal{D}/S}\otimes_{\mathcal{O}_\mathcal{X}} E_1\otimes\mathcal{L} & \xrightarrow{\Phi'\otimes\mathrm{id}} & E_2'\otimes_S\mathcal{L}
\end{array}$$

commutes.

3.2. Construction of the moduli space. We recall the construction of the moduli space of $\overline{\mathcal{MF}}^{\mathcal{D},\boldsymbol{\alpha}',\boldsymbol{\beta},\gamma}_{\mathcal{X}/S}(n,d,\{d_i\})$ due to Inaba–Iwasaki–Saito [**23**]. Let $P(m) := nd_\mathcal{X}m + d + r(1-g)$ where $d_\mathcal{X} = \mathcal{O}_{\mathcal{X}_s}(1)$ for $s\in S$ and g is genus of $\mathcal{X}_s$. Fix a integer m_0. The family of geometric points of $\overline{\mathcal{MF}}^{\mathcal{D},\boldsymbol{\alpha}',\boldsymbol{\beta},\gamma}_{\mathcal{X}/S}(n,d,\{d_i\})$ is bounded (see [**23**, Proposition 5.1]). Then, by replacing m_0, we may assume that for any $m\geq m_0$,

- $h^j(F_i(E_1)(m)) = h^j(E_1(m-\gamma)) = 0$ for $j>0$, $i=1,\dots,l+1$,
- $E_2(m-\gamma)$, $F_i(E_1)(m)$, $(i=1,\dots,l+1)$ are generated by their global sections

for any geometric point $(E_1,E_2,\Phi,F_*(E_1))$ of $\overline{\mathcal{MF}}^{\mathcal{D},\boldsymbol{\alpha}',\boldsymbol{\beta},\gamma}_{\mathcal{X}/S}(n,d,\{d_i\})$.

Put $n_1 = P(m_0)$ and $n_2 = P(m_0-\gamma)$. Let V_1 and V_2 be free $\mathcal{O}_S$-modules of rank n_1 and rank n_2, respectively. Let Q_1 be the Quot-scheme $\operatorname{Quot}_{\mathcal{X}/S}(V_1\otimes\mathcal{O}_\mathcal{X}(-m_0),P(m))$ and

$$V_1\otimes\mathcal{O}_{\mathcal{X}_{Q_1}}(-m_0)\longrightarrow\mathcal{E}_1$$

be the universal quotient sheaf. Similarly, let Q_2 be the Quot-scheme $\operatorname{Quot}_{\mathcal{X}/S}(V_2\otimes\mathcal{O}_\mathcal{X}(-m_0+\gamma),P(m))$ and

$$V_2\otimes\mathcal{O}_{\mathcal{X}_{Q_2}}(-m_0+\gamma)\longrightarrow\mathcal{E}_2$$

be the universal quotient sheaf. We put $Q_1^{(i)} := \mathrm{Quot}_{\mathcal{X}_{Q_1}/Q_1}(\mathcal{E}_1, d_i)$ and we denote by $F_{i+1}(\mathcal{E}_1) \subset (\mathcal{E}_1)_{Q_1^{(i)}}$ be the universal subsheaf. Let Q be the maximal closed subscheme of $Q_1^{(1)} \times_{Q_1} \cdots \times_{Q_1} Q_1^{(l)} \times Q_2$ such that there are factorizations

$$(\mathcal{E}_1)_Q \otimes \mathcal{O}_{\mathcal{X}_Q}(-\mathcal{D}_Q) \longrightarrow F_{i+1}(\mathcal{E}_1)_Q \hookrightarrow F_1(\mathcal{E}_1)_Q \subset (\mathcal{E}_1)$$

for $i = 1, \dots, l$ where $F_1(\mathcal{E}_1) = \mathcal{E}_1$. Since $(\mathcal{E}_2)_Q$ is flat over Q, there is a coherent sheaf $\mathcal{H}$ on Q such that there is a functorial isomorphism

$$\mathrm{Hom}_{\mathcal{X}_T}(\Lambda^1_{\mathcal{D}/S} \otimes_{\mathcal{O}_{\mathcal{X}}} (\mathcal{E}_1)_T, (\mathcal{E}_2)_T \otimes \mathcal{L}) \cong \mathrm{Hom}(\mathcal{H} \otimes \mathcal{O}_T, \mathcal{L})$$

for any noetherian scheme T over Q and any quasi-coherent sheaf $\mathcal{L}$ on T (see [**1**, Section 1]). Put $\mathbf{V}^*(\mathcal{H}) := \mathrm{Spec} S(\mathcal{H})$ where $S(\mathcal{H})$ is the symmetric algebra of $\mathcal{H}$ over $\mathcal{O}_Q$. Note that the scheme $\mathbf{V}^*(\mathcal{H})$ represents a functor

$$(Sch/Q) \ni T \longmapsto \mathrm{Hom}_{\mathcal{X}_T}(\Lambda^1_{\mathcal{D}/S} \otimes_{\mathcal{O}_{\mathcal{X}}} (\mathcal{E}_1)_T, (\mathcal{E}_2)_T).$$

Let

$$\widetilde{\Phi} \colon \Lambda^1_{\mathcal{D}/S} \otimes_{\mathcal{O}_{\mathcal{X}}} (\mathcal{E}_1)_{\mathbf{V}^*(\mathcal{H})} \longrightarrow (\mathcal{E}_2)_{\mathbf{V}^*(\mathcal{H})}$$

be the universal homomorphism. Then, we define the open subscheme R^s of $\mathbf{V}^*(\mathcal{H})$ by

$$R^s := \left\{ s \in \mathbf{V}^*(\mathcal{H}) \;\middle|\; \begin{array}{l} (1)\ (V_1)_s \to H^0((\mathcal{E}_1)_s(m_0)), (V_2)_s \to H^0((\mathcal{E}_2)_s(m_0-\gamma)) \\ \quad\ \text{are bijective,} \\ (2)\ F_i(\mathcal{E}_1)_s(m_0), (\mathcal{E}_2)_s(m_0-\gamma) \text{ are generated by their} \\ \quad\ \text{global sections,} \\ (3)\ h^j(F_i(\mathcal{E}_1)_s(m_0)) = h^j((\mathcal{E}_2)_s(m_0-\gamma)) = 0 \\ \quad\ \text{for } j>0,\ 1 \le i \le l+1, \\ (4)\ ((\mathcal{E}_1)_s, (\mathcal{E}_2)_s, \widetilde{\Phi}_s, F_*(\mathcal{E}_1)_s) \text{ is } (\boldsymbol{\alpha}', \boldsymbol{\beta})\text{-stable.} \end{array} \right\}.$$

Note that the openness of stability is shown in [**23**, Proposition 5.3].

We construct the immersion of the open subscheme R^s into a projective scheme as follows. Fix a integer m_1 which is large enough (see [**23**, p.1035–p.1036]). The composite

$$V_1 \otimes \Lambda^1_{\mathcal{D}/S} \otimes \mathcal{O}_{\mathcal{X}}(-m_0)_{R^s} \longrightarrow \Lambda^1_{\mathcal{D}/S} \otimes (\mathcal{E}_1)_{R^s} \xrightarrow{\widetilde{\Phi}} (\mathcal{E}_2)_{R^s}$$

induces a homomorphism

$$V_1 \otimes W_1 \otimes \mathcal{O}_{R^s} \longrightarrow (\pi_{R^s})_*(\mathcal{E}_2(m_0+m_1-\gamma)_{R^s}) \tag{3.2.1}$$

where $W_1 := (\pi_S)_*(\mathcal{O}_{\mathcal{X}}(m_0+m_1-\gamma) \otimes \Lambda^1_{\mathcal{D}/S} \otimes \mathcal{O}_{\mathcal{X}}(-m_0))$, $\pi_S \colon \mathcal{X} \to S$, and $\pi_{R^s} \colon \mathcal{X}_{R^s} \to R^s$. The quotient $V_2 \otimes \mathcal{O}_{\mathcal{X}}(-m_0+\gamma) \to \mathcal{E}_2$ induces a homomorphism

$$V_2 \otimes W_2 \otimes \mathcal{O}_{R^s} \longrightarrow (\pi_{R^s})_*(\mathcal{E}_2(m_0+m_1-\gamma)_{R^s}) \tag{3.2.2}$$

where $W_2 := (\pi_S)_*(\mathcal{O}_{\mathcal{X}}(m_1))$. Those homomorphism induce a quotient bundle

$$(V_1 \otimes W_1 \oplus V_2 \otimes W_2) \otimes \mathcal{O}_{R^s} \longrightarrow (\pi_{R^s})_*(\mathcal{E}_2(m_0+m_1-\gamma)_{R^s}) \tag{3.2.3}$$

Moreover, we have the canonical quotient bundles

$$V_1 \otimes W_2 \otimes \mathcal{O}_{R^s} = V_1 \otimes (\pi_S)_*(\mathcal{O}_{\mathcal{X}}(m_1)) \otimes \mathcal{O}_{R^s} \longrightarrow (\pi_{R^s})_*(\mathcal{E}_2(m_0+m_1-\gamma)_{R^s}) \tag{3.2.4}$$

and

$$V_1 \otimes \mathcal{O}_{R^s} \longrightarrow (\pi_{R^s})_*(\mathcal{E}_1/F_{i+1}(\mathcal{E}_1)(m_0)_{R^s}) \quad (i=1,\dots,l). \tag{3.2.5}$$

We put

$$\begin{aligned}\mathrm{Gr}_1 :=&\mathrm{Grass}_S(V_1 \otimes W_1 \oplus V_2 \otimes W_2, r_2),\\ \mathrm{Gr}_2 :=&\mathrm{Grass}_S(V_1 \otimes W_2, r_1), \text{ and}\\ \mathrm{Gr}_3^i :=&\mathrm{Grass}_S(V_1, d_i)\end{aligned}$$

where $r_1 := h^0(\mathcal{E}_1(m_0+m_1)_s)$, $r_2 := h^0(\mathcal{E}_2(m_0+m_1-\gamma)_s)$ for any point $s \in R^s$. Then we obtain the following morphism

$$\iota\colon R^s \longrightarrow \mathrm{Gr}_1 \times \mathrm{Gr}_2 \times \prod_{i=1}^{l} \mathrm{Gr}_3^i. \tag{3.2.6}$$

We may check that ι is an immersion.

Put

$$G := (\mathrm{GL}(V_1) \times \mathrm{GL}(V_2))/(\mathbb{G}_m \times S) \tag{3.2.7}$$

where $\mathbb{G}_m \times S$ is contained in $(\mathrm{GL}(V_1) \times \mathrm{GL}(V_2))$ as scalar matrices. Then G acts canonically on R^s and on $\mathrm{Gr}_1 \times \mathrm{Gr}_2 \times \mathrm{Gr}_3$. We may show that ι is a G-equivariant immersion. We construct a G-linearized S-ample line bundle on R^s as follows. There are S-ample line bundles $\mathcal{O}_{\mathrm{Gr}_i}(1)$ on Gr_i and $\mathcal{O}_{\mathrm{Gr}_3^j}(1)$ on Gr_3^j induced by Plücker embedding for $i = 1, 2$, $j = 1, \dots, l$. We define positive rational numbers ν_1, ν_2, $\nu_1^{(i)}$ $(1 \leq i \leq l)$ by

$$\begin{aligned}\nu_1 :=&\beta_1(\beta_1 P(m_0) + \beta_2 P(m_0-\gamma) - \sum_{i=1}^{l} \beta_1 \epsilon_i d_i),\\ \nu_2 :=&\beta_2(\beta_1 P(m_0) + \beta_2 P(m_0-\gamma) - \sum_{i=1}^{l} \beta_1 \epsilon_i d_i),\\ \nu_1^{(i)} :=&(\beta_1+\beta_2)\beta_1 n d_{\mathcal{X}} m_1 \epsilon_i.\end{aligned}$$

Let us consider the $\mathbb{Q}$-line bundle

$$L := \iota^*(\mathcal{O}_{\mathrm{Gr}_1}(\nu_1) \otimes \mathcal{O}_{\mathrm{Gr}_2}(\nu_2) \otimes \bigotimes_{i=1}^{l} \mathrm{Gr}_3^i(\nu_1^{(i)})) \tag{3.2.8}$$

on R^s. Then for some positive integer N, $L^{\otimes N}$ becomes a G-linearized S-ample line bundle on R^s.

By hard computation of the Mumford–Hilbert criterion, we have the following proposition

Proposition 3.2.1 ([**23**, Proposition 5.4]). *All points of R^s are stable (in the sense of the GIT) with respect to the action of G and the G-linearized S-ample line bundle $L^{\otimes N}$.*

Then we have the following theorem

Theorem 3.2.2 ([**23**, Theorem 5.1]). *The scheme $\overline{\mathcal{M}}_{\mathcal{X}/S}^{\mathcal{D},\boldsymbol{\alpha}',\boldsymbol{\beta},\gamma}(n,d,\{d_i\}) := R^s/G$ is a coarse moduli scheme of $\overline{\mathcal{MF}}_{\mathcal{X}/S}^{\mathcal{D},\boldsymbol{\alpha}',\boldsymbol{\beta},\gamma}(n,d,\{d_i\})$.*

4. Character varieties

4.1. Character varieties. We fix integers $g \geq 0, k \geq 0$ and $n > 0$. We also fix a k-tuple of partition of n, denoted by $\boldsymbol{\mu} = (\mu^1, \dots, \mu^k)$, that is, $\mu^i = (\mu^i_1, \dots, \mu^i_{r_i})$ such that $\mu^i_1 \geq \mu^i_2 \geq \cdots$ are non-negative integers and $\sum_j \mu^i_j = n$. Let Σ be a smooth complex projective curve of genus g. We fix k-distinct points $p_1, \dots, p_k$ in Σ and we define a divisor by $D := p_1 + \cdots + p_k$. We put $\Sigma_0 = \Sigma \setminus D$.

We now construct a variety, called a *character variety*, whose points parametrize representation of the fundamental group of Σ_0 into $\mathrm{GL}(n, \mathbb{C})$ with prescribed images in semisimple conjugacy classes $\mathcal{C}_1, \dots, \mathcal{C}_k$ at each puncture. Assume that

$$\prod_{i=1}^{k} \det \mathcal{C}_i = 1 \tag{4.1.1}$$

and that $(\mathcal{C}_1, \dots, \mathcal{C}_k)$ has type $\boldsymbol{\mu} = (\mu^1, \dots, \mu^k)$; that is, $\mathcal{C}_i$ has type μ^i for each $i = 1, \dots, k$, where the type of the semisimple conjugacy class $\mathcal{C}_i \subset \mathrm{GL}(n, \mathbb{C})$ is defined as the partition $\mu^i = (\mu^i_1, \dots, \mu^i_{r_i})$ describing the multiplicities of the eigenvalues of any matrix in $\mathcal{C}_i$. Let $\nu^i = (\nu^i_1, \dots, \nu^i_{r_i}) \in (\mathbb{C}^\times)^{r_i}$ be the eigenvalues of $\mathcal{C}_i$. We denote the k-tuple $(\nu^1, \dots, \nu^k)$ by $\boldsymbol{\nu}$.

Definition 4.1.1. The k-tuple $(\mathcal{C}_1, \dots, \mathcal{C}_k)$ is *generic* if the following holds. If $V \subset \mathbb{C}^n$ is a subset which is stable by some $X_i \in \mathcal{C}_i$ for each i such that

$$\prod_{i=1}^{k} \det(X_i|_V) = 1,$$

then either $V = 0$ or $V = \mathbb{C}^n$.

Lemma 4.1.2 ([**15**, Lemma 2.1.2]). *For any $\boldsymbol{\mu}$, there exists a generic k-tuple of semisimple conjugacy classes $(\mathcal{C}_1, \dots, \mathcal{C}_k)$ of type $\boldsymbol{\mu}$ over $\mathbb{C}$.*

Definition 4.1.3. For a k-tuple of generic semisimple conjugacy classes $(\mathcal{C}_1, \dots, \mathcal{C}_k)$ of type $\boldsymbol{\mu}$, we define a subvariety of $\mathrm{GL}(n, \mathbb{C})^{2g+n}$ by

$$\begin{aligned} \mathcal{U}^{\boldsymbol{\mu}}(\boldsymbol{\nu}) :=\{ &(A_1, B_1, \dots, A_g, B_g; X_1, \dots, X_k) \in \mathrm{GL}(n, \mathbb{C})^{2g} \times \mathcal{C}_1 \times \cdots \times \mathcal{C}_k \\ &\mid (A_1, B_1) \cdots (A_g, B_g) X_1 \cdots X_k = I_n \}, \end{aligned}$$

where $(A, B) := ABA^{-1}B^{-1}$. The group $\mathrm{GL}(n, \mathbb{C})$ acts by conjugation on $\mathrm{GL}(n, \mathbb{C})^{2g+n}$. As the center acts trivially, the action induces that of $\mathrm{PGL}(n, \mathbb{C})$. The action induces that of $\mathrm{PGL}(n, \mathbb{C})$ on $\mathcal{U}^{\boldsymbol{\mu}}(\boldsymbol{\nu})$. We call the affine GIT quotient

$$\mathcal{M}^{\boldsymbol{\mu}}_B(\boldsymbol{\nu}) := \mathcal{U}^{\boldsymbol{\mu}}(\boldsymbol{\nu}) /\!/ \mathrm{PGL}(n, \mathbb{C})$$

a *generic character variety of type $\boldsymbol{\mu}$*. We denote by $\pi_{\boldsymbol{\mu}}$ the quotient morphism

$$\pi_{\boldsymbol{\mu}} \colon \mathcal{U}^{\boldsymbol{\mu}}(\boldsymbol{\nu}) \longrightarrow \mathcal{M}^{\boldsymbol{\mu}}_B(\boldsymbol{\nu}). \tag{4.1.2}$$

Proposition 4.1.4 ([**15**, Proposition 2.1.4]). *If $(\mathcal{C}_1, \dots, \mathcal{C}_k)$ is generic of type $\boldsymbol{\mu}$, then the group $\mathrm{PGL}(n, \mathbb{C})$ acts set-theoretically freely on $\mathcal{U}^{\boldsymbol{\mu}}(\boldsymbol{\nu})$ and every point of $\mathcal{U}^{\boldsymbol{\mu}}(\boldsymbol{\nu})$ corresponds to an irreducible representation of $\pi_1(\Sigma_0)$.*

Theorem 4.1.5 ([**15**, Theorem 2.1.5]). *If* $(\mathcal{C}_1, \ldots, \mathcal{C}_k)$ *is a generic type* $\boldsymbol{\mu}$, *then the quotient*

$$\pi_{\boldsymbol{\mu}} \colon \mathcal{U}^{\boldsymbol{\mu}}(\boldsymbol{\nu}) \longrightarrow \mathcal{M}_B^{\boldsymbol{\mu}}(\boldsymbol{\nu})$$

is a geometric quotient and a principal $\mathrm{PGL}(n, \mathbb{C})$*-bundle.*

Theorem 4.1.6 ([**16**, Theorem 1.1.1]). *If non-empty, the generic character variety* $\mathcal{M}_B^{\boldsymbol{\mu}}(\boldsymbol{\nu})$ *is a connected non-singular variety of dimension*

$$n^2(2g - 2 + k) - \sum_{i,j} (\mu_j^i)^2 + 2.$$

CHAPTER 3

Hodge Theory

We recall topics of the Hodge theory which are *variations of Hodge structures* and *mixed Hodge structures*. In Chapter V, we compute the types of the mixed Hodge structures of the moduli space of semistable parabolic Higgs and the moduli spaces of semistable parabolic connections. In Chapter VI, we compute the mixed Hodge structures of a few examples of character varieties. Those moduli spaces are open smooth varieties, and there are compactifications whose boundaries are varieties with normal crossing. Therefore, in this chapter, we give a detailed description of mixed Hodge structures of varieties with normal crossing and open smooth varieties. Main references are Deligne's papers [**6**], [**7**] and Griffiths-Schmid's paper [**14**].

1. Variation of Hodge structure

1.1. Hodge structure. Let X be a compact Kähler manifold. Then, the cohomology of X carries a *Hodge structure of weight* k:

$$H^k(X,\mathbb{C}) = \bigoplus_{p+q=k} H^{p,q}(V)$$

where $H^{p,q} = \overline{H^{q,p}}$ and $H^{p,q} \cong H^q(V, \Omega^p_X)$. Alternatively, the Hodge structure on $H^k(X,\mathbb{C})$ is determined by the *Hodge filtration*

$$\{F^p\} = \{F^p H^k(X,\mathbb{C})\}$$

which is a decreasing filtration satisfying

$$F^0 = H^k(X,\mathbb{C}), \quad F^k = H^0(X,\Omega^k_X) = H^{k,0}(X)$$

and

$$F^r = \bigoplus_{p+q=k, p\geq r} H^{p,q}(X).$$

The F^p satisfy in addition:

$$F^p \oplus \overline{F^{k-p+1}} = H^k(X,\mathbb{C}), \quad H^{p,q} = F^p \cap \overline{F^{k-p}}.$$

One way to obtain the Hodge filtration on $H^k(X,\mathbb{C})$ as follows: let $\Omega^\bullet_X$ be the complex of sheaves

$$\mathcal{O}_X \longrightarrow \Omega^1_X \longrightarrow \Omega^2_X \longrightarrow \cdots .$$

By the holomorphic Poincaré lemma, the complex $\Omega^\bullet_X$ is a resolution of the constant sheaf $\mathbb{C}_X$ on X [**5**]. Hence, we have

$$\mathbb{H}^k(X,\Omega^\bullet_X) \cong H^k(X,\mathbb{C}).$$

There is an associated spectral sequence, the *Hodge-de Rham spectral sequence*:

$$H_1^{p,q} = H^q(X, \Omega_X^p) \Longrightarrow H^{p+q}(X, \mathbb{C}),$$

which degenerates at E_1 term by classical Hodge theory. Hence, there exists a filtration on $H^k(X, \mathbb{C})$ whose associated graded space is $\bigoplus_{p+q=k} H^{p,q}(V)$ and this filtration is $\{F^p\}$.

1.2. Polarization of Hodge structure. Let X be a compact Kähler manifold. The cohomology ring $H^*(X, \mathbb{C})$ has a non-degenerate pairing defined over the integers and unimodular on $H^*(X, \mathbb{Z})$ mod torsion via

$$(x, y) \longmapsto (x \cup y)[X]$$

where $[X]$ is the positively oriented fundamental class of X.

We polarize the Hodge structure of X as follows: let $[\omega]$ be the cohomology class associated to a Kähler metric on X. The class $[\omega]$ lives in $H^2(X, \mathbb{C})$.

Remark 1.2.1. If X is a smooth projective variety, then $[\omega]$ is the first Chern class of some ample line bundle on X. Hence, the class $[\omega]$ lives in $H^2(X, \mathbb{Z})$.

We define the Lefschetz operator as follows:

$$L\colon H^*(X, \mathbb{C}) \longrightarrow H^{*+2}(X, \mathbb{C})$$
$$\alpha \longmapsto [\omega] \cup \alpha.$$

If $\dim X = n$, the Hard Lefschetz theorem asserts the isomorphism

$$L^k\colon H^{n-k}(X, \mathbb{C}) \xrightarrow{\cong} H^{n+k}(X, \mathbb{C}).$$

We define the *primitive cohomology* $H_0^{n-k}(X, \mathbb{C})$ by

$$H_0^{n-k}(X, \mathbb{C}) = \mathrm{Ker}\{L^{k+1}\colon H^{n-k}(X, \mathbb{C}) \to H^{n+k+2}(X, \mathbb{C})\}.$$

and similarly for $H_0^{p,q}(X)$. Then, we have

$$H_0^{n-k}(X, \mathbb{C}) = \bigoplus_{p+q=n-k} H_0^{p,q}(V).$$

Given a choice of $[\omega]$, define the following modified intersection pairing on $H_0^{n-k}(X, \mathbb{C})$

$$Q\colon H_0^{n-k}(X, \mathbb{C}) \times H_0^{n-k}(X, \mathbb{C}) \longrightarrow \mathbb{C}$$
$$(x, y) \longmapsto ([\omega]^k \cup x \cup y)[X].$$

The pairing Q is non-degenerate, symmetric or alternating as $n-k$ is even or odd, and defined over $\mathbb{Q}$ if $[\omega] \in H^2(X, \mathbb{Q})$. Moreover, the pairing Q satisfies the *Hodge–Riemann bilinear relation*, that is, satisfies

(1) *first bilinear relation*: $H^{p,q}$ is Q-orthogonal to $H^{p',q'}$ unless $p' = q$ (and hence $p = q'$);
(2) *second bilinear relation*: if $x \neq 0$, $x \in H_0^{p,q}$, $p + q = k$, then

$$(\sqrt{-1})^{p-q}(-1)^{\frac{k(k-1)}{2}} Q(x, \overline{x}) > 0.$$

A reference for these facts is, for example, [**13**]. We call the paring Q a *polarization of the Hodge structure of* X.

1.3. Variation of Hodge structure. Let $\pi\colon X \to S$ be a smooth proper map if complex manifold with fibers $X_i = \pi^{-1}(t)$. Assume that π is projective, that is, there exists an embedding $X \hookrightarrow \mathbb{P}^N \times S$ for some N such that the following diagram commutes:

$$\begin{array}{ccc} X & \longrightarrow & \mathbb{P}^N \times S \\ & \overset{\pi}{\searrow} & \downarrow \\ & & S. \end{array}$$

The higher direct image sheaves associated to π yield a formalism for fitting together the cohomology group $H^k(X_t, \mathbb{C})$ and $H^{p,q}(X_t)$ to obtain global object over S.

The higher direct image $R^k\pi_*\mathbb{C}_X$ is a local system of complex vector space. We put

$$\mathcal{H}^k := R^k\pi_*\mathbb{C}_X \otimes_{\mathbb{C}} \mathcal{O}_S,$$

which is a locally free sheaf of germs of holomorphic sections of $R^k\pi_*\mathbb{C}_X$. The inclusion of sheaves

$$R^k\pi_*\mathbb{C}_X \subset H^k$$

determines an unique connection ∇^{GM} on H^k, the *Gauss–Manin connection*, by decreeing that the local sections of $R^k\pi_*\mathbb{C}_X$ shall be annihilated by ∇^{GM}.

We ask how the Hodge structures vary with t. Let H^k be the holomorphic vector bundle associated to $\mathcal{H}^k$. It turns out that the subbundles $\{H_t^{p,q}\}_{t\in S} = \{H^{p,q}(X_t)\}_{t\in S}$ do *not* in general come from holomorphic subbundle of H^k, but only C^∞ subbundle. On the other hand, the subbundle $\{F_t^p\}_{t\in S} = \{F^p(X_t)\}_{t\in S}$ are holomorphic subbundle of H^k (see [**12**]).

Theorem 1.3.1 (Griffiths transversality).

$$\nabla^{GM}(F^p) \subset \Omega_S^1 \otimes F^{p-1} \text{ for any } 0 \le p \le k.$$

The general set-up of the *period map* is as follows. Let H_0^k be the primitive cohomology bundle. Here, primitive means with respect to the metric induced by $X \hookrightarrow \mathbb{P}^N \times S$. We only consider primitive variations of Hodge structure. Let $\widetilde{S}$ be a universal covering of S. The bundle H_0^k can be trivialized after lifting it to $\widetilde{S}$, denoted by $\widetilde{H}_0^k$. We fix points $t \in S$ and $\tilde{t} \in \widetilde{S}$ lifting t. We have canonically identification of fibers

$$(\widetilde{H}_0^k)_{\tilde{s}} \cong (\widetilde{H}_0^k)_{\tilde{t}} \tag{1.3.1}$$

for any point $\tilde{s} \in \widetilde{S}$. By the identification (1.3.1), the Hodge bundle $\{(\widetilde{F}_0^p)_{\tilde{s}}\}_{\tilde{s}\in\widetilde{S}}$ lifted to $\widetilde{S}$ may be viewed as a filter of the fixed vector space $H_0^k(X_t, \mathbb{C})$ ($\cong (\widetilde{H}_0^k)_{\tilde{t}}$). Then we have the following holomorphic map

$$\widetilde{S} \longrightarrow \mathcal{F} := \{\{F^p\}_{p=1}^k \mid F^p \in \mathrm{Grass}(f^p, H_0^k(X_t, \mathbb{C})), F^1 \supset F^2 \supset \cdots \supset F^k\} \tag{1.3.2}$$

where $f^p := \dim H^p$. By the two bilinear relations, we put

$$\check{D} := \{\{F^p\} \in \mathcal{F} \mid Q(F^p, F^{k-p+1}) = 0,\ 1 < p < k\}$$

and

$$D := \left\{ \{F^p\} \in \check{D} \;\middle|\; \begin{array}{l} (\sqrt{-1})^{p-q}(-1)^{\frac{k(k-1)}{2}} Q(x, \bar{x}) > 0, \\ \forall x \in H^{p,q}\ (x \neq 0,\ p+q=k) \end{array} \right\}.$$

We call D a *Griffiths period domain.* Let Γ be the arithmetic subgroup of G defined as follows:

$$\Gamma = \{\gamma \in \mathrm{Aut}_{\mathbb{Z}} H^k(X_t, \mathbb{Z}) \mid \gamma \text{ fixes the polarization}\}.$$

Then Γ acts properly discontinuously on D and hence $\Gamma\backslash D$ is naturally a complex analytic space. Then the *period map* is defined by the induced map

$$\phi\colon S \longrightarrow \Gamma\backslash D. \tag{1.3.3}$$

We consider the derivation of the period map:

$$d\phi_s\colon T_sS \longrightarrow T_{\phi(s)}(\Gamma\backslash D) = T_{\phi(s)}D \subset \bigoplus_{p=1}^{k} \mathrm{Hom}(F^p, H_0^k(X_t, \mathbb{C})/F^p). \tag{1.3.4}$$

By the Griffiths transversality, we may show that the image of $d\phi_s$ is contained in the subspace $\bigoplus_{p=1}^k (F^p/F^{p+1}, F^{p-1}/F^p)$ of $\bigoplus_{p=1}^k (F^p, H_0^k(X_t, \mathbb{C})/F^p)$, i.e.,

$$d\phi_s\colon T_sS \longrightarrow (T_{\phi(s)}D) \cap \bigoplus_{p=1}^{k} \mathrm{Hom}(F^p/F^{p+1}, F^{p-1}/F^p). \tag{1.3.5}$$

We recall the relation between the derivation of the period map and the *Kodaira–Spenser map.* Let Θ_X be the tangent bundle of X. The Kodaira–Spenser map $\theta\colon \Theta_{S,s} \to H^1(X_s, \Theta_{X_s})$ is described as follows. We consider the extension of $\mathcal{O}_X$-module

$$0 \longrightarrow \pi^*(\Omega_S^1) \longrightarrow \Omega_X^1 \longrightarrow \Omega_{X/S}^1 \longrightarrow 0. \tag{1.3.6}$$

We have the element of $\mathrm{Ext}^1(\Omega^1_{X/S}, \pi^*(\Omega^1_S)) \cong H^1(M, \underline{Hom}(\Omega^1_{X/S}, \pi^*(\Omega^1_S)))$ which is the class of the extension (1.3.6). We take the image of this element via the edge homomorphism of the Leray spectral sequence of π

$$\begin{aligned} &H^1(X, \Theta_{X/S} \otimes \pi^*(\Omega_S^1)) \longrightarrow \\ &\qquad H^0(S, R^1\pi_*(\Theta_{X/S} \otimes \pi^*(\Omega_S^1))) \cong H^0(S, \Omega_S^1 \otimes R^1\pi_*(\Theta_{X/S})), \end{aligned}$$

denoted by $\rho_{X/S} \in H^0(S, \Omega^1_S \otimes R^1\pi_*(\Theta_{X/S}))$. The class $\rho_{X/S}$ is viewed as the element of $\mathrm{Hom}(\Theta_S, R^1\pi_*(\Theta_{X/S}))$. Then, for any $s \in S$, we obtain the map

$$\rho_{X/S,s}\colon \Theta_{S,s} \to H^1(X_s, \Theta_{X_s}),$$

which is called the *Kodaira–Spenser map.*

Note that $F^p/F^{p+1} \cong H_0^q(X_t, \Omega_X^p)$ where $p + q = k$. Therefore, for any $\xi \in T_sS$, the image of the derivation of the period map $d\phi_s(\xi)$ is view as the element of

$$\bigoplus_{p=1}^{k} \mathrm{Hom}(H_0^q(X_t, \Omega_X^p), H_0^{q+1}(X_t, \Omega_X^{p-1})).$$

We put

$$d\phi_s(\xi) = (d\phi_s(\xi)_1, d\phi_s(\xi)_2, \ldots, d\phi_s(\xi)_k).$$

Then we have the following theorem

Theorem 1.3.2. *The element* $d\phi_s(\xi)_p \in \mathrm{Hom}(H_0^q(X_t, \Omega_X^p), H_0^{q+1}(X_t, \Omega_X^{p-1}))$ *coincides with the map*

$$H_0^q(X_t, \Omega_X^p) \longrightarrow H^{q+1}(X_t, \Theta_{X_t} \otimes \Omega_X^p) \longrightarrow H^{q+1}(X_t, \Omega_X^{p-1})$$

where the first map is the cup-product of $\rho_{X/S,t}(\xi)$ *and the second map is the contraction.*

2. Mixed Hodge structure

2.1. Definition and Properties.

Definition 2.1.1. A *Hodge structure of weight* k consists of:

(1) a finitely generated free $\mathbb{Z}$-module $H^{\mathbb{Z}}$ and
(2) a decreasing filtration $F^p \supset F^{p+1}$of $H^{\mathbb{C}} = H^{\mathbb{Z}} \otimes_{\mathbb{Z}} \mathbb{C}$, satisfying

$$F^p \oplus \overline{F^{k-p+1}} \cong H^{\mathbb{C}}.$$

Example 2.1.2 (Tate twist). We put $\mathbb{Z}(k) := (2\pi\sqrt{-1})^k\mathbb{Z}$ is a Hodge structure of weight $-2k$ and of pure Hodge type $(-k, -k)$, that is, $F^{-k} \cap \overline{F^{-k}} = \mathbb{Z}(k)^{\mathbb{C}}$. We call $\mathbb{Z}(k)$ *the Tate twist.* Note that $H^2(\mathbb{P}^1_{\mathbb{C}}, \mathbb{Z}) = \mathbb{Z}(-1)$.

Definition 2.1.3. A *mixed Hodge structure* consists of:

(1) a finitely generated free $\mathbb{Z}$-module $H^{\mathbb{Z}}$;
(2) an increasing filtration $W_k \subset W_{k+1}$ of $H^{\mathbb{Q}} = H^{\mathbb{Z}} \otimes \mathbb{Q}$
(3) a decreasing filtration $F^p \supset F^{p+1}$ of $H^{\mathbb{C}}$ such that the filtration induced by $F^\bullet$ on $\mathrm{Gr}^k W_\bullet := W_k/W_{k-1}$ is a Hodge structure of weight k.

The mixed Hodge structure arises in natural is shown by the next basic result.

Theorem 2.1.4 ([**6**], [**7**]). *Let* X *be a complex algebraic variety. Then, there exists a mixed Hodge structure on* $H^j(X, \mathbb{Q})$. *Moreover, the weight filtration satisfies*

$$0 = W_{-1} \subseteq W_0 \subseteq \cdots \subseteq W_{2j} = H^j(X, \mathbb{Q})$$

and the Hodge filtration satisfies

$$H^j(X, \mathbb{C}) = F^0 \supseteq F^1 \supseteq \cdots \supseteq F^m \supseteq F^{m+1} = 0$$

Moreover, the structure is functorial.

Remark 2.1.5. If X is smooth and projective, the above mixed Hodge structure on $H^j(X, \mathbb{Q})$ agrees the usual Hodge structure.

One can define a mixed Hodge structure on the compactly supported cohomology $H_c^*(X) := H_c^*(X, \mathbb{Q})$.

Definition 2.1.6. The *mixed Hodge number* $h^{p,q;j}(X)$ is defined by $\dim_{\mathbb{C}}(\mathrm{Gr}_p^F \mathrm{Gr}_W^{p+q} H^j(X)^{\mathbb{C}})$. The *compactly supported mixed Hodge number* $h_c^{p,q;j}$ is defined by $\dim_{\mathbb{C}}(\mathrm{Gr}_p^F \mathrm{Gr}_W^{p+q} H_c^j(X)^{\mathbb{C}})$. We call the polynomials

$$H(X; x, y, t) := \sum h^{p,q;j}(X) x^p y^q t^j \text{ and} \tag{2.1.1}$$

$$H_c(X; x, y, t) := \sum h_c^{p,q;j}(X) x^p y^q t^j \tag{2.1.2}$$

the *mixed Hodge polynomial* and the *compactly supported mixed Hodge polynomial*, respectively.

We consider the two especially cases of Theorem 2.1.4 where X is a variety with normal crossing and where X is a open smooth variety.

2.2. Varieties with normal crossing. Let $X = \bigcup_{i=1}^{l} X_i$ be a union of smooth projective varieties meeting transversally; And locally X is defined by a single equation of the form

$$\{z_1 \cdots z_k = 0\}.$$

Definition 2.2.1. The *dual graph* Γ of X is a simplicial complex defined as follows:

- the point p_i of Γ correspond to components X_i;
- there exists a 1-simplex σ_{ij} connecting p_i and p_j if and only if $X_i \cap X_j \neq \emptyset$;
- σ_{ij}, σ_{jk}, σ_{ki} bounded a 2-simplex if and only if $X_i \cap X_j \cap X_k \neq \emptyset$, and so on.

We put $X^{[1]} := \coprod_i X_i$ and in general

$$X^{[k]} := \coprod_{i_1 < \cdots < i_k} X_{i_1} \cap \cdots \cap X_{i_k}$$

so that $X^{[k]}$ is smooth for all $k \geq 1$.

Consider the double complex

$$\bigoplus_{p,q} \mathcal{F}^{p,q} = \bigoplus_{p,q} \Omega^q_{X^{[p]}}$$

with the following differentials:

- the de Rham d
$$d \colon \Omega^q_{X^{[p]}} \longrightarrow \Omega^{q+1}_{X^{[p]}};$$
- the Čech boundary operator (an alternating sum of restriction maps)
$$\delta \colon \Omega^q_{X^{[p]}} \longrightarrow \Omega^q_{X^{[p+1]}};$$
- $D := d + \delta$.

We put

$$\mathcal{F}^k := \bigoplus_{p+q=k} \mathcal{F}^{p,q}.$$

Then $(\mathcal{F}^\bullet)$ is a complex.

Proposition 2.2.2. *The complex $(\mathcal{F}^\bullet)$ is resolution of the locally constant sheaf $\mathbb{C}_X$ on X. Hence $\mathbb{H}^k(\mathcal{F}^\bullet) = H^k(X, \mathbb{C})$ where $\mathbb{H}^k(\mathcal{F}^\bullet)$ is a hypercohomology of $(\mathcal{F}^\bullet)$.*

There is then a spectral sequence associated to $\mathbb{H}^k(\mathcal{F}^\bullet)$:

$$E_1^{p,q} := H^q(X^{[p+1]}, \mathbb{C}) \Longrightarrow H^{p+q}(X, \mathbb{C}).$$

We have an increasing filtration

$$W_k E_1 = \bigoplus_{q \le k} E_1^{p,q}.$$

Theorem 2.2.3. (1) *$E_r^{p,q}$ degenerates at the $r = 2$ term, i.e. $E_2^{p,q} = E_\infty^{p,q}$;*

(2) *W_k induce a filtration on E_∞ and hence on $H^*(X, \mathbb{C})$ defined over $\mathbb{Q}$. Moreover $W_n H^n(X, \mathbb{C}) = H^n(X, \mathbb{C})$ for all n;*

(3) *The natural Hodge filtrations on $H^q(X^{[p]}, \mathbb{C})$ induce a decreasing filtration $\{F^p\}$ of $H^{p+q}(X, \mathbb{C})$;*

(4) *the filtrations $\{W_k\}$ and $\{F^p\}$ on $H^{p+q}(X, \mathbb{C})$ are a mixed Hodge structure.*

Proof. For example, see [**14**, §4]. □

Example 2.2.4 (Curves). Let $C = \bigcup C_i$ be a curve with normal crossing. Everything still work if we drop the requirement of global normal crossings. We will only assume that locally X looks like $\{z_1 z_2 = 0\}$. A double point on an irreducible component of C contributes to a loop in the dual graph Γ. We consider the mixed Hodge structure on $H^1(C, \mathbb{C})$. The part of pure weight 0 is the following. We put $\widetilde{C} := C^{[1]} := \coprod C_i$ and $C^{[2]} := \coprod C_i \cap C_j = \{\text{the double points on } C\}$. We have

$$\begin{aligned} W_0 H^1(C, \mathbb{C}) &= \operatorname{Coker}\{H^0(\widetilde{C}, \mathbb{C}) \longrightarrow H^0(C^{[2]}, \mathbb{C})\} \\ &\cong H^1(\Gamma, \mathbb{C}). \end{aligned}$$

On the other hand, by the Mayer-Vietoris exact sequence, we have

$$W_0 H^1(C, \mathbb{C}) = \operatorname{Ker}\{H^1(C, \mathbb{C}) \longrightarrow H^1(\widetilde{C}, \mathbb{C})\}.$$

Hence,

$$\begin{aligned} W_0 &\cong H^1(\Gamma, \mathbb{C}) \\ W_1/W_0 &\cong H^1(\widetilde{C}, \mathbb{C}), \end{aligned}$$

which is a pure Hodge structure of weight one.

Example 2.2.5 (Surfaces). Let $X = \bigcup X_i$ be a surface with normal crossing. We put We put $X^{[1]} := \coprod X_i$ and $X^{[2]} := \coprod X_i \cap X_j$. Then, for $H^2(X, \mathbb{C})$ we have

$$\begin{aligned} W_2/W_1 &= \operatorname{Ker}\{H^2(X^{[1]}, \mathbb{C}) \longrightarrow H^2(X^{[2]}, \mathbb{C})\}, \\ W_1/W_0 &= \operatorname{Coker}\{H^1(X^{[1]}, \mathbb{C}) \longrightarrow H^1(X^{[2]}, \mathbb{C})\}, \\ W_0 &= H^2(\Gamma, \mathbb{C}). \end{aligned}$$

2.3. Open smooth varieties. Let X be a smooth quasi-projective variety. By resolution of singularities, we can assume that $X \subset \overline{X}$ where $\overline{X}$ is a smooth projective variety and the divisor $D := \overline{X} \setminus X$ is a normal crossing divisor. We put $j \colon X \hookrightarrow \overline{X}$ which is the natural inclusion.

We put

$$\Omega^1_{\overline{X}}(\log D) := \left\{ \phi \in j_* \Omega^1_X \mid \mathcal{I}_D \phi,\ \mathcal{I}_D d\phi \subset \Omega^1_{\overline{X}} \right\}.$$

We call a section of this sheaf a *logarithmic differential form of degree* 1 *along* D. The condition means that ϕ and the exterior differential ϕ have at worst simple poles along D. Moreover, we define the sheaf of *logarithmic differential form of degree* p *along* D by

$$\Omega^p_{\overline{X}}(\log D) := \bigwedge^k \Omega^1_{\overline{X}}(\log D).$$

If the divisor D is defined by $\{z_1 \cdots z_m = 0\}$ in some neighborhood of a point of D, then the element $\phi \in \Omega^p_{\overline{X}}(\log D)$ is represented by

$$\psi \wedge \frac{dz_{i_1}}{z_{i_1}} \wedge \cdots \wedge \frac{dz_{i_k}}{z_{i_k}}, \quad \psi \in \Omega^{p-k}_{\overline{X}}$$

locally. Exterior differential makes $\Omega^\bullet_{\overline{X}}(\log D)$ into a complex, called *log complex.*

Theorem 2.3.1. (1) $\mathbb{H}^k(\overline{X}, \Omega^\bullet_{\overline{X}}(\log D)) = H^k(X, \mathbb{C})$;
(2) *the associated spectral sequence*

$${}_F E_1^{p,q} := H^q(\overline{X}, \Omega^p_{\overline{X}}(\log D)) \Longrightarrow H^{p+q}(X, \mathbb{C})$$

is degenerate at the E_1 *term, and induce a decreasing filtration* $\{F^p\}$ *on* $H^k(X, \mathbb{C})$.

Proof. See [**6**, (3.2.2) and (3.2.13)]. □

To describe the weight filtration, we put

$$W_l \Omega^p_{\overline{X}}(\log D) := \Omega^{p-l}_{\overline{X}} \wedge \Omega^l_{\overline{X}}(\log D).$$

Locally, we can define the *Poincaré residue operator* as

$$\psi \wedge \frac{dz_{i_1}}{z_{i_1}} \wedge \cdots \wedge \frac{dz_{i_l}}{z_{i_l}} \longmapsto \psi|_{D_{i_1} \cap \ldots \cap D_{i_l}}$$

where D_i is defined by $z_i = 0$, $D_i \subset D = \{z_1 \cdots z_k = 0\}$. By the operator, we have the homomorphism $W_l \Omega^\bullet_{\overline{X}}(\log D) \longrightarrow \Omega^{\bullet - l}_{D_{i_1} \cap \ldots \cap D_{i_l}}$ *locally.* To obtain the *global* Poincaré residue operator, we introduce the following local system ϵ^l: We put $D^{[l]} := \coprod_{i_1 < \cdots < i_l} X_{i_1} \cap \cdots \cap X_{i_l}$ and $D^{[0]} := \overline{X}$. Let i_l be the natural map $D^{[l]} \to X$. For each $p \in D^{[l]}$, we take the set of l-th components of D which contain the image in X of an open neighborhood of p in $D^{[l]}$ locally. Then, we define the local system E_l over $D^{[l]}$ such that each fiber is this set of l-th elements. The local system of orientation of the set of l-th components is a $\mathbb{Z}/(2)$-torsor over $D^{[l]}$. The $\mathbb{Z}/(2)$-torsor defines, via the inclusion of $\mathbb{Z}/(2)$ in $\mathbb{C}^*$, a complex local system ϵ^l of rank 1 over $D^{[l]}$ with the isomorphism $(\epsilon^l)^{\otimes 2} \cong \mathbb{C}$. We have

$$\epsilon^l \cong \bigwedge^l \mathbb{C}^{E_n}.$$

Locally, ϵ^l has the two isomorphism $\pm\alpha \colon \epsilon^l \cong \mathbb{C}$ which are opposites of one another. We put

$$\epsilon^l_{\mathbb{Z}} := \alpha^{-1}((2\pi\sqrt{-1})^{-l}\mathbb{Z}).$$

The tuple $(\epsilon^l_{\mathbb{Z}}, \epsilon^l)$ is a local system of $\mathbb{Z}(-l)$, the Tate Hodge structure. We define the Poincaré residue operator

$$PR_l \colon W_l \Omega^\bullet_{\overline{X}}(\log D) \longrightarrow i_{l*}(\Omega^{\bullet - l}_{D^{[l]}} \otimes \epsilon^l)$$

via

$$PR_l\left(\psi\wedge\frac{dz_{i_1}}{z_{i_1}}\wedge\cdots\wedge\frac{dz_{i_l}}{z_{i_l}}\right)=\psi|_{D_{i_1}\cap\ldots\cap D_{i_l}}\otimes(\text{orientation }\{i_1,\ldots,i_l\})$$

where D_i is defined by $z_i = 0$, $D_i \subset D = \{z_1 \cdots z_k = 0\}$. Then PR_l is well-defined globally.

Lemma 2.3.2. *The Poincaré residue operator PR_l induce the following isomorphism of the complexes*

$$Gr_l^W \Omega^\bullet_{\overline{X}}(\log D) \longrightarrow i_{l*}(\Omega^{\bullet-l}_{D^{[l]}}\otimes\epsilon^l).$$

Proof. See [**6**, (3.1.5)]. □

Moreover, since $D^{[l]}$ is proper and smooth, we have $\mathbb{C}_{D^{[l]}} \cong \Omega^\bullet_{D^{[l]}}$. Then, the morphism

$$Gr_l^W \Omega^\bullet_{\overline{X}}(\log D) \longrightarrow \epsilon^l_{\overline{X}}[-l] := i_{l*}(\epsilon^l)[-l]$$

is a quasi-isomorphism.

Proposition 2.3.3. *The spectral sequence defined by $\Omega^\bullet_{\overline{X}}(\log D)$ with the filtration $W_\bullet$:*

$$\begin{aligned}{}_WE_1^{-l,k+l} &:= \mathbb{H}^k(\overline{X}, Gr_l^W\Omega^\bullet_{\overline{X}}(\log D))\\ &\cong H^{k-l}(D^{[l]},\epsilon^l)\Longrightarrow H^k(X,\mathbb{C})\end{aligned}$$

is degenerate at E_2 term. Then we have

$$Gr_l^W H^k(X,\mathbb{C}) = \bigoplus_{q-l=k} {}_WE_\infty^{-l,q}.$$

Proof. See [**6**, (3.2.13)]. □

Theorem 2.3.4. (1) *The weight filtration $W_\bullet$ of $H^k(X,\mathbb{C})$ is defined over $\mathbb{Q}$;*
(2) *the weight range from $0 = W_{k-1} \subset \cdots \subset W_{2k} = H^k(X,\mathbb{C})$;*
(3) *the filtration $\{F^\bullet\}$ and $\{W_\bullet\}$ define a mixed Hodge structure on $H^k(X,\mathbb{C})$;*
(4) *the resulting mixed Hodge structure is independent of the choice of $\overline{X}$ and functorial.*

Proof. For example, see [**14**, §5]. □

Example 2.3.5. Let $\overline{C}$ be a smooth complete curve, $S \subset \overline{C}$ a finite set of points, and $C = \overline{C} \setminus S$.

The weight filtration on $H^1(C,\mathbb{C})$ is as follows:

$$\begin{aligned}W_1 &= \mathrm{Ker}\{H^1(S^{[0]},\epsilon^0)\longrightarrow H^1(S^{[1]},\epsilon^1)\}\\ &= \mathrm{Image}\{H^1(\overline{C},\mathbb{C})\longrightarrow H^1(C,\mathbb{C})\},\\ W_2/W_1 &= \mathrm{Coker}\{H^0(S^{[0]},\epsilon^0)\longrightarrow H^0(S^{[1]},\epsilon^1)\}\\ &\cong \{\text{divisor of degree 0 supported on } S\},\end{aligned}$$

where W_2/W_1 has dimension $\sharp S - 1$.

To describe the Hodge filtration, let $\Omega^1_{\overline{C}}(S)$ be the sheaf of holomorphic 1-form on C with at worst simple poles along S. The short complex

$$\mathcal{C}^\bullet = \{\mathcal{O}_{\overline{C}} \longrightarrow \Omega^1_{\overline{C}}(S)\}$$

fits into an exact sequence

$$0 \longrightarrow \Omega^\bullet_{\overline{C}} \longrightarrow \mathcal{C}^\bullet \xrightarrow{2\pi\sqrt{-1}\mathrm{Res}} \mathbb{C}_S[1] \longrightarrow 0$$

where $\mathbb{C}_S$ is a skyscraper sheaf of $\mathbb{C}$ concentrated at the points of S and $\mathbb{C}_S[1]$ means that the complex consists of the single term $\mathbb{C}_S$ in dimension 1. Now, we have

$$F^1 = H^0(\overline{C}, \Omega^1_{\overline{C}}(S)) \subset H^1(C, \mathbb{C}).$$

It is a standard fact that $\varphi \in F^1$ can have arbitrary residues at $p \in S$, subject only to the condition

$$\sum_{p \in S} \mathrm{Res}_p(\varphi) = 0$$

(for example, see [**13**, p.233]). Thus, $F^1 \xrightarrow{2\pi\sqrt{-1}\mathrm{Res}} W_2/W_1$ is surjective. In fact, the condition that $H^1(C, \mathbb{C})$ is a mixed Hodge structure means that

$$F^1 \cap \overline{F^1} \cong W_2/W_1.$$

Classically, this is the statement that, given two points p and q on the complex Riemann surface $\overline{C}$, there exists a differential of the third kind with residue $\frac{1}{2\pi\sqrt{-1}}$ at p and $-\frac{1}{2\pi\sqrt{-1}}$ at q which is regular elsewhere and has real periods on $\overline{C}$ (see [**43**, vol2, p.104]).

CHAPTER 4

Nonabelian Hodge theory and Riemann-Hilbert correspondence

In this chapter, we describe relationships between the moduli space of semistable parabolic Higgs bundles, the moduli space of semistable $\boldsymbol{\xi}$-parabolic connections, and character varieties.

1. Nonabelian Hodge theory

The nonabelian Hodge theory for noncompact curve is due to Simpson [**44**]. The nonabelian Hodge theory gives a natural one-to-one correspondence between stable parabolic Higgs bundles, and stable $\boldsymbol{\xi}$-parabolic connections. This correspondence arises from the fact that both sides are shown to correspond to *irreducible tame harmonic bundles.*

Here, we consider (*parabolic*) λ-*connections* due to Deligne. We construct the moduli space of semistable parabolic λ-connection. By the moduli space, we have the deformation between the moduli space of semistable parabolic Higgs bundles and the moduli space of semistable $\boldsymbol{\xi}$-parabolic connections.

1.1. Parabolic λ-connection. We fix integers $g \geq 0, k \geq 0$ and $n > 0$. We also fix a k-tuple of partition of n, denoted by $\boldsymbol{\mu} = (\mu^1, \dots, \mu^k)$, that is, $\mu^i = (\mu^i_1, \dots, \mu^i_{r_i})$ such that $\mu^i_1 \geq \mu^i_2 \geq \cdots$ are non-negative integers and $\sum_j \mu^i_j = n$. Let Σ be a smooth complex projective curve of genus g. We fix k-distinct points $p_1, \dots, p_k$ in Σ and we define a divisor by $D := p_1 + \cdots + p_k$. We put $\Sigma_0 = \Sigma \setminus D$. For integer d, we put

$$\Xi^{\boldsymbol{\mu},d}_n := \left\{ \left(\lambda, (\xi^i_j)^{1\leq i\leq k}_{1\leq j\leq r_i}\right) \in \mathbb{C} \times \mathbb{C}^r \;\middle|\; \lambda d + \sum_{i,j} \mu^i_j \xi^i_j = 0 \right\}$$

where $r := \sum r_i$. We take $(\lambda, \boldsymbol{\xi}) \in \Xi^{\boldsymbol{\mu},d}_n$ where $\boldsymbol{\xi} = (\xi^i_j)^{1\leq i\leq k}_{1\leq j\leq r_i}$.

Definition 1.1.1. For $(\lambda, \boldsymbol{\xi}) \in \Xi^{\boldsymbol{\mu},d}_n$, we call $(\lambda, E, \nabla, \{l^{(i)}_*\}_{1\leq i\leq k})$ a *$\boldsymbol{\xi}$-parabolic λ-connection of rank n, of degree d, and of type $\boldsymbol{\mu}$* if

(1) E is an algebraic vector bundle on Σ of rank n and of degree d,
(2) $\nabla \colon E \to E \otimes \Omega^1_\Sigma(D)$ is a λ-connection, that is, ∇ is a homomorphism of sheaves satisfying $\nabla(fa) = \lambda a \otimes df + f\nabla(a)$ for $f \in \mathcal{O}_\Sigma$ and $a \in E$, and
(3) for each p_i, $l^{(i)}_*$ is a filtration $E|_{p_i} = l^{(i)}_1 \supset l^{(i)}_2 \supset \cdots \supset l^{(i)}_{r_i} \supset l^{(i)}_{r_i+1} = 0$ such that $\dim(l^{(i)}_j / l^{(i)}_{j+1}) = \mu^i_j$ and $(\mathrm{Res}_{p_i}(\nabla) - \xi^i_j \mathrm{id}_{E|_{p_i}})(l^{(i)}_j) \subset l^{(i)}_{j+1}$ for $j = 1, \dots, r_i$.

For $\lambda = 1$, this is a $\boldsymbol{\xi}$-parabolic connection of spectral type $\boldsymbol{\mu}$ (defined in the introduction). For $\lambda = 0$ and $\boldsymbol{\xi} = 0$, this is a parabolic Higgs bundle (defined in the introduction).

Remark 1.1.2. For $\lambda \neq 0$, we have

$$\deg E = \deg(\det(E)) = -\sum_{i=1}^{k} \mathrm{Res}_{p_i}((\lambda^{-1}\nabla)_{\det E}) = -\sum_{i=1}^{k}\sum_{j=0}^{n-1} \frac{\xi^i_j}{\lambda} = d.$$

We take rational numbers

$$0 < \alpha^{(i)}_1 < \alpha^{(i)}_2 < \cdots < \alpha^{(i)}_{r_i} < 1$$

for $i = 1, \ldots, k$ satisfying $\alpha^{(i)}_j \neq \alpha^{(i')}_{j'}$ for $(i, j) \neq (i', j')$. We choose $\boldsymbol{\alpha} = (\alpha^{(i)}_j)$ sufficiently generic.

We define the *parabolic degree* and *parabolic slope* of E by

$$\mathrm{pardeg}(E) := \deg(E) + \sum_{i=1}^{k}\sum_{j=1}^{r_i} \alpha^{(i)}_j \dim(l^{(i)}_j / l^{(i)}_{j+1}),$$

$$\mathrm{par}\mu(E) := \frac{\mathrm{pardeg}(E)}{\mathrm{rk}(E)}.$$

Definition 1.1.3. A parabolic λ-connection $(\lambda, E, \nabla, \{l^{(i)}_*\}_{1\leq i\leq k})$ is *$\boldsymbol{\alpha}$-stable* (resp. *$\boldsymbol{\alpha}$-semistable*) if for any proper nonzero subbundle $F \subset E$ satisfying $\nabla(F) \subset F \otimes \Omega^1_\Sigma(D)$, the inequality

$$\mathrm{par}\mu(F) < \mathrm{par}\mu(E) \quad (\text{resp. } \leq)$$

holds.

Remark 1.1.4 ([**21**, Remark 2.2]). We chose $\boldsymbol{\alpha} = (\alpha^{(i)}_j)$ sufficiently generic. Then, a parabolic λ-connection $(\lambda, E, \nabla, \{l^{(i)}_*\})$ is $\boldsymbol{\alpha}$-stable if and only if $(\lambda, E, \nabla, \{l^{(i)}_*\})$ is $\boldsymbol{\alpha}$-semistable.

1.2. Construction of the moduli space. The argument in this subsection is almost the same as in [**21**]. The difference from [**21**] is that we fix the k-distinct points $\{p_1, \ldots, p_k\}$, the flag $\{l^{(i)}_*\}$ is not necessarily full flag, and we construct the moduli space of $\boldsymbol{\alpha}$-semistable parabolic λ-connections instead of $\boldsymbol{\alpha}$-semistable parabolic connections.

Definition 1.2.1. We put $\mathcal{C} = \Sigma \times \Xi^{\boldsymbol{\mu},d}_n$, $S = \Xi^{\boldsymbol{\mu},d}_n$, $\tilde{p}_i = p_i \times \Xi^{\boldsymbol{\mu},d}_n$ (for $i = 1, \ldots, k$) and $\mathcal{D} = \tilde{p}_1 + \cdots + \tilde{p}_k$. We define a functor $\mathcal{MF}^{\mathcal{D},\boldsymbol{\alpha}}_{\mathcal{C}/S,Hod}(n, d, \boldsymbol{\mu})$ of category of locally noetherian schemes to the category of sets by

$$\mathcal{MF}^{\mathcal{D},\boldsymbol{\alpha}}_{\mathcal{C}/S,Hod}(n, d, \boldsymbol{\mu})(T) := \left\{(\lambda, E, \nabla, \{l^{(i)}_*\}_{1\leq i\leq k})\right\} / \sim,$$

for a locally noetherian scheme T over S where

(1) E is a vector bundle on $\mathcal{C}_T$ of rank n,
(2) $\nabla : E \to E \otimes \Omega^1_{\mathcal{C}_T}((\mathcal{D}_T))$ is a relative $(\lambda)_T$-connection,
(3) for each $p_i \times T$, $l^{(i)}_*$ is a filtration $E|_{(\tilde{p}_i)_T} = l^{(i)}_1 \supset l^{(i)}_2 \supset \cdots \supset l^{(i)}_{r_i} \supset l^{(i)}_{r_i+1} = 0$ such that $\dim(l^{(i)}_j / l^{(i)}_{j+1}) = \mu^i_j$ and $(\mathrm{Res}_{(\tilde{p}_i)_T}(\nabla) - (\xi^i_j)_T) \subset l^{(i)}_{j+1}$ for $j = 1, \ldots, r_i$,
(4) for any geometric point $t \in T$, $\dim(l^i_j / l^i_{j+1}) \otimes k(t) = \mu^i_j$ for any i, j and $(\lambda, E, \nabla, \{l^{(i)}_*\}) \otimes k(t)$ is $\boldsymbol{\alpha}$-stable.

Proposition 1.2.2. *There exists a relative coarse moduli scheme*

$$\pi\colon \mathcal{M}^{\boldsymbol{\mu}}_{Hod} \longrightarrow \Xi^{\boldsymbol{\mu},d}_n$$
$$(\lambda, E, \nabla, \{l^{(i)}_*\}_{1\le i\le r_i}) \longmapsto (\lambda, \boldsymbol{\xi})$$

of $\boldsymbol{\alpha}$-stable parabolic λ-connections of rank r, of degree d, and of type $\boldsymbol{\mu}$. For simplicity, we drop $\boldsymbol{\alpha}$ and d from the notation of the moduli space.

If n and d are coprime, then $\mathcal{M}^{\boldsymbol{\mu}}_{Hod}$ is a relative fine moduli scheme, that is, there is a universal family over $\mathcal{M}^{\boldsymbol{\mu}}_{Hod}$.

Proof. Fix a weight $\boldsymbol{\alpha}$ which determines the stability of parabolic λ-connections. We take positive integers β_1, β_2, γ and rational numbers $0 < \tilde{\alpha}^{(i)}_1 < \cdots < \tilde{\alpha}^{(i)}_{r_i} < 1$ satisfying $(\beta_1+\beta_2)\alpha^{(i)}_j = \beta_1\tilde{\alpha}^{(i)}_j$ for any i, j. We assume $\gamma \gg 0$. We take an increasing sequence $0 < \alpha'_1 < \cdots < \alpha'_r < 1$ such that

$$\{\alpha'_i \mid 1 \le i \le r\} = \{\tilde{\alpha}^{(i)}_j \mid 1 \le i \le k, 1 \le j \le r_i\}$$

where we put $r = \sum_{i=1}^k r_i$. We take any member $(\lambda, E, \nabla, \{l^{(i)}_*\}_{1\le i\le r_i}) \in \mathcal{MF}^{\mathcal{D},\boldsymbol{\alpha}}_{\mathcal{C}/S,Hod}(n, d, \boldsymbol{\mu})(T)$. For each $1 \le p \le r$, there exist i, j satisfying $\tilde{\alpha}^{(i)}_j = \alpha'_p$. We put $F_1(E) := E$ and define inductively

$$F_p(E) := \mathrm{Ker}(F_{p-1}(E) \longrightarrow E|_{(\tilde{p}_i)_T}/l_p)$$

for $p = 1, \dots, r$. Here, we put $l_p = l^{(i)}_j$ satisfying $p = j + \sum_{l=1}^{i-1} r_l$. We also put $d_p :=$ length$((E/F_{p+1}(E)) \otimes k(t))$ and $t \in T$. Then, $(\lambda, E, \nabla, \{l^{(i)}_*\}) \mapsto (E, E, \lambda\mathrm{id}, \nabla, F_*(E))$ determines the morphism

$$\iota\colon \mathcal{MF}^{\mathcal{D},\boldsymbol{\alpha}}_{\mathcal{C}/S,Hod}(n, d, \boldsymbol{\mu}) \longrightarrow \overline{\mathcal{MF}}^{\mathcal{D},\boldsymbol{\alpha}',\boldsymbol{\beta},\gamma}_{\mathcal{C}/S}(n, d, \{d_i\})$$

where $\overline{\mathcal{MF}}^{\mathcal{D},\boldsymbol{\alpha}',\boldsymbol{\beta},\gamma}_{\mathcal{C}/S}(n, d, \{d_i\})$ is the moduli functor of $(\boldsymbol{\alpha}, \boldsymbol{\beta}, \gamma)$-stable Λ^1_D-triples whose coarse moduli scheme exists by Theorem 3.2.2 of Chapter II. Then we have that a certain subscheme $\mathcal{M}^{\boldsymbol{\mu}}_{Hod}$ of $\overline{\mathcal{M}}^{\mathcal{D},\boldsymbol{\alpha}',\boldsymbol{\beta},\gamma}_{\mathcal{C}/S}(n, d, \{d_i\})$ is just the coarse moduli scheme of $\mathcal{MF}^{\mathcal{D},\boldsymbol{\alpha}}_{\mathcal{C}/S,Hod}(n, d, \boldsymbol{\mu})$ in the same way as in [**23**, Theorem 2.1] and [**21**, Theorem 2.1].

If n and d are coprime, then there is a universal family on $\mathcal{M}^{\boldsymbol{\mu}}_{Hod} \times \Sigma$ (see [**20**, Theorem 4.6.5] and the proof of [**21**, Theorem 2.1]). □

We denote the fibers of $\mathcal{M}^{\boldsymbol{\mu}}_{Hod}$ over $\lambda = 0$ and $\lambda = 1$ by $\mathcal{M}^{\boldsymbol{\mu}}_{DR}$ and $\mathcal{M}^{\boldsymbol{\mu}}_{Dol}$, respectively. Let $\mathcal{M}^{\boldsymbol{\mu}}_{Hod}(\lambda, \boldsymbol{\xi})$ be the fiber of $(\lambda, \boldsymbol{\xi})$. Let $\mathcal{M}^{\boldsymbol{\mu}}_{DR}(\boldsymbol{\xi})$ and $\mathcal{M}^{\boldsymbol{\mu}}_{Dol}(\mathbf{0})$ be the fibers of $(1, \boldsymbol{\xi})$ and $(0, \mathbf{0})$, respectively. The fiber $\mathcal{M}^{\boldsymbol{\mu}}_{DR}(\boldsymbol{\xi})$ is the moduli space of $\boldsymbol{\alpha}$-semistable $\boldsymbol{\xi}$-parabolic connections of spectral type $\boldsymbol{\mu}$ (constructed in [**22**]), and the fiber $\mathcal{M}^{\boldsymbol{\mu}}_{Dol}(\mathbf{0})$ is the moduli space of $\boldsymbol{\alpha}$-semistable parabolic Higgs bundles of rank n and of degree d (constructed as a hyperkähler quotient using gauge theory in [**29**] or as a closed subvariety of the moduli space of parabolic Higgs sheaves constructed in [**54**]).

Proposition 1.2.3. *The morphism*

$$\pi\colon \mathcal{M}^{\boldsymbol{\mu}}_{Hod} \longrightarrow \Xi^{\boldsymbol{\mu},d}_n$$
$$(\lambda, E, \nabla, \{l^{(i)}_*\}_{1\le i\le r_i}) \longmapsto (\lambda, \boldsymbol{\xi})$$

is smooth. Moreover, $\mathcal{M}^{\boldsymbol{\mu}}_{Hod}$ is nonsingular.

PROOF. ([**21**, Theorem 2.1] and [**22**]). At first, we prove that $\pi : \mathcal{M}^{\boldsymbol{\mu}}_{Hod} \to \Xi^{\boldsymbol{\mu},d}_n$ is smooth. Let $\mathcal{M}^1_{Hod}$ be the moduli space of tuples (λ, L, ∇_L) where L is a line bundle of degree d on Σ and $\nabla_L\colon L \to L \otimes \Omega^1_\Sigma(D)$ is a λ-connection. We put

$$\Xi^{k,d} := \left\{ (\lambda, (\xi^i)) \in \mathbb{C} \times \mathbb{C}^k \;\middle|\; \lambda d + \sum_{i=1}^{k} \xi^i = 0 \right\}.$$

Let $\Xi^{k,d}_{\lambda=1}$ be the subset of $\Xi^{k,d}$ where $\lambda = 1$ and let $\mathcal{M}^1_{DR}$ be the inverse image of the subset $\Xi^{k,d}_{\lambda=1}$. Since $\mathcal{M}^1_{DR} \to \Xi^{k,d}_{\lambda=1}$ is smooth (see [**21**] and [**22**]), $\mathcal{M}^1_{Hod} \to \Xi^{k,d}$ is smooth (see [**49**, Lemma 6.1]). We consider the morphism

$$\begin{aligned} \det\colon \mathcal{M}^{\boldsymbol{\mu}}_{Hod} &\longrightarrow \mathcal{M}^1_{Hod} \times_{\Xi^{k,d}} \Xi^{\boldsymbol{\mu},d}_n \\ (\lambda, E, \nabla, \{l^{(i)}_j\}) &\longmapsto ((\lambda, \det(E), \det(\nabla)), \pi(\lambda, E, \nabla, \{l^{(i)}_j\})). \end{aligned}$$

It is sufficient to show that the morphism det is smooth. Let A be an artinian local ring over $\mathcal{M}^1_{Hod} \times_{\Xi^{k,d}} \Xi^{\boldsymbol{\mu},d}_n$ with the maximal ideal m and I be an ideal of A such that $mI = 0$. Let $(\lambda, L, \nabla) \in \mathcal{M}^1_{Hod}(A)$ and $(\lambda, \boldsymbol{\xi}) \in \Xi^{\boldsymbol{\mu},d}_n(A)$ be the elements corresponding to the morphism

$$\mathrm{Spec} A \longrightarrow \mathcal{M}^1_{Hod} \times_{\Xi^{k,d}} \Xi^{\boldsymbol{\mu},d}_n.$$

We take any member $(\lambda, E, \nabla, \{l^{(i)}_j\}) \in \mathcal{M}^{\boldsymbol{\mu}}_{Hod}(A/I)$ such that $\det(\lambda, E, \nabla, \{l^{(i)}_j\}) \cong ((\lambda, L, \nabla), (\lambda, \boldsymbol{\xi})) \otimes A/I$. It is sufficient to show that $(\lambda, E, \nabla, \{l^{(i)}_j\})$ may be lifted to a flat family $(\tilde{\lambda}, \tilde{E}, \tilde{\nabla}, \{\tilde{l}^{(i)}_j\})$ over A such that $\det(\tilde{\lambda}, \tilde{E}, \tilde{\nabla}, \{\tilde{l}^{(i)}_j\}) \cong ((\lambda, L, \nabla), (\lambda, \boldsymbol{\xi}))$. The obstructions lie in the hypercohomology $\mathbb{H}^2(\Sigma, \mathcal{F}^\bullet_0 \otimes I)$. Here, $\mathcal{F}^\bullet_0$ is the complex of sheaves defined by $\mathcal{F}^i_0 = 0$ for $i \neq 0, 1$,

$$\mathcal{F}^0_0 := \left\{ s \in \mathcal{E}nd(E \otimes A/m) \;\middle|\; \begin{array}{l} \mathrm{Tr}(s) = 0 \text{ and} \\ s_{|_{p_i \otimes A/m}}(l^i_j)_{A/m} \subset (l^i_j)_{A/m} \text{ for any } i, j \end{array} \right\},$$

$$\mathcal{F}^1_0 := \left\{ s \in \mathcal{E}nd(E \otimes A/m) \otimes \Omega^1_\Sigma(D) \;\middle|\; \begin{array}{l} \mathrm{Tr}(s) = 0 \text{ and} \\ \mathrm{Res}_{p_i \otimes A/m}(s)(l^i_j)_{A/m} \subset (l^i_{j+1})_{A/m} \text{ for any } i, j \end{array} \right\},$$

and $d : \mathcal{F}^0_0 \to \mathcal{F}^1_0$ maps s to $s\nabla - \nabla s$. From the spectral sequence $H^q(\mathcal{F}^p_0) \Rightarrow \mathbb{H}^{p+q}(\mathcal{F}^\bullet_0)$, there is an isomorphism

$$\mathbb{H}^2(\mathcal{F}^\bullet_0) \cong \mathrm{Coker}\left(H^1(\mathcal{F}^0_0) \xrightarrow{H^1(d)} H^1(\mathcal{F}^1_0) \right).$$

Since $(\mathcal{F}_0^0)^\vee \otimes \Omega^1_\Sigma \cong \mathcal{F}_0^1$ and $(\mathcal{F}_0^1)^\vee \otimes \Omega^1_\Sigma \cong \mathcal{F}_0^0$, we have

$$\begin{aligned}\mathbb{H}^2(\mathcal{F}_0^\bullet) &\cong \mathrm{Coker}\left(H^1(\mathcal{F}_0^0) \xrightarrow{H^1(d)} H^1(\mathcal{F}_0^1)\right)\\ &\cong \mathrm{Ker}\left(H^1(\mathcal{F}_0^1)^\vee \xrightarrow{H^1(d)} H^1(\mathcal{F}_0^0)^\vee\right)^\vee\\ &\cong \mathrm{Ker}\left(H^0((\mathcal{F}_0^1)^\vee \otimes \Omega^1_\Sigma) \xrightarrow{-H^1(d)} H^0((\mathcal{F}_0^0)^\vee \otimes \Omega^1_\Sigma)\right)^\vee\\ &\cong \mathrm{Ker}\left(H^0(\mathcal{F}_0^0) \xrightarrow{-H^1(d)} H^0(\mathcal{F}_0^1)\right)^\vee.\end{aligned}$$

We take any element $s \in \mathrm{Ker}\left(H^0(\mathcal{F}_0^0) \xrightarrow{-H^1(d)} H^0(\mathcal{F}_0^1)\right)$, which may be regarded as an element of $\mathrm{End}((\lambda, E, \nabla, \{l_j^{(i)}\}))$. Since $(\lambda, E, \nabla, \{l_j^{(i)}\})$ is $\boldsymbol{\alpha}$-stable, the endmorphism s is a scalar multiplication. By $\mathrm{Tr}(s) = 0$, we have $s = 0$. Hence, $\mathbb{H}^2(\mathcal{F}_0^\bullet) = 0$.

Secondly, we prove that $\mathcal{M}^{\boldsymbol{\mu}}_{Hod}$ is nonsingular. (see [**23**, Remark 6.1]). It is enough to show $\lambda : \mathcal{M}^{\boldsymbol{\mu}}_{Hod} \to \mathbb{C}$ given by $(\lambda, E, \nabla, \{l_*^{(i)}\}) \mapsto \lambda$ is smooth. In this case, the obstructions of the extensions lie in the hypercohomology $\mathbb{H}^2(\Sigma, \mathcal{F}_0^{\bullet,+} \otimes I)$. Here, $\mathcal{F}_0^{\bullet,+}$ is the complexes of sheaves defined by $\mathcal{F}_0^{i,+} = 0$ for $i \neq 0, 1$, $\mathcal{F}_0^{0,+} := \mathcal{F}_0^0$,

$$\mathcal{F}_0^{1,+} := \left\{ s \in \mathcal{E}nd(E \otimes A/m) \otimes \Omega^1_\Sigma(D) \;\middle|\; \begin{array}{l} \mathrm{Tr}(s) = 0, \\ \mathrm{Res}_{p_i(s)\otimes A/m}(l_j^{(i)})_{A/m} \subset (l_j^{(i)})_{A/m} \text{ for any } i, j \\ \text{and the element of } \mathrm{End}((l_j^{(i)})_{A/m}/(l_{j+1}^{(i)})_{A/m}) \\ \text{induced by } \mathrm{Res}_{p_i(s)\otimes A/m} \text{is a scalar.} \end{array} \right\},$$

and $d^+ : \mathcal{F}_0^{0,+} \to \mathcal{F}_0^{1,+}$ maps s to $s\nabla - \nabla s$. We put $\mathcal{T}_0^1 = \mathcal{F}_0^{1,+}/\mathcal{F}_0^1$ and $\mathcal{T}_0^\bullet = [0 \to \mathcal{T}_0^1]$. Then, we have the following exact sequence of the complex on Σ:

$$0 \longrightarrow \mathcal{F}_0^\bullet \longrightarrow \mathcal{F}_0^{\bullet,+} \longrightarrow \mathcal{T}_0^\bullet \longrightarrow 0.$$

Note that $\mathcal{T}_0^1$ is a skyscraper sheaf. We consider the long exact sequence. Since $\mathbb{H}^2(\mathcal{F}_0^\bullet) = \mathbb{H}^2(\mathcal{T}_0^\bullet) = 0$, we obtain $\mathbb{H}^2(\mathcal{F}_0^{\bullet,+}) = 0$. $\square$

2. Riemann-Hilbert correspondence

2.1. Riemann Problem. We fix integers $g \geq 0$, $k \geq 0$ and $n > 0$. Let Σ be a smooth complex projective curve of genus g, and let $D := p_1 + \cdots + p_k$ be a divisor on Σ where $p_1, \ldots, p_k$ are discrete points on Σ.

We fix a point $* \in \Sigma \setminus D$. The fundamental group of $\Sigma \setminus D$ is described as follows:

$$\pi_1(\Sigma \setminus D, *) = \langle \alpha_1, \beta_1, \ldots, \alpha_g, \beta_g, \gamma_1, \ldots, \gamma_k \mid (\alpha_1, \beta_1) \cdots (\alpha_g, \beta_g)\gamma_1 \cdots \gamma_k \rangle$$

where $(\alpha, \beta) := \alpha\beta\alpha^{-1}\beta^{-1}$. Let us consider a representation

$$\rho \colon \pi_1(\Sigma \setminus D, *) \longrightarrow \mathrm{GL}(n, \mathbb{C}).$$

We put $A_i := \rho(\alpha_i)$, $B_i := \rho(\beta_i)$ and $X_j := \rho(\gamma_j)$ for $i = 1, \ldots, g$, $j = 1, \ldots, k$. These matrices satisfy the relation $(A_1, B_1) \cdots (A_g, B_g) X_1 \cdots X_k = I_n$. There exists a locally constant sheaf $\mathcal{F}$ on $\Sigma \setminus D$ corresponding to the representation ρ. Moreover, there is a holomorphic bundle with connection (E, ∇) on $\Sigma \setminus D$ such that $E^{\nabla} = \mathcal{F}$, namely $E = \mathcal{O}_{\Sigma \setminus D} \otimes_{\mathbb{C}} \mathcal{F}$. Here, E^{∇} is the sheaf of ∇-horizontal holomorphic sections of E.

Problem. *Does there exist a meromorphic bundle (M, ∇) with flat connection on M which has a regular singularity at each point $p_1, \ldots, p_k$ and whose restriction to $\Sigma \setminus D$ is isomorphic to (E, ∇)?*

Theorem 2.1.1. *Given a locally constant sheaf $\mathcal{F}$ on $M \setminus \Sigma$, there exists a holomorphic bundle E on M and a connection $\nabla \colon E \to \Omega^1_M(\Sigma) \otimes_{\mathcal{O}_M} E$ with logarithmic poles, such that the locally constant sheaf E^{∇} is isomorphic to $\mathcal{F}$ on $\Sigma \setminus D$.*

Proof. We put $E' = \mathcal{O}_{\Sigma \setminus D} \otimes_{\mathbb{C}} \mathcal{F}$, and we define ∇' by the condition $(E')^{\nabla'} = \mathcal{F}$. We take a sufficiently small open neighborhood U_i of p_i such that $U_i \cap D = \{p_i\}$ where $i = 1, \ldots, k$. Then, by Lemma 3.2.7 in Chapter I, there exists a holomorphic bundle (E_i, ∇_i) with meromorphic connection on U_i with logarithmic pole at p_i such that $(E_i|_{U_i \setminus p_i})^{\nabla_i} = \mathcal{F}|_{U_i \setminus p_i}$, namely $E_i = \mathcal{O}_{U_i} \otimes_{\mathbb{C}} \mathcal{F}_t$ where t is a point of $U_i \setminus p_i$. We glue the bundle with connections $(E', \nabla'), (E_1, \nabla_1), \ldots, (E_k, \nabla_k)$. Then we have a desired bundle. □

We put $M := \mathcal{O}_{\Sigma}(*D) \otimes_{\mathcal{O}_{\Sigma}} E$ where (E, ∇) is given by Theorem 2.1.1. Then (M, ∇) is a desired meromorphic bundle with connection.

2.2. Riemann-Hilbert correspondence as a functor. We consider the correspondence

$$(M, \nabla) \longmapsto M^{\nabla}_{|\Sigma \setminus D}. \tag{2.2.1}$$

Here, (M, ∇) is a meromorphic bundles with connection having poles at the points of D and having regular singularity at any point of Σ, and $M^{\nabla}_{|\Sigma \setminus D}$ is the constant sheaf of horizontal sections. We call the correspondence (2.2.1) the *Riemann–Hilbert correspondence.* The Riemann–Hilbert correspondence is functorial, that is, we have a functor from the category of meromorphic bundles with connection having poles at the points of D and having regular singularity at any point of D to the category of locally constant sheaves of $\mathbb{C}$-vector space on $\Sigma \setminus D$. We recall the following theorem due to Deligne.

Theorem 2.2.1 (Curve case of [**5**, II, Theorem 5.9]). *The functor induced by the Riemann–Hilbert correspondence is equivalence.*

Proof. We obtain already that the functor is essentially surjective. We show that it is fully faithful, that is, for $\varphi \colon M^{\nabla}_{|\Sigma \setminus D} \to M'^{\nabla'}_{|\Sigma \setminus D}$, there exists a unique homomorphism $\psi \colon (M, \nabla) \to (M', \nabla')$ which induces the homomorphism φ. The homomorphism φ can be viewed as a horizontal section of the bundle $(\mathcal{H}om_{\mathcal{O}(M,M')_{\Sigma \setminus D}}, \nabla)$. We show that this horizontal section is the restriction to $\Sigma \setminus D$ of a unique meromorphic horizontal section of $(\mathcal{H}om_{\mathcal{O}(M,M')_{\Sigma}}, \nabla)$. Since (M, ∇) and (M', ∇') have regular singularity, $(\mathcal{H}om_{\mathcal{O}(M,M')_{\Sigma}}, \nabla)$ has regular singularity. The problem of the existence of a unique meromorphic horizontal section is local We may show the existence by Proposition 3.2.5 and Example 3.2.1 in Chapter I. □

2.3. Moduli theoretic Riemann-Hilbert correspondence. We put

$$\text{g.c.d.}(\boldsymbol{\mu}) := \text{g.c.d.}(\mu^1_1, \ldots, \mu^1_{r_1}, \mu^2_1, \ldots, \mu^2_{r_2}, \ldots, \mu^k_1, \ldots, \mu^k_{r_k}). \tag{2.3.1}$$

We take an integer d such that d and g.c.d.$(\boldsymbol{\mu})$ are coprime. For an integer d, we put

$$\Xi^{\boldsymbol{\mu},d}_{n,\lambda=1} := \left\{ \boldsymbol{\xi} = (\xi^i_j)^{1\le i\le k}_{1\le j\le r_i} \in \mathbb{C}^r \;\middle|\; d + \sum_{i,j} \mu^i_j \xi^i_j = 0 \right\}$$

where $r := \sum r_i$.

Definition 2.3.1. Take an element $\boldsymbol{\xi} \in \Xi^{\boldsymbol{\mu},d}_{n,\lambda=1}$. We call $\boldsymbol{\xi}$ *generic* if

(1) $\xi^i_j - \xi^i_k \notin \mathbb{Z}$ for any i and $j \neq k$, and
(2) there exist no integer s with $0 < s < n$, integers s_i with $1 \le s_i \le r_i$, and subsets $\{j^i_1, \ldots, j^i_{s_i}\} \subset \{1, \ldots, r_i\}$ for each $1 \le i \le k$ such that

$$\sum_{i=1}^{k} \sum_{l=1}^{s_i} \upsilon^i_{j^i_l} \xi^i_{j^i_l} \notin \mathbb{Z},$$

for any tuple of integers $\boldsymbol{\upsilon} = (\upsilon^i_j)$ with $0 \le \upsilon^i_j \le \mu^i_j$ where $\upsilon^i_{j^i_1} + \cdots + \upsilon^i_{j^i_{s_i}} = s$ for $i = 1, \ldots, k$.

Let $\Xi^{\boldsymbol{\mu},d,irr}_{n,\lambda=1}$ be the locus of generic elements in $\Xi^{\boldsymbol{\mu},d}_{n,\lambda=1}$, and let $\mathcal{M}^{\boldsymbol{\mu},irr}_{DR}$ be the inverse image of $\Xi^{\boldsymbol{\mu},d,irr}_{n,\lambda=1}$ via $\mathcal{M}^{\boldsymbol{\mu}}_{DR} \to \Xi^{\boldsymbol{\mu},d}_{n,\lambda=1}$.

Remark 2.3.2. If d and g.c.d.$(\boldsymbol{\mu})$ have the greatest common divisor $r' \neq 1$, then $\Xi^{\boldsymbol{\mu},d,irr}_{n,\lambda=1} = \emptyset$, since

$$\sum_{i,j} \frac{\mu^i_j}{r'} \xi^i_j = -\frac{d}{r'} \in \mathbb{Z}$$

for any $\boldsymbol{\xi} \in \Xi^{\boldsymbol{\mu},d}_{n,\lambda=1}$.

Conversely, if d and g.c.d.$(\boldsymbol{\mu})$ are coprime, then $\Xi^{\boldsymbol{\mu},d,irr}_{n,\lambda=1}$ is non-empty. (see Remark 2.3.4 as below and the proof of [**15**, Lemma 2.1.2]).

Remark 2.3.3 (see [**21**, Section 2]). For generic $\boldsymbol{\xi}$, any $\boldsymbol{\xi}$-parabolic connection $(E, \nabla, \{l^{(i)}_*\})$ is *irreducible.* Here, we call $(E, \nabla, \{l^{(i)}_*\})$ *reducible* if there is a non-trivial subbundle $0 \neq F \subsetneq E$ such that $\nabla(F) \subset F \otimes \Omega^1_\Sigma(D)$. We call $(E, \nabla, \{l^{(i)}_*\})$ *irreducible* if it is not reducible. In particular, for generic $\boldsymbol{\xi}$, any $(E, \nabla, \{l^{(i)}_*\})$ is semistable.

We construct a family of all generic character varieties of type $\boldsymbol{\mu}$. We put $r := \sum r_i$ and

$$N^{\boldsymbol{\mu}}_n := \left\{ \boldsymbol{\nu} = (\nu^i_j)^{1\le i\le k}_{1\le j\le r_i} \in \mathbb{C}^r \;\middle|\; \prod_{i,j} {\nu^i_j}^{\mu^i_j} = 1 \right\},$$

which is the set of eigenvalues of k-tuple of semisimple conjugacy classes $(\mathcal{C}_1, \ldots, \mathcal{C}_k)$. We denote by $\mathcal{U}^{\boldsymbol{\mu}}$ the following subvariety of $N_n^{\boldsymbol{\mu}} \times \mathrm{GL}(n, \mathbb{C})^{2g+n}$

$$\left\{ (\boldsymbol{\nu}, A_1, B_1, \ldots, A_g, B_g, X_1, \ldots, X_k) \;\middle|\; \begin{array}{l} (1)\ (A_1, B_1) \cdots (A_g, B_g) X_1 \cdots X_k = I_n, \\ (2)\ \text{For each } i, \text{ there is a filtration} \\ \quad\ \mathbb{C}^n = W_1^i \supset W_2^i \supset \cdots \supset W_{r_i+1}^i = 0 \\ \quad\ \text{such that } \dim W_j^i / W_{j+1}^i = \mu_j^i \\ \quad\ \text{and } (X_i - \nu_j^i \mathrm{id})(W_j^i) \subset W_{j+1}^i \text{ for any } i, j \end{array} \right\}$$

where $(\boldsymbol{\nu}, A_1, B_1, \ldots, A_g, B_g, X_1, \ldots, X_k) \in N_n^{\boldsymbol{\mu}} \times \mathrm{GL}(n, \mathbb{C})^{2g+n}$. The group $\mathrm{PGL}(n, \mathbb{C})$ acts on $N_n^{\boldsymbol{\mu}} \times \mathrm{GL}(n, \mathbb{C})^{2g+n}$ which is trivial on $N_n^{\boldsymbol{\mu}}$ and conjugation on $\mathrm{GL}(n, \mathbb{C})^{2g+n}$. We take the categorical quotient of $\mathcal{U}^{\boldsymbol{\mu}}$ by the $\mathrm{PGL}(n, \mathbb{C})$-action;

$$\begin{aligned} \mathcal{M}_B^{\boldsymbol{\mu}} &:= \mathcal{U}^{\boldsymbol{\mu}} /\!/ \mathrm{PGL}(n, \mathbb{C}) \\ &= \mathrm{Spec}(\mathbb{C}[\mathcal{U}^{\boldsymbol{\mu}}]^{\mathrm{PGL}(n, \mathbb{C})}). \end{aligned}$$

The map

$$\begin{aligned} \mathcal{M}_B^{\boldsymbol{\mu}} &\longrightarrow N_n^{\boldsymbol{\mu}} \\ [(\boldsymbol{\nu}, A_1, B_1, \ldots, A_g, B_g, X_1, \ldots, X_k)] &\longmapsto \boldsymbol{\nu} \end{aligned}$$

is well-defined. Let $N_n^{\boldsymbol{\mu}, irr} \subset N_n^{\boldsymbol{\mu}}$ be the set of generic eigenvalues in the sense of Definition 4.1.1. Then, we take the base change of $\mathcal{M}_B^{\boldsymbol{\mu}} \to N_n^{\boldsymbol{\mu}}$ via inclusion map $N_n^{\boldsymbol{\mu}, irr} \hookrightarrow N_n^{\boldsymbol{\mu}}$, denoted by

$$\mathcal{M}_B^{\boldsymbol{\mu}, irr} \longrightarrow N_n^{\boldsymbol{\mu}, irr},$$

which is a family of any generic character varieties of type $\boldsymbol{\mu}$. We denote the fiber of $\boldsymbol{\nu}$ by $\mathcal{M}_B^{\boldsymbol{\mu}}(\boldsymbol{\nu})$, which is a generic character variety of type $\boldsymbol{\mu}$.

We define the morphism

$$rh_d \colon \Xi_{n, \lambda=1}^{\boldsymbol{\mu}, d} \ni \boldsymbol{\xi} \longmapsto \boldsymbol{\nu} \in N_n^{\boldsymbol{\mu}}$$

by $\nu_j^i = \exp(-2\pi\sqrt{-1}\xi_j^i)$ for any i, j.

Remark 2.3.4 (see the proof of [**15**, Lemma 2.1.2]). Let $(\mathcal{C}_1, \ldots, \mathcal{C}_k)$ be a k-tuple of semisimple conjugacy classes such that the eigenvalue of any matrix in $\mathcal{C}_i$ is

$$(\exp(-2\pi\sqrt{-1}\xi_1^i), \ldots, \exp(-2\pi\sqrt{-1}\xi_{r_i}^i))$$

where the multiplicity of $\exp(-2\pi\sqrt{-1}\xi_j^i)$ is μ_j^i. If $\boldsymbol{\xi}$ is generic, then $(\mathcal{C}_1, \ldots, \mathcal{C}_k)$ is generic in the sense of Definition 4.1.1.

For each member $(E, \nabla, \{l_j^i\}) \in \mathcal{M}_{DR}^{\boldsymbol{\mu}, irr}$, $\mathrm{Ker}(\nabla^{an}|_{\Sigma_0})$ becomes a local system on Σ_0, where ∇^{an} means the analytic connection corresponding to ∇. The local system $\mathrm{Ker}(\nabla^{an}|_{\Sigma_0})$ corresponds to a representation of $\pi_1(\Sigma_0)$. Let γ_i be a loop around p_i. The representation of γ_i is semisimple for $i = 1, \ldots, k$, and the eigenvalues of the representation of γ_i are

$$\exp(-2\pi\sqrt{-1}\xi_1^i), \ldots, \exp(-2\pi\sqrt{-1}\xi_{r_i}^i)$$

where the multiplicities are $\mu_1^i, \ldots, \mu_{r_i}^i$, respectively. Then, we can define the morphism

$$\mathbf{RH}_{\boldsymbol{\xi}} \colon \mathcal{M}_{DR}^{\boldsymbol{\mu}}(\boldsymbol{\xi}) \longrightarrow \mathcal{M}_B^{\boldsymbol{\mu}}(\boldsymbol{\nu})$$

where $\boldsymbol{\nu} = rh_d(\boldsymbol{\xi})$. Then, $\{\mathbf{RH}_{\boldsymbol{\xi}}\}$ induces a morphism

$$\mathbf{RH}\colon \mathcal{M}_{DR}^{\boldsymbol{\mu},irr} \longrightarrow \mathcal{M}_{B}^{\boldsymbol{\mu},irr}, \tag{2.3.2}$$

which gives the commutative diagram

$$\begin{array}{ccc} \mathcal{M}_{DR}^{\boldsymbol{\mu},irr} & \xrightarrow{\mathbf{RH}} & \mathcal{M}_{B}^{\boldsymbol{\mu},irr} \\ \downarrow & & \downarrow \\ \Xi_{n,\lambda=1}^{\boldsymbol{\mu},d,irr} & \xrightarrow{rh_d} & N_n^{\boldsymbol{\mu},irr}. \end{array}$$

The following theorem follows from the result of Inaba [**21**, Theorem 2.2] and Inaba-Saito [**22**].

Theorem 2.3.5 ([**21**, Theorem 2.2] and [**22**]). *The morphism*

$$\mathbf{RH}_{\boldsymbol{\xi}}\colon \mathcal{M}_{DR}^{\boldsymbol{\mu}}(\boldsymbol{\xi}) \longrightarrow \mathcal{M}_{B}^{\boldsymbol{\mu}}(rh_d(\boldsymbol{\xi}))$$

is an analytic isomorphism for any $\boldsymbol{\xi} \in \Xi_{n,\lambda=1}^{\boldsymbol{\mu},d,irr}$.

Proof. We take any point $\rho \in \mathcal{M}_{B}^{\boldsymbol{\mu}}(rh_d(\boldsymbol{\xi}))$ where $\boldsymbol{\xi}$ is generic. By [**21**, Proposition 3.1], we obtain the following isomorphism,

$$\mathcal{M}_{DR}^{\boldsymbol{\mu}}(\boldsymbol{\xi}) \cong \mathcal{M}_{DR}^{\boldsymbol{\mu}}(\boldsymbol{\xi}')$$

where $0 \leq \mathrm{Re}(\xi_j'^i) < 1$ for any i, j. Hence, we assume that $\boldsymbol{\xi}$ satisfy $0 \leq \mathrm{Re}(\xi_j^i) < 1$ for any i, j. By Corollary 2.2.1 in Chapter 4, there is a unique pair (E, ∇_E) where E is a vector bundle on Σ and $\nabla_E\colon E \to E \otimes \Omega_{\Sigma}^1(D)$ is a logarithmic connection, such that the local system $\mathrm{Ker}(\nabla_E^{an})|_{\Sigma\setminus\{p_1,\ldots,p_k\}}$ corresponds to the representation ρ and all the eigenvalue of $\mathrm{Res}_{p_i}(\nabla_E)$ lie in $\{z \in \mathbb{C} \mid 0 \leq \mathrm{Re}(z) < 1\}$. Since $\boldsymbol{\xi}$ is generic, we can define a parabolic structure of (E, ∇_E), uniquely. Therefore $\mathbf{RH}_{\boldsymbol{\xi}}$ gives a one to one correspondence between the points of $\mathcal{M}_{DR}^{\boldsymbol{\mu},irr}(\boldsymbol{\xi})$ and the points of $\mathcal{M}_{B}^{\boldsymbol{\mu},irr}(rh_d(\boldsymbol{\xi}))$. We can define this correspondence between flat families. Hence,

$$\mathbf{RH}_{\boldsymbol{\xi}}\colon \mathcal{M}_{DR}^{\boldsymbol{\mu}}(\boldsymbol{\xi}) \longrightarrow \mathcal{M}_{B}^{\boldsymbol{\mu}}(rh_d(\boldsymbol{\xi}))$$

is an analytic isomorphism. □

Remark 2.3.6. For any generic $\boldsymbol{\nu}_1$ and $\boldsymbol{\nu}_2 \in N_n^{\boldsymbol{\mu},irr}$, there exist integers d_1 and d_2 with $0 \leq d_1, d_2 < \mathrm{g.c.d.}(\boldsymbol{\mu})$ such that $\boldsymbol{\nu}_1$ and $\boldsymbol{\nu}_2$ are contained in the images of the morphisms

$$rh_{d_1}\colon \Xi_{n,\lambda=1}^{\boldsymbol{\mu},d_1,irr} \longrightarrow N_n^{\boldsymbol{\mu},irr} \text{ and } rh_{d_2}\colon \Xi_{n,\lambda=1}^{\boldsymbol{\mu},d_2,irr} \longrightarrow N_n^{\boldsymbol{\mu},irr},$$

respectively, that is, $\boldsymbol{\nu}_1 \in \mathrm{Im}(rh_{d_1})$ and $\boldsymbol{\nu}_2 \in \mathrm{Im}(rh_{d_2})$.

CHAPTER 5

Mixed Hodge structures of the moduli spaces of parabolic connections

In this chapter, we prove Theorem A in the introduction.

1. Relative compactification of moduli spaces

1.1. Relative compactification. We consider the natural $\mathbb{C}^\times$-action on $\mathcal{M}^{\boldsymbol{\mu}}_{Hod}$

$$\begin{aligned}\mathbb{C}^\times \times \mathcal{M}^{\boldsymbol{\mu}}_{Hod} &\longrightarrow \mathcal{M}^{\boldsymbol{\mu}}_{Hod} \\ (t, (\lambda, E, \nabla, \{l^{(i)}_*\})) &\longmapsto (t\lambda, E, t\nabla, \{l^{(i)}_*\}).\end{aligned}$$

Since the relation between λ and $\boldsymbol{\xi}$ is $\lambda d + \sum \mu^i_j \xi^i_j = 0$, the following $\mathbb{C}^\times$ action on $\Xi^{\boldsymbol{\mu},d}_n$ is well-defined,

$$\begin{aligned}\mathbb{C}^\times \times \Xi^{\boldsymbol{\mu},d}_n &\longrightarrow \Xi^{\boldsymbol{\mu},d}_n \\ (t, (\lambda, \boldsymbol{\xi})) &\longmapsto (t\lambda, t\boldsymbol{\xi}).\end{aligned}$$

Clearly, $\pi \colon \mathcal{M}^{\boldsymbol{\mu}}_{Hod} \to \Xi^{\boldsymbol{\mu},d}_n$ is a $\mathbb{C}^\times$-equivariant morphism.

Lemma 1.1.1. *The fixed point set* $(\mathcal{M}^{\boldsymbol{\mu}}_{Hod})^{\mathbb{C}^\times}$ *is proper over* $\Xi^{\boldsymbol{\mu},d}_n$, *and for any* $(\lambda, E, \nabla, \{l^{(i)}_*\})$ *the limit* $\lim_{t\to 0} t \cdot (\lambda, E, \nabla, \{l^{(i)}_*\})$ *exists in* $(\mathcal{M}^{\boldsymbol{\mu}}_{Hod})^{\mathbb{C}^\times}$.

Proof. The fixed point set lies over the origin $(0, \mathbf{0}) \in \Xi^{\boldsymbol{\mu},d}_n$. Therefore, this fixed point is just the fixed point set of the moduli space of semistable parabolic Higgs bundles, which is a closed subvariety of the moduli space of parabolic Higgs sheaves. Then, the fixed point set is proper by [**54**, Theorem 5.12].

The second part follows from Langton's type theorem [**23**, Proposition 5.5] in the same way as in [**48**, Corollary 10.2]. (also see [**50**, Lemma 4.1 and Section 6] and [**32**, Proposition 4.1]) □

We construct a relative compactification of $\mathcal{M}^{\boldsymbol{\mu}}_{Hod}$ over $\Xi^{\boldsymbol{\mu},d}_n$. Let $\mathbb{C}^\times$ act on $\mathbb{C} \times \Xi^{\boldsymbol{\mu},d}_n$ by $t \cdot (x, (\lambda, \boldsymbol{\xi})) = (tx, (\lambda, \boldsymbol{\xi}))$. Then, $\mathbb{C} \times \Xi^{\boldsymbol{\mu},d}_n \to \Xi^{\boldsymbol{\mu},d}_n$ given by $(x, (\lambda, \boldsymbol{\xi})) \mapsto (x\lambda, x\boldsymbol{\xi})$ is $\mathbb{C}^\times$-equivariant with the standard action on $\mathbb{C}$. Let $\mathcal{M}'$ denote the base change of $\mathcal{M}^{\boldsymbol{\mu}}_{Hod}$ via this map; in other wards,

$$\mathcal{M}' = \left\{ ((\lambda, E, \nabla, \{l^{(i)}_*\}), x, (\lambda', \boldsymbol{\xi})) \;\middle|\; \begin{array}{l} (\lambda, E, \nabla, \{l^{(i)}_*\}) \in \mathcal{M}^{\boldsymbol{\mu}}_{Hod}, \\ (x, (\lambda', \boldsymbol{\xi})) \in \mathbb{C} \times \Xi^{\boldsymbol{\mu},d}_n \text{ and} \\ \pi((\lambda, E, \nabla, \{l^{(i)}_*\})) = (x\lambda', x\boldsymbol{\xi}) \end{array} \right\}.$$

Then, $\mathcal{M}'$ inherits the $\mathbb{C}^\times$-action given by

$$t \cdot ((\lambda, E, \nabla, \{l_*^{(i)}\}), x, (\lambda', \boldsymbol{\xi})) = ((t\lambda, E, t\nabla, \{l_*^{(i)}\}), tx, (\lambda', \boldsymbol{\xi})),$$

and π induces the map $\pi' \colon \mathcal{M}' \to \Xi_n^{\boldsymbol{\mu},d}$ by

$$\pi'((\lambda, E, \nabla, \{l_*^{(i)}\}), x, (\lambda', \boldsymbol{\xi})) = (\lambda', \boldsymbol{\xi}),$$

which is equivarinat with respect to the trivial action on the base. By [**48**, Theorem 11.2], the set $U \subset \mathcal{M}'$ of points $u \in U$ such that $\lim_{t\to\infty} t \cdot (\lambda, E, \nabla, \{l_*^{(i)}\})$ does not exist is open, and there exists a geometric quotient $\overline{\mathcal{M}} := U//\mathbb{C}^\times$, which is proper over $\Xi_n^{\boldsymbol{\mu},d}$ via the induce map $\bar{\pi} \colon \overline{\mathcal{M}} \to \Xi_n^{\boldsymbol{\mu},d}$. Indeed, it is a relative compactification of $\mathcal{M}_{Hod}^{\boldsymbol{\mu}}$ over $\Xi_n^{\boldsymbol{\mu},d}$ by the embedding

$$(\lambda, E, \nabla, \{l_*^{(i)}\}) \longmapsto \mathbb{C}^\times \cdot ((\lambda, E, \nabla, \{l_*^{(i)}\}), 1, (\lambda', \boldsymbol{\xi})).$$

1.2. Main results.

The following theorems are our main Theorem A.

Theorem 1.2.1. *There are isomorphisms between rational cohomology groups with compact support of fibers of* $\pi \colon \mathcal{M}_{Hod}^{\boldsymbol{\mu}} \to \Xi_n^{\boldsymbol{\mu},d}$ *which preserve the mixed Hodge structures. The mixed Hodge structures on these cohomology groups of the fibers are pure.*

Proof. Let us show that for any non-empty fiber $\mathcal{M}_{Hod}^{\boldsymbol{\mu}}(\lambda, \boldsymbol{\xi})$ of $(\lambda, \boldsymbol{\xi}) \in \Xi_n^{\boldsymbol{\mu},d}$ via π, there exists an isomorphism

$$H_c^\bullet(\mathcal{M}_{Hod}^{\boldsymbol{\mu}}(\lambda, \boldsymbol{\xi}), \mathbb{Q}) \cong H_c^\bullet(\mathcal{M}_{Dol}^{\boldsymbol{\mu}}(\mathbf{0}), \mathbb{Q}),$$

and this isomorphism preserves the mixed Hodge structures. First, for the pair $(\lambda, \boldsymbol{\xi}) \in \Xi_n^{\boldsymbol{\mu},d}$, we consider the following subset of $\Xi_n^{\boldsymbol{\mu},d}$

$$\Xi_{(\lambda,\boldsymbol{\xi})} := \{(t\lambda, t\boldsymbol{\xi}) \mid t \in \mathbb{C}\} \cong \mathbb{C}.$$

Let

$$\bar{\pi}_{(\lambda,\boldsymbol{\xi})} \colon \overline{\mathcal{M}}_{\Xi_{(\lambda,\boldsymbol{\xi})}} \longrightarrow \Xi_{(\lambda,\boldsymbol{\xi})}$$

be the base change of $\overline{\mathcal{M}}$ via $\Xi_{(\lambda,\boldsymbol{\xi})} \hookrightarrow \Xi_n^{\boldsymbol{\mu},d}$. Let $\overline{\mathcal{M}}(t\lambda, t\boldsymbol{\xi})$ be the fiber of $(t\lambda, t\boldsymbol{\xi})$ via $\bar{\pi}_{(\lambda,\boldsymbol{\xi})}$. The map $\bar{\pi}_{(\lambda,\boldsymbol{\xi})}$ is a proper surjective morphism. Moreover, $\bar{\pi}_{(\lambda,\boldsymbol{\xi})}$ is topologically trivial (see [**15**, Theorem B.1] or [**18**, Lemma 6.1]). Then, the map $H^*(\overline{\mathcal{M}}_{\Xi_{(\lambda,\boldsymbol{\xi})}}) \to H^*(\overline{\mathcal{M}}(t\lambda, t\boldsymbol{\xi}))$ is an isomorphism. On the other hand, the boundary

$$Z := \bar{\mathcal{M}} \setminus \mathcal{M} = \left\{ \mathbb{C}^\times((\lambda, E, \nabla, \{l_*^{(i)}\}), 0, (\lambda', \boldsymbol{\xi})) \;\middle|\; \lim_{t\to\infty} t \cdot (\lambda, E, \nabla, \{l_*^{(i)}\}) \text{ exists} \right\}$$

is trivial over $\Xi_n^{\boldsymbol{\mu},d}$. Let $\bar{\pi}_{(\lambda,\boldsymbol{\xi})}|_Z \colon Z_{\Xi_{(\lambda,\boldsymbol{\xi})}} \to \Xi_{(\lambda,\boldsymbol{\xi})}$ be the restriction. Then, we have $H^*(Z_{\Xi_{(\lambda,\boldsymbol{\xi})}}) \cong H^*(Z(t\lambda, t\boldsymbol{\xi}))$. Here, $Z(t\lambda, t\boldsymbol{\xi})$ is the fiber of $(t\lambda, t\boldsymbol{\xi})$ via $\bar{\pi}_{(\lambda,\boldsymbol{\xi})}|_Z$. Applying the five Lemma to the long exact sequences of the pairs $(\overline{\mathcal{M}}_{\Xi_{(\lambda,\boldsymbol{\xi})}}, Z_{\Xi_{(\lambda,\boldsymbol{\xi})}})$ and $(\overline{\mathcal{M}}(t\lambda, t\boldsymbol{\xi}), Z(t\lambda, t\boldsymbol{\xi}))$, we obtain the isomorphism

$$H^\bullet(\overline{\mathcal{M}}_{\Xi_{(\lambda,\boldsymbol{\xi})}}, Z_{\Xi_{(\lambda,\boldsymbol{\xi})}}) \cong H^\bullet(\overline{\mathcal{M}}(t\lambda, t\boldsymbol{\xi}), Z(t\lambda, t\boldsymbol{\xi})) \cong H_c^\bullet(\mathcal{M}_{Hod}^{\boldsymbol{\mu}}(t\lambda, t\boldsymbol{\xi}))$$

for any $t \in \mathbb{C}$. Thus,

$$H_c^\bullet(\mathcal{M}_{Hod}^{\boldsymbol{\mu}}(\lambda, \boldsymbol{\xi})) \cong H_c^\bullet(\mathcal{M}_{Dol}^{\boldsymbol{\mu}}(\mathbf{0})),$$

and this isomorphism preserves the mixed Hodge structures.

Next, we show that $H^\bullet(\mathcal{M}_{Dol}^{\boldsymbol{\mu}}(\mathbf{0}))$ has the pure mixed Hodge structure. We may show that the mixed Hodge structure of $H^\bullet(\overline{\mathcal{M}}_{Dol}^{\boldsymbol{\mu}}(\mathbf{0}))$ is pure and the restriction map

$H^\bullet(\overline{\mathcal{M}}^{\boldsymbol{\mu}}_{Dol}(\mathbf{0})) \to H^\bullet(\mathcal{M}^{\boldsymbol{\mu}}_{Dol}(\mathbf{0}))$ is surjective in the same way as in [**15**, Theorem B.1]. Here, $\overline{\mathcal{M}}^{\boldsymbol{\mu}}_{Dol}(\mathbf{0})$ is the fiber of $(0,\mathbf{0})$ via $\bar{\pi}_{(\lambda,\boldsymbol{\xi})}$. Thus, for any fiber of π, the mixed Hodge structure of the cohomology group of the fiber is also pure. □

Theorem 1.2.2. *With the notation of the proof of Theorem* 1.2.1, *we put* $\mathcal{M}_{\Xi_{(\lambda,\boldsymbol{\xi})}} := \overline{\mathcal{M}}_{\Xi_{(\lambda,\boldsymbol{\xi})}} \setminus Z_{\Xi_{(\lambda,\boldsymbol{\xi})}}$. *The restriction map of the ordinary rational cohomology groups*

$$H^\bullet(\mathcal{M}_{\Xi_{(\lambda,\boldsymbol{\xi})}}) \longrightarrow H^\bullet(\mathcal{M}^{\boldsymbol{\mu}}_{Hod}(\lambda,\boldsymbol{\xi}))$$

is an isomorphism. In particular, the ordinary rational cohomology groups of the fibers of $\pi\colon \mathcal{M}^{\boldsymbol{\mu}}_{Hod} \to \Xi^{\boldsymbol{\mu},d}_n$ *are isomorphic.*

Proof. By the Gysin map and the Poincaré duality, we have

$$\begin{array}{ccc} H^\bullet_c(\mathcal{M}^{\boldsymbol{\mu}}_{Hod}(\lambda,\boldsymbol{\xi}),\mathbb{Q}) & \longrightarrow & H^{\bullet+2}_c(\mathcal{M}_{\Xi_{(\lambda,\boldsymbol{\xi})}},\mathbb{Q}) \\ \downarrow \cong & & \downarrow \cong \\ H_{2d_{\boldsymbol{\mu}}-\bullet}(\mathcal{M}^{\boldsymbol{\mu}}_{Hod}(\lambda,\boldsymbol{\xi}),\mathbb{Q}) & \longrightarrow & H_{2d_{\boldsymbol{\mu}}-\bullet}(\mathcal{M}_{\Xi_{(\lambda,\boldsymbol{\xi})}},\mathbb{Q}) \end{array}$$

where $d_{\boldsymbol{\mu}} := \dim \mathcal{M}^{\boldsymbol{\mu}}_{Hod}(\lambda,\boldsymbol{\xi})$. The top map is an isomorphism, since the map is given by the composition

$$H^{\bullet+2}_c(\mathcal{M}_{\Xi_{(\lambda,\boldsymbol{\xi})}},\mathbb{Q}) \cong H^{\bullet+2}_c(\mathcal{M}^{\boldsymbol{\mu}}_{Hod}(\lambda,\boldsymbol{\xi}) \times \Xi_{(\lambda,\boldsymbol{\xi})},\mathbb{Q}) \cong H^\bullet_c(\mathcal{M}^{\boldsymbol{\mu}}_{Hod}(\lambda,\boldsymbol{\xi}),\mathbb{Q}).$$

Then, the bottom map $H_\bullet(\mathcal{M}^{\boldsymbol{\mu}}_{Hod}(\lambda,\boldsymbol{\xi}),\mathbb{Q}) \to H_\bullet(\mathcal{M}_{\Xi_{(\lambda,\boldsymbol{\xi})}},\mathbb{Q})$ is an isomorphism. We take the dual of the map. Then, the corollary follows. □

By the Riemann-Hilbert correspondence, we obtain the following corollary:

Corollary 1.2.3. *For* $\boldsymbol{\mu} = (\mu^i_j)$ *such that* $\text{g.c.d.}(\mu^i_j) = 1$, *the Poincaré polynomials of character varieties* $\mathcal{M}^{\boldsymbol{\mu}}_B(\boldsymbol{\nu})$ *are independent of the choice of generic eigenvalues.*

Proof. It follows from Theorem 1.2.2 and Theorem 2.3.5 of Chapter 4. □

2. An example: The moduli space of Type ((11), (11), (11), (11))

Inaba-Iwasaki-Saito [**24**] proved that the moduli space of $\boldsymbol{\alpha}$-stable parabolic connections of rank 2 degree -1 over $\mathbb{P}^1$ with type $\boldsymbol{\mu} = ((11),(11),(11),(11))$ is isomorphic to an open part of Okamoto-Painlevé pair of type $D_4^{(1)}$ ([**40**]) or generalized Halphen surface of type $D_4^{(1)}$ ([**41**]).

This was done by an explicit description of the moduli space as follows.

Let p_1,p_2,p_3,p_4 be distinct points on $\mathbb{P}^1$ and we normalize them as $(p_1,p_2,p_3,p_4) = (0,1,t,\infty)$. We denote by $\mathcal{M}^{\boldsymbol{\mu}}_{DR}(\boldsymbol{\xi},t)$ the fine moduli space of the $\boldsymbol{\alpha}$-stable parabolic connections (E,∇,l_*) of rank 2 and degree -1 with 4-regular singular points $(0,1,t,\infty)$ with the data $\boldsymbol{\xi}$ of eigenvalues of the residues of ∇ at p_i. We can normalize them as $\boldsymbol{\xi} = (\pm\xi_1,\pm\xi_2,\pm\xi_3,\xi_4,1-\xi_4)$.

Let $\pi : \mathbb{F}_2 = \mathbb{P}(\mathcal{O}\oplus\mathcal{O}(-2)) \to \mathbb{P}^1$ be the ruled surface of degree 2 and let C be a unique section whose self-intersection number is -2. Let us take the fibers $F_i = \pi^{-1}(p_i)$,

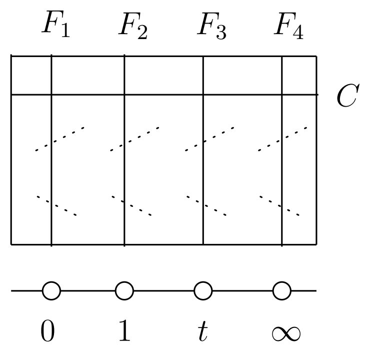


FIGURE 1

then we can take two distinct points on F_i determined by the eigenvalue ξ_i. Blowing up these two points on each fiber F_i, then we obtain the 8-times blowing up $h : S_{\boldsymbol{\xi},t} \longrightarrow \mathbb{F}_2$. Let us denote by $F_i'$the proper transform of F_i by the blowing up h. Then it is easy to see that the anti-canonical divisor $-K_S$ of the projective surface $S = S_{\boldsymbol{\xi},t}$ is given by $-K_S = Y = 2C + F_1' + F_2' + F_3' + F_4'$ such that (S, Y) satisfy the condition of Okamoto-Painlevé pair, that is, $-K_S \cdot C = -K_S \cdot F_i' = 0$. Note that the configuration of $-K_S = Y$ is a tree of smooth rational curves of type $D_4^{(1)}$.

Inaba-Iwasaki-Saito [**24**] showed that there exists an isomorphism

$$\mathcal{M}_{DR}^{\boldsymbol{\mu}}(\boldsymbol{\xi}, t) \simeq S_{\boldsymbol{\xi},t} \setminus Y_{red}. \tag{2.0.1}$$

Moreover a natural compactification $\overline{\mathcal{M}_{DR}^{\boldsymbol{\mu}}(\boldsymbol{\xi}, t)}$ of $\mathcal{M}_{DR}^{\boldsymbol{\mu}}(\boldsymbol{\xi}, t)$ is given by the coarse moduli space of stable parabolic ϕ-connections and moreover we have an isomorphism

$$\overline{\mathcal{M}_{DR}^{\boldsymbol{\mu}}(\boldsymbol{\xi}, t)} \simeq S_{\boldsymbol{\xi},t} \tag{2.0.2}$$

The second homology group of $\mathcal{M}_{DR}^{\boldsymbol{\mu}}(\boldsymbol{\xi}, t) \simeq S_{\boldsymbol{\xi},t} \setminus Y_{red}$ fits into the exact sequence of mixed Hodge structure ([**41**, Section 5])

$$0 \longrightarrow H^1(Y_{red}, \mathbb{Q}) \longrightarrow H^2(S_{\boldsymbol{\xi},t} \setminus Y_{red}, \mathbb{Q}) \longrightarrow H_2(S_{\boldsymbol{\xi},t}, \mathbb{Q}) \longrightarrow H^2(Y_{red}, \mathbb{Q}) \longrightarrow$$

It is easy to see that $H^1(Y_{red}, \mathbb{Q}) = 0$ and $H_2(S_{\boldsymbol{\xi},t}, \mathbb{Q}) \simeq \mathbb{Q}^{10}$, so hence

$$H^2(S_{\boldsymbol{\xi},t} \setminus Y_{red}, \mathbb{Q}) \simeq \ker\left(H_2'(S_{\boldsymbol{\xi},t}, \mathbb{Q}) \longrightarrow H^2(Y_{red}, \mathbb{Q})\right) \simeq \mathbb{Q}^5.$$

This isomorphism shows that the mixed Hodge structure on $H^2(S_{\boldsymbol{\xi},t} \setminus Y_{red}, \mathbb{Q})$ is pure of weight 2 and of type $(1, 1)$ (Theorem A). On the other hand, the complex deformations of Okamoto-Painlevé pair $(S_{\boldsymbol{\xi},t}, Y)$ have 5 parameters which correspond to $\boldsymbol{\xi}$ and t. Moreover the mixed Hodge structure does not depend on the parameters $\boldsymbol{\xi}$ and t. Note that this t is the time variable of Painlevé equaions of type VI.

CHAPTER 6

Mixed Hodge structures of character varieties

In this chapter, we compute the mixed Hodge structures of certain character varieties. It is done by explicit descriptions of the character varieties by the classical invariant theory [**10**], [**25**], [**26**], [**30**]. Next, we construct compactifications of character varieties and investigate the configurations of their boundaries. As the result, we obtain Theorem B in the introduction.

1. Explicit descriptions of certain character varieties

Let $(\mathcal{C}_1, \dots, \mathcal{C}_k)$ be a k-tuple of generic semisimple conjugacy classes of $\mathrm{SL}(n, \mathbb{C})$ of type $\boldsymbol{\mu}$. We put

$$\begin{aligned}\hat{\mathcal{U}}^{\boldsymbol{\mu}}(\boldsymbol{\nu}) :=& \{(A_1, B_1, \dots, A_g, B_g; X_1, \dots, X_k) \in \mathrm{SL}(n, \mathbb{C})^{2g} \times \mathcal{C}_1 \times \cdots \times \mathcal{C}_k \\ & \mid (A_1, B_1) \cdots (A_g, B_g) X_1 \cdots X_k = I_n\},\end{aligned}$$

where $(A, B) := ABA^{-1}B^{-1}$. The group $\mathrm{SL}(n, \mathbb{C})$ acts by conjugation on $\mathrm{SL}(n, \mathbb{C})^{2g+n}$. The action induces that of $\mathrm{SL}(n, \mathbb{C})$ on $\hat{\mathcal{U}}^{\boldsymbol{\mu}}(\boldsymbol{\nu})$. We call the affine GIT quotient

$$\hat{\mathcal{M}}_B^{\boldsymbol{\mu}}(\boldsymbol{\nu}) := \hat{\mathcal{U}}^{\boldsymbol{\mu}}(\boldsymbol{\nu}) // \mathrm{SL}(n, \mathbb{C})$$

a *generic* $\mathrm{SL}(n, \mathbb{C})$*-character variety of type* $\boldsymbol{\mu}$. Note that for $g = 0$, $\mathrm{GL}(n, \mathbb{C})$-character varieties and $\mathrm{SL}(n, \mathbb{C})$-character varieties are same. We consider $\mathrm{SL}(n, \mathbb{C})$-character varieties of $g = 0$, since the description of the invariant rings is simple.

In the following, we only consider the following types

$$\boldsymbol{\mu} = ((1, \dots, 1), \dots, (1, \dots, 1)).$$

We denote by R_{k-1}^n be the affine coordinate ring of $\mathrm{SL}(n, \mathbb{C})^{k-1}$ on which $\mathrm{SL}(n, \mathbb{C})$ acts by simultaneous adjoint action. Let $(R_{k-1}^n)^{\mathrm{SL}(n,\mathbb{C})}$ be the invariant subring of R_{k-1}^n. Moreover let us fix a data $\boldsymbol{\nu}$ of eigenvalues in $(\mathcal{C}_1, \dots, \mathcal{C}_k)$ of type $\boldsymbol{\mu} = ((1, \dots, 1), \dots, (1, \dots, 1))$. Let $(R_{k-1}^n)_{\boldsymbol{\nu}}^{\mathrm{SL}(n,\mathbb{C})}$ denote the affine coordinate ring of $\hat{\mathcal{M}}_B^{\boldsymbol{\mu}}(\boldsymbol{\nu})$, i.e.,

$$\hat{\mathcal{M}}_B^{\boldsymbol{\mu}}(\boldsymbol{\nu}) = \mathrm{Spec}(R_{k-1}^n)_{\boldsymbol{\nu}}^{\mathrm{SL}(n,\mathbb{C})}.$$

Note that we have the natural surjective homomorphism $(R_{k-1}^n)^{\mathrm{SL}(n,\mathbb{C})} \to (R_{k-1}^n)_{\boldsymbol{\nu}}^{\mathrm{SL}(n,\mathbb{C})}$.

1.1. Classical invariant theory. Now, we recall the explicit description of the invariant ring $(R_{k-1}^n)^{\mathrm{SL}(n,\mathbb{C})}$ for the two cases $n = 2, k = 4$ and $n = 3, k = 3$. The following proposition follows from the fundamental theorem for matrix invariants. (See [**9**] or [**38**]).

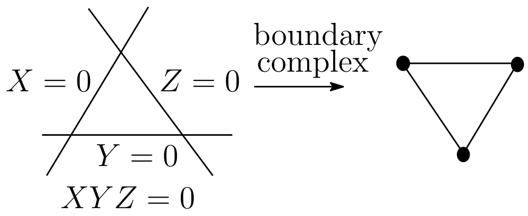


FIGURE 1

Proposition 1.1.1.

$$(R^n_{k-1})^{\mathrm{SL}(n,\mathbb{C})} = \mathbb{C}[\mathrm{Tr}(M_{i_1}M_{i_2}\cdots M_{i_k}) \mid 1 \le i_1, \ldots, i_k \le n-1].$$

In particular, for $n = 2$, *the elements* $\mathrm{Tr}(M_{i_1}M_{i_2}\cdots M_{i_k})$ *of degree* $k \le 3$ *generate the invariant ring, that is,*

$$(R^2_{k-1})^{\mathrm{SL}(2,\mathbb{C})} = \mathbb{C}[\mathrm{Tr}(M_i), \mathrm{Tr}(M_iM_j), \mathrm{Tr}(M_iM_jM_k) \mid 1 \le i,j,k \le n-1].$$

First, we consider the case where $n = 2, k = 4$, namely $\boldsymbol{\mu} = ((1,1),(1,1),(1,1),(1,1))$. We denote by (i,j,k) a cyclic permutation of $(1,2,3)$. Put

(1.1.1) $\quad x_i := \mathrm{Tr}(M_kM_j)\ (i = 1,2,3), a_i := \mathrm{Tr}M_i\ (i = 1,2,3), a_4 := \mathrm{Tr}(M_3M_2M_1).$

The following proposition is due to Frike-Klein, Iwasaki, and Jimbo ([**10**], [**25**], [**26**]).

Proposition 1.1.2. *The invariant ring* $(R^2_3)^{\mathrm{SL}(2,C)}$ *is generated by seven elements* $x_1, x_2, x_3, a_1, a_2, a_3, a_4$ *and there exists a relation*

$$f_{\boldsymbol{a}}(x) := x_1x_2x_3 + x_1^2 + x_2^2 + x_3^2 - \theta_1(\boldsymbol{a})x_1 - \theta_2(\boldsymbol{a})x_2 - \theta_3(\boldsymbol{a})x_3 + \theta_4(\boldsymbol{a}) = 0,$$

where

$$\theta_i(\boldsymbol{a}) = a_ia_4 - a_ja_k \quad (i,j,k),$$
$$\theta_4(\boldsymbol{a}) = a_1a_2a_3a_4 + a_1^2 + a_2^2 + a_3^2 + a_4^2 - 4.$$

Therefore we have an isomorphism

$$(R^2_3)^{\mathrm{SL}(2,\mathbb{C})} \cong \mathbb{C}[x_1, x_2, x_3, a_1, a_2, a_3, a_4]/(f_{\boldsymbol{a}}(x)).$$

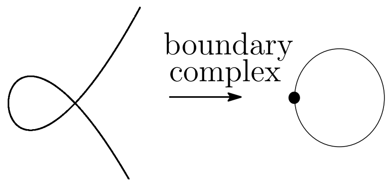
Hence, the $\mathrm{SL}(2,\mathbb{C})$*-character variety* $\hat{\mathcal{M}}^{\boldsymbol{\mu}}_B(\boldsymbol{\nu})$ *of type* $\boldsymbol{\mu} = ((1,1),(1,1),(1,1),(1,1))$ *is an affine cubic hypersurface on* $\mathbb{C}^3$. *Here,* $\nu^i_1 + \nu^i_2 = a_i$ *for* $i = 1,\ldots,4$. *The affine cubic hypersurface is called a Fricke-Klein cubic surface.*

We consider the natural compactification $\mathbb{C}^3 \hookrightarrow \mathbb{P}^3$ as follow, setting $x_1 = X/W, x_2 = Y/W, x_3 = Z/W$, then we obtain the following homogeneous polynomial,

$$XYZ + X^2W + Y^2W + Z^2W - \theta_1(a)XW^2 - \theta_2(a)YW^2 - \theta_3(a)ZW^2 + \theta_4(a)W^3 = 0.$$

Substitute $W = 0$ to this equation, then we obtain the equation $XYZ = 0$. Hence, the boundary components consist of three lines. The dual graph is the following. The dual graph is a simplicial decomposition of S^1.

Next, we consider the case where $n = 3, k = 3$, namely $\boldsymbol{\mu} = ((1,1,1),(1,1,1),(1,1,1))$. We describe generators and defining relations for the invariant ring $(R^3_2)^{\mathrm{SL}(3,\mathbb{C})}$. The following proposition is due to Lawton [**30**].



$X^3 + Y^3 - XYZ = 0$

FIGURE 2

Proposition 1.1.3. *The invariant ring* $(R_2^3)^{\mathrm{SL}(3,\mathbb{C})}$ *is generated by*

$$\begin{aligned} a_1 &:= \mathrm{Tr}(M_1) & a_2 &:= \mathrm{Tr}(M_1^{-1}) \\ b_1 &:= \mathrm{Tr}(M_2) & b_2 &:= \mathrm{Tr}(M_2^{-1}) \\ c_1 &:= \mathrm{Tr}(M_1^{-1}M_2^{-1}) = \mathrm{Tr}(M_3) & c_2 &:= \mathrm{Tr}(M_1M_2) = \mathrm{Tr}(M_3^{-1}) \\ x_1 &:= \mathrm{Tr}(M_1M_2^{-1}) & x_2 &:= \mathrm{Tr}(M_1^{-1}M_2) \\ x_3 &:= \mathrm{Tr}(M_1M_2M_1^{-1}M_2^{-1}). \end{aligned}$$

and there exist a relation

$$x_3^2 - fx_3 + g = 0.$$

where f, g *are polynomials of* x_1, x_2 *over* $\mathbb{C}[a_1, a_2, b_1, b_2, c_1, c_2]$*, more precisely,*

$$f = x_1x_2 - a_2b_1x_1 - a_1b_2x_2 + (\textit{constant terms in } x_1, x_2)$$
$$g = x_1^3 + x_2^3 + (\textit{terms that order is at most 2 in } x_1, x_2).$$

Hence, the $\mathrm{SL}(3,\mathbb{C})$*-character variety* $\hat{\mathcal{M}}_B^{\boldsymbol{\mu}}(\boldsymbol{\nu})$ *of type* $\boldsymbol{\mu} = ((1,1,1),(1,1,1),(1,1,1))$ *is an affine cubic hypersurface on* $\mathbb{C}^3$*. Here,*

$$\begin{aligned} \nu_1^1 + \nu_2^1 + \nu_i^1 = a_1, &\quad \frac{1}{\nu_1^1} + \frac{1}{\nu_2^1} + \frac{1}{\nu_i^1} = a_1, \\ \nu_1^2 + \nu_2^2 + \nu_i^2 = b_1, &\quad \frac{1}{\nu_1^2} + \frac{1}{\nu_2^2} + \frac{1}{\nu_i^2} = b_1, \\ \nu_1^3 + \nu_2^3 + \nu_i^3 = c_1, &\quad \frac{1}{\nu_1^3} + \frac{1}{\nu_2^3} + \frac{1}{\nu_i^3} = c_1. \end{aligned}$$

We consider the compactification $\mathbb{C}^3 \hookrightarrow \mathbb{P}^3$ as follow, setting $x_1 = X/W, x_2 = Y/W, x_3 = Z/W$, then, we obtain the following homogeneous polynomial

$$X^3 + Y^3 - XYZ + (\text{term containing } W) = 0.$$

We substitute $W = 0$ to this equation. Then we obtain the equation $X^3+Y^3-XYZ = 0$. This equation define a plane cubic curve having a node.

Corollary 1.1.4. The dual graph of the boundary divisor of $\hat{\mathcal{M}}_B^{\boldsymbol{\mu}}(\boldsymbol{\nu})$ is a simplicial decomposition of S^1. The dual graph is the following.

1.2. Mixed Hodge structure of the two examples. First, we consider the case where $\boldsymbol{\mu} = ((11),(11),(11),(11))$. The character variety $\hat{\mathcal{M}}_B^{\boldsymbol{\mu}}(\boldsymbol{\nu})$ is a nonsingular affine cubic surface in $\mathbb{C}^3$. Let $\overline{\hat{\mathcal{M}}_B^{\boldsymbol{\mu}}(\boldsymbol{\nu})}$ be a natural compactification of $\hat{\mathcal{M}}_B^{\boldsymbol{\mu}}(\boldsymbol{\nu})$ in $\mathbb{P}^3$, which is a blowing up 6-points in $\mathbb{P}^2$. Put $D := \overline{\hat{\mathcal{M}}^{\boldsymbol{\mu}}_B(\boldsymbol{\nu})} \setminus \hat{\mathcal{M}}^{\boldsymbol{\mu}}_B(\boldsymbol{\nu})$, which is triangle of $\mathbb{P}^1$. From $\overline{\hat{\mathcal{M}}^{\boldsymbol{\mu}}_B(\boldsymbol{\nu})} \hookrightarrow \hat{\mathcal{M}}^{\boldsymbol{\mu}}_B(\boldsymbol{\nu}) \hookleftarrow D$, we obtain the following spectral sequence:

$$\begin{aligned} & H^1(\overline{\mathcal{M}_B}) \to H^1(\mathcal{M}_B) \\ \to H^2(\overline{\mathcal{M}_B}, \mathcal{M}_B) & \xrightarrow{\varphi} H^2(\overline{\mathcal{M}_B}) \to H^2(\mathcal{M}_B) \\ \to H^3(\overline{\mathcal{M}_B}, \mathcal{M}_B) & \to H^3(\overline{\mathcal{M}_B}) = 0. \end{aligned}$$

Then we have the following extension

$$0 \to \operatorname{Coker}\varphi \to H^2(\mathcal{M}_B) \to H^3(\overline{\mathcal{M}_B}, \mathcal{M}_B) \to 0.$$

By the Poincaré duality

$$H^k(\overline{\mathcal{M}_B}, \mathcal{M}_B) \otimes H^{4-k}(D) \to \mathbb{C}(-4),$$

the rank of $\operatorname{Coker}\varphi$ is 4 and the weight of elements of $\operatorname{Coker}\varphi$ are type $(1,1)$, and the rank of $H^3(\overline{\mathcal{M}_B}, \mathcal{M}_B)$ is 1 and the weight of elements of $H^3(\overline{\mathcal{M}_B}, \mathcal{M}_B)$ are type $(2,2)$. The weight filtration is as follows

$$0 \subset W_2 = W_3 \subset W_4 = H^2(\hat{\mathcal{M}}^{\boldsymbol{\mu}}_B(\boldsymbol{\nu}))$$

where $W_2 = \operatorname{Coker}\varphi$.

Second, we consider the case where $\boldsymbol{\mu} = ((111), (111), (111))$. The character variety $\hat{\mathcal{M}}^{\boldsymbol{\mu}}_B(\boldsymbol{\nu})$ is a nonsingular affine cubic surface in $\mathbb{C}^3$. Let $\overline{\hat{\mathcal{M}}^{\boldsymbol{\mu}}_B(\boldsymbol{\nu})}$ be a natural compactification of $\hat{\mathcal{M}}^{\boldsymbol{\mu}}_B(\boldsymbol{\nu})$ in $\mathbb{P}^3$, which is a blowing up 6-points in $\mathbb{P}^2$. Put $D := \overline{\hat{\mathcal{M}}^{\boldsymbol{\mu}}_B(\boldsymbol{\nu})} \setminus \hat{\mathcal{M}}^{\boldsymbol{\mu}}_B(\boldsymbol{\nu})$, which is nodal $\mathbb{P}^1$. By the same way as above, we obtain the following weight filtration:

$$0 \subset W_2 = W_3 \subset W_4 = H^2(\hat{\mathcal{M}}^{\boldsymbol{\mu}}_B(\boldsymbol{\nu}))$$

where the rank of W_2 is 6, elements of W_2 are type $(1,1)$, and elements of W_4/W_3 are type $(2,2)$.

2. Certain compactifications of character varieties and their boundaries

2.1. Construction of a compactification of the character varieties. We construct a compactification of the $\mathrm{SL}(2,\mathbb{C})$-character variety $\hat{\mathcal{M}}^{\boldsymbol{\mu}}_B(\boldsymbol{\nu})$ of type $\boldsymbol{\mu} = ((1,1), \ldots, (1,1))$. For $\boldsymbol{\nu} = (\nu^i_j)^{1 \le i \le k}_{j=1,2}$, we put

$$\kappa_i := \nu^i_1 + \nu^i_2 \quad \text{for } i = 1, \ldots, k$$

and $\boldsymbol{\kappa} := (\kappa_1, \ldots, \kappa_k)$. We may denote by $\hat{\mathcal{M}}^{\boldsymbol{\mu}}_B(\boldsymbol{\kappa})$ the $\mathrm{SL}(2,\mathbb{C})$-character variety of type $\boldsymbol{\mu} = ((1,1), \ldots, (1,1))$ instead of $\hat{\mathcal{M}}^{\boldsymbol{\mu}}_B(\boldsymbol{\nu})$.

The construction of the compactification is done by the geometric invariant theory for a compactification of the following variety

Definition 2.1.1. We put

$$\begin{aligned} \hat{\mathcal{U}}^{\boldsymbol{\mu}}(\boldsymbol{\kappa}) &:= \{(M_1, \ldots, M_{k-1}) \in \mathcal{C}_1 \times \cdots \times \mathcal{C}_{k-1} | M^{-1}_{k-1} \cdots M^{-1}_1 \in \mathcal{C}_k\} \\ &= \{(M_1, \ldots, M_{k-1}) \in \mathcal{C}_1 \times \cdots \times \mathcal{C}_{k-1} | \operatorname{Tr}(M^{-1}_{k-1} \cdots M^{-1}_1) = \kappa_k\} \end{aligned} \tag{2.1.1}$$

where $\mathcal{C}_i = \{M \in \mathrm{SL}(2,\mathbb{C}) \mid \mathrm{Tr}(M) = \kappa_i\}$ and $\boldsymbol{\kappa} = (\kappa_1, \ldots, \kappa_k) \in \mathbb{C}^k$. The affine variety $\mathrm{Rep}_{k,\boldsymbol{\kappa}}$ is said to the $\mathrm{SL}(2,\mathbb{C})$-*representation variety of type* $\boldsymbol{\mu} = ((1,1), \ldots, (1,1))$.

We will introduce a compactification of the representation variety due to Benjamin [**3**]. First, we consider a construction of a compactification of the algebraic group $SL_2(\mathbb{C})$. We pick an embedding $\alpha\colon \mathrm{SL}(2,\mathbb{C}) \hookrightarrow \mathrm{PGL}(3,\mathbb{C})$. Such an embedding always exists: we consider the natural embedding $\mathrm{SL}(2,\mathbb{C}) \to \mathrm{GL}(2,\mathbb{C})$ and we take the composition of the embedding and the map $\mathrm{GL}(2,\mathbb{C}) \xrightarrow{\xi} \mathrm{GL}(3,\mathbb{C}) \to \mathrm{PGL}(3,\mathbb{C})$ where

$$\xi(A) = \left(\begin{array}{c|c} A & \\ \hline & 1 \end{array}\right)$$

and the second arrow is the canonical projection. We regard $\mathrm{PGL}(3,\mathbb{C})$ as an open subvariety of $\mathbb{P}(\mathrm{M}(3,\mathbb{C}))$, and define the compactification $\overline{\mathrm{SL}(2,\mathbb{C})}$ of $\mathrm{SL}(2,\mathbb{C})$ as the closure of $\alpha(\mathrm{SL}(2,\mathbb{C}))$ in $\mathbb{P}(\mathrm{M}(3,\mathbb{C}))$, that is,

$$\overline{\mathrm{SL}(2,\mathbb{C})} = \left\{ \left(\begin{array}{cc|c} a & b & \\ c & d & \\ \hline & & e \end{array}\right) \in \mathbb{P}(\mathrm{M}(3,\mathbb{C})) \;\middle|\; ad - bc = e^2 \right\}.$$

Then, we obtain a compactification of the semisimple conjugacy class $\mathcal{C}_i$, denoted by $\overline{\mathcal{C}_i}$, that is,

$$\overline{\mathcal{C}_i} = \left\{ \left(\begin{array}{cc|c} a & b & \\ c & d & \\ \hline & & e \end{array}\right) \in \mathbb{P}(\mathrm{M}(3,\mathbb{C})) \;\middle|\; ad - bc = e^2,\ a + d = \kappa_i e \right\}.$$

Lemma 2.1.2.

$$\overline{\mathcal{C}_i} \cong \mathbb{P}^1 \times \mathbb{P}^1$$

Proof. By the equations $ad - bc = e^2$ and $a + d = \kappa_i e$, we obtain the equation

$$(-a^2 + \kappa_i ae - e^2) - bc = 0.$$

Note that the equation define a hypersurface of degree 2 in $\mathbb{P}^3$, which is isomorphic to $\mathbb{P}^1 \times \mathbb{P}^1$. We put the coordinate $([S : T], [U : V]) \in \mathbb{P}^1 \times \mathbb{P}^1$ such that

$$\begin{aligned} (SU)(TV) &= -a^2 + \kappa_i ae - e^2 = -(a - \nu_1^i e)(a - \nu_2^i e) \\ SV &= b \\ TU &= c \end{aligned}$$

where ν_1^i, ν_2^i are eigenvalues of a matrix of the semisimple conjugacy class $\mathcal{C}_i$. Then, we obtain the following transformation from $\mathbb{P}^1 \times \mathbb{P}^1$ to the hypersurface of degree 2 on $\mathbb{P}^3$:

$$\begin{aligned} a &= \frac{\nu_2^i SU + \nu_1^i TV}{\nu_1^i - \nu_2^i}, & b &= SV, \\ c &= TU, & d_i &= \frac{\nu_1^i SU + \nu_2^i TV}{\nu_1^i - \nu_2^i}, \\ e &= \frac{SU + TV}{\nu_1^i - \nu_2^i}. \end{aligned} \tag{2.1.2}$$

□

We can define a compactification of the representation variety.

Definition 2.1.3. We put

$$\overline{\hat{\mathcal{U}}^{\boldsymbol{\mu}}(\boldsymbol{\kappa})} := \{(M_1,\dots,M_{k-1}) \in \overline{\mathcal{C}}_1 \times \cdots \times \overline{\mathcal{C}}_{n-1} \mid \mathrm{Tr}(A_1\cdots A_{k-1}) = \kappa_k e_1 \cdots e_{k-1}\} \tag{2.1.3}$$

where

$$M_1 = \left(\begin{array}{c|c} A_1 & \\ \hline & e_1 \end{array}\right), \dots, M_{k-1} = \left(\begin{array}{c|c} A_{k-1} & \\ \hline & e_{k-1} \end{array}\right).$$

Remark 2.1.4. In general, for $X \in \overline{\mathrm{SL}(2,\mathbb{C})}$, there is no inverse. Since

$$\mathrm{Tr}(A_{k-1}^{-1}\cdots A_1^{-1}) = \mathrm{Tr}(A_1\cdots A_{k-1})$$

for $\forall A_i \in \mathrm{SL}(2,\mathbb{C})$, we use the condition $\mathrm{Tr}(A_1\cdots A_{k-1}) = \kappa_k$, instead of $\mathrm{Tr}(A_{k-1}^{-1}\cdots A_1^{-1}) = \kappa_k$.

We have the following action of $\mathrm{SL}(2,\mathbb{C})$ on $\overline{\hat{\mathcal{U}}^{\boldsymbol{\mu}}(\boldsymbol{\kappa})}$, which is compatible with the simultaneous action of $\mathrm{SL}(2,\mathbb{C})$ on $\hat{\mathcal{U}}^{\boldsymbol{\mu}}(\boldsymbol{\kappa})$

$$\begin{aligned} P \curvearrowright & \left(\left(\begin{array}{c|c} A_1 & \\ \hline & e_1 \end{array}\right), \dots, \left(\begin{array}{c|c} A_{k-1} & \\ \hline & e_{k-1} \end{array}\right)\right) \\ \longmapsto & \left(\left(\begin{array}{c|c} PA_1P^{-1} & \\ \hline & e_1 \end{array}\right), \dots, \left(\begin{array}{c|c} PA_{k-1}P^{-1} & \\ \hline & e_{k-1} \end{array}\right)\right). \end{aligned} \tag{2.1.4}$$

We regard $\overline{\mathrm{Rep}_{k,\boldsymbol{\kappa}}} \subset \overline{\mathcal{C}_1} \times \cdots \times \overline{\mathcal{C}_{k-1}}$ as the closed subset in $\mathbb{P}^4 \times \cdots \times \mathbb{P}^4$. Then, we obtain an embedding in the projective space by the Segre embedding. Let L be an ample line bundle associated with this embedding, that is,

$$L = \bigotimes_{i=1}^{k-1} p_i^*(\mathcal{O}_{\mathbb{P}^4}(1))$$

where $p_i\colon \overline{\mathrm{Rep}_{k,\boldsymbol{\kappa}}} \to \mathbb{P}^4$ is the i-th projection. Then, L admits the $\mathrm{SL}(2,\mathbb{C})$-linearization with respect to the action.

For $x = (M_1,\dots,M_{k-1}) \in \overline{\hat{\mathcal{U}}^{\boldsymbol{\mu}}(\boldsymbol{\kappa})}$, we put

$$I^{nil} := \{i \in \{1,\dots,k-1\} \mid M_i \text{ is nilpotent i.e. } e_i = 0\ \}.$$

If I^{nil} is not empty, we decompose

$$I^{nil} = I_1^{nil} \cup \cdots \cup I_r^{nil} \tag{2.1.5}$$

where the index set $I_l^{nil} \subset I^{nil}$ $(1 \leq l \leq r)$ consists of indexes of same matrices, that is, matrices indexed by elements of I_l^{nil} are same each other and two matrices which respectively have indexes in I_l^{nil} and $I_{l'}^{nil}$ where $l \neq l'$ are not equal. Let $\sharp I_l^{nil}$ be the cardinality of I_l^{nil}, and let m_1 be a maximum value in $\sharp I_1^{nil},\dots,\sharp I_r^{nil}$. We put

$$J_l := \{j \in \{1,\dots,k-1\} \mid M_j \text{ is not nilpotent, } M_j * M_i = M_i * M_j = M_i, i \in I_l^{nil}\}.$$

Here, we define the product $*$ as

$$M * M' := \left(\begin{array}{c|c} AA' & \\ \hline & e \end{array}\right) \in \mathbb{P}\mathrm{M}(3,\mathbb{C}) \quad \text{for } M := \left(\begin{array}{c|c} A & \\ \hline & e \end{array}\right) \text{ and } M' := \left(\begin{array}{c|c} A' & \\ \hline & e' \end{array}\right).$$

Note that the product $*$ is well-defined in the case where M (resp. M') is nilpotent and M' (resp. M) is not nilpotent where $M \in \overline{\mathcal{C}}$ and $M' \in \overline{\mathcal{C}'}$. Let m_2 be a maximum value in $\{\sharp J_l \mid l \text{ is satisfied } \sharp I_l^{nil} = m_1, 1 \leq l \leq r\}$. If I^{nil} is empty, then we put $m_1 = m_2 = 0$.

Remark 2.1.5. Let $(M_1, \ldots, M_{k-1}) \in \overline{\hat{\mathcal{U}}^{\boldsymbol{\mu}}(\boldsymbol{\kappa})}$. Suppose that $i \in I^{nil}$. We normalize the nilpotent matrix M_i:

$$M_i = \left(\begin{array}{cc|c} 0 & 1 & \\ 0 & 0 & \\ \hline & & 0 \end{array}\right). \tag{2.1.6}$$

For a matrix M_j $(j \neq i)$, the condition which, by this transformation, the matrix M_j is transformed to the following form

$$\begin{pmatrix} a_j & b_j \\ 0 & d_i \end{pmatrix}$$

is equivalent to the condition $M_j * M_i = M_i * M_j = M_i$.

Proposition 2.1.6. *The point $x = (M_1, \ldots, M_{k-1})$ is semi-stable (resp. stable) point if and only if x is satisfied the following condition,*

$$k - 1 \geq 2m_1 + m_2 \quad (\textit{resp.} >). \tag{2.1.7}$$

Proof. For any integer $m > 0$, let λ_m be the 1-parameter subgroup (1-PS) of $\mathrm{SL}(2, \mathbb{C})$ given by

$$\lambda_m \colon t \longmapsto \begin{pmatrix} t^m & 0 \\ 0 & t^{-m} \end{pmatrix}, t \in \mathbb{C}^{\times}. \tag{2.1.8}$$

The matrix $\lambda_r(t)$ acts on $\overline{\hat{\mathcal{U}}^{\boldsymbol{\mu}}(\boldsymbol{\kappa})}$ as follows.

$$\begin{aligned} \begin{pmatrix} t^m & 0 \\ 0 & t^{-m} \end{pmatrix} \curvearrowright & \left(\left(\begin{array}{cc|c} a_1 & b_1 & \\ c_1 & d_1 & \\ \hline & & e_1 \end{array}\right), \ldots, \left(\begin{array}{cc|c} a_{k-1} & b_{k-1} & \\ c_{k-1} & d_{k-1} & \\ \hline & & e_{k-1} \end{array}\right) \right) \\ \longmapsto & \left(\left(\begin{array}{cc|c} a_1 & t^{2m} b_1 & \\ t^{-2m} c_1 & d_1 & \\ \hline & & e_1 \end{array}\right), \ldots, \left(\begin{array}{cc|c} a_{k-1} & t^{2m} b_{k-1} & \\ t^{-2m} c_{k-1} & d_{k-1} & \\ \hline & & e_{k-1} \end{array}\right) \right). \end{aligned}$$

We put $k' := 5^{k-1}$. Let $\mathbb{A}^{k'}$ be the affine cone over the projective space $\mathbb{P}^{k'-1}$ which is the target space of the Segre embedding. We take a base change of the affine cone $\mathbb{A}^{k'}$ via $\overline{\hat{\mathcal{U}}^{\boldsymbol{\mu}}(\boldsymbol{\kappa})} \hookrightarrow \mathbb{P}^{k'-1}$, denoted by the same notation $\mathbb{A}^{k'}$. Let $x^* = (M_1^*, \ldots, M_{k-1}^*)$ be the closed point of $\mathbb{A}^{k'}$ lying over $x \in \overline{\hat{\mathcal{U}}^{\boldsymbol{\mu}}(\boldsymbol{\kappa})}$, that is, $x^* \neq 0$ and x^* projects to x. The action (2.1.4) and the linearization L define a linear action of $\mathrm{SL}(2, \mathbb{C})$ on $\mathbb{A}^{k'}$. In particular, the matrix $\lambda_m(t)$ acts on $\mathbb{A}^{k'}$ as follows. For each $i = 1, \ldots, k-1$, let $e_1^{(i)}, \ldots, e_5^{(i)}$ be a basis of $\mathbb{A}^5$ such that the matrix

$$M_i^* = \left(\begin{array}{cc|c} a_i & b_i & \\ c_i & d_i & \\ \hline & & e_i \end{array}\right)$$

is describe by

$$M_i^* = a_i e_1^{(i)} + b_i e_2^{(i)} + c_i e_3^{(i)} + d_i e_4^{(i)} + e_i e_5^{(i)}.$$

Let $e_{i_1,\ldots,i_{k-1}}$ be the base $e^{(1)}_{i_1} \otimes \cdots \otimes e^{(n-1)}_{i_{k-1}}$ of $\mathbb{A}^{k'}$ where $i_1, \ldots, i_{k-1} \in \{1, \ldots, 5\}$. Then, the action of $\lambda_r(t)$ on $\mathbb{A}^5$ is given by

$$\lambda_m(t) \cdot e_{i_1,\ldots,i_{k-1}} = t^{2m(m^+_{i_1,\ldots,i_{n-1}} - m^-_{i_1,\ldots,i_{n-1}})} e_{i_1,\ldots,i_{k-1}}$$

where $i_1, \ldots, i_{n-1} \in \{1, \ldots, 5\}$ and $m^+_{i_1,\ldots,i_{k-1}}$ (resp. $m^-_{i_1,\ldots,i_{k-1}}$) is the number of 2 (resp. 3) in the index set $\{i_1, \ldots, i_{k-1}\}$. For $x^* \in \mathbb{A}^{k'}$ lying over $x \in \overline{\mathrm{Rep}_{k,\boldsymbol{\kappa}}}$, we write $x^* = \sum x^*_{i_1,\ldots,i_{k-1}} e_{i_1,\ldots,i_{k-1}}$, so that

$$\lambda_m(t) \cdot x^* = \sum t^{2m m_{i_1,\ldots,i_{k-1}}} x^*_{i_1,\ldots,i_{k-1}} e_{i_1,\ldots,i_{k-1}}$$

where $m_{i_1,\ldots,i_{k-1}} = m^+_{i_1,\ldots,i_{k-1}} - m^-_{i_1,\ldots,i_{k-1}}$, and we put

$$\begin{aligned}\mu^L(x, \lambda_m) :=\ & \max\{-m_{i_1,\ldots,i_{k-1}} \mid i_1, \ldots, i_{k-1} \text{ such that } x^*_{i_1,\ldots,i_{k-1}} \neq 0\} \\ =\ & \sharp\left\{ i \;\middle|\; M_i = \begin{pmatrix} a_i & b_i \\ c_i & d_i \end{pmatrix}, c_i \neq 0 \right\} - \sharp\left\{ i \;\middle|\; M_i = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, e_i = 0 \right\}.\end{aligned} \tag{2.1.9}$$

On the other hand, we have

$$\begin{aligned}&\sharp\left\{ i \;\middle|\; M_i = \begin{pmatrix} a_i & b_i \\ c_i & d_i \end{pmatrix}, c_i \neq 0 \right\} \\ &= (k-1) - \sharp\left\{ i \;\middle|\; M_i = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, e_i = 0 \right\} - \sharp\left\{ i \;\middle|\; M_i = \begin{pmatrix} a_i & b_i \\ 0 & d_i \end{pmatrix}, e_i \neq 0 \right\}.\end{aligned}$$

Then, we have

$$\begin{aligned}&\mu^L(x, \lambda_r) \\ &= (k-1) - 2\sharp\left\{ i \;\middle|\; M_i = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, e_i = 0 \right\} - \sharp\left\{ i \;\middle|\; M_i = \begin{pmatrix} a_i & b_i \\ 0 & d_i \end{pmatrix}, e_i \neq 0 \right\}.\end{aligned} \tag{2.1.10}$$

By the Hilbert-Mumford criterion (see [**35**, Theorem 2.1] or [**36**, Proposition 4.11]), the point x is stable (resp. semi-stable) for this action if and only if $\mu^L(g \cdot x, \lambda_m) > 0$ (resp. ≥ 0) for every $g \in \mathrm{SL}(2, \mathbb{C})$ and every 1-PS λ_m of the form (2.1.8). If the point x satisfies the condition $2m_1 < \sharp I^{nil}$, then we have $\mu^L(g \cdot x, \lambda_m) > 0$ for any $g \in \mathrm{SL}(2, \mathbb{C})$. On the other hand, we consider the case where the point x satisfies the condition $2m_1 \geq \sharp I^{nil}$. There are at most two components of the decomposition (2.1.5) of I^{nil} such that the cardinalities are m_1. We denote by I^{nil}_{max} the union of the components. If the index set $I^{nil} \setminus I^{nil}_{max}$ is nonempty, then we have $\mu^L(g \cdot x, \lambda_m) > 0$ for $g \in \mathrm{SL}(2, \mathbb{C})$ such that gM_ig^{-1} is the matrix (2.1.6) where $i \in I^{nil} \setminus I^{nil}_{max}$. For $g \in \mathrm{SL}(2, \mathbb{C})$ such that gM_ig^{-1} is the matrix (2.1.6) where $i \in I^{nil}_{max}$, we have

$$\mu^L(g \cdot x, \lambda_r) \geq (k-1) - (2m_1 + m_2). \tag{2.1.11}$$

If the index $i \in I^{nil}_{max}$ of the normalized matrix is a element of I^{nil}_l such that $\sharp I^{nil}_l = m_1$ and $\sharp J_l = m_2$, then the equality of (2.1.11) holds. For the other matrix $g \in \mathrm{SL}(2, \mathbb{C})$, we have $\mu^L(g \cdot x, \lambda_m) > 0$. We have thus proved the proposition. □

We obtain a compactification of the character variety $\hat{\mathcal{M}}^{\boldsymbol{\mu}}_B(\boldsymbol{\kappa})$.

Definition 2.1.7.

$$\overline{\hat{\mathcal{M}}^{\boldsymbol{\mu}}_B(\boldsymbol{\kappa})} := \mathrm{Proj}\ H^0(\overline{\hat{\mathcal{U}}^{\boldsymbol{\mu}}(\boldsymbol{\kappa})}, L^{\otimes r})^{\mathrm{SL}(2,\mathbb{C})}.$$

The variety $\overline{\hat{\mathcal{M}}^{\boldsymbol{\mu}}_B(\boldsymbol{\kappa})}$ is a projective algebraic variety. This variety may have singular points on the boundary. Then, we should take a resolution of singular points of $\overline{\hat{\mathcal{M}}^{\boldsymbol{\mu}}_B(\boldsymbol{\kappa})}$. In general, it is not easy to give a systematic resolution of singularities for any n. On the following sections, we treat the cases for $n = 4, 5$. We will show that $\overline{\hat{\mathcal{M}}^{\boldsymbol{\mu}}_B(\boldsymbol{\kappa})}$ is non-singular and the boundary divisor is a triangle of $\mathbb{P}^1$. On Section 2.3, we will treat the case for $n = 5$.

2.2. Type $\boldsymbol{\mu} = ((1,1),(1,1),(1,1),(1,1))$. Let

$$\left(\left(\begin{array}{cc|c} a_1 & b_1 & \\ c_1 & d_1 & \\ \hline & & e_1 \end{array}\right), \left(\begin{array}{cc|c} a_2 & b_2 & \\ c_2 & d_2 & \\ \hline & & e_2 \end{array}\right), \left(\begin{array}{cc|c} a_3 & b_3 & \\ c_3 & d_3 & \\ \hline & & e_3 \end{array}\right)\right) \in \overline{\hat{\mathcal{U}}^{\boldsymbol{\mu}}(\boldsymbol{\kappa})}. \tag{2.2.1}$$

The compactification $\overline{\hat{\mathcal{U}}^{\boldsymbol{\mu}}(\boldsymbol{\kappa})}$ is defined by the following equations in $\mathbb{P}^4 \times \mathbb{P}^4 \times \mathbb{P}^4$

$$a_i + d_i = \kappa_i e_i, \ (i = 1,2,3), \tag{2.2.2}$$

$$a_i d_i - b_i c_i = e_i^2, \ (i = 1,2,3), \tag{2.2.3}$$

$$\mathrm{Tr}\left(\begin{pmatrix} a_1 & b_1 \\ c_1 & d_1 \end{pmatrix}\begin{pmatrix} a_2 & b_2 \\ c_2 & d_2 \end{pmatrix}\begin{pmatrix} a_3 & b_3 \\ c_3 & d_3 \end{pmatrix}\right) = \kappa_4 e_1 e_2 e_3. \tag{2.2.4}$$

We analyze the stability. If $e_i = 0$ and $e_j e_k \neq 0$ $(j,k \in \{1,2,3\} \setminus \{i\})$, then x is an unstable point if and only if x is a point of the orbit of (M_1, M_2, M_3) where

$$M_i = \left(\begin{array}{cc|c} 0 & 1 & \\ 0 & 0 & \\ \hline & & 0 \end{array}\right), M_j = \left(\begin{array}{cc|c} a_j & b_j & \\ 0 & d_j & \\ \hline & & e_j \end{array}\right), M_k = \left(\begin{array}{cc|c} a_k & b_k & \\ 0 & d_k & \\ \hline & & e_k \end{array}\right).$$

If $e_i = 0, e_j = 0$, then x is an unstable point if and only if x is a point of the orbit of (M_1, M_2, M_3) where two matrices in M_1, M_2, M_3 are

$$\left(\begin{array}{cc|c} 0 & 1 & \\ 0 & 0 & \\ \hline & & 0 \end{array}\right).$$

Lemma 2.2.1. *The point $x \in \overline{\hat{\mathcal{U}}^{\boldsymbol{\mu}}(\boldsymbol{\kappa})}$ where $\boldsymbol{\mu} = ((1,1),(1,1),(1,1),(1,1))$ is stable if and only if x is semistable.*

Proof. The point $x = (M_1, M_2, M_3)$ is not stable if only x is normalized as follows.

$$M_i = \left(\begin{array}{cc|c} 0 & 1 & \\ 0 & 0 & \\ \hline & & 0 \end{array}\right), M_j = \left(\begin{array}{cc|c} a_j & b_j & \\ c_j & d_j & \\ \hline & & e_j \end{array}\right), M_k = \left(\begin{array}{cc|c} a_k & b_k & \\ 0 & d_k & \\ \hline & & e_k \end{array}\right), \text{ where } c_j \neq 0,$$

or

$$M_i = \left(\begin{array}{cc|c} 0 & 1 & \\ 0 & 0 & \\ \hline & & 0 \end{array}\right), M_j = \left(\begin{array}{cc|c} 0 & 0 & \\ 1 & 0 & \\ \hline & & 0 \end{array}\right), M_k = \left(\begin{array}{cc|c} a_k & b_k & \\ 0 & d_k & \\ \hline & & e_k \end{array}\right).$$

However, the matrices do not satisfied the equation (2.2.4). Then, there are no strictly semistable points. □

The following theorem shows that our compactification $\overline{\hat{\mathcal{M}}_B^{\boldsymbol{\mu}}(\boldsymbol{\kappa})}$ of $\hat{\mathcal{M}}_B^{\boldsymbol{\mu}}(\boldsymbol{\kappa})$ has the same configuration of the boundary divisor as the natural compactification of the Fricke-Klein cubic surface.

Theorem 2.2.2. *The boundary divisor of the compactification* $\overline{\hat{\mathcal{M}}_B^{\boldsymbol{\mu}}(\boldsymbol{\kappa})}$ *where* $\boldsymbol{\mu} = ((1,1),(1,1),(1,1),(1,1))$ *is a triangle of three projective lines.*

PROOF. We describe the boundary divisor explicitly. Let E_i be the image of the divisor $[e_i = 0]$ on $\overline{\hat{\mathcal{U}}^{\boldsymbol{\mu}}(\boldsymbol{\kappa})}$ by the quotient $\overline{\hat{\mathcal{U}}^{\boldsymbol{\mu}}(\boldsymbol{\kappa})} \to \overline{\hat{\mathcal{M}}_B^{\boldsymbol{\mu}}(\boldsymbol{\kappa})}$ $(i = 1, 2, 3)$. First, we describe $[e_1 = 0]$. We normalize M_1 by the $\mathrm{SL}(2,\mathbb{C})$-conjugate action as the matrix (2.1.6). The stabilizer subgroup of the matrix is $\left\{\begin{pmatrix} a & b \\ 0 & 1/a \end{pmatrix}\right\}$.

By the stability, we obtain $c_2 \neq 0$ and $c_3 \neq 0$. Since $c_2 \neq 0$, the matrices of the component $[e_1 = 0]$ are normalized by the action of this stabilizer subgroup:

$$\left(\left(\begin{array}{cc|c} 0 & 1 & \\ 0 & 0 & \\ \hline & & 0 \end{array}\right), \left(\begin{array}{cc|c} 0 & -e_2^2 & \\ c_2^2 & k_2c_2e_2 & \\ \hline & & c_2e_2 \end{array}\right), \left(\begin{array}{cc|c} a_3 & b_3 & \\ c_3 & d_3 & \\ \hline & & e_3 \end{array}\right)\right). \tag{2.2.5}$$

The stabilizer subgroup of the normalized matrices is the torus group $\left\{\begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix}\right\}$.

Before we consider the quotient by the torus group, we consider the normalized matrices (2.2.5). The normalized matrices are defined by the following equations

$$\begin{cases} a_3 + d_3 = \kappa_3 e_3, \\ a_3d_3 - b_3c_3 = e_3^2, \\ c_2a_3 + k_2e_2c_3 = 0 \end{cases} \tag{2.2.6}$$

in the Zariski open set $c_2c_3 \neq 0$ of $\mathbb{P}^1 \times \mathbb{P}^4$. By Lemma 2.1.2, the normalized matrices are defined by

$$c_2(\nu_2^3 S_3U_3 + \nu_1^3 T_3V_3) + \kappa_2(\nu_1^3 - \nu_2^3)e_2(T_3U_3) = 0 \tag{2.2.7}$$

in the Zariski open set $c_2T_3U_3 \neq 0$ of $\mathbb{P}^1 \times (\mathbb{P}^1 \times \mathbb{P}^1)$.

We consider the quotient by the torus group. The torus action on $\mathbb{P}^1 \times (\mathbb{P}^1 \times \mathbb{P}^1)$ is

$$\begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix} \curvearrowright ([c_2 : e_2], [S_3 : T_3], [U_3 : V_3]) \longmapsto ([a^{-1}c_2 : ae_2], [aS_3 : a^{-1}T_3], [a^{-1}U_3 : aV_3]).$$

We consider the $\mathrm{SL}(2,\mathbb{C})$-linearization $L = \bigotimes_{i=1}^3 p_i^*(\mathcal{O}_{\mathbb{P}^4}(1))$ on $\overline{\hat{\mathcal{U}}^{\boldsymbol{\mu}}(\boldsymbol{\kappa})}$. We take a pull-back of L via the embedding

$$p_{e_1} : \mathbb{P}^1 \times (\mathbb{P}^1 \times \mathbb{P}^1) \hookrightarrow \overline{\hat{\mathcal{U}}^{\boldsymbol{\mu}}(\boldsymbol{\kappa})} \tag{2.2.8}$$

defined by the matrices (2.2.5) and the transform (2.1.2). Let L_{e_1} be the pull-back of L on $\mathbb{P}^1 \times (\mathbb{P}^1 \times \mathbb{P}^1)$. We obtain the T-linearization on L_{e_1} induced by the $\mathrm{SL}(2,\mathbb{C})$-linearization

L on $\overline{\hat{\mathcal{U}}^{\boldsymbol{\mu}}(\boldsymbol{\kappa})}$. We consider the dual action on $H^0(\mathbb{P}^1\times(\mathbb{P}^1\times\mathbb{P}^1), L_{e_1})$. We have the following basis of the subspace consisting of invariant sections:

$$\begin{aligned} & s_1 = b_1 \otimes c_2^2 \otimes S_3U_3, \quad s_2 = b_1 \otimes c_2^2 \otimes T_3V_3, \\ & s_3 = b_1 \otimes c_2e_2 \otimes T_3U_3 \end{aligned} \tag{2.2.9}$$

where $b_1 \in H^0(\mathbb{P}^1 \times (\mathbb{P}^1 \times \mathbb{P}^1), (p_{e_1} \circ p_1)^*(\mathcal{O}_{\mathbb{P}^4}(1)))$ corresponding to the $(1,2)$-entry of the matrix M_1. The sections have the relation

$$\nu_2^3 s_1 + \nu_1^3 s_2 + \kappa_2(\nu_1^3 - \nu_2^3)s_3 = 0$$

by the equation (2.2.7). Therefore, we obtain $E_1 \cong \mathbb{P}^1$. In the same way, we also obtain $E_i \cong \mathbb{P}^1$ $(i = 2, 3)$.

We show that E_1 and E_2 intersect at one point. We substitute $e_2 = 0$ for (2.2.6). Then, we have the following equations

$$\begin{cases} a_3 + d_3 = \kappa_3 e_3, \\ a_3d_3 - b_3c_3 = e_3^2, \\ a_3 = 0. \end{cases}$$

The locus defined by the equations above is a quadric curve in $\mathbb{P}^2$, which is isomorphic to $\mathbb{P}^1$. There are two unstable points in the locus, $[b_3 : c_3 : e_3] = [0 : 1 : 0]$ and $[b_3 : c_3 : e_3] = [1 : 0 : 0]$. The intersection is the quotient of $\mathbb{P}^1$ minus the two points by the torus action. Then, the intersection is a point. In the same way, the intersection of E_2 and E_3 (resp. E_3 and E_1) is a point. □

2.3. Type $\boldsymbol{\mu} = ((1,1),(1,1),(1,1),(1,1))$. Let

$$\left(\left(\begin{array}{cc|c} a_1 & b_1 & \\ c_1 & d_1 & \\ \hline & & e_1 \end{array}\right), \left(\begin{array}{cc|c} a_2 & b_2 & \\ c_2 & d_2 & \\ \hline & & e_2 \end{array}\right), \left(\begin{array}{cc|c} a_3 & b_3 & \\ c_3 & d_3 & \\ \hline & & e_3 \end{array}\right), \left(\begin{array}{cc|c} a_4 & b_4 & \\ c_4 & d_4 & \\ \hline & & e_4 \end{array}\right)\right) \in \overline{\mathrm{Rep}_{5,\boldsymbol{\kappa}}}.$$

The compactification $\overline{\hat{\mathcal{U}}^{\boldsymbol{\mu}}(\boldsymbol{\kappa})}$ is defined by the following equations in $(\mathbb{P}^4)^4$

$$a_i + d_i = \kappa_i e_i, \ (i = 1, 2, 3, 4), \tag{2.3.1}$$

$$a_id_i - b_ic_i = e_i^2, \ (i = 1, 2, 3, 4), \tag{2.3.2}$$

$$\mathrm{Tr}\left(\begin{pmatrix} a_1 & b_1 \\ c_1 & d_1 \end{pmatrix}\begin{pmatrix} a_2 & b_2 \\ c_2 & d_2 \end{pmatrix}\begin{pmatrix} a_3 & b_3 \\ c_3 & d_3 \end{pmatrix}\begin{pmatrix} a_4 & b_4 \\ c_4 & d_4 \end{pmatrix}\right) = \kappa_5 e_1e_2e_3e_4. \tag{2.3.3}$$

We consider the stability condition.

Lemma 2.3.1. *The closures of orbits of properly semistable points contain the point*

$$s_1 = \left(\left(\begin{array}{cc|c} 0 & 1 & \\ 0 & 0 & \\ \hline & & 0 \end{array}\right), \left(\begin{array}{cc|c} 0 & 1 & \\ 0 & 0 & \\ \hline & & 0 \end{array}\right), \left(\begin{array}{cc|c} 0 & 0 & \\ 1 & 0 & \\ \hline & & 0 \end{array}\right), \left(\begin{array}{cc|c} 0 & 0 & \\ 1 & 0 & \\ \hline & & 0 \end{array}\right)\right) \tag{2.3.4}$$

or

$$s_2 = \left(\left(\begin{array}{cc|c} 0 & 1 & \\ 0 & 0 & \\ \hline & & 0 \end{array}\right), \left(\begin{array}{cc|c} 0 & 0 & \\ 1 & 0 & \\ \hline & & 0 \end{array}\right), \left(\begin{array}{cc|c} 0 & 0 & \\ 1 & 0 & \\ \hline & & 0 \end{array}\right), \left(\begin{array}{cc|c} 0 & 1 & \\ 0 & 0 & \\ \hline & & 0 \end{array}\right)\right). \tag{2.3.5}$$

Expect for the points of the orbits of s_1 and s_2, the stabilizer groups of every points are finite. Each stabilizer group of the orbits of s_1 and s_2 is conjugate to the torus group $T = \left\{ \begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix} \right\}$.

Proof. Let $x = (M_1, \ldots, M_4)$ be a property semistable point. By Proposition 2.1.6, we have $2m_1 + m_2 = 4$. First, we consider the case where $m_1 = 1, m_2 = 2$. We put

$$M_{i_1} = \left(\begin{array}{cc|c} 0 & 1 & \\ 0 & 0 & \\ \hline & & 0 \end{array}\right), M_{i_2} = \left(\begin{array}{cc|c} * & * & \\ 0 & * & \\ \hline & & * \end{array}\right), M_{i_3} = \left(\begin{array}{cc|c} * & * & \\ 0 & * & \\ \hline & & * \end{array}\right), M_{i_4} = \left(\begin{array}{cc|c} * & * & \\ c_{i_4} & * & \\ \hline & & * \end{array}\right) \tag{2.3.6}$$

where $\{i_1, \ldots, i_4\} = \{1, \ldots, 4\}$ and $c_{i_4} \neq 0$. However, by the condition $c_{i_4} \neq 0$, the matrices do not satisfy the equation (2.3.3).

Second, we consider the case where $m_1 = 2, m_2 = 0$. We put

$$M_{i_1} = \left(\begin{array}{cc|c} 0 & 1 & \\ 0 & 0 & \\ \hline & & 0 \end{array}\right), M_{i_2} = \left(\begin{array}{cc|c} 0 & 1 & \\ 0 & 0 & \\ \hline & & 0 \end{array}\right), M_{i_3} = \left(\begin{array}{cc|c} * & * & \\ c_{i_3} & * & \\ \hline & & * \end{array}\right), M_{i_4} = \left(\begin{array}{cc|c} * & * & \\ c_{i_4} & * & \\ \hline & & * \end{array}\right) \tag{2.3.7}$$

where $\{i_1, \ldots, i_4\} = \{1, \ldots, 4\}$, $c_{i_3} \neq 0$, and $c_{i_3} \neq 0$. If $(i_1, i_2) = (1, 3)$ or $(2, 4)$, then the matrices do not satisfy the equation (2.3.3). Therefore, we consider the case where $(i_1, i_2) = (1, 2)$, $(2, 3)$, or $(3, 4)$. The 1-parameter subgroup (2.1.8) acts on the matrices (2.3.7). For the matrices M_{i_1} and M_{i_2}, the action is trivial. The actions of the 1-parameter subgroup $\lambda_m(t)$ on M_{i_3} and M_{i_4} are

$$\begin{aligned} \lambda_m(t) \cdot M_{i_3} &= \left(\begin{array}{cc|c} * & t^{2m}* & \\ t^{-2m}c_{i_3} & * & \\ \hline & & * \end{array}\right) & \lambda_m(t) \cdot M_{i_4} &= \left(\begin{array}{cc|c} * & t^{2m}* & \\ t^{-2m}c_{i_4} & * & \\ \hline & & * \end{array}\right) \\ &= \left(\begin{array}{cc|c} t^{2m}* & t^{4m}* & \\ c_{i_3} & t^{2m}* & \\ \hline & & t^{2m}* \end{array}\right), & &= \left(\begin{array}{cc|c} t^{2m}* & t^{4m}* & \\ c_{i_4} & t^{2m}* & \\ \hline & & t^{2m}* \end{array}\right). \end{aligned} \tag{2.3.8}$$

Then, the limit $\lim_{m \to 0} \lambda_r \cdot M$ is the matrix (2.3.4) or (2.3.5).

Since the orbits of the points s_1 and s_2 are closed, the orbits have the maximum dimension of the stabilizer group, which is one dimension. □

We consider a resolution of properly semistable points. We take the blowing up along the orbits of s_1 and s_2:

$$\overline{\widetilde{\hat{\mathcal{U}}^{\boldsymbol{\mu}}(\boldsymbol{\kappa})}} \longrightarrow \overline{\hat{\mathcal{U}}^{\boldsymbol{\mu}}(\boldsymbol{\kappa})}. \tag{2.3.9}$$

The simultaneous action of $\mathrm{SL}(2, \mathbb{C})$ on $\overline{\hat{\mathcal{U}}^{\boldsymbol{\mu}}(\boldsymbol{\kappa})}$ induces an action on $\overline{\widetilde{\hat{\mathcal{U}}^{\boldsymbol{\mu}}(\boldsymbol{\kappa})}}$. By taking the blowing up (2.3.9), the condition for stability and unstability is unchanging. On the other hand, the points of the exceptional divisors are stable points. The points of orbits which are not closed are unstable points. Hence, there is no properly semistable point in $\overline{\widetilde{\hat{\mathcal{U}}^{\boldsymbol{\mu}}(\boldsymbol{\kappa})}}$. (See [**27**, Section 6]). We will show that the quotient of the blowing up is non-singular. First, we describe the blowing up of $\overline{\hat{\mathcal{U}}^{\boldsymbol{\mu}}(\boldsymbol{\kappa})}$ along the orbit of s_1. Let U_1 and U_2 be the Zariski open sets $U_1 = [b_1 \neq 0, b_2 \neq 0, c_3 \neq 0, c_4 \neq 0]$ and $U_2 = [c_1 \neq 0, c_2 \neq 0, b_3 \neq 0, b_4 \neq 0]$ of $\overline{\hat{\mathcal{U}}^{\boldsymbol{\mu}}(\boldsymbol{\kappa})} \subset \overline{\mathcal{C}}_1 \times \cdots \times \overline{\mathcal{C}}_4$. Note that the orbit of s_1 is contained in $U_1 \cup U_2$. Since $\overline{\mathcal{C}}_i \cong \mathbb{P}^1 \times \mathbb{P}^1$ for $i = 1, \ldots, 4$ by the transformation (2.1.2), we have

$$U_i \subset \overline{\hat{\mathcal{U}}^{\boldsymbol{\mu}}(\boldsymbol{\kappa})} \subset (\mathbb{P}^1 \times \mathbb{P}^1)^4 \text{ for } i = 1, 2. \tag{2.3.10}$$

In the open sets U_1 and U_2, we put the following affine coordinates

$$([1 : x_1], [y_1 : 1]), ([1 : x_2], [y_2 : 1]), ([x_3 : 1], [1 : y_3]), ([x_4 : 1], [1 : y_4]),$$

and

$$([z_1 : 1], [1 : w_1]), ([z_2 : 1], [1 : w_2]), ([1 : z_3], [w_3 : 1]), ([1 : z_4], [w_4 : 1]),$$

respectively. In the open set U_1, the ideal of the orbit of s_1 is $(X_1, X_2, X_3, X_4, X_5)$ where

$$\begin{aligned} X_0 &:= e_1 = \frac{y_1 + x_1}{\nu_1^1 - \nu_2^1}, & X_1 &:= e_2 = \frac{y_2 + x_2}{\nu_1^2 - \nu_2^2}, \\ X_2 &:= e_3 = \frac{y_3 + x_3}{\nu_1^3 - \nu_2^3}, & X_3 &:= e_4 = \frac{y_4 + x_4}{\nu_1^4 - \nu_2^4}, \\ X_4 &:= x_1 - x_2, & X_5 &:= x_3 - x_4. \end{aligned}$$

We can extend the torus action on $\overline{\hat{\mathcal{U}}^{\boldsymbol{\mu}}(\boldsymbol{\kappa})}$ to the torus action on $\widetilde{\overline{\hat{\mathcal{U}}^{\boldsymbol{\mu}}(\boldsymbol{\kappa})}}$ by

$$\begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix} \curvearrowright [X_0 : X_1 : X_2 : X_3 : X_4 : X_5] \longmapsto [a^{-2}X_0 : a^{-2}X_1 : a^2X_2 : a^2X_3 : a^{-2}X_4 : a^2X_5].$$

On the other hand, in the open set U_2, the ideal of the orbit of s_1 is $(Y_1, Y_2, Y_3, Y_4, Y_5)$ where

$$\begin{aligned} Y_0 &:= e_1 = \frac{z_1 + w_1}{\nu_1^1 - \nu_2^1}, & Y_1 &:= e_2 = \frac{z_2 + w_2}{\nu_1^2 - \nu_2^2}, \\ Y_2 &:= e_3 = \frac{z_3 + w_3}{\nu_1^3 - \nu_2^3}, & Y_3 &:= e_4 = \frac{z_4 + w_4}{\nu_1^4 - \nu_2^4}, \\ Y_4 &:= z_1 - z_2, & Y_5 &:= z_3 - z_4. \end{aligned}$$

We can extend the torus action on $\overline{\hat{\mathcal{U}}^{\boldsymbol{\mu}}(\boldsymbol{\kappa})}$ to the torus action on $\widetilde{\overline{\hat{\mathcal{U}}^{\boldsymbol{\mu}}(\boldsymbol{\kappa})}}$ by

$$\begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix} \curvearrowright [Y_0 : Y_1 : Y_2 : Y_3 : Y_4 : Y_5] \longmapsto [a^2Y_0 : a^2Y_1 : a^{-2}Y_2 : a^{-2}Y_3 : a^2Y_4 : a^{-2}Y_5].$$

Hence, we have

$$\widetilde{\overline{\hat{\mathcal{U}}^{\boldsymbol{\mu}}(\boldsymbol{\kappa})}}_{s_1} \hookrightarrow (\overline{\hat{\mathcal{U}}^{\boldsymbol{\mu}}(\boldsymbol{\kappa})} \setminus U_1 \cup U_2) \cup (U_1 \times \mathbb{P}^5) \cup (U_2 \times \mathbb{P}^5)$$

where $\widetilde{\overline{\hat{\mathcal{U}}^{\boldsymbol{\mu}}(\boldsymbol{\kappa})}}_{s_1}$ is the blowing up along the orbit of s_1. The stabilizer group of any point in the exceptional divisor is

$$\left\{ \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} \right\}.$$

This action is trivial. In the same way, we can describe the blowing up along the orbit of s_2.

Theorem 2.3.2. *In the case of $n = 5$, there exists a non-singular compactification of $\hat{\mathcal{M}}^{\boldsymbol{\mu}}_B(\boldsymbol{\kappa})$ where $\boldsymbol{\mu} = ((1,1),(1,1),(1,1),(1,1),(1,1))$ such that the boundary complex is a simplicial decomposition of sphere S^3.*

Proof. The outline of the proof is as follows. We put

$$\overline{\widetilde{\hat{\mathcal{M}}^{\boldsymbol{\mu}}_B(\boldsymbol{\kappa})}} := \overline{\widetilde{\hat{\mathcal{U}}^{\boldsymbol{\mu}}(\boldsymbol{\kappa})}} /\!/ \mathrm{SL}(2,\mathbb{C}).$$

We have the six components of the boundary divisor of $\overline{\widetilde{\hat{\mathcal{M}}^{\boldsymbol{\mu}}_B(\boldsymbol{\kappa})}}$: the quotients of the proper transformations of the divisors $[e_1 = 0], [e_2 = 0], [e_3 = 0], [e_4 = 0]$ of $\overline{\hat{\mathcal{U}}^{\boldsymbol{\mu}}(\boldsymbol{\kappa})}$ and the quotients of the exceptional divisors associated with blowing up along s_1 and s_2. We denote by E_1, E_2, E_3, E_4 and ex_1, ex_2 each component. In Step 1, we describe the components E_1, E_2, E_3 and E_4 explicitly. In Step 2, we describe the intersections $E_i \cap E_j,\ i \neq j$. In particular, the intersections $E_i \cap E_{i+1}, i = 1,2,3,4$ (where E_5 implies E_1) are nonempty and irreducible. On the other hand, the intersections $E_i \cap E_{i+2}, i = 1,2$ are not irreducible. The intersection $E_i \cap E_{i+2}$ consists of two components, denoted by $E^+_{i,i+2}, E^-_{i,i+2}$. Then, we take the blowing up along the components $E^+_{1,3}, E^-_{1,3}, E^+_{2,4}, E^-_{2,4}$:

$$\widetilde{X} \longrightarrow X := \overline{\widetilde{\hat{\mathcal{M}}^{\boldsymbol{\mu}}_B(\boldsymbol{\kappa})}}. \tag{2.3.11}$$

We use the same notation E_i which is the proper transform of E_i. We denote by $ex^+_{1,3}, ex^-_{1,3}, ex^+_{2,4}, ex^-_{2,4}$ the exceptional divisors associated with the blowing up (2.3.11). Consequently, the components of the boundary divisor of the compactification $\widetilde{X}$ of $\hat{\mathcal{M}}^{\boldsymbol{\mu}}_B(\boldsymbol{\kappa})$ are

$$E_1, E_2, E_3, E_4, ex_1, ex_2, ex^+_{1,3}, ex^-_{1,3}, ex^+_{2,4}, ex^-_{2,4}.$$

Next, we see how ex_i and the other components intersect. In Step 3, we describe the 2-dimensional simplices and the 3-dimensional simplices. Finally, we can describe the boundary complex of the boundary divisor of the compactification of the character variety.

Step 1. We describe the component E_i (i.e. $[e_i = 0]/\!/\mathrm{SL}(2,\mathbb{C})$) explicitly. We consider the case where $e_1 = 0$. Let D_i be the divisor $[e_i = 0]$ on $\overline{\hat{\mathcal{U}}^{\boldsymbol{\mu}}(\boldsymbol{\kappa})}$ for $i = 1, \ldots, 4$. Let $(M_1, \ldots, M_4)$ be a point on D_1. We normalize the matrix M_1 by the $\mathrm{SL}(2,\mathbb{C})$-conjugate action as the matrix (2.1.6). The stabilizer subgroup of the matrix is the group of upper triangular matrices. From the stability, we obtain $c_2 \neq 0$, $c_3 \neq 0$ or $c_4 \neq 0$. In the case of $c_2 \neq 0$, the matrices of the divisor D_1 are normalized by the action of this stabilizer subgroup:

$$\left(\left(\begin{array}{cc|c} 0 & 1 & \\ 0 & 0 & \\ \hline & & 0 \end{array}\right), \left(\begin{array}{cc|c} 0 & -e_2^2 & \\ c_2^2 & k_2c_2e_2 & \\ \hline & & c_2e_2 \end{array}\right), \left(\begin{array}{cc|c} a_3 & b_3 & \\ c_3 & d_3 & \\ \hline & & e_3 \end{array}\right), \left(\begin{array}{cc|c} a_4 & b_4 & \\ c_4 & d_4 & \\ \hline & & e_4 \end{array}\right) \right). \tag{2.3.12}$$

Then, we have the locus defined by the following equations

$$
(2.3.13) \qquad \begin{cases} a_3 + d_3 = \kappa_3 e_3 \\ a_3 d_3 - b_3 c_3 = e_3^2 \\ a_4 + d_4 = \kappa_4 e_4 \\ a_4 d_4 - b_4 c_4 = e_4^2 \\ c_2 a_3 a_4 + \kappa_2 e_2 c_3 a_4 + c_2 b_3 c_4 + \kappa_2 e_2 d_3 c_4 = 0 \end{cases}
$$

in $(\mathbb{P}^1 \times (\mathbb{P}^4 \times \mathbb{P}^4)) \cap [c_2 \neq 0]$. The locus defined by $a_i + d_i = \kappa_i e_i$ and $a_i d_i - b_i c_i = e_i^2$ in $\mathbb{P}^4$ is isomorphic to $\mathbb{P}^1 \times \mathbb{P}^1$. We put the coordinates S_3, T_3, U_3, V_3 and S_4, T_4, U_4, V_4 of $(\mathbb{P}^1 \times \mathbb{P}^1)^2$ in the same way as in Section 2.2. Then, the locus of the normalized matrices is defined by the following equation

$$
\begin{aligned}
c_2(\nu_2^3 S_3 U_3 + \nu_1^3 T_3 V_3)(\nu_2^4 S_4 U_4 + \nu_1^4 T_4 V_4) \\
+ \kappa_2 e_2 (\nu_1^3 - \nu_2^3)(T_3 U_3)(\nu_2^4 S_4 U_4 + \nu_1^4 T_4 V_4) \\
+ c_2 (\nu_1^3 - \nu_2^3)(\nu_1^4 - \nu_2^4)(S_3 V_3)(T_4 U_4) \\
+ \kappa_2 e_2 (\nu_1^4 - \nu_2^4)(\nu_1^3 S_3 U_3 + \nu_2^3 T_3 V_3)(T_4 U_4) = 0
\end{aligned}
$$

in $(\mathbb{P}^1)^5 \cap [c_2 \neq 0]$. Let $D_1^{c_2 \neq 0}$ be the Zariski open set of the hypersrface in $(\mathbb{P}^1)^5$. The torus action on $D_1^{c_2 \neq 0}$ is the following action:

$$
\begin{aligned}
\begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix} &\curvearrowright ([c_2 : e_2], [S_3 : T_3], [U_3 : V_3], [S_4 : T_4], [U_4 : V_4]) \\
&\mapsto ([a^{-1} c_2 : a e_2], [a S_3 : a^{-1} T_3], [a^{-1} U_3 : a V_3], [a S_3 : a^{-1} T_3], [a^{-1} U_3 : a V_3]).
\end{aligned}
$$

In the same way as in the case $c_2 \neq 0$, we have the Zariski open sets of the hypersurfaces in $(\mathbb{P}^1)^5$ corresponding to $c_3 \neq 0$ and $c_4 \neq 0$, denoted by $D_1^{c_3 \neq 0}$ and $D_1^{c_4 \neq 0}$. We glue $D_1^{c_2 \neq 0}$, $D_1^{c_3 \neq 0}$ and $D_1^{c_4 \neq 0}$, denoted by D_1'. We take the blowing up (2.3.9). Let $\widetilde{D}_1'$ be the proper transform of D_1'. Then, the component of the boundary divisor E_1 is the quotient of $\widetilde{D}_1'$ by the torus action. Similarly, we may describe the components E_j $(j = 2, 3, 4)$.

Step 2. We denote by $D_{i,j}$ the intersection of the divisors $[e_i = 0]$ and $[e_j = 0]$ on $\widehat{\mathcal{U}^{\boldsymbol{\mu}}}(\boldsymbol{\kappa})$. First, we consider the intersection of E_1 and E_2. We substitute $e_2 = 0$ for (2.3.13). Then, we have the locus defined by the following equations

$$
\begin{cases} a_3 + d_3 = \kappa_3 e_3 \\ a_3 d_3 - b_3 c_3 = e_3^2 \\ a_4 + d_4 = \kappa_4 e_4 \\ a_4 d_4 - b_4 c_4 = e_4^2 \\ a_3 a_4 + b_3 c_4 = 0 \end{cases}
$$

in $(\mathbb{P}^1 \times (\mathbb{P}^4)^2) \cap [c_2 \neq 0]$. By the transformation (2.1.2), we have the Zariski open set of the hypersurface in $(\mathbb{P}^1)^5$, denoted by $D_{12}^{c_2 \neq 0}$. Next, we consider the case where $c_3 \neq 0$. In the same way as in the case where $c_2 \neq 0$, we have the locus defined by the following equations

$$\begin{cases} a_2 + d_2 = 0 \\ a_2 d_2 - b_2 c_2 = 0 \\ a_4 + d_4 = \kappa_4 e_4 \\ a_4 d_4 - b_4 c_4 = e_4^2 \\ d_2 c_3^2 a_4 - c_2 e_3^2 c_4 + \kappa_3 d_2 c_3 e_3 c_4 = 0 \end{cases}$$

in $(\mathbb{P}^1 \times (\mathbb{P}^4)^2) \cap [c_3 \neq 0]$. Since we may put $\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} st & s^2 \\ -t^2 & -st \end{pmatrix}$ where $a + d = 0, ad - be = 0$, we have

$$\begin{cases} a_4 + d_4 = \kappa_4 e_4 \\ a_4 d_4 - b_4 c_4 = e_4^2 \\ t(s c_3^2 a_4 - t e_3^2 c_4 + \kappa_3 s c_3 e_3 c_4) = 0. \end{cases}$$

By the transform (2.1.2), we have the Zariski open set of the hypersurface in $(\mathbb{P}^1)^5$, denoted by $D_{1,2}^{c_3 \neq 0}$. The locus $D_{1,2}^{c_3 \neq 0}$ is not irreducible. Now, we take the blowing up along the orbits of s_1 and s_2. Let $\widetilde{D}_{1,2}^{c_3 \neq 0}$ be the proper transform of $D_{1,2}^{c_3 \neq 0}$. Since an orbit of a point of

$$[t = 0] \setminus ([t = 0] \cap [s c_3^2 a_4 - t e_3^2 c_4 + \kappa_3 s c_3 e_3 c_4]) \subset D_{1,2}^{c_3 \neq 0} \tag{2.3.14}$$

are not closed, the points of the inverse image of (2.3.14) on $\widetilde{D}_{1,2}^{c_3 \neq 0}$ are unstable (see [**27**, Lemma 6.6]). Then, the quotient of $\widetilde{D}_{1,2}^{c_3 \neq 0}$ by the torus action is irreducible. Next, we consider the case where $c_4 \neq 0$. In the same way as in the case where $c_3 \neq 0$, we have the Zariski open set of the hypersurface in $(\mathbb{P}^1)^5$, denoted by $D_{1,2}^{c_4 \neq 0}$. We glue $D_{1,2}^{c_2 \neq 0}$, $D_{1,2}^{c_3 \neq 0}$ and $D_{1,2}^{c_4 \neq 0}$, denoted by $D'_{1,2}$. We take the proper transform of $D'_{1,2}$ of the blowing up along the orbits of s_1 and s_2, denoted by $\widetilde{D}'_{1,2}$. Then, the intersection of E_1 and E_2 is the quotient of $\widetilde{D}'_{1,2}$ by the torus action, denoted by $E_{1,2}$. The intersection $E_{1,2}$ is irreducible.

Second, we consider the intersection of E_1 and E_3. We substitute $e_3 = 0$ for (2.3.13). Then, we have the locus defined by the following equations

$$\begin{cases} a_3 + d_3 = 0 \\ a_3 d_3 - b_3 c_3 = 0 \\ a_4 + d_4 = \kappa_4 e_4 \\ a_4 d_4 - b_4 c_4 = e_4^2 \\ c_2 a_3 a_4 + \kappa_2 e_2 c_3 a_4 + c_2 b_3 c_4 + \kappa_2 e_2 d_3 c_4 = 0 \end{cases}$$

in $(\mathbb{P}^1 \times (\mathbb{P}^4)^2) \cap [c_2 \neq 0]$. We put $a_3 = st, b_3 = s^2, c_3 = -t^2, d_3 = -st$. Then, we have the equations

$$\begin{cases} a_4 + d_4 = \kappa_4 e_4 \\ a_4 d_4 - b_4 c_4 = e_4^2 \\ (t a_4 + s c_4)(c_2 s - \kappa_2 e_2 t) = 0. \end{cases} \tag{2.3.15}$$

We denote the two components $[ta_4 + sc_4 = 0]$ and $[c_2 s - \kappa_2 e_2 t = 0]$ by $D_{1,3}^{c_2 \neq 0,+}$ and $D_{1,3}^{c_2 \neq 0,-}$.

Remark 2.3.3. Any point (M_1, M_2, M_3, M_4) on $D_{1,3}^{c_2 \neq 0,+}$ is conjugate to the following matrices

$$\left(\left(\begin{array}{cc|c} 0 & 1 & \\ 0 & 0 & \\ \hline & & 0 \end{array}\right), \left(\begin{array}{cc|c} a_2 & b_2 & \\ c_2 & d_2 & \\ \hline & & e_2 \end{array}\right), \left(\begin{array}{cc|c} 0 & 0 & \\ 1 & 0 & \\ \hline & & 0 \end{array}\right), \left(\begin{array}{cc|c} 0 & b_4 & \\ c_4 & d_4 & \\ \hline & & e_4 \end{array}\right) \right). \tag{2.3.16}$$

In fact, we normalize the third matrix M_3 instead of M_2. Then, we have

$$M_3 = \left(\begin{array}{cc|c} 0 & 1 & \\ 0 & 0 & \\ \hline & & 0 \end{array}\right) \quad \text{or} \quad \left(\begin{array}{cc|c} 0 & 0 & \\ 1 & 0 & \\ \hline & & 0 \end{array}\right).$$

In the former case, by the stability, we have $c_4 \neq 0$. However, the matrices do not satisfy the condition (2.3.3). In the latter case, the equation $ta_4 + sc_4 = 0$ implies that $a_4 = 0$. On the other hand, any point on $D_{1,3}^{c_2 \neq 0,-}$ is conjugate to the following matrices

$$\left(\left(\begin{array}{cc|c} 0 & 1 & \\ 0 & 0 & \\ \hline & & 0 \end{array}\right), \left(\begin{array}{cc|c} a_2 & b_2 & \\ c_2 & 0 & \\ \hline & & e_2 \end{array}\right), \left(\begin{array}{cc|c} 0 & 0 & \\ 1 & 0 & \\ \hline & & 0 \end{array}\right), \left(\begin{array}{cc|c} a_4 & b_4 & \\ c_4 & d_4 & \\ \hline & & e_4 \end{array}\right) \right). \tag{2.3.17}$$

We consider the cases where $c_3 \neq 0$ and $c_4 \neq 0$. In the same way as in the case where $c_2 \neq 0$, we have the Zariski open sets

$$D_{1,3}^{c_3 \neq 0,+}, D_{1,3}^{c_3 \neq 0,-}, D_{1,3}^{c_4 \neq 0,+}, D_{1,3}^{c_4 \neq 0,-}$$

of the hypersurfaces in $(\mathbb{P}^1)^5$. We glue $D_{1,3}^{c_2 \neq 0,+}$, $D_{1,3}^{c_3 \neq 0,+}$ and $D_{1,3}^{c_4 \neq 0,+}$ (resp. $D_{1,3}^{c_2 \neq 0,-}$, $D_{1,3}^{c_3 \neq 0,-}$ and $D_{1,3}^{c_4 \neq 0,-}$), denoted by $'D_{1,3}^{+}$ (resp. $'D_{1,3}^{-}$). We take the blowing up (2.3.9). Let $'\widetilde{D}_{1,3}^{+}$ and $'\widetilde{D}_{1,3}^{-}$ be the proper transforms of $'D_{1,3}^{+}$ and $'D_{1,3}^{-}$, respectively. Then, the intersections of E_1 and E_3 are the quotients of $'\widetilde{D}_{1,3}^{+}$ and $'\widetilde{D}_{1,3}^{-}$ by the torus action, denoted by $E_{1,3}^{+}$ and $E_{1,3}^{-}$.

We consider the intersections $E_2 \cap E_3$, $E_3 \cap E_4$ and $E_1 \cap E_4$. In the same way as in the case $E_1 \cap E_2$, the intersections are irreducible, denoted by $E_{2,3}$, $E_{3,4}$ and $E_{1,4}$.

We consider the intersection of E_2 and E_4. In the same way as in the case $E_1 \cap E_3$, the intersection $E_2 \cap E_4$ is not irreducible. The intersection has two components, denoted by $E_{2,4}^{+}$ and $E_{2,4}^{-}$. Here, the components $E_{2,4}^{+}$ and $E_{2,4}^{-}$ correspond respectively to the following matrices

$$\left(\left(\begin{array}{cc|c} a_1 & b_1 & \\ c_1 & d_1 & \\ \hline & & e_1 \end{array}\right), \left(\begin{array}{cc|c} 0 & 1 & \\ 0 & 0 & \\ \hline & & 0 \end{array}\right), \left(\begin{array}{cc|c} a_3 & b_3 & \\ c_3 & 0 & \\ \hline & & e_3 \end{array}\right), \left(\begin{array}{cc|c} 0 & 0 & \\ 1 & 0 & \\ \hline & & 0 \end{array}\right) \right)$$

and

$$\left(\left(\begin{array}{cc|c} 0 & b_1 & \\ c_1 & d_1 & \\ \hline & & e_1 \end{array}\right), \left(\begin{array}{cc|c} 0 & 1 & \\ 0 & 0 & \\ \hline & & 0 \end{array}\right), \left(\begin{array}{cc|c} a_3 & b_3 & \\ c_3 & d_3 & \\ \hline & & e_3 \end{array}\right), \left(\begin{array}{cc|c} 0 & 0 & \\ 1 & 0 & \\ \hline & & 0 \end{array}\right) \right).$$

Now, we take the blowing up along the components $E_{1,3}^+, E_{1,3}^-, E_{2,4}^+, E_{2,4}^-$:

$$\widetilde{X} \longrightarrow X := \overline{\widetilde{\hat{\mathcal{M}}_B^{\boldsymbol{\mu}}(\boldsymbol{\kappa})}}.$$

We use the same notation E_i which is the proper transforms of E_i. We denote by $ex_{1,3}^+$, $ex_{1,3}^-$, $ex_{2,4}^+$, $ex_{2,4}^-$ the quotients of the exceptional divisors associated with this blowing up. Consequently, we have the ten components of the boundary divisor of the compactification $\widetilde{X}$ of $\hat{\mathcal{M}}_B^{\boldsymbol{\mu}}(\boldsymbol{\kappa})$

$$E_1, E_2, E_3, E_4, ex_1, ex_2, ex_{1,3}^+, ex_{1,3}^-, ex_{2,4}^+, ex_{2,4}^-,$$

and we obtain that the intersections

$$E_1 \cap E_2, \quad E_2 \cap E_3, \quad E_3 \cap E_4, \quad E_4 \cap E_1$$

and

$$E_1 \cap ex_{1,3}^\pm,\ E_3 \cap ex_{1,3}^\pm,\ E_2 \cap ex_{2,4}^\pm,\ E_4 \cap ex_{2,4}^\pm$$

are nonempty and irreducible.

We describe the intersections of the other pairs. We consider the intersection of $ex_{1,3}^+$ and E_4. If we substitute $e_4 = 0$ for the matrix (2.3.16), then we have $d_4 = 0$. Moreover, we have $b_4 = 0$ or $c_4 = 0$. Then, we obtain that

$$'D_{1,3}^+ \cap [e_4 = 0] = \{s_1, s_2\} \cup [\text{points whose orbits are not closed}].$$

By the blowing up along s_1 and s_2, we obtain that the intersection of $E_{1,3}^+$ and E_4 is empty (see [**27**, Lemma 6.6]). Then, the intersection of $ex_{1,3}^+$ and E_4 is empty. In the same way as above, the intersections

$$ex_{1,3}^- \cap E_2, \quad ex_{2,4}^+ \cap E_3, \quad ex_{2,4}^- \cap E_1$$

are empty. On the other hand, the intersections

$$ex_{1,3}^+ \cap E_2,\ ex_{1,3}^- \cap E_4,\ ex_{2,4}^+ \cap E_1,\ ex_{2,4}^- \cap E_3,\ ex_{1,3}^+ \cap ex_{1,3}^-,\ ex_{2,4}^+ \cap ex_{2,4}^-$$

are nonempty and irreducible. Next, we consider the intersections of the pairs containing ex_1 or ex_2. The orbit of the point s_1 (resp. s_2) is contained in the components $D_1, \ldots, D_4$ and $D_{1,3}^\pm, D_{2,4}^\pm$, respectively. Here, $D_{1,3}^\pm$ and $D_{2,4}^\pm$ are the irreducible components of $D_{1,3}$ and $D_{2,4}$. Then, the intersections $ex_i \cap E_j$ and $ex_i \cap ex_{k,k+2}^\pm$ are nonempty and irreducible for $i = 1, 2$, $j = 1, \ldots, 4$ and $k = 1, 2$. On the other hand, the orbits of the point s_1 and s_2 are not intersect. Then, the intersection of ex_1 and ex_2 is empty.

Step 3. We draw the vertexes and the 1-dimensional simplices except ex_1 and ex_2. Then, we obtain the following figure We consider the following sphere

$$\mathbb{R}^4 \supset S^3 = \{(x, y, z, w) \in \mathbb{R}^4 \mid x^2 + y^2 + z^2 + w^2 = 1\}.$$

We arrange the vertexes except ex_1 and ex_2 on $S^2 = S^3 \cap [w = 0]$ and arrange the vertexes ex_1 and ex_2 at $(0, 0, 0, 1)$ and $(0, 0, 0, -1)$ respectively. We glue together the vertex ex_i $(i = 1, 2)$ and each vertex on $S^2 = S^3 \cap [w = 0]$.

Next, we describe the 2-dimensional simplices. First, we consider the intersections $E_1 \cap E_2 \cap ex_{1,3}^+$ and $E_2 \cap E_3 \cap ex_{1,3}^+$. The intersection $E_1 \cap E_2 \cap E_3 = E_{1,3}^+ \cap E_2$ is nonempty and irreducible in $\overline{\widetilde{\hat{\mathcal{M}}_B^{\boldsymbol{\mu}}(\boldsymbol{\kappa})}}$. We take the blowing up along $E_{1,3}^+$. Then, the intersections $E_1 \cap E_2 \cap ex_{1,3}^+$ and $E_2 \cap E_3 \cap ex_{1,3}^+$ are irreducible. Second, we consider the intersections

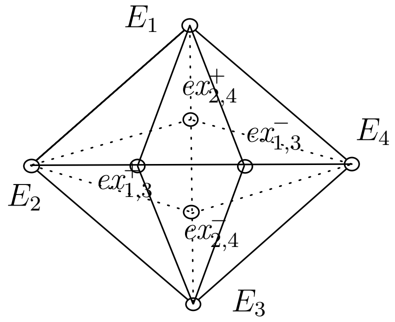


FIGURE 3

$E_1 \cap ex^+_{1,3} \cap ex^-_{1,3}$ and $E_3 \cap ex^+_{1,3} \cap ex^-_{1,3}$. We substitute $d_2 = 0$ for the matrices (2.3.16). Then, we have that $D^{c_2 \neq 0,+}_{1,3} \cap [d_2 = 0]$ is irreducible. Therefore, the intersection $E^+_{1,3} \cap E^-_{1,3}$ is irreducible. We take the blowing up along $E^+_{1,3}$. Then, the intersections $E_1 \cap ex^+_{1,3} \cap ex^-_{1,3}$ and $E_3 \cap ex^+_{1,3} \cap ex^-_{1,3}$ are irreducible. Then, we glue together the triangles

$$(E_1, E_2, ex^+_{1,3}), (E_2, E_3, ex^+_{1,3}), (E_1, ex^+_{1,3}, ex^-_{1,3}) \text{ and } (E_3, ex^+_{1,3}, ex^-_{1,3})$$

in the graph of Figure 3. In the same way as above, we glue together each triangle. Then, we obtain that the complex of Figure 3 is a simplicial decomposition of S^2. Third, we consider the intersection of 3-tuple of components of the boundary divisor containing ex_1 or ex_2. The divisors ex_1 and ex_2 are the exceptional divisors of the blowing up along the orbits of s_1 and s_2. The orbits of s_1 and s_2 are contained in $D_i \cap D_{i+1}$ $(i = 1, \dots, 4)$, $D^+_{1,3}$, $D^-_{1,3}$, $D^+_{2,4}$ and $D^-_{2,4}$, respectively. Then, the intersections $E_i \cap E_{i+1} \cap ex_j$, $E^+_{k,k+2} \cap ex_j$ and $E^-_{k,k+2} \cap ex_j$ are nonempty and irreducible for $i = 1, \dots, 4$, $j = 1, 2$, and $k = 1, 2$. We take the blowing up along $E^+_{1,3}$ and $E^-_{1,3}$. Then, we can glue together the 3-tuples which have either ex_i or ex_i in the graph.

Lastly, we describe the 3-dimensional simplices. We can glue together the 4-tuples of components of the boundary divisor such that the 4-tuples have either ex_i or ex_i and 3-tuples expect ex_i or ex_i are glued together. On the other hand, the intersections of the 4-tuples which have the vertexes expect ex_i or ex_i are empty. Then, we obtain that the boundary complex of the compactification $\widetilde{X}$ of $\hat{\mathcal{M}}^{\boldsymbol{\mu}}_B(\boldsymbol{\kappa})$ is a simplicial decomposition of S^3. $\square$

Bibliography

[1] A. B. Altman, S. L. Kleiman, *Compactifying the Picard scheme.* Adv. in Math. **35** (1980), no. 1, 50–112.

[2] D. Arinkin, *Orthogonality of natural sheaves on moduli stacks of* SL(2)*-bundles with connections on* $\mathbb{P}_1$ *minus 4 points.* Selecta Math. (N.S.) **7** (2001), no. 2, 213–239.

[3] M. S. M. Benjamin, *Compactifications of a representation variety*, J. Group Theory **14** (2011), no. 6, 947-963.

[4] M. A. A. de Cataldo, T. Hausel, L. Migliorini, *Topology of Hitchin systems and Hodge theory of character varieties: the case* A_1, Ann. of Math. **175** (2012), no. 3, 1329-1407.

[5] P. Deligne, *Équations différentielles à points singuliers réguliers.* Lecture Notes in Mathematics, Vol. **163**. Springer-Verlag, Berlin-New York, 1970.

[6] P. Deligne, *Theorie de Hodge. II.* Inst. Hautes Études Sci. Publ. Math. No. **40** (1971), 5–57.

[7] P. Deligne, *Theorie de Hodge. III.* Inst. Hautes Études Sci. Publ. Math. No. **44** (1974), 5–77.

[8] P. Etingof, W. L. Gan, A. Oblomkov, *Generalized double affine Hecke algebras of higher rank.* J. Reine Angew. Math. **600** (2006), 177–201.

[9] E. Formanek, *The invariants of* $n \times n$ *matrices. Invariant theory*, 18-43, Lecture Notes in Math., 1278, Springer, Berlin, 1987.

[10] R. Fricke, F. Klein, *Vorlesungen uber die Theorie der automorphen Funktionen. Band 1: Die gruppentheoretischen Grundlagen. Band II: Die funktionentheoretischen Ausfuhrungen und die Andwendungen.* Bibliotheca Mathematica Teubneriana, Bande 3, 4 Johnson Reprint Corp., New York; B. G. Teubner Verlagsgesellschaft, Stuttg art 1965 Band I: xiv+634 pp.; Band II: xiv+668 pp.

[11] R. D. Friedman, *Hodge theory, degenerations, and the Global Torelli problem.* Thesis (Ph.D.)-Harvard University. (1981).

[12] P. A. Griffiths, *Periods of integrals on algebraic manifolds. I, II.* Amer. J. Math. **90** (1968) 568–626 and 805–865.

[13] P. A. Griffiths, J. Harris, *Principles of algebraic geometry.* Pure and Applied Mathematics. Wiley-Interscience [John Wiley & Sons], New York, (1978).

[14] P. A. Griffiths, W. Schmid, *Recent developments in Hodge theory: a discussion of techniques and results.* Discrete subgroups of Lie groups and applicatons to moduli (Internat. Colloq., Bombay, 1973), pp. 31–127. Oxford Univ. Press, Bombay, 1975.

[15] T. Hausel, E. Letellier, F. Rodriguez-Villegas, *Arithmetic harmonic analysis on character and quiver varieties*, Duke Math. Journal, vol. **160** (2011) 323–400.

[16] T. Hausel, E. Letellier, F. Rodriguez-Villegas, *Arithmetic harmonic analysis on character and quiver varieties II*, Adv. Math. **234** (2013), 85–128.

[17] T. Hausel, F. Rodriguez-Villegas, *Mixed Hodge polynomials of character varieties*, with an appendix by N. M. Katz, Invent. Math. **174** (2008), 555–624.

[18] T. Hausel and M. Thaddeus, *Mirror symmetry, Langlands duality, and Hitchin system*, Invent. Math. **153** (2003), 197–229.

[19] T. Hausel, M. Thaddeus, *Generators for the cohomology ring of the moduli space of rank 2 Higgs bundles.* Proc. London Math. Soc. (3) **88** (2004), no. 3, 632–658.

[20] D. Huybrechts, M. Lehn, *The geometry of moduli spaces of sheaves.* Second edition. Cambridge Mathematical Library. Cambridge University Press, Cambridge, 2010. xviii+325 pp.

[21] M.-A. Inaba, *Moduli of parabolic connections on curves and the Riemann-Hilbert correspondence.* J. Algebraic Geom. **22** (2013), no. 3, 407–480.

[22] M.-A. Inaba, M.-H. Saito, *Moduli of regular singular parabolic connection of spectral type on smooth projective curves.* in preparation.

[23] M.-A. Inaba, K. Iwasaki, M.-H. Saito, *Moduli of stable parabolic connections, Riemann-Hilbert correspondence and geometry of Painlevé equation of type VI. I* . Publ. Res. Inst. Math. Sci. **42** (2006), no. 4, 987–1089.

[24] M.-A. Inaba, K. Iwasaki, M.-H. Saito, *Moduli of stable parabolic connections, Riemann-Hilbert correspondence and geometry of Painleve equation of type VI. II.* Moduli spaces and arithmetic geometry, 387–432, Adv. Stud. Pure Math., **45**, Math. Soc. Japan, Tokyo, 2006.

[25] K. Iwasaki, *An area-preserving action of the modular group on cubic Surfaces and the Painlevé VI Equations* , Comm. Math. Phys., **242** (2003), 185-219.

[26] M. Jimbo, *Monodromy problem and the boundary condition for some Painlevé equations.* Publ. Res. Inst. Math. Sci. **18** (1982), no. 3, 1137-1161.

[27] F. Kirwan, *Partial desingularisations of quotients of nonsingular varieties and their Betti numbers*, Ann. of Math. **122** (1985), no. 1, 41-85.

[28] A. Komyo, *On compactifications of character varieties of n-punctured projective line*, to appear in Ann. Inst. Fourier (Grenoble), **64**, (2014).

[29] H. Konno, *Construction of the moduli space of stable parabolic Higgs bundles on a Riemann surface.* J. Math. Soc. Japan **45** (1993), no. 2, 253–276.

[30] S. Lawton, *Generators, relations and symmetries in pairs of 3×3 unimodular matrices* , J. Algebra **313** (2) (2007) 782-801.

[31] E. Looijenga, *Rational surfaces with an anticanonical cycle.* Ann. of Math. (2) **114** (1981), no. 2, 267–322.

[32] F. Loray, M.-H. Saito, C. Simpson, *Foliations on the moduli space of rank two connections on the projective line minus four points*, Seminaire et Congres **27** (2012), 115–168.

[33] S. Mac Lane, *Categories for the working mathematician.* Second edition. Graduate Texts in Mathematics, 5. Springer-Verlag, New York, 1998. xii+314 pp.

[34] S. Mukai, *An introduction to invariants and moduli.* Translated from the 1998 and 2000 Japanese editions by W. M. Oxbury. Cambridge Studies in Advanced Mathematics, **81**. Cambridge University Press, Cambridge, 2003. xx+503 pp.

[35] D. Mumford, J. Fogarty, F. Kirwan, *Geometric invariant theory*, Third edition. Ergebnisse der Mathematik und ihrer Grenzgebiete (2) , 34. Springer-Verlag, Berlin, 1994.

[36] P. E. Newstead, *Introduction to moduli problems and orbit spaces.* Tata Institute of Fundamental Research Lectures on Mathematics and Physics, **51**. Tata Institute of Fundamental Research, Bombay; by the Narosa Publishing House, New Delhi, 1978. vi+183 pp.

[37] S. Payne, *Boundary complexes and weight filtrations.* arXiv:1109.4286.

[38] C. Procesi, *The invariant theory of $n \times n$ matrices*, Advances in Math. **19** (1976), no. 3, 306-381.

[39] C. Sabbah, *Isomonodromic deformations and Frobenius manifolds. An introduction.* Translated from the 2002 French edition. Universitext. Springer-Verlag London, Ltd., London; EDP Sciences, Les Ulis, 2007. xiv+279 pp.

[40] M.-H. Saito, T. Takebe and H. Terajima, *Deformation of Okamoto-Painlevé pairs and Painlevé equations*, J. Algebraic Geom. **11** (2002), no. 2, 311–362.

[41] H. Sakai, *Rational surfaces associated with affine root systems and geometry of the Painlevé equations.* Comm. Math. Phys. **220** (2001), no. 1, 165–229.

[42] Y. Shimizu, K. Ueno, *Advances in moduli theory.* Translated from the 1999 Japanese original. Translations of Mathematical Monographs, **206**. Iwanami Series in Modern Mathematics. American Mathematical Society, Providence, RI, 2002. xx+300 pp.

[43] C. L. Siegel *Topics in Complex Function Theory.* (3 vol.), New York, John Wiley, (1969).

[44] C. Simpson, *Harmonic bundles on noncompact curves.* J. Amer. Math. Soc. **3** (1990), no. 3, 713–770.

[45] C. T. Simpson, *Nonabelian Hodge theory.* Proceedings of the International Congress of Mathematicians, Vol. I, II (Kyoto, 1990), 747-756, Math. Soc. Japan, Tokyo, 1991.

[46] C. Simpson, *Moduli of representations of the fundamental group of a smooth projective variety. I*, Inst. Hautes Études Sci. Publ. Math. No. **79** (1994), 47-129.

[47] C. Simpson, *Moduli of representations of the fundamental group of a smooth projective variety. II*, Inst. Hautes Études Sci. Publ. Math. No. **80** (1994), 5-79 (1995).

[48] C. T. Simpson, *The Hodge filtration on nonabelian cohomology.* Algebraic geometry–Santa Cruz 1995, 217-281, Proc. Sympos. Pure Math., **62**, Part 2, Amer. Math. Soc., Providence, RI, 1997.

[49] C. T. Simpson, *A weight two phenomenon for the moduli of rank one local systems on open varieties.* From Hodge theory to integrability and TQFT tt^*-geometry, 175-214, Proc. Sympos. Pure Math., **78**, Amer. Math. Soc., Providence, RI, 2008.

[50] C. T. Simpson, *Iterated destabilizing modifications for vector bundles with connection.* Vector bundles and complex geometry, 183-206, Contemp. Math., **522**, Amer. Math. Soc., Providence, RI, 2010.

[51] C. Simpson, *Towards the boundary of the character variety*, Recent progress in the theory of Painlevé equations: algebraic, asymptotic and topological aspects, CNRS-JSPS, IRMA, Strasbourg, 4–8 november 2013.

[52] D. A. Stepanov, *A remark on the dual complex of a resolution of singularities.* Uspekhi Mat. Nauk **61** (2006), no. 1(367), 185–186.

[53] A. Thuillier, *Géométrie toroïdale et géométrie analytique non archimédienne. Application au type d'homotopie de certains schémas formels.* Manuscripta Math. **123** (2007), no. 4, 381–451.

[54] K. Yokogawa, *Compactification of moduli of parabolic sheaves and moduli of parabolic Higgs sheaves.* J. Math. Kyoto Univ. **33** (1993), no. 2, 451–504.