



Padé method to q-Painlevé equations

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PADÉ METHOD TO q -PAINLEVÉ EQUATIONS

HIDEHITO NAGAO

ABSTRACT. Recently, we studied Padé interpolation problems of the q -grid, related to all the q -Painlevé equations of types from $E_7^{(1)}$ to $(A_2 + A_1)^{(1)}$. Then, by solving those problems, we could derive the evolution equations, the scalar Lax pairs and the determinant formulae of special solutions for the corresponding q -Painlevé equations. We can choose another grid: the differential grid, namely the usual Padé approximation. In this paper, we will review the Padé interpolation method of the q -grid, and apply the Padé approximation method, related to all the q -Painlevé equations of types from $E_6^{(1)}$ to $(A_2 + A_1)^{(1)}$.

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1. INTRODUCTION

1.1. The background of differential Painlevé equations.

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In this subsection, we will give a rough explanation of the background of differential Painlevé equations. For the general background, we refer the reader to [2] [12] [15] [38] [42] [55].

1.1.1. *Two origins.*

It is well known that the differential Painlevé equations have two independent origins.

The first is the "Painlevé property" of second order nonlinear ordinary differential equations. In original works [44] and [4], P. Painlevé and B. Gambier discovered the Painlevé equations as the differential equations whose solutions do not have movable singularities other than poles. This property, now called the Painlevé property, has become a guiding principle in the theory of integrable systems.

The second is the "monodromy preserving deformation" of second order linear ordinary differential equations (see [3] [17] [18] [19]). In [3], R. Fuchs arrived at the sixth Painlevé equation from a completely different problem, namely the deformation theory of linear equations. Going into more detail, we see that the sixth equation appears as the condition for moving the coefficients of the second order Fuchsian equation having four regular singularities without changing its monodromy.

As integrable systems, differential Painlevé equations have been studied from various points of view. For example, Lax pairs, special solutions, Bäcklund transformations, bilinear forms, degenerations between the discrete Painlevé equations, spaces of initial conditions, higher order system, Garnier (multivariable) system and quantum system, etc.

1.1.2. *The classification.*

In the first origin, we illustrate the differential Painlevé equations. We consider a second order nonlinear ordinary differential equation

$$(1.1) \quad \lambda'' = R(t; \lambda, \lambda')$$

for the dependent variable $\lambda = \lambda(t)$, where $R(t; \lambda, \eta)$ is a rational function in (t, λ, η) and the prime $' = d/dt$ is the derivation with respect to the independent variable t . The differential equation (1.1) is said to have the Painlevé property, if any solution of (1.1) has no movable singular point except for poles.

As is well known, any rational differential equation of second order having the Painlevé property is reduced to one of the six Painlevé equations P_J ($J = \text{I, II, III, IV, V, VI}$), unless it can be algebraically integrated, or transformed into the linear differential equations or the differential equations of the elliptic functions. The six Painlevé equations are as follows:

$$\begin{aligned}
(1.2) \quad & P_I : \lambda'' = 6\lambda^2 + t, \\
& P_{II} : \lambda'' = 2\lambda^3 + t\lambda + \alpha, \\
& P_{III} : \lambda'' = \frac{1}{\lambda}(\lambda')^2 - \frac{1}{t}\lambda' + \frac{1}{t}(\alpha\lambda^2 + \beta) + \gamma\lambda^3 + \frac{\delta}{\lambda}, \\
& P_{IV} : \lambda'' = \frac{1}{2\lambda}(\lambda')^2 + \frac{3}{2}\lambda^3 + 4t\lambda^2 + 2(t^2 - \alpha)\lambda + \frac{\beta}{\lambda}, \\
& P_V : \lambda'' = \left(\frac{1}{2\lambda} + \frac{1}{\lambda-1}\right)(\lambda')^2 - \frac{1}{t}\lambda' + \frac{(\lambda-1)^2}{t^2}\left(\alpha\lambda + \frac{\beta}{\lambda}\right) + \frac{\gamma}{t}\lambda + \delta\frac{\lambda(\lambda+1)}{\lambda-1}, \\
& P_{VI} : \lambda'' = \frac{1}{2}\left(\frac{1}{\lambda} + \frac{1}{\lambda-1} + \frac{1}{\lambda-t}\right)(\lambda')^2 - \left(\frac{1}{t} + \frac{1}{t-1} + \frac{1}{\lambda-t}\right)\lambda' \\
& \quad + \frac{\lambda(\lambda-1)(\lambda-t)}{t^2(t-1)^2}\left(\alpha + \beta\frac{t}{\lambda^2} + \gamma\frac{t-1}{(\lambda-1)^2} + \delta\frac{t(t-1)}{(\lambda-t)^2}\right).
\end{aligned}$$

Here, $\alpha, \beta, \dots$ are complex parameters.

We remark that the six Painlevé equations are related by the following degeneration diagram (or the coalescence cascade):

$$(1.3) \quad \begin{array}{ccccc} P_{VI} & \rightarrow & P_V & \rightarrow & P_{III} \\ & & \searrow & & \searrow \\ & & & P_{IV} & \rightarrow P_{II} \rightarrow P_I. \end{array}$$

In section 2, we will consider the differential Painlevé equations of types from VI to IV. The main results of this section are the differential equations, the scalar Lax pairs and the determinant formulae of special solutions for the corresponding differential Painlevé equations.

1.1.3. Special solutions.

Generic solutions of the differential Painlevé equations are known to be very transcendental (See Umemura's survey [55]). Also, these equations are known to have hypergeometric solutions and algebraic solutions.

We recall the definition of the hypergeometric series [5] [6]. The hypergeometric functions are defined by

$$\begin{aligned}
(1.4) \quad & {}_rF_s \left(\begin{matrix} a_1, & \dots, & a_r \\ b_1, & \dots, & b_s \end{matrix} ; x \right) = \sum_{n=0}^{\infty} \frac{(a_1, \dots, a_r)_n}{(b_1, \dots, b_s)_n} \frac{x^n}{n!}, \\
& (a_1, a_2, \dots, a_i)_j = \prod_{k=1}^i a_k(a_k + 1)(a_k + 2) \cdots (a_k + j - 1).
\end{aligned}$$

The degeneration diagram (1.3) of the Painlevé equations is now mapped to the confluence diagram of their hypergeometric solutions

$$(1.5) \quad \begin{array}{ccccc} \text{Gauss} & \rightarrow & \text{Kummer} & \rightarrow & \text{Bessel} \\ {}_2F_1 & & {}_1F_1 & & \\ & & \searrow & & \searrow \\ & & \text{Hermite} - \text{Weber} & \rightarrow & \text{Airy} \rightarrow \dots \\ & & {}_2F_0 & & \end{array}$$

Besides the hypergeometric solutions along the walls, the Painlevé equations occur to have algebraic solutions. They typically arise when the parameters take values corresponding to some fixed point of the extended affine Weyl group. For the algebraic solutions of the Painlevé

equations up to P_{VI} , we refer the reader [11] [21] [29] [34].

1.2. The background of discrete Painlevé equations.

In this subsection, we will roughly explain the background of discrete Painlevé equations. For the general background, we refer the reader to [7] [9] [39] [50] [54].

1.2.1. The origin.

Discrete Painlevé equations are discrete equations which reduce to differential Painlevé equations by a suitable limiting process. In the pioneering works of Grammaticos, Ramani, Papageorgiou and Hietarinta [10] [47] in early 1990s, discrete Painlevé equations were discovered through the studies of the singularity confinement property, called discrete analogue of the Painlevé property.

As integrable systems, as well as the differential Painlevé equations, discrete (or difference) Painlevé equations have been studied from various points of view. For example, Lax pairs, special solutions, Bäcklund transformations, bilinear forms, degenerations between the discrete Painlevé equations, spaces of initial conditions, higher order system, Garnier (multivariable) system and quantum system, etc.

1.2.2. Sakai's classification.

As for discrete Painlevé equations of second order, it seems to be standard nowadays to refer to Sakai's classification [50] defined by means of geometry of rational surfaces. Sakai's framework not only fits nicely with discrete Painlevé equations discovered earlier by different approaches, but also clarifies their connection with the geometry of plane curves and Cremona transformations. Each equation in this class is defined by an affine Weyl group of Cremona transformations on a certain family of rational surfaces obtained from P_2 by blowing up.

Sakai's list of discrete Painlevé equations consists of elliptic (e -), multiplicative (q -) and additive (d -) difference equations. The single elliptic difference Painlevé equations is associated with the affine Weyl group of type $E_8^{(1)}$. The diagrams of affine Weyl groups for the single e -Painlevé, for the q -Painlevé equations and for the d -Painlevé equations are given as follows, respectively.

(1.6)

$$\begin{array}{ccccccccccc}
 \text{ell.}(e-) & E_8^{(1)} & & & & & & & & & \mathbb{Z} \\
 & & & & & & & & & \nearrow & \\
 \text{mul.}(q-) & E_8^{(1)} \rightarrow E_7^{(1)} \rightarrow E_6^{(1)} \rightarrow D_5^{(1)} \rightarrow A_4^{(1)} \rightarrow (A_2 + A_1)^{(1)} \rightarrow (A_1 + A_1')^{(1)} \rightarrow A_1^{(1)} \rightarrow \mathcal{D}_6 \\
 & & & (q-P_{VI}) & (q-P_V) & (q-P_{IV}, q-P_{III}) & (q-P_{II}) & (q-P_I) & & & \\
 \text{add.}(d-) & E_8^{(1)} \rightarrow E_7^{(1)} \rightarrow E_6^{(1)} & \rightarrow & D_4^{(1)} & \rightarrow & A_3^{(1)} & \rightarrow (A_1 + A_1')^{(1)} \rightarrow A_1^{(1)} \rightarrow \mathbb{Z}_2 \\
 & & & (P_{VI}) & & (P_V) & (P_{III}) & & & \searrow & \downarrow \\
 & & & & & & & & & A_2^{(1)} & \rightarrow A_1^{(1)} \rightarrow 1 \\
 & & & & & & & & & (P_{IV}) & (P_{II}) & (P_I)
 \end{array}$$

We remark two points. Firstly, the d -Painlevé equations of type $D_4^{(1)}$ and its degeneration arise as Bäcklund (Schlesinger) transformations of differential Painlevé equations, namely P_J ($J = \text{I, II, III, IV, V, VI}$). Secondly, there are the symbols (q - P_J) under the q -Painlevé equation of types from $D_5^{(1)}$ to $A_1^{(1)}$ in the classification (1.6). The symbols have traditionally been used before Sakai's classification. For example, the q -Painlevé equation of type $D_5^{(1)}$ is the q -Painlevé VI equation of Jimbo-Sakai [20]. other q -Painlevé equation are similar.

For the forms of all the discrete Painlevé equations, we refer the reader to [50] [53]. For example, the q -Painlevé equations of types from $E_8^{(1)}$ to $(A_1 + A_1)^{(1)}$ are as follows:

The q -Painlevé equation of type $E_8^{(1)}$ ([23] [43] [50] [53]) is

$$(1.7) \quad \frac{(\bar{g}st - f)(gst - f) - (\bar{s}^2t^2 - 1)(s^2t^2 - 1)}{\left(\frac{\bar{g}}{\bar{s}t} - f\right)\left(\frac{g}{st} - f\right) - \left(1 - \frac{1}{\bar{s}^2t^2}\right)\left(1 - \frac{1}{s^2t^2}\right)} = \frac{P(f, t, m_1, \dots, m_7)}{P(f, t^{-1}, m_7, \dots, m_1)},$$

$$\frac{(fst - g)(fst - g) - (s^2\bar{t}^2 - 1)(s^2t^2 - 1)}{\left(\frac{f}{st} - g\right)\left(\frac{f}{st} - g\right) - \left(1 - \frac{1}{s^2\bar{t}^2}\right)\left(1 - \frac{1}{s^2t^2}\right)} = \frac{P(g, s, m_7, \dots, m_1)}{P(g, s^{-1}, m_1, \dots, m_7)},$$

where

$$(1.8) \quad P(f, t, m_1, \dots, m_7) = f^4 - m_1tf^3 + (m_2t^2 - 3 - t^8)f^2 + (m_7t^7 - m_3t^3 + 2m_1t)f + (t^8 - m_6t^6 + m_4t^4 - m_2t^2 + 1),$$

and m_k ($k = 1, 2, \dots, 7$) are the elementary symmetric functions of k -th degree in b_i ($i = 1, 2, \dots, 8$) with

$$(1.9) \quad b_1b_2 \cdots b_8 = 1.$$

Moreover,

$$(1.10) \quad \bar{t} = qt, \quad t = q^{1/2}s.$$

The q -Painlevé equation of type $E_7^{(1)}$ ([8] [23] [48] [50] [53]) is

$$(1.11) \quad \begin{cases} \frac{(\bar{g}f - t\bar{t})(gf - t^2)}{(\bar{g}f - 1)(gf - 1)} = \frac{(f - b_1t)(f - b_2t)(f - b_3t)(f - b_4t)}{(f - b_5)(f - b_6)(f - b_7)(f - b_8)}, \\ \frac{(gf - t^2)(g\bar{f} - t\bar{t})}{(gf - 1)(g\bar{f} - 1)} = \frac{(g - t/b_1)(g - t/b_2)(g - t/b_3)(g - t/b_4)}{(g - 1/b_5)(g - 1/b_6)(g - 1/b_7)(g - 1/b_8)}, \end{cases}$$

where

$$(1.12) \quad \bar{t} = qt, \quad b_1b_2b_3b_4 = q, \quad b_5b_6b_7b_8 = 1.$$

The q -Painlevé equation of type $E_6^{(1)}$ ([14] [23] [48] [50] [53]) is

$$(1.13) \quad \begin{cases} (\bar{g}f - 1)(gf - 1) = t\bar{t} \frac{(f - b_1)(f - b_2)(f - b_3)(f - b_4)}{(f - b_5t)(f - t/b_5)}, \\ (gf - 1)(g\bar{f} - 1) = t^2 \frac{(g - 1/b_1)(g - 1/b_2)(g - 1/b_3)(g - 1/b_4)}{(g - b_6t)(g - t/b_6)}, \end{cases}$$

where

$$(1.14) \quad \bar{t} = qt, \quad b_1b_2b_3b_4 = 1.$$

The q -Painlevé equation of type $D_5^{(1)}$ (q - P_{VI} equation) ([20] [23] [50] [53]) is

$$(1.15) \quad \begin{cases} \bar{g}g = \frac{(f - a_1t)(f - a_2t)}{(f - a_3)(f - a_4)}, \\ f\bar{f} = \frac{(g - b_1t/q)(g - b_2t/q)}{(g - b_3)(g - b_4)}, \end{cases}$$

where

$$(1.16) \quad \frac{b_1b_2}{b_3b_4} = q \frac{a_1a_2}{a_3a_4}.$$

The q -Painlevé equation of type $A_4^{(1)}$ (q - P_V equation) ([23] [28] [50] [53]) is

$$(1.17) \quad \begin{cases} \bar{g}g = \frac{(f + b_1/t)(f + 1/b_1t)}{1 + b_3f}, \\ f\bar{f} = \frac{(g + b_2/s)(g + 1/b_2s)}{1 + g/b_3}, \end{cases}$$

where

$$(1.18) \quad \bar{t} = qt, \quad t = q^{\frac{1}{2}}s.$$

The q -Painlevé equation of type $(A_2 + A_1)^{(1)}$ (q - P_{IV} equation) ([26] [50] [53]) is

$$(1.19) \quad \begin{cases} \bar{f}f = -\left(a_2b_0 - \frac{a_0b_1}{g}\right)(1 - g), \\ a_1b_0\bar{g}g = -\frac{\bar{f} - a_2b_0}{\bar{f} + a_2}, \end{cases}$$

where

$$(1.20) \quad q = a_1a_2a_0 = b_1b_2.$$

The q -Painlevé equation of type $(A_2 + A_1)^{(1)}$ (q - P_{III} equation) ([23] [46] [48] [50] [53]) is

$$(1.21) \quad \begin{cases} \bar{g}gf = b_0 \frac{1 + a_0tf}{a_0t + f}, \\ gff = b_0 \frac{a_1/t + g}{1 + ga_1/t}, \end{cases}$$

where

$$(1.22) \quad \bar{t} = qt.$$

The q -Painlevé equation of type $(A_1 + A_1')^{(1)}$ (q - P_{II} equation) ([23] [28] [50]) is

$$(1.23) \quad (\bar{f}f - 1)(f\bar{f} - 1) = \frac{at^2f}{f + t},$$

where

$$(1.24) \quad \bar{t} = qt.$$

In sections 3 and 4, we will consider the q -Painlevé equations of types from $E_7^{(1)}$ to $(A_2 + A_1)^{(1)}$. The main results of these section are the evolution equations, the scalar Lax pairs and the determinant formulae of special solutions for the corresponding q -Painlevé equations ((1.11),

(1.13), (1.15), (1.17), (1.19), (1.21)).

1.2.3. Hypergeometric solutions of the q -Painlevé equations.

The q -Painlevé equations are known to have hypergeometric solutions (e.g. [23] [39] [48]).

We recall the definition of the q -hypergeometric series [5] [6]. The q -hypergeometric series ${}_r\varphi_s$ is defined by

$$(1.25) \quad {}_r\varphi_s \left(\begin{matrix} a_1, \dots, a_r \\ b_1, \dots, b_s \end{matrix}; q, z \right) = \sum_{n=0}^{\infty} \frac{(a_1; q)_n \cdots (a_r; q)_n}{(b_1; q)_n \cdots (b_s; q)_n (q; q)_n} \left[(-1)^n q^{\binom{n}{2}} \right]^{1+s-r} z^n,$$

$$(a; q)_n = (1-a)(1-qa) \cdots (1-q^{n-1}a).$$

The q -hypergeometric series ${}_{r+1}\varphi_r$ is called *balanced* if the condition

$$(1.26) \quad qa_1 a_2 \cdots a_{r+1} = b_1 b_2 \cdots b_r, \quad z = q$$

is satisfied, and is called *well-poised* if the condition

$$(1.27) \quad qa_1 = a_2 b_1 = \cdots = a_{r+1} b_r$$

is satisfied. Moreover, it is called *very-well-poised* if it satisfies

$$(1.28) \quad a_2 = qa_1^{1/2}, \quad a_3 = -qa_1^{1/2},$$

in addition to equation (1.27), and denoted as ${}_{r+1}W_r$:

$$(1.29) \quad {}_{r+1}W_r(a_1; a_4, \dots, a_{r+1}; q, z) = {}_r\varphi_s \left(\begin{matrix} a_1, qa_1^{1/2}, -qa_1^{1/2}, a_4, \dots, a_{r+1} \\ a_1^{1/2}, -a_1^{1/2}, qa_1/a_4, \dots, qa_1/a_{r+1} \end{matrix}; q, z \right).$$

As reviewed in [39], it is shown explicitly in [23] [24] that q -hypergeometric functions in Table (1.30) arise as Riccati solutions to the q -Painlevé equations of each type from $q-E_8^{(1)}$ to $q-(A_1 + A'_1)$.

	Weyl Group symmetry	q -Hypergeometric function	Terminating case
(1.30)	$q-E_8^{(1)}$	Balanced ${}_{10}W_9$	Ismail-Masson-Rahman
	$q-E_7^{(1)}$	${}_8W_7, {}_4\varphi_3$	Askey-Wilson polynomials
	$q-E_6^{(1)}$	${}_3\varphi_2$	Big q -Jacobi polynomials
	$q-D_5^{(1)} (q-P_{VI})$	${}_2\varphi_1$	Little q -Jacobi polynomials
	$q-A_4^{(1)} (q-P_V)$	${}_1\varphi_1, {}_2\varphi_1 \left(\begin{smallmatrix} a, b \\ 0 \end{smallmatrix}; q, z \right)$	q -Laguerre polynomials
	$q-(A_2 + A_1)^{(1)} (q-P_{III,IV})$	${}_1\varphi_1 \left(\begin{smallmatrix} a \\ 0 \end{smallmatrix}; q, z \right), {}_1\varphi_1 \left(\begin{smallmatrix} 0 \\ b \end{smallmatrix}; q, z \right)$	Stieltjes-Wigert polynomials
	$q-(A_1 + A'_1)^{(1)} (q-P_{II})$	${}_1\varphi_1 \left(\begin{smallmatrix} 0 \\ -q \end{smallmatrix}; q, z \right)$	

Besides, concerning the results of the elliptic difference Painlevé equation (of type $E_8^{(1)}$) and its hypergeometric solutions, we refer the reader to [22] [25] [45].

1.3. The background of the Padé method.

In this subsection, we will briefly explain the background of the Padé method (See also [40] [58]).

1.3.1. What is the Padé method?

The Padé method has been presented by applying Padé approximation to differential Painlevé equations of types from P_{VI} to P_{IV} in [58] (see section 2).

The method, which we call the "Padé method" in this paper, is a method for giving the Painlevé equations, the scalar Lax pairs and the determinant formulae of special solutions simultaneously, by starting from suitable problems of Padé approximation (of the differential grid)/interpolation (of the difference grid).

The Padé method is closely related to the theory of orthogonal polynomials (e.g. [1] [30] [56] [58] [63]). By both approaches, we can obtain Painlevé equations, Lax pairs and special solutions. (The theory of orthogonal polynomials is more general and the Padé method is simpler.)

1.3.2. Previous works on the Padé method.

Concerning all the q -Painlevé equations of types from $E_7^{(1)}$ to $(A_2 + A_1)^{(1)}$ in the classification (1.6), the q -Painlevé equations, the scalar Lax pairs and the determinant formulae of special solutions have already been derived by various methods. References are written in the following Table:

		$q-E_7^{(1)}$	$q-E_6^{(1)}$	$q-D_5^{(1)}$	$q-A_4^{(1)}$	$q-(A_2 + A_1)^{(1)}$
(1.31)	q -Painlevé	[8]	[48]	[20]	[28]	[26] [28] [50]
	Lax pair	[59]	[52]	[20]	[35]	[35]
	special solutions	[33]	[14]	[49]	[13]	[36]

Here, amongst the references in the Table (1.31), only [14] is directly related to the Padé method.

The Padé method for discrete Painlevé equations has been applied to the following types:

		$e-E_8^{(1)}$	$q-E_8^{(1)}$	$q-E_7^{(1)}$	$q-E_6^{(1)}$	$q-D_5^{(1)}$	$q-A_4^{(1)}$	$q-(A_2 + A_1)^{(1)}$
(1.32)		[40]	[60]	[37]	[14][37]	[14] [37] [62]	[37]	[37]
	grid	elliptic	q -quadric	q	q	differential, q	q	q

Here, the type $q-D_5^{(1)}$ is studied as both the q -grid and the differential grid (i.e. Padé approximation).

1.4. The purpose and the organization of this paper.

- The purpose of this paper is to apply the Padé interpolation/approximation method to all the q -Painlevé equations of types from $q-E_7^{(1)}$ to $q-(A_2 + A_1)^{(1)}$. As the main results, the following items are presented for each type.

(a) Setting of the Padé interpolation/approximation problem,

- (b) Contiguity relations,
- (c) The Painlevé equation,
- (d) The Lax pair,
- (e) Special solutions.

• The main part of this paper is composed of section 2, section 3 and section 4. Because each section is written independently, subsection 3.1 (b), (c), (d) and subsection 4.1 (b), (c), (d) overlap almost completely each other. Therefore, they may be skipped over.

This paper is organized as follows:

In section 2, we will reconsider the results of previous works [58] [61] [62]. In subsection 2.1, we will explain the Padé approximation method applied to the differential Painlevé equations. Namely, we will explain the method for items (a)–(e). In subsection 2.2, we will present these main results for all the differential Painlevé equations of types from P_{VI} to P_{IV} . The contents of this section is the same as the results of [58] [61] [62].

In section 3, we will reconsider the results of the previous work [37]. In subsection 3.1, we will explain the Padé interpolation method applied to the q -Painlevé equations. Namely, we will explain the methods for items (a)–(e). In subsection 3.2, we will present these main results for all the q -Painlevé equations of types from $E_7^{(1)}$ to $(A_2 + A_1)^{(1)}$. The contents of this section is the same as the results of [37].

In section 4, we will consider the Padé approximation method. In subsection 4.1, we will explain the Padé approximation method applied to the q -Painlevé equations. Namely, we will explain the methods for items (a)–(e). In subsection 4.2, we will present these main results for all the q -Painlevé equations of types from $E_6^{(1)}$ to $(A_2 + A_1)^{(1)}$.

In section 5, we will give a summary and discuss some future problems.

2. PADÉ APPROXIMATION METHOD TO DIFFERENTIAL PAINLEVÉ EQUATIONS

In this section, we will reconsider the results of previous works [58] [61] [62]. In subsection 2.1, we will explain the Padé approximation method applied to the differential Painlevé equations. Namely, we will explain the method for items (a)–(e). In subsection 2.2, we will present these main results for all the differential Painlevé equations of types from P_{VI} to P_{IV} . The contents of this section is the same as the results of [58] [61] [62].

2.1. Padé approximation method.

In this subsection, we will explain the methods for deriving the items (a)–(e) in the main results given in section 2.2.

2.1.1. (a) Setting of the Padé approximation problem.

• In this section, we will consider the following approximation problem (of the differential grid):

For a given function $Y(x)$, we look for functions

$$(2.1) \quad P_m(x) = \sum_{s=0}^m a_s x^s, \quad Q_n(x) = \sum_{s=0}^n b_s x^s, \quad a_0 = b_0$$

which are polynomials of degree m and n , satisfying the approximation condition

$$(2.2) \quad Y(x) \equiv \frac{P_m(x)}{Q_n(x)} \pmod{x^{m+n+1}}.$$

We call this problem the "Padé approximation problem (of the differential grid)". Then, the function $Y(x)$ is called the "generating function" (because the $Y(x)$ generates the p_k in equation (2.13) in item (e) below), and the polynomials $P_m(x)$ and $Q_n(x)$ are called "approximating functions" respectively.

The common normalization factor of the polynomials $P_m(x)$ and $Q_n(x)$ is not determined by the condition (2.2). However, this normalization factor is not essential for our arguments, i.e. the main results in subsection 2.2. The explicit expressions of $P_m(x)$ and $Q_n(x)$, which will be used in the computations for the item (e) above, were essentially given in equation (2.16) (see item (e) below).

• In this section, we will establish the approximation problems (2.2) by specifying the generating functions $Y(x)$ as follows:

		P_{VI}	P_V	P_{IV}
(2.3)	$Y(x)$	$(1-x)^a \left(1 - \frac{x}{t}\right)^b$	$(1-x)^{-b} \exp\left(\frac{xt}{1-x}\right)$	$\exp(2tx - x^2)$

2.1.2. (b) Computation of contiguity relations.

• We will consider the following pair of linear differential equations L_1 and L_2 for an unknown function y , which are the main object in our study.

$$(2.4) \quad L_1 : \begin{vmatrix} y & y' & y'' \\ u & u' & u'' \\ v & v' & v'' \end{vmatrix} = 0, \quad L_2 : \begin{vmatrix} y & y' & \dot{y} \\ u & u' & \dot{u} \\ v & v' & \dot{v} \end{vmatrix} = 0,$$

where $u = P_m, v = YQ_n$ and $' = d/dx, \dot{} = d/dt$. Then, the pair of the differential equations (2.4) are called the "contiguity relations".

• We will show the method of computation of the Lax pair L_1 and L_2 .

Set $\mathbf{y} = \begin{bmatrix} P_m \\ YQ_n \end{bmatrix}$ and define Wronskian determinants D_i by

$$(2.5) \quad D_1 = \det[\mathbf{y}, \mathbf{y}'], \quad D_2 = \det[\mathbf{y}', \mathbf{y}''], \quad D_3 = \det[\mathbf{y}, \dot{\mathbf{y}}], \quad D_4 = \det[\mathbf{y}', \dot{\mathbf{y}}].$$

Then, the Lax pair (2.4) can be rewritten as follows:

$$(2.6) \quad L_1 : y'' + A_1 y' + A_2 y = 0, \quad L_2 : \dot{y} + B_1 y' + B_2 y = 0,$$

where

$$(2.7) \quad A_1 = -D_1'/D_1, \quad A_2 = D_2/D_1, \quad B_1 = -D_3/D_1, \quad B_2 = D_4/D_1.$$

We define basic quantities G , K and L (e.g. (3.20), (3.33)) by

$$(2.8) \quad G = Y'/Y, \quad K = Y''/Y, \quad L = \dot{Y}/Y,$$

where G_{den} and G_{num} are defined as the polynomials of the denominator and the numerator of G respectively, and K_{den} , K_{num} , L_{den} and L_{num} are similarly defined. Substituting these quantities into the equations (2.5), we obtain the following determinants:

$$(2.9) \quad \begin{aligned} D_1 &= \frac{Y}{G_{\text{den}}} \{G_{\text{den}} P_m Q'_n - G_{\text{num}} P_m Q_n - G_{\text{den}} P'_m Q_n\}, \\ D_2 &= \frac{Y}{K_{\text{den}}} \{K_{\text{num}} P'_m Q_n + 2GK_{\text{den}} P'_m Q'_n + K_{\text{den}} P'_m Q''_n - GK_{\text{den}} P''_m Q_n - K_{\text{den}} P''_m Q'_n\}, \\ D_3 &= \frac{Y}{L_{\text{den}}} \{L_{\text{num}} P_m Q_n + L_{\text{den}} P_m \dot{Q}_n - L_{\text{den}} \dot{P}_m Q_n\}, \\ D_4 &= \frac{Y}{G_{\text{den}}} \{LG_{\text{den}} P'_m Q_n + G_{\text{den}} P'_m \dot{Q}_n - G_{\text{num}} \dot{P}_m Q_n - G_{\text{num}} \dot{P}_m Q'_n\}. \end{aligned}$$

Using the approximation condition (2.2) and the form of the basic quantities (e.g. (2.25), (2.39)), we can investigate positions of zeros and degrees of the polynomials within braces $\{ \}$ of the equations (2.9). Then, we can simply compute the determinants D_i (e.g. (2.26), (2.40)) except for some factors such as $\lambda - x$ in D_1 , $\bigcirc \lambda + c_0 x + \bigcirc x^2$ in D_2 (Both of two symbols $\bigcirc$ depend on parameters a, b, m, n), where λ and c_0 , etc. are constants with respect to x . In this way, we obtain the contiguity relations L_1 and L_2 (e.g. (2.27), (2.41)), whose coefficients A_i and B_i (e.g. (2.28), (2.42)) are expressed in terms of three unknown constants λ, μ, H and three unknown constants λ, M_1, M_2 (e.g. (2.29), (2.43)) respectively. In item (c), the λ and the μ are expressed as the unknown functions of the differential Painlevé equation. Similarly, the H is expressed as a Hamiltonian (2.11) (e.g. (2.32), (2.46)), and the M_1 and the M_2 are expressed in terms of λ and μ (e.g. (2.33), (2.47)).

2.1.3. (c) Computation of the q -Painlevé equation.

We can derive the differential Painlevé equation from the compatibility condition of the contiguity relations L_1 and L_2 . Computing the compatibility condition

$$(2.10) \quad \dot{A}_1 = B''_1 + 2B'_2 - A_1 B'_1 - A'_1 B_1, \quad \dot{A}_2 - A_1 B'_2 = B''_2 - 2A_2 B'_1 - A'_2 B_1,$$

we determine five quantities $\dot{\lambda}, \dot{\mu}, H, M_1$ and M_2 . Expressions for these quantities (e.g. (2.32), (2.33), (2.46), (2.47)) are obtained in terms of λ and μ .

Eliminating μ and $\dot{\mu}$ from two expressions for the $\dot{\lambda}$ and the $\dot{\mu}$, we can obtain an expression for $\ddot{\lambda}$ in terms of λ and $\dot{\lambda}$, namely the differential Painlevé equations (e.g. (2.30), (2.44)). The expression for the H is a Hamiltonian, namely it satisfies

$$(2.11) \quad \frac{d\lambda}{dt} = \frac{\partial H}{\partial \mu}, \quad \frac{d\mu}{dt} = -\frac{\partial H}{\partial \lambda}.$$

Remark 1. on two meanings of λ, μ, m and n

We use the variables λ and μ with two different meanings. The first meaning is the λ and μ which are explicitly determined in terms of parameters a, b, t, m and n by the Padé problem.

The second meaning is the λ and μ which are unknown functions in the differential Painlevé equation. In items (c) and (d), we consider λ and μ in the second meaning.

Similarly, we use the parameters m and n with two meanings. In the first meaning, m and n are integers. In the second meaning, m and n are generic complex parameters. In items (c) and (d), we consider m and n in the second meaning. Then, the result of the compatibility of $L_1(x)$ and $L_2(x)$ also holds with respect to the second meaning. $\square$

2.1.4. (d) Computation of the Lax pair.

We will consider the following pair of linear differential equations for the unknown function y :

$$(2.12) \quad L_1 : y'' + A_1 y' + A_2 y = 0, \quad L_2 : \dot{y} + B_1 y' + B_2 y = 0,$$

such that their compatibility condition gives a q -Painlevé equation. Then, the pair of differential equations is called the "scalar Lax pair". Substituting the expression for H , M_1 and M_2 (e.g. (2.32), (2.33), (2.46), (2.47)), derived in item (c), into the contiguity relations L_1 and L_2 (e.g. (2.27), (2.41)) in item (b), we can obtain the scalar Lax pair L_1 and L_2 (e.g. (2.34), (2.48)).

2.1.5. (e) Computation of special solutions.

By construction, expressions for λ as in the first meaning in Remark 1 give a special solution for the differential Painlevé equation. We will present how to compute determinant formulae of the special solutions.

- We will derive formulae (2.16) which are convenient for deriving the special solution λ . We can assume that

$$(2.13) \quad Y(x) = \sum_{k=0}^{\infty} p_k x^k, \quad p_0 = 1, \quad p_i = 0 \quad (i < 0),$$

near $x = 0$ without loss of generality. Then, for each type of $Y(x)$ in Table (2.3), the p_k are given respectively as a HGF (the hypergeometric functions [5] [6]) defined by

$$(2.14) \quad {}_rF_s \left(\begin{matrix} a_1, \dots, a_r \\ b_1, \dots, b_s \end{matrix}; x \right) = \sum_{n=0}^{\infty} \frac{(a_1, \dots, a_r)_n}{(b_1, \dots, b_s)_n} \frac{x^n}{n!}.$$

Here, the Pochhammer symbol is defined by

$$(2.15) \quad (a_1, a_2, \dots, a_i)_j = \prod_{k=1}^j a_k(a_k + 1)(a_k + 2) \cdots (a_k + j - 1).$$

The formulae (2.16) were described in [58]. For a given function $Y(x)$, the polynomials $P_m(x)$ and $Q_n(x)$ of degree m and n for an approximation condition (2.2) are given by the following determinant expressions:

$$(2.16) \quad P_m(x) = \sum_{i=0}^m s_{(m^n, i)} x^i, \quad Q_n(x) = \sum_{i=0}^n s_{((m+1)^i, m^n - i)} (-x)^i,$$

where s_λ is the Schur function defined by the Jacobi-Trudi formula

$$(2.17) \quad s_{(\lambda_1, \dots, \lambda_l)} = \det(p_{\lambda_i - i + j})_{i,j=1}^l.$$

The derivation of (2.16) is as follows:
By using the relation

$$(2.18) \quad x^n Y(x) = \sum_{k=0}^{\infty} p_k x^{k+n} = \sum_{k=0}^{\infty} p_{k-n} x^k,$$

the formulae (2.16) is given by the following computation:

$$(2.19) \quad \begin{aligned} Y(x)Q_n(x) &= Y(x) \begin{vmatrix} 1 & p_{m+1} & \cdots & p_{m+n} \\ x & p_m & \ddots & \vdots \\ \vdots & \vdots & \ddots & p_{m+1} \\ x^n & p_{m-n+1} & \cdots & p_m \end{vmatrix} = \begin{vmatrix} p_m & p_{m+1} & \cdots & p_{m+n} \\ p_{m-1} & \ddots & \ddots & \vdots \\ \vdots & \vdots & p_m & p_{m+1} \\ x^n Y(x) & \cdots & xY(x) & Y(x) \end{vmatrix} \\ &= \sum_{k=0}^{\infty} \begin{vmatrix} p_m & p_{m+1} & \cdots & p_{m+n} \\ \vdots & \ddots & \ddots & \vdots \\ p_{m-n+1} & \cdots & p_m & p_{m+1} \\ p_{k-n} & \cdots & p_{k-1} & p_k \end{vmatrix} x^k. \end{aligned}$$

Here, we note that

$$(2.20) \quad \sum_{k=m+1}^{m+n} \begin{vmatrix} p_m & p_{m+1} & \cdots & p_{m+n} \\ \vdots & \ddots & \ddots & \vdots \\ p_{m-n+1} & \cdots & p_m & p_{m+1} \\ p_{k-n} & \cdots & p_{k-1} & p_k \end{vmatrix} x^k = 0.$$

Substituting equation (2.20) into equation (2.19), we obtain

$$(2.21) \quad Y(x)Q_n(x) = \left(\sum_{k=0}^m + \sum_{k=m+n+1}^{\infty} \right) s_{(m^n, k)} x^k = P_m(x) + \sum_{k=m+n+1}^{\infty} s_{(m^n, k)} x^k. \quad \square$$

- We will show the method of computation of the special solution λ .

We can derive the expression for the λ by comparing D_1 in (2.5) and D_1 in item (b) (e.g. (2.26), (2.40)) as the identity with respect to the variable x , when we apply (2.21) to the first D_1 and apply the formulae (2.16) to the second D_1 above.

For example, the computation for the case P_{VI} is as follows: Applying (2.21) to D_1 in (2.5), we obtain the coefficient of minimum degree x^{m+n} in D_1 . Similarly, applying (2.16) to D_1 in (2.26), we obtain the coefficient of minimum degree x^{m+n} in D_1 . Therefore, we obtain the expression for the λ by comparing each coefficient of x^{m+n} .

2.2. Main results.

In this subsection, we will present the results obtained through the method, which is explained for each case from P_{VI} to P_{IV} in subsection 2.1.

We use the following notations:

$$(2.22) \quad \begin{aligned} a_1 a_2 \cdots a_n / b_1 b_2 \cdots b_n &= \frac{a_1 a_2 \cdots a_n}{b_1 b_2 \cdots b_n}, \\ \tau_{m,n} &= s_{(m^n)}. \end{aligned}$$

Here, by definition (2.17), the Schur functions $s_{(m^n)}$ is expressed as

$$(2.23) \quad s_{(m^n)} = \begin{vmatrix} p_m & p_{m+1} & \cdots & p_{m+n-1} \\ p_{m-1} & \ddots & & \vdots \\ \vdots & \cdots & \ddots & \vdots \\ p_{m-n+1} & \cdots & \cdots & p_m \end{vmatrix}$$

where the p_k are defined in (2.13).

2.2.1. Case P_{VI} .

(a) Setting of the Padé approximation problem

The generating function (2.3):

$$(2.24) \quad Y(x) = (1-x)^a \left(1 - \frac{x}{t}\right)^b.$$

(b) Contiguity relations

The basic quantities:

$$(2.25) \quad G = \frac{a}{x-1} + \frac{b}{x-t}, \quad K = \frac{a(a-1)}{(x-1)^2} + \frac{b(b-1)}{(x-t)^2} + \frac{2ab}{(x-1)(x-t)}, \quad L = \frac{-bx/t}{x-t}.$$

The Wronskian determinants:

$$(2.26) \quad \begin{aligned} D_1 &= \frac{Y}{G_{\text{den}}} (m-n-a-b) a_m b_n x^{m+n} (\lambda-x), \\ D_2 &= \frac{Y}{K_{\text{den}}} (m-n-a-b) a_m b_n x^{m+n-1} \{(m+n)a_1 \lambda t + c_0 x - m(n+a+b)x^2\}, \\ D_3 &= \frac{Y}{L_{\text{den}}} x^{m+n+1} \left(-\dot{a}_m b_n - \frac{b}{t} a_m b_n + a_m \dot{b}_n \right), \\ D_4 &= \frac{Y}{G_{\text{den}}} x^{m+n+1} \{m(a_m \dot{b}_n - \frac{b}{t} a_m b_n) - (n+a+b)\dot{a}_m b_n\}, \end{aligned}$$

where λ and c_0 are constants with respect to x .

The contiguity relations:

$$(2.27) \quad L_1 : y'' + A_1 y' + A_2 y = 0, \quad L_2 : \dot{y} + B_1 y' + B_2 y = 0,$$

where

$$(2.28) \quad \begin{aligned} A_1 &= \frac{-(m+n)}{x} + \frac{1-a}{x-1} + \frac{1-b}{x-t} - \frac{1}{x-\lambda}, \\ A_2 &= \frac{m(n+a+b)}{x(x-1)} - \frac{t(t-1)H}{x(x-1)(x-t)} + \frac{\lambda(\lambda-1)\mu}{x(x-1)(x-\lambda)}, \\ B_1 &= M_1 \frac{x(x-1)}{x-\lambda}, \quad B_2 = M_2 \frac{x}{x-\lambda}. \end{aligned}$$

Here,

$$\begin{aligned}
 t(t-1)H &= \frac{mt^2(a+b+n) - a_1\lambda t(m+n) - c_0t}{\lambda - t}, \\
 \mu &= \frac{\lambda m(a+b+n) - a_1t(m+n) - c_0}{(\lambda-1)(\lambda-t)}, \\
 M_1 &= -\frac{\dot{a}_m b_n + \frac{b}{t}a_m b_n - a_m \dot{b}_n}{(m-n-a-b)a_m b_n}, \\
 M_2 &= \frac{(n+a+b)\dot{a}_m b_n + m\left(\frac{b}{t}a_m b_n - a_m \dot{b}_n\right)}{(m-n-a-b)a_m b_n}.
 \end{aligned}
 \tag{2.29}$$

(c) Differential Painlevé equation

The compatibility gives the following equations:

$$\begin{aligned}
 \frac{d^2\lambda}{dt^2} &= \frac{1}{2}\left(\frac{1}{\lambda} + \frac{1}{\lambda-1} + \frac{1}{\lambda-t}\right)\left(\frac{d\lambda}{dt}\right)^2 - \left(\frac{1}{t} + \frac{1}{t-1} + \frac{1}{y-t}\right)\frac{d\lambda}{dt} \\
 &\quad + \frac{\lambda(\lambda-1)(\lambda-t)}{t^2(t-1)^2}\left(\alpha + \beta\frac{t}{\lambda^2} + \gamma\frac{t-1}{(\lambda-1)^2} + \delta\frac{t(t-1)}{(\lambda-t)^2}\right),
 \end{aligned}
 \tag{2.30}$$

where

$$\alpha = \frac{(m-n-a-b)^2}{2}, \quad \beta = -\frac{(m+n+1)^2}{2}, \quad \gamma = \frac{a^2}{2}, \quad \delta = \frac{1-b^2}{2}.
 \tag{2.31}$$

This equation is equivalent to the differential Painlevé equation of type VI (1.2).

The Hamiltonian H :

$$\begin{aligned}
 t(t-1)H &= \lambda(\lambda-1)(\lambda-t)\mu^2 \\
 &\quad + \{(1-b)\lambda(\lambda-1) - (m+n+1)(\lambda-1)(\lambda-t) - a\lambda(\lambda-t)\}\mu \\
 &\quad + (a+b+n)m(\lambda-t), \\
 \frac{d\lambda}{dt} &= \frac{1}{t(t-1)}\{2\lambda^3\mu + \lambda^2(-a-b-2\mu-m-n-2\mu t) \\
 &\quad + \lambda(at+b+mt+m+nt+n+2\mu t+t) - t(m+n+1)\}, \\
 \frac{d\mu}{dt} &= \frac{1}{t(t-1)}\{\mu(2a\lambda-at+2b\lambda-b+2\lambda m-mt-m+2\lambda n-nt-n-t) \\
 &\quad - m(a+b+n) + \mu^2(-3\lambda^2+2\lambda+2\lambda t-t)\}.
 \end{aligned}
 \tag{2.32}$$

The Hamiltonian H (2.32) is the same as that in [58]. (Note that there is a typographical error in equation (19) in [58], namely the expression $a+m+n$ should read $a+b+n$.)

The part of B_1 and B_2 :

$$M_1 = -\frac{\lambda-t}{t(t-1)}, \quad M_2 = \frac{\mu(\lambda-1)(\lambda-t)}{t(t-1)}.
 \tag{2.33}$$

(d) Lax pair

$$L_1 : y'' + A_1 y' + A_2 y = 0, \quad L_2 : \dot{y} + B_1 y' + B_2 y = 0,
 \tag{2.34}$$

where

$$(2.35) \quad \begin{aligned} A_1 &= \frac{-(m+n)}{x} + \frac{1-a}{x-1} + \frac{1-b}{x-t} - \frac{1}{x-\lambda}, \\ A_2 &= \frac{m(n+a+b)}{x(x-1)} - \frac{1}{t(t-1)H} + \frac{\lambda(\lambda-1)\mu}{x(x-1)(x-\lambda)}, \\ B_1 &= -\frac{x(x-1)}{t(t-1)} \frac{x(x-1)}{x-\lambda}, \quad B_2 = \frac{\mu(\lambda-1)(\lambda-t)}{t(t-1)} \frac{x}{x-\lambda}. \end{aligned}$$

Here, the Hamiltonian H in A_2 is given by (2.32). The scalar Lax pair (2.34) is the same as that in [58].

(e) The special solution

$$(2.36) \quad \lambda = \frac{(m+n+1)t}{m-n-a-b} \frac{\tau_{m,n}\tau_{m+1,n+1}}{\tau_{m,n+1}\tau_{m+1,n}},$$

where the p_k in the $\tau_{m,n}$ (2.22) are given by

$$(2.37) \quad p_k = \frac{(-a-b)_k}{k!} {}_2F_1\left(\begin{matrix} -k, -b \\ -a-b \end{matrix}; 1 - \frac{1}{t}\right).$$

The p_k are expressed in terms of the terminating ${}_2F_1$ (2.14) series (Jacobi polynomials [5] [6] [27]). The corresponding Schur functions, defined in (2.17), were obtained in [31].

2.2.2. Case P_V .

(a) Setting of the Padé approximation problem

The generating function (2.3):

$$(2.38) \quad Y(x) = (1-x)^{-b} \exp\left(\frac{xt}{1-x}\right).$$

(b) Contiguity relations

The basic quantities:

$$(2.39) \quad G = \frac{b}{1-x} + \frac{t}{(1-x)^2}, \quad K = \frac{b(b+1)}{(1-x)^2} + \frac{2t(1+b)}{(1-x)^3} + \frac{t^2}{(1-x)^4}, \quad L = \frac{-x}{x-1}.$$

The Wronskian determinants:

$$(2.40) \quad \begin{aligned} D_1 &= \frac{Y}{G_{\text{den}}}(m-n+b)a_mb_nx^{m+n}(\lambda-x), \\ D_2 &= \frac{Y}{K_{\text{den}}}(m-n+b)a_mb_nx^{m+n-1}\{(m+n)a_1\lambda + c_0x - m(n-b)x^2\}, \\ D_3 &= \frac{Y}{L_{\text{den}}}x^{m+n+1}(-\dot{a}_mb_n - a_m\dot{b}_n + a_m\dot{b}_n), \\ D_4 &= \frac{Y}{G_{\text{den}}}x^{m+n+1}\{m(a_m\dot{b}_n - a_m\dot{b}_n) - (n-b)\dot{a}_mb_n\}, \end{aligned}$$

where λ and c_0 are constants with respect to x .

The contiguity relations:

$$(2.41) \quad L_1 : y'' + A_1 y' + A_2 y = 0, \quad L_2 : \dot{y} + B_1 y' + B_2 y = 0,$$

where

$$(2.42) \quad \begin{aligned} A_1 &= \frac{-(m+n)}{x} - \frac{t}{(x-1)^2} + \frac{b+2}{x-1} - \frac{1}{x-\lambda}, \\ A_2 &= \frac{m(n-b)}{x(x-1)} - \frac{tH}{x(x-1)^2} + \frac{\lambda(\lambda-1)\mu}{x(x-1)(x-\lambda)}, \\ B_1 &= M_1 \frac{x(x-1)}{x-\lambda}, \quad B_2 = M_2 \frac{x}{x-\lambda}. \end{aligned}$$

Here,

$$(2.43) \quad \begin{aligned} tH &= \frac{m(n-b) - a_1 \lambda(m+n) - c_0}{\lambda-1}, \quad \mu = \frac{\lambda m(n-b) - a_1(m+n) - c_0}{(\lambda-1)^2}, \\ M_1 &= -\frac{\dot{a}_m b_n + a_m \dot{b}_n - a_m \dot{b}_n}{(m-n+b)a_m b_n}, \\ M_2 &= \frac{(n-b)\dot{a}_m b_n + m(a_m \dot{b}_n - a_m \dot{b}_n)}{(m-n+b)a_m b_n}. \end{aligned}$$

(c) Differential Painlevé equation

The compatibility gives the following equations:

$$(2.44) \quad \frac{d^2 \lambda}{dt^2} = \left(\frac{1}{2\lambda} + \frac{1}{\lambda-1} \right) \left(\frac{d\lambda}{dt} \right)^2 - \frac{1}{t} \frac{d\lambda}{dt} + \frac{(\lambda-1)^2}{t^2} \left(\alpha\lambda + \frac{\beta}{\lambda} \right) + \frac{\gamma}{t} \lambda + \delta \frac{\lambda(\lambda+1)}{\lambda-1},$$

where

$$(2.45) \quad \alpha = \frac{(m-n+b)^2}{2}, \quad \beta = -\frac{(m+n+1)^2}{2}, \quad \gamma = b, \quad \delta = -\frac{1}{2}.$$

This equation is equivalent to the differential Painlevé equation of type V (1.2).

The Hamiltonian H :

$$(2.46) \quad \begin{aligned} tH &= (n-b)m(\lambda-1) + (\lambda\mu - m - n)\mu(\lambda-1)^2 + b\lambda\mu(\lambda-1) + \mu(\lambda-1) - \lambda\mu t, \\ \frac{d\lambda}{dt} &= \frac{1}{t} \{ 2\lambda^3 \mu + \lambda^2(b - 4\mu - m - n) + \lambda(-b + 2\mu + 2m + 2n - t + 1) - m - n - 1 \}, \\ \frac{d\mu}{dt} &= \frac{1}{t} \{ -(\lambda-1)(3\lambda-1)\mu^2 + \mu(-2b\lambda + b + 2\lambda m - 2m + 2\lambda n - 2n + t - 1) + m(b-n) \}. \end{aligned}$$

The Hamiltonian H (2.46) is the same as that in [58].

The part of B_1 and B_2 :

$$(2.47) \quad M_1 = \frac{1-\lambda}{t}, \quad M_2 = \frac{\mu(\lambda-1)^2}{t}.$$

(d) Lax pair

$$(2.48) \quad L_1 : y'' + A_1 y' + A_2 y = 0, \quad L_2 : \dot{y} + B_1 y' + B_2 y = 0,$$

where

$$(2.49) \quad \begin{aligned} A_1 &= \frac{-(m+n)}{x} - \frac{t}{(x-1)^2} + \frac{b+2}{x-1} - \frac{1}{x-\lambda}, \\ A_2 &= \frac{m(n-b)}{x(x-1)} - \frac{tH}{x(x-1)^2} + \frac{\lambda(\lambda-1)\mu}{x(x-1)(x-\lambda)}, \\ B_1 &= \frac{1-\lambda}{t} \frac{x(x-1)}{x-\lambda}, \quad B_2 = \frac{\mu(\lambda-1)^2}{t} \frac{x}{x-\lambda}. \end{aligned}$$

Here, the Hamiltonian H in A_2 is given by (2.46). The scalar Lax pair (2.48) is the same as that in [58].

(e) The special solution

$$(2.50) \quad \lambda = \frac{m+n+1}{m-n+b} \frac{\tau_{m,n}\tau_{m+1,n+1}}{\tau_{m,n+1}\tau_{m+1,n}},$$

where the p_k in the $\tau_{m,n}$ (2.22) are given by

$$(2.51) \quad p_k = \frac{(b)_k}{k!} {}_1F_1\left(\begin{matrix} -k \\ b \end{matrix}; -t\right).$$

The p_k are expressed in terms of the terminating ${}_1F_1$ (2.14) series (Laguerre polynomials [5] [6] [27]). The corresponding Schur functions, defined in (2.17), were obtained in [31] [32].

2.2.3. Case P_{IV} .

(a) Setting of the Padé approximation problem

The generating function (2.3):

$$(2.52) \quad Y(x) = \exp(2tx - x^2).$$

(b) Contiguity relations

The basic quantities:

$$(2.53) \quad G = 2t - 2x, \quad K = (2t - 2x)^2, \quad L = 2x.$$

The Wronskian determinants:

$$(2.54) \quad \begin{aligned} D_1 &= \frac{Y}{G_{\text{den}}} 2a_m b_n x^{m+n} (\lambda - x), \\ D_2 &= \frac{Y}{K_{\text{den}}} 2a_m b_n x^{m+n-1} \{(m+n)a_1 \lambda + c_0 x + 2mx^2\}, \\ D_3 &= \frac{Y}{L_{\text{den}}} 2a_m b_n x^{m+n+1}, \quad D_4 = \frac{Y}{G_{\text{den}}} 2\dot{a}_m b_n x^{m+n+1}, \end{aligned}$$

where λ and c_0 are constants with respect to x .

The contiguity relations:

$$(2.55) \quad L_1 : y'' + A_1 y' + A_2 y = 0, \quad L_2 : \dot{y} + B_1 y' + B_2 y = 0,$$

where

$$(2.56) \quad \begin{aligned} A_1 &= 2(x-t) - \frac{m+n}{x} - \frac{1}{x-\lambda}, & A_2 &= -2m - \frac{H}{x} - \frac{\lambda\mu}{x(x-\lambda)}, \\ B_1 &= \frac{x}{x-\lambda}, & B_2 &= \frac{\mu x}{x-\lambda}. \end{aligned}$$

Here,

$$(2.57) \quad H = \mu - (m+n)a_1, \quad \mu = (m+n)a_1 + c_0 + 2m\lambda.$$

(c) Differential Painlevé equation

The compatibility gives the following equations

$$(2.58) \quad \frac{d^2\lambda}{dt^2} = \frac{1}{2\lambda} \left(\frac{d\lambda}{dt} \right)^2 + 6\lambda^3 - 8t\lambda^2 + 2(t^2 - \alpha)\lambda + \frac{\beta}{\lambda},$$

where

$$(2.59) \quad \alpha = -m + n, \quad \beta = -\frac{(m+n+1)^2}{2}.$$

This equation is equivalent to the differential Painlevé equation of type IV (1.2) by rescaling $\lambda \mapsto -\lambda/2, \beta \mapsto \beta/4$.

The Hamiltonian H :

$$(2.60) \quad \begin{aligned} H &= \lambda\mu(\mu - 2\lambda + 2t) + (m+n+1)\mu - 2m\lambda, \\ \frac{d\lambda}{dt} &= -2\lambda^2 + m + n + 2\lambda(\mu + t) + 1, \\ \frac{d\mu}{dt} &= -\mu^2 + 2m - 2\mu(t - 2\lambda). \end{aligned}$$

The Hamiltonian H (2.60) is equivalent to that in [58] by taking transformations $H \mapsto -H$ and $\mu \mapsto -\mu$.

(d) Lax pair

$$(2.61) \quad L_1 : y'' + A_1 y' + A_2 y = 0, \quad L_2 : \dot{y} + B_1 y' + B_2 y = 0,$$

where

$$(2.62) \quad \begin{aligned} A_1 &= 2(x-t) - \frac{m+n}{x} - \frac{1}{x-\lambda}, & A_2 &= -2m - \frac{H}{x} - \frac{\lambda\mu}{x(x-\lambda)}, \\ B_1 &= \frac{x}{x-\lambda}, & B_2 &= \frac{\mu x}{x-\lambda}. \end{aligned}$$

Here, the Hamiltonian H in A_2 is given by (2.60). The scalar Lax pair (2.61) is equivalent to that in [58] by taking transformations $H \mapsto -H$ and $\mu \mapsto -\mu$.

(e) The special solution

$$(2.63) \quad \lambda = \frac{m+n+1}{2} \frac{\tau_{m,n} \tau_{m+1,n+1}}{\tau_{m,n+1} \tau_{m+1,n}},$$

where the p_k in the $\tau_{m,n}$ (2.22) are given by

$$(2.64) \quad p_k = \frac{(2t)_k}{k!} {}_2F_0 \left(\begin{matrix} -k/2, -(k-1)/2 \\ - \end{matrix} ; -\frac{1}{t^2} \right).$$

The p_k are expressed in terms of the terminating ${}_2F_0$ (2.14) series (Hermite polynomials [5] [6] [27]). The corresponding Schur functions, defined in (2.17), were obtained in [41].

3. PADÉ INTERPOLATION METHOD TO q -PAINLEVÉ EQUATIONS

In this section, we will reconsider the case of the Padé interpolation method of the q -grid. In subsection 3.1, we will explain the Padé interpolation method applied to the q -Painlevé equations. Namely, we will explain the methods for items (a)–(e). In subsection 3.2, we will present these main results for all the q -Painlevé equations of types from $E_7^{(1)}$ to $(A_2 + A_1)^{(1)}$. The contents of this section is the same as the results of [37].

3.1. Padé interpolation method.

In this subsection, we will explain the methods for deriving the items (a)–(e) in the main results given in subsection 3.2.

3.1.1. (a) Setting of the Padé interpolation problem.

- In this section, we will consider the following interpolation problem (of the q -grid):

For a given function $Y(x)$, we look for functions $P_m(x)$ and $Q_n(x)$ which are polynomials of degree m and n , satisfying the interpolation condition

$$(3.1) \quad Y(q^s) = P_m(q^s)/Q_n(q^s) \quad (s = 0, 1, \dots, m+n).$$

We call this problem the "Padé interpolation problem (of the q -grid)". Then, the function $Y(x)$ is called the "interpolated function". Correspondingly, the polynomials $P_m(x)$ and $Q_n(x)$ are called "interpolating functions" respectively.

Here, the "interpolated function" and the "interpolating function" in this section correspond to the "generating function" and the "approximating function" in section 2 respectively.

The common normalization factor of the polynomials $P_m(x)$ and $Q_n(x)$ is not determined by the condition (3.1). However, this normalization factor is not essential for our arguments, i.e. the main results in section 3.2 (see Remark 3). The explicit expressions of $P_m(x)$ and $Q_n(x)$, which will be used in the computations for the item (e) above, were essentially given in [16] (see item (e) below).

- In this section, we will establish the interpolation problems (3.1) by specifying the interpolated functions $Y(x)$ and the interpolated sequences $Y_s = Y(q^s)$ as follows:

(3.2)

	$q-E_7^{(1)}$	$q-E_6^{(1)}$	$q-D_5^{(1)}$	$q-A_4^{(1)}$	$q-(A_2 + A_1)^{(1)}$
$Y(x)$	$\prod_{i=1}^3 \frac{(a_i x, b_i; q)_\infty}{(a_i, b_i x; q)_\infty}$ $\frac{a_1 a_2 a_3 q^m}{b_1 b_2 b_3 q^n} = 1$	$\prod_{i=1}^2 \frac{(a_i x, b_i; q)_\infty}{(a_i, b_i x; q)_\infty}$	$c^{\log_q x} \frac{(a_1 x, b_1; q)_\infty}{(a_1, b_1 x; q)_\infty}$	$c^{\log_q x} \frac{(b_1; q)_\infty}{(b_1 x; q)_\infty}$	$(d \sqrt{x/q})^{\log_q x}$
Y_s	$\prod_{i=1}^3 \frac{(b_i; q)_s}{(a_i; q)_s}$	$\prod_{i=1}^2 \frac{(b_i; q)_s}{(a_i; q)_s}$	$c^s \frac{(b_1; q)_s}{(a_1; q)_s}$	$c^s (b_1; q)_s$	$q^{\binom{s}{2}} d^s$
q -HGF	$4\varphi_3$	$3\varphi_2$	$2\varphi_1$	$2\varphi_1$	$1\varphi_1$

where $\frac{a_1 a_2 a_3 q^m}{b_1 b_2 b_3 q^n} = 1$ is a constraint for the parameters in the case $q-E_7^{(1)}$, and the q -shifted factorials are defined by

$$(3.3) \quad (a_1, a_2, \dots, a_i; q)_j = \prod_{k=0}^{j-1} (1 - a_1 q^k)(1 - a_2 q^k) \cdots (1 - a_i q^k),$$

and the q -HGF (the q -hypergeometric functions [5] [6]) is defined by

$$(3.4) \quad {}_k\varphi_l \left(\begin{matrix} a_1, \dots, a_k \\ b_1, \dots, b_l \end{matrix}; q; x \right) = \sum_{s=0}^{\infty} \frac{(a_1, \dots, a_k; q)_s}{(b_1, \dots, b_l, q; q)_s} \left[(-1)^s q^{\binom{s}{2}} \right]^{1+l-k} x^s,$$

with $\binom{s}{2} = s(s-1)/2$.

Remark 2. on the choice of Y_s and $Y(x)$

In Table (3.2), the expressions for the interpolated sequences Y_s are closely related to those of q -HGF ${}_r\varphi_s$. In fact, we chose the Y_s by comparing the coefficient of x^s in ${}_r\varphi_s$ (3.4) and the expression for $Y_s(q^{-(m+n)}; q)_s/(q; q)_s$ in (3.15). Then, we also chose the interpolated functions $Y(x)$, which are equal to the Y_s at $x = q^s$. $\square$

• In this section, we will consider yet another Padé problem where some parameters a_i, b_i, c, d, m and n in $Y(x)$ are shifted. The parameter shift operators T are given as follows:

	parameters
$q-E_7^{(1)}$	$(a_1, a_2, a_3, b_1, b_2, b_3, m, n) \mapsto (qa_1, a_2, qa_3, b_1, b_2, qb_3, m-1, n)$
$q-E_6^{(1)}$	$(a_1, a_2, b_1, b_2, m, n) \mapsto (qa_1, a_2, b_1, b_2, m-1, n)$
$q-D_5^{(1)}$	$(a_1, b_1, c, m, n) \mapsto (qa_1, b_1, c, m-1, n)$
$q-A_4^{(1)}$	$(b_1, c, m, n) \mapsto (b_1, c, m-1, n)$
$q-(A_2 + A_1)^{(1)}$	$(d, m, n) \mapsto (d, m-1, n)$

Here, the operators T are called the "time evolutions", because they specify the directions of the time evolutions for q -Painlevé equations.

3.1.2. (b) Computation of contiguity relations.

• We will consider the following pair of linear q -difference equations $L_2(x)$ and $L_3(x)$ for unknown function $y(x)$, which are the main object in our study.

$$(3.6) \quad \begin{aligned} L_2(x) : & \begin{vmatrix} y(x) & y(qx) & \bar{y}(x) \\ P_m(x) & P_m(qx) & \bar{P}_m(x) \\ Y(x)Q_n(x) & Y(qx)Q_n(qx) & \bar{Y}(x)\bar{Q}_n(x) \end{vmatrix} = 0, \\ L_3(x) : & \begin{vmatrix} y(x) & \bar{y}(x) & \bar{y}(x/q) \\ P_m(x) & \bar{P}_m(x) & \bar{P}_m(x/q) \\ Y(x)Q_n(x) & \bar{Y}(x)\bar{Q}_n(x) & \bar{Y}(x/q)\bar{Q}_n(x/q) \end{vmatrix} = 0. \end{aligned}$$

Then, the pair of q -difference equations (3.6) is called the "contiguity relations". Here, $\bar{F}$ and $\underline{F}$ denote $T(F)$ and $T^{-1}(F)$ respectively. T is the shift operator acting on parameters given in

Table (3.5).

- We will show the method of computation of the contiguity relations $L_2(x)$ and $L_3(x)$.

Set $\mathbf{y}(x) = \begin{bmatrix} P_m(x) \\ Y(x)Q_n(x) \end{bmatrix}$ and define Casorati determinants $D_i(x)$ by

$$(3.7) \quad D_1(x) = \det[\mathbf{y}(x), \mathbf{y}(qx)], \quad D_2(x) = \det[\mathbf{y}(x), \bar{\mathbf{y}}(x)], \quad D_3(x) = \det[\mathbf{y}(qx), \bar{\mathbf{y}}(x)].$$

Then, the contiguity relations (3.6) can be rewritten as follows:

$$(3.8) \quad \begin{aligned} L_2(x) : D_1(x)\bar{y}(x) - D_2(x)y(qx) + D_3(x)y(x) &= 0, \\ L_3(x) : \bar{D}_1(x/q)y(x) + D_3(x/q)\bar{y}(x) - D_2(x)\bar{y}(x/q) &= 0. \end{aligned}$$

We define basic quantities $G(x)$, $K(x)$ and $H(x)$ (e.g. (3.20), (3.33)) by

$$(3.9) \quad G(x) = Y(qx)/Y(x), \quad K(x) = \bar{Y}(x)/Y(x), \quad H(x) = \text{L.C.M.}(G_{\text{den}}(x), K_{\text{den}}(x)),$$

where $G_{\text{den}}(x)$ and $G_{\text{num}}(x)$ are defined as the polynomials of the denominator and the numerator of $G(x)$ respectively, and $K_{\text{den}}(x)$ and $K_{\text{num}}(x)$ are similarly defined. Substituting these quantities into the equations (3.7), we obtain the following determinants:

$$(3.10) \quad \begin{aligned} D_1(x) &= \frac{Y(x)}{G_{\text{den}}(x)} \{G_{\text{num}}(x)P_m(x)Q_n(qx) - G_{\text{den}}(x)P_m(qx)Q_n(x)\}, \\ D_2(x) &= \frac{Y(x)}{K_{\text{den}}(x)} \{K_{\text{num}}(x)P_m(x)\bar{Q}_n(x) - K_{\text{den}}(x)\bar{P}_m(x)Q_n(x)\}, \\ D_3(x) &= \frac{Y(x)}{H(x)} \left\{ \frac{H(x)}{K_{\text{den}}(x)} K_{\text{num}}(x)P_m(qx)\bar{Q}_n(x) - \frac{H(x)}{G_{\text{den}}(x)} G_{\text{num}}(x)\bar{P}_m(x)Q_n(qx) \right\}. \end{aligned}$$

Using the interpolation condition (3.1) and the form of the basic quantities (e.g. (3.20), (3.33)), we can investigate positions of zeros and degrees of the polynomials within braces $\{ \}$ of the equations (3.10). Then, we can simply compute the determinants $D_i(x)$ (e.g. (3.21), (3.34)) except for some factors such as $1 - fx$ in $D_1(x)$ and $1 - x/g$ in $D_3(x)$, where f and g , etc, are constants with respect to x . In this way, we obtain the contiguity relations $L_2(x)$ and $L_3(x)$ (e.g. (3.22), (3.35)).

Remark 3. on the gauge invariance of C_0C_1

When the common normalization factor of $P_m(x)$ and $Q_n(x)$ is changed, an x -independent gauge transformation of $y(x)$ is induced in $L_2(x)$ and $L_3(x)$. Under the x -independent gauge transformation of $y(x)$: $y(x) \mapsto Gy(x)$, the coefficients of $\bar{y}(x)$, $y(x/q)$, $y(x)$ and $y(x)$, $\bar{y}(x)$, $\bar{y}(x/q)$ in (3.8) change as follows:

$$(3.11) \quad \begin{aligned} (D_1(x) : D_2(x) : D_3(x)) &\mapsto (\bar{G}D_1(x)/G : D_2(x) : D_3(x)) \\ (\bar{D}_1(x/q) : D_3(x/q) : D_2(x)) &\mapsto (G\bar{D}_1(x/q)/\bar{G} : D_3(x/q) : D_2(x)). \end{aligned}$$

The coefficients C_0 and C_1 in $L_2(x)$ and $L_3(x)$ (e.g. (3.22), (3.35)) are defined as the normalization factors of the coefficients of $\bar{y}(x)$ and $y(x)$ respectively. Then, C_0 and C_1 change under the gauge transformation, but the product C_0C_1 is a gauge invariant quantity. Moreover, C_0 and C_1 do not appear in the final form of the q -Painlevé equations. $\square$

3.1.3. (c) Computation of the q -Painlevé equation.

We can derive the q -Painlevé equation from the compatibility condition of the contiguity relations $L_2(x)$ and $L_3(x)$ (e.g. (3.22), (3.35)). Computing the compatibility condition, we determine three quantities $\underline{g}$, $\bar{f}$ and C_0C_1 . Expressions for $\underline{g}$ and $\bar{f}$ are obtained in terms of f and g . An expression for C_0C_1 is obtained in terms of f , g and $\bar{f}$ (and hence in terms of f and g).

The first and the second expressions are the q -Painlevé equation (e.g. (3.24), (3.36)). The third expression is a constraint for the product C_0C_1 (e.g. (3.26), (3.38)).

Remark 4. on two meanings of f, g, m and n

We use the variables f and g with two different meanings. The first meaning is the f and g which are explicitly determined in terms of parameters a_i, b_i, m and n by the Padé problem. The second meaning is the f and g which are unknown functions in the q -Painlevé equation. In items (c) and (d), we consider f and g in the second meaning.

Similarly, we use the parameters m and n with two meanings. In the first meaning, m and n are integers. In the second meaning, m and n are generic complex parameters. In items (c) and (d), we consider m and n in the second meaning. Then, the result of the compatibility of $L_2(x)$ and $L_3(x)$ also holds with respect to the second meaning. $\square$

3.1.4. (d) Computation of the Lax pair.

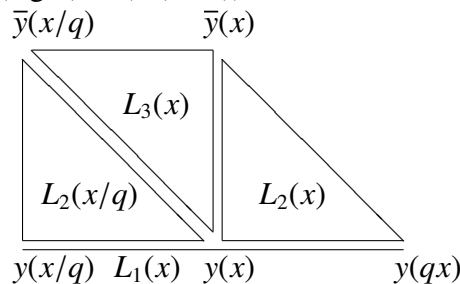
- We will consider the following pair of linear q -difference equations for the unknown function $y(x)$:

$$(3.12) \quad \begin{aligned} L_1(x) : A_1(x)y(x/q) + A_2(x)y(x) + A_3(x)y(qx) &= 0, \\ L_2(x) : A_4(x)\bar{y}(x) + A_5(x)y(x) + A_6(x)y(qx) &= 0, \end{aligned}$$

such that their compatibility condition gives a q -Painlevé equation. Then, the pair of q -difference equations is called the "scalar Lax pair".

- We will show the method of computation of the scalar Lax pair (3.12).

The Lax pair $L_1(x)$ and $L_2(x)$, which satisfies the compatibility condition, is derived by using the results of items (a)–(c) as follows: The $L_2(x)$ equation in item (d) (e.g. (3.27), (3.39)) is the same as the $L_2(x)$ in the item (b). We can obtain the Lax equation $L_1(x)$ as follows: First, combining the contiguity relations $L_2(x)$ and $L_3(x)$ (e.g. (3.27), (3.39), etc.), one obtains an equation between the three terms $y(qx)$, $\bar{y}(x)$ and $y(x/q)$ (See the figure below), whose coefficient functions depend on the variables $f, g, \bar{f}, C_0$ and C_1 . However, the variables C_0 and C_1 appear through the product C_0C_1 . Therefore, expressing $\bar{f}$ and C_0C_1 in terms of f and g only, one obtains the Lax equation $L_1(x)$ (e.g. (3.27), (3.39)).



3.1.5. (e) Computation of special solutions.

By construction, expressions for f and g as in the first meaning in Remark 4 give a special solution for the q -Painlevé equation. We will present how to compute determinant formulae of the special solutions.

- We will derive formulae (3.15) which are convenient for computing the special solutions f and g . The Cauchy-Jacobi formulae (3.14) are essentially presented in [16]. For a given sequence Y_s , the polynomials $P_m(x)$ and $Q_n(x)$ of degree m and n for an interpolation problem

$$(3.13) \quad Y_s = P_m(x_s)/Q_n(x_s) \quad (s = 0, 1, \dots, m+n)$$

are given by the following determinant expressions:

$$(3.14) \quad P_m(x) = F(x) \det \left[\sum_{s=0}^{m+n} u_s \frac{x_s^{i+j}}{x - x_s} \right]_{i,j=0}^n, \quad Q_n(x) = \det \left[\sum_{s=0}^{m+n} u_s x_s^{i+j} (x - x_s) \right]_{i,j=0}^{n-1},$$

where $u_s = Y_s/F'(x_s)$ and $F(x) = \prod_{i=0}^{m+n} (x - x_i)$.

In the q -grid case of problem (3.13) (i.e. the case of problem (3.1)), the formulae (3.14) take the following form:

$$(3.15) \quad \begin{aligned} P_m(x) &= \frac{F(x)}{(q; q)_{m+n}^{n+1}} \det \left[\sum_{s=0}^{m+n} Y_s \frac{(q^{-(m+n)}; q)_s}{(q; q)_s} \frac{q^{s(i+j+1)}}{x - q^s} \right]_{i,j=0}^n, \\ Q_n(x) &= \frac{1}{(q; q)_{m+n}^n} \det \left[\sum_{s=0}^{m+n} Y_s \frac{(q^{-(m+n)}; q)_s}{(q; q)_s} q^{s(i+j+1)} (x - q^s) \right]_{i,j=0}^{n-1}. \end{aligned}$$

In the derivation of (3.15), we have used the following relations:

$$(3.16) \quad \begin{aligned} F'(x_s) &= (x_s - x_0) \cdots (x_s - x_{s-1})(x_s - x_{s+1}) \cdots (x_s - x_{m+n}) \\ &= (-1)^s q^{(s-1)s/2} (q; q)_s q^{s(m+n-s)} (q; q)_{m+n-s} \\ &= (q; q)_s (q; q)_{m+n} / q^s (q^{-(m+n)}; q)_s. \end{aligned}$$

Moreover, substituting the values of Y_s (3.2) and $F'(x_s)$ (3.16) into the formulae (3.14), one obtains the determinant formulae (3.15).

- We will show the method of computation of the special solutions f and g .

The expressions for the f and g can be derived by comparing $D_i(x)$ in (3.10) and $D_i(x)$ (e.g. (3.21), (3.34)) in item (b) as the identity with respect to the variable x and applying the formulae (3.15).

For example, the computation for the case q - $E_7^{(1)}$ is as follows: Substituting $x = 1/a_i$ into $D_1(x)$ in (3.10) and $D_1(x)$ in (3.21), we obtain an expression for the f in the first equation of (3.28) by comparing the two expressions for $D_1(x)$ and applying the formulae (3.15). Similarly, substituting $x = 1/b_i$ into the third equation of (3.10) and $D_3(x)$ in (3.21), we obtain an expression for the g in the second equation of (3.28) by comparing the two expressions for $D_3(x)$ and applying the formulae (3.15).

3.2. Main results.

In this subsection, we will present the results obtained through the method, which is explained for each case from q - $E_7^{(1)}$ to q -($A_2 + A_1$)⁽¹⁾ in subsection 3.1.

We use the following notations:

$$(3.17) \quad \begin{aligned} a_1 a_2 \cdots a_n / b_1 b_2 \cdots b_n &= \frac{a_1 a_2 \cdots a_n}{b_1 b_2 \cdots b_n}, \\ \mathcal{N}(x) &= \prod_{i=0}^{m+n-1} (1 - x/q^i), \\ T_{a_i}(F) &= F|_{a_i \rightarrow qa_i}, \quad T_{a_i}^{-1}(F) = F|_{a_i \rightarrow a_i/q}, \end{aligned}$$

for any quantity (or function) F depending on variables a_i and b_i .

3.2.1. Case q - $E_7^{(1)}$.

(a) Setting of the Padé interpolation problem

The interpolated function, the interpolated sequence and the constraint (3.2):

$$(3.18) \quad Y(x) = \prod_{i=1}^3 \frac{(a_i x, b_i; q)_\infty}{(a_i, b_i x; q)_\infty}, \quad Y_s = \prod_{i=1}^3 \frac{(b_i; q)_s}{(a_i; q)_s}, \quad \frac{a_1 a_2 a_3 q^m}{b_1 b_2 b_3 q^n} = 1.$$

The time evolution (3.5):

$$(3.19) \quad T : (a_1, a_2, a_3, b_1, b_2, b_3, m, n) \mapsto (qa_1, a_2, qa_3, b_1, b_2, qb_3, m-1, n).$$

(b) Contiguity relations

The basic quantities:

$$(3.20) \quad G(x) = \prod_{i=1}^3 \frac{(1 - b_i x)}{(1 - a_i x)}, \quad K(x) = \frac{1 - b_3 x}{1 - b_3} \prod_{i=1,3} \frac{(1 - a_i)}{(1 - a_i x)}, \quad H(x) = (1 - b_3) \prod_{i=1}^3 (1 - a_i x).$$

The Casorati determinants:

$$(3.21) \quad \begin{aligned} D_1(x) &= \frac{c_0 x (1 - x f) \mathcal{N}(x) Y(x)}{G_{\text{den}}(x)}, \quad D_2(x) = \frac{c_1 (1 - b_3 x / a_2 q^m g) \mathcal{N}(x) Y(x)}{K_{\text{den}}(x)}, \\ D_3(x) &= \frac{c_1 (1 - b_3 x) (1 - x/g) \mathcal{N}(x) Y(x)}{H(x)}, \end{aligned}$$

where f, g, c_0 and c_1 are constants with respect to x .

The contiguity relations:

$$(3.22) \quad \begin{aligned} L_2(x) : C_0 x (1 - x f) \bar{y}(x) - (1 - a_2 x) (1 - b_3 x / a_2 q^m g) y(qx) + (1 - b_3 x) (1 - x/g) y(x) &= 0, \\ L_3(x) : C_1 x (1 - x \bar{f}/q) y(x) + \frac{A_2(x)}{1 - b_3 x} (1 - \frac{x}{qg}) \bar{y}(x) - \frac{A_1(x/q)}{1 - a_2 x/q} (1 - b_3 x / a_2 q^m g) \bar{y}(x/q) &= 0, \end{aligned}$$

where

$$\begin{aligned}
 (3.23) \quad A_1(x) &= (1 - a_2x)(1 - qx) \prod_{i=1,2} (1 - b_i x), \\
 A_2(x) &= (1 - b_3x)(1 - x/q^{m+n}) \prod_{i=1,3} (1 - a_i x), \\
 C_0 &= c_0(1 - b_3)/c_1, \quad C_1 = \bar{c}_0(1 - a_1)(1 - a_3)/qc_1.
 \end{aligned}$$

(c) The q -Painlevé equation

Compatibility gives the following equations:

$$\begin{aligned}
 (3.24) \quad & \frac{(fg - 1)(\underline{f}\underline{g} - 1)}{(fg - b_3/a_2q^m)(\underline{f}\underline{g} - b_3/a_2q^{m+1})} = \frac{A_1(1/f)}{A_2(1/f)}, \\
 & \frac{(1 - fg)(1 - \bar{f}\bar{g})}{(1 - a_2q^m fg/b_3)(1 - a_2q^{m-1} \bar{f}\bar{g}/b_3)} = \frac{A_1(g)}{A_2(a_2q^m g/b_3)}.
 \end{aligned}$$

These equations (3.24) are equivalent to the q -Painlevé equation of type $E_7^{(1)}$ given in [8] [23] [48]. The 8 singular points are on the two curves $fg = 1$ and $fg = b_3/a_2q^m$.

$$\begin{aligned}
 (3.25) \quad (f, g) &= (a_2, 1/a_2), (b_1, 1/b_1), (b_2, 1/b_2), (q, 1/q), \\
 & (a_1, b_3/a_1a_2q^m), (b_3, 1/a_2q^m), (1/q^{m+n}, b_3q^n/a_2), (a_3, b_3/a_2a_3q^m).
 \end{aligned}$$

The product C_0C_1 :

$$(3.26) \quad C_0C_1 = \frac{A_1(g)(1 - b_3/a_2q^m)(1 - b_3/a_2q^{m-1})}{q(1 - fg)(1 - \bar{f}\bar{g})g^2}.$$

(d) The Lax pair

$$\begin{aligned}
 (3.27) \quad L_1(x) : & \frac{(b_3 - a_2q^m)x^2}{a_2b_3q^mg} \left[\frac{A_1(g)}{(fg - 1)(qg - x)} - \frac{A_2(a_2q^mg/b_3)}{(a_2q^m fg/b_3 - 1)(a_2q^mg/b_3 - x)} \right] y(x) \\
 & + \frac{q^2(q - b_3x)A_1(x/q)}{(q - a_2x)(q - fx)} \left[y(x/q) - \frac{b_3(a_2q^{m+1}g/b_3 - x)(q - a_2x)}{a_2q^m(qg - x)(q - b_3x)} y(x) \right] \\
 & + \frac{(1 - a_2x)A_2(x)}{(1 - b_3x)(1 - fx)} \left[y(qx) - \frac{a_2q^m(g - x)(1 - b_3x)}{b_3(a_2q^mg/b_3 - x)(1 - a_2x)} y(x) \right] = 0, \\
 L_2(x) : & C_0x(1 - xf)\bar{y}(x) - (1 - a_2x)(1 - b_3x/a_2q^mg)y(qx) \\
 & + (1 - b_3x)(1 - x/g)y(x) = 0.
 \end{aligned}$$

The scalar Lax pair (3.27) is equivalent to that in [59] by using a suitable gauge transformation of $y(x)$. (Note that there is a typographical error in the second equation of (36) in [59], namely the expression $\bar{f}g - t^2$ should read $\bar{f}gq - t^2$.)

(e) Special solutions

$$(3.28) \quad \frac{1 - f/a_1}{1 - f/a_2} = \frac{\gamma_1 T_{a_1}(\tau_{m,n}) T_{a_1}^{-1}(\tau_{m+1,n-1})}{\gamma_2 T_{a_2}(\tau_{m,n}) T_{a_2}^{-1}(\tau_{m+1,n-1})}, \quad \frac{1 - 1/b_1g}{1 - 1/b_2g} = \frac{\omega_1 T_{b_1}^{-1}(\tau_{m,n}) T_{b_1}(\bar{\tau}_{m+1,n-1})}{\omega_2 T_{b_2}^{-1}(\tau_{m,n}) T_{b_2}(\bar{\tau}_{m+1,n-1})},$$

where

$$(3.29) \quad \tau_{m,n} = \det \left[{}_4\varphi_3 \left(\begin{matrix} b_1, b_2, b_3, q^{-(m+n)} \\ a_1, a_2, a_3 \end{matrix} ; q, q^{i+j+1} \right) \right]_{i,j=0}^n,$$

$$(3.30) \quad \gamma_i = \frac{a_i(1 - a_i q^{m+n})(1 - a_i/q)^n \prod_{k=1}^3 (1 - b_k/a_i)}{(1 - a_i)^{n+1}}, \quad \omega_i = \frac{(1 - a_2/b_i)(1 - b_i)^n}{(1 - b_i/q)^n}, \quad \text{for } i = 1, 2.$$

These determinant formulae of hypergeometric solutions (3.28) are expressed in terms of the terminating balanced ${}_4\varphi_3$ (3.4) series (Askey-Wilson polynomials [23] [27] [39]), and they are expected to be equivalent to the terminating case of that in [33].

3.2.2. Case q - $E_6^{(1)}$.

The contents of this subsubsection is the same as [14].

(a) Setting of the Padé interpolation problem

The interpolated function and the interpolated sequence (3.2):

$$(3.31) \quad Y(x) = \prod_{i=1}^2 \frac{(a_i x, b_i; q)_\infty}{(a_i, b_i x; q)_\infty}, \quad Y_s = \prod_{i=1}^2 \frac{(b_i; q)_s}{(a_i; q)_s}.$$

The time evolution (3.5):

$$(3.32) \quad T : (a_1, a_2, b_1, b_2, m, n) \mapsto (qa_1, a_2, b_1, b_2, m-1, n).$$

(b) Contiguity relations

The basic quantities:

$$(3.33) \quad G(x) = \prod_{i=1}^2 \frac{(1 - b_i x)}{(1 - a_i x)}, \quad K(x) = \frac{1 - a_1}{1 - a_1 x}, \quad H(x) = \prod_{i=1}^2 (1 - a_i x).$$

The Casorati determinants:

$$(3.34) \quad D_1(x) = \frac{c_0 x(1 - xf)\mathcal{N}(x)Y(x)}{G_{\text{den}}(x)}, \quad D_2(x) = \frac{c_1 \mathcal{N}(x)Y(x)}{K_{\text{den}}(x)}, \quad D_3(x) = \frac{c_1(1 - x/g)\mathcal{N}(x)Y(x)}{H(x)},$$

where f, g, c_0 and c_1 are constants with respect to x .

The contiguity relations:

$$(3.35) \quad \begin{aligned} L_2(x) : C_0 x(1 - xf)\bar{y}(x) - (1 - a_2 x)y(qx) + (1 - x/g)y(x) &= 0, \\ L_3(x) : C_1 x(1 - x\bar{f}/q)y(x) + (1 - a_1 x)(1 - x/q^{m+n})(1 - x/qg)\bar{y}(x) \\ &\quad - (1 - x)(1 - b_1 x/q)(1 - b_2 x/q)\bar{y}(x/q) = 0, \end{aligned}$$

where $C_0 = c_0/c_1$ and $C_1 = \bar{c}_0(1 - a_1)/qc_1$.

(c) The q -Painlevé equation

Compatibility gives the following equations:

$$(3.36) \quad \frac{(fg-1)(\underline{fg}-1)}{g\underline{g}} = \frac{(f-a_2)(f-b_1)(f-b_2)(f-q)}{(f-a_1)(f-1/q^{m+n})},$$

$$\frac{(fg-1)(\overline{fg}-1)}{f\overline{f}} = \frac{(g-1/a_2)(g-1/b_1)(g-1/b_2)(g-1/q)}{(g-1/a_2q^m)(g-a_1/b_1b_2q^n)}.$$

These equations (3.36) are equivalent to the q -Painlevé equation of type $E_6^{(1)}$ given in [14] [23] [48]. The 8 singular points are on the two lines $f = 0$ and $g = 0$ and one curve $fg = 1$.

$$(3.37) \quad (f, g) = (a_2, 1/a_2), (b_1, 1/b_1), (b_2, 1/b_2), (q, 1/q),$$

$$(a_1, 0), (1/q^{m+n}, 0), (0, 1/a_2q^m), (0, a_1/b_1b_2q^n).$$

The product C_0C_1 :

$$(3.38) \quad C_0C_1 = \frac{(1-a_2g)(1-b_1g)(1-b_2g)(1-qg)}{qg^2(1-fg)(1-\overline{fg})}.$$

(d) The Lax pair

$$(3.39) \quad L_1(x) : \frac{q^{m+n}g(1-x)(q-b_1x)(q-b_2x)}{q-fx} \left[y(x/q) - \frac{g(q-a_2x)}{qg-x} y(x) \right]$$

$$+ \frac{g(q^{m+n}-x)(1-a_1x)(1-a_2x)}{1-fx} \left[y(qx) - \frac{(g-x)}{g(1-a_2x)} y(x) \right]$$

$$+ x^2 \left[\frac{(a_2q^mg-1)(b_1b_2q^ng-a_1)}{f} - \frac{q^{m+n}(a_2g-1)(b_1g-1)(b_2g-1)(qg-1)}{(fg-1)(qg-x)} \right] y(x) = 0,$$

$$L_2(x) : C_0x(1-xf)\overline{y}(x) - (1-a_2x)y(qx) + (1-x/g)y(x) = 0.$$

The scalar Lax pair (3.39) is equivalent to the 2×2 matrix ones in [52] [57] and the scalar ones in [14] [59] by using suitable gauge transformations of $y(x)$. (Note that there are some typographical errors in equations (30) and (31) in [14], namely the expressions $(b_4q' - z)$ and $Y(x)$ should read $(b_4q' - z)t^2$ and $Y(z)$ respectively.)

(e) Special solutions

$$(3.40) \quad \frac{1-f/a_1}{1-f/a_2} = \frac{\gamma_1 T_{a_1}(\tau_{m,n}) T_{a_1}^{-1}(\tau_{m+1,n-1})}{\gamma_2 T_{a_2}(\tau_{m,n}) T_{a_2}^{-1}(\tau_{m+1,n-1})}, \quad \frac{1-1/b_1g}{1-1/b_2g} = \frac{\omega_1 T_{b_1}^{-1}(\tau_{m,n}) T_{b_1}(\overline{\tau}_{m+1,n-1})}{\omega_2 T_{b_2}^{-1}(\tau_{m,n}) T_{b_2}(\overline{\tau}_{m+1,n-1})},$$

where

$$(3.41) \quad \tau_{m,n} = \det \left[{}_3\varphi_2 \left(\begin{matrix} b_1, b_2, q^{-(m+n)} \\ a_1, a_2 \end{matrix}; q, q^{i+j+1} \right) \right]_{i,j=0}^n,$$

$$(3.42) \quad \gamma_i = \frac{a_i(1-a_iq^{m+n})(1-a_i/q)^n \prod_{k=1}^2 (1-b_k/a_i)}{(1-a_i)^{n+1}}, \quad \omega_i = \frac{(1-a_2/b_i)(1-b_i)^n}{(1-b_i/q)^n}, \quad \text{for } i = 1, 2.$$

These determinant formulae of hypergeometric solutions (3.40) are expressed in terms of the terminating ${}_3\varphi_2$ (3.4) series (big q -Jacobi polynomials [23] [27] [39]), and they are equivalent to that in [14]. (Note that there is a typographical error in equation (38) in [14], namely the

expression $T_{a_2}T_{a_3}(\tau_{m,n-1})$ should read $T_{a_2}T_{a_4}(\tau_{m,n-1})$.

3.2.3. Case q - $D_5^{(1)}$.

(a) Setting of the Padé interpolation problem

The interpolated function and the interpolated sequence (3.2):

$$(3.43) \quad Y(x) = c^{\log_q x} \frac{(a_1x, b_1; q)_\infty}{(a_1, b_1x; q)_\infty}, \quad Y_s = c^s \frac{(b_1; q)_s}{(a_1; q)_s}.$$

The time evolution (3.5):

$$(3.44) \quad T : (a_1, b_1, c, m, n) \mapsto (qa_1, b_1, c, m-1, n).$$

(b) Contiguity relations

The basic quantities:

$$(3.45) \quad G(x) = \frac{(1-b_1x)c}{1-a_1x}, \quad K(x) = \frac{1-a_1}{1-a_1x}, \quad H(x) = 1-a_1x.$$

The Casorati determinants:

$$(3.46) \quad D_1(x) = \frac{c_0(1-xf)N(x)Y(x)}{G_{\text{den}}(x)}, \quad D_2(x) = \frac{c_1N(x)Y(x)}{K_{\text{den}}(x)}, \quad D_3(x) = \frac{c_2N(x)Y(x)}{H(x)},$$

where f, c_0, c_1 and c_2 are constants with respect to x .

The contiguity relations:

$$(3.47) \quad \begin{aligned} L_2(x) : C_0(1-xf)\bar{y}(x) - y(qx) + y(x)/g &= 0, \\ L_3(x) : C_1(1-x\bar{f}/q)y(x) - (1-a_1x)(1-x/q^{m+n})\bar{y}(x)/g \\ &\quad + c(1-x)(1-b_1x/q)\bar{y}(x/q) = 0, \end{aligned}$$

where $C_0 = c_0/c_1, C_1 = -\bar{c}_0(1-a_1)/c_1$ and $g = c_1/c_2$.

(c) The q -Painlevé equation

Compatibility gives the following equations:

$$(3.48) \quad g\underline{g} = \frac{(f-a_1)(f-1/q^{m+n})}{c(f-b_1)(f-q)}, \quad f\bar{f} = \frac{qb_1(g-1/q^m)(g-a_1/b_1q^n c)}{(g-1)(g-1/c)}.$$

These equations (3.48) are equivalent to the q -Painlevé equation of type $D_5^{(1)}$ given in [20] [23]. The 8 singular points are on the four lines $f = 0, f = \infty, g = 0$ and $g = \infty$.

$$(3.49) \quad (f, g) = (a_1, 0), (1/q^{m+n}, 0), (0, 1/q^m), (0, a_1/b_1cq^n), (\infty, 1), (\infty, 1/c), (b_1, \infty), (q, \infty).$$

The product C_0C_1 :

$$(3.50) \quad C_0C_1 = -(1-g)(1-cg)/g^2.$$

(d) The Lax pair

$$\begin{aligned}
(3.51) \quad L_1(x) : & \frac{g(x - q^{m+n})(a_1x - 1)}{fx - 1} \left[y(qx) - \frac{y(x)}{g} \right] + \frac{cq^{m+n}g(x-1)(b_1x - q)}{fx - q} \left[y\left(\frac{x}{q}\right) - gy(x) \right] \\
& + \left[\frac{(q^m g - 1)(b_1cq^n g - a_1)x}{f} - q^{m+n}(g-1)(cg-1) \right] y(x) = 0, \\
L_2(x) : & C_0(1 - xf)\bar{y}(x) - y(qx) + y(x)/g = 0.
\end{aligned}$$

The scalar Lax pair (3.51) is equivalent to the 2×2 matrix ones in [20] [35] and the scalar one in [59] by using suitable gauge transformations of $y(x)$.

(e) Special solutions

$$\begin{aligned}
(3.52) \quad \frac{f}{a_1} - 1 &= \frac{c(1 - b_1/a_1)(1 - a_1q^{m+n})(1 - a_1/q)^n}{q^m(1 - c)(1 - a_1)^{n+1}} \frac{T_{a_1}(\tau_{m,n,1})T_{a_1}^{-1}(\tau_{m+1,n-1,1})}{\tau_{m,n,0}\tau_{m+1,n-1,2}}, \\
g &= \frac{(1 - a_1q^{m+n})(1 - b_1/q)^n}{q^m(1 - a_1)(1 - b_1)^n} \frac{T_{a_1}(\tau_{m,n,1})\tau_{m,n-1,1}}{T_{b_1}^{-1}(\tau_{m,n,1})T_{a_1b_1}(\tau_{m,n-1,1})},
\end{aligned}$$

where

$$(3.53) \quad \tau_{m,n,k} = \det \left[{}_2\varphi_1 \left(\begin{matrix} b_1, q^{-(m+n)} \\ a_1 \end{matrix} ; q, cq^{i+j+k} \right) \right]_{i,j=0}^n.$$

These determinant formulae of hypergeometric solutions (3.52) are expressed in terms of the terminating ${}_2\varphi_1$ (3.4) series (little q -Jacobi polynomials [23] [27] [39]), and they are expected to be equivalent to the terminating case of that in [49].

3.2.4. Case q - $A_4^{(1)}$.**(a) Setting of the Padé interpolation problem**

The interpolated function and the interpolated sequence (3.2):

$$(3.54) \quad Y(x) = c^{\log_q x} \frac{(b_1; q)_\infty}{(b_1x; q)_\infty}, \quad Y_s = c^s (b_1; q)_s.$$

The time evolution (3.5):

$$(3.55) \quad T : (b_1, c, m, n) \mapsto (b_1, c, m-1, n).$$

(b) Contiguity relations

The basic quantities:

$$(3.56) \quad G(x) = (1 - b_1x)c, \quad K(x) = 1, \quad H(x) = 1.$$

The Casorati determinants:

$$(3.57) \quad D_1(x) = \frac{c_0(1 - xf)\mathcal{N}(x)Y(x)}{G_{\text{den}}(x)}, \quad D_2(x) = \frac{c_1N(x)Y(x)}{K_{\text{den}}(x)}, \quad D_3(x) = \frac{c_2N(x)Y(x)}{H(x)},$$

where f, c_0, c_1 and c_2 are constants with respect to x .

The contiguity relations:

$$(3.58) \quad \begin{aligned} L_2(x) : C_0(1 - xf)\bar{y}(x) - y(qx) + y(x)/g &= 0, \\ L_3(x) : C_1(1 - xf/q)y(x) - (1 - x/q^{m+n})\bar{y}(x)/g + c(1 - x)(1 - b_1x/q)\bar{y}(x/q) &= 0, \end{aligned}$$

where $C_0 = c_0/c_1$, $C_1 = -\bar{c}_0/c_1$ and $g = c_1/c_2$.

(c) The q -Painlevé equation

Compatibility gives the following equations:

$$(3.59) \quad g\underline{g} = \frac{f(f - 1/q^{m+n})}{c(f - b_1)(f - q)}, \quad f\bar{f} = \frac{qb_1g(g - 1/q^m)}{(g - 1)(g - 1/c)}.$$

These equations (3.59) are equivalent to the q -Painlevé equation of type $A_4^{(1)}$ given in [23] [28]. The 8 singular points are on the four lines $f = 0, f = \infty, g = 0$ and $g = \infty$. $(0, 0)$ is a double point.

$$(3.60) \quad \begin{aligned} (f, g) &= (1/q^{m+n}, 0), (0, 1/q^m), (\infty, 1), (\infty, 1/c), (b_1, \infty), (q, \infty), \\ (0, 0) &\leftarrow g/f = -1/b_1cq^n. \end{aligned}$$

The product C_0C_1 :

$$(3.61) \quad C_0C_1 = -(1 - g)(1 - cg)/g^2.$$

(d) The Lax pair

$$(3.62) \quad \begin{aligned} L_1(x) : & \frac{cgq^{m+n}(1 - x)(q - b_1x)}{q - fx} [y(x/q) - gy(x)] + \frac{g(q^{m+n} - x)}{1 - fx} [y(qx) - y(x)/g] \\ & + q^n [q^m(1 - g)(1 - cg) + b_1cg(1 - q^mg)x/f] y(x) = 0, \\ L_2(x) : & C_0(1 - xf)\bar{y}(x) - y(qx) + y(x)/g = 0. \end{aligned}$$

The scalar Lax pair (3.62) is equivalent to the 2×2 matrix one for the q -Painlevé equation of type q - $P(A_4)$ in [35] by using a suitable gauge transformation of $y(x)$.

(e) Computation of special solutions

$$(3.63) \quad f = \frac{b_1c}{(c - 1)q^m} \frac{\tau_{m,n,1}\tau_{m+1,n-1,1}}{\tau_{m,n,0}\tau_{m+1,n-1,2}}, \quad g = \frac{(1 - b_1/q)^n}{q^m(1 - b_1)^n} \frac{\tau_{m,n,1}\tau_{m,n-1,1}}{T_{b_1}^{-1}(\tau_{m,n,1})T_{b_1}(\tau_{m,n-1,1})},$$

where

$$(3.64) \quad \tau_{m,n,k} = \det \left[{}_2\varphi_1 \left(\begin{matrix} b_1, q^{-(m+n)} \\ 0 \end{matrix}; q, cq^{i+j+k} \right) \right]_{i,j=0}^n.$$

These determinant formulae of hypergeometric solutions (3.63) are expressed in terms of the terminating ${}_2\varphi_1$ (3.4) series (q -Laguerre polynomials [23] [27] [39]), and they are expected to be equivalent to the terminating case of that in [13].

3.2.5. Case q -($A_2 + A_1$)⁽¹⁾.

(a) Setting of the Padé interpolation problem

The interpolated function and the interpolated sequence (3.2):

$$(3.65) \quad Y(x) = (d\sqrt{x/q})^{\log_q x}, \quad Y_s = q^{\binom{s}{2}} d^s.$$

The time evolution (3.5):

$$(3.66) \quad T : (d, m, n) \mapsto (d, m-1, n).$$

(b) Contiguity relations

The basic quantities:

$$(3.67) \quad G(x) = dx, \quad K(x) = 1, \quad H(x) = 1.$$

The Casorati determinants:

$$(3.68) \quad D_1(x) = \frac{c_0(1-xf)N(x)Y(x)}{G_{\text{den}}(x)}, \quad D_2(x) = \frac{c_1N(x)Y(x)}{K_{\text{den}}(x)}, \quad D_3(x) = \frac{c_2N(x)Y(x)}{H(x)},$$

where f, c_0, c_1 and c_2 are constants with respect to x .

The contiguity relations:

$$(3.69) \quad \begin{aligned} L_2(x) : C_0(1-xf)\bar{y}(x) - y(qx) + y(x)/g &= 0, \\ L_3(x) : C_1(1-x\bar{f}/q)y(x) + (1-x/q^{m+n})\bar{y}(x)/g - dx(1-x)\bar{y}(x/q)/q &= 0, \end{aligned}$$

where $C_0 = c_0/c_1$, $C_1 = \bar{c}_0/c_1$ and $g = c_1/c_2$.

(c) The q -Painlevé equation

Compatibility gives the following equation:

$$(3.70) \quad g\underline{g} = \frac{f(f-1/q^{m+n})}{d(f-q)}, \quad f\bar{f} = \frac{qdg(g-1/q^m)}{(g-1)}.$$

These equations (3.70) are equivalent to the q -Painlevé equation of type $(A_2 + A_1)^{(1)}$, namely P_{III} , given in [23] [28]. The 8 singular points are on the four lines $f = 0, f = \infty, g = 0$ and $g = \infty$. The two points $(0, 0)$ and (∞, ∞) are double points.

$$(3.71) \quad \begin{aligned} (f, g) &= (1/q^{m+n}, 0), (0, 1/q^m), (\infty, 1), (q, \infty), \\ (0, 0) &\leftarrow g/f = 1/dq^n, \\ (\infty, \infty) &\leftarrow g/f = 1/d. \end{aligned}$$

The product C_0C_1 :

$$(3.72) \quad C_0C_1 = (1-g)/g^2.$$

(d) The Lax pair

$$(3.73) \quad \begin{aligned} L_1(x) : & \frac{dgq^{m+n}(1-x)x}{fx-q} [y(x/q) - gy(x)] + \frac{g(q^{m+n}-x)}{fx-1} [y(qx) - y(x)/g] \\ & + q^n [q^m(g-1) - dg(gq^m-1)x/f] y(x) = 0, \\ L_2(x) : & C_0(1-xf)\bar{y}(x) - y(qx) + y(x)/g = 0. \end{aligned}$$

The scalar Lax pair (3.73) is equivalent to the 2×2 matrix one for the q -Painlevé equation of type q - $P(A_5)^\#$ in [35] by using a suitable gauge transformation of $y(x)$.

(e) Special solutions

$$(3.74) \quad f = \frac{d}{q^m} \frac{\tau_{m,n,1}\tau_{m+1,n-1,1}}{\tau_{m,n,0}\tau_{m+1,n-1,2}}, \quad g = \frac{1}{q^{m+n}} \frac{\tau_{m,n,1}\tau_{m,n-1,1}}{\tau_{m,n,0}\tau_{m,n-1,2}},$$

where

$$(3.75) \quad \tau_{m,n,k} = \det \left[{}_1\varphi_1 \left(\begin{matrix} q^{-(m+n)} \\ 0 \end{matrix} ; q, -dq^{i+j+k} \right) \right]_{i,j=0}^n.$$

These determinant formulae of hypergeometric solutions (3.74) are expressed in terms of the terminating ${}_1\varphi_1$ (3.4) series (Stieltjes-Wigert polynomials [23] [27] [39]), and they are expected to be equivalent to the terminating case of that in [36].

4. PADÉ APPROXIMATION METHOD TO q -PAINLEVÉ EQUATIONS

In this section, we will consider the Padé approximation method of the differential grid. In subsection 4.1, we will explain the Padé approximation method applied to the q -Painlevé equations. Namely, we will explain the methods for items (a)–(e). In subsection 4.2, we will present these main results for all the q -Painlevé equations of types from $E_6^{(1)}$ to $(A_2 + A_1)^{(1)}$.

4.1. Padé approximation method.

In this subsection, we will explain the methods for deriving the items (a)–(e) in the main results given in subsection 4.2.

4.1.1. (a) Setting of the Padé approximation problem.

• In this section, we will consider the following approximation problem (of the differential grid):

For a given function $Y(x)$, we look for functions $P_m(x)$ and $Q_n(x)$ which are polynomials of degree m and n , satisfying the approximation condition

$$(4.1) \quad Y(x) \equiv \frac{P_m(x)}{Q_n(x)} \pmod{x^{m+n+1}}.$$

We call this problem the "Padé approximation problem (of the differential grid)". Then, the function $Y(x)$ is called the "generating function" (because the $Y(x)$ generates the p_k in equation (4.13) in item (e) below), and the polynomials $P_m(x)$ and $Q_n(x)$ are called "approximating functions" respectively.

Here, the "generating function" and the "approximating function" in this section correspond to the "interpolated function" and the "interpolating function" in section 3 respectively.

The common normalization factor of the polynomials $P_m(x)$ and $Q_n(x)$ is not determined by the condition (4.1). However, this normalization factor is not essential for our arguments, i.e. the main results in subsection 4.2 (see Remark 6). The explicit expressions of $P_m(x)$ and $Q_n(x)$, which will be used in the computations for the item (e) above, were essentially given in equation (4.15) (see item (e) below).

• In this section, we will establish the approximation problems (4.1) by specifying the generating functions $Y(x)$ as follows:

$$(4.2) \quad \begin{array}{|c|c|c|c|c|} \hline & q-E_6^{(1)} & q-D_5^{(1)} & q-A_4^{(1)} & q-(A_2 + A_1)^{(1)} \\ \hline Y(x) & \prod_{i=1}^3 \frac{(a_i x; q)_\infty}{(b_i x; q)_\infty} \frac{a_1 a_2 a_3 q^m}{b_1 b_2 b_3 q^n} = 1 & \prod_{i=1}^2 \frac{(a_i x; q)_\infty}{(b_i x; q)_\infty} & \frac{(a_1 x, a_2 x; q)_\infty}{(b_1 x; q)_\infty} & (a_1 x, a_2 x; q)_\infty \\ \hline \end{array}$$

where $\frac{a_1 a_2 a_3 q^m}{b_1 b_2 b_3 q^n} = 1$ is a constraint for the parameters in the case $q-E_6^{(1)}$, and the q -shifted factorials are defined by

$$(4.3) \quad (a_1, a_2, \dots, a_i)_j = \prod_{k=0}^{j-1} (1 - a_1 q^k)(1 - a_2 q^k) \cdots (1 - a_i q^k).$$

Remark 5. on the choice of $Y(x)$

In [58] [61] [62], for differential Painlevé equations of types from P_{VI} to P_{IV} , the Padé approximation problems (4.1) were established by specifying the generating functions $Y(x)$ as follows:

$$(4.4) \quad \begin{array}{|c|c|c|c|} \hline & P_{VI} & P_V & P_{IV} \\ \hline Y(x) & (1-x)^a \left(1 - \frac{x}{t}\right)^b & (1-x)^{-b} \exp\left(\frac{xt}{1-x}\right) & \exp(2tx - x^2) \\ \hline \end{array}$$

The expressions for the generating functions $Y(x)$ in Table (4.2) are closely related to those in Table (4.4). In fact, we chose the $Y(x)$ in the Table (4.2) as q -analogs of those in the Table (4.4). $\square$

• In this section, we will consider yet another Padé problem where some parameters a_i, b_i, m and n in $Y(x)$ are shifted. The parameter shift operators T are given as follows:

$$(4.5) \quad \begin{array}{|c|c|c|} \hline & \text{parameter} & \\ \hline q-E_6^{(1)} & (a_1, a_2, a_3, b_1, b_2, b_3, m, n) & \mapsto (qa_1, a_2, a_3, qb_1, b_2, b_3, m, n) \\ q-D_5^{(1)} & (a_1, a_2, b_1, b_2, m, n) & \mapsto (qa_1, a_2, qb_1, b_2, m, n) \\ q-A_4^{(1)} & (a_1, a_2, b_1, m, n) & \mapsto (qa_1, a_2, qb_1, m, n) \\ q-(A_2 + A_1)^{(1)} & (a_1, a_2, m, n) & \mapsto (qa_1, a_2, m, n) \\ \hline \end{array}$$

Here, the operators T are called the "time evolutions", because they specify the directions of the time evolutions for q -Painlevé equations.

4.1.2. (b) Computation of contiguity relations.

The following description is almost the same as that in subsection 3.1 (b).

- We will consider the following pair of linear q -difference equations $L_2(x)$ and $L_3(x)$ for unknown function $y(x)$, which are the main object in our study.

$$(4.6) \quad \begin{aligned} L_2(x) : & \begin{vmatrix} y(x) & y(qx) & \bar{y}(x) \\ P_m(x) & P_m(qx) & \bar{P}_m(x) \\ Y(x)Q_n(x) & Y(qx)Q_n(qx) & \bar{Y}(x)\bar{Q}_n(x) \end{vmatrix} = 0, \\ L_3(x) : & \begin{vmatrix} y(x) & \bar{y}(x) & \bar{y}(x/q) \\ P_m(x) & \bar{P}_m(x) & \bar{P}_m(x/q) \\ Y(x)Q_n(x) & \bar{Y}(x)\bar{Q}_n(x) & \bar{Y}(x/q)\bar{Q}_n(x/q) \end{vmatrix} = 0. \end{aligned}$$

Then, the pair of q -difference equations (4.6) is called the "contiguity relations". Here, $\bar{F}$ and $\underline{F}$ denote $T(F)$ and $T^{-1}(F)$ respectively. T is the shift operator acting on parameters given in Table (4.5).

- We will show the method of computation of the contiguity relations $L_2(x)$ and $L_3(x)$.

Set $\mathbf{y}(x) = \begin{bmatrix} P_m(x) \\ Y(x)Q_n(x) \end{bmatrix}$ and define Casorati determinants $D_i(x)$ by

$$(4.7) \quad D_1(x) = \det[\mathbf{y}(x), \mathbf{y}(qx)], \quad D_2(x) = \det[\mathbf{y}(x), \bar{\mathbf{y}}(x)], \quad D_3(x) = \det[\mathbf{y}(qx), \bar{\mathbf{y}}(x)].$$

Then, the contiguity relations (4.6) can be rewritten as follows:

$$(4.8) \quad \begin{aligned} L_2(x) : & D_1(x)\bar{y}(x) - D_2(x)y(qx) + D_3(x)y(x) = 0, \\ L_3(x) : & \bar{D}_1(x/q)y(x) + D_3(x/q)\bar{y}(x) - D_2(x)\bar{y}(x/q) = 0. \end{aligned}$$

We define basic quantities $G(x)$, $K(x)$ and $H(x)$ (e.g. (3.20), (3.33)) by

$$(4.9) \quad G(x) = Y(qx)/Y(x), \quad K(x) = \bar{Y}(x)/Y(x), \quad H(x) = \text{L.C.M.}(G_{\text{den}}(x), K_{\text{den}}(x)),$$

where $G_{\text{den}}(x)$ and $G_{\text{num}}(x)$ are defined as the polynomials of the denominator and the numerator of $G(x)$ respectively, and $K_{\text{den}}(x)$ and $K_{\text{num}}(x)$ are similarly defined. Substituting these quantities into the equations (4.7), we obtain the following determinants:

$$(4.10) \quad \begin{aligned} D_1(x) &= \frac{Y(x)}{G_{\text{den}}(x)} \{G_{\text{num}}(x)P_m(x)Q_n(qx) - G_{\text{den}}(x)P_m(qx)Q_n(x)\}, \\ D_2(x) &= \frac{Y(x)}{K_{\text{den}}(x)} \{K_{\text{num}}(x)P_m(x)\bar{Q}_n(x) - K_{\text{den}}(x)\bar{P}_m(x)Q_n(x)\}, \\ D_3(x) &= \frac{Y(x)}{H(x)} \left\{ \frac{H(x)}{K_{\text{den}}(x)} K_{\text{num}}(x)P_m(qx)\bar{Q}_n(x) - \frac{H(x)}{G_{\text{den}}(x)} G_{\text{num}}(x)\bar{P}_m(x)Q_n(qx) \right\}. \end{aligned}$$

Using the approximation condition (4.1) and the form of the basic quantities (e.g. (4.26), (4.36)), we can investigate positions of zeros and degrees of the polynomials within braces $\{ \}$ of the equations (4.10). Then, we can simply compute the determinants $D_i(x)$ (e.g. (4.27), (4.37)) except for some factors such as $1 - fx$ in $D_1(x)$ and $1 - x/g$ in $D_3(x)$, where f and g , etc,

are constants with respect to x . In this way, we obtain the contiguity relations $L_2(x)$ and $L_3(x)$ (e.g. (4.28), (4.38)).

Remark 6. on the gauge invariance of C_0C_1

When the common normalization factor of $P_m(x)$ and $Q_n(x)$ is changed, an x -independent gauge transformation of $y(x)$ is induced in $L_2(x)$ and $L_3(x)$. Under the x -independent gauge transformation of $y(x)$: $y(x) \mapsto Gy(x)$, the coefficients of $\bar{y}(x)$, $y(x/q)$, $y(x)$ and $y(x)$, $\bar{y}(x)$, $\bar{y}(x/q)$ in (4.8) change as follows:

$$(4.11) \quad \begin{aligned} (D_1(x) : D_2(x) : D_3(x)) &\mapsto (\bar{G}D_1(x)/G : D_2(x) : D_3(x)) \\ (\bar{D}_1(x/q) : D_3(x/q) : D_2(x)) &\mapsto (G\bar{D}_1(x/q)/\bar{G} : D_3(x/q) : D_2(x)). \end{aligned}$$

The coefficients C_0 and C_1 in $L_2(x)$ and $L_3(x)$ (e.g. (4.28), (4.38)) are defined as the normalization factors of the coefficients of $\bar{y}(x)$ and $y(x)$ respectively. Then, C_0 and C_1 change under the gauge transformation, but the product C_0C_1 is a gauge invariant quantity. Moreover, C_0 and C_1 do not appear in the final form of the q -Painlevé equations. $\square$

4.1.3. (c) *Computation of the q -Painlevé equation.*

Also the following description is almost the same as that in subsection 3.1 (c).

We can derive the q -Painlevé equation from the compatibility condition of the contiguity relations $L_2(x)$ and $L_3(x)$. Computing the compatibility condition, we determine three quantities $\underline{g}$, $\bar{f}$ and C_0C_1 . Expressions for $\underline{g}$ and $\bar{f}$ are obtained in terms of f and g . An expression for C_0C_1 is obtained in terms of f , g and $\bar{f}$ (and hence in terms of f and g).

The first and the second expressions are the q -Painlevé equation (e.g. (4.29), (4.39)). The third expression is a constraint for the product C_0C_1 (e.g. (4.31), (4.41)).

Remark 7. on two meanings of f, g, m and n

We use the variables f and g with two different meanings. The first meaning is the f and g which are explicitly determined in terms of parameters a_i, b_i, m and n by the Padé problem. The second meaning is the f and g which are unknown functions in the q -Painlevé equation. In items (c) and (d), we consider f and g in the second meaning.

Similarly, we use the parameters m and n with two meanings. In the first meaning, m and n are integers. In the second meaning, m and n are generic complex parameters. In items (c) and (d), we consider m and n in the second meaning. Then, the result of the compatibility of $L_2(x)$ and $L_3(x)$ also holds with respect to the second meaning. $\square$

4.1.4. (d) *Computation of the Lax pair.*

The following description is almost the same as that in subsection 3.1 (d).

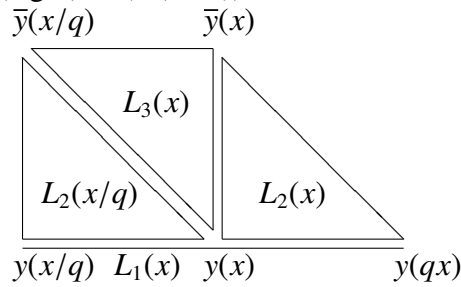
- We will consider the following pair of linear q -difference equations for the unknown function $y(x)$:

$$(4.12) \quad \begin{aligned} L_1(x) : A_1(x)y(x/q) + A_2(x)y(x) + A_3(x)y(qx) &= 0, \\ L_2(x) : A_4(x)\bar{y}(x) + A_5(x)y(x) + A_6(x)y(qx) &= 0, \end{aligned}$$

such that their compatibility condition gives a q -Painlevé equation. Then, the pair of q -difference equations is called the "scalar Lax pair".

- We will show the method of computation of the scalar Lax pair $L_1(x)$ and $L_2(x)$.

The Lax pair $L_1(x)$ and $L_2(x)$, which satisfies the compatibility condition, is derived by using the results of items (a)–(c) as follows: The $L_2(x)$ equation in item (d) (e.g. (4.32), (4.42)) is the same as the $L_2(x)$ in the item (b). We can obtain the Lax equation $L_1(x)$ as follows: First, combining the contiguity relations $L_2(x)$ and $L_3(x)$ (e.g. (4.32), (4.42), etc.), one obtains an equation between the three terms $y(qx)$, $\bar{y}(x)$ and $y(x/q)$ (See the figure below), whose coefficient functions depend on the variables $f, g, \bar{f}, C_0$ and C_1 . However, the variables C_0 and C_1 appear through the product C_0C_1 . Therefore, expressing $\bar{f}$ and C_0C_1 in terms of f and g only, one obtains the Lax equation $L_1(x)$ (e.g. (4.32), (4.42)).



4.1.5. (e) Computation of special solutions.

By construction, expressions for f and g as in the first meaning in Remark 7 give a special solution for the q -Painlevé equation. We will present how to compute determinant formulae of the special solutions.

- We will derive the formulae (4.15) which are convenient for computing the special solutions f and g . We can assume that

$$(4.13) \quad Y(x) = \sum_{k=0}^{\infty} p_k x^k, \quad p_0 = 1, \quad p_i = 0 \quad (i < 0),$$

near $x = 0$ without loss of generality. Then, for each type of $Y(x)$ in Table (4.2), the p_k (e.g. (4.44), (4.55)) are given respectively as a q -HGF (the q -hypergeometric functions [5] [6]) defined by

$$(4.14) \quad {}_k\varphi_l \left(\begin{matrix} a_1, \dots, a_k \\ b_1, \dots, b_l \end{matrix}; q, x \right) = \sum_{s=0}^{\infty} \frac{(a_1, \dots, a_k; q)_s}{(b_1, \dots, b_l, q; q)_s} \left[(-1)^s q^{\binom{s}{2}} \right]^{1+l-k} x^s,$$

with $\binom{s}{2} = s(s-1)/2$.

The formulae (4.15) were described in [58]. For a given function $Y(x)$, the polynomials $P_m(x)$ and $Q_n(x)$ of degree m and n for an approximation condition (4.1) are given by the following determinant expressions:

$$(4.15) \quad P_m(x) = \sum_{i=0}^m s_{(m^n, i)} x^i, \quad Q_n(x) = \sum_{i=0}^n s_{((m+1)^i, m^n-i)} (-x)^i,$$

where s_{λ} is the Schur function defined by the Jacobi-Trudi formula

$$(4.16) \quad s_{(\lambda_1, \dots, \lambda_l)} = \det(p_{\lambda_i - i + j})_{i,j=1}^l.$$

The derivation of (4.15) is as follows:
By using the relation

$$(4.17) \quad x^n Y(x) = \sum_{k=0}^{\infty} p_k x^{k+n} = \sum_{k=0}^{\infty} p_{k-n} x^k,$$

the formulae (4.15) are given by the following computation:

$$(4.18) \quad \begin{aligned} Y(x)Q_n(x) &= Y(x) \begin{vmatrix} 1 & p_{m+1} & \cdots & p_{m+n} \\ x & p_m & \ddots & \vdots \\ \vdots & \vdots & \ddots & p_{m+1} \\ x^n & p_{m-n+1} & \cdots & p_m \end{vmatrix} = \begin{vmatrix} p_m & p_{m+1} & \cdots & p_{m+n} \\ p_{m-1} & \ddots & \ddots & \vdots \\ \vdots & \vdots & p_m & p_{m+1} \\ x^n Y(x) & \cdots & xY(x) & Y(x) \end{vmatrix} \\ &= \sum_{k=0}^{\infty} \begin{vmatrix} p_m & p_{m+1} & \cdots & p_{m+n} \\ \vdots & \ddots & \ddots & \vdots \\ p_{m-n+1} & \cdots & p_m & p_{m+1} \\ p_{k-n} & \cdots & p_{k-1} & p_k \end{vmatrix} x^k. \end{aligned}$$

Here, we note that

$$(4.19) \quad \sum_{k=m+1}^{m+n} \begin{vmatrix} p_m & p_{m+1} & \cdots & p_{m+n} \\ \vdots & \ddots & \ddots & \vdots \\ p_{m-n+1} & \cdots & p_m & p_{m+1} \\ p_{k-n} & \cdots & p_{k-1} & p_k \end{vmatrix} x^k = 0.$$

Substituting equation (4.19) into equation (4.18), we obtain

$$(4.20) \quad Y(x)Q_n(x) = \left(\sum_{k=0}^m + \sum_{k=m+n+1}^{\infty} \right) s_{(m^n, k)} x^k = P_m(x) + \sum_{k=m+n+1}^{\infty} s_{(m^n, k)} x^k. \quad \square$$

Furthermore, in [58], interesting formulae were also written as follows: The polynomials $P_m(x)$ and $Q_n(x)$ can be expressed in terms of a single determinant as

$$(4.21) \quad P_m(x) = x^m s_{(m^{n+1})} |_{p_i \rightarrow \sum_{j=0}^i x^{-j} p_{i-j}}, \quad Q_n(x) = (-x)^n s_{((m+1)^n)} |_{p_i \rightarrow p_i - x^{-1} p_{i-1}}.$$

• We will show the method of computation of the special solutions f and g .

The expressions for the f and g can be derived by comparing $D_i(x)$ in (4.10) and $D_i(x)$ (e.g. (4.27), (4.37)) in item (b) as the identity with respect to the variable x and applying the formulae (4.15) and (4.21).

For example, the computation for the case $q-E_6^{(1)}$ is as follows: Substituting $x = 1/a_i$ into $D_1(x)$ in (4.10) and $D_1(x)$ in (4.27), we obtain an expression for the f in the first equation of (4.33) by comparing the two expressions for $D_1(x)$ and applying the formulae (4.21). Similarly, substituting $x = 1/b_i$ into the third equation of (4.10) and $D_3(x)$ in (4.27), we obtain an expression for the g in the second equation of (4.33) by comparing the two expressions for $D_3(x)$ and applying the formulae (4.21).

4.2. Main results.

In this subsection, we will present the results obtained through the method, which is explained for each case from $q-E_6^{(1)}$ to $q-(A_2 + A_1)^{(1)}$ in subsection 4.1.

We use the following notations:

$$\begin{aligned}
 (4.22) \quad & a_1 a_2 \cdots a_n / b_1 b_2 \cdots b_n = \frac{a_1 a_2 \cdots a_n}{b_1 b_2 \cdots b_n}, \\
 & \mathcal{N}(x) = x^{m+n+1}, \\
 & \tau_{m,n} = s_{(m^n)}, \\
 & T_{a_i}(F) = F|_{a_i \rightarrow qa_i}, \quad T_{a_i}^{-1}(F) = F|_{a_i \rightarrow a_i/q}.
 \end{aligned}$$

Here, any quantity (or function) F depends on variables a_i and b_i , and by definition (4.16), the Schur functions $s_{(m^n)}$ is expressed as

$$(4.23) \quad s_{(m^n)} = \begin{vmatrix} p_m & p_{m+1} & \cdots & p_{m+n-1} \\ p_{m-1} & \ddots & & \vdots \\ \vdots & \cdots & \ddots & \vdots \\ p_{m-n+1} & \cdots & \cdots & p_m \end{vmatrix}$$

where the p_k are defined in (4.13).

4.2.1. Case q - $E_6^{(1)}$.

(a) Setting of the Padé approximation problem

The generating function and the constraint (4.2):

$$(4.24) \quad Y(x) = \frac{(a_1 x, a_2 x, a_3 x; q)_\infty}{(b_1 x, b_2 x, b_3 x; q)_\infty}, \quad \frac{a_1 a_2 a_3 q^m}{b_1 b_2 b_3 q^n} = 1.$$

The time evolution (4.5):

$$(4.25) \quad T : (a_1, a_2, a_3, b_1, b_2, b_3, m, n) \mapsto (qa_1, a_2, a_3, qb_1, b_2, b_3, m, n).$$

(b) Contiguity relations

The basic quantities:

$$(4.26) \quad G(x) = \prod_{i=1}^3 \frac{(1 - b_i x)}{(1 - a_i x)}, \quad K(x) = \frac{1 - b_1 x}{1 - a_1 x}, \quad H(x) = \prod_{i=1}^3 (1 - a_i x).$$

The Casorati determinants:

$$\begin{aligned}
 (4.27) \quad & D_1(x) = c_0(1 - xf)\mathcal{N}(x)Y(x)/G_{\text{den}}(x), \quad D_2(x) = c_1\mathcal{N}(x)Y(x)/K_{\text{den}}(x), \\
 & D_3(x) = c_1 \frac{a_2 a_3 q^m g}{b_1} (1 - x/g)\mathcal{N}(x)Y(x)/H(x),
 \end{aligned}$$

where f, g, c_0 and c_1 are constants with respect to x .

The contiguity relations:

$$\begin{aligned}
 L_2(x) : C_0(1 - xf)\bar{y}(x) - (1 - a_2x)(1 - a_3x)y(qx) \\
 + \frac{a_2a_3q^mg}{b_1}(1 - b_1x)(1 - x/g)y(x) = 0, \\
 (4.28) \quad L_3(x) : C_1(1 - x\bar{f}/q)y(x) + \frac{a_2a_3q^mg}{b_1}(1 - a_1x)(1 - x/qg)\bar{y}(x) \\
 - q^{m+n+1}(1 - b_2x/q)(1 - b_3x/q)\bar{y}(x/q) = 0,
 \end{aligned}$$

where $C_0 = c_0/c_1$ and $C_1 = \bar{c}_0/c_1$.

(c) The q -Painlevé equation

Compatibility gives the following equations:

$$\begin{aligned}
 (4.29) \quad (fg - 1)(f\underline{g} - 1) &= \frac{b_1^2}{a_2^2a_3^2q^{m-n}} \frac{(f - a_2)(f - a_3)(f - b_2)(f - b_3)}{(f - a_1)(f - b_1)}, \\
 (fg - 1)(\bar{f}g - 1) &= qa_1b_1 \frac{(g - 1/a_2)(g - 1/a_3)(g - 1/b_2)(g - 1/b_3)}{(g - b_1/a_2a_3q^m)(g - b_1q^{n+1}/a_2a_3)}.
 \end{aligned}$$

These equations (4.29) are equivalent to the q -Painlevé equation of type $E_6^{(1)}$ given in [14] [23] [48]. The 8 singular points are on the two lines $f = \infty$ and $g = \infty$ and one curve $fg = 1$.

$$\begin{aligned}
 (4.30) \quad (f, g) &= (1/a_2, a_2), (1/a_3, a_3), (1/b_2, b_2), (1/b_3, b_3), \\
 & (a_1, \infty), (b_1, \infty), (\infty, b_1/a_2a_3q^m), (\infty, b_1q^{n+1}/a_2a_3).
 \end{aligned}$$

The product C_0C_1 :

$$(4.31) \quad C_0C_1 = q^m(b_1 - a_2a_3gq^m)(b_1q^{1+n} - a_2a_3g)/b_1^2.$$

(d) The Lax pair

$$\begin{aligned}
 (4.32) \quad L_1(x) : & \frac{(q - b_1x)(q - b_2x)(q - b_3x)q^{m+n-1}}{q - fx} \left[y(x/q) - \frac{a_1(q - a_2x)(q - a_3x)}{b_2b_3q^n(q - b_1x)(gq - x)} y(x) \right] \\
 & + \frac{(1 - a_1x)(1 - a_2x)(1 - a_3x)}{1 - fx} \left[y(qx) - \frac{b_2b_3q^n(1 - b_1x)(g - x)}{a_1(1 - a_2x)(1 - a_3x)} y(x) \right] \\
 & + \frac{a_1q^{m+1}}{b_2b_3g} \left[\left(1 - \frac{b_2b_3g}{a_1q^{m+1}} \right) \left(1 - \frac{b_2b_3gq^n}{a_1} \right) + \frac{x(1 - a_2g)(1 - a_3g)(1 - b_2g)(1 - b_3g)}{(1 - fg)(gq - x)} \right] y(x) \\
 L_2(x) : & (1 - xf)\bar{y}(x) - (1 - a_2x)(1 - a_3x)y(qx) + \frac{a_2a_3q^mg}{b_1}(1 - b_1x)(1 - x/g)y(x) = 0.
 \end{aligned}$$

The scalar Lax pair (4.32) is equivalent to the 2×2 matrix ones in [52] [57] and the scalar ones in [14] [59] by using suitable gauge transformations of $y(x)$ (see the comment below equation (3.39)).

(e) Special solutions

$$(4.33) \quad \begin{aligned} \frac{1 - f/a_1}{1 - f/a_2} &= \frac{a_1 \prod_{i=1}^3 (1 - b_i/a_1) T_{a_1}(\tau_{m,n+1}) T_{a_1}^{-1}(\tau_{m+1,n})}{a_2 \prod_{i=1}^3 (1 - b_i/a_2) T_{a_2}(\tau_{m,n+1}) T_{a_2}^{-1}(\tau_{m+1,n})}, \\ \frac{1 - 1/b_2 g}{1 - 1/b_3 g} &= \frac{b_2 \prod_{i=2}^3 (1 - a_i/b_2) T_{b_2}^{-1}(\tau_{m,n+1}) T_{b_2}(\bar{\tau}_{m+1,n})}{b_3 \prod_{i=2}^3 (1 - a_i/b_3) T_{b_3}^{-1}(\tau_{m,n+1}) T_{b_3}(\bar{\tau}_{m+1,n})}, \end{aligned}$$

where the p_k in the $\tau_{m,n}$ (3.17) are expressed in terms of the terminating ${}_3\phi_2$ (4.14) series (big q -Jacobi polynomials [23] [27] [39]) under the transformation of $b_3 = q^{n-1}a_1a_2$ and $a_3 = b_1b_2$. These determinant formulae of hypergeometric solutions (4.33) are equivalent to that in [14] (see the comment below equation (3.42)).

4.2.2. Case q - $D_5^{(1)}$.

The contents of this subsection (a) (b) (c) is the same as [14].

(a) Setting of the Padé approximation problem

The generating function (4.2):

$$(4.34) \quad Y(x) = \frac{(a_1x, a_2x; q)_\infty}{(b_1x, b_2x; q)_\infty}.$$

The time evolution (4.5):

$$(4.35) \quad T : (a_1, a_2, b_1, b_2, m, n) \mapsto (qa_1, a_2, qb_1, b_2, m, n).$$

(b) Contiguity relations

The basic quantities:

$$(4.36) \quad G(x) = \prod_{i=1}^2 \frac{(1 - b_i x)}{(1 - a_i x)}, \quad K(x) = \frac{1 - b_1 x}{1 - a_1 x}, \quad H(x) = \prod_{i=1}^2 (1 - a_i x).$$

The Casorati determinants:

$$(4.37) \quad D_1(x) = \frac{c_0(1 - xf)\mathcal{N}(x)Y(x)}{G_{\text{den}}(x)}, \quad D_2(x) = \frac{c_1N(x)Y(x)}{K_{\text{den}}(x)}, \quad D_3(x) = \frac{c_2(1 - b_1x)N(x)Y(x)}{H(x)},$$

where f, c_0, c_1 and c_2 are constants with respect to x .

The contiguity relations:

$$(4.38) \quad \begin{aligned} L_2(x) : C_0(1 - xf)\bar{y}(x) - (1 - a_2x)y(qx) + (1 - b_1x)y(x)/g &= 0, \\ L_3(x) : C_1(1 - xf/q)y(x) + (1 - a_1x)\bar{y}(x)/g - q^{m+n+1}(1 - b_2x/q)\bar{y}(x/q) &= 0, \end{aligned}$$

where $C_0 = c_0/c_1, C_1 = \bar{c}_0/c_1$ and $g = c_1/c_2$.

(c) The q -Painlevé equation

Compatibility gives the following equations:

$$(4.39) \quad g\bar{g} = \frac{1}{q^{m+n+1}} \frac{(f-a_1)(f-b_1)}{(f-a_2)(f-b_2)}, \quad f\bar{f} = a_2 b_2 \frac{(g-b_1/a_2 q^m)(g-a_1/b_2 q^n)}{(g-1)(g-1/q^{m+n+1})}.$$

These equations (4.39) are equivalent to the q -Painlevé equation of type $D_5^{(1)}$ given in [20] [23]. The 8 singular points are on the four lines $f = 0$, $f = \infty$, $g = 0$ and $g = \infty$.

(4.40)

$$(f, g) = (a_1, 0), (b_1, 0), (0, b_1/a_2 q^m), (0, a_1/b_2 q^n), (\infty, 1), (\infty, 1/q^{m+n+1}), (a_2, \infty), (b_2, \infty).$$

The product $C_0 C_1$:

$$(4.41) \quad C_0 C_1 = (1-g)(1-q^{m+n+1}g)/g^2.$$

(d) The Lax pair

$$(4.42) \quad \begin{aligned} L_1(x) : & \left[(1-g)(1-gq^{m+n+1}) - \frac{x(a_1-b_2 g q^n)(b_1-a_2 g q^m)}{f} \right] y(x) \\ & + \frac{g(1-a_1 x)(1-a_2 x)}{1-fx} \left[y(qx) - \frac{1-b_1 x}{g(1-a_2 x)} y(x) \right] \\ & + \frac{g(q-b_1 x)(q-b_2 x)q^{m+n}}{q-fx} \left[y(x/q) - \frac{g(q-a_2 x)}{q-b_1 x} y(x) \right], \\ L_2(x) : & (1-xf)\bar{y}(x) - (1-a_2 x)y(qx) + (1-b_1 x)y(x)/g = 0. \end{aligned}$$

The scalar Lax pair (3.51) is equivalent to the 2×2 matrix ones in [20] [35] and the scalar one in [59] by using suitable gauge transformations of $y(x)$.

(e) Special solutions

$$(4.43) \quad \begin{aligned} \frac{1-f/a_1}{1-f/a_2} &= \frac{a_1 \prod_{i=1}^2 (1-b_i/a_1)}{a_2 \prod_{i=1}^2 (1-b_i/a_2)} \frac{T_{a_1}(\tau_{m,n+1})T_{a_1}^{-1}(\tau_{m+1,n})}{T_{a_2}(\tau_{m,n+1})T_{a_2}^{-1}(\tau_{m+1,n})}, \\ g &= \frac{a_1(1-b_1/a_1)}{q^n a_2(1-b_2/a_2)} \frac{T_{a_1}(\tau_{m,n+1})T_{a_1}^{-1}(\bar{\tau}_{m+1,n})}{T_{a_2}(\bar{\tau}_{m,n+1})T_{a_2}^{-1}(\tau_{m+1,n})}, \end{aligned}$$

where the p_k in the $\tau_{m,n}$ (3.17) are given by

$$(4.44) \quad p_k = b_2^k \frac{(a_2/b_2; q)_k}{(q; q)_k} {}_2\varphi_1 \left(\begin{matrix} q^{-k}, a_1/b_1 \\ b_2 q^{-k+1}/a_2 \end{matrix}; q, \frac{b_1}{a_2} q \right).$$

The p_k are expressed in terms of the terminating ${}_2\varphi_1$ (4.14) series (little q -Jacobi polynomials [23] [27] [39]). These determinant formulae of hypergeometric solutions (4.43) are expected to be equivalent to the terminating case of that in [49].

4.2.3. Case q - $A_4^{(1)}$.

(a) Setting of the Padé approximation problem

The generating function (4.2):

$$(4.45) \quad Y(x) = \frac{(a_1 x, a_2 x; q)_\infty}{(b_1 x; q)_\infty}.$$

The time evolution (4.5):

$$(4.46) \quad T : (a_1, a_2, b_1, m, n) \mapsto (qa_1, a_2, qb_1, m, n).$$

(b) Contiguity relations

The basic quantities:

$$(4.47) \quad G(x) = \frac{1 - b_1x}{(1 - a_1x)(1 - a_2x)}, \quad K(x) = \frac{1 - b_1x}{1 - a_1x}, \quad H(x) = (1 - a_1x)(1 - a_2x).$$

The Casorati determinants:

$$(4.48) \quad D_1(x) = \frac{c_0(1 - xf)N(x)Y(x)}{G_{\text{den}}(x)}, \quad D_2(x) = \frac{c_1N(x)Y(x)}{K_{\text{den}}(x)}, \quad D_3(x) = \frac{c_2N(x)Y(x)}{H(x)},$$

where f, c_0, c_1 and c_2 are constants with respect to x .

The contiguity relations:

$$(4.49) \quad \begin{aligned} L_2(x) : C_0(1 - xf)\bar{y}(x) - (1 - a_2x)y(qx) + (1 - b_1x)y(x)/g &= 0, \\ L_3(x) : C_1(1 - x\bar{f}/q)y(x) + (1 - a_1x)\bar{y}(x)/g - q^{m+n+1}\bar{y}(x/q) &= 0, \end{aligned}$$

where $C_0 = c_0/c_1, C_1 = \bar{c}_0/c_1$ and $g = c_1/c_2$.

(c) The q -Painlevé equation

Compatibility gives the following equations:

$$(4.50) \quad g\underline{g} = \frac{1}{q^{m+n+1}} \frac{(f - a_1)(f - b_1)}{f(f - a_2)}, \quad f\bar{f} = -\frac{a_1a_2}{q^n} \frac{g - b_1/a_2q^m}{(g - 1)(g - 1/q^{m+n+1})}.$$

These equations (4.50) are equivalent to the q -Painlevé equation of type $A_4^{(1)}$ given in [23] [28]. The 8 singular points are on the four lines $f = 0, f = \infty, g = 0$ and $g = \infty$. $(0, \infty)$ is a double point.

$$(4.51) \quad \begin{aligned} (f, g) &= (a_1, 0), (b_1, 0), (0, a_2/b_1q^m), (\infty, 1), (\infty, 1/q^{m+n+1}), (a_2, \infty), \\ &(0, \infty) \leftarrow fg = -a_1/q^n. \end{aligned}$$

The product C_0C_1 :

$$(4.52) \quad C_0C_1 = (1 - g)(1 - q^{m+n+1}g)/g^2.$$

(d) The Lax pair

$$(4.53) \quad \begin{aligned} L_1(x) : & \frac{gq^{m+n+1}(q - b_1x)}{fx - q} \left[y(x/q) - \frac{g(q - a_2x)}{q - b_1x} y(x) \right] \\ & + \frac{g(1 - a_1x)(1 - a_2x)}{fx - 1} \left[y(qx) - \frac{1 - b_1x}{g(1 - a_2x)} y(x) \right] \\ & + \left[\frac{a_1(b_1 - a_2q^mg)x}{f} - (g - 1)(q^{m+n+1} - 1) \right] y(x) = 0, \\ L_2(x) : & (1 - xf)\bar{y}(x) - (1 - a_2x)y(qx) + (1 - b_1x)y(x)/g = 0. \end{aligned}$$

The scalar Lax pair (4.53) is equivalent to the 2×2 matrix one for the q -Painlevé equation of type q - $P(A_4)$ in [35] by using a suitable gauge transformation of $y(x)$.

(e) Computation of special solutions

$$(4.54) \quad \begin{aligned} \frac{1 - f/a_1}{1 - f/a_2} &= \frac{a_1(1 - b_1/a_1)}{a_2(1 - b_1/a_2)} \frac{T_{a_1}(\tau_{m,n+1})T_{a_1}^{-1}(\tau_{m+1,n})}{T_{a_2}(\tau_{m,n+1})T_{a_2}^{-1}(\tau_{m+1,n})}, \\ g &= \frac{a_1}{q^n a_2} \frac{T_{a_1}(\tau_{m,n+1})T_{a_1}^{-1}(\bar{\tau}_{m+1,n})}{T_{a_2}(\bar{\tau}_{m,n+1})T_{a_2}^{-1}(\tau_{m+1,n})}, \end{aligned}$$

where the p_k in the $\tau_{m,n}$ (3.17) are given by

$$(4.55) \quad p_k = (-1)^k q^{\binom{k}{2}} \frac{(a_2; q)_k}{(q; q)_k} {}_2\varphi_1 \left(\begin{matrix} q^{-k}, a_1/b_1 \\ 0 \end{matrix}; q, \frac{b_1}{a_2} q \right).$$

The p_k are expressed in terms of the terminating ${}_2\varphi_1$ (4.14) series (q -Laguerre polynomials [23] [27] [39]). These determinant formulae of hypergeometric solutions (4.54) are expected to be equivalent to the terminating case of that in [13].

4.2.4. Case q -($A_2 + A_1$)⁽¹⁾.

(a) Setting of the Padé approximation problem

The generating function (4.2):

$$(4.56) \quad Y(x) = (a_1 x, a_2 x; q)_\infty$$

The time evolution (4.5):

$$(4.57) \quad T : (a_1, a_2, m, n) \mapsto (qa_1, a_2, m, n).$$

(b) Contiguity relations

The basic quantities:

$$(4.58) \quad G(x) = \frac{1}{(1 - a_1 x)(1 - a_2 x)}, \quad K(x) = \frac{1}{1 - a_1 x}, \quad H(x) = (1 - a_1 x)(1 - a_2 x).$$

The Casorati determinants:

$$(4.59) \quad D_1(x) = \frac{c_0(1 - xf)N(x)Y(x)}{G_{\text{den}}(x)}, \quad D_2(x) = \frac{c_1 N(x)Y(x)}{K_{\text{den}}(x)}, \quad D_3(x) = \frac{c_2 N(x)Y(x)}{H(x)},$$

where f, c_0, c_1 and c_2 are constants with respect to x .

The contiguity relations:

$$(4.60) \quad \begin{aligned} L_2(x) : C_0(1 - xf)\bar{y}(x) - (1 - a_2 x)y(qx) + y(x)/g &= 0, \\ L_3(x) : C_1(1 - xf/q)y(x) + (1 - a_1 x)\bar{y}(x)/g - q^{m+n+1}\bar{y}(x/q) &= 0, \end{aligned}$$

where $C_0 = c_0/c_1, C_1 = \bar{c}_0/c_1$ and $g = c_1/c_2$.

(c) The q -Painlevé equation

Compatibility gives the following equation:

$$(4.61) \quad g\underline{g} = \frac{1}{q^{m+n+1}} \frac{f - a_1}{f - a_2}, \quad f\bar{f} = -\frac{a_1 a_2}{q^n} \frac{g}{(g - 1)(g - 1/q^{m+n+1})}.$$

These equations (4.61) are equivalent to the q -Painlevé equation of type $(A_2 + A_1)^{(1)}$, namely P_{IV} , given in [26] [50]. The 8 singular points are on the four lines $f = 0, f = \infty, g = 0$ and $g = \infty$. Two points $(0, 0)$ and $(0, \infty)$ are double points.

$$(4.62) \quad \begin{aligned} (f, g) &= (a_1, 0), (0, a_2/b_1 q^m), (\infty, 1), (\infty, 1/q^{m+n+1}), (a_2, \infty), \\ (0, \infty) &\leftarrow fg = -a_1/q^n \\ (0, 0) &\leftarrow g/f = -1/a_2 q^m. \end{aligned}$$

The product $C_0 C_1$:

$$(4.63) \quad C_0 C_1 = (1 - g)(1 - q^{m+n+1}g)/g^2.$$

(d) The Lax pair

$$(4.64) \quad \begin{aligned} L_1(x) : & \frac{gq^{m+n+2}}{q - fx} [y(x/q) - g(1 - a_2 x/q)y(x)] \\ & + \frac{g(1 - a_1 x)(1 - a_2 x)}{1 - fx} \left[y(qx) - \frac{1}{g(1 - a_2 x)} y(x) \right] \\ & + \left[\frac{a_1 a_2 q^m g x}{f} + (g - 1)(q^{m+n+1} - 1) \right] y(x) = 0, \\ L_2(x) : & (1 - xf)\bar{y}(x) - (1 - a_2 x)y(qx) + y(x)/g = 0. \end{aligned}$$

The scalar Lax pair (4.64) is equivalent to the 2×2 matrix one for the q -Painlevé equation of type q - $P(A_5)^\#$ in [35] by using a suitable gauge transformation of $y(x)$.

(e) Special solutions

$$(4.65) \quad \begin{aligned} \frac{1 - f/a_1}{1 - f/a_2} &= \frac{a_1}{a_2} \frac{T_{a_1}(\tau_{m,n+1})T_{a_1}^{-1}(\tau_{m+1,n})}{T_{a_2}(\tau_{m,n+1})T_{a_2}^{-1}(\tau_{m+1,n})}, \\ g &= \frac{a_1}{q^n a_2} \frac{T_{a_1}(\tau_{m,n+1})T_{a_1}^{-1}(\bar{\tau}_{m+1,n})}{T_{a_2}(\bar{\tau}_{m,n+1})T_{a_2}^{-1}(\tau_{m+1,n})}, \end{aligned}$$

where the p_k in the $\tau_{m,n}$ (3.17) are given by

$$(4.66) \quad p_k = (-1)^k q^{\binom{k}{2}} \frac{(a_2; q)_k}{(q; q)_k} {}_1\varphi_1 \left(\begin{matrix} q^{-k} \\ 0 \end{matrix}; q, \frac{a_1}{a_2} q \right)$$

The p_k are expressed in terms of the terminating ${}_2\varphi_1$ (4.14) series (Stieltjes-Wigert polynomials [23] [27] [39]). These determinant formulae of hypergeometric solutions (4.65) are expected to be equivalent to the terminating case of that in [36].

5. CONCLUSION

5.1. Summary.

In this paper, we reviewed the Padé interpolation method of the q -grid in [37], and applied the approximation method of the differential grid, related to all the q -Painlevé equations of types from $E_6^{(1)}$ to $(A_2 + A_1)^{(1)}$. The main results of this paper are the following two provisions:

- By applying the Padé interpolation method of the q -grid to all the q -Painlevé equations of types from $q-E_7^{(1)}/q-E_6^{(1)}$ to $q-(A_2 + A_1)^{(1)}$, we could derive the evolution equations, the scalar Lax pairs and the determinant formulae of special solutions for the corresponding q -Painlevé equations. These results were presented in [37].
- By applying the Padé approximation method of the differential grid to all the q -Painlevé equations of types from $q-E_6^{(1)}$ to $q-(A_2 + A_1)^{(1)}$, we could derive the same as the first item above. These are the new results in this paper.

5.2. Problems. Some open problems related to the results in this paper are as follows:

1. In this paper, by choosing one time evolution T , we applied the Padé method to each type from $q-E_7^{(1)}$ to $q-(A_2 + A_1)^{(1)}$. By choosing other time evolutions, we can perform similar computations. It may be interesting to study the relations between the Padé method for the various time evolutions and Bäcklund transformations of the affine Weyl group, for example the case $q-E_6^{(1)}$ in [14].
2. By the results of this paper and previous ones [14] [40] [60], it turned out that the Padé method could be applied to all the discrete Painlevé equations of types from $e-E_8^{(1)}$ to $q-(A_2 + A_1)^{(1)}$. It may be interesting to study the degenerations between these results.
3. In previous papers [14] [62], the Padé approximation method of the differential grid was applied to the q -Painlevé equation of type $D_5^{(1)}$. In this paper, we applied the approximation method to all the q -Painlevé equation of types from $E_6^{(1)}$ to $(A_2 + A_1)^{(1)}$. It may be interesting to study whether the approximation method can be also applied to the q -Painlevé equations of other types.
4. In previous papers [58] [61] [62], the Padé approximation method of the differential grid was applied to the d -Painlevé equation of types $D_4^{(1)}$, $A_3^{(1)}$ and $A_2^{(1)}$. It may be interesting to study whether the Padé method can be also applied to the d -Painlevé equations of other types.
5. It may be interesting to study whether the Padé method can be further applied to the other generalized Painlevé systems, for example the q -Garnier system in [51].

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