An Integrable Systems Approach to Constant Mean Curvature Surfaces and their Singularities

Ogata, Yuta

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博　士　論　文

An Integrable Systems Approach to
Constant Mean Curvature Surfaces and their Singularities

平均曲率一定曲面に対する可積分系的アプローチとその特異点

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神戸大学大学院理学研究科

学籍番号　　125S006S
氏　　名　緒方 勇太 (Yuta Ogata)

Introduction

In this thesis, we study the construction of constant mean curvature (CMC) surfaces and the theory for their singularities, and we analyze these surfaces via an integrable systems approach, which involves, for example, the DPW method and Lax pairs. CMC surfaces are well known as mathematical models of soap bubbles, and many researchers have been studying them over a long period of time. In 1866, K. T. Weierstrass showed that one can construct CMC $H = 0$ (minimal) surfaces in the 3-dimensional Euclidean space $\mathbb{R}^3$ by using an integral formula involving a pair of holomorphic functions satisfying certain conditions. This famous formula is called the *Weierstrass representation formula*, and many examples of minimal surfaces have been constructed via this formula:

Fact 0.0.1 (Weierstrass representation)**.** Any minimal surface $X : \Sigma \subset \mathbb{C} \to \mathbb{R}^3$ can be locally represented as

$$X = \mathrm{Re} \int \left(1 - h^2, i(1 + h^2), 2h\right) \eta \, \mathrm{d}z$$

over a simply-connected domain Σ on which h is meromorphic, while η and $h^2\eta$ are holomorphic.

In 1998, J. Dorfmeister, F. Pedit and H. Wu discovered a generalization of the Weierstrass representation formula. They showed that one can construct CMC $H \neq 0$ surfaces in $\mathbb{R}^3$ by using holomorphic data satisfying certain conditions and using a matrix loop splitting called the *Iwasawa splitting*. This method is called the *DPW method*. Recently, this DPW method has been applied to construct surfaces in various non-Euclidean spaces.

In the first half of this thesis, we apply the DPW method for spacelike CMC surfaces in semi-Riemannian spaceforms, i.e. Lorentz 3-space, de Sitter 3-space and anti-de Sitter 3-space (see also [70, 71]). We also study some criteria for the types of their singularities. In the last half of this thesis, we study the corank two singularities of CMC surfaces, the transformation theory for them and their discretization (see also [22, 72, 73, 74]).

This thesis has the following chapters:

- **Chapters** 1 **and** 2**.** In previous works ([17], [24], [29], [30], [86], etc), Brander, Dorfmeister, Inoguchi, Kobayashi, Kilian, Rossman and Schmitt constructed new examples of CMC surfaces in the 3-dimensional sphere $\mathbb{S}^3$ and hyperbolic space $\mathbb{H}^3$ (non-Euclidean positive definite spaceforms) via the DPW method. In [17], Brander, Rossman and Schmitt constructed spacelike CMC surfaces in the 3-dimensional Lorentzian space $\mathbb{R}^{2,1}$, and they classified the spacelike rotationally symmetric surfaces. (See also [33] and [34].) In the appendix of [24] (arXiv version), spacelike CMC $|H| < 1$ surfaces in the de Sitter space $\mathbb{S}^{2,1}$ are considered.

  Here we apply this method to spacelike CMC surfaces in $\mathbb{R}^{2,1}$, $\mathbb{S}^{2,1}$ and anti de-Sitter space $\mathbb{H}^{2,1}$, and we give some examples of spacelike CMC surfaces in those Lorentzian spaceforms. We can see this theory as a special case of [33], however here we also study the singularities of the resulting CMC surfaces, and look at some examples in detail (totally umbilical surfaces, round cylinders, and more interestingly, analogs of Smyth surfaces).

  Section 1.1 explains the DPW method for spacelike CMC surfaces in $\mathbb{R}^{2,1}$, as in [17] and [33], using Lax pairs and loop groups. In Sections 1.2 and 1.3, we describe the DPW method for spacelike CMC surfaces in $\mathbb{S}^{2,1}$ and $\mathbb{H}^{2,1}$, as in [33]. Section 1.4 introduces some models of $\mathbb{S}^{2,1}$ and $\mathbb{H}^{2,1}$ for visualization. We use the *hollow ball model* for $\mathbb{S}^{2,1}$, and we use the *cylindrical models* for $\mathbb{H}^{2,1}$. In Section 1.5, as applications, we describe the most basic examples of spacelike CMC surfaces in $\mathbb{S}^{2,1}$ and $\mathbb{H}^{2,1}$. In Section 2.1, we explore singularities on CMC surfaces in Lorentzian spaceforms, and give criteria for determining certain types of singularities (cuspidal edges, swallowtails and cuspidal cross caps) in the context of Lax representations. Finally, in Section 2.2, we apply the methods and results in this thesis to the analogs of Smyth surfaces in Lorentzian spaceforms, including results about their symmetries, singularities, and relations with the Painleve III equation.

- **Chapters** 3 **and** 4**.** In the previous chapters, we studied spacelike CMC surfaces in $\mathbb{R}^{2,1}$, $\mathbb{S}^{2,1}$ and $\mathbb{H}^{2,1}$, and created criteria for singularities on these surfaces, by using the frame-change method, called the s-spectral deformation. However, as in [70], we omitted the case of spacelike CMC H surfaces with $0 \leq H < 1$ in $\mathbb{S}^{2,1}$ because the Lax pair and Iwasawa splitting are quite different from the case of $H > 1$. On the other hand, in the appendix of [24] (arXiv version), spacelike CMC H surfaces with $0 \leq H < 1$ in the de Sitter space $\mathbb{S}^{2,1}$with no umbilics are considered by using the normal vector of the parallel surfaces of CMC H surfaces with $0 \leq H < 1$ in $\mathbb{H}^3$. See Proposition 3.1.2 in the present thesis.

  Here, rather, to allow for umbilics as one of the reasons, we give a DPW method for spacelike CMC H surfaces with $0 \leq H < 1$ in $\mathbb{S}^{2,1}$ by using Iwasawa splitting. (See Theorems 3.2.1, 3.3.1 and 3.3.2.) We also study the singularities of these surfaces, and specify types of singularities, focusing on the asymptotic behavior of Iwasawa splitting and using the s-spectral deformation. (See Theorems 3.4.1, 4.1.3.) In particular we look at singularities of Smyth-type surfaces with $0 \leq H < 1$ in $\mathbb{S}^{2,1}$. The data for such surfaces in $\mathbb{H}^3$ was given by Dorfmeister, Inoguchi and Kobayashi in [24] - however, in that $\mathbb{H}^3$ case the Iwasawa splitting is more easily extendable beyond the Iwasawa core, and so singularities do not appear on these surfaces. Here instead we go to the $\mathbb{S}^{2,1}$ case to examine singularities.

  Section 3.1 explains Lax pairs and the immersion formula as in [33]. In Section 3.2, we prove the existence and uniqueness of $\Lambda\mathrm{SL}_2(\mathbb{C})_\tau$-Iwasawa splitting. In Section 3.3, we apply the DPW method for spacelike CMC H surfaces with $0 \leq H < 1$ in $\mathbb{S}^{2,1}$, and in Section 3.4 we consider the behavior of the frames and surfaces when approaching the $\Lambda\mathrm{SL}_2(\mathbb{C})_\tau$-Iwasawa small cell $\mathcal{P}$. In Section 4.1, we introduce criteria of singularities on these surfaces, including cuspidal edges, swallowtails and cuspidal cross caps, and in the last Section 4.2 we introduced Smyth-type surfaces with umbilics and singularities. Here, we show these Smyth-type surfaces have three types of singularities, i.e. cuspidal edges, swallowtails and cuspidal cross caps (see Theorem 4.2.2).

- **Chapter** 5**.** The hyperbolic 3-space $\mathbb{H}^3$ is a 3-dimensional Riemannian spaceform with constant sectional curvature -1 and the de Sitter 3-space $\mathbb{S}^{2,1}$ is a 3-dimensional Lorentzian spaceform with constant sectional curvature 1 in Lorentz-Minkowski 4-space $\mathbb{R}^{3,1}$. There are numerous articles on the study of surfaces in $\mathbb{H}^3$ and $\mathbb{S}^{2,1}$ (for example, [29, 31, 32, 36, 50, 52, 53, 57, 58, 60]). Gálvez, Martínez and Milán [36] showed the representation formula for flat surfaces in $\mathbb{H}^3$. For flat fronts in $\mathbb{H}^3$, Kokubu, Rossman, Saji, Umehara and Yamada [57] showed that generic singularities on flat fronts are cuspidal edges and swallowtails (see also [47]). Moreover, the criteria for cuspidal edges and swallowtails were obtained. Recently, differential geometric properties of fronts were studied (cf. [63, 64, 68, 83, 84]). In particular, normal forms and isometric deformations of cuspidal edges were obtained in [63, 68].

  On the other hand, the Gauss map plays an important role in investigating differential geometry of surfaces in the Euclidean 3-space $\mathbb{R}^3$. Singularities and stability of Gauss maps for surfaces in $\mathbb{R}^3$ were studied in [4, 7]. Bleecker and Wilson [7] showed that generic singularities of the Gauss map are fold and cusp singularities. Banchoff, Gaffney and McCrory

[4] studied geometric meanings of cusp singularities of Gauss maps. The differential of the Gauss map, that is, the Weingarten map, gives the Gaussian curvature, the mean curvature and the principal curvatures for the surface. Therefore, for surfaces in $\mathbb{H}^3$, such maps also play important roles. In [36, 58, 60], flat fronts in $\mathbb{H}^3$ and their hyperbolic Gauss maps were studied. Izumiya, Pei and Sano [50] studied singularities of hyperbolic Gauss maps and hyperbolic Gauss indicatrices for surfaces in $\mathbb{H}^3$. They regard hyperbolic Gauss indicatrices as wave fronts and showed that generic singularities of hyperbolic Gauss map images are cuspidal edges and swallowtails, by applying the Legendrian singularity theory. Moreover, Izumiya [49] introduced a Legendrian duality theorem for pseudo-spheres in the Lorentz-Minkowski space. Using this duality theorem, the de Sitter Gauss map image and the lightcone Gauss map image of surfaces in $\mathbb{H}^3$ can be constructed. Thus one can study extrinsic differential geometry of surfaces in $\mathbb{H}^3$.

In this chapter, we clarify relations between differential geometric properties called ridge points of cuspidal edges in $\mathbb{H}^3$ and singularities of de Sitter Gauss map images in $\mathbb{S}^{2,1}$. In Section 5.1, we consider local differential geometric properties of cuspidal edges in $\mathbb{H}^3$. We shall define the principal curvature, the principal direction and ridge points for cuspidal edges. In Section 5.2, we show relations between differential geometric properties of cuspidal edges and singularities of de Sitter Gauss map images (Theorem 5.2.1). In Section 5.3, we consider the normal form of cuspidal edges in $\mathbb{H}^3$. The normal form of cuspidal edges in $\mathbb{R}^3$ was introduced in [63]. We give a condition that the origin is a ridge point in the context of the coefficients of the normal form. In the last section, we apply results obtained in previous sections to flat fronts in $\mathbb{H}^3$ and $\mathbb{S}^{2,1}$. We show conditions for singularities of de Sitter Gauss map images for flat fronts in the context of the data which appear in the representation theorem for flat fronts in $\mathbb{H}^3$ given by [36] (Theorem 5.4.1). Furthermore, we consider the Enneper-type flat fronts as global examples. We give the duality between Enneper-type flat fronts and their de Sitter Gauss map images (Theorem 5.4.2).

- **Chapter 6.** Singularities of wave fronts can appear on surfaces via parallel transformations. It is known that generic singularities of wave fronts in 3-spaces are cuspidal edges and swallowtails. Moreover, the singularities of the bifurcations in generic one-parameter families of wave fronts in 3-spaces are cuspidal lips, cuspidal beaks, cuspidal butterflies and D_4-singularities, in addition to the above two (see [3, 51]). There are criteria for these singularities in [52, 53, 57, 82]. On the other hand, in [35], Fukui and Hasegawa studied singularities of the parallel surfaces of regular surfaces in Euclidean 3-space $\mathbb{R}^3$. They gave criteria for cuspidal edges, swallowtails, cuspidal lips, cuspidal beaks, cuspidal butterflies and D_4-singularities by using geometric invariants of the original surfaces, for example principal curvatures.

  For constant mean curvature (CMC) surfaces in Riemannian and semi-Riemannian spaceforms, there are several studies. In general, CMC surfaces in semi-Riemannian spaceforms have singularities (see [16, 32, 46, 70, 71, 91, 92], for example). In [92], criteria for cuspidal edges and swallowtails of maximal surfaces were obtained by using Weierstrass data. Similarly, for maximal surfaces and CMC 1 surfaces, criteria for corank one singularities were given in [32]. Umeda [91] gave criteria for cuspidal edges, swallowtails and cuspidal cross caps of extended CMC surfaces in Minkowski 3-space $\mathbb{R}^{2,1}$, and in [70, 71], the present author also studied the analogues of Umeda's criteria for these singularities of extended CMC surfaces in other semi-Riemanian spaceforms.

  However, the above previous studies did not consider corank two singularities of such surfaces. It is well-known that the corank two singularities appear even on CMC surfaces in

Riemannian spaceforms. Thus in this thesis, we consider the criteria for the corank two singularities, especially D_4-singularities.

For CMC $H \neq 0$ surfaces in $\mathbb{R}^3$ and $\mathbb{R}^{2,1}$, the following fact is known. (For the definition of $\hat{f}^t$, see (6.1.3).)

Fact 0.0.2 ([35, 45])**.** Let f be a conformal (spacelike) CMC surface in $\mathbb{R}^3$ (or $\mathbb{R}^{2,1}$) with mean curvature $H > 0$, unit normal vector ν and Hopf differential factor Q, and let p be an umbilic point of f. Then for $t = 1/H$, the parallel transform $\hat{f}^t$ of f becomes a conformal (spacelike) CMC surface in $\mathbb{R}^3$ (or $\mathbb{R}^{2,1}$) with mean curvature $-H$ and Hopf differential factor $-Q$. Moreover, $\hat{f}^t$ has a corank two singularity at p.

For CMC surfaces in the spherical 3-space $\mathbb{S}^3$, hyperbolic 3-space $\mathbb{H}^3$, de Sitter 3-space $\mathbb{S}^{2,1}$ and anti-de Sitter 3-space $\mathbb{H}^{2,1}$, the following is known. (For definitions of $\hat{f}^t$ and $\check{f}^t$, see (6.1.4) and (6.1.5).)

Fact 0.0.3 ([20, 24])**.** Let $f : U \to M^3$ be a conformal (spacelike) CMC H immersion and ν its unit normal vector.

(1) If $M^3 = \mathbb{H}^3$ or $\mathbb{S}^{2,1}$ with $H > 1$, then for $t = \operatorname{arccoth}(H)$, $\hat{f}^t$ becomes a conformal (spacelike) CMC surface in M^3 with mean curvature $-H$.

(2) If $M^3 = \mathbb{H}^3$ (resp. $\mathbb{S}^{2,1}$) with $0 < H < 1$, then for $t = \operatorname{arctanh}(H)$, $\check{f}^t = \hat{\nu}^t$ becomes a conformal spacelike CMC surface in $\mathbb{S}^{2,1}$ (resp. $\mathbb{H}^3$) with mean curvature $-H$.

(3) If $M^3 = \mathbb{S}^3$ or $\mathbb{H}^{2,1}$ with mean curvature $H > 0$, then for $t = \operatorname{arccot}(H)$, $\hat{f}^t$ is a conformal (spacelike) CMC surface in M^3 with mean curvature $-H$. Moreover, if f is a conformally immersed minimal (resp. maximal) surface, then ν is a conformal minimal (resp. maximal) surface in M^3 and f is a unit normal vector to ν.

By these facts, considering parallel transforms of CMC surfaces naturally emphasizes their umbilic points, and parallel transforms play an important role in understanding relations between umbilic points and corank two singularities. Moreover, by Facts 0.0.2 and 0.0.3, we can start to consider a regular CMC surface f with an umbilic point instead of a CMC surface $\hat{f}$ (or $\check{f}$) with a D_4-singularity via parallel transform. Thus, in this thesis, we study (spacelike) CMC surfaces with D_4-singularities in Riemannian and semi-Riemannian spaceforms, and we give criteria for D_4-singularities in terms of the Hopf differential factors (Theorem 6.2.1). For minimal surfaces, we give conditions for which they have D_4-singularities by using Weierstrass data (Theorem 6.2.2).

For surfaces in $\mathbb{S}^3$, $\mathbb{H}^3$, $\mathbb{S}^{2,1}$ or $\mathbb{H}^{2,1}$, their unit normal vectors form surfaces. Thus we can compare curvatures of CMC surfaces with curvatures of their unit normal vectors (Proposition 6.3.1). Moreover, we show a kind of duality between parallel transforms of CMC surfaces and their unit normal vectors (Proposition 6.3.2).

- **Chapter 7.** The study of minimal surfaces with planar curvature lines is a classical subject, having been studied by Bonnet, Enneper, and Eisenhart in the late 19th century, as recorded in [14], [26], and [27], respectively. The subject was further studied by Nitsche in [69], where he gave a full classification of such surfaces. Nitsche showed that families of planar curvature lines transform into orthogonal families of circles on the unit sphere $\mathbb{S}^2$ under the Gauss map, and by analyzing orthogonal systems of circles, he classified different types of minimal surfaces with planar curvature lines, which we will state here along with their respective Weierstrass data.

Fact 0.0.4. A minimal surface in Euclidean space $\mathbb{R}^3$ with planar curvatures lines must be a piece of one, and only one, of

- a plane $(0, 1\,\mathrm{d}z)$,
- a catenoid $(e^z, e^{-z}\,\mathrm{d}z)$,
- an Enneper surface $(z, 1\,\mathrm{d}z)$, or
- a surface in the Bonnet family $\{(e^z + t, e^{-z}\,\mathrm{d}z), t > 0\}$

up to isometries and homotheties of $\mathbb{R}^3$.

In fact, planes can also be considered as spheres with infinite radius. Thus, minimal surfaces with planar curvature lines can be thought of as a special case of minimal surfaces with spherical curvature lines. There are many works focused on such surfaces, including [23] and [95].

On the other hand, Thomsen studied surfaces with zero mean curvature which are also affine minimal. In [88], he classified such surfaces using the fact that the families of asymptotic lines of these surfaces transformed into orthogonal families of circles on the unit sphere under the Gauss map. He also mentioned that since the asymptotic lines correspond to the curvature lines of the conjugate surface, minimal surfaces that are also affine minimal are conjugate surfaces of minimal surfaces with planar curvature lines. Building on the result of Thomsen, Schaal showed that the plane and the Enneper surface can be attained as a limit of Thomsen surfaces in [85], and Barthel, Volkmer, and Haubitz were able to join these surfaces into a one-parameter family of surfaces using an analytical approach in [5].

It is also possible to attain a deformation between minimal surfaces with planar curvature lines using the Goursat transformation introduced in [37] and [38]. The Goursat transformation transforms minimal surfaces to minimal surfaces; additionally, the transformation not only maps curvature lines to curvature lines but also preserves the planar curvature line condition, as seen in [42], [66], or [75]. However, one may not transform a non-planar minimal surface into a plane via Goursat transformations alone.

Meanwhile, non-zero constant mean curvature (CMC) surfaces with planar curvature lines also have been studied extensively, as in [1], [93], [94]. In particular, Wente constructed nontrivial examples of compact CMC surfaces, called Wente tori in [94]. Then in [1], Abresch gave a classification of all CMC surfaces with planar curvature lines including the Wente tori and cylindrical ended bubbletons, by solving a system of partial differential equations. Similarly, in [93] Walter considered explicit parametrizations of Wente tori, by showing the existence a notion of axes.

In this thesis, to classify minimal surfaces in $\mathbb{R}^3$ with planar curvature lines, we propose an alternative method to using orthogonal systems of circles. In particular, we utilize the method analogous to the approach taken in [1], [5], and [93], modified for the subject at hand. In Section 7.1, we first obtain and solve a system of partial differential equations describing the metric function, similar to the method used in [1] and [5]. Then, following [93], we prove the existence of axial directions of these surfaces, using them to recover the Weierstrass data, and ultimately their parametrizations. In Section 7.2, we use the results from the previous section to obtain a single-parameter deformation of all minimal surfaces preserving the planar curvature line condition. Finally as the main result, we state the classification, parametrization, and deformation of all minimal surfaces with planar curvature lines (see Theorem 7.2.1 and Figure 7.2).

- **Chapter** 8. In the smooth (or continuous) case, surfaces governed by some integrable equation, like constant negative Gaussian curvature surfaces and non-zero constant mean curvature surfaces, have been well studied. When studying such surfaces, it is useful to describe 2×2 matrix representations and matrix-valued partial differential equations called Lax pairs. In particular, as mentioned before, applying matrix-splitting theorems, Dorfmeister, Pedit, Wu [25] established the generalized Weierstrass representation for smooth CMC surfaces in Euclidean 3-space $\mathbb{R}^3$ (regarding the cases of smooth CMC surfaces in spherical 3-space $\mathbb{S}^3$ and hyperbolic 3-space $\mathbb{H}^3$, see [10], [30] for example), now called the *DPW method* for smooth CMC surfaces.

  Stepping away from smooth surface theory, there has been recent progress on discrete surface theory. In the last three decades, using integrable systems techniques, discrete surface theory has been developed. Burstall, Hertrich-Jeromin, Rossman, Santos [19] described discrete CMC surfaces in any 3-dimensional Riemannian spaceform and gave several new examples of discrete CMC surfaces, and Bobenko, Hertrich-Jeromin, Lukyanenko [11] gave a curvature theory for discrete surfaces in Riemannian spaceforms (see also [18]). Due to these works, we are able to treat discrete surfaces in 3-dimensional Riemannian spaceforms. In particular, constructing discrete CMC surfaces is one of the central topics in the study of discrete surface theory. In fact, Bobenko and Pinkall [9] introduced a Weierstrass representation for discrete minimal surfaces in $\mathbb{R}^3$, and Hertrich-Jeromin [40] derived a Weierstrass-type representation for discrete CMC 1 surfaces in $\mathbb{H}^3$.

  Bobenko, Pinkall [12] described Lax pairs for discrete CMC surfaces in $\mathbb{R}^3$ and gave a Cauchy problem for them (see also [41]). Applying matrix-splitting formulae (see also Propositions 8.3.1, 8.3.2 here), Hoffmann [43] gave a construction for discrete non-zero CMC surfaces in $\mathbb{R}^3$. This method is called the *discrete DPW method* for discrete CMC surfaces in $\mathbb{R}^3$. On the other hand, although discrete CMC surfaces became treatable recently, the discrete DPW method for discrete CMC surfaces in other 3-dimensional Riemannian spaceforms had not yet been considered.

  In this chapter, we give the discrete DPW method for discrete CMC surfaces in $\mathbb{S}^3$ and $\mathbb{H}^3$, which is a generalization of the work by Hoffmann [43], and give several examples. In the smooth case, we can choose a common Lax pair for $\mathbb{R}^3$, $\mathbb{S}^3$ and $\mathbb{H}^3$. Also in the discrete case, using the same Lax pair as in $\mathbb{R}^3$, we will show that we can construct discrete CMC surfaces in $\mathbb{S}^3$ and $\mathbb{H}^3$ and that discrete CMC surfaces given by Lax pairs have mean curvatures (in the sense of [11] and [18]) that are constant. Our construction covers discrete isothermic surfaces in $\mathbb{S}^3$ with any constant mean curvature H, and the discrete isothermic surfaces in $\mathbb{H}^3$ with constant mean curvature H satisfying $|H| > 1$.

  As an application, we will also construct discrete constant positive Gaussian curvature surfaces by taking parallel surfaces of discrete CMC surfaces, and look at their singularities. In the smooth case, constant Gaussian curvature surfaces generally have singularities (for example, see [47]), so it is natural to expect that discrete constant positive Gaussian curvature surfaces have certain configurations of singularities. Based on work by Rossman and Yasumoto [81], we will analyze singularities of such discrete constant positive Gaussian curvature surfaces in $\mathbb{R}^3$, $\mathbb{S}^3$ and $\mathbb{H}^3$.

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Contents

Chapter 1

The DPW method for spacelike CMC surfaces

1.1 The loop group method in $\mathbb{R}^{2,1}$

We consider the DPW method for constructing spacelike CMC surfaces in $\mathbb{R}^{2,1}$ as in [17], [33].

Let $\mathbb{R}^{2,1}$ be the 3-dimensional Lorentz space with Lorentz metric

$$\langle x, y\rangle_{\mathbb{R}^{2,1}} := x_1y_1 + x_2y_2 - x_0y_0 \quad \text{for} \quad x = (x_1, x_2, x_0), y = (y_1, y_2, y_0) \in \mathbb{R}^{2,1}.$$

We simplify the notation $\langle\cdot,\cdot\rangle$ to $\langle\cdot,\cdot\rangle_{\mathbb{R}^{2,1}}$, and use its bilinear extension to $\mathbb{C}^3$. Let Σ be a simply-connected domain in $\mathbb{C}$ with the usual complex coordinate $z = x + iy$, and let $f : \Sigma \longrightarrow \mathbb{R}^{2,1}$ be a conformally immersed spacelike surface. Since f is conformal,

$$\langle f_z, f_z\rangle = \langle f_{\bar z}, f_{\bar z}\rangle = 0, \quad \langle f_z, f_{\bar z}\rangle := 2e^{2u} \tag{1.1.1}$$

for some function $u : \Sigma \longrightarrow \mathbb{R}$. We choose the unit timelike normal vector field $N : \Sigma \longrightarrow \mathbb{H}^2$ ($\mathbb{H}^2$ is the hyperbolic 2-space in $\mathbb{R}^{2,1}$) of f, and then the mean curvature and Hopf differential are

$$H = \frac{1}{2e^{2u}}\langle f_{z\bar z}, N\rangle, \quad Q := \langle f_{zz}, N\rangle. \tag{1.1.2}$$

The Gauss-Codazzi equations are the following form in the CMC cases:

$$4u_{z\bar z} + Q\bar Q e^{-2u} - 4H^2e^{2u} = 0, \quad Q_{\bar z} = 0. \tag{1.1.3}$$

The Codazzi equation in (1.1.3) is equivalent to the Hopf differential Q being holomorphic, and (1.1.3) is invariant with respect to the transformation $Q \to \lambda^{-2}Q$ for $\lambda \in \mathbb{S}^1$. When $f(x, y)$ is a spacelike CMC in $\mathbb{R}^{2,1}$, the spectral parameter $\lambda \in \mathbb{S}^1$ allows us to create a 1-parameter family of CMC surfaces $f^\lambda = f(x, y, \lambda)$ associated to $f(x, y)$.

To describe the 2×2 matrix representation of $\mathbb{R}^{2,1}$ as in [17], [33], writing $su_{1,1}$ for the Lie algebra of the Lie group

$$\mathrm{SU}_{1,1} \;:=\; \left\{ \left(\begin{array}{cc} \alpha & \beta \\ \bar\beta & \bar\alpha \end{array} \right) \middle| \alpha, \beta \in \mathbb{C}, \alpha\bar\alpha - \beta\bar\beta = 1 \right\}, \tag{1.1.4}$$

we identify $\mathbb{R}^{2,1}$ with $su_{1,1}$ via

$$\mathbb{R}^{2,1} \ni x = (x_1, x_2, x_0) \quad \longmapsto \quad \begin{pmatrix} ix_0 & x_1 - ix_2 \\ x_1 + ix_2 & -ix_0 \end{pmatrix} \in su_{1,1}.$$

The metric becomes, under this identification, $\langle X, Y \rangle = \frac{1}{2}\mathrm{trace}(XY)$ for $X, Y \in su_{1,1}$.

Let f be a conformal immersed spacelike surface in $\mathbb{R}^{2,1}$ with associated family f^λ, and let the identity matrix and Pauli matrices be as follows:

$$I := \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \sigma_1 := \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 := \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_3 := \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{1.1.5}$$

Then $\{\sigma_1, \sigma_2, i\sigma_3\}$ is an orthogonal basis for $su_{1,1} \approx \mathbb{R}^{2,1}$. We can define

$$e_1 := \frac{f^\lambda_x}{|f^\lambda_x|} = \frac{f^\lambda_x}{2e^u} = \hat{F}\sigma_1\hat{F}^{-1}, \quad e_2 := \frac{f^\lambda_y}{|f^\lambda_y|} = \frac{f^\lambda_y}{2e^u} = \hat{F}\sigma_2\hat{F}^{-1}, \quad N := \hat{F}i\sigma_3\hat{F}^{-1} \tag{1.1.6}$$

for $\hat{F} = \hat{F}(z, \bar{z}, \lambda) \in \mathrm{SU}_{1,1}$. For this $\hat{F}$, we get the *untwisted* 2×2 *Lax pair* in $\mathbb{R}^{2,1}$ as follows:

$$\hat{F}_z = \hat{F}\hat{U}, \ \hat{F}_{\bar{z}} = \hat{F}\hat{V}, \ \text{where} \quad \hat{U} = \frac{1}{2}\begin{pmatrix} -u_z & -i\lambda^{-2}Qe^{-u} \\ 2iHe^u & u_z \end{pmatrix}, \ \hat{V} = \frac{1}{2}\begin{pmatrix} u_{\bar{z}} & -2iHe^u \\ i\lambda^2\bar{Q}e^{-u} & -u_{\bar{z}} \end{pmatrix} \tag{1.1.7}$$

We change the "untwisted" setting to the "twisted" setting by the following transformation (3.1.4). Let F be defined by

$$\hat{F} = -\sigma_3 \left(F^{-1}\right)^t \begin{pmatrix} \sqrt{\lambda} & 0 \\ 0 & \frac{1}{\sqrt{\lambda}} \end{pmatrix} \sigma_3, \tag{1.1.8}$$

producing the *twisted* 2×2 *Lax pair* of f in $\mathbb{R}^{2,1}$,

$$F_z = FU, \ F_{\bar{z}} = FV, \ \text{where} \quad U = \frac{1}{2}\begin{pmatrix} u_z & 2i\lambda^{-1}He^u \\ -i\lambda^{-1}Qe^{-u} & -u_z \end{pmatrix}, \ V = \frac{1}{2}\begin{pmatrix} -u_{\bar{z}} & i\lambda\bar{Q}e^{-u} \\ -2i\lambda He^u & u_{\bar{z}} \end{pmatrix}. \tag{1.1.9}$$

The following Proposition 1.1.1 gives us a method for determining spacelike CMC $H \neq 0$ surfaces in $\mathbb{R}^{2,1}$ from given data u and Q, by choosing a solution F of (1.1.9) and inserting F into the Sym-Bobenko type formula (1.1.10).

Proposition 1.1.1 (Sym-Bobenko type formula for spacelike CMC surfaces in $\mathbb{R}^{2,1}$ [17], [33])**.** *Let Σ be a simply-connected domain in $\mathbb{C}$. Let u and Q solve* (1.1.3)*, and let $F = F(z, \bar{z}, \lambda)$ be a solution of the system* (1.1.9)*. Suppose $F \in \mathrm{SU}_{1,1}$ for all $\lambda \in \mathbb{S}^1$ and one value of z. Then $F \in \mathrm{SU}_{1,1}$ for all z. Defining the following Sym-Bobenko type formulas*

$$f = \left[\frac{1}{2H}Fi\sigma_3F^{-1} + \frac{i}{H}\lambda(\partial_\lambda F)F^{-1}\right]\Bigg|_{\lambda=1}, \ N = -\left[Fi\sigma_3F^{-1}\right]\Big|_{\lambda=1}, \tag{1.1.10}$$

f is a conformally parametrized spacelike CMC $H \neq 0$ surface in $\mathbb{R}^{2,1}$ with normal N.

1.2 The loop group method in $\mathbb{S}^{2,1}$ and $\mathbb{H}^{2,1}$

We apply the DPW method to construct spacelike CMC surfaces in $\mathbb{S}^{2,1}$ and $\mathbb{H}^{2,1}$, as in [33].

Let $\mathbb{R}^{3,1}$, resp. $\mathbb{R}^{2,2}$, be the 4-dimensional space with metric $\langle (x_1,x_2,x_3,x_4),(y_1,y_2,y_3,y_4)\rangle := x_1y_1+x_2y_2+\varepsilon\cdot x_3y_3-x_4y_4$, where $\varepsilon=1$ for $\mathbb{R}^{3,1}$, resp. $\varepsilon=-1$ for $\mathbb{R}^{2,2}$. We define the spaceform $\mathcal{S}:=\{x|\langle x,x\rangle=\varepsilon\}$. Thus we obtain $\mathcal{S}=\mathbb{S}^{2,1}$ (resp. $\mathcal{S}=\mathbb{H}^{2,1}$) when $\varepsilon=1$ (resp. $\varepsilon=-1$).

Let Σ be a simply-connected domain in $\mathbb{C}$ with the usual complex coordinate $w=x+iy$. Let $f:\Sigma\longrightarrow\mathcal{S}$ be a conformally immersed spacelike surface. Since f is conformal,

$$\langle f_w,f_w\rangle=\langle f_{\bar w},f_{\bar w}\rangle=0,\quad \langle f_w,f_{\bar w}\rangle=2e^{2u} \tag{1.2.1}$$

for some function $u:\Sigma\longrightarrow\mathbb{R}$. For the unit normal vector field N of f satisfying $\langle N,N\rangle=-1$, $\langle f_w,N\rangle=\langle f_{\bar w},N\rangle=0$, we define the mean curvature H and Hopf differential $\mathcal{A}$ as follows:

$$H:=\frac{1}{2e^{2u}}\langle f_{w\bar w},N\rangle,\quad \mathcal{A}:=\langle f_{ww},N\rangle. \tag{1.2.2}$$

The Gauss-Codazzi equations are of the following form in the CMC ($H^2>\varepsilon$) cases:

$$2u_{w\bar w}-2e^{2u}(H^2-\varepsilon)+\frac{1}{2}\mathcal{A}\bar{\mathcal{A}}e^{-2u}=0,\quad \mathcal{A}_{\bar w}=0. \tag{1.2.3}$$

Making the change of parameter $z:=2\sqrt{H^2-\varepsilon}\cdot w$ and defining Q by $\mathcal{A}=-2\sqrt{H^2-\varepsilon}\cdot e^{-i\psi}Q$ for a real constant ψ, we have equation (1.1.3) with $H=\pm\frac{1}{2}$:

$$4u_{z\bar z}+Q\bar Q e^{-2u}-e^{2u}=0,\quad Q_{\bar z}=0. \tag{1.2.4}$$

$\mathbb{S}^{2,1}$ and $\mathbb{H}^{2,1}$ and their matrix group representations. We identify $\mathbb{R}^{3,1}$, resp. $\mathbb{R}^{2,2}$, with the Hermitian symmetric group $\left\{X\in M_{2\times 2}\,\middle|\, X=\bar X^t\right\}$, resp. another matrix group, as follows:

$$\mathbb{R}^{3,1},\text{resp. }\mathbb{R}^{2,2}\ni x=(x_1,x_2,x_3,x_4)\longmapsto\begin{pmatrix} x_4+\nu\cdot x_3 & x_1-ix_2\\ x_1+ix_2 & x_4-\nu\cdot x_3\end{pmatrix}, \tag{1.2.5}$$

with $\nu=1$ for $\mathbb{R}^{3,1}$ and $\nu=i$ for $\mathbb{R}^{2,2}$. The metric becomes, under this identification, $\langle X,Y\rangle=-\frac{1}{2}\mathrm{trace}(X\sigma_2Y^t\sigma_2)$. In particular, $\langle X,X\rangle=-\det(X)$, and we can identify $\mathbb{S}^{2,1}$, resp. $\mathbb{H}^{2,1}$, with

$$\left\{X\in M_{2\times 2}\,\middle|\, X=\bar X^t,\det(X)=-1\right\}=\left\{F\sigma_3\bar F^t\,\middle|\, F\in\mathrm{SL}_2(\mathbb{C})\right\}, \tag{1.2.6}$$

respectively, with $\mathrm{SU}_{1,1}$, as in (1.1.4) via

$$\mathbb{H}^{2,1}\ni x=(x_1,x_2,x_3,x_4)\longmapsto\begin{pmatrix} x_4+ix_3 & x_1-ix_2\\ x_1+ix_2 & x_4-ix_3\end{pmatrix}. \tag{1.2.7}$$

The Sym-Bobenko type formula in $\mathbb{S}^{2,1}$ and $\mathbb{H}^{2,1}$. Defining $\hat F$ and f^λ in the same way as in (3.1.3), once again we change the "untwisted" setting to the "twisted" setting by the transformation (3.1.4) defining F. We define the *twisted* 2×2 *Lax pair* of f in $\mathcal{S}$ as

$$F_z=FU,\quad F_{\bar z}=FV, \tag{1.2.8}$$

where U and V are as in (1.1.9) with H fixed to be $\frac{1}{2}$.

The following Proposition 1.2.1 gives us a method for determining spacelike CMC H surfaces in $\mathbb{S}^{2,1}$ with $|H|>1$, resp. CMC H surfaces in $\mathbb{H}^{2,1}$ for any value of H, from given data u and Q.

Proposition 1.2.1 (Sym-Bobenko type formula for spacelike CMC surfaces in $\mathbb{S}^{2,1}$, $\mathbb{H}^{2,1}$ [33]). *Let Σ be a simply-connected domain in $\mathbb{C}$. Let u and Q solve (1.2.4), and let $F = F(z, \bar{z}, \lambda) \in \mathrm{SL}_2(\mathbb{C})$ be a solution of the system (1.2.8) such that $F(z, \bar{z}, \lambda) \in \mathrm{SU}_{1,1}$ when $\lambda \in \mathbb{S}^1$.*

- *In the case of $\mathbb{S}^{2,1}$, set $F_0 = F|_{\lambda = e^{\frac{q}{2}} e^{i\psi}}$ for $q, \psi \in \mathbb{R}$, $q \neq 0$. We define the following Sym-Bobenko type formulas*

$$f = F_0 \begin{pmatrix} e^{\frac{1}{2}q} & 0 \\ 0 & -e^{-\frac{1}{2}q} \end{pmatrix} \overline{F_0}^t, \quad N = -F_0 \begin{pmatrix} e^{\frac{1}{2}q} & 0 \\ 0 & e^{-\frac{1}{2}q} \end{pmatrix} \overline{F_0}^t. \tag{1.2.9}$$

  Then, f is a spacelike CMC $H = -\coth(-q)$ surface in $\mathbb{S}^{2,1}$ with normal N.

- *In the case of $\mathbb{H}^{2,1}$, set $F_1 = F|_{\lambda = e^{i\gamma_1}}$ and $F_2 = F|_{\lambda = e^{i\gamma_2}}$ for $\gamma_1, \gamma_2 \in \mathbb{R}$ and $\gamma_1 - \gamma_2 \neq n\pi$ ($n \in \mathbb{Z}$). We define the following Sym-Bobenko type formulas*

$$f = iF_1 \begin{pmatrix} e^{\frac{1}{2}i(\gamma_1-\gamma_2)} & 0 \\ 0 & -e^{-\frac{1}{2}i(\gamma_1-\gamma_2)} \end{pmatrix} \overline{F_2}^t, \quad N = -F_1 \begin{pmatrix} e^{\frac{1}{2}i(\gamma_1-\gamma_2)} & 0 \\ 0 & e^{-\frac{1}{2}i(\gamma_1-\gamma_2)} \end{pmatrix} \overline{F_2}^t. \tag{1.2.10}$$

  Then, f is a spacelike CMC $H = -\cot(\gamma_1 - \gamma_2)$ surface in $\mathbb{H}^{2,1}$ with normal N.

1.3 Application of the DPW method to $\mathbb{S}^{2,1}$ and $\mathbb{H}^{2,1}$

In this section, we give a description of the DPW method, and apply this method to spacelike CMC surfaces in $\mathbb{S}^{2,1}$ and $\mathbb{H}^{2,1}$ as in [17], [33]. First, we define the potential ξ:

Definition 1.3.1 (holomorphic potential [17], [25], [30]). Let Σ be a simply-connected domain, $z \in \Sigma$ and $\lambda \in \mathbb{C}$. A *holomorphic potential* ξ is of the form

$$\xi := Adz, \quad A = A(z, \lambda) = \sum_{j=-1}^{\infty} A_j(z)\lambda^j, \tag{1.3.1}$$

where each $A_j(z)$ is a 2×2 matrix that is independent of λ, is holomorphic in $z \in \Sigma$, is traceless, is a diagonal (resp. off-diagonal) matrix when j is even (resp. odd), and the upper-right entry of $A_{-1}(z)$ is never zero.

Given a holomorphic potential ξ, we then solve the equation

$$d\varphi = \varphi\xi, \quad \varphi(z_*) = I \quad \text{for} \quad \varphi \in \Lambda\mathrm{SL}_2(\mathbb{C}), \tag{1.3.2}$$

where $\Lambda\mathrm{SL}_2(\mathbb{C}) = \left\{ \varphi(\lambda) \in M_{2\times 2} \middle| \varphi : \mathbb{S}^1 \xrightarrow{C^\infty} \mathrm{SL}_2(\mathbb{C}), \varphi(-\lambda) = \sigma_3\varphi(\lambda)\sigma_3 \right\}$ for some choice of initial point $z_* \in \Sigma$.

We will use the following "$\mathrm{SU}_{1,1}$-Iwasawa splitting" defined on an open dense subset $\mathcal{B}_{1,1}$, called the *Iwasawa big cell*, of this loop group $\Lambda\mathrm{SL}_2(\mathbb{C})$. The following proposition was proven in [17].

Proposition 1.3.1 ($\mathrm{SU}_{1,1}$-Iwasawa splitting [17]). *For all $\varphi \in \mathcal{B}_{1,1}$, there exist unique loops F and B such that*

$$\varphi = F \cdot B, \tag{1.3.3}$$

where
$$F \in \Lambda\mathrm{SU}_{1,1} \cup \left\{ \begin{pmatrix} 0 & \lambda i \\ \lambda^{-1} i & 0 \end{pmatrix} \cdot \Lambda\mathrm{SU}_{1,1} \right\}, \quad B \in \Lambda^+_{\mathbb{R}}\mathrm{SL}_2(\mathbb{C}),$$
$$\Lambda\mathrm{SU}_{1,1} = \left\{ \varphi(\lambda) \in M_{2\times 2} \middle| \varphi : \mathbb{S}^1 \xrightarrow{C^\infty} \mathrm{SU}_{1,1}, \varphi(-\lambda) = \sigma_3 \varphi(\lambda) \sigma_3 \right\},$$
$$\Lambda^+_{\mathbb{R}}\mathrm{SL}_2(\mathbb{C}) = \left\{ B_+(\lambda) \in \Lambda\mathrm{SL}_2(\mathbb{C}) \middle| \begin{array}{c} B_+ \text{ extends holomorphically to } \mathbb{D}, \\ B_+(0) = \begin{pmatrix} \rho & 0 \\ 0 & \rho^{-1} \end{pmatrix} \text{ for some } \rho > 0. \end{array} \right\}.$$

After obtaining a solution φ of (1.3.2), we restrict to $\mathcal{B}_{1,1}$ and split φ as in (1.3.3). We then input F into the "Sym-Bobenko type formula" in (1.1.10) (resp. (1.2.9) or (1.2.10)), and the following proposition tells us we have a conformally immersed CMC surface $f = f(z, \bar{z})$ in $\mathbb{R}^{2,1}$ (resp. $\mathbb{S}^{2,1}$ or $\mathbb{H}^{2,1}$).

Proposition 1.3.2 ([17])**.** *Let ξ be a holomorphic potential as in* (3.3.1) *over a simply-connected domain Σ in $\mathbb{C}$, and let $\varphi : \Sigma \longrightarrow \Lambda\mathrm{SL}_2(\mathbb{C})$ be a solution of* (1.3.2)*. Define the open set $\Sigma^o := \varphi^{-1}(\mathcal{B}_{1,1}) \subset \Sigma$, and consider the unique $\mathrm{SU}_{1,1}$-Iwasawa splitting on Σ^o as in* (1.3.3)*.*

Then, after a conformal change of parameter z and appropriate choice of definitions for u, H and Q, we have that F satisfies the Lax pair (1.1.9)*.*

The converse of this recipe, that any conformally immersed CMC $H \neq 0$ (resp. $H = -\coth(-q)$ or $H = -\cot(\gamma_1 - \gamma_2)$) surface in $\mathbb{R}^{2,1}$ (resp. $\mathbb{S}^{2,1}$ or $\mathbb{H}^{2,1}$) has a holomorphic potential, also holds, but we will not prove that here, see [17] [33] for details.

1.4 Visualization of surfaces in $\mathbb{S}^{2,1}$ and $\mathbb{H}^{2,1}$

In the next section, we introduce some examples of CMC surfaces in $\mathbb{S}^{2,1}$ and $\mathbb{H}^{2,1}$. At that point, we wish to visualize CMC surfaces in $\mathbb{S}^{2,1}$ and $\mathbb{H}^{2,1}$, using appropriate models for $\mathbb{S}^{2,1}$ and $\mathbb{H}^{2,1}$, and we consider those models here. These models are already known, and we describe them explicitly here, in the context of our setting.

The hollow ball model of $\mathbb{S}^{2,1}$. To visualize CMC surfaces in $\mathbb{S}^{2,1}$, we use the *hollow ball model* of $\mathbb{S}^{2,1}$, as in [29], [54], [61], [96]. We get the following identification (bijection):

$$\mathbb{S}^{2,1} \ni (x_1, x_2, x_3, x_4) \longmapsto \left(\frac{e^{\arctan(x_4)}}{\sqrt{1+x_4^2}} x_1, \frac{e^{\arctan(x_4)}}{\sqrt{1+x_4^2}} x_2, \frac{e^{\arctan(x_4)}}{\sqrt{1+x_4^2}} x_3 \right) \in \mathcal{H}. \tag{1.4.1}$$

In this way, $\mathbb{S}^{2,1}$ is identified with the hollow ball $\mathcal{H}$. The hollow ball model is the set $\{(y_1, y_2, y_3) \in \mathbb{R}^3 | e^{-\pi} < y_1^2 + y_2^2 + y_3^2 < e^{\pi}\}$.

The cylindrical model of $\mathbb{H}^{2,1}$. To visualize CMC surfaces in $\mathbb{H}^{2,1}$, we use a model for $\mathbb{H}^{2,1}$, which we call the *cylindrical model*, like in [15] and [45]. It is actually only a model for the universal cover of $\mathbb{H}^{2,1}$, but has the advantage that certain symmetries become more apparent. We have the following homeomorphism and universal covering: $\mathbb{H}^{2,1} \approx \mathbb{H}^2 \times \mathbb{S}^1 \subset \mathbb{H}^2 \times \mathbb{R} =: \mathcal{C}$. This means that we have the following covering:

$$\mathbb{H}^{2,1} \ni (x_1, x_2, x_3, x_4) \longmapsto \left(\frac{x_1}{1+\sqrt{x_3^2+x_4^2}}, \frac{x_2}{1+\sqrt{x_3^2+x_4^2}}, \mathrm{Arg}\left(\frac{x_3 + i x_4}{\sqrt{x_3^2+x_4^2}} \right) \right) \in \mathcal{C}. \tag{1.4.2}$$

The cylindrical model is the set $\left\{(y_1, y_2, y_3) \in \mathbb{R}^3 | 0 \leqq y_1^2 + y_2^2 < 1\right\}$.

1.5 Examples

Here we introduce some examples of CMC surfaces in $\mathbb{S}^{2,1}$ and $\mathbb{H}^{2,1}$, using the DPW method.

Round cylinders. Here we show how the DPW method makes round cylinders. Defining

$$\xi := \lambda^{-1} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} dz \tag{1.5.1}$$

for $z = x + iy \in \Sigma = \mathbb{C}$ and $\lambda \in \mathbb{S}^1$, we solve $d\varphi = \varphi\xi$ and determine F, obtaining

$$\varphi = \begin{pmatrix} \cosh(\lambda^{-1}z) & \sinh(\lambda^{-1}z) \\ \sinh(\lambda^{-1}z) & \cosh(\lambda^{-1}z) \end{pmatrix}, \ F = \begin{pmatrix} \cosh(\lambda^{-1}z + \bar{z}\lambda) & \sinh(\lambda^{-1}z + \bar{z}\lambda) \\ \sinh(\lambda^{-1}z + \bar{z}\lambda) & \cosh(\lambda^{-1}z + \bar{z}\lambda) \end{pmatrix}. \tag{1.5.2}$$

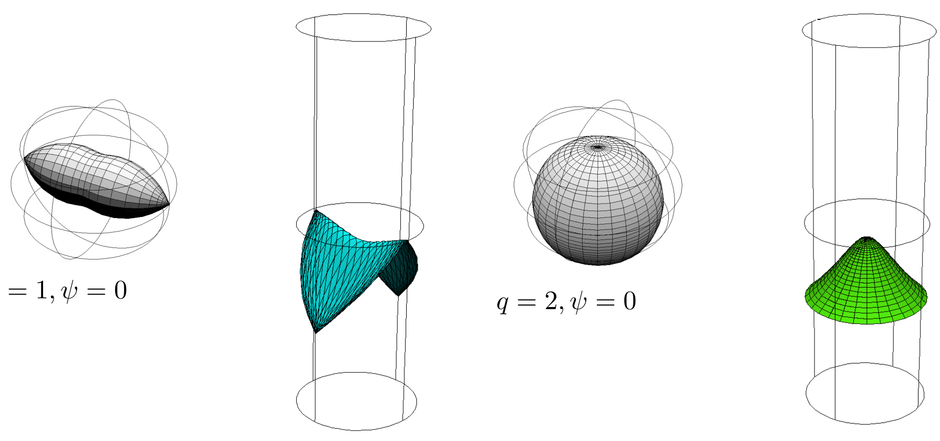
Totally umbilical surfaces. The DPW method produces totally umbilical surfaces via

$$\xi := \lambda^{-1} \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} dz, \tag{1.5.3}$$

for $z \in \Sigma = \mathbb{C}$ and $\lambda \in \mathbb{S}^1$, we solve $d\varphi = \varphi\xi$ and determine F ($\varphi \in \mathcal{B}_{1,1}$ when $|z| \neq 1$), obtaining

$$\varphi = \exp\left(z\lambda^{-1} \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}\right) = \begin{pmatrix} 1 & z\lambda^{-1} \\ 0 & 1 \end{pmatrix}, \ F = \frac{1}{\sqrt{1-|z|^2}} \begin{pmatrix} 1 & z\lambda^{-1} \\ \bar{z}\lambda & 1 \end{pmatrix} \in \Lambda \mathrm{SU}_{1,1}.$$

Remark 1.5.1. These surfaces in $\mathbb{S}^{2,1}$ and $\mathbb{H}^{2,1}$ are not compact, but are totally umbilic.



$q = 1, \psi = 0$ $\gamma_1 = \frac{\pi}{3}, \gamma_2 = \frac{\pi}{6}$ $q = 2, \psi = 0$ $\gamma_1 = \frac{\pi}{3}, \gamma_2 = \frac{\pi}{6}$

Fig. 1.1: The left two images are round cylinders in $\mathbb{S}^{2,1}$ (resp. $\mathbb{H}^{2,1}$), and right two image are totally umbilical surfaces in $\mathbb{S}^{2,1}$ (resp. $\mathbb{H}^{2,1}$).

Chapter 2

Singularity theory for spacelike CMC surfaces

2.1 Theory for CMC surfaces with singularities

In this section, we consider the singularities of CMC surfaces in $\mathbb{R}^{2,1}$, $\mathbb{S}^{2,1}$ and $\mathbb{H}^{2,1}$, and we show criteria for cuspidal edges, swallowtails and cuspidal cross caps, using some geometric notions.
Relationships between Iwasawa splitting and singularities. We first make some remarks about relationships between the DPW method and singularities. We define *small cells* as follows:

Definition 2.1.1 ([16], [17])**.** Define, for a positive integer $m \in \mathbb{Z}$,

$$\omega_m = \begin{pmatrix} 1 & 0 \\ \lambda^{-m} & 1 \end{pmatrix} : \ m \text{ odd}, \quad \omega_m = \begin{pmatrix} 1 & \lambda^{1-m} \\ 0 & 1 \end{pmatrix} : \ m \text{ even}.$$

Then the group $\Lambda\mathrm{SL}_2(\mathbb{C})$ is a disjoint union

$$\Lambda\mathrm{SL}_2(\mathbb{C}) = \mathcal{B}_{1,1} \bigsqcup_{m \in \mathbb{Z}^+} \mathcal{P}_m, \text{ where } \mathcal{B}_{1,1} := \left(\Lambda\mathrm{SU}_{1,1} \cup \left\{ \begin{pmatrix} 0 & \lambda i \\ \lambda^{-1} i & 0 \end{pmatrix} \cdot \Lambda\mathrm{SU}_{1,1} \right\} \right) \cdot \Lambda^+_{\mathbb{R}}\mathrm{SL}_2(\mathbb{C}),$$

is called the Iwasawa big cell, and the m-th small cell is $\mathcal{P}_m := \Lambda\mathrm{SU}_{1,1} \cdot \omega_m \cdot \Lambda^+_{\mathbb{R}}\mathrm{SL}_2(\mathbb{C})$.

Using these small cells, we have the following proposition for the case of $\mathbb{R}^{2,1}$, as in [16], [17].

Proposition 2.1.1 ([16], [17] Theorem 4.2)**.** *Let Σ be a simply connected domain, and let $\varphi : \Sigma \longrightarrow \Lambda\mathrm{SL}_2(\mathbb{C})$. Define $\Sigma^0 = \varphi^{-1}(\mathcal{B}_{1,1})$, $C_1 = \varphi^{-1}(\mathcal{P}_1)$ and $C_2 = \varphi^{-1}(\mathcal{P}_2)$. Then:*

1. *The sets $\Sigma^0 \cup C_1$ and $\Sigma^0 \cup C_2$ are both open subsets of Σ. The sets C_i are each locally given as the zero set of a non-constant real analytic function $\mathbb{R}^2 \longrightarrow \mathbb{R}$.*

2. *All components of the matrix F obtained by Proposition* 1.3.1 *on Σ^0, and evaluated at $\lambda_0 \in \mathbb{S}^1$, blow up as z approaches a point z_0 in either C_1 or C_2. In the limit, the unit normal vector N, to the corresponding surface in $\mathbb{R}^{2,1}$, becomes asymptotically lightlike, i.e. its length in the Euclidean $\mathbb{R}^3$ metric approaches infinity.*

3. *The CMC surface $f \in \mathbb{R}^{2,1}$ given by the DPW method extends to a real analytic map $\Sigma^0 \cup C_1 \longrightarrow \mathbb{R}^{2,1}$, but is not immersed as points $z_0 \in C_1$.*

4. *The CMC surface $f \in \mathbb{R}^{2,1}$ given by the DPW method diverges to ∞ as $z \to z_0 \in C_2$. Moreover, the induced metric on the surface blows up as such a point in the coordinate domain is approached.*

By the above proposition, at a singular point, we cannot split φ to $F \cdot B$ for $F \in \Lambda \mathrm{SU}_{1,1}$, and the procedure of the DPW method does not work. Thus, we consider the claims of Proposition 1.1.1 and 1.2.1 directly, without using the Iwasawa splitting. Our recipe is as follows:

1. First we choose a real constant $H \neq 0$ (resp. $H = 1/2$) and holomorphic function Q, for the case of $\mathbb{R}^{2,1}$ (resp. $\mathbb{S}^{2,1}$ and $\mathbb{H}^{2,1}$).

2. We define the metric function $g = e^u$ satisfying (1.1.3).

3. We obtain F by solving the system (1.1.9).

4. Finally, we get a conformal spacelike CMC surface f in $\mathbb{R}^{2,1}$ (resp. $\mathbb{S}^{2,1}$ or $\mathbb{H}^{2,1}$) by inputting F into the Sym-Bobenko type formula (1.1.10) (resp. (1.2.9) or (1.2.10)).

Criteria for singularities of spacelike CMC $H \neq 0$ surfaces in $\mathbb{R}^{2,1}$. Here we introduce criteria for cuspidal edges, swallowtails and cuspidal cross caps of CMC $H \neq 0$ surfaces in $\mathbb{R}^{2,1}$, as in [91]. However, we use a different approach from [91] because we start not with harmonic maps, but with H, Q and $g = e^u$.

Let $H \neq 0$ be a real constant, and let Q be a holomorphic function. Let $g = e^u$ be a solution of (1.1.3). Here, in order to match [91], we use the untwisted setting. Thus we define $\hat{F}$ satisfying the system (1.1.7) for $\lambda = 1$. For this $\hat{F}$, we have the following untwisted version of Proposition 1.1.1:

Proposition 2.1.2 (untwisted version of Proposition 1.1.1)**.** *Let Σ be a simply-connected domain in $\mathbb{C}$. Let u and Q solve* (1.1.3)*, and let $\hat{F} = \hat{F}(z, \bar{z}, \lambda)$ be a solution of the system* (1.1.7)*. Suppose $\hat{F} \in \mathrm{SU}_{1,1}$ for all $\lambda \in \mathbb{S}^1$ and one value of z. Then $\hat{F} \in \mathrm{SU}_{1,1}$ for all z where $\hat{F}$ is bounded, and $\hat{F}$ is bounded wherever u and Q are bounded. Defining the following Sym-Bobenko type formulas*

$$f = \left[-\frac{1}{H} \hat{F} i\sigma_3 \hat{F}^{-1} + \frac{i}{H} \lambda (\partial_\lambda \hat{F}) \hat{F}^{-1} \right] \Bigg|_{\lambda=1}, \quad N = \left[\hat{F} i\sigma_3 \hat{F}^{-1} \right] \Big|_{\lambda=1}. \tag{2.1.1}$$

f is a conformally parametrized spacelike CMC $H \neq 0$ surface in $\mathbb{R}^{2,1}$ with normal N.

We denote $\hat{F} = \hat{F}(z, \bar{z}) = e^{-\frac{u}{2}} \begin{pmatrix} \bar{a} & b \\ \bar{b} & a \end{pmatrix} \in \mathrm{SU}_{1,1}$, where $|a|^2 - |b|^2 = e^u = g$. So we define h and ω such that $h := -\frac{ia}{b}$ and $\omega := -2b^2$, and metric is

$$ds^2 = 4g^2 dz d\bar{z} = (1 - |h|^2)^2 |\omega|^2 dz d\bar{z}. \tag{2.1.2}$$

This implies that, wherever ds^2 is finite, f has a singularity if and only if $h \in \mathbb{S}^1$ or $\omega = 0$. However, by (2.1.1) we have

$$f_z = \frac{\omega}{2} \left(1 + h^2, i(1 - h^2), -2h \right), \quad f_{\bar{z}} = \frac{\bar{\omega}}{2} \left(1 + \bar{h}^2, -i(1 - \bar{h}^2), -2\bar{h} \right), \tag{2.1.3}$$

and $\omega = 0$ means that f has an isolated singular point there. Here we consider only *extended CMC surfaces* defined by the following, as in [91].

Definition 2.1.2 ([91])**.** A CMC surface f restricted to the subdomain $\mathcal{D} = \{p \in \Sigma \mid ds^2 < \infty\}$ is called an extended CMC surface if ω, resp. $h^2\omega$, is never zero on $\mathcal{D}$ when $|h| < \infty$, resp. $|h| = \infty$.

Remark 2.1.1. By this definition, any point $p \in \Sigma$ is singular only when $|h(p)| = 1$. (See [91].)

Remark 2.1.2. By [65], the normal vector N of spacelike CMC surface f satisfies that $N_{z\bar{z}}$ is parallel to N at each regular point p. This implies that $h_{z\bar{z}} + \frac{2\bar{h}}{1-|h|^2} h_z h_{\bar{z}} = 0$ meaning that h is a harmonic map at each regular point p. Similarly, $N_z = -Hf_z - \frac{1}{2}Qe^{-2u}f_{\bar{z}}$ implies $\omega = \frac{\bar{h}_z}{(1-|h|^2)^2}$ at each regular point p. However, these Equations do not necessarily hold at singular points.

Now we have the following criteria for singularities of spacelike extended CMC $H \neq 0$ surfaces, as in [91]. However, since we use different notations and a different approach, we give a sketch of the proof here. The notion of $\mathcal{A}$-equivarece used in this theorem is fundamental in singularity theory, and is explained in [32], [91], [92].

Theorem 2.1.1 ([91])**.** *Let Σ be a simply connected domain, and let $f : \Sigma \longrightarrow \mathbb{R}^{2,1}$ be a spacelike extended CMC $H \neq 0$ surface. Then:*

1. *f is a front at a singular point $p \in \Sigma$ (i.e. $h(p) \in \mathbb{S}^1$) if and only if $\mathrm{Re}\left(\frac{h_z}{h^2\omega}\right)\Big|_p \neq 0$. If this is the case, p is a non-degenerate singular point.*
2. *f is $\mathcal{A}$-equivalent to a cuspidal edge at a singular point $p \in \Sigma$ if and only if*
$$\mathrm{Re}\left(\frac{h_z}{h^2\omega}\right)\Bigg|_p \neq 0 \quad \textit{and} \quad \mathrm{Im}\left(\frac{h_z}{h^2\omega}\right)\Bigg|_p \neq 0.$$
3. *f is $\mathcal{A}$-equivalent to a swallowtail at a singular point $p \in \Sigma$ if and only if*
$$\mathrm{Re}\left(\frac{h_z}{h^2\omega}\right)\Bigg|_p \neq 0,\ \mathrm{Im}\left(\frac{h_z}{h^2\omega}\right)\Bigg|_p = 0 \text{ and } \mathrm{Re}\left\{\overline{\left(\frac{h_z}{h}\right)}\left(\frac{h_z}{h^2\omega}\right)_z\right\}\Bigg|_p \neq \mathrm{Re}\left\{\left(\frac{h_z}{h}\right)\left(\frac{h_z}{h^2\omega}\right)_{\bar{z}}\right\}\Bigg|_p.$$
4. *f is $\mathcal{A}$-equivalent to a cuspidal cross cap at a singular point $p \in \Sigma$ if and only if*
$$\mathrm{Re}\left(\frac{h_z}{h^2\omega}\right)\Bigg|_p = 0,\ \mathrm{Im}\left(\frac{h_z}{h^2\omega}\right)\Bigg|_p \neq 0 \text{ and } \mathrm{Im}\left\{\overline{\left(\frac{h_z}{h}\right)}\left(\frac{h_z}{h^2\omega}\right)_z\right\}\Bigg|_p \neq \mathrm{Im}\left\{\left(\frac{h_z}{h}\right)\left(\frac{h_z}{h^2\omega}\right)_{\bar{z}}\right\}\Bigg|_p.$$

Proof. The essential idea behind proving this is to input h, ω into the Kenmotsu type representation as in [2] and to compute the same way as in [91]. □

Here we give equivalent conditions for Theorem 2.1.1, as follows (in this corollary $\langle f_z, f_z\rangle_{\mathbb{C}^3} := \frac{1}{4}(|f_x|^2_{\mathbb{R}^3} - |f_y|^2_{\mathbb{R}^3}) - \frac{i}{2}\langle f_x, f_y\rangle_{\mathbb{R}^3}$ is the bilinear extension of the $\mathbb{R}^3$ inner product) :

Corollary 2.1.1. *Let Σ be a simply connected domain, and let $f : \Sigma \longrightarrow \mathbb{R}^{2,1}$ be a spacelike extended CMC $H \neq 0$ surface, given by a real constant H, a holomorphic function Q and a metric function g. Then:*

1. *f is a front at a singular point $p \in \Sigma$ if and only if $\mathrm{Re}\left(\frac{Q}{\langle f_z, f_z\rangle_{\mathbb{C}^3}}\right)\Big|_p \neq 0$. If this is the case, p is a non-degenerate singular point.*

2. f is $\mathcal{A}$-equivalent to a cuspidal edge at a singular point $p \in \Sigma$ if and only if

$$\mathrm{Re}\left(\frac{Q}{\langle f_z, f_z\rangle_{\mathbb{C}^3}}\right)\bigg|_p \neq 0 \quad \text{and} \quad \mathrm{Im}\left(\frac{Q}{\langle f_z, f_z\rangle_{\mathbb{C}^3}}\right)\bigg|_p \neq 0.$$

3. f is $\mathcal{A}$-equivalent to a swallowtail at a singular point $p \in \Sigma$ if and only if

$$\mathrm{Re}\left(\frac{Q}{\langle f_z, f_z\rangle_{\mathbb{C}^3}}\right)\bigg|_p \neq 0, \quad \mathrm{Im}\left(\frac{Q}{\langle f_z, f_z\rangle_{\mathbb{C}^3}}\right)\bigg|_p = 0 \quad \text{and}$$

$$\mathrm{Re}\left[\frac{Q_z\langle f_z, f_z\rangle_{\mathbb{C}^3} - 2Q\langle f_{zz}, f_z\rangle_{\mathbb{C}^3}}{\langle f_z, f_z\rangle^2_{\mathbb{C}^3}} \cdot \frac{\overline{Q}}{\langle f_{\bar{z}}, f_{\bar{z}}\rangle_{\mathbb{C}^3}}\right]\bigg|_p \neq \mathrm{Re}\left[\frac{-2Q\langle f_{z\bar{z}}, f_z\rangle_{\mathbb{C}^3}}{\langle f_z, f_z\rangle^2_{\mathbb{C}^3}} \cdot \frac{Q}{\langle f_z, f_z\rangle_{\mathbb{C}^3}}\right]\bigg|_p.$$

4. f is $\mathcal{A}$-equivalent to a cuspidal cross cap at a singular point $p \in \Sigma$ if and only if

$$\mathrm{Re}\left(\frac{Q}{\langle f_z, f_z\rangle_{\mathbb{C}^3}}\right)\bigg|_p = 0, \quad \mathrm{Im}\left(\frac{Q}{\langle f_z, f_z\rangle_{\mathbb{C}^3}}\right)\bigg|_p \neq 0 \quad \text{and}$$

$$\mathrm{Im}\left[\frac{Q_z\langle f_z, f_z\rangle_{\mathbb{C}^3} - 2Q\langle f_{zz}, f_z\rangle_{\mathbb{C}^3}}{\langle f_z, f_z\rangle^2_{\mathbb{C}^3}} \cdot \frac{\overline{Q}}{\langle f_{\bar{z}}, f_{\bar{z}}\rangle_{\mathbb{C}^3}}\right]\bigg|_p \neq \mathrm{Im}\left[\frac{-2Q\langle f_{z\bar{z}}, f_z\rangle_{\mathbb{C}^3}}{\langle f_z, f_z\rangle^2_{\mathbb{C}^3}} \cdot \frac{Q}{\langle f_z, f_z\rangle_{\mathbb{C}^3}}\right]\bigg|_p.$$

Proof. Using the Hopf differential $Q = \langle f_{zz}, N\rangle_{\mathbb{R}^{2,1}} = \omega h_z$ and $\langle f_z, f_z\rangle_{\mathbb{C}^3} = 2\omega^2 h^2$, we draw the conclusions from Theorem 2.1.1. □

Criteria for singularities of spacelike CMC $H^2 > 1$ surfaces in $\mathbb{S}^{2,1}$. In this section, we study singularities of spacelike CMC H ($H^2 > 1$) surfaces f in $\mathbb{S}^{2,1}$, similarly to Theorem 2.1.1 for spacelike CMC H ($H \neq 0$) surfaces in $\mathbb{R}^{2,1}$. However, in $\mathbb{R}^{2,1}$ we used $\lambda \in \mathbb{S}^1$ in the Sym-Bobenko type formula and thus the solution $\hat{F}$ in (1.1.7) is in $\mathrm{SU}_{1,1}$ for that λ, while in $\mathbb{S}^{2,1}$ this will not be the case. In the case of $\mathbb{S}^{2,1}$, the λ we use in the Sym-Bobenko type formula is not in $\mathbb{S}^1$, and so $\hat{F}|_\lambda \notin \mathrm{SU}_{1,1}$, and this creates complications for attempting to imitate Theorem 2.1.1. We remedy this problem by using the s-spectral deformation to shift to a new CMC surface $\hat{f}$ in $\mathbb{S}^{2,1}$ of the same type where arguments like those proving Theorem 2.1.1 can be used. Noting that, by Lemma 4.1.1, as we have not restricted the full class of surfaces being considered, we are still proving a result (Theorem 4.1.3) that applies to all spacelike CMC surfaces in $\mathbb{S}^{2,1}$ with constant mean curvature greater than 1 in absolute value. Here we introduce criteria for cuspidal edges, swallowtails and cuspidal cross caps on CMC $H^2 > 1$ surfaces in $\mathbb{S}^{2,1}$.

Let $f : \Sigma \longrightarrow \mathbb{S}^{2,1}$ be a spacelike CMC surface for a simply-connected domain $\Sigma \subset \mathbb{C}$, with the metric $ds^2 = 4g^2 dw d\bar{w} = 4e^{2u} dw d\bar{w}$ and unit normal vector N. First, we consider the *moving frame* $\mathfrak{F}$ such that

$$f = \mathfrak{F}\sigma_3\overline{\mathfrak{F}}^t, \ \frac{f_x}{2e^u} = \mathfrak{F}\sigma_1\overline{\mathfrak{F}}^t, \ \frac{f_y}{2e^u} = \mathfrak{F}\sigma_2\overline{\mathfrak{F}}^t, \ N = \mathfrak{F}\overline{\mathfrak{F}}^t.$$

Then, we have

$$\mathfrak{F}_w = \mathfrak{F}\boldsymbol{A}, \ \mathfrak{F}_{\bar{w}} = \mathfrak{F}\boldsymbol{B}, \ \text{where } \boldsymbol{A} = \frac{1}{2}\begin{pmatrix} -u_w & -e^{-u}\mathcal{A} \\ 2e^u(1-H) & u_w \end{pmatrix}, \ \boldsymbol{B} = \frac{1}{2}\begin{pmatrix} u_{\bar{w}} & -2e^u(1+H) \\ -e^{-u}\bar{\mathcal{A}} & -u_{\bar{w}} \end{pmatrix}. \tag{2.1.4}$$

For this $\mathfrak{F}$, the compatibility condition implies the Gauss and Codazzi equations (1.2.3). We define *s-spectral deformations* as follows:

Definition 2.1.3 ([90])**.** The *s-spectral deformation* of the CMC surface f in $\mathbb{S}^{2,1}$ is the deformation defined by $(1+H) \to s(1+H)$, $(1-H) \to s^{-1}(1-H)$ in Equations (4.1.1) for the parameter $s > 0$.

The s-spectral deformation maps CMC surfaces to other CMC surfaces conformally, as follows (the analogous result in the case of $\mathbb{H}^3$ was proven in [90]):

Theorem 2.1.2. *For all $s \in \mathbb{R}_{>0}$, the s-spectral deformation deforms a surface f in $\mathbb{S}^{2,1}$, with mean curvature H, metric $ds^2 = 4e^{2u}dwd\bar{w}$ and Hopf differential $\mathcal{A}$, into a surface f^s with mean curvature $H^s = \frac{s(1+H)-s^{-1}(1-H)}{s(1+H)+s^{-1}(1-H)}$, metric $4e^{2u^s}dwd\bar{w} = 4k^2e^{2u}dwd\bar{w}$ and Hopf differential $\mathcal{A}^s = k\mathcal{A}$ for $k = \frac{s(1+H)+s^{-1}(1-H)}{2}$.*

Proof. We notice that the s-spectral deformation implies that

$$\begin{cases} k(1+H^s) = s(1+H) \\ k(1-H^s) = s^{-1}(1-H) \end{cases}, \text{ equivalently } \begin{cases} k = \frac{s(1+H)+s^{-1}(1-H)}{2} \\ H^s = \frac{s(1+H)-s^{-1}(1-H)}{s(1+H)+s^{-1}(1-H)}. \end{cases}$$

Then (1.2.3) holds with $\varepsilon = 1$, and u, $\mathcal{A}$ replaced by $u^s = u + \log|k|$, $\mathcal{A}^s = k\mathcal{A}$. Thus the deformation family of surfaces exists. ☐

Lemma 2.1.1. $(f^s)^{\frac{1}{s}} = f$.

Proof. The $\frac{1}{s}$-spectral deformation of f^s is

$$\begin{cases} k^{\frac{1}{s}}\left(1+(H^s)^{\frac{1}{s}}\right) = \frac{1}{s}(1+H^s) \\ k^{\frac{1}{s}}\left(1-(H^s)^{\frac{1}{s}}\right) = s(1-H^s) \end{cases}, \text{ equivalently } \begin{cases} k^{\frac{1}{s}} = \frac{s^{-1}(1+H^s)+s(1-H^s)}{2} = \frac{1}{k} \\ (H^s)^{\frac{1}{s}} = \frac{s^{-1}(1+H^s)-s(1-H^s)}{s^{-1}(1+H^s)+s(1-H^s)} = H. \end{cases}$$

Similarly we have $(\mathcal{A}^s)^{\frac{1}{s}} = \mathcal{A}$, $(u^s)^{\frac{1}{s}} = u$. ☐

We define the (twisted) s-spectral Lax pair.

Definition 2.1.4 (s-spectral Lax pair)**.** We define $\mathfrak{F}^s$ as a solution of the following system:

$$\mathfrak{F}^s_w = \mathfrak{F}^s \boldsymbol{A}^s, \quad \mathfrak{F}^s_{\bar{w}} = \mathfrak{F}^s \boldsymbol{B}^s,$$

where

$$\boldsymbol{A}^s := \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \frac{1}{2} \begin{pmatrix} -u_w & -e^{-u}\mathcal{A} \\ 2e^u s^{-1}(1-H) & u_w \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \frac{1}{2} \begin{pmatrix} u_w & 2e^u s^{-1}(1-H) \\ -e^{-u}\mathcal{A} & -u_w \end{pmatrix},$$
$$\boldsymbol{B}^s := \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \frac{1}{2} \begin{pmatrix} u_{\bar{w}} & -2e^u s(1+H) \\ -e^{-u}\bar{\mathcal{A}} & -u_{\bar{w}} \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \frac{1}{2} \begin{pmatrix} -u_{\bar{w}} & -e^{-u}\bar{\mathcal{A}} \\ -2e^u s(1+H) & u_{\bar{w}} \end{pmatrix}.$$

Further, we define the form $\Omega^s := (\mathfrak{F}^s)^{-1} d\mathfrak{F}^s$.

Theorem 2.1.3. *For f given by the frame $\mathfrak{F}$, and mean curvature $H^2 > 1$, there exists a member of the s-spectral deformation, for the special value $s = s_0 := \sqrt{\frac{H-1}{H+1}}$, that generates a frame $\tilde{\mathfrak{F}} = \mathfrak{F}^{s_0} \in \mathrm{SU}_{1,1}$. This frame $\tilde{\mathfrak{F}}$ represents the lift of a harmonic map in $\mathbb{H}^2$.*

Proof. It is easy to see that choosing $s = s_0 := \sqrt{\frac{H-1}{H+1}}$ gives the only deformation that makes the Maurer-Cartan form become an $\mathrm{su}_{1,1}$-valued form. $\square$

As s approaches s_0, the mean curvature goes to infinity, and $\tilde{f} := \tilde{\mathfrak{F}}\sigma_3\bar{\tilde{\mathfrak{F}}}^t$ degenerates to a point, but there still exists a map $\tilde{\mathfrak{F}}$ from Σ to $\mathrm{SU}_{1,1}$ such that $\tilde{\mathfrak{F}}^{-1}d\tilde{\mathfrak{F}} = \tilde{\Omega}$ defined by the following (4.1.2). The harmonic map is the natural projection of the adjusted frame $\tilde{\mathfrak{F}}$ (defined just below) to $\mathbb{H}^2$.

Definition 2.1.5. We call $\tilde{\mathfrak{F}} : \Sigma \longrightarrow \mathrm{SU}_{1,1}$ the *adjusted frame* of $\mathfrak{F}$ and the form $\tilde{\Omega} = \tilde{\mathfrak{F}}^{-1}d\tilde{\mathfrak{F}}$ the *adjusted Maurer-Cartan form*, where

$$\tilde{\Omega} = \frac{1}{2}\begin{pmatrix} u_w & -2e^u\sqrt{H^2-1} \\ -e^{-u}\mathcal{A} & -u_w \end{pmatrix} dw + \frac{1}{2}\begin{pmatrix} -u_{\bar{w}} & -e^{-u}\bar{\mathcal{A}} \\ -2e^u\sqrt{H^2-1} & u_{\bar{w}} \end{pmatrix} d\bar{w} =: \tilde{\boldsymbol{A}}dw + \tilde{\boldsymbol{B}}d\bar{w}. \quad (2.1.5)$$

Theorem 2.1.4. *Let Σ be a simply-connected domain. Let $a > 0$ ($a \neq 1$) be an arbitrary real constant, and let*

$$\beta_1(a) := \frac{a^{-1}-1}{2}\begin{pmatrix} 0 & -2e^u\sqrt{H^2-1} \\ 0 & 0 \end{pmatrix} dw, \quad \beta_2(a) := \frac{a-1}{2}\begin{pmatrix} 0 & 0 \\ -2e^u\sqrt{H^2-1} & 0 \end{pmatrix} d\bar{w}.$$

Define $\hat{\Omega} := \tilde{\Omega} + \beta_1(a) + \beta_2(a)$. Then we have the following:

1. $d\hat{\Omega} + \frac{1}{2}[\hat{\Omega} \wedge \hat{\Omega}] = 0$.

2. *If $\hat{\mathfrak{F}}$ is a $\mathrm{SL}_2(\mathbb{C})$-valued solution of $\hat{\Omega} = \hat{\mathfrak{F}}^{-1}d\hat{\mathfrak{F}}$, then $\hat{f} = \hat{\mathfrak{F}}\sigma_3\bar{\hat{\mathfrak{F}}}^t$ is a conformal spacelike CMC surface with $\hat{H} = \frac{a^2+1}{a^2-1}$.*

Proof. For $s = \sqrt{\frac{H-1}{H+1}}a \in \mathbb{R}_{>0}$, we have $\Omega^s = \hat{\Omega}$, by direct computation. Thus we have existence of $\hat{f}$, and $\hat{H} = H^s = \frac{a^2+1}{a^2-1}$. $\square$

Remark 2.1.3. Defining $G := \hat{\mathfrak{F}} \cdot \tilde{\mathfrak{F}}^{-1}$, we have $G\sigma_3\bar{G}^t = \hat{\mathfrak{F}}\sigma_3\bar{\hat{\mathfrak{F}}}^t = \hat{f}$.

As noted previously, we will consider the criteria for singularities of $\hat{f}$ instead of f.

We denote $\tilde{\mathfrak{F}} = \tilde{\mathfrak{F}}(w, \bar{w}) = e^{-\frac{u}{2}}\begin{pmatrix} u_1 & u_2 \\ \overline{u_2} & \overline{u_1} \end{pmatrix} \in \mathrm{SU}_{1,1}$, where $g = e^u$ is the metric function of f, and $|u_1|^2 - |u_2|^2 = g$. So we define $h := \frac{\overline{u_2}}{u_1}$ and $\omega := u_1^2$. By Remark 4.1.1, we have $\hat{f} = \hat{\mathfrak{F}}\sigma_3\bar{\hat{\mathfrak{F}}}^t = G\sigma_3\bar{G}^t$. Setting $\hat{k} = \frac{(a-a^{-1})\sqrt{H^2-1}}{2}$, we get that $\hat{f}$ has metric

$$ds^2 := 4\hat{g}^2 dw d\bar{w} = 4\hat{k}^2(1-|h|^2)^2|\omega|^2 dw d\bar{w}.$$

Thus this implies that, wherever ds^2 is finite, $\hat{f}$ has a singularity if and only if $h \in \mathbb{S}^1$ or $\omega = 0$. However we consider only extended CMC surfaces $\hat{f}$ defined in the same way as Definition 2.1.2.

We have the following criteria for singularities of spacelike extended CMC $\hat{H}^2 > 1$ surfaces in $\mathbb{S}^{2,1}$. The proof of Theorem 4.1.3 is parallel to the proof of Theorem 3.1 in [32]. (Also see [91], [92].)

Theorem 2.1.5. *Let Σ be a simply connected domain, and let $\hat{f} : \Sigma \longrightarrow \mathbb{S}^{2,1}$ be a spacelike extended CMC $\hat{H}^2 > 1$ surface, given by Theorem 2.1.4. Then:*

1. *A point $p \in \Sigma$ is a singular point if and only if $h \in \mathbb{S}^1$.*

2. *$\hat{f}$ is a front at a singular point $p \in \Sigma$ if and only if $\mathrm{Re}\left(\frac{h_w}{h^2\omega}\right)\Big|_p \neq 0$. If this is the case, p is non-degenerate singular point.*

3. *$\hat{f}$ is $\mathcal{A}$-equivalent to a cuspidal edge at a singular point $p \in \Sigma$ if and only if*
$$\mathrm{Re}\left(\frac{h_w}{h^2\omega}\right)\Bigg|_p \neq 0 \quad \text{and} \quad \mathrm{Im}\left(\frac{h_w}{h^2\omega}\right)\Bigg|_p \neq 0.$$

4. *$\hat{f}$ is $\mathcal{A}$-equivalent to a swallowtail at a singular point $p \in \Sigma$ if and only if*
$$\mathrm{Re}\left(\frac{h_w}{h^2\omega}\right)\Bigg|_p \neq 0, \quad \mathrm{Im}\left(\frac{h_w}{h^2\omega}\right)\Bigg|_p = 0 \text{ and } \mathrm{Re}\left\{\overline{\left(\frac{h_w}{h}\right)}\left(\frac{h_w}{h^2\omega}\right)_w\right\}\Bigg|_p \neq \mathrm{Re}\left\{\left(\frac{h_w}{h}\right)\left(\frac{h_w}{h^2\omega}\right)_{\bar{w}}\right\}\Bigg|_p.$$

5. *$\hat{f}$ is $\mathcal{A}$-equivalent to a cuspidal cross cap at a singular point $p \in \Sigma$ if and only if*
$$\mathrm{Re}\left(\frac{h_w}{h^2\omega}\right)\Bigg|_p = 0, \quad \mathrm{Im}\left(\frac{h_w}{h^2\omega}\right)\Bigg|_p \neq 0 \text{ and } \mathrm{Im}\left\{\overline{\left(\frac{h_w}{h}\right)}\left(\frac{h_w}{h^2\omega}\right)_w\right\}\Bigg|_p \neq \mathrm{Im}\left\{\left(\frac{h_w}{h}\right)\left(\frac{h_w}{h^2\omega}\right)_{\bar{w}}\right\}\Bigg|_p.$$

Proof. (1) This is clear (like in Remark 4.1.2).
(2) First we define
$$\nu := G\begin{pmatrix} 1+|h|^2 & 2\bar{h} \\ 2h & 1+|h|^2 \end{pmatrix}\bar{G}^t, \tag{2.1.6}$$
and this ν is the Lorentz normal vector field of $\hat{f}$ on the regular set of $\hat{f}$. This is not a unit vector, but extends smoothly across the singular set. By Lemma 1.6 of [32], $\hat{f}$ is a front at a singular point p if and only if ν is not proportional to $D_\eta^{\mathbb{R}^{3,1}}\nu$ at p, for the null direction η of $\hat{f}$ and the canonical connection $D^{\mathbb{R}^{3,1}}$. Since $d\hat{f} = G\left\{2\hat{k}\begin{pmatrix} -h & -1 \\ -h^2 & -h \end{pmatrix}\omega dw - 2\hat{k}\begin{pmatrix} \bar{h} & \bar{h}^2 \\ 1 & \bar{h} \end{pmatrix}\bar{\omega}d\bar{w}\right\}\bar{G}^t$, the null direction is $\eta = \frac{i}{h\omega}\partial_w - \frac{i}{\bar{h}\bar{\omega}}\partial_{\bar{w}}$ at a singular point p. We also have, at p,
$$D_\eta^{\mathbb{R}^{3,1}}\nu = G\left\{\frac{i}{h\omega}\begin{pmatrix} h_w\bar{h} & 0 \\ 2h_w & h_w\bar{h} \end{pmatrix} - \frac{i}{\bar{h}\bar{\omega}}\begin{pmatrix} h\bar{h}_{\bar{w}} & 2\bar{h}_{\bar{w}} \\ 0 & h\bar{h}_{\bar{w}} \end{pmatrix}\right\}\bar{G}^t.$$

On the other hand, we have $\langle \nu, \nu\rangle = \langle D_\eta^{\mathbb{R}^{3,1}}\nu, \nu\rangle = 0$ at p, thus $D_\eta^{\mathbb{R}^{3,1}}\nu$ is proportional to ν if and only if $D_\eta^{\mathbb{R}^{3,1}}\nu$ is a null vector, which is equivalent to
$$0 = \det(D_\eta^{\mathbb{R}^{3,1}}\nu) = 4\left(\mathrm{Re}\left(\frac{h_w}{h^2\omega}\right)\right)^2. \tag{2.1.7}$$

By the identification between $\mathbb{R}^{3,1}$ and the Hermitian symmetric group (1.2.5), we have
$$G^{-1}\hat{f}(\bar{G}^t)^{-1} = (0,0,1,0), \quad G^{-1}\hat{f}_w(\bar{G}^t)^{-1} = -\hat{k}\omega(h^2+1, -i(h^2-1), 0, 2h),$$
$$G^{-1}\hat{f}_{\bar{w}}(\bar{G}^t)^{-1} = -\hat{k}\bar{\omega}(\bar{h}^2+1, -i(1-\bar{h}^2), 0, 2\bar{h}),$$

and we define $\mathbf{n} := GI\bar{G}^t$. This implies that $G^{-1}\mathbf{n}(\bar{G}^t)^{-1} = (0,0,0,1)$. By Proposition 1.2 of [32], we can define

$$\boldsymbol{\lambda} := \det(\hat{f}, \hat{f}_x, \hat{f}_y, \mathbf{n}) = 4\hat{k}^2|\omega|^2(1+|h|^2)(1-|h|^2).$$

Then, at a singular point p, $d\boldsymbol{\lambda} = -8\hat{k}^2|\omega|^2(dh\cdot\bar{h} + h\cdot d\bar{h})$, and thus p is a non-degenerate point if and only if $h_w \neq 0$.
(3)&(4) These cases are proven analogously to the proofs of (3)&(4) in Theorem 2.1.1.
(5) We define the limiting tangent bundle, as in [32], as $\left\{X \in T\mathbb{S}^{2,1}|_{\hat{f}(\Sigma)}; \langle X, \nu\rangle = 0\right\}$. By direct computation, we notice any section X of the limiting tangent bundle is parametrized by

$$X = G\begin{pmatrix} \bar{\zeta}\bar{h} + \zeta h & \zeta(|h|^2+1) \\ \bar{\zeta}(|h|^2+1) & \bar{\zeta}\bar{h} + \zeta h \end{pmatrix}\bar{G}^t$$

for some $\zeta : \Sigma \longrightarrow \mathbb{C}$, and $X \not\parallel \nu$ exactly when $\mathrm{Im}\,(\zeta h)\,|_p \neq 0$ at a singular point p. For such X such that $\mathrm{Im}\,(\zeta h)\,|_p \neq 0$, we define $\boldsymbol{\psi} := \langle D^{\mathbb{R}^{3,1}}_\eta X, \nu\rangle = -4i\mathrm{Re}\left(\frac{h_w}{h^2\omega}\right)\mathrm{Im}\,(\zeta h)$. We can apply Theorem 1.4 of [32], and the conditions to have a cuspidal cross cap are

$$\begin{aligned} &\det(\gamma_t, \eta) \neq 0,\ \boldsymbol{\psi} = 0 \text{ and } \partial_t\boldsymbol{\psi} \neq 0 \\ \iff\ &\mathrm{Im}\left(\frac{h_w}{h^2\omega}\right) \neq 0,\ \mathrm{Re}\left(\frac{h_w}{h^2\omega}\right) = 0 \text{ and } \mathrm{Im}\left\{\overline{\left(\frac{h_w}{h}\right)}\left(\frac{h_w}{h^2\omega}\right)_w\right\} \neq \mathrm{Im}\left\{\left(\frac{h_w}{h}\right)\left(\frac{h_w}{h^2\omega}\right)_{\bar{w}}\right\}. \end{aligned}$$

□

Criteria for singularities of spacelike CMC surfaces in $\mathbb{H}^{2,1}$. Here we introduce criteria for cuspidal edges, swallowtails and cuspidal cross caps of CMC surfaces in $\mathbb{H}^{2,1}$. In this case as well, like in the previous case of $\mathbb{S}^{2,1}$, we will shift from one surface f to another surface $\hat{f}$, again without causing any restriction on the class of surfaces involved. However, in the case of $\mathbb{H}^{2,1}$, the reason for doing this is different, because now we will indeed have the frames $\mathfrak{F}_1$, $\mathfrak{F}_2 \in \mathrm{SU}_{1,1}$. But the fact that two frames $\mathfrak{F}_1$, $\mathfrak{F}_2$ are now involved again causes us to switch from f to $\hat{f}$.

Let $f : \Sigma \longrightarrow \mathbb{H}^{2,1}$ be a spacelike CMC surface for a simply-connected domain $\Sigma \subset \mathbb{C}$, with the metric $ds^2 = 4g^2 dw d\bar{w} = 4e^{2u}dwd\bar{w}$ and unit normal vector N. First, we consider frames $\mathfrak{F}_1, \mathfrak{F}_2 \in \mathrm{SU}_{1,1}$ such that

$$f = \mathfrak{F}_1\overline{\mathfrak{F}_2}^t,\ \frac{f_x}{2e^u} = \mathfrak{F}_1\sigma_1\overline{\mathfrak{F}_2}^t,\ \frac{f_y}{2e^u} = \mathfrak{F}_1\sigma_2\overline{\mathfrak{F}_2}^t,\ N = \mathfrak{F}_1 i\sigma_3\overline{\mathfrak{F}_2}^t.$$

Then, we have

$$\Omega_1 := (\mathfrak{F}_1)^{-1}d\mathfrak{F}_1 = \frac{1}{2}\begin{pmatrix} -u_w & -ie^{-u}\mathcal{A} \\ 2e^u(1+iH) & u_w \end{pmatrix}dw + \frac{1}{2}\begin{pmatrix} u_{\bar{w}} & 2e^u(1-iH) \\ ie^{-u}\bar{\mathcal{A}} & -u_{\bar{w}} \end{pmatrix}d\bar{w} =: \boldsymbol{A}_1 dw + \boldsymbol{B}_1 d\bar{w}, \quad (2.1.8)$$

$$\Omega_2 := (\mathfrak{F}_2)^{-1}d\mathfrak{F}_2 = \frac{1}{2}\begin{pmatrix} -u_w & ie^{-u}\mathcal{A} \\ 2e^u(1-iH) & u_w \end{pmatrix}dw + \frac{1}{2}\begin{pmatrix} u_{\bar{w}} & 2e^u(1+iH) \\ -ie^{-u}\bar{\mathcal{A}} & -u_{\bar{w}} \end{pmatrix}d\bar{w} =: \boldsymbol{A}_2 dw + \boldsymbol{B}_2 d\bar{w}, \quad (2.1.9)$$

For these $\mathfrak{F}_1$ and $\mathfrak{F}_2$, the compatibility condition implies the Gauss and Codazzi equations (1.2.3). We define *s-spectral deformations* as follows, now using $s \in \mathbb{S}^1$ and the terms $1 \pm iH$ (in Definition 4.1.1 we used $s > 0$ and the terms $1 \pm H$):

Definition 2.1.6. The *s-spectral deformation* of the CMC surface f in $\mathbb{H}^{2,1}$ is the deformation defined by $(1+iH) \longrightarrow s(1+iH)$ and $(1-iH) \longrightarrow \bar{s}(1-iH)$ in the equations (2.1.8) and (2.1.9) for the complex parameter $s \in \mathbb{S}^1$.

The s-spectral deformation maps CMC surfaces to other CMC ones conformally, as in the following Theorem 2.1.6. The proof is analogous to the one of Theorem 4.1.1, but s^{-1} becomes $\bar{s}$.

Theorem 2.1.6. *For all $s \in \mathbb{S}^1$, the s-spectral deformation deforms a surface f in $\mathbb{H}^{2,1}$, with mean curvature H, metric $ds^2 = 4e^{2u}dwd\bar{w}$ and Hopf differential $\mathcal{A}$, into a surface f^s with mean curvature $H^s = \frac{s(1+iH)-\bar{s}(1-iH)}{i\{s(1+iH)+\bar{s}(1-iH)\}}$, metric $4e^{2u^s}dwd\bar{w} = 4k^2e^{2u}dwd\bar{w}$ and Hopf differential $\mathcal{A}^s = k\mathcal{A}$ for $k = \frac{s(1+iH)+\bar{s}(1-iH)}{2}$.*

The next lemma is proven like for Lemma 4.1.1:

Lemma 2.1.2. $(f^s)^{\bar{s}} = f$.

We define the (twisted) s-spectral Lax pair.

Definition 2.1.7 (s-spectral Lax pair)**.** We define $\mathfrak{F}_1^s$ and $\mathfrak{F}_2^s$ as solutions of the following systems:

$$(\mathfrak{F}_j^s)_w = \mathfrak{F}_j^s \boldsymbol{A}_j^s, \quad (\mathfrak{F}_j^s)_{\bar{w}} = \mathfrak{F}_j^s \boldsymbol{B}_j^s \quad (j = 1, 2),$$

where

$$\begin{aligned}
\boldsymbol{A}_1^s &:= \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \frac{1}{2} \begin{pmatrix} -u_w & -ie^{-u}\mathcal{A} \\ 2e^u s(1+iH) & u_w \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \frac{1}{2} \begin{pmatrix} u_w & 2e^u s(1+iH) \\ -ie^{-u}\mathcal{A} & -u_w \end{pmatrix}, \\
\boldsymbol{B}_1^s &:= \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \frac{1}{2} \begin{pmatrix} u_{\bar{w}} & 2e^u \bar{s}(1-iH) \\ ie^{-u}\bar{\mathcal{A}} & -u_{\bar{w}} \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \frac{1}{2} \begin{pmatrix} -u_{\bar{w}} & ie^{-u}\bar{\mathcal{A}} \\ 2e^u \bar{s}(1-iH) & u_{\bar{w}} \end{pmatrix}, \\
\boldsymbol{A}_2^s &:= \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \frac{1}{2} \begin{pmatrix} -u_w & ie^{-u}\mathcal{A} \\ 2e^u \bar{s}(1-iH) & u_w \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \frac{1}{2} \begin{pmatrix} u_w & 2e^u \bar{s}(1-iH) \\ ie^{-u}\mathcal{A} & -u_w \end{pmatrix}, \\
\boldsymbol{B}_2^s &:= \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \frac{1}{2} \begin{pmatrix} u_{\bar{w}} & 2e^u s(1+iH) \\ -ie^{-u}\bar{\mathcal{A}} & -u_{\bar{w}} \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \frac{1}{2} \begin{pmatrix} -u_{\bar{w}} & -ie^{-u}\bar{\mathcal{A}} \\ 2e^u s(1+iH) & u_{\bar{w}} \end{pmatrix}.
\end{aligned}$$

Further, we define the forms $\Omega_1^s := (\mathfrak{F}_1^s)^{-1}d\mathfrak{F}_1^s$ and $\Omega_2^s := (\mathfrak{F}_2^s)^{-1}d\mathfrak{F}_2^s$.

Definition 2.1.8. For all f given by $\mathfrak{F}_1$ and $\mathfrak{F}_2$, and mean curvature H, there exists a member of the s-spectral deformation, for the special value $s = s_0 := i\sqrt{\frac{1-iH}{1+iH}}$, that generates frames $\tilde{\mathfrak{F}}_1 = \mathfrak{F}_1^{s_0}$, $\tilde{\mathfrak{F}}_2 = \mathfrak{F}_2^{s_0} \in \mathrm{SU}_{1,1}$. We call $\tilde{\mathfrak{F}}_1$ and $\tilde{\mathfrak{F}}_2$ the *adjusted frames* of $\mathfrak{F}_1$ and $\mathfrak{F}_2$, and the forms $\tilde{\Omega}_1 = \tilde{\mathfrak{F}}_1^{-1}d\tilde{\mathfrak{F}}_1$ and $\tilde{\Omega}_2 = \tilde{\mathfrak{F}}_2^{-1}d\tilde{\mathfrak{F}}_2$ the *adjusted Maurer-Cartan forms*, where

$$\tilde{\Omega}_1 = \frac{1}{2} \begin{pmatrix} u_w & 2e^u i\sqrt{H^2+1} \\ -ie^{-u}\mathcal{A} & -u_w \end{pmatrix} dw + \frac{1}{2} \begin{pmatrix} -u_{\bar{w}} & ie^{-u}\bar{\mathcal{A}} \\ -2e^u i\sqrt{H^2+1} & u_{\bar{w}} \end{pmatrix} d\bar{w} =: \tilde{\boldsymbol{A}}_1 dw + \tilde{\boldsymbol{B}}_1 d\bar{w}, \quad (2.1.10)$$

$$\tilde{\Omega}_2 = \frac{1}{2} \begin{pmatrix} u_w & -2e^u i\sqrt{H^2+1} \\ ie^{-u}\mathcal{A} & -u_w \end{pmatrix} dw + \frac{1}{2} \begin{pmatrix} -u_{\bar{w}} & -ie^{-u}\bar{\mathcal{A}} \\ 2e^u i\sqrt{H^2+1} & u_{\bar{w}} \end{pmatrix} d\bar{w} =: \tilde{\boldsymbol{A}}_2 dw + \tilde{\boldsymbol{B}}_2 d\bar{w}. \quad (2.1.11)$$

These forms satisfy $\tilde{\Omega}_1 = \sigma_3 \tilde{\Omega}_2 \sigma_3$.

Theorem 2.1.7. *Let $\tilde{\mathfrak{F}}_1, \tilde{\mathfrak{F}}_2 : \Sigma \longrightarrow \mathrm{SU}_{1,1}$, where Σ is a simply-connected domain. Let $a \in \mathbb{S}^1$ ($a \neq 1$) be an arbitrary constant, and let*

$$\beta_1 := \begin{pmatrix} 0 & 2e^u i\sqrt{H^2+1} \\ 0 & 0 \end{pmatrix} dw, \quad \beta_2 := \begin{pmatrix} 0 & 0 \\ -2e^u i\sqrt{H^2+1} & 0 \end{pmatrix} d\bar{w}.$$

Define $\hat{\Omega}_1 := \tilde{\Omega}_1 + \frac{a-1}{2}\beta_1 + \frac{\bar{a}-1}{2}\beta_2$ and $\hat{\Omega}_2 := \tilde{\Omega}_2 + \frac{-\bar{a}+1}{2}\beta_1 + \frac{-a+1}{2}\beta_2$. Then we have the following:

1. *$d\hat{\Omega}_1 + \frac{1}{2}[\hat{\Omega}_1 \wedge \hat{\Omega}_1] = 0$, $d\hat{\Omega}_2 + \frac{1}{2}[\hat{\Omega}_2 \wedge \hat{\Omega}_2] = 0$.*
2. *If $\hat{\mathfrak{F}}_1$ and $\hat{\mathfrak{F}}_2$ are $\mathrm{SU}_{1,1}$-valued solutions of $\hat{\Omega}_1 = \hat{\mathfrak{F}}_1^{-1} d\hat{\mathfrak{F}}_1$ and $\hat{\Omega}_2 = \hat{\mathfrak{F}}_2^{-1} d\hat{\mathfrak{F}}_2$, then $\hat{f} = \hat{\mathfrak{F}}_1 \overline{\hat{\mathfrak{F}}_2}^t$ is a conformal spacelike CMC surface with $\hat{H} = \frac{a+\bar{a}}{i(a-\bar{a})}$.*

Proof. For $s = ia\sqrt{\frac{1-iH}{1+iH}} \in \mathbb{S}^1$, we have $\Omega_1^s = \hat{\Omega}_1$ and $\Omega_2^s = \hat{\Omega}_2$, by direct computations. Thus we have existence of $\hat{f}$, and $\hat{H} = H^s = \frac{a+\bar{a}}{i(a-\bar{a})}$. □

Remark 2.1.4. Defining $G_1 := \hat{\mathfrak{F}}_1 \cdot \tilde{\mathfrak{F}}_1^{-1}$ and $G_2 := \hat{\mathfrak{F}}_2 \cdot \tilde{\mathfrak{F}}_2^{-1}$, then $G_1 \overline{G_2}^t = \hat{\mathfrak{F}}_1 \overline{\hat{\mathfrak{F}}_2}^t = \hat{f}$.

As noted previously, again we will consider the criteria for singularities of $\hat{f}$ instead of f.

We denote $\tilde{\mathfrak{F}}_1 = \tilde{\mathfrak{F}}_1(w, \bar{w}) = e^{-\frac{u}{2}} \begin{pmatrix} u_1 & u_2 \\ \overline{u_2} & \overline{u_1} \end{pmatrix} \in \mathrm{SU}_{1,1}$, where $g = e^u$ is the metric function of f, and $|u_1|^2 - |u_2|^2 = g$. So we define $h := \frac{\overline{u_2}}{u_1}$ and $\omega := u_1^2$. By Remark 2.1.2, we have $\hat{f} = \hat{\mathfrak{F}}_1 \overline{\hat{\mathfrak{F}}_2}^t = G_1 \overline{G_2}^t$. By the definition of G_1 and G_2, we have

$$G_1^{-1} dG_1 = (a-1)i\sqrt{H^2+1} \begin{pmatrix} -h & 1 \\ -h^2 & h \end{pmatrix} \omega dw - (\bar{a}-1)i\sqrt{H^2+1} \begin{pmatrix} \bar{h} & -\bar{h}^2 \\ 1 & -\bar{h} \end{pmatrix} \bar{\omega} d\bar{w},$$

$$G_2^{-1} dG_2 = (-\bar{a}+1)i\sqrt{H^2+1} \begin{pmatrix} h & 1 \\ -h^2 & -h \end{pmatrix} \omega dw - (-a+1)i\sqrt{H^2+1} \begin{pmatrix} -\bar{h} & -\bar{h}^2 \\ 1 & \bar{h} \end{pmatrix} \bar{\omega} d\bar{w}.$$

Setting $\hat{k} = \frac{(a-\bar{a})i\sqrt{H^2+1}}{2}$, $\hat{f}$ has metric $ds^2 := 4\hat{g}^2 dw d\bar{w} = 4\hat{k}^2(1-|h|^2)^2|\omega|^2 dw d\bar{w}$. Thus this implies that, wherever ds^2 is finite, $\hat{f}$ has a singularity if and only if $h \in \mathbb{S}^1$ or $\omega = 0$. However we consider only extended CMC surfaces $\hat{f}$ defined in the same way as Definition 2.1.2.

Now we have the following criteria for singularities of spacelike extended CMC surfaces in $\mathbb{H}^{2,1}$. The proof of Theorem 2.1.8 is parallel to the proof of Theorem 4.1.3.

Theorem 2.1.8. *Let Σ be a simply connected domain, and let $\hat{f} : \Sigma \longrightarrow \mathbb{H}^{2,1}$ be a spacelike extended CMC surface, given by Theorem 2.1.7. Then:*

1. *A point $p \in \Sigma$ is a singular point if and only if $h \in \mathbb{S}^1$.*
2. *$\hat{f}$ is a front at a singular point $p \in \Sigma$ if and only if $\mathrm{Im}\left(\frac{h_w}{h^2\omega}\right)\Big|_p \neq 0$. If this is the case, p is non-degenerate singular point.*
3. *$\hat{f}$ is $\mathcal{A}$-equivalent to a cuspidal edge at a singular point $p \in \Sigma$ if and only if*

$$\mathrm{Im}\left(\frac{h_w}{h^2\omega}\right)\Big|_p \neq 0 \quad \text{and} \quad \mathrm{Re}\left(\frac{h_w}{h^2\omega}\right)\Big|_p \neq 0.$$

4. $\hat{f}$ is $\mathcal{A}$-equivalent to a swallowtail at a singular point $p \in \Sigma$ if and only if

$$\mathrm{Im}\left(\frac{h_w}{h^2\omega}\right)\bigg|_p \neq 0,\ \mathrm{Re}\left(\frac{h_w}{h^2\omega}\right)\bigg|_p = 0 \text{ and } \mathrm{Im}\left\{\overline{\left(\frac{h_w}{h}\right)}\left(\frac{h_w}{h^2\omega}\right)_w\right\}\bigg|_p \neq \mathrm{Im}\left\{\left(\frac{h_w}{h}\right)\left(\frac{h_w}{h^2\omega}\right)_{\bar{w}}\right\}\bigg|_p.$$

5. $\hat{f}$ is $\mathcal{A}$-equivalent to a cuspidal cross cap at a singular point $p \in \Sigma$ if and only if

$$\mathrm{Im}\left(\frac{h_w}{h^2\omega}\right)\bigg|_p = 0,\ \mathrm{Re}\left(\frac{h_w}{h^2\omega}\right)\bigg|_p \neq 0 \text{ and } \mathrm{Re}\left\{\overline{\left(\frac{h_w}{h}\right)}\left(\frac{h_w}{h^2\omega}\right)_w\right\}\bigg|_p \neq \mathrm{Re}\left\{\left(\frac{h_w}{h}\right)\left(\frac{h_w}{h^2\omega}\right)_{\bar{w}}\right\}\bigg|_p.$$

Proof. We can prove this theorem by computing the same way as in the proof of Theorem 4.1.3. □

2.2 Analogues of Smyth surfaces in Lorentzian spaceforms and their singularities

B. Smyth studied a generalization of Delaunay surfaces in $\mathbb{R}^3$, which are CMC surfaces with rotationally invariant metrics, in [87]. These surfaces are called *Smyth surfaces*, and there are numerous studies about them. For example, in [89], Timmreck et al. showed properness of Smyth surfaces in $\mathbb{R}^3$, and A. I. Bobenko and A. Its studied relationships between Smyth surfaces and Painleve III equations in [8]. The DPW method was applied to Smyth surfaces in Riemannian spaceforms, in [8], [25], [30] for example. (See Figure 2.1.) Recently, in [17], D. Brander et al. constructed the analogue of Smyth surfaces in $\mathbb{R}^{2,1}$.

Here we will construct the analogues of Smyth surfaces in $\mathbb{S}^{2,1}$ and $\mathbb{H}^{2,1}$, in addition to $\mathbb{R}^{2,1}$, and show that there are different kinds of Smyth surfaces in semi-Riemannian spaceforms, some which have singularities before reaching an end, and some which do not. We also identify the types of singularities on Smyth surfaces, using the criteria in Section 2.1.

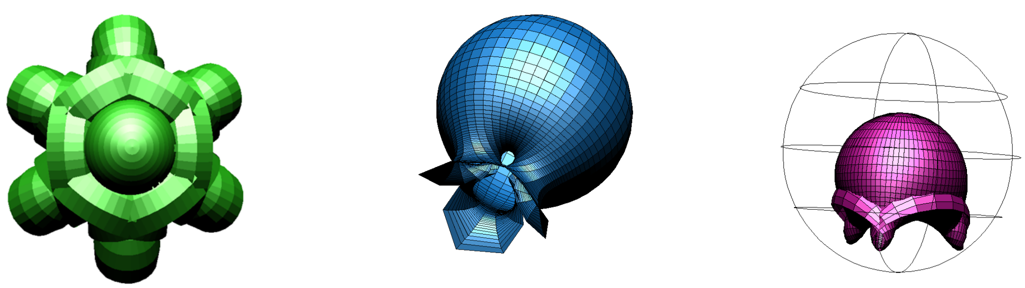


Fig. 2.1: The left image is a 6-legged Smyth surface in $\mathbb{R}^3$, the middle is a 3-legged Smyth surface in $\mathbb{S}^3$, and the right is a 3-legged Smyth surface in $\mathbb{H}^3$.

Reflective symmetry of Smyth surfaces in $\mathbb{R}^{2,1}$, $\mathbb{S}^{2,1}$ and $\mathbb{H}^{2,1}$. Define

$$\xi = \lambda^{-1}\begin{pmatrix} 0 & 1 \\ cz^k & 0 \end{pmatrix} dz, \quad c \in \mathbb{C}, \quad z \in \Sigma = \mathbb{C}, \tag{2.2.1}$$

and take a solution φ such that $d\varphi = \varphi\xi$ and $\varphi_{z=0} = I$. If $k = 0$ and $c \in \mathbb{S}^1$, then we have a round cylinder, as in Section 1.5. However, when $k \neq 0$ or $c \notin \mathbb{S}^1 \cup \{0\}$, Iwasawa splitting of φ is not so simple, and the surface f has singularities in some cases where the $\Lambda\mathrm{SU}_{1,1}$-Iwasawa splitting of φ approaches small cells.

Now we can assume $c \in \mathbb{R}_{>0}$ using a reparametrization of z and a rigid motion of f, as in [17].

Theorem 2.2.1 ([17])**.** *The surface $f : \Sigma^0 = \varphi^{-1}(\mathcal{B}_{1,1}) \longrightarrow \mathbb{R}^{2,1}$, produced via the DPW method, from ξ in* (4.2.1)*, with $\varphi|_{z=0} = I$ and $\lambda = 1$, has reflective symmetry with respect to $k+2$ geodesic planes that meet equiangularly along a geodesic line.*

Proof. Consider the reflections $R_l(z) = e^{\frac{2\pi i l}{k+2}}\bar{z}$ of the domain $\Sigma = \mathbb{C}$, for $l \in \{0, 1, \cdots, k+1\}$. Note that $\xi(R_l(z), \lambda) = A_l \cdot \xi(\bar{z}, \lambda) \cdot A_l^{-1}$, where

$$A_l := \begin{pmatrix} e^{\frac{\pi i l}{k+2}} & 0 \\ 0 & e^{-\frac{\pi i l}{k+2}} \end{pmatrix} : \text{ constant in } z, \lambda.$$

Since $d\left(\varphi(R_l(z), \lambda) \cdot A_l\right) = d\varphi(R_l(z), \lambda) \cdot A_l \cdot \xi(\bar{z}, \lambda)$, and since any solutions of this equation differ by a factor that is constant in z, we have $\varphi(R_l(z), \lambda) \cdot A_l = \mathfrak{A} \cdot \varphi(\bar{z}, \lambda)$. However the initial condition $\varphi(z, \lambda)|_{z=0} = I$ implies that $\mathfrak{A} = A_l$, so $\varphi(R_l(z), \lambda) = A_l \cdot \varphi(\bar{z}, \lambda) \cdot A_l^{-1}$. It is easy to see that this relation extends to the factors F and B in the Iwasawa splitting $\varphi = F \cdot B$, and so we have a frame F which satisfies $F(R_l(z), \overline{R_l(z)}, \lambda) = A_l \cdot F(\bar{z}, z, \lambda) \cdot A_l^{-1}$. Note that $c \in \mathbb{R}_{>0}$ implies $\xi(\bar{z}, \lambda) = \overline{\xi(z, \bar{\lambda})}$, $\varphi(\bar{z}, \lambda) = \overline{\varphi(z, \bar{\lambda})}$ and $F(\bar{z}, \lambda) = \overline{F(z, \bar{\lambda})}$, thus we have $F(R_l(z), \overline{R_l(z)}, \lambda) = A_l \cdot \overline{F(z, \bar{z}, \bar{\lambda})} \cdot A_l^{-1}$. Inserting this into the Sym-Bobenko type formula, we have

$$f(R_l(z), \overline{R_l(z)}) = -A_l \cdot \overline{f(z, \bar{z})} \cdot A_l^{-1}. \tag{2.2.2}$$

The transformation $f(z, \bar{z}) \longrightarrow -\overline{f(z, \bar{z})}$ represents a reflection across the plane $\{x_1 = 0\}$ of $\mathbb{R}^{2,1}$, and conjugation by A_l represents a rotation by angle $\frac{2\pi l}{k+2}$ about the x_0-axis, proving the result. □

Theorem 2.2.2. *The surfaces $f : \varphi^{-1}(\mathcal{B}_{1,1}) \longrightarrow \mathbb{S}^{2,1}$, produced via the DPW method, from ξ in* (4.2.1)*, with $\varphi|_{z=0} = I$ and $F_0 = F|_{\lambda = e^{\frac{q}{2}}}$ for $q \in \mathbb{R}$, $q \neq 0$, has reflective symmetry with respect to $k+2$ geodesic planes that meet equiangularly along a geodesic line.*

Proof. As the proof of Theorem 2.2.1, we have $F(R_l(z), \overline{R_l(z)}, \lambda) = A_l \cdot \overline{F(z, \bar{z}, \bar{\lambda})} \cdot A_l^{-1}$. Inserting this into the Sym-Bobenko type formula, we have $f(R_l(z), \overline{R_l(z)}) = A_l \cdot \overline{f(z, \bar{z})} \cdot \overline{A_l}^t$. The transformation $f(z, \bar{z}) \longrightarrow \overline{f(z, \bar{z})}$ represents a reflection across the plane $\{x_2 = 0\}$ of $\mathbb{R}^{3,1}$, and conjugation by A_l represents a rotation by angle $\frac{2\pi l}{k+2}$ about the x_4-axis. □

Similarly, we can prove:

Theorem 2.2.3. *The surfaces $f : \varphi^{-1}(\mathcal{B}_{1,1}) \to \mathbb{H}^{2,1}$, produced via the DPW method, from ξ in* (4.2.1)*, with $\varphi|_{z=0} = I$, $F_1 = F|_{\lambda = e^{i\gamma_1}}$ and $F_2 = F|_{\lambda = e^{i\gamma_2}}$ for $\gamma_1, \gamma_2 \in \mathbb{R}\backslash\{0\}$ and $\gamma_1 = -\gamma_2$, has reflective symmetry with respect to $k+2$ geodesic planes that meet equiangularly along a geodesic line.*

Remark 2.2.1. The surfaces f in Theorems 2.2.1, 2.2.2 and 2.2.3 extend to $\varphi^{-1}(C_1)$ at singularities (see Proposition 2.1.1, (3)).

The Gauss equation of Smyth surfaces. Here we assume the mean curvature is $H = \frac{1}{2}$, as in [8], [17], [89], and then we show that the metric of Smyth surfaces is rotational invariant. Before giving the theorems, we create the notation $g := e^u$ for the metric function, and we have the following Lax pair

$$F_z = FU, \quad F_{\bar{z}} = FV, \text{ where } U = \frac{1}{2}\begin{pmatrix} \frac{g_z}{g} & i\lambda^{-1}g \\ -i\lambda^{-1}Qg^{-1} & -\frac{g_z}{g} \end{pmatrix}, \; V = \frac{1}{2}\begin{pmatrix} -\frac{g_{\bar{z}}}{g} & i\lambda\bar{Q}g^{-1} \\ -i\lambda g & \frac{g_{\bar{z}}}{g} \end{pmatrix}, \tag{2.2.3}$$

and Gauss equation

$$4(g_{z\bar{z}} \cdot g - g_z \cdot g_{\bar{z}}) + Q\bar{Q} - g^4 = 0. \tag{2.2.4}$$

Theorem 2.2.4 ([17]). *The Gauss equation* (2.2.4) *for a surface in $\mathbb{R}^{2,1}$, $\mathbb{S}^{2,1}$ or $\mathbb{H}^{2,1}$ generated by ξ in* (4.2.1)*, with $\varphi|_{z=0} = I$, is equivalent to a special case of the Painleve III equations, and the metric function g, which is a solution of* (2.2.4)*, is rotational invariant.*

Using polar coordinates $z = re^{i\theta}$, (2.2.4) and the suitable choice of the initial conditions are

$$g\left(g_{rr} + \frac{g_r}{r}\right) - (g_r)^2 + c^2 r^{2k} - g^4 = 0, \; g|_{r=0} = 1, \; g_r|_{r=0} = 0. \tag{2.2.5}$$

We can assume $c = 1$ in (2.2.5) by a change of coordinate:

Lemma 2.2.1. *After an appropriate change of coordinate z, the Gauss equation* (2.2.5) *for (k+2)-legged Smyth surfaces in $\mathbb{R}^{2,1}$, $\mathbb{S}^{2,1}$ or $\mathbb{H}^{2,1}$, becomes*

$$g\left(g_{rr} + \frac{g_r}{r}\right) - (g_r)^2 + r^{2k} - g^4 = 0, \quad g|_{r=0} = q^2, \quad g_r|_{r=0} = 0 \tag{2.2.6}$$

for a real constant $q > 0$.

Proof. Let φ be a solution of $d\varphi = \varphi\xi$ for $\xi = \begin{pmatrix} 0 & \lambda^{-1} \\ \lambda^{-1}cz^k & 0 \end{pmatrix} dz$ and $\varphi|_{z=0} = I$ with $\Lambda\mathrm{SU}_{1,1}$-Iwasawa splitting $\varphi = F \cdot B$. We define $\breve{\varphi} = \varphi \begin{pmatrix} q & 0 \\ 0 & q^{-1} \end{pmatrix}$. By the uniqueness of the Iwasawa splitting $\breve{\varphi} = \breve{F} \cdot \breve{B}$ of $\breve{\varphi}$, we have $\breve{F} = F$ and $\breve{B} = B \begin{pmatrix} q & 0 \\ 0 & q^{-1} \end{pmatrix}$. Furthermore, we notice that $\breve{B}|_{\lambda=0,z=0} = \begin{pmatrix} q & 0 \\ 0 & q^{-1} \end{pmatrix}$, since $\breve{F}|_{z=0} = I$, and this implies that $\breve{g}|_{r=0} = q^2$. On the other hand, we have

$$\breve{\xi} = \breve{\varphi}^{-1} d\breve{\varphi} = \begin{pmatrix} 0 & q^{-2}\lambda^{-1} \\ q^2\lambda^{-1}c^2 z^k & 0 \end{pmatrix} dz = \begin{pmatrix} 0 & \lambda^{-1} \\ q^{2k+4}\lambda^{-1}c^2 \breve{z}^k & 0 \end{pmatrix} d\breve{z}$$

for $\breve{z} := q^{-2}z$, and we can let $\breve{c} = q^{2k+4}c$. In this way, we can change c to 1. □

Some examples of the metric function g are seen in Figure 2.2. By the previous studies [39] and [48], there are at least three kinds of solutions g of this special case of Painleve III equations (2.2.6). This implies that (spacelike) Smyth surfaces in $\mathbb{R}^{2,1}$, $\mathbb{S}^{2,1}$ and $\mathbb{H}^{2,1}$, near the origin $\mathbf{0} \in \mathbf{\Sigma}$, are classified into the following three cases (See Figures 2.2, 2.3, 2.5 and 2.7.):

- The first kind does not have singularities and g diverges to ∞.

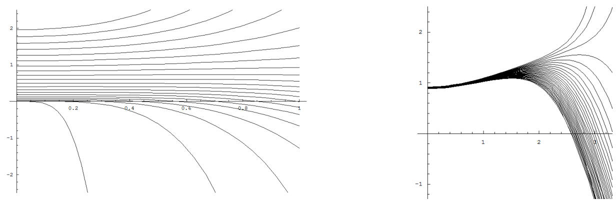


Fig. 2.2: Solutions of a special case of Painleve III (near the origin)

- The second kind does not have singularities and g does not diverge to $\pm\infty$. This case is unique, and given by g with the following initial condition:

$$g|_{r=0} = q_0^2, \quad g_r|_{r=0} = 0 \quad \text{for} \quad q_0 = \left(1 + \frac{k}{2}\right)^{\frac{k}{4+2k}} 2^{\frac{k}{2+k}} \sqrt{\frac{\Gamma\left(\frac{1}{2} + \frac{k}{4+2k}\right)}{\Gamma\left(\frac{1}{2} - \frac{k}{4+2k}\right)}}.$$

- The third kind has singularities before g diverges to $-\infty$.

Smyth surfaces with singularities. Here we only consider Smyth surfaces that have singularities before g diverges to $-\infty$. By numerical calculation, we know that these Smyth surfaces have cuspidal edges, swallowtails and cuspidal cross caps, using criteria as in Section 2.1, see Figures 2.3∼2.8.

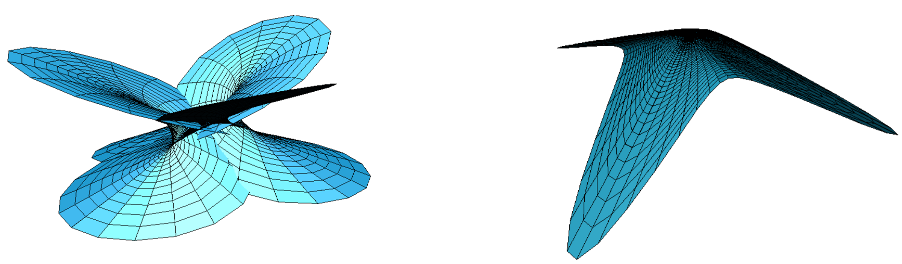


Fig. 2.3: The left image is a 3-legged Smyth surface with singularities in $\mathbb{R}^{2,1}$, and the right image is one with no singularities in $\mathbb{R}^{2,1}$.

Fact 2.2.1. There exist Smyth surfaces in $\mathbb{R}^{2,1}$, $\mathbb{S}^{2,1}$ and $\mathbb{H}^{2,1}$ which have singularities before g diverges to $-\infty$, and which have cuspidal edges, swallowtails and cuspidal cross caps. (See Figures 2.4, 2.6 and 2.8.)

Here we show, for the surfaces in Fact 2.2.1, that there are at least $2(k+2)$-swallowtails for the case of $\mathbb{R}^{2,1}$, without relying on numerical calculation, and using only geometric properties. Before doing that, we have some lemmas.

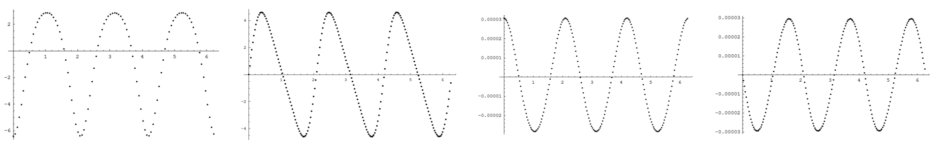


Fig. 2.4: The values of The values of $\mathrm{Re}\left(\frac{h_z}{h^2\omega}\right)$, $\mathrm{Im}\left(\frac{h_z}{h^2\omega}\right)$, $\mathrm{Re}\left\{\overline{\left(\frac{h_z}{h}\right)}\left(\frac{h_z}{h^2\omega}\right)_z\right\} - \mathrm{Re}\left\{\left(\frac{h_z}{h}\right)\left(\frac{h_z}{h^2\omega}\right)_{\bar{z}}\right\}$ and $\mathrm{Im}\left\{\overline{\left(\frac{h_z}{h}\right)}\left(\frac{h_z}{h^2\omega}\right)_z\right\} - \mathrm{Im}\left\{\left(\frac{h_z}{h}\right)\left(\frac{h_z}{h^2\omega}\right)_{\bar{z}}\right\}$ for a 3-legged Smyth surface in $\mathbb{R}^{2,1}$ at (r_0, θ) such that $g(r_0) = 0$ and $0 \leq \theta \leq 2\pi$. (left to right)

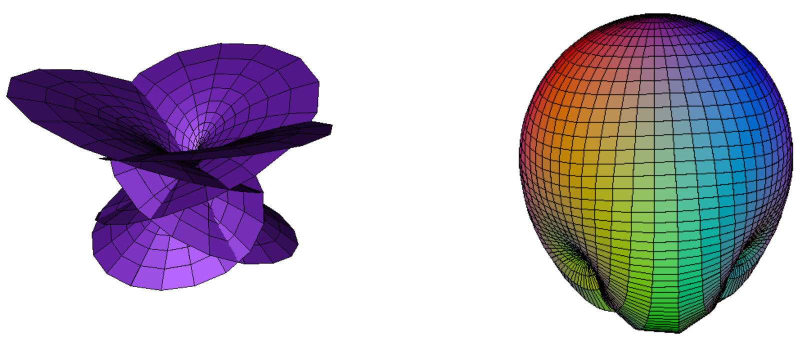


Fig. 2.5: The left image is a 3-legged Smyth surface with singularities in $\mathbb{S}^{2,1}$, and the right image is one with no singularities in $\mathbb{S}^{2,1}$.

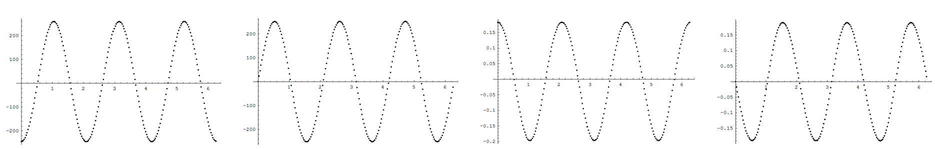


Fig. 2.6: The values of $\mathrm{Re}\left(\frac{h_w}{h^2\omega}\right)$, $\mathrm{Im}\left(\frac{h_w}{h^2\omega}\right)$, $\mathrm{Re}\left\{\overline{\left(\frac{h_w}{h}\right)}\left(\frac{h_w}{h^2\omega}\right)_w\right\} - \mathrm{Re}\left\{\left(\frac{h_w}{h}\right)\left(\frac{h_w}{h^2\omega}\right)_{\bar{w}}\right\}$ and $\mathrm{Im}\left\{\overline{\left(\frac{h_w}{h}\right)}\left(\frac{h_w}{h^2\omega}\right)_w\right\} - \mathrm{Im}\left\{\left(\frac{h_w}{h}\right)\left(\frac{h_w}{h^2\omega}\right)_{\bar{w}}\right\}$ for a 3-legged Smyth surface in $\mathbb{S}^{2,1}$ at (r_0, θ) such that $g(r_0) = 0$ and $0 \leq \theta \leq 2\pi$. (left to right)

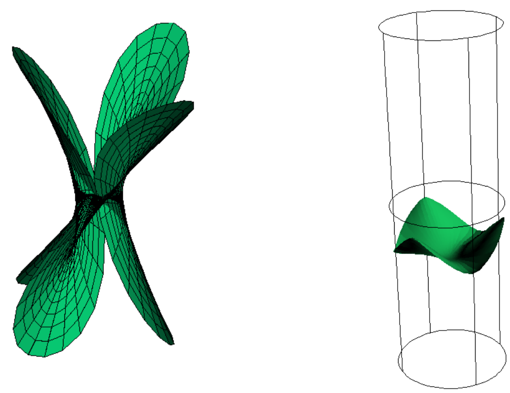


Fig. 2.7: The left image is a 3-legged Smyth surface with singularities in $\mathbb{H}^{2,1}$, and the right image is one with no singularities in $\mathbb{H}^{2,1}$.

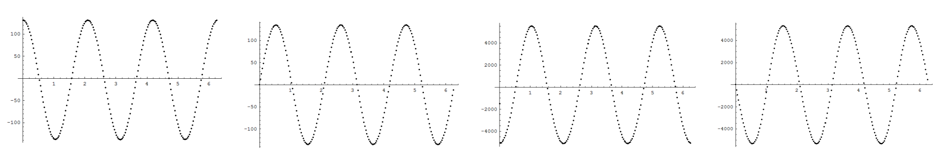


Fig. 2.8: The values of $\mathrm{Im}\left(\frac{h_w}{h^2\omega}\right)$, $\mathrm{Re}\left(\frac{h_w}{h^2\omega}\right)$, $\mathrm{Im}\left\{\overline{\left(\frac{h_w}{h}\right)}\left(\frac{h_w}{h^2\omega}\right)_w\right\} - \mathrm{Im}\left\{\left(\frac{h_w}{h}\right)\left(\frac{h_w}{h^2\omega}\right)_{\bar{w}}\right\}$ and $\mathrm{Re}\left\{\overline{\left(\frac{h_w}{h}\right)}\left(\frac{h_w}{h^2\omega}\right)_w\right\} - \mathrm{Re}\left\{\left(\frac{h_w}{h}\right)\left(\frac{h_w}{h^2\omega}\right)_{\bar{w}}\right\}$ for a 3-legged Smyth surface in $\mathbb{H}^{2,1}$ at (r_0, θ) such that $g(r_0) = 0$ and $0 \leq \theta \leq 2\pi$. (left to right)

Lemma 2.2.2. *Let $\hat{F} = \hat{F}(z, \bar{z}, \lambda)$ be the solution of the untwisted Lax pair (1.1.7) with $\hat{F}|_{z=0} = I$ for the case of a Smyth surface. Then $\hat{F}(z) = \sigma_3 \overline{\hat{F}(\bar{z})} \sigma_3$ for $\lambda = 1$.*

Proof. By direct computation, we have $\hat{U}(z) = -\hat{V}(\bar{z})^t$ and $\hat{V}(z) = -\hat{U}(\bar{z})^t$. By this equation and $\hat{F}(z)|_{z=0} = I$, we get the conclusion. □

Corollary 2.2.1.
(1) $h(z) = -\overline{h(\bar{z})}$ and $\omega(z) = \overline{\omega(\bar{z})}$.
(2) At $(r, \theta) = (r_0, 0)$ for r_0 such that $g(r_0) = 0$, we have $h(r_0, 0) = \pm i$ (i.e. $h(r_0, 0) \in i\mathbb{R} \cap \mathbb{S}^1$), $\omega(r_0, 0) \in \mathbb{R} \setminus \{0\}$ and $\omega_z(r_0, 0) = \omega_{\bar{z}}(r_0, 0)$.

Theorem 2.2.5. *Let $\hat{f}(z) = \hat{f}(r, \theta)$ be a $(k+2)$-legged Smyth surface in $\mathbb{R}^{2,1}$, and let r_0 satisfy $g(r_0) = 0$. Then $\hat{f}$ has a swallowtail at $(r_0, 0)$.*

Proof. We will use the criteria of Theorem 2.1.1. First we check that

$$\operatorname{Re}\left(\frac{h_z}{h^2\omega}\right) = \operatorname{Re}\left(\frac{Q}{h^2\omega^2}\right) \neq 0 \tag{2.2.7}$$

at $(r_0, 0)$. By using $Q(r, 0) = -r^k \in \mathbb{R}_{<0}$ and Corollary 2.2.1, we notice that (2.2.7) holds. Similarly, we also have

$$\operatorname{Im}\left(\frac{h_z}{h^2\omega}\right) = \operatorname{Im}\left(\frac{Q}{h^2\omega^2}\right) = 0 \tag{2.2.8}$$

at $(r_0, 0)$. Lastly, we check that

$$\operatorname{Re}\left\{\overline{\left(\frac{h_z}{h}\right)}\left(\frac{h_z}{h^2\omega}\right)_z\right\} \neq \operatorname{Re}\left\{\left(\frac{h_z}{h}\right)\left(\frac{h_z}{h^2\omega}\right)_{\bar{z}}\right\} \tag{2.2.9}$$

at $(r_0, 0)$. By direct computation, this is equivalent to

$$\operatorname{Re}\left\{\overline{\left(\frac{Q}{h\omega}\right)}\left(\frac{Q_z h^2\omega^2 - Q(2Qh\omega + 2h^2\omega\omega_z)}{h^4\omega^4}\right)\right\} \neq \operatorname{Re}\left\{\left(\frac{Q}{h\omega}\right)\left(\frac{-2Qh^2\omega\omega_{\bar{z}}}{h^4\omega^4}\right)\right\}. \tag{2.2.10}$$

Applying Corollary 2.2.1 to (2.2.10), we have $h^4\omega^4 \in \mathbb{R} \setminus \{0\}$ and $\frac{Q}{h\omega} = -\overline{\left(\frac{Q}{h\omega}\right)} \in i\mathbb{R} \setminus \{0\}$. Thus, (2.2.10) is equivalent to

$$-\operatorname{Im}\{Q_z h^2\omega - 2Q^2h - 2Qh^2\omega_z\} \neq \operatorname{Im}\left\{-2Qh^2\omega_{\bar{z}}\right\}. \tag{2.2.11}$$

Using $Q(r, 0) = -r^k \in \mathbb{R}_{<0}$, $Q_z(r, 0) = -kr^{k-1} \in \mathbb{R}_{<0}$ and Corollary 2.2.1, (2.2.11) becomes

$$\pm 2r_0^{2k} \neq 0. \tag{2.2.12}$$

As this is clear, we concluded that (2.2.9) holds. □

Similarly, we have the same conclusion when $\theta = \frac{\pi}{k+2}$.

Theorem 2.2.6. *Let $\hat{f}(z) = \hat{f}(r, \theta)$ be a $(k+2)$-legged Smyth surface in $\mathbb{R}^{2,1}$, and let r_0 satisfy $g(r_0) = 0$ and $g_r(r_0) = -r_0^k$. Then $\hat{f}$ has a swallowtail at $(r_0, \frac{\pi}{k+2})$.*

By above two theorems and reflective symmetry, we get the following main result:

Theorem 2.2.7. *If a (k+2)-legged Smyth surface in* $\mathbb{R}^{2,1}$ *has singularities before* g *diverges to* $-\infty$*, then it has at least* $2(k+2)$ *swallowtails.*

Remark 2.2.2. We have checked numerically that there are cuspidal cross caps along the cuspidal edges between each adjacent pair of swallowtails, using item (4) of Corollary 2.1.1, item (5) of Theorem 4.1.3 and item (5) of Theorem 2.1.8. Thus the surface as in Theorem 4.2.2 will also have at least $2(k+2)$ cuspidal cross caps.

Chapter 3

Spacelike CMC surfaces with $0 \leq H < 1$ in $\mathbb{S}^{2,1}$ via Iwasawa splitting

3.1 The Lax pair in $\mathbb{S}^{2,1}$

3.1.1 The 3-dimensional de Sitter space

Let $\mathbb{R}^{3,1}$ be the Cartesian 4-space with metric
$\langle (x_1, x_2, x_3, x_4), (y_1, y_2, y_3, y_4) \rangle := x_1y_1 + x_2y_2 + x_3y_3 - x_4y_4$. We define the 3-dimensional de Sitter space as the hyperquadric $\mathbb{S}^{2,1} := \{x | \langle x, x \rangle = 1\} \subset \mathbb{R}^{3,1}$.

Let Σ be a simply-connected domain in $\mathbb{C}$ with the usual complex coordinate $w = x + iy$. Let $f : \Sigma \longrightarrow \mathbb{S}^{2,1}$ be a conformally immersed spacelike surface. Since f is conformal,

$$\langle f_w, f_w \rangle = \langle f_{\bar{w}}, f_{\bar{w}} \rangle = 0, \quad \langle f_w, f_{\bar{w}} \rangle = 2e^{2u}$$

for some function $u : \Sigma \longrightarrow \mathbb{R}$. For the unit normal vector field N of f satisfying $\langle N, N \rangle = -1$, $\langle f_w, N \rangle = \langle f_{\bar{w}}, N \rangle = 0$, we define the mean curvature H and Hopf differential $\mathcal{A}dw^2$ as follows:

$$H := \frac{1}{2e^{2u}} \langle f_{w\bar{w}}, N \rangle, \quad \mathcal{A} := \langle f_{ww}, N \rangle.$$

The Gauss-Codazzi equations are of the following form in the CMC cases:

$$2u_{w\bar{w}} - 2e^{2u}(H^2 - 1) + \frac{1}{2}\mathcal{A}\bar{\mathcal{A}}e^{-2u} = 0, \quad \mathcal{A}_{\bar{w}} = 0. \tag{3.1.1}$$

The Codazzi equation in (3.1.1) is equivalent to the Hopf differential $\mathcal{A}$ being holomorphic, and (3.1.1) is invariant under the deformation $\mathcal{A} \mapsto \mu^{-2}\mathcal{A}$ for $\mu \in \mathbb{S}^1$. When $f(x, y)$ is a spacelike CMC in $\mathbb{S}^{2,1}$, the spectral parameter $\mu \in \mathbb{S}^1$ allows us to create a 1-parameter family of CMC surfaces $f^\mu = f(x, y, \mu)$ associated to $f(x, y)$.

3.1.2 The 2×2 matrix model of $\mathbb{S}^{2,1}$

We identify $\mathbb{R}^{3,1}$ with the space $\left\{X \in M_{2\times 2} \,\middle|\, X = \bar{X}^t\right\}$ of all 2 by 2 Hermitian matrices as follows:

$$\mathbb{R}^{3,1} \ni x = (x_1, x_2, x_3, x_4) \longmapsto \begin{pmatrix} x_4 + x_3 & x_1 - ix_2 \\ x_1 + ix_2 & x_4 - x_3 \end{pmatrix}. \tag{3.1.2}$$

The metric becomes, under this identification, $\langle X, Y\rangle = -\frac{1}{2}\mathrm{trace}(X\sigma_2 Y^t \sigma_2)$. In particular, $\langle X, X\rangle = -\det(X)$, and we can identify $\mathbb{S}^{2,1}$ with

$$\left\{X \in M_{2\times 2} \,\middle|\, X = \bar{X}^t, \det(X) = -1\right\} = \left\{F\sigma_3 \overline{F^t} \,|\, F \in \mathrm{SL}_2(\mathbb{C})\right\}.$$

Let f be a conformal spacelike CMC surface in $\mathbb{S}^{2,1}$ with associated family f^μ, and let the identity matrix and Pauli matrices be as follows:

$$I := \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \sigma_1 := \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 := \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_3 := \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

Then $\{I, \sigma_1, \sigma_2, \sigma_3\}$ is an orthogonal basis for $\mathbb{R}^{3,1}$. We can define

$$f^\mu := \hat{F}\sigma_3\overline{\hat{F}^t}, \quad e_1 := \frac{f^\mu_x}{|f^\mu_x|} = \frac{f^\mu_x}{2e^u} = \hat{F}\sigma_1\overline{\hat{F}^t}, \quad e_2 := \frac{f^\mu_y}{|f^\mu_y|} = \frac{f^\mu_y}{2e^u} = \hat{F}\sigma_2\overline{\hat{F}^t}, \quad N := \hat{F}\overline{\hat{F}^t}$$

for $\hat{F} = \hat{F}(w, \bar{w}, \mu) \in \mathrm{SL}_2(\mathbb{C})$. For this $\hat{F}$, we get the *untwisted* 2×2 *Lax pair* in $\mathbb{S}^{2,1}$ as follows:

$$\hat{F}_w = \hat{F}\hat{S},\ \hat{F}_{\bar{w}} = \hat{F}\hat{T},\ \text{where}\quad \hat{S} = \frac{1}{2}\begin{pmatrix} -u_w & -\mu^{-2}\mathcal{A}e^{-u} \\ 2(1-H)e^u & u_w \end{pmatrix},\ \hat{T} = \frac{1}{2}\begin{pmatrix} u_{\bar{w}} & -2(1+H)e^u \\ -\mu^2\bar{\mathcal{A}}e^{-u} & -u_{\bar{w}} \end{pmatrix}. \tag{3.1.3}$$

We change the "untwisted" setting to the "twisted" setting by the following transformation (3.1.4). Let $\tilde{F}$ be defined by

$$\hat{F} = -\sigma_3\left(\tilde{F}^{-1}\right)^t \begin{pmatrix} \sqrt{\mu} & 0 \\ 0 & \frac{1}{\sqrt{\mu}} \end{pmatrix}\sigma_3, \tag{3.1.4}$$

producing the *twisted* 2×2 *Lax pair* of f^μ in $\mathbb{S}^{2,1}$,

$$\tilde{F}_w = \tilde{F}\tilde{S},\ \tilde{F}_{\bar{w}} = \tilde{F}\tilde{T},\ \text{where}\quad \tilde{S} = \frac{1}{2}\begin{pmatrix} u_w & 2\mu^{-1}(1-H)e^u \\ -\mu^{-1}\mathcal{A}e^{-u} & -u_w \end{pmatrix},\ \tilde{T} = \frac{1}{2}\begin{pmatrix} -u_{\bar{w}} & -\mu\bar{\mathcal{A}}e^{-u} \\ -2\mu(1+H)e^u & u_{\bar{w}} \end{pmatrix}. \tag{3.1.5}$$

Now we consider $0 \le H < 1$ case, and we set $H := \tanh(-q)$ for $q \le 0$. We change $\tilde{F}$ to a new frame F, as follows:

$$F = \tilde{F}\begin{pmatrix} e^{\frac{q}{4}} & 0 \\ 0 & e^{-\frac{q}{4}} \end{pmatrix}.$$

We call this F the *extended frame* of spacelike CMC H surfaces with $0 \le H < 1$. Moreover we set

$$\mathcal{H} := -ie^{-q}(1-H) = -ie^q(1+H) \in i\mathbb{R},\ Q := -i\mathcal{A},\ \nu := e^{-\frac{q}{2}}\mu, \tag{3.1.6}$$

and we have the following:

$$F_w = FU,\ F_{\bar{w}} = FV,\ \text{where}\quad U = \frac{1}{2}\begin{pmatrix} u_w & 2i\nu^{-1}\mathcal{H}e^u \\ -i\nu^{-1}Qe^{-u} & -u_w \end{pmatrix},\ V = \frac{1}{2}\begin{pmatrix} -u_{\bar{w}} & i\nu\bar{Q}e^{-u} \\ -2i\nu\mathcal{H}e^u & u_{\bar{w}} \end{pmatrix}. \tag{3.1.7}$$

We call (3.1.7) the *extended Lax pair*, and $F = F(w, \bar{w}, \nu)$ is in the following loop group

$$\Lambda\mathrm{SL}_2(\mathbb{C})_\tau = \left\{ F(\lambda) \in M_{2\times 2} \middle| F : \mathbb{S}^1 \xrightarrow{C^\infty} \mathrm{SL}_2(\mathbb{C}), F(-\lambda) = \sigma_3 F(\lambda)\sigma_3, \tau(F(\lambda)) = F(\lambda) \right\}, \tag{3.1.8}$$

where $\tau(F(\lambda)) := \mathrm{Ad}\left(\begin{pmatrix} e^{\frac{\pi}{4}i} & 0 \\ 0 & e^{-\frac{\pi}{4}i} \end{pmatrix}\right) \cdot \left(\overline{F(i\bar{\lambda}^{-1})^t}\right)^{-1}$. For simplicity, we set $R = \begin{pmatrix} e^{-\frac{\pi}{4}i} & 0 \\ 0 & e^{\frac{\pi}{4}i} \end{pmatrix}$ as in [24] and the symbol $*$ is defined by $F^*(\lambda) := \left(\overline{F(i\bar{\lambda}^{-1})^t}\right)^{-1}$, and we can rewrite $\tau(F(\lambda)) = \mathrm{Ad}(R^{-1})F^*(\lambda)$.

The following Proposition 3.1.1 gives us a method for determining spacelike CMC H surfaces with $0 \leq H < 1$ in $\mathbb{S}^{2,1}$, from given data u and $\mathcal{A}$.

Proposition 3.1.1 (The immersion formula for spacelike CMC H surfaces with $0 \leq H < 1$ in $\mathbb{S}^{2,1}$)**.** *Let Σ be a simply-connected domain in $\mathbb{C}$. Let u and $\mathcal{A}$ solve* (3.1.1). *Set $Q = -i\mathcal{A}$ and $\mathcal{H} := -ie^{-q}(1 - \tanh(-q))$ for $q \leq 0$. Let $F = F(w, \bar{w}, \nu) \in \Lambda\mathrm{SL}_2(\mathbb{C})_\tau$ be a solution of the system* (3.1.7). *Set $F_0 = F|_{\nu = e^{-\frac{q}{2}}}$. We define the following immersion formulas*

$$f = F_0 \begin{pmatrix} e^{-\frac{1}{2}q} & 0 \\ 0 & -e^{\frac{1}{2}q} \end{pmatrix} \overline{F_0}^t, \quad N = F_0 \begin{pmatrix} -e^{-\frac{1}{2}q} & 0 \\ 0 & -e^{\frac{1}{2}q} \end{pmatrix} \overline{F_0}^t. \tag{3.1.9}$$

Then, f is a spacelike CMC $H = \tanh(-q)$ surface in $\mathbb{S}^{2,1}$ with unit normal N.

Proof. Applying a frame change in Theorem 8.5 in [33], we can prove this proposition. We can also prove it by the same argument as in the proof of Proposition 4.1 in [24]. So we omit the proof. □

By the above Proposition 3.1.1, we can construct all simply-connected spacelike CMC H surfaces with $0 \leq H < 1$. However, here we also have the following proposition in Appendix E of [24], and it says that spacelike CMC H surfaces with $0 \leq H < 1$ with no umbilics can be constructed as the normal vector of parallel transformation of CMC H surfaces with $0 \leq H < 1$ in the 3-dimensional hyperbolic space $\mathbb{H}^3$. In this proposition, we need the following extended Lax pair for CMC H surfaces with $0 \leq H < 1$ in $\mathbb{H}^3$:

$$(F_{\mathbb{H}^3})_w = F_{\mathbb{H}^3} U_{\mathbb{H}^3},\ (F_{\mathbb{H}^3})_{\bar{w}} = F_{\mathbb{H}^3} V_{\mathbb{H}^3}, \text{ where } U_{\mathbb{H}^3} = \frac{1}{2}\begin{pmatrix} (u_{\mathbb{H}^3})_w & -2\nu^{-1}\mathcal{H}_{\mathbb{H}^3} e^{u_{\mathbb{H}^3}} \\ \nu^{-1} Q_{\mathbb{H}^3} e^{-u_{\mathbb{H}^3}} & -(u_{\mathbb{H}^3})_w \end{pmatrix},\ V_{\mathbb{H}^3} = \frac{1}{2}\begin{pmatrix} -(u_{\mathbb{H}^3})_{\bar{w}} & -\nu \overline{Q_{\mathbb{H}^3}} e^{-u_{\mathbb{H}^3}} \\ 2\nu \mathcal{H}_{\mathbb{H}^3} e^{u_{\mathbb{H}^3}} & (u_{\mathbb{H}^3})_{\bar{w}} \end{pmatrix},$$

where $u_{\mathbb{H}^3}$ is the metric function for $ds^2 = 4e^{2u_{\mathbb{H}^3}} dw d\bar{w}$, and $\mathcal{H}_{\mathbb{H}^3} = ie^{-q}(1 - H_{\mathbb{H}^3})$ for mean curvature $H_{\mathbb{H}^3} = \tanh(-q)$, and $Q_{\mathbb{H}^3} = i\mathcal{A}_{\mathbb{H}^3}$ for Hopf differential $\mathcal{A}_{\mathbb{H}^3}$.

Proposition 3.1.2 ([24])**.** *Let Σ be a simply-connected domain in $\mathbb{C}$. Let $F_{\mathbb{H}^3}$ be the extended frame of some CMC $0 \leq H_{\mathbb{H}^3} = \tanh(-q) < 1$ surface $f_{\mathbb{H}^3}(w, \bar{w})$ in $\mathbb{H}^3$ with unit normal $N_{\mathbb{H}^3}$. Set $(F_{\mathbb{H}^3})_0 = (F_{\mathbb{H}^3})|_{\nu = e^{-\frac{q}{2}}}$ for $q < 0$. Then, by changing parameter such that $z = iw$,*

$$f(z, \bar{z}) = \sinh(-q) f_{\mathbb{H}^3} + \cosh(-q) N_{\mathbb{H}^3} = (F_{\mathbb{H}^3})_0 \begin{pmatrix} -e^{\frac{1}{2}q} & 0 \\ 0 & e^{-\frac{1}{2}q} \end{pmatrix} \overline{(F_{\mathbb{H}^3})_0}^t \tag{3.1.10}$$

is a spacelike CMC $H = \tanh(-q)$ surface in $\mathbb{S}^{2,1}$, and $ds^2 = e^{-2u_{\mathbb{H}^3}} \cosh^2(-q) |\mathcal{A}_{\mathbb{H}^3}|^2 dz d\bar{z}$, $\mathcal{A} = \mathcal{A}_{\mathbb{H}^3}$.

3.2 Iwasawa splitting for $\Lambda\mathrm{SL}_2(\mathbb{C})_\tau$

In this section, we introduce the *Birkhoff splitting* and prove the existence and uniqueness of the *(twisted)* $\Lambda\mathrm{SL}_2(\mathbb{C})_\tau$*-Iwasawa splitting*, by using the Birkhoff splitting. First we define the notations for loop groups and algebras.

Definition 3.2.1.

$$\Lambda\mathrm{SL}_2(\mathbb{C}) = \left\{\varphi : \mathbb{S}^1 \xrightarrow{C^\infty} SL_2(\mathbb{C}) \middle| \varphi(-\lambda) = \sigma_3\varphi(\lambda)\sigma_3\right\},\ \Lambda sl_2(\mathbb{C}) = \left\{A : \mathbb{S}^1 \xrightarrow{C^\infty} sl_2(\mathbb{C}) \middle| A(-\lambda) = \sigma_3 A(\lambda)\sigma_3\right\},$$

$$\Lambda^+_\triangle\mathrm{SL}_2(\mathbb{C}) = \left\{B_+(\lambda) \in \Lambda\mathrm{SL}_2(\mathbb{C}) \middle| \ B_+ \text{ extends holomorphically to } \mathbb{D}.\right\},$$

$$\Lambda^-_\triangle\mathrm{SL}_2(\mathbb{C}) = \left\{B_-(\lambda) \in \Lambda\mathrm{SL}_2(\mathbb{C}) \middle| \ B_- \text{ extends holomorphically to } \mathbb{C}\cup\{\infty\}\backslash\overline{\mathbb{D}}.\right\},$$

$$\Lambda^+_\mathbb{R}\mathrm{SL}_2(\mathbb{C}) = \left\{B_+(\lambda) \in \Lambda^+_\triangle\mathrm{SL}_2(\mathbb{C}) \middle| \ B_+(0) = \begin{pmatrix} \rho & 0 \\ 0 & \rho^{-1} \end{pmatrix} \text{ for } \rho \in \mathbb{R}, \rho > 0.\right\},$$

where $\mathbb{S}^1 = \left\{\lambda \in \mathbb{C}\cup\{\infty\} | \lambda\bar{\lambda} = 1\right\}$, $\mathbb{D} = \left\{\lambda \in \mathbb{C}\cup\{\infty\} | \lambda\bar{\lambda} < 1\right\}$. We can also define $\Lambda\mathrm{GL}_2(\mathbb{C})$, etc., in a similar way.

Here we introduce the twisted version of the Birkhoff splitting as follows:

Proposition 3.2.1 ([79])**.** *For all $\varphi \in \Lambda\mathrm{GL}_2(\mathbb{C})$, there exist $\varphi_+ \in \Lambda^+\mathrm{GL}_2(\mathbb{C})$, $\varphi_- \in \Lambda^-\mathrm{GL}_2(\mathbb{C})$ and $a_1, a_2 \in \mathbb{Z}$ such that*

$$\varphi = \varphi_- \begin{pmatrix} \lambda^{2a_1} & 0 \\ 0 & \lambda^{2a_2} \end{pmatrix} \varphi_+. \tag{3.2.1}$$

The middle term is uniquely determined by φ, and the big cell $\mathcal{B}$, where $a_1 = a_2 = 0$, is an open dense subset in $\Lambda\mathrm{GL}_2(\mathbb{C})$. When $\varphi \in \mathcal{B}$, we have a unique splitting such that $\varphi_+ \in \Lambda^+_I\mathrm{GL}_2(\mathbb{C})$, $\varphi_- \in \Lambda^-\mathrm{GL}_2(\mathbb{C})$.

We also introduce the specialized version of the Birkhoff splitting for $\Lambda\mathrm{SL}_2(\mathbb{C})$, as in [17].

Proposition 3.2.2 (the specialized version of the Birkhoff splitting [17])**.** *For all $\varphi \in \Lambda\mathrm{SL}_2(\mathbb{C})$, there exist $B_+ \in \Lambda^+_\triangle\mathrm{SL}_2(\mathbb{C})$, $B_- \in \Lambda^-_\triangle\mathrm{SL}_2(\mathbb{C})$ and $k \in \mathbb{Z}$ such that*

$$\varphi = B_- M B_+, \tag{3.2.2}$$

where either

$$M = \begin{pmatrix} \lambda^{2k} & 0 \\ 0 & \lambda^{-2k} \end{pmatrix}, \text{ or } M = \begin{pmatrix} 0 & \lambda^{2k+1} \\ -\lambda^{-2k-1} & 0 \end{pmatrix}. \tag{3.2.3}$$

Remark 3.2.1 ([17])**.**
(1) The factor M is uniquely determined by φ. For $\varphi \in \mathcal{B}$, where $k = 0$, there is a unique splitting $\varphi = B_- B_+$ with $B_- \in \Lambda^-_\triangle\mathrm{SL}_2(\mathbb{C})$, $B_+ \in \Lambda^+_I\mathrm{SL}_2(\mathbb{C})$, with $M = I$.
(2) For the $\Lambda\mathrm{SL}_2(\mathbb{C})_\tau$-Iwasawa splitting, we define

$$\mathcal{P} = \left\{\varphi \in \Lambda\mathrm{SL}_2(\mathbb{C}) \middle| M \text{ of the Birkhoff splitting of } \varphi \text{ is } \begin{pmatrix} 0 & \lambda \\ -\lambda^{-1} & 0 \end{pmatrix}\right\},$$

and we call this $\mathcal{P}$ the *Birkhoff first small cell.*

In order to prove the $\Lambda\mathrm{SL}_2(\mathbb{C})_\tau$-Iwasawa splitting, we need some definitions and lemmas.

Definition 3.2.2 (operator τ)**.** For $\varphi(\lambda) \in \Lambda\mathrm{SL}_2(\mathbb{C})$, we define an operator τ by

$$\tau(\varphi(\lambda)) := R^{-1}\left(\overline{\varphi(i\overline{\lambda}^{-1})}^{t}\right)^{-1} R \quad \text{for} \quad R = \begin{pmatrix} e^{-\frac{\pi}{4}i} & 0 \\ 0 & e^{\frac{\pi}{4}i} \end{pmatrix}.$$

Remark 3.2.2. For $\varphi \in \Lambda\mathrm{SL}_2(\mathbb{C})$, the following two statements hold:

- $\tau(\tau(\varphi(\lambda))) = \sigma_3\varphi(-\lambda)\sigma_3 = \varphi(\lambda)$.
- $\tau(\varphi(\lambda)) = \varphi(\lambda)$ for all $\lambda \in \mathbb{S}^1 \iff \varphi(\lambda) \in \Lambda\mathrm{SL}_2(\mathbb{C})_\tau$.
  Thus, τ is an automorphism on $\Lambda\mathrm{SL}_2(\mathbb{C})_\tau$.

Lemma 3.2.1. *Let $\psi \in \Lambda\mathrm{SL}_2(\mathbb{C})$ such that $(\tau(\psi))^{-1} = \psi$. If $\psi \in \mathcal{B} \cup \mathcal{P}$, then we have*

$$\psi = \tau(B_+)^{-1}\cdot(\pm I)\cdot B_+ \quad \text{or} \quad \psi = \tau(B_+)^{-1}\cdot\begin{pmatrix} 0 & \lambda \\ -\lambda^{-1} & 0 \end{pmatrix}\cdot B_+ \quad \text{for} \quad B_+ \in \Lambda^+_\triangle\mathrm{SL}_2(\mathbb{C}).$$

However, if $\psi \notin \mathcal{B} \cup \mathcal{P}$, then we do not have

$$\psi = \tau(B_+)^{-1}\cdot\begin{pmatrix} \lambda^{2k} & 0 \\ 0 & \lambda^{-2k} \end{pmatrix}\cdot B_+ \quad \text{or} \quad \psi = \tau(B_+)^{-1}\cdot\begin{pmatrix} 0 & \lambda^{2k+1} \\ -\lambda^{-2k-1} & 0 \end{pmatrix}\cdot B_+$$

for $B_+ \in \Lambda^+_\triangle\mathrm{SL}_2(\mathbb{C})$.

Proof. (i) Noting the result in Proposition 3.2.2, suppose

$$\psi = B_-\begin{pmatrix} \lambda^{2k} & 0 \\ 0 & \lambda^{-2k} \end{pmatrix}B_+$$

for some $B_+ \in \Lambda^+_\triangle\mathrm{SL}_2(\mathbb{C})$ and $B_- \in \Lambda^-_\triangle\mathrm{SL}_2(\mathbb{C})$ and some $k \in \mathbb{Z}$. By $\tau(\psi)^{-1} = \psi$, we have

$$\delta\cdot\tau(B_+\cdot\tau(B_-))^{-1}\cdot\begin{pmatrix} -\lambda^{-2k} & 0 \\ 0 & -\lambda^{2k} \end{pmatrix} = \begin{pmatrix} \lambda^{2k} & 0 \\ 0 & \lambda^{-2k} \end{pmatrix}\cdot B_+\cdot\tau(B_-)$$

for $\delta = 1$ (resp. -1) if $k \in 2\mathbb{Z}+1$ (resp. $k \in 2\mathbb{Z}$). Setting $\mathbb{B}(\lambda) = B_+\cdot\tau(B_-)$, this equals

$$\delta\cdot\tau(\mathbb{B}(\lambda))^{-1}\cdot\begin{pmatrix} -\lambda^{-2k} & 0 \\ 0 & -\lambda^{2k} \end{pmatrix} = \begin{pmatrix} \lambda^{2k} & 0 \\ 0 & \lambda^{-2k} \end{pmatrix}\cdot\mathbb{B}(\lambda), \tag{3.2.4}$$

Now we note that $\mathbb{B}(\lambda) \in \Lambda^+_\triangle\mathrm{SL}_2(\mathbb{C})$, and we let $\mathbb{B}(\lambda) = \begin{pmatrix} a(\lambda) & b(\lambda) \\ c(\lambda) & d(\lambda) \end{pmatrix}$. Then, (3.2.4) equals

$$\delta\cdot\begin{pmatrix} -\lambda^{-2k}\overline{a^*(\lambda)} & -i\lambda^{2k}\overline{c^*(\lambda)} \\ i\lambda^{-2k}\overline{b^*(\lambda)} & -\lambda^{2k}\overline{d^*(\lambda)} \end{pmatrix} = \begin{pmatrix} \lambda^{2k}a(\lambda) & \lambda^{2k}b(\lambda) \\ \lambda^{-2k}c(\lambda) & \lambda^{-2k}d(\lambda) \end{pmatrix}, \tag{3.2.5}$$

where $a^*(\lambda) = a(i\overline{\lambda}^{-1})$.

If $k > 0$, then the upper-left component of (3.2.5) implies $-\delta\overline{a^*(\lambda)} = \lambda^{4k}a(\lambda)$, and this a has the power series expansion $\sum_{j=0}^{\infty} a_j\lambda^j$, where the a_j do not depend on λ since $\mathbb{B}(\lambda) \in \Lambda^+_\triangle\mathrm{SL}_2(\mathbb{C})$.

By these conditons, we know $a \equiv 0$. Similarly, the upper-right and lower-left component of (3.2.5) imply that

$$b, c : \text{complex constants} \quad \text{and} \quad \delta i b = c.$$

However, by the twisted property, we have $b \equiv c \equiv 0$, and this contradicts $\det(\mathbb{B}(\lambda)) = 1$. Similarly, if $k < 0$, then we get the contradiction. Thus, we conclude $k = 0$.

By $k = 0$, we have $\delta = -1$, $b \equiv c \equiv 0$, $a \in \mathbb{R} \setminus \{0\}$: constant, $d = a^{-1}$. Here we notice that $\sqrt{a}$ is well-defined, and we get the following:

$$\tau\left(\begin{pmatrix} \sqrt{a} & 0 \\ 0 & \sqrt{a}^{-1} \end{pmatrix}\right)^{-1} = \begin{pmatrix} \sqrt{a} & 0 \\ 0 & \sqrt{a}^{-1} \end{pmatrix} \quad \text{if} \quad a > 0 \tag{3.2.6}$$

$$\tau\left(\begin{pmatrix} \sqrt{a} & 0 \\ 0 & \sqrt{a}^{-1} \end{pmatrix}\right)^{-1} = \begin{pmatrix} -\sqrt{a} & 0 \\ 0 & -\sqrt{a}^{-1} \end{pmatrix} \quad \text{if} \quad a < 0. \tag{3.2.7}$$

Thus, we separately consider these cases to finish the case (i).

(i)-(1) If $a > 0$, then

$$\mathbb{B} = B_+ \cdot \tau(B_-) = \begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix} \quad \text{for} \quad B_\pm \in \Lambda_\triangle^\pm \mathrm{SL}_2(\mathbb{C}).$$

This implies that $\begin{pmatrix} \sqrt{a}^{-1} & 0 \\ 0 & \sqrt{a} \end{pmatrix} B_+ = \tau\left(B_- \begin{pmatrix} \sqrt{a} & 0 \\ 0 & \sqrt{a}^{-1} \end{pmatrix}\right)^{-1}$.

Now let $\beta_+ := \begin{pmatrix} \sqrt{a}^{-1} & 0 \\ 0 & \sqrt{a} \end{pmatrix} B_+ \in \Lambda_\triangle^+ \mathrm{SL}_2(\mathbb{C})$ and $\beta_- := B_- \begin{pmatrix} \sqrt{a} & 0 \\ 0 & \sqrt{a}^{-1} \end{pmatrix} \in \Lambda_\triangle^- \mathrm{SL}_2(\mathbb{C})$, then

$$\psi = B_- B_+ = \beta_- \beta_+ = \tau(\beta_+)^{-1} \cdot \beta_+.$$

(i)-(2) If $a < 0$, similarly we have

$$\mathbb{B} = B_+ \cdot \tau(B_-) = \begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix} \quad \text{for} \quad B_\pm \in \Lambda_\triangle^\pm \mathrm{SL}_2(\mathbb{C}).$$

Thus this implies that $\begin{pmatrix} \sqrt{a}^{-1} & 0 \\ 0 & \sqrt{a} \end{pmatrix} B_+ = \tau\left(B_- \begin{pmatrix} -\sqrt{a} & 0 \\ 0 & -\sqrt{a}^{-1} \end{pmatrix}\right)^{-1}$. Now let $\beta_+ := \begin{pmatrix} \sqrt{a}^{-1} & 0 \\ 0 & \sqrt{a} \end{pmatrix} B_+ \in \Lambda_\triangle^+ \mathrm{SL}_2(\mathbb{C})$ and $\beta_- := B_- \begin{pmatrix} -\sqrt{a} & 0 \\ 0 & -\sqrt{a}^{-1} \end{pmatrix} \in \Lambda_\triangle^+ \mathrm{SL}_2(\mathbb{C})$, then

$$\psi = B_- B_+ = \beta_-(-I)\beta_+ = \tau(\beta_+)^{-1}(-I)\beta_+. \tag{3.2.8}$$

This completes the proof of the case (i).

(ii) Again noting the result in Proposition 3.2.2, suppose

$$\psi = B_- \begin{pmatrix} 0 & \lambda^{2k+1} \\ -\lambda^{-2k-1} & 0 \end{pmatrix} B_+$$

for some $B_+ \in \Lambda^+_\triangle \mathrm{SL}_2(\mathbb{C})$ and $B_- \in \Lambda^-_\triangle \mathrm{SL}_2(\mathbb{C})$. By $\tau(\psi)^{-1} = \psi$, we have

$$\delta \cdot \begin{pmatrix} i\lambda^{-2k-1}\overline{c^*(\lambda)} & -\lambda^{2k+1}\overline{a^*(\lambda)} \\ \lambda^{-2k-1}\overline{d^*(\lambda)} & i\lambda^{2k+1}\overline{b^*(\lambda)} \end{pmatrix} = \begin{pmatrix} \lambda^{2k+1}c(\lambda) & \lambda^{2k+1}d(\lambda) \\ -\lambda^{-2k-1}a(\lambda) & -\lambda^{-2k-1}b(\lambda) \end{pmatrix} \tag{3.2.9}$$

for $\mathbb{B}(\lambda) = B_+ \cdot \tau(B_-) = \begin{pmatrix} a(\lambda) & b(\lambda) \\ c(\lambda) & d(\lambda) \end{pmatrix}$. As in the case (i), by the form of the power series expansion of $b(\lambda)$ and $c(\lambda)$, we conclude

$$c \equiv 0,\ b(\lambda) = b_1\lambda^1 + b_3\lambda^3 + \cdots + b_{4k+1}\lambda^{4k+1},\ -\delta i\lambda^{4k+2}\overline{b^*(\lambda)} = b(\lambda) \quad \text{if} \quad k \geq 0, \tag{3.2.10}$$

$$b \equiv 0,\ c(\lambda) = c_1\lambda^1 + c_3\lambda^3 + \cdots + c_{-4k-3}\lambda^{-4k-3},\ \delta i\lambda^{-4k-2}\overline{c^*(\lambda)} = c(\lambda) \quad \text{if} \quad k < 0. \tag{3.2.11}$$

Similarly, by the off-diagonal components of (3.2.9), we have

$$\delta = -1 \ (\Longleftrightarrow k \in 2\mathbb{Z}), \quad a \in \mathbb{S}^1 : \text{complex constant}, \quad d = a^{-1}. \tag{3.2.12}$$

Finally, we need to change B_+ and B_- to $\beta_+ = YB_+ \in \Lambda^+_\triangle \mathrm{SL}_2(\mathbb{C})$ and $\beta_- = B_-X^{-1} \in \Lambda^-_\triangle \mathrm{SL}_2(\mathbb{C})$ by using $X \in \Lambda^-_\triangle \mathrm{SL}_2(\mathbb{C})$ and $Y \in \Lambda^+_\triangle \mathrm{SL}_2(\mathbb{C})$. To complete the requirement, these matrices X and Y should satisfy the following condition:

$$\mathbb{B} = Y^{-1}\tau(X) \tag{3.2.13}$$

and

$$X^{-1}\begin{pmatrix} \lambda^{2l} & 0 \\ 0 & \lambda^{-2l} \end{pmatrix} Y = \begin{pmatrix} 0 & \lambda^{2k+1} \\ -\lambda^{-2k-1} & 0 \end{pmatrix} \text{ or } X^{-1}\begin{pmatrix} 0 & \lambda^{2l+1} \\ -\lambda^{-2l-1} & 0 \end{pmatrix} Y = \begin{pmatrix} 0 & \lambda^{2k+1} \\ -\lambda^{-2k-1} & 0 \end{pmatrix} \tag{3.2.14}$$

for some $l \in \mathbb{Z}$. By direct computation with (3.2.10), (3.2.11), (3.2.12) and (3.2.13), we know that the diagonal case in (3.2.14) does not occur for any k, and that the off-diagonal case in (3.2.14) occurs only for $k = 0$. For $k = 0$, as in (i), we set $\beta_+ := \begin{pmatrix} \sqrt{a}^{-1} & -\frac{1}{2}b\sqrt{a}\lambda \\ 0 & \sqrt{a} \end{pmatrix} B_+ \in \Lambda^+_\triangle \mathrm{SL}_2(\mathbb{C})$ and $\beta_- := B_- \begin{pmatrix} \sqrt{a}^{-1} & 0 \\ -\frac{1}{2}b\sqrt{a}\lambda^{-1} & \sqrt{a} \end{pmatrix} \in \Lambda^-_\triangle \mathrm{SL}_2(\mathbb{C})$, then

$$\psi = B_- \begin{pmatrix} 0 & \lambda \\ -\lambda^{-1} & 0 \end{pmatrix} B_+ = \beta_- \begin{pmatrix} 0 & \lambda \\ -\lambda^{-1} & 0 \end{pmatrix} \beta_+ = \tau(\beta_+)^{-1} \begin{pmatrix} 0 & \lambda \\ -\lambda^{-1} & 0 \end{pmatrix} \beta_+.$$

This completes the proof of Lemma 3.2.1. □

Here we define the $\Lambda\mathrm{SL}_2(\mathbb{C})_\tau$-*Iwasawa big cell* $\mathcal{B}_\tau$ and $\Lambda\mathrm{SL}_2(\mathbb{C})_\tau$-*Iwasawa first small cell* $\mathcal{P}_\tau$.

Definition 3.2.3.

$$\mathcal{B}_\tau = \left\{\varphi \in \Lambda\mathrm{SL}_2(\mathbb{C}) \middle| \tau(\varphi)^{-1}\varphi \in \mathcal{B}\right\},\ \mathcal{P}_\tau = \left\{\varphi \in \Lambda\mathrm{SL}_2(\mathbb{C}) \middle| \tau(\varphi)^{-1}\varphi \in \mathcal{P}\right\}.$$

We introduce one of the main theorems here, and this Theorem 3.2.1 plays an important role in the next section about the DPW method, in order to construct the solution of the extended Lax pair which was introduced in the previous section. We will also study the asymptotic behavior of F near $\mathcal{P}_\tau$ related to singularities of surfaces in the two latter sections. In Theorem 3.2.1, we use the map

$$\Psi\left(\begin{pmatrix} a(\lambda) & b(\lambda) \\ c(\lambda) & d(\lambda) \end{pmatrix}\right) = \begin{pmatrix} a(\lambda^2) & \lambda b(\lambda^2) \\ \lambda^{-1}c(\lambda^2) & d(\lambda^2) \end{pmatrix}. \tag{3.2.15}$$

Theorem 3.2.1 ($\Lambda\mathrm{SL}_2(\mathbb{C})_\tau$-Iwasawa splitting)**.**
(1) *For all $\varphi \in \mathcal{B}_\tau$, there exist $F \in \Lambda\mathrm{SL}_2(\mathbb{C})_\tau \cup \Psi(i\sigma_2) \cdot \Lambda\mathrm{SL}_2(\mathbb{C})_\tau$ and $B \in \Lambda^+_\triangle\mathrm{SL}_2(\mathbb{C})$ such that*

$$\varphi = FB. \tag{3.2.16}$$

We can choose $B \in \Lambda^+_\mathbb{R}\mathrm{SL}_2(\mathbb{C})$, and then F and B are uniquely determined. We call this unique splitting "normalized."
(2) *For all $\varphi \in \mathcal{P}_\tau$, there exist $F \in \Lambda\mathrm{SL}_2(\mathbb{C})_\tau \cup \Psi(i\sigma_2) \cdot \Lambda\mathrm{SL}_2(\mathbb{C})_\tau$ and $B \in \Lambda^+_\mathbb{R}\mathrm{SL}_2(\mathbb{C})$ such that*

$$\varphi = FCB, \quad C = \begin{pmatrix} \frac{1}{2} & \lambda \\ -\frac{1}{2}\lambda^{-1} & 1 \end{pmatrix}. \tag{3.2.17}$$

Proof. Take any $\varphi \in \Lambda\mathrm{SL}_2(\mathbb{C})$, and set $\psi = \tau(\varphi)^{-1}\varphi$. Then, we have $\tau(\psi)^{-1} = \psi$, thus we can apply Lemma 3.2.1 for this ψ. This implies that

$$\psi = \tau(B_+)^{-1}\tau(W)^{-1}WB_+ = \tau(WB_+)^{-1}WB_+$$

for $W = I$ or $W = \begin{pmatrix} 0 & \lambda \\ -\lambda^{-1} & 0 \end{pmatrix}$ or $W = \begin{pmatrix} \frac{1}{2} & \lambda \\ -\frac{1}{2}\lambda^{-1} & 1 \end{pmatrix}$, and for some $B_+ \in \Lambda^+_\triangle\mathrm{SL}_2(\mathbb{C})$. The first two cases occur when $\psi \in \mathcal{B}$ (*i.e.* $\varphi \in \mathcal{B}_\tau$), and the third case occurs when $\psi \in \mathcal{P}$ (i.e. $\varphi \in \mathcal{P}_\tau$).

First we show that $\hat{F} := \tau(\varphi)\tau(WB_+)^{-1} \in \Lambda\mathrm{SL}_2(\mathbb{C})_\tau$. The twisted property is automatically satisfied by definition, and we need to check $\tau(\hat{F}) = \hat{F}$. However, it is also clear because $\tau(\hat{F}) = \hat{F}$ is equivalent to $\varphi = \tau(\varphi)\tau(WB_+)^{-1}WB_+$.

Next we will consider cases:
(i) The case $W = I$, $W \in \Lambda\mathrm{SL}_2(\mathbb{C})_\tau$. We set $F := \hat{F}W \in \Lambda\mathrm{SL}_2(\mathbb{C})_\tau$ and $B := B_+ \in \Lambda^+_\triangle\mathrm{SL}_2(\mathbb{C})$, and this is the required splitting. Moreover, about normalization, we set

$$B|_{\lambda=0} = \begin{pmatrix} \alpha & 0 \\ 0 & \alpha^{-1} \end{pmatrix}$$

for $\alpha \in \mathbb{C} \setminus \{0\}$. Splitting to

$$\begin{pmatrix} \alpha & 0 \\ 0 & \alpha^{-1} \end{pmatrix} = \begin{pmatrix} \sqrt{\frac{\alpha}{\bar{\alpha}}} & 0 \\ 0 & \sqrt{\frac{\bar{\alpha}}{\alpha}} \end{pmatrix} \begin{pmatrix} \sqrt{\alpha\bar{\alpha}} & 0 \\ 0 & \frac{1}{\sqrt{\alpha\bar{\alpha}}} \end{pmatrix},$$

we notice that $\begin{pmatrix} \sqrt{\frac{\alpha}{\bar{\alpha}}} & 0 \\ 0 & \sqrt{\frac{\bar{\alpha}}{\alpha}} \end{pmatrix} \in \Lambda\mathrm{SL}_2(\mathbb{C})_\tau$, and that $\sqrt{\alpha\bar{\alpha}} \in \mathbb{R}_{>0}$. Hence, we can split to $\varphi = F'B'$ for $F' := F\begin{pmatrix} \sqrt{\frac{\alpha}{\bar{\alpha}}} & 0 \\ 0 & \sqrt{\frac{\bar{\alpha}}{\alpha}} \end{pmatrix} \in \Lambda\mathrm{SL}_2(\mathbb{C})_\tau$ and $B' := \begin{pmatrix} \sqrt{\frac{\bar{\alpha}}{\alpha}} & 0 \\ 0 & \sqrt{\frac{\alpha}{\bar{\alpha}}} \end{pmatrix} B \in \Lambda^+_\mathbb{R}\mathrm{SL}_2(\mathbb{C})$. So the uniqueness of this splitting follows from uniqueness of the Birkhoff splitting on $\mathcal{B}$.

(ii) The case $W = \begin{pmatrix} 0 & \lambda \\ -\lambda^{-1} & 0 \end{pmatrix}$, $\hat{F}W \in \Psi(i\sigma_2) \cdot \Lambda\mathrm{SL}_2(\mathbb{C})_\tau$. We can prove the existence and normalization of splitting in the same way as in (i).

(iii) The case $W = \begin{pmatrix} \frac{1}{2} & \lambda \\ -\frac{1}{2}\lambda^{-1} & 1 \end{pmatrix}$, we set $F := \hat{F}$, $C := W$ and $B := B_+$, and this is the required splitting. Moreover, about normalization, we set $B|_{\lambda=0} = \begin{pmatrix} \alpha & 0 \\ 0 & \alpha^{-1} \end{pmatrix}$ for $\alpha \in \mathbb{C} \setminus \{0\}$. Splitting to the same form as in (i), we have

$$C \begin{pmatrix} \sqrt{\frac{\alpha}{\bar{\alpha}}} & 0 \\ 0 & \sqrt{\frac{\bar{\alpha}}{\alpha}} \end{pmatrix} = \frac{1}{2} \begin{pmatrix} \sqrt{\frac{\alpha}{\bar{\alpha}}} + \sqrt{\frac{\bar{\alpha}}{\alpha}} & \left(\sqrt{\frac{\bar{\alpha}}{\alpha}} - \sqrt{\frac{\alpha}{\bar{\alpha}}}\right)\lambda \\ \left(\sqrt{\frac{\bar{\alpha}}{\alpha}} - \sqrt{\frac{\alpha}{\bar{\alpha}}}\right)\lambda^{-1} & \sqrt{\frac{\alpha}{\bar{\alpha}}} + \sqrt{\frac{\bar{\alpha}}{\alpha}} \end{pmatrix} C,$$

and we notice that $\frac{1}{2} \begin{pmatrix} \sqrt{\frac{\alpha}{\bar{\alpha}}} + \sqrt{\frac{\bar{\alpha}}{\alpha}} & \left(\sqrt{\frac{\bar{\alpha}}{\alpha}} - \sqrt{\frac{\alpha}{\bar{\alpha}}}\right)\lambda \\ \left(\sqrt{\frac{\bar{\alpha}}{\alpha}} - \sqrt{\frac{\alpha}{\bar{\alpha}}}\right)\lambda^{-1} & \sqrt{\frac{\alpha}{\bar{\alpha}}} + \sqrt{\frac{\bar{\alpha}}{\alpha}} \end{pmatrix} \in \Lambda\mathrm{SL}_2(\mathbb{C})_\tau$. Hence we can split to $\varphi = F'CB'$ for $F' := F\frac{1}{2} \begin{pmatrix} \sqrt{\frac{\alpha}{\bar{\alpha}}} + \sqrt{\frac{\bar{\alpha}}{\alpha}} & \left(\sqrt{\frac{\bar{\alpha}}{\alpha}} - \sqrt{\frac{\alpha}{\bar{\alpha}}}\right)\lambda \\ \left(\sqrt{\frac{\bar{\alpha}}{\alpha}} - \sqrt{\frac{\alpha}{\bar{\alpha}}}\right)\lambda^{-1} & \sqrt{\frac{\alpha}{\bar{\alpha}}} + \sqrt{\frac{\bar{\alpha}}{\alpha}} \end{pmatrix} \in \Lambda\mathrm{SL}_2(\mathbb{C})_\tau$ and $B' := \begin{pmatrix} \sqrt{\frac{\bar{\alpha}}{\alpha}} & 0 \\ 0 & \sqrt{\frac{\alpha}{\bar{\alpha}}} \end{pmatrix} B \in \Lambda^+_{\mathbb{R}}\mathrm{SL}_2(\mathbb{C})$. □

Remark 3.2.3. $\mathcal{B}_\tau$ becomes an open dense subset of $\Lambda\mathrm{SL}_2(\mathbb{C})$ because we can use the same argument as in the proof of Theorem 1.2 (4) in [17].

3.3 The DPW method for spacelike CMC H surfaces with $0 \leq H < 1$ in $\mathbb{S}^{2,1}$

3.3.1 Holomorphic potential

Here we will show that all simply-connected spacelike CMC H surfaces with $0 \leq H < 1$ in $\mathbb{S}^{2,1}$ are given by a holomorphic potential, defined in Definition 3.3.1.

Definition 3.3.1 (holomorphic potential [17], [25])**.** Let Σ be a simply-connected domain, $z \in \Sigma$ and $\lambda \in \mathbb{C}$. A *holomorphic potential* ξ is of the form

$$\xi := Adz, \quad A = A(z, \lambda) = \sum_{j=-1}^{\infty} A_j(z)\lambda^j, \tag{3.3.1}$$

where each $A_j(z)$ is a 2×2 matrix that is independent of λ, is holomorphic in $z \in \Sigma$, is traceless, is a diagonal (resp. off-diagonal) matrix when j is even (resp. odd), and the upper-right entry of $A_{-1}(z)$ is never zero.

3.3.2 The inverse problem of the DPW method

We will show that given a holomorphic potential ξ, then we get a conformal spacelike CMC H surface f with $0 \leq H < 1$ in $\mathbb{S}^{2,1}$. However, in Theorem 3.3.1, we will see that finding spacelike CMC H surfaces with $0 \leq H < 1$ is equivalent to finding the solution of the extended Lax pair of the form (3.1.7), and then the surface is found by using the immersion formula (3.1.9). So to prove that the DPW method finds all spacelike CMC H surfaces with $0 \leq H < 1$, we want to prove that the DPW method produces all integrable Lax pairs of the form (3.1.7) and all their solutions F.

Theorem 3.3.1 (The inverse problem of the DPW method)**.** *Let $\xi := A(\lambda)dz = \sum_{j=-1}^{\infty} A_j(z)\lambda^j dz$ be a holomorphic potential over a simply-connected domain Σ in $\mathbb{C}$ including the origin, and let $\varphi : \Sigma \longrightarrow \Lambda\mathrm{SL}_2(\mathbb{C})$ be a solution of*

$$d\varphi = \varphi\xi \text{ and } \varphi(z)|_{z=0} = I. \tag{3.3.2}$$

Define the open set $\Sigma^o := \varphi^{-1}(\mathcal{B}_\tau) \subset \Sigma$, and take the unique $\Lambda\mathrm{SL}_2(\mathbb{C})_\tau$-Iwasawa splitting on Σ^o:

$$\varphi = FB, \tag{3.3.3}$$

where

$$F \in \Lambda\mathrm{SU}_2(\mathbb{C})_\tau, \quad B \in \Lambda^+_{\mathbb{R}}\mathrm{SL}_2(\mathbb{C}). \tag{3.3.4}$$

Then, after a change of coordinates and notations, F satisfies the extended Lax pair in (3.1.7)*.*

We can prove Theorem 3.3.1 in the same way as in [17], [24]. We omit the proof.

3.3.3 The ordinary problem of the DPW method

We will consider the converse of Theorem 3.3.1, in the same way as in [24].

Theorem 3.3.2 (The ordinary problem of the DPW method)**.** *Let $f : \Sigma \longrightarrow \mathbb{S}^{2,1}$ be a spacelike CMC H surfaces with $0 \le H < 1$ for simply-connected domain Σ, and let F be the extended frame of f satisfying* (3.1.7)*. Then, there exist φ such that $\varphi = FB$ for some $B \in \Lambda^+_{\triangle}\mathrm{SL}_2(\mathbb{C})$ and holomorphic potential ξ such that $\xi = \varphi^{-1}d\varphi$.*

We can also prove Theorem 3.3.2 in the same way as in [17], [24]. Again we omit the proof.

3.4 Behavior of the frame and surface when approaching $\mathcal{P}_\tau$

We have introduced $\Lambda\mathrm{SL}_2(\mathbb{C})_\tau$-Iwasawa splitting and defined the $\Lambda\mathrm{SL}_2(\mathbb{C})_\tau$-Iwasawa big cell $\mathcal{B}_\tau$ and small cell $\mathcal{P}_\tau$. In the previous section, we also showed that on $\Sigma^0 = \varphi^{-1}(\mathcal{B}_\tau)$ the surface f is immeresed since the metric function e^u is positive definite. However, here we will show that on $\varphi^{-1}(\mathcal{P}_\tau)$ f is not immersed, since the metric function e^u is approaching zero there.

First, we show the following lemma.

Lemma 3.4.1. *Let $B = \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} \sum_{j=0}^{\infty} a_j\lambda^j & \sum_{j=1}^{\infty} b_j\lambda^j \\ \sum_{j=1}^{\infty} c_j\lambda^j & \sum_{j=0}^{\infty} d_j\lambda^j \end{pmatrix} \in \Lambda^+_{\triangle}\mathrm{SL}_2(\mathbb{C})$, and let $C = \begin{pmatrix} \frac{1}{2} & \lambda \\ -\frac{1}{2}\lambda^{-1} & 1 \end{pmatrix}$. Then, there exist three factorizations:*

(1) *If $|2a_0 + b_1| > |d_0|$, then*

$$BC^{-1} = K_1\check{B}, \quad K_1 := \begin{pmatrix} u & v\lambda \\ \bar{v}\lambda^{-1} & \bar{u} \end{pmatrix} \in \Lambda\mathrm{SL}_2(\mathbb{C})_\tau, \quad \check{B} \in \Lambda^+_{\triangle}\mathrm{SL}_2(\mathbb{C}),$$

where u and v are constant in λ, and determined by a, b, c, d.

(2) *If $|2a_0 + b_1| < |d_0|$, then*

$$BC^{-1} = K_2\check{B}, \quad K_2 := \begin{pmatrix} u & v\lambda \\ -\bar{v}\lambda^{-1} & -\bar{u} \end{pmatrix} \in \Psi(i\sigma_2)\Lambda\mathrm{SL}_2(\mathbb{C})_\tau, \quad \check{B} \in \Lambda^+_{\triangle}\mathrm{SL}_2(\mathbb{C}),$$

where u and v are constant in λ, and determined by a, b, c, d.
(3) *If $|2a_0 + b_1| = |d_0|$, then*

$$BC^{-1} = K_3 C\check{B}, \quad K_3 := \begin{pmatrix} \sqrt{\alpha} & 0 \\ 0 & \sqrt{\alpha}^{-1} \end{pmatrix} \in \Lambda\mathrm{SL}_2(\mathbb{C})_\tau, \quad \check{B} \in \Lambda^+_\triangle \mathrm{SL}_2(\mathbb{C}),$$

where $\alpha := -\frac{2a_0+b_1}{d_0}$.

First two cases imply $BC^{-1} \in \mathcal{B}_\tau$, and third one implies that $BC^{-1} \in \mathcal{P}_\tau$.

Proof. By direct computation, we have the following results for the three cases:
(1) $K_1 = \begin{pmatrix} u & v\lambda \\ \bar{v}\lambda^{-1} & \bar{u} \end{pmatrix}$, $\check{B} = \begin{pmatrix} a\bar{u} - cv\lambda + \frac{b}{2}\bar{u}\lambda^{-1} - \frac{d}{2}v & -a\bar{u}\lambda + cv\lambda^2 + \frac{b}{2}\bar{u} - \frac{d}{2}v\lambda \\ -a\bar{v}\lambda^{-1} + cu - \frac{b}{2}\bar{v}\lambda^{-2} + \frac{d}{2}u\lambda^{-1} & a\bar{v} - cu\lambda - \frac{b}{2}\bar{v}\lambda^{-1} + \frac{d}{2}u \end{pmatrix}$,
where u and v are the solutions of the following equations:

$$|u|^2 - |v|^2 = 1, \quad 2(2a_0 + b_1)\bar{v} - d_0 u = 0.$$

(2) $K_2 = \begin{pmatrix} u & v\lambda \\ -\bar{v}\lambda^{-1} & -\bar{u} \end{pmatrix}$, $\check{B} = \begin{pmatrix} -a\bar{u} - cv\lambda - \frac{b}{2}\bar{u}\lambda^{-1} - \frac{d}{2}v & a\bar{u}\lambda + cv\lambda^2 - \frac{b}{2}\bar{u} - \frac{d}{2}v\lambda \\ a\bar{v}\lambda^{-1} + cu + \frac{b}{2}\bar{v}\lambda^{-2} + \frac{d}{2}u\lambda^{-1} & -a\bar{v} - cu\lambda + \frac{b}{2}\bar{v}\lambda^{-1} + \frac{d}{2}u \end{pmatrix}$,
where u and v are the solutions of the following equations:

$$-|u|^2 + |v|^2 = 1, \quad 2(2a_0 + b_1)\bar{v} + d_0 u = 0.$$

(3) $K_3 = \begin{pmatrix} \sqrt{\alpha} & 0 \\ 0 & \sqrt{\alpha}^{-1} \end{pmatrix}$,
$\check{B} = \begin{pmatrix} \sqrt{2\alpha}e^{-i\theta} & 0 \\ 0 & \sqrt{2\alpha}e^{i\theta} \end{pmatrix} \begin{pmatrix} a\bar{u} - cv\lambda + \frac{b}{2}\bar{u}\lambda^{-1} - \frac{d}{2}v & -a\bar{u}\lambda + cv\lambda^2 + \frac{b}{2}\bar{u} - \frac{d}{2}v\lambda \\ a\bar{v}\lambda^{-1} + cu + \frac{b}{2}\bar{v}\lambda^{-2} + \frac{d}{2}u\lambda^{-1} & -a\bar{v} - cu\lambda + \frac{b}{2}\bar{v}\lambda^{-1} + \frac{d}{2}u \end{pmatrix}$,
where $u = \frac{\alpha}{\sqrt{2}}e^{-i\theta}$ and $v = \frac{1}{\sqrt{2}}e^{i\theta}$. $\square$

This is one of our main theorems:

Theorem 3.4.1. *Let φ_n be a sequence in $\mathcal{B}_\tau$, with $\lim_{n\to\infty}\varphi_n = \varphi_0 \in \mathcal{P}_\tau$. Let $\varphi_n = F_n B_n$ be a $\Lambda\mathrm{SL}_2(\mathbb{C})_\tau$-Iwasawa splitting. Then,*
(1) *Writing F_n as*

$$F_n = \begin{pmatrix} x_n & y_n \\ \pm i\overline{y_n^*} & \pm\overline{x_n^*} \end{pmatrix} \in \Lambda\mathrm{SL}_2(\mathbb{C})_\tau \cup \Psi(i\sigma_2)\cdot\Lambda\mathrm{SL}_2(\mathbb{C})_\tau \quad \text{for} \quad \overline{x_n^*} := \overline{x_n(i\overline{\lambda^{-1}})},$$

we have $\lim_{n\to\infty}|x_n| = \lim_{n\to\infty}|y_n| = \infty$ for all λ.
(2) *Writing the constant term of B_n with respect to λ as*

$$B_n|_{\lambda=0} = \begin{pmatrix} \rho_n & 0 \\ 0 & \rho_n^{-1} \end{pmatrix},$$

we have $\lim_{n\to\infty}|\rho_n| = 0$. This implies that f has singularities on $\mathcal{P}_\tau$, since the metric function u is defined as $u = 2\log(\rho)$ (i.e. the metric $ds^2 = 4e^{2u}dwd\bar{w} \to 0$ if $n \to \infty$).

Proof. (1) By $\Lambda\mathrm{SL}_2(\mathbb{C})_\tau$-Iwasawa splitting in Theorem 3.2.1, we have $\varphi_0 = F_0 C B_0$. Expressing φ_n as

$$\varphi_n = \hat{\varphi}_n C B_0, \quad \hat{\varphi}_n := \varphi_n B_0^{-1} C^{-1},$$

we have $\lim_{n\to\infty} \hat{\varphi}_n = F_0$. So, for sufficiently large n, we have $\hat{\varphi}_n \in \mathcal{B}_\tau$. Thus, for these n, $\hat{\varphi}_n$ is $\Lambda\mathrm{SL}_2(\mathbb{C})_\tau$-Iwasawa split into $\hat{\varphi}_n = \hat{F}_n \hat{B}_n$, and $\lim_{n\to\infty} \hat{F}_n = F_0$, $\lim_{n\to\infty} \hat{B}_n = I$. Applying Lemma 3.4.1 with λ replacing $-\lambda$, we have

$$\varphi_n = \hat{F}_n \hat{B}_n C B_0 = \hat{F}_n X_n \check{B}_n B_0,$$

where $X_n = K_1(-\lambda)$ or $K_2(-\lambda)$. Thus, $X_n = \begin{pmatrix} u_n & v_n\lambda \\ \pm\overline{v_n}\lambda^{-1} & \pm\overline{u_n} \end{pmatrix}$ for u_n, v_n: constant in λ. By the computaion in the proof of Lemma 3.4.1, we also have

$$\frac{|u_n|}{|v_n|} = \frac{|2\hat{a}_{0,n} + \hat{b}_{1,n}|}{|\hat{d}_{0,n}|},$$

where $\hat{a}_{0,n}, \hat{b}_{1,n}, \hat{d}_{0,n}$ are determined by the components of $\hat{B}_n = \begin{pmatrix} \sum_{j=0}^\infty \hat{a}_{j,n}\lambda^j & \sum_{j=1}^\infty \hat{b}_{j,n}\lambda^j \\ \sum_{j=1}^\infty \hat{c}_{j,n}\lambda^j & \sum_{j=0}^\infty \hat{d}_{j,n}\lambda^j \end{pmatrix}$. So

$$\lim_{n\to\infty} \hat{b}_{1,n} = 0 \quad \text{and} \quad \lim_{n\to\infty} \hat{a}_{0,n} = \lim_{n\to\infty} \hat{d}_{0,n} = 1 \tag{3.4.1}$$

because $\lim_{n\to\infty} \hat{B}_n = I$. By (3.4.1) and $|u_n|^2 - |v_n|^2 = \pm 1$, we get $\lim_{n\to\infty} |u_n| = \lim_{n\to\infty} |v_n| = \lim_{n\to\infty} ||X_n|| = \infty$ for some suitable matrix norm $||\cdot||$. Now the uniqueness of the $\Lambda\mathrm{SL}_2(\mathbb{C})_\tau$-Iwasawa splitting says that

$$F_n = \hat{F}_n X_n D_n$$

for some diagonal matrix D_n which is constant in λ. Then we have

$$||X_n|| = ||\hat{F}^{-1} F_n|| \leq ||\hat{F}^{-1}|| \cdot ||F_n||,$$

so $\lim_{n\to\infty} ||\hat{F}^{-1}|| \cdot ||F_n|| = \lim_{n\to\infty} ||F_n|| = \infty$, since $\lim_{n\to\infty} ||\hat{F}^{-1}|| = ||F_0||$ is finite. Because $|x_n|^2 - |y_n|^2 = \pm 1$, we get the conclusion $\lim_{n\to\infty} |x_n| = \lim_{n\to\infty} |y_n| = \infty$.
(2) In the same way as in the proof of (1), we get $\lim_{n\to\infty} |\rho_n^{-1}| = \infty$, by using (3.4.1). $\square$

Chapter 4

Singularity theory for spacelike CMC surfaces with $0 \leq H < 1$ in $\mathbb{S}^{2,1}$

4.1 Criteria for singularities of spacelike CMC H surfaces with $0 \leq H < 1$ in $\mathbb{S}^{2,1}$.

In this section, we study singularities of spacelike CMC H surfaces f with $0 \leq H < 1$ in $\mathbb{S}^{2,1}$, similarly to our previous work [70] of spacelike CMC H surfaces with $H > 1$ in $\mathbb{S}^{2,1}$. We will use the frame-changing method, called the s-spectral deformation, in order to specify the types of singularities. Here we introduce criteria for cuspidal edges, swallowtails and cuspidal cross caps on spacelike CMC H surfaces with $0 \leq H < 1$ in $\mathbb{S}^{2,1}$.

Let $f : \Sigma \longrightarrow \mathbb{S}^{2,1}$ be a spacelike CMC immersion of a simply-connected domain $\Sigma \subset \mathbb{C}$, with the metric $ds^2 = 4g^2 dw d\bar{w} = 4e^{2u} dw d\bar{w}$ and unit normal vector N. First, we consider the frame $\mathfrak{F}$ such that

$$f = \mathfrak{F}\sigma_3\overline{\mathfrak{F}}^t,$$

$$\mathfrak{F}_w = \mathfrak{F}\boldsymbol{A},\ \mathfrak{F}_{\bar{w}} = \mathfrak{F}\boldsymbol{B},\ \text{where } \boldsymbol{A} = \frac{1}{2}\begin{pmatrix} u_w & 2e^u(1-H) \\ -e^{-u}\mathcal{A} & -u_w \end{pmatrix},\ \boldsymbol{B} = \frac{1}{2}\begin{pmatrix} -u_{\bar{w}} & -e^{-u}\bar{\mathcal{A}} \\ -2e^u(1+H) & u_{\bar{w}} \end{pmatrix}. \quad (4.1.1)$$

This $\mathfrak{F}$ is related to $\tilde{F}$ in (3.1.5) such that $\mathfrak{F} = \tilde{F}|_{\mu=1}$. For this $\mathfrak{F}$, the compatibility condition implies the Gauss and Codazzi equations (3.1.1). We define *s-spectral deformations* as follows:

Definition 4.1.1. The *s-spectral deformation* of the spacelike CMC surface f in $\mathbb{S}^{2,1}$ is the deformation defined by $(1+H) \to is(1+H)$, $(1-H) \to is^{-1}(1-H)$ in Equations (4.1.1) for the parameter $s > 0$.

The s-spectral deformation maps spacelike CMC surfaces to other spacelike CMC surfaces conformally, as follows:

Theorem 4.1.1. *For all $s \in \mathbb{R}_{>0}$, the s-spectral deformation deforms a surface f in $\mathbb{S}^{2,1}$, with mean curvature H, metric $ds^2 = 4e^{2u}dwd\bar{w}$ and Hopf differential $\mathcal{A}$, into a surface f^s with mean curvature $H^s = \frac{s(1+H)-s^{-1}(1-H)}{s(1+H)+s^{-1}(1-H)}$, metric $4e^{2u^s}dwd\bar{w} = 4(k^s)^2e^{2u}dwd\bar{w}$ and Hopf differential $\mathcal{A}^s = -ik^s\mathcal{A}$ for $k^s = \frac{s(1+H)+s^{-1}(1-H)}{2}$.*

Proof. Checking the Gauss-Weingarten equations for f^s, we get the conclusion. □

Lemma 4.1.1. $(f^s)^{-\frac{1}{s}} = f$.

Proof. Direct computation implies $(H^s)^{-\frac{1}{s}} = H, \quad (k^s)^{-\frac{1}{s}} = -\frac{1}{k^s}, \quad (\mathcal{A}^s)^{-\frac{1}{s}} = \mathcal{A},\ (u^s)^{-\frac{1}{s}} = u.$ □

We define the s-spectral Lax pair.

Definition 4.1.2 (s-spectral Lax pair)**.** We define $\mathfrak{F}^s$ as a solution of the following system:

$$\mathfrak{F}^s_w = \mathfrak{F}^s\boldsymbol{A}^s, \quad \mathfrak{F}^s_{\bar{w}} = \mathfrak{F}^s\boldsymbol{B}^s, \quad \text{where } \boldsymbol{A}^s := \frac{1}{2}\begin{pmatrix} u_w & 2e^u is^{-1}(1-H) \\ -e^{-u}\mathcal{A} & -u_w \end{pmatrix},\ \boldsymbol{B}^s := \frac{1}{2}\begin{pmatrix} -u_{\bar{w}} & -e^{-u}\bar{\mathcal{A}} \\ -2e^u is(1+H) & u_{\bar{w}} \end{pmatrix}.$$

Further, we define the form $\Omega^s := (\mathfrak{F}^s)^{-1}d\mathfrak{F}^s$.

Theorem 4.1.2. *For f given by the frame $\mathfrak{F}$, and mean curvature $0 \leq H < 1$, there exists a member of the s-spectral deformation, for the special value $s = s_0 := \sqrt{\frac{1-H}{H+1}}$, that generates a frame $\tilde{\mathfrak{F}} = \mathfrak{F}^{s_0} \in \mathrm{SU}_{1,1}$ (defined in p.3 of [70], etc.).*

Proof. It is easy to see that choosing $s = s_0 := \sqrt{\frac{1-H}{H+1}}$ gives the only deformation that makes the Maurer-Cartan form become an $\mathrm{su}_{1,1}$-valued form. □

As s approaches s_0, the mean curvature goes to infinity, and $\tilde{f} := \tilde{\mathfrak{F}}\sigma_3\bar{\tilde{\mathfrak{F}}}^t$ degenerates to a point, but there still exists a map $\tilde{\mathfrak{F}}$ from Σ to $\mathrm{SU}_{1,1}$ such that $\tilde{\mathfrak{F}}^{-1}d\tilde{\mathfrak{F}} = \tilde{\Omega}$ defined by the following (4.1.2).

Definition 4.1.3. We call $\tilde{\mathfrak{F}} : \Sigma \longrightarrow \mathrm{SU}_{1,1}$ the *adjusted frame* of $\mathfrak{F}$ and the form $\tilde{\Omega} = \tilde{\mathfrak{F}}^{-1}d\tilde{\mathfrak{F}}$ the *adjusted Maurer-Cartan form*, where

$$\tilde{\Omega} = \frac{1}{2}\begin{pmatrix} u_w & -2ie^u\sqrt{1-H^2} \\ -e^{-u}\mathcal{A} & -u_w \end{pmatrix}dw + \frac{1}{2}\begin{pmatrix} -u_{\bar{w}} & -e^{-u}\bar{\mathcal{A}} \\ -2ie^u\sqrt{1-H^2} & u_{\bar{w}} \end{pmatrix}d\bar{w} =: \tilde{\boldsymbol{A}}dw + \tilde{\boldsymbol{B}}d\bar{w}. \quad (4.1.2)$$

Remark 4.1.1. Defining $G := \mathfrak{F} \cdot \tilde{\mathfrak{F}}^{-1}$, we have $G\sigma_3\bar{G}^t = \mathfrak{F}\sigma_3\bar{\mathfrak{F}}^t = f$.

By the above remark, we can use a new frame G instead of $\mathfrak{F}$, to construct the criteria for singularities of f.

We denote $\tilde{\mathfrak{F}} = \tilde{\mathfrak{F}}(w,\bar{w}) = e^{-\frac{u}{2}}\begin{pmatrix} u_1 & u_2 \\ \overline{u_2} & \overline{u_1} \end{pmatrix} \in \mathrm{SU}_{1,1}$, where $g = e^u$ is the metric function of f, and $|u_1|^2 - |u_2|^2 = g$. So we define $h := \frac{\overline{u_2}}{u_1}$ and $\omega := u_1^2$. By Remark 4.1.1, we have $f = \mathfrak{F}\sigma_3\bar{\mathfrak{F}}^t = G\sigma_3\bar{G}^t$. Setting $\alpha := (1-H) - i\sqrt{1-H^2}$ and $\beta := -(1+H) + i\sqrt{1-H^2}$, we get

$$ds^2 := 4g^2dwd\bar{w} = (1-|h|^2)^2|\omega|^2dwd\bar{w},$$
$$G^{-1}dG = \alpha\begin{pmatrix} -h & 1 \\ -h^2 & h \end{pmatrix}\omega dw + \beta\begin{pmatrix} \bar{h} & -\bar{h}^2 \\ 1 & -\bar{h} \end{pmatrix}\bar{\omega}d\bar{w}.$$

This implies that, wherever ds^2 is finite, f has a singularity if and only if $h \in \mathbb{S}^1$ or $\omega = 0$. However we consider only extended CMC surfaces f defined in the following Definition 4.1.4.

Definition 4.1.4 ([91])**.** A CMC surface f restricted to the subdomain $\mathcal{D} = \{p \in \Sigma \mid ds^2 < \infty\}$ is called an extended CMC surface if ω, resp. $h^2\omega$, is never zero on $\mathcal{D}$ when $|h| < \infty$, resp. $|h| = \infty$.

Remark 4.1.2. By this definition, any point $p \in \Sigma$ is singular only when $|h(p)| = 1$. (See [91].)

We have the following criteria for singularities of spacelike extended CMC H surfaces with $0 \le H < 1$ in $\mathbb{S}^{2,1}$. The proof of Theorem 4.1.3 is parallel to the proof of Theorem 7.5 in [70]. (See also [32], [91], [92].)

Theorem 4.1.3. *Let Σ be a simply connected domain, and let $f : \Sigma \longrightarrow \mathbb{S}^{2,1}$ be a spacelike extended CMC H surface with $0 \le H < 1$. Then:*

1. *A point $p \in \Sigma$ is a singular point if and only if $h \in \mathbb{S}^1$.*
2. *f is a front at a singular point $p \in \Sigma$ if and only if $\mathrm{Re}\left(\frac{\mathcal{A}}{h^2\omega^2}\right)\Big|_p \neq 0$. If this is the case, p is a non-degenerate singular point.*
3. *f has a cuspidal edge at a singular point $p \in \Sigma$ if and only if*
$$\mathrm{Re}\left(\frac{\mathcal{A}}{h^2\omega^2}\right)\Bigg|_p \neq 0 \quad \text{and} \quad \mathrm{Im}\left(\frac{\mathcal{A}}{h^2\omega^2}\right)\Bigg|_p \neq 0.$$
4. *f has a swallowtail at a singular point $p \in \Sigma$ if and only if*
$$\mathrm{Re}\left(\frac{\mathcal{A}}{h^2\omega^2}\right)\Bigg|_p \neq 0, \quad \mathrm{Im}\left(\frac{\mathcal{A}}{h^2\omega^2}\right)\Bigg|_p = 0$$
and
$$\mathrm{Re}\left\{\overline{\left(\frac{\mathcal{A}}{h\omega}\right)}\left(\frac{\mathcal{A}_w h\omega - \mathcal{A}(2\mathcal{A} + 2h\omega_w)}{h^3\omega^3}\right)\right\}\Bigg|_p \neq \mathrm{Re}\left\{\left(\frac{\mathcal{A}}{h\omega}\right)\left(\frac{-2\mathcal{A}\omega_{\bar{w}}}{h^2\omega^3}\right)\right\}\Bigg|_p.$$
5. *f has a cuspidal cross cap at a singular point $p \in \Sigma$ if and only if*
$$\mathrm{Re}\left(\frac{\mathcal{A}}{h^2\omega^2}\right)\Bigg|_p = 0, \quad \mathrm{Im}\left(\frac{\mathcal{A}}{h^2\omega^2}\right)\Bigg|_p \neq 0$$
and
$$\mathrm{Im}\left\{\overline{\left(\frac{\mathcal{A}}{h\omega}\right)}\left(\frac{\mathcal{A}_w h\omega - \mathcal{A}(2\mathcal{A} + 2h\omega_w)}{h^3\omega^3}\right)\right\}\Bigg|_p \neq \mathrm{Im}\left\{\left(\frac{\mathcal{A}}{h\omega}\right)\left(\frac{-2\mathcal{A}\omega_{\bar{w}}}{h^2\omega^3}\right)\right\}\Bigg|_p.$$

Proof. We can use the same argument as in the proof of Theorem 7.5 of [70], which is the case of CMC $H > 1$. Then, we get the exactly the same claim as Theorem 7.5 of [70], for example:
" f has a swallowtail at a singular point $p \in \Sigma$ if and only if

$$\mathrm{Re}\left(\frac{h_w}{h^2\omega}\right)\Bigg|_p \neq 0,\ \mathrm{Im}\left(\frac{h_w}{h^2\omega}\right)\Bigg|_p = 0 \text{ and } \mathrm{Re}\left\{\overline{\left(\frac{h_w}{h}\right)}\left(\frac{h_w}{h^2\omega}\right)_w\right\}\Bigg|_p \neq \mathrm{Re}\left\{\left(\frac{h_w}{h}\right)\left(\frac{h_w}{h^2\omega}\right)_{\bar{w}}\right\}\Bigg|_p."$$

However, we also have $\mathcal{A} = -2h_w\omega$, $\mathcal{A}_{\bar{w}} = 0$ and $h_{\bar{w}}|_p = 0$, since H is constant. Applying these, we get the conclusion. ∎

4.2 Example: Smyth-type surfaces with $0 \leq H < 1$ in $\mathbb{S}^{2,1}$ and their singularities

4.2.1 Smyth surfaces

Smyth studied a generalization of Delaunay surfaces in $\mathbb{R}^3$, which are CMC surfaces with rotationally invariant metrics, in [87]. These surfaces are called *Smyth surfaces*, and there are numerous studies about them. For example, Bobenko and Its studied relationships between Smyth surfaces and Painleve III equations in [8]. The DPW method was also applied to Smyth surfaces in Riemannian spaceforms, in [8], [25]. In our previous work of [70], we constructed the analogue of Smyth surfaces in $\mathbb{R}^{2,1}$, $\mathbb{S}^{2,1}$ and $\mathbb{H}^{2,1}$, and specified the types of singularities they have. However, in [70] we omitted the case of $0 \leq H < 1$ in $\mathbb{S}^{2,1}$ because the Iwasawa splitting given here becomes quite different from the $H > 1$ case.

Here we will construct Smyth-type surfaces with $0 \leq H < 1$ in $\mathbb{S}^{2,1}$. In Equation (4.2.3), we will notice that Smyth-type surfaces with $0 \leq H < 1$ in $\mathbb{S}^{2,1}$ have an umbilic point at the origin, thus Smyth-type surfaces can be constructed by applying only Proposition 3.1.1, not by Proposition 3.1.2. Hence, Smyth-type surfaces with $0 \leq H < 1$ are good examples of applying the DPW method introduced in this paper. We also identify the types of singularities on Smyth-type surfaces, using the criteria in Section 4.1.

4.2.2 Reflective symmetry of Smyth-type surfaces

Define

$$\xi = \lambda^{-1} \begin{pmatrix} 0 & 1 \\ cw^k & 0 \end{pmatrix} dw, \quad c \in \mathbb{C}, \quad w \in \Sigma = \mathbb{C}, \tag{4.2.1}$$

and take a solution φ such that $d\varphi = \varphi\xi$ and $\varphi_{w=0} = I$. Now we can assume $c \in \mathbb{R}_{>0}$ using a reparametrization of w and a rigid motion of f, as in [17].

The following Proposition 4.2.1 is proven in the same way as Theorem 8.2 in [70].

Proposition 4.2.1. *The surfaces $f : \varphi^{-1}(\mathcal{B}_\tau) \longrightarrow \mathbb{S}^{2,1}$, produced via the DPW method, from ξ in* (4.2.1)*, with $\varphi|_{w=0} = I$ and $F_0 = F|_{\lambda = e^{-\frac{q}{2}}}$ for $q < 0$, has reflective symmetry with respect to $k+2$ geodesic planes that meet equiangularly along a geodesic line.*

4.2.3 The Gauss equation of Smyth-type surfaces

Here we assume the mean curvature is $\mathcal{H} = \frac{i}{2}$ for simplicity, and then we show that the metric of Smyth-type surfaces is rotationally invariant.

Theorem 4.2.1. *The Gauss equation* (3.1.1) *for a surface in $\mathbb{S}^{2,1}$ generated by ξ in* (4.2.1)*, with $\varphi|_{w=0} = I$, is equivalent to a special case of the Painleve III equations, and the metric function u is rotationally invariant.*

Proof. When $\mathcal{H} = \frac{i}{2}$, the Gauss equation is of the following form:

$$4u_{w\bar{w}} + e^{2u} + |Q|^2 e^{-2u} = 0. \tag{4.2.2}$$

By the proof of Theorem 3.3.1, we have

$$Q = -2\mathcal{H}\frac{b_{-1}}{a_{-1}} = -icw^k \ (i.e.\ \mathcal{A} = cw^k). \tag{4.2.3}$$

Set $v := u - \frac{1}{2}\log|Q|$, and (4.2.2) is equivalent to

$$4v_{w\bar{w}} + 2|c| \cdot |w|^k \cosh(2v) = 0. \tag{4.2.4}$$

Using $v_{w\bar{w}} = \frac{1}{4}\partial_r^2 v + \frac{1}{4r}\partial_r v$ for $r := |w|$, (4.2.4) becomes

$$\partial_r^2 v + \frac{1}{r}\partial_r v + 2|c| \cdot r^k \cosh(2v) = 0. \tag{4.2.5}$$

Next we set $x := \frac{1}{1+\frac{k}{2}} r^{1+\frac{k}{2}} \sqrt{|c|}$, and (4.2.5) is equivalent to

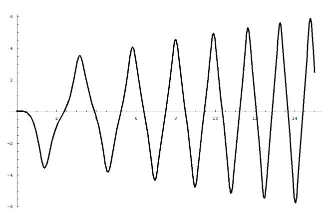
$$\partial_x^2 v + \frac{1}{x}\partial_x v + 2\cosh(2v) = 0. \tag{4.2.6}$$

(4.2.6) is a special case of the Painleve III equation that is $y_{xx} = \frac{1}{y}(y_x)^2 - \frac{1}{x}y_x - \frac{1}{x}(\alpha y^2 + \beta) + \gamma y^3 + \frac{\delta}{y}$, for $y = e^v$, $\alpha = \beta = 0$, $\gamma = \delta = -1$. $\square$

Using polar coordinates $w = re^{i\theta}$ and setting $g := e^u$, the Gauss equation and the suitable choice of the initial conditions are as follows:

$$g(g_{rr} + \frac{g_r}{r}) - g_r^2 + c^2 r^{2k} - 4\mathcal{H}^2 g^4 = 0, \quad g|_{r=0} = 1, \quad g_r|_{r=0} = 0. \tag{4.2.7}$$

This solution g depends only on r, and some examples of g are seen in Figure 4.1. The singular set $S(f) := \{(r,\theta) \in \mathbb{C} | g(r) = 0\}$ corresponding to this data of g is seen in the right-side of Figure 4.1. As in Figure 4.2, Smyth-type surfaces with $\mathcal{H} = \frac{i}{2}$ in $\mathbb{S}^{2,1}$ arrive at the singular set S(f) repeatedly before they diverge to infinity. This phenomenon does not occur in the case of $H > 1$. (See [70].)



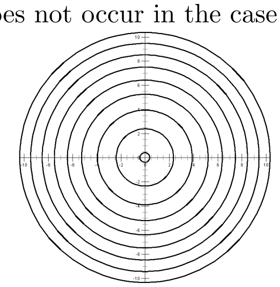


Fig. 4.1: The left image is a solution g of Equation (4.2.7), and the right image is the corresponding singular set.

4.2.4 The types of singularities on Smyth-type surfaces

By numerical calculation, we know that these Smyth surfaces have cuspidal edges, swallowtails and cuspidal cross caps, using the criteria as in Section 4.1, see Figure 4.3.

Fact 4.2.1. There exist Smyth-type surfaces in $\mathbb{S}^{2,1}$ which have cuspidal edges, swallowtails and cuspidal cross caps. (See Figures 4.3.)

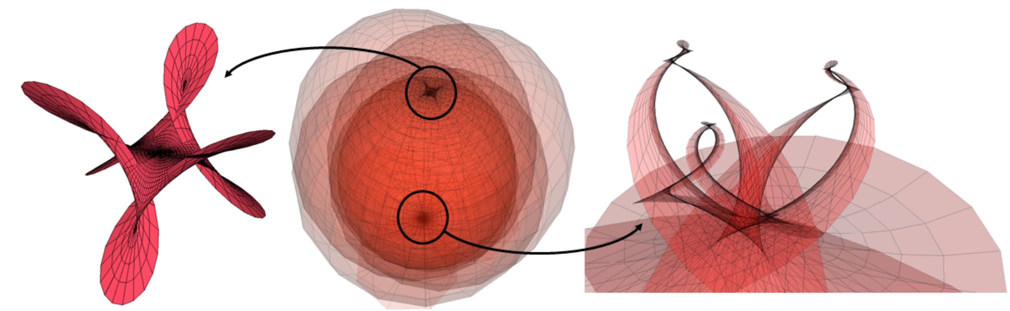


Fig. 4.2: The middle image is a 3-legged Smyth-type surface with $\mathcal{H} = \frac{i}{2}$, and the left image is part of the middle one, from the origin to the first singular set. The right image is a part of the middle one, near second singular set (using the hollow ball model as in [29]).

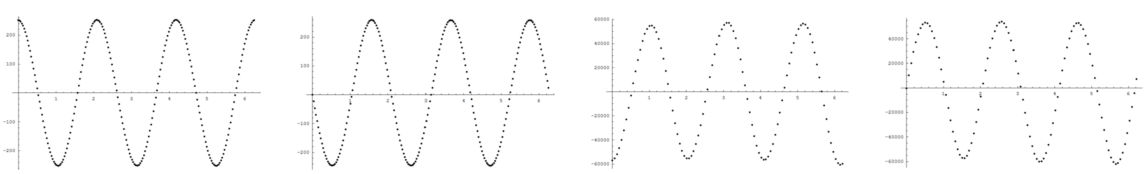


Fig. 4.3: The values of $\mathrm{Re}\left(\frac{\mathcal{A}}{h^2\omega^2}\right)\Big|_p$, $\mathrm{Im}\left(\frac{\mathcal{A}}{h^2\omega^2}\right)\Big|_p$, $\mathrm{Re}\left\{\overline{\left(\frac{\mathcal{A}}{h\omega}\right)}\left(\frac{\mathcal{A}_w h\omega - \mathcal{A}(2\mathcal{A}+2h\omega_w)}{h^3\omega^3}\right)\right\}\Big|_p$ $-\mathrm{Re}\left\{\left(\frac{\mathcal{A}}{h\omega}\right)\left(\frac{-2\mathcal{A}\omega_{\bar{w}}}{h^2\omega^3}\right)\right\}\Big|_p$ and $\mathrm{Im}\left\{\overline{\left(\frac{\mathcal{A}}{h\omega}\right)}\left(\frac{\mathcal{A}_w h\omega - \mathcal{A}(2\mathcal{A}+2h\omega_w)}{h^3\omega^3}\right)\right\}\Big|_p - \mathrm{Im}\left\{\left(\frac{\mathcal{A}}{h\omega}\right)\left(\frac{-2\mathcal{A}\omega_{\bar{w}}}{h^2\omega^3}\right)\right\}\Big|_p$ for a 3-legged Smyth-type surface with $\mathcal{H} = \frac{i}{2}$ in $\mathbb{S}^{2,1}$ at (r_0, θ) such that $g(r_0) = 0$ and $0 \leq \theta \leq 2\pi$ (left to right).

Here we show, for the surfaces in Fact 4.2.1, that there are at least $2(k+2)$-swallowtails, without relying on numerical calculation, and using only geometric properties. Before doing that, we have a lemma.

Lemma 4.2.1. *Let $\tilde{\mathfrak{F}} = \tilde{\mathfrak{F}}(w, \bar{w})$ be the solution of the adjusted Lax pair* (4.1.2) *with $\tilde{\mathfrak{F}}_{w=0} = I$ for the case of a Smyth-type surface. Then $\tilde{\mathfrak{F}}(w) = \sigma_3 \overline{\tilde{\mathfrak{F}}(\bar{w})} \sigma_3$.*

Proof. By direct computation, we have $\tilde{A}(w) = -\tilde{B}(\bar{w})^t$. By this equation and $\tilde{\mathfrak{F}}_{w=0} = I$, we get the conclusion. □

Corollary 4.2.1.
(1) $h(w) = -\overline{h(\bar{w})}$, $\omega(w) = \overline{\omega(\bar{w})}$.
(2) *At $(r, \theta) = (r_0, 0)$ for $g(r_0) = 0$, we have $h(r_0, 0) = \pm i$, $\omega(r_0, 0) \in \mathbb{R} \setminus \{0\}$, $\omega_w(r_0, 0) = \overline{\omega_{\bar{w}}(r_0, 0)}$.*

Proposition 4.2.2. *Let $f(w) = f(r, \theta)$ be a $(k+2)$-legged Smyth-type surface in $\mathbb{S}^{2,1}$, and let r_0 satisfy $g(r_0) = 0$. Then f has a swallowtail at $(r_0, 0)$.*

Proof. We will use the criteria of Theorem 4.1.3, and by the data of the above Corollary 4.2.1 we can get the conclusion. □

Similarly, we have the same conclusion when $\theta = \frac{\pi}{k+2}$.

Proposition 4.2.3. *Let $f(w) = f(r, \theta)$ be a $(k+2)$-legged Smyth-type surface in $\mathbb{S}^{2,1}$, and let r_0 satisfy $g(r_0) = 0$. Then f has a swallowtail at $(r_0, \frac{\pi}{k+2})$.*

By the above two propositions and the reflective symmetry, we get the following main result:

Theorem 4.2.2. *If a (k+2)-legged Smyth-type surface in $\mathbb{S}^{2,1}$ has singularities, then it has at least $2(k+2)$ swallowtails.*

Remark 4.2.1. We have checked numerically that there are cuspidal cross caps along the cuspidal edges between each adjacent pair of swallowtails. Thus the surface as in Theorem 4.2.2 will also have at least $2(k+2)$ cuspidal cross caps.

Chapter 5

Gauss maps of cuspidal edges in $\mathbb{H}^3$

5.1 Local differential geometry of cuspidal edges in $\mathbb{H}^3$

Let $\mathbb{R}^{3,1}$ denote the space $\mathbb{R}^4$ of 4-tuples the real numbers, equipped with the signature $(-,+,+,+)$ and symmetric bilinear form $\langle , \rangle$ defined by

$$\langle (x_0, x_1, x_2, x_3), (y_0, y_1, y_2, y_3) \rangle = -x_0 y_0 + x_1 y_1 + x_2 y_2 + x_3 y_3$$

for any $\boldsymbol{x} = (x_0, x_1, x_2, x_3), \boldsymbol{y} = (y_0, y_1, y_2, y_3)$. We call this space $\mathbb{R}^{3,1}$ the *Lorentz–Minkowski* 4-*space* or briefly the *Minkowski* 4-*space*.

For $\boldsymbol{x} \in \mathbb{R}^{3,1} \setminus \{\boldsymbol{0}\}$, there are three kinds of vector called *spacelike, lightlike* or *timelike* and defined by $\langle \boldsymbol{x}, \boldsymbol{x} \rangle > 0, = 0$ or < 0 respectively. The norm of $\boldsymbol{x} \in \mathbb{R}^{3,1}$ is defined by $||\boldsymbol{x}|| = \sqrt{|\langle \boldsymbol{x}, \boldsymbol{x} \rangle|}$. Especially, for a spacelike vector $\boldsymbol{x}$, the norm of $\boldsymbol{x}$ is $||\boldsymbol{x}|| = \sqrt{\langle \boldsymbol{x}, \boldsymbol{x} \rangle}$. We now define the *pseudo wedge product* $\boldsymbol{a}_1 \wedge \boldsymbol{a}_2 \wedge \boldsymbol{a}_3$ as follows:

$$\boldsymbol{a}_1 \wedge \boldsymbol{a}_2 \wedge \boldsymbol{a}_3 = \begin{vmatrix} -\boldsymbol{e}_0 & \boldsymbol{e}_1 & \boldsymbol{e}_2 & \boldsymbol{e}_3 \\ a_0^1 & a_1^1 & a_2^1 & a_3^1 \\ a_0^2 & a_1^2 & a_2^2 & a_3^2 \\ a_0^3 & a_1^3 & a_2^3 & a_3^3 \end{vmatrix},$$

where $\{\boldsymbol{e}_0, \boldsymbol{e}_1, \boldsymbol{e}_2, \boldsymbol{e}_3\}$ is the canonical basis of $\mathbb{R}^{3,1}$, $\boldsymbol{a}_i = (a_0^i, a_1^i, a_2^i, a_3^i) \in \mathbb{R}^{3,1}$ $(i = 1, 2, 3)$. We can easily check that

$$\langle \boldsymbol{a}, \boldsymbol{a}_1 \wedge \boldsymbol{a}_2 \wedge \boldsymbol{a}_3 \rangle = \det(\boldsymbol{a}, \boldsymbol{a}_1, \boldsymbol{a}_2, \boldsymbol{a}_3)$$

holds. Hence $\boldsymbol{a}_1 \wedge \boldsymbol{a}_2 \wedge \boldsymbol{a}_3$ is pseudo orthogonal to $\boldsymbol{a}_i, i = 1, 2, 3$.

There are three kinds of pseudo-spheres in $\mathbb{R}^{3,1}$: the *hyperbolic* 3-*space* $\mathbb{H}^3$ is defined by

$$\mathbb{H}^3 = \{\boldsymbol{x} \in \mathbb{R}^{3,1} | \langle \boldsymbol{x}, \boldsymbol{x} \rangle = -1, x_0 > 0\},$$

the *de Sitter* 3-*space* $\mathbb{S}^{2,1}$ is defined by

$$\mathbb{S}^{2,1} = \{\boldsymbol{x} \in \mathbb{R}^{3,1} | \langle \boldsymbol{x}, \boldsymbol{x} \rangle = 1\}$$

and *open lightcone* LC^* is defined by

$$LC^* = \{\boldsymbol{x} \in \mathbb{R}^{3,1} \setminus \{\boldsymbol{0}\} | \langle \boldsymbol{x}, \boldsymbol{x} \rangle = 0\}.$$

We recall that wave fronts in $\mathbb{H}^3$. Let $f : U \to \mathbb{H}^3 \subset \mathbb{R}^{3,1}$ be a smooth map, where $U \subset \mathbb{R}^2$ is a simply-connected domain with local coordinates u, v. We call f a *wave front* (or *front*, for short) if there exists a unit vector field ν along f such that the following conditions hold:

(a) $\langle df(X_p), \nu(p)\rangle = 0$, for any $X_p \in T_pU,\ p \in U$, and

(b) the pair $L_f = (f, \nu) : U \to T_1\mathbb{H}^3 = \{(\boldsymbol{v}, \boldsymbol{w}) \in \mathbb{H}^3 \times \mathbb{S}^{2,1} | \langle \boldsymbol{v}, \boldsymbol{w}\rangle = 0\}$ is an immersion, where $T_1\mathbb{H}^3$ is the unit tangent bundle over $\mathbb{H}^3$ equipped with the canonical contact structure.

Here, we call this vector ν a *unit pseudo normal vector* of f and L_f a *Legendrian lift* (cf. [3], see also [57, 83]). A map f is called a *frontal* if (a) of the above condition is satisfied. A front f might have singularities. Arnol'd and Zakalyukin showed that the generic singularities of fronts in $\mathbb{R}^3$ are cuspidal edges and swallowtails (for example, see [3]). A cuspidal edge is a map-germ $f : (\mathbb{R}^2, \mathbf{0}) \to (\mathbb{R}^3, \mathbf{0})$ $\mathcal{A}$-equivalent to the germ $(u, v) \mapsto (u, v^2, v^3)$ and a swallowtail is a map-germ $f : (\mathbb{R}^2, \mathbf{0}) \to (\mathbb{R}^3, \mathbf{0})$ $\mathcal{A}$-equivalent to the germ $(u, v) \mapsto (u, 4v^3 + 2uv, 3v^4 + uv^2)$ at the origin, where two map-germs $f, g : (\mathbb{R}^2, \mathbf{0}) \to (\mathbb{R}^3, \mathbf{0})$ are $\mathcal{A}$-equivalent if there exist diffeomorphism-germs $\Xi_s : (\mathbb{R}^2, \mathbf{0}) \to (\mathbb{R}^2, \mathbf{0})$ on the source and $\Xi_t : (\mathbb{R}^3, \mathbf{0}) \to (\mathbb{R}^3, \mathbf{0})$ on the target such that $f \circ \Xi_s = \Xi_t \circ g$ holds (see Fig. 5.1).

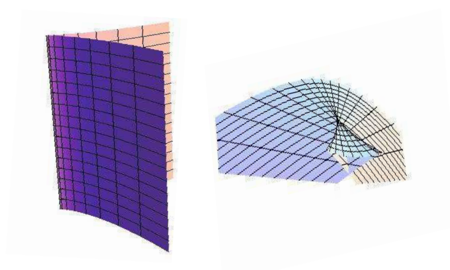


Fig. 5.1: Cuspidal edge (left) and swallowtail (right).

For a front f, we define a function called the *signed area density function* λ as follows: $\lambda = \det(f_u, f_v, \nu, f)$, where $f_u = \partial f/\partial u$ and $f_v = \partial f/\partial v$ respectively. We denote by $S(f)$ the singular set of f. By definition of the signed area density function, the relation $S(f) = \lambda^{-1}(0)$ holds. For a singular point $p \in S(f)$, we say that p is *non-degenerate* if the condition $d\lambda(p) \neq 0$ holds. Let $p \in S(f)$ be a non-degenerate singular point of f. Then, by the implicit function theorem, there exists a regular curve $\gamma : (-\varepsilon, \varepsilon) \to U$ $(\varepsilon > 0)$ with $\gamma(0) = p$ such that γ locally parametrizes $S(f)$. Since non-degenerate singular points are corank 1 singular points, there exists a vector field η on $S(f)$ such that $df(\eta) = \mathbf{0}$ holds. We call such a vector field the *null vector field*. Under the above situation, the following criteria are known.

Theorem 5.1.1 ([57, Propositin 1.3]). *Let $f : U \to \mathbb{H}^3$ be a front and $p \in U$ be a non-degenerate singular point of f. Then*

(1) f at p is $\mathcal{A}$-equivalent to a cuspidal edge if and only if $\eta\lambda(p) \neq 0$ holds.

(2) f at p is $\mathcal{A}$-equivalent to a swallowtail if and only if $\eta\lambda(p) = 0$ and $\eta\eta\lambda(p) \neq 0$ hold.

We consider (extrinsic) differential geometric properties of cuspidal edges in $\mathbb{H}^3$. Let $f : U \to \mathbb{H}^3$ be a front and $p \in U$ a cuspidal edge. In this case, we can take a special local coordinate system called the *adapted coordinate system* (see [63, 64], for example).

Definition 5.1.1. A coordinate system $(U; u, v)$ is called *adapted* if the following conditions hold:

1) the u-axis is the singular curve,

2) $\eta = \partial_v$ gives a null vector field along the u-axis, and

3) there are no singular points other than the u-axis.

We use this coordinate system in Sections 2, 3 and 4. With this coordinate, $df(\eta) = f_v = \mathbf{0}$ and $f_{vv} \neq \mathbf{0}$ hold along the u-axis. Thus there exists a smooth map $\varphi : U \to \mathbb{R}^{3,1} \setminus \{\mathbf{0}\}$ such that $f_v = v\varphi$ holds. We define a map

$$\nu = \frac{f_u \wedge \varphi \wedge f}{||f_u \wedge \varphi \wedge f||} : U \to \mathbb{S}^{2,1}.$$

From the definition of ν, we have

$$\langle f_u, \nu\rangle = \langle \varphi, \nu\rangle = \langle f, \nu\rangle = 0, \quad \langle \nu, \nu\rangle = 1.$$

We call this map ν the *de Sitter Gauss map image* or the *de Sitter Gauss image* of f.

The signed area density of f is given by

$$\lambda = \det(f_u, f_v, \nu, f) = v \det(f_u, \varphi, \nu, f) = v\tilde{\lambda}.$$

Since $\eta = \partial_v$, a point p is a cuspidal edge of f if and only if $\eta\lambda = \tilde{\lambda} \neq 0$ holds along the u-axis. Thus f, f_u, φ and ν are linearly independent.

We define the following functions:

$$\hat{E} = \langle f_u, f_u\rangle, \qquad \hat{F} = \langle f_u, \varphi\rangle, \qquad \hat{G} = \langle \varphi, \varphi\rangle, \tag{5.1.1}$$

$$\hat{L} = -\langle f_u, \nu_u\rangle, \qquad \hat{M} = -\langle \varphi, \nu_u\rangle, \qquad \hat{N} = -\langle \varphi, \nu_v\rangle. \tag{5.1.2}$$

We note that $\hat{E}\hat{G} - \hat{F}^2 \neq 0$ near p and $-\langle f_u, \nu_v\rangle = v\hat{M}$ holds. Using these functions, we have the following.

Lemma 5.1.1. *The differentials ν_u and ν_v can be written as*

$$\nu_u = \frac{\hat{F}\hat{M} - \hat{G}\hat{L}}{\hat{E}\hat{G} - \hat{F}^2} f_u + \frac{\hat{F}\hat{L} - \hat{E}\hat{M}}{\hat{E}\hat{G} - \hat{F}^2}\varphi, \;\; \nu_v = \frac{\hat{F}\hat{N} - v\hat{G}\hat{M}}{\hat{E}\hat{G} - \hat{F}^2} f_u + \frac{v\hat{F}\hat{M} - \hat{E}\hat{N}}{\hat{E}\hat{G} - \hat{F}^2}\varphi.$$

We set

$$\psi(t) = \det(\hat{\gamma}, \hat{\gamma}', D^f_\eta(\nu \circ \gamma), \nu \circ \gamma)(t),$$

where $\hat{\gamma} = f \circ \gamma$, D^f is the canonical covariant derivative along a map f induced from the Levi-Civita connection on $\mathbb{H}^3$ and $' = d/dt$. We note that $\psi(0) \neq 0$ if and only if (f, ν) is a Legendre immersion at p, that is, f is a front at p when $\hat{\gamma}'(0) \neq 0$ (see [32, 53]). Taking an adapted coordinate system $(U; u, v)$ around p, $D^f_\eta \nu = \nu_v$ holds. Thus we have

$$\begin{aligned}\psi(u) &= \det(f(u,0), f_u(u,0), \nu_v(u,0), \nu(u,0)) \\ &= -\frac{\hat{E}\hat{N}}{\hat{E}\hat{G} - \hat{F}^2} \det(f(u,0), f_u(u,0), \varphi(u,0), \nu(u,0))\end{aligned}$$

by Lemma 5.1.1, and we see that $\hat{N}$ does not vanish along the u-axis.

Remark 5.1.1. Let $f : U \to (M^3, g)$ be a front with non-degenerate singular points, where (M^3, g) is an oriented 3-dimensional Riemannian manifold. We set

$$f_\eta = df(\eta), \quad f_{\eta\eta} = \nabla_\eta f_\eta, \quad f_{\eta\eta\eta} = \nabla_\eta f_{\eta\eta},$$

where ∇ is the Levi-Civita connection of (M^3, g). In [64], a differential geometric invariant κ_c called the *cuspidal curvature* is defined by

$$\kappa_c(t) = \frac{|\hat{\gamma}'(t)|^{3/2} \det_g(\hat{\gamma}'(t), f_{\eta\eta}(\gamma(t)), f_{\eta\eta\eta}(\gamma(t)))}{|\hat{\gamma}'(t) \times_g f_{\eta\eta}(\gamma(t))|^{5/2}}$$

along the singular curve γ, where $\hat{\gamma} = f \circ \gamma$, $\det_g$ is the Riemannian volume element of (M^3, g) and $\langle \boldsymbol{a} \times_g \boldsymbol{b}, \boldsymbol{c} \rangle = \det_g(\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c})$ for each $\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c} \in T_qM^3$ ($q \in M^3$). If $M = \mathbb{R}^3$, then $\det_g$ can be identified with the usual determinant. By [83, Corollary 3.5], f at $p \in S(f)$ is a cuspidal edge if and only if $\kappa_c(p) \neq 0$ holds. For details about the cuspidal curvature κ_c, see [64].

Here the Gauss-Kronecker curvature function K_{ext} and the mean curvature function H of f are given by

$$K_{\text{ext}} = \frac{\hat{L}\hat{N} - v\hat{M}^2}{v(\hat{E}\hat{G} - \hat{F}^2)}, \quad H = \frac{\hat{E}\hat{N} - 2v\hat{F}\hat{M} + v\hat{G}\hat{L}}{2v(\hat{E}\hat{G} - \hat{F}^2)}. \tag{5.1.3}$$

We define the matrix

$$S^d = -\begin{pmatrix} \hat{E} & \hat{F} \\ \hat{F} & \hat{G} \end{pmatrix}^{-1} \begin{pmatrix} \hat{L} & v\hat{M} \\ \hat{M} & \hat{N} \end{pmatrix},$$

called the *modified de Sitter shape operator* of f. Then the relation

$$\det S^d = vK_{\text{ext}} = \hat{K}_{\text{ext}}$$

holds. By definition of K_{ext} given by (5.1.3), $\hat{K}_{\text{ext}}$ is C^∞-function of u, v.

We consider the principal curvatures for cuspidal edges. We now define two functions as follows:

$$\kappa_1 = \frac{A + B}{2v(\hat{E}\hat{G} - \hat{F}^2)}, \quad \kappa_2 = \frac{A - B}{2v(\hat{E}\hat{G} - \hat{F}^2)}. \tag{5.1.4}$$

Here $A = \hat{E}\hat{N} - 2v\hat{F}\hat{M} + v\hat{G}\hat{L}$, $B = \sqrt{A^2 - 4v(\hat{E}\hat{G} - \hat{F}^2)(\hat{L}\hat{N} - v\hat{M}^2)}$. By definitions of κ_1 and κ_2, we have $K_{\text{ext}} = \kappa_1\kappa_2$ and $2H = \kappa_1 + \kappa_2$. Thus we may regard κ_1 and κ_2 as principal curvatures of f. However, we note that one of κ_i ($i = 1, 2$) may not be well-defined along the singular curve. Two functions κ_1 and κ_2 in (5.1.4) can be rewritten as

$$\kappa_1 = \frac{2(\hat{L}\hat{N} - v\hat{M}^2)}{A - B}, \quad \kappa_2 = \frac{2(\hat{L}\hat{N} - v\hat{M}^2)}{A + B}. \tag{5.1.5}$$

Here, $A = \hat{E}\hat{N}$ and $B = \hat{E}|\hat{N}|$ hold on the singular set $\{v = 0\}$. Thus $A + B \neq 0$ (resp. $A - B \neq 0$) holds on $\{v = 0\}$ if $\hat{N} > 0$ (resp. $\hat{N} < 0$). By the above arguments, if $\hat{N}$ is positive (resp. negative) along the singular curve, κ_2 (resp. κ_1) is well-defined along the singular curve. So we can regard κ_2 as the *principal curvature* for the cuspidal edge if and only if $\hat{N}(u, 0)$ is a non-zero positive C^∞-function of u along the u-axis. We note that Murata and Umehara [67] introduced the notion of *principal curvature maps* for fronts in $\mathbb{R}^3$.

Let us consider the principal direction $\boldsymbol{v} = (\xi, \zeta)$ with respect to the principal curvature κ_2. In this case, $\boldsymbol{v}$ satisfies the following equation:

$$\begin{pmatrix} \hat{L} & v\hat{M} \\ v\hat{M} & v\hat{N} \end{pmatrix} \begin{pmatrix} \xi \\ \zeta \end{pmatrix} = \kappa_2 \begin{pmatrix} \hat{E} & v\hat{F} \\ v\hat{F} & v^2\hat{G} \end{pmatrix} \begin{pmatrix} \xi \\ \zeta \end{pmatrix}.$$

Thus the principal direction $\boldsymbol{v} = (\xi, \zeta)$ can be taken as

$$\boldsymbol{v} = (\hat{N} - v\kappa_2\hat{G}, -\hat{M} + \kappa_2\hat{F}). \tag{5.1.6}$$

Since $\hat{N} \neq 0$ holds on $\{v = 0\}$, the principal direction $\boldsymbol{v}$ is also well-defined along the u-axis.

Using the principal curvature κ_2 and the principal direction $\boldsymbol{v}$ corresponding to κ_2, we define a notion of ridge point.

Definition 5.1.2. Let $f : U \to \mathbb{H}^3$ be a cuspidal edge, κ_2 the principal curvatures of f and $\boldsymbol{v}$ the principal directions with respect to κ_2. The point $f(p)$ is called a *ridge point* relative to $\boldsymbol{v}$ if $\boldsymbol{v}\kappa_2(p) = 0$, where $\boldsymbol{v}\kappa_2$ is the directional derivative of κ_2 in direction $\boldsymbol{v}$. Moreover, $f(p)$ is called a *k-th order ridge point* relative to $\boldsymbol{v}$ if $\boldsymbol{v}^{(m)}\kappa_2(p) = 0$ $(1 \leq m \leq k)$ and $\boldsymbol{v}^{(k+1)}\kappa_2(p) \neq 0$, where $\boldsymbol{v}^{(m)}\kappa_2$ is the directional derivative of κ_2 with respect to $\boldsymbol{v}$ applied m times.

Properties of ridge points for regular surfaces in $\mathbb{R}^3$ were first studied by Porteous to investigate caustics of them. For details, see [77, 78].

5.2 Singularities of the de Sitter Gauss map image

In this section, we consider the de Sitter Gauss image $\nu : U \to \mathbb{S}^{2,1}$ of f. From arguments in Section 2, the pair $L = (f, \nu)$ gives a Legendrian immersion. Thus ν can be regarded as a (wave) front in $\mathbb{S}^{2,1}$ with unit normal vector f.

We consider the signed area density function λ^ν of ν. Using ν, ν_u, ν_v and f, λ^ν is given by

$$\lambda^\nu = \det(\nu_u, \nu_v, \nu, f). \tag{5.2.1}$$

By Lemma 5.1.1 and (5.1.3), it can be written as

$$\lambda^\nu = vK_{\text{ext}} \det(f_u, \varphi, f, \nu). \tag{5.2.2}$$

We note that $\det(f_u, \varphi, f, \nu)$ is a non-zero function. By definitions of Gauss-Kronecker curvature and κ_1 and κ_2, we have

$$vK_{\text{ext}} = (v\kappa_1)\kappa_2. \tag{5.2.3}$$

In this case, $v\kappa_1$ is a non-zero function on U. From (5.2.1), (5.2.2) and (5.2.3), we can consider the signed area density as

$$\hat{\lambda}^\nu = \kappa_2. \tag{5.2.4}$$

Thus we have the following:

Proposition 5.2.1. *Under the above conditions, a point $p \in U$ is a singular point of the de Sitter Gauss image ν if and only if $\kappa_2(p) = 0$ holds.*

We denote by $S(\nu) = \{q \in U | \kappa_2(q) = 0\}$ the set of singular points of ν and we call a point $p \in S(\nu)$ a *parabolic point* for cuspidal edges. For $p \in S(\nu)$, a point p is a non-degenerate singular point of ν if and only if $(\kappa_2)_u(p) \neq 0$ or $(\kappa_2)_v(p) \neq 0$ hold, that is, p is not a critical point of κ_2.

We consider the case that $p \in S(\nu)$ is non-degenarate. In this case, there exists a vector field η^ν on $S(\nu)$ such that $d\nu(\eta^\nu) = \mathbf{0}$. We will find a concrete form for η^ν. Let us take $\eta^\nu = \eta_1^\nu \partial_u + \eta_2^\nu \partial_v$. Then

$$d\nu(\eta^\nu) = \left(\frac{\hat{F}\hat{M} - \hat{G}\hat{L}}{\hat{E}\hat{G} - \hat{F}^2} f_u + \frac{\hat{F}\hat{L} - \hat{E}\hat{M}}{\hat{E}\hat{G} - \hat{F}^2} \varphi \right) \eta_1^\nu + \left(\frac{\hat{F}\hat{N} - v\hat{G}\hat{M}}{\hat{E}\hat{G} - \hat{F}^2} f_u + \frac{v\hat{F}\hat{M} - \hat{E}\hat{N}}{\hat{E}\hat{G} - \hat{F}^2} \varphi \right) \eta_2^\nu = \mathbf{0} \quad (5.2.5)$$

holds on $S(\nu)$ by Lemma 5.1.1. Since f_u and φ are linearly independent, this equation is equivalent to the following:

$$\begin{pmatrix} \hat{E} & \hat{F} \\ \hat{F} & \hat{G} \end{pmatrix}^{-1} \begin{pmatrix} \hat{L} & v\hat{M} \\ \hat{M} & \hat{N} \end{pmatrix} \begin{pmatrix} \eta_1^\nu \\ \eta_2^\nu \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad (5.2.6)$$

holds on $S(\nu)$. Now $\hat{N} \neq 0$, so we can take η^ν as

$$\eta^\nu = \hat{N}\partial_u - \hat{M}\partial_v \quad (5.2.7)$$

on $S(\nu)$. Moreover, η^ν can be extended on U by the form

$$\eta^\nu = (\hat{N} - v\kappa_2\hat{G})\partial_u + (-\hat{M} + \kappa_2\hat{F})\partial_v = \boldsymbol{v},$$

where $\boldsymbol{v}$ is the principal direction with respect to κ_2. Thus we can regard the principal direction $\boldsymbol{v}$ as a null vector field η^ν.

Using Theorem 5.1.1, the signed area density $\hat{\lambda}^\nu$ and null vector field η^ν, we obtain conditions for singularities of ν.

Proposition 5.2.2. *Let $f : U \to \mathbb{H}^3$ be a cuspidal edge and $\nu : U \to \mathbb{S}^{2,1}$ a de Sitter Gauss image of f. Suppose that $p \in S(\nu)$ is a non-degenerate singular point of ν. Then the following assertions hold.*

(1) ν at p is a cuspidal edge if and only if $\boldsymbol{v}\kappa_2(p) \neq 0$ holds.

(2) ν at p is a swallowtail if and only if $\boldsymbol{v}\kappa_2(p) = 0$ and $\boldsymbol{v}^{(2)}\kappa_2(p) \neq 0$ hold.

Combining the results obtained in Sections 2 and 3, we have relations between singularities of ν and the differential geometric properties of f.

Theorem 5.2.1. *Let $f : U \to \mathbb{H}^3$ be a cuspidal edge and $\nu : U \to \mathbb{S}^{2,1}$ a de Sitter Gauss image of f. Then the following properties hold.*

(1) A point $p \in U$ is a singular point of ν if and only if $\kappa_2(p) = 0$ holds.

(2) *A point $p \in S(\nu)$ is non-degenerate if and only if p is not a critical point of κ_2.*

(3) *For a non-degenerate singular point $p \in S(\nu)$, ν at p is a cuspidal edge if and only if p is not a ridge point of f.*

(4) *For a non-degenerate singular point $p \in S(\nu)$, ν at p is a swallowtail if and only if p is a first order ridge point of f.*

In [84], duality between A_{k+1}-inflection point ($k \leq n$) of immersed C^∞-hypersurfaces $f : M^n \to P(\mathbb{R}^{n+2})$ and A_k-singularity of dual front $g : M^n \to P((\mathbb{R}^{n+2})^*)$ were shown, where M^n is a C^∞-manifold of dimension n and A_2-singularity and A_3-singularity correspond to a cuspidal edge and a swallowtail for fronts respectively. We also remark that cusp singularities of Gauss maps of regular surfaces in $\boldsymbol{R}^3$ are related to parabolic points and ridge points for surfaces (see [4, Theorem 3.1]).

5.3 Normal form of cuspidal edges in $\mathbb{H}^3$

In this section, we consider the normal form of cuspidal edges in $\mathbb{H}^3$. For cuspidal edges in $\mathbb{R}^3$, the following normal form obtained by Martins and Saji in [63] is known.

Proposition 5.3.1 ([63]). *Let $f : (\mathbb{R}^2, \mathbf{0}) \to (\mathbb{R}^3, \mathbf{0})$ be a smooth map-germ and $\mathbf{0}$ a cuspidal edge. Then there exist a diffeomorphism-germ $\psi : (\mathbb{R}^2, \mathbf{0}) \to (\mathbb{R}^2, \mathbf{0})$ on the source and an isometry-germ $\Psi : (\mathbb{R}^3, \mathbf{0}) \to (\mathbb{R}^3, \mathbf{0})$ on the target such that*

$$\begin{aligned}&\Psi \circ f \circ \psi(u,v)\\ &\qquad = \left(u, \frac{a_{20}}{2}u^2 + \frac{a_{30}}{6}u^3 + \frac{v^2}{2}, \frac{b_{20}}{2}u^2 + \frac{b_{30}}{6}u^3 + \frac{b_{12}}{2}uv^2 + \frac{b_{03}}{6}v^3\right) + h(u,v) \qquad (5.3.1)\end{aligned}$$

with $b_{20} \geq 0$ and $b_{03} \neq 0$, where

$$h(u,v) = (0, u^4h_1(u), u^4h_2(u) + u^2v^2h_3(u) + uv^3h_4(u) + v^4h_5(u,v)),$$

with $h_i(u)$ $(1 \leq i \leq 4)$, $h_5(u,v)$ smooth functions.

See [63] for detailed descriptions and geometric properties of the coefficients in (5.3.1). We extend (5.3.1) to the case of $\mathbb{H}^3$ by analogy to the *hyperbolic-Monge form* (or the *H-Monge form*, for short) for regular surfaces which is introduced by Izumiya, Pei and Sano (see [50, Section 8]). Let $f : U \to \mathbb{H}^3$ be a cuspidal edge. Then we have H-Monge form $f = (f_0, f_1, f_2, f_3)$ for surfaces with cuspidal edge as follows:

$$\begin{aligned}
f_0 &= \sqrt{1 + f_1^2 + f_2^2 + f_3^2},\\
f_1 &= \frac{b_{20}}{2}u^2 + \frac{b_{30}}{6}u^3 + \frac{b_{12}}{2}uv^2 + \frac{b_{03}}{6}v^3\\
&\qquad + u^4h_2(u) + u^2v^2h_3(u) + uv^3h_4(u) + v^4h_5(u,v),\\
f_2 &= u,\\
f_3 &= \frac{a_{20}}{2}u^2 + \frac{a_{30}}{6}u^3 + \frac{v^2}{2} + u^4h_1(u).
\end{aligned}$$

Using H-Monge form of cuspidal edge, we consider the conditions of ridge for cuspidal edge in terms of the coefficients. Let us assume that $b_{03} > 0$ holds. Here f, f_u and φ are

$$f = (1,0,0,0),\ f_u = (0,0,1,0) \text{ and } \varphi = (0,0,0,1)$$

at the origin. We can take de Sitter Gauss image ν of f with $\nu = (0,1,0,0)$ at the origin. In this case, the coefficients of the first and the second fundamental forms are $\hat{E} = 1, \hat{F} = 0, \tilde{G} = 1, \hat{L} = b_{20}, \hat{M} = b_{12}, \hat{N} = b_{03}/2$ at the origin. Moreover, the principal curvature satisfies $\kappa_2 = \hat{L}/\hat{E} = b_{20}, (\kappa_2)_u = b_{30} - a_{20}b_{12}$ and $(\kappa_2)_v = -(4b_{12}^2 + a_{20}b_{03}^2)/2b_{03}$ at the origin. Thus we have the following lemma.

Lemma 5.3.1. *Let $f : U \to \mathbb{H}^3$ be a H-Monge form of cuspidal edge. Then $\kappa_2(\mathbf{0}) = 0$ if and only if $b_{20} = 0$. Moreover the origin $\mathbf{0}$ is not critical point of κ_2 if and only if $b_{30} - a_{20}b_{12} \neq 0$ or $4b_{12}^2 + a_{20}b_{03}^2 \neq 0$ hold.*

We assume that $b_{20} = 0$ holds in what follows.

Lemma 5.3.2. *Let $f : U \to \mathbb{H}^3$ be the H-Monge form of a cuspidal edge, κ_2 the principal curvature and $\boldsymbol{v}$ the principal direction corresponding to κ_2. Then the following assertions hold.*

(1) The origin is not a ridge point if and only if $4b_{12}^3 + b_{30}b_{03}^2 \neq 0$ holds.

(2) The origin is a first order ridge point if and only if $4b_{12}^3 + b_{30}b_{03}^2 = 0$ and

$$b_{03}^4 h_2(0) + 4b_{12}^2 b_{03}^2 h_3(0) - 8b_{12}^3 b_{03} h_4(0) + 16b_{12}^4 h_5(0,0) \neq 0.$$

hold.

Proof. First we show the condition (1) of Lemma 5.3.2. By defunitions of the principal curvature and the principal direction, we have $(\kappa_2)_u = b_{30} - a_{20}b_{12}, (\kappa_2)_v = -(4b_{12}^2 + a_{20}b_{03}^2)/2b_{03}, \xi = b_{03}/2, \zeta = -b_{12}$ at the origin. Using these conditions,

$$\boldsymbol{v}\kappa_2(\mathbf{0}) = \xi(\mathbf{0})(\kappa_2)_u(\mathbf{0}) + \zeta(\mathbf{0})(\kappa_2)_v(\mathbf{0}) = \frac{4b_{12}^3 + b_{30}b_{03}^2}{2b_{03}}$$

holds. On the other hand, the condition that the origin $\mathbf{0}$ is not a ridge point is $\boldsymbol{v}\kappa_2(\mathbf{0}) \neq 0$. Thus it follows that the first assertion holds.

Next we show (2) of Lemma 5.3.2. Let us assume that

$$\boldsymbol{v}\kappa_2(\mathbf{0}) = \frac{4b_{12}^3 + b_{30}b_{03}^2}{2b_{03}} = 0$$

holds, that is, the origin is a ridge point. Then we see that $\xi_u(\mathbf{0}) = 3h_4(0), \xi_v(\mathbf{0}) = 8h_5(0,0), \zeta_u(\mathbf{0}) = -4h_3(0), \zeta_v(\mathbf{0}) = -3h_4(0)$ and

$$\begin{aligned}
(\kappa_2)_{uu}(\mathbf{0}) &= -2a_{30}b_{12} + 24h_2(0) - 4a_{20}h_3(0)),\\
(\kappa_2)_{uv}(\mathbf{0}) &= \frac{1}{2b_{03}^2}(-a_{30}b_{03}^3 - 32b_{12}b_{03}h_3(0) + 6(4b_{12}^2 - a_{20}b_{03}^2)h_4(0)),\\
(\kappa_2)_{vv}(\mathbf{0}) &= \frac{4}{b_{03}^2}(b_{03}^2 h_3(0) - 6b_{12}b_{03}h_4(0) + 2(8b_{12}^2 - a_{20}b_{03}^2)h_5(0,0)).
\end{aligned}$$

Thus we obtain

$$\boldsymbol{v}^{(2)}\kappa_2(\mathbf{0}) = \frac{1}{4b_{03}^2}(6b_{03}(4b_{03}^3h_2(0) + 16b_{12}^2b_{03}h_3(0) - (28b_{12}^3 - b_{30}b_{03}^2)h_4(0)) \\ + 32b_{12}(8b_{12}^3 - b_{30}b_{03}^2)h_5(0,0)).$$

Since the condition $b_{30} = -4b_{12}^3/b_{03}^2$ holds, we have

$$\boldsymbol{v}^{(2)}\kappa_2(\mathbf{0}) = \frac{6}{b_{03}^2}(b_{03}^4h_2(0) + 4b_{12}^2b_{03}^2h_3(0) - 8b_{12}^3b_{03}h_4(0) + 16b_{12}^4h_5(0,0))).$$

This shows that (2) of Lemma 5.3.2 holds. □

We show some examples of cuspidal edge in $\mathbb{H}^3$ and corresponding de Sitter Gauss image. To visualize surface in $\mathbb{H}^3$ and $\mathbb{S}^{2,1}$, we will use the *Poincaré model* $\mathcal{P}$ and the *hollow ball model* $\mathcal{H}$ in what follows. For details of the hollow ball model, see [29]. Here the Poincaré model $\mathcal{P}$ is given by a map

$$\mathbb{H}^3 \ni (x_0, x_1, x_2, x_3) \mapsto \left(\frac{x_1}{1+x_0}, \frac{x_2}{1+x_0}, \frac{x_3}{1+x_0}\right) \in \mathcal{P}$$

and the hollow ball model $\mathcal{H}$ is given by

$$\mathbb{S}^{2,1} \ni (x_0, x_1, x_2, x_3) \mapsto \left(\frac{e^{\arctan x_0}}{\sqrt{1+x_0^2}}x_1, \frac{e^{\arctan x_0}}{\sqrt{1+x_0^2}}x_2, \frac{e^{\arctan x_0}}{\sqrt{1+x_0^2}}x_3\right) \in \mathcal{H}.$$

Then we can view the Poincaré model $\mathcal{P}$ as the Euclidean unit ball

$$B^3 = \{(x_1, x_2, x_3) \in \mathbb{R}^3 \mid x_1^2 + x_2^2 + x_3^2 < 1\}$$

in $\mathbb{R}^3$. Moreover the hollow ball model $\mathcal{H}$ can be viewed as

$$\mathcal{H} = \{(x_1, x_2, x_3) \in \mathbb{R}^3 \mid e^{-\pi} < x_1^2 + x_2^2 + x_3^2 < e^{\pi}\}.$$

Example 5.3.1. Let $f = (f_0, f_1, f_2, f_3) : U \to \mathbb{H}^3$ be a cuspidal edge with

$$f_0 = \sqrt{1 + f_1^2 + f_2^2 + f_3^2}, f_1 = u^3 + \frac{v^3}{3}, f_2 = u, f_3 = \frac{u^2}{2} + \frac{u^3}{3} + \frac{v^2}{2} + u^4.$$

This form satisfies the conditions of Lemma 5.3.1 and (1) of Lemma 5.3.2. Thus, by Theorem 5.2.1, de Sitter Gauss image of f has cuspidal edge at the origin.

Example 5.3.2. Let $f = (f_0, f_1, f_2, f_3) : U \to \mathbb{H}^3$ be a cuspidal edge with

$$f_0 = \sqrt{1 + f_1^2 + f_2^2 + f_3^2}, f_1 = \frac{v^3}{3} + u^4, f_2 = u, f_3 = \frac{u^2}{2} + \frac{u^3}{3} + \frac{v^2}{2}.$$

This form satisfies the conditions of Lemma 5.3.1 and (2) of Lemma 5.3.2. Thus, by Theorem 5.2.1, de Sitter Gauss image of f has swallowtail singularity at the origin.

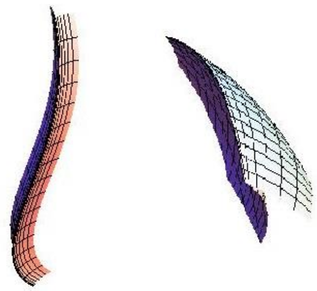


Fig. 5.2: Pictures of Example 5.3.1. Cuspidal edge in $\mathbb{H}^3$ (left) and its de Sitter Gauss image in $\mathbb{S}^{2,1}$ (right).

5.4 Application to flat fronts

In this section, we consider constant Gauss-Kronecker curvature $K_{\mathrm{ext}} = 1$ surfaces, called *flat fronts* in $\mathbb{H}^3$ and $\mathbb{S}^{2,1}$. First we introduce explicit formula for flat fronts in $\mathbb{H}^3$, called the Bryant-type representation, as in [36].

Proposition 5.4.1 ([36])**.** *Let U be a simply-connected domain in $\mathbb{C}$ with the usual complex coordinate $z = u + iv$. Then, any flat front $f : U \to \mathbb{H}^3$ is given by*

$$f = \mathcal{F}\overline{\mathcal{F}^t}, \quad \text{where } d\mathcal{F} = \mathcal{F}\begin{pmatrix} 0 & h(z) \\ g(z) & 0 \end{pmatrix} dz \tag{5.4.1}$$

for some holomorphic functions g and h.

We also have the following well-known fact as in Proposition 5.4.2:

Proposition 5.4.2. *For flat fronts f in $\mathbb{H}^3$ as given in Proposition 5.4.1, the unit normal vector ν of f becomes (spacelike) flat fronts in $\mathbb{S}^{2,1}$, and ν can be described as*

$$\nu = \mathcal{F}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\overline{\mathcal{F}^t}. \tag{5.4.2}$$

5.4.1 The singularity theory of flat fronts in $\mathbb{H}^3$ and $\mathbb{S}^{2,1}$.

Here we introduce criteria for singularities of flat fronts f in $\mathbb{H}^3$ as in [57], and we give criteria for singularities of the de Sitter Gauss image ν (i.e. flat fronts in $\mathbb{S}^{2,1}$) of f in the same way.

Proposition 5.4.3 ([57])**.** *Let U be a simply connected domain, and let $f : U \to \mathbb{H}^3$ be a flat front given as in Proposition 5.4.1.*

1. *A point $p \in U$ is a non-degenerate singular point if and only if*

$$|g(p)| = |h(p)| \quad \text{and} \quad (g_z h - g h_z)\big|_p \neq 0.$$

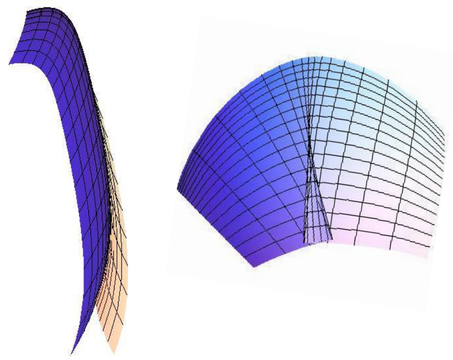


Fig. 5.3: Pictures of Example 5.3.2. Cuspidal edge in $\mathbb{H}^3$ (left) and its de Sitter Gauss image in $\mathbb{S}^{2,1}$ (right).

2. f has a cuspidal edge at a non-degenerate singular point $p \in U$ if and only if

$$\operatorname{Im}\left(\frac{g_z h - g h_z}{(gh)^{\frac{3}{2}}}\right)\bigg|_p \neq 0.$$

3. f has a swallowtail at a non-degenerate singular point $p \in U$ if and only if

$$\operatorname{Im}\left(\frac{g_z h - g h_z}{(gh)^{\frac{3}{2}}}\right)\bigg|_p = 0 \quad \textit{and} \quad \operatorname{Re}\left\{\left(\frac{g_z h - g h_z}{(gh)^{\frac{3}{2}}}\right)_z \overline{\left(\frac{g_z}{g} - \frac{h_z}{h}\right)}\right\}\bigg|_p \neq 0.$$

Here we give criteria for singularities of the de Sitter Gauss image ν. The following theorem can be proven by the same way as in the proof of the above proposition. Thus, we omit that proof.

Theorem 5.4.1. *Let U be a simply connected domain, and let $\nu : U \to \mathbb{S}^{2,1}$ be a flat front given in Proposition 5.4.2.*

1. A point $p \in U$ is a non-degenerate singular point if and only if

$$|g(p)| = |h(p)| \quad \textit{and} \quad (g_z h - g h_z)\big|_p \neq 0.$$

2. ν has a cuspidal edge at a non-degenerate singular point $p \in U$ if and only if

$$\operatorname{Re}\left(\frac{g_z h - g h_z}{(gh)^{\frac{3}{2}}}\right)\bigg|_p \neq 0.$$

3. ν has a swallowtail at a non-degenerate singular point $p \in U$ if and only if

$$\operatorname{Re}\left(\frac{g_z h - g h_z}{(gh)^{\frac{3}{2}}}\right)\bigg|_p = 0 \quad \textit{and} \quad \operatorname{Im}\left\{\left(\frac{g_z h - g h_z}{(gh)^{\frac{3}{2}}}\right)_z \overline{\left(\frac{g_z}{g} - \frac{h_z}{h}\right)}\right\}\bigg|_p \neq 0.$$

We note that Fujimori, Noro, Saji, Sasaki and Yoshida [31] study fronts in $\mathbb{S}^{2,1}$ from the viewpoint of the de Sitter Schwarz map and they give criteria which correspond to Theorem 5.4.1 for singularities of flonts in $\mathbb{S}^{2,1}$ in terms of de Sitter Schwarz map (For details, see [31, Proposition 6]).

By the above Proposition 5.4.3 and Theorem 5.4.1, we get the following Corollary 5.4.1.

Corollary 5.4.1.

1. *The singular set of f coincides with the singular set of ν, i.e. $S(f) = S(\nu)$.*
2. *We define the set of non-degenerate singular points of f as $\Sigma(f)$, which is the subset of $S(f)$. We define $\Sigma(\nu)$ similarly. Then, $\Sigma(f) = \Sigma(\nu)$.*

We note that (1) of Corollary 5.4.1 is a special case of [64, Corollary C] (see also [31]).

5.4.2 Global example: Enneper-type flat fronts and their singularities

Here we introduce Enneper-type flat fronts in $\mathbb{H}^3$ and $\mathbb{S}^{2,1}$, which are flat fronts and have reflective symmetry (see Fig 5.4 and 5.5), as global application of Theorem 5.2.1

Define $g(z) = z^k$ and $h(z) = 1$ for $k \in \mathbb{N}$. Applying Proposition 5.4.1, we get $(k+2)$-legged Enneper-type flat fronts in $\mathbb{H}^3$. See Fig 5.4. We also have $(k+2)$-legged Enneper-type flat fronts in $\mathbb{S}^{2,1}$ applying Proposition 5.4.2. See Fig 5.5. Using polar coordinates $z = re^{i\theta}$ and applying Proposition 5.4.3, we get the following lemma:

Lemma 5.4.1. *Let f be a $(k+2)$-legged Enneper-type flat front in $\mathbb{H}^3$. Then:*

1. $S(f) = \left\{(r_0,\theta) \middle| r_0^k = 1, 0 \le \theta < 2\pi\right\}$.
2. *f has $(k+2)$-swallowtails at (r_0,θ_0) such that $r_0^k = 1$ and $\theta_0 = \frac{2i\pi}{k+2}$ $(i = 0,1,2,\cdots,k+1)$.*
3. *f has cuspidal edges at (r_0,θ_1) such that $r_0^k = 1$ and $\theta_1 \neq \frac{2i\pi}{k+2}$ $(i = 0,1,2,\cdots,k+1)$.*

Similarly we also get the following lemma by applying Theorem 5.4.1.

Lemma 5.4.2. *Let ν be a $(k+2)$-legged Enneper-type flat front in $\mathbb{S}^{2,1}$. Then:*

1. $S(\nu) = S(f) = \left\{(r_0,\theta) \middle| r_0^k = 1, 0 \le \theta < 2\pi\right\}$.
2. *ν has $(k+2)$-swallowtails at (r_0,θ_2) such that $r_0^k = 1$ and $\theta_2 = \frac{(2i+1)\pi}{k+2}$ $(i = 0,1,2,\cdots,k+1)$.*
3. *ν has cuspidal edges at (r_0,θ_3) such that $r_0^k = 1$ and $\theta_3 \neq \frac{(2i+1)\pi}{k+2}$ $(i = 0,1,2,\cdots,k+1)$.*

By the above two lemmas and Theorem 5.2.1, we get the following theorem:

Theorem 5.4.2 (Duality of Enneper-type flat fronts)**.** *Let f be a $(k+2)$-legged Enneper-type flat front in $\mathbb{H}^3$, and let ν be a de Sitter Gauss image of f. Then:*

1. *Points (r_0,θ_2) such that $r_0^k = 1$ and $\theta_2 = \frac{(2i+1)\pi}{k+2}$ $(i = 0,1,2,\cdots,k+1)$ are first order ridge points of f.*
2. *Points (r_0,θ_3) such that $r_0^k = 1$ and $\theta_3 \neq \frac{(2i+1)\pi}{k+2}$ $(i = 0,1,2,\cdots,k+1)$ are not ridge points of f.*

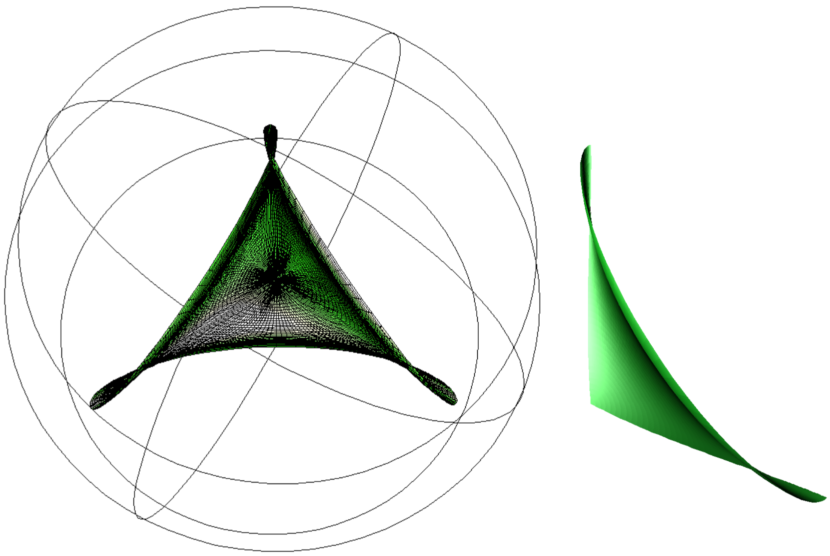


Fig. 5.4: The left image is a 3-legged Enneper-type flat front in $\mathbb{H}^3$, and the right is one portion of it between $0 \leq \theta \leq \frac{2\pi}{3}$.

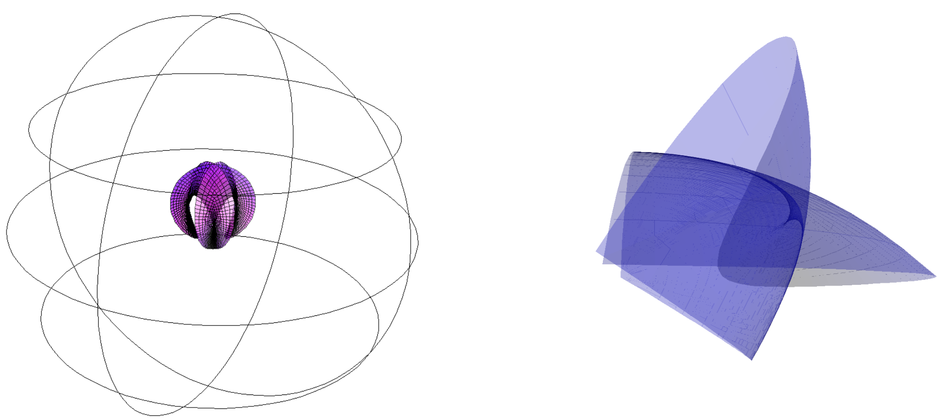


Fig. 5.5: The left image is the de Sitter Gauss image of a 3-legged Enneper-type flat front, and the right is one portion of it between $0 \leq \theta \leq \frac{2\pi}{3}$.

Chapter 6

CMC surfaces and D_4-singularities

6.1 Preliminaries

6.1.1 Surfaces in spaceforms

We recall some properties of surfaces in several spaceforms. For more details, see [51, 52, 53].

Let $\mathbb{R}^n = \{(x_1, \dots, x_n) \mid x_i \in \mathbb{R},\ 1 \leq i \leq n\}$ be an n-dimensional vector space. For any $\boldsymbol{x} = (x_1, \dots, x_n)$ and $\boldsymbol{y} = (y_1, \dots, y_n) \in \mathbb{R}^n$, we define the *pseudo inner product with the signature* $(n-k, k)$ $(0 \leq k < n)$ by

$$\langle \boldsymbol{x}, \boldsymbol{y} \rangle = \sum_{i=1}^{n-k} x_i y_i - \sum_{j=n-k+1}^{n} x_j y_j.$$

We denote $\mathbb{R}^{n-k,k} = (\mathbb{R}^n, \langle , \rangle)$. We say that a vector $\boldsymbol{x} \in \mathbb{R}^n \setminus \{\boldsymbol{0}\}$ is *spacelike*, *timelike* or *lightlike* if $\langle \boldsymbol{x}, \boldsymbol{x} \rangle > 0$, < 0 or $= 0$ respectively. We note that $\mathbb{R}^{n,0} = \mathbb{R}^n$ is the *Euclidean n-space*. If $k = 1$, we call the space $\mathbb{R}^{n-1,1}$ the *Minkowski n-space*.

Let $\boldsymbol{e}_1, \dots, \boldsymbol{e}_n$ be the pseudo orthonormal basis of $\mathbb{R}^{n-k,k}$ and $\boldsymbol{x}^i = (x^i_1, \dots, x^i_n) \in \mathbb{R}^{n-k,k}$ $(1 \leq i \leq n-1)$. Then we define the *wedge product $\boldsymbol{x}^1 \wedge \cdots \wedge \boldsymbol{x}^{n-1}$ with respect to the signature* $(n-k, k)$ by

$$\boldsymbol{x}^1 \wedge \cdots \wedge \boldsymbol{x}^{n-1} = \begin{vmatrix} \boldsymbol{e}_1 & \cdots & \boldsymbol{e}_{n-k} & -\boldsymbol{e}_{n-k+1} & \cdots & -\boldsymbol{e}_n \\ x^1_1 & \cdots & x^1_{n-k} & x^1_{n-k+1} & \cdots & x^1_n \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ x^{n-1}_1 & \cdots & x^{n-1}_{n-k} & x^{n-1}_{n-k+1} & \cdots & x^{n-1}_n \end{vmatrix}.$$

One can check that $\langle \boldsymbol{x}^i, \boldsymbol{x}^1 \wedge \cdots \wedge \boldsymbol{x}^{n-1} \rangle = 0$ holds for $1 \leq i \leq n-1$.

Let $n = 4$. Then we define the following spaceforms:

$$\mathbb{S}^3 = \{\boldsymbol{x} \in \mathbb{R}^4 \mid \langle \boldsymbol{x}, \boldsymbol{x} \rangle = 1\}, \qquad \mathbb{H}^3 = \{\boldsymbol{x} \in \mathbb{R}^{3,1} \mid \langle \boldsymbol{x}, \boldsymbol{x} \rangle = -1\},$$
$$\mathbb{S}^{2,1} = \{\boldsymbol{x} \in \mathbb{R}^{3,1} \mid \langle \boldsymbol{x}, \boldsymbol{x} \rangle = 1\}, \qquad \mathbb{H}^{2,1} = \{\boldsymbol{x} \in \mathbb{R}^{2,2} \mid \langle \boldsymbol{x}, \boldsymbol{x} \rangle = -1\}.$$

We call $\mathbb{S}^3$, $\mathbb{H}^3$, $\mathbb{S}^{2,1}$ and $\mathbb{H}^{2,1}$ the *spherical* 3*-space*, the *hyperbolic* 3*-space*, the *de Sitter* 3*-space* and the *anti-de Sitter* 3*-space*, respectively. It is known that $\mathbb{S}^3$ and $\mathbb{S}^{2,1}$ (resp. $\mathbb{H}^3$ and $\mathbb{H}^{2,1}$) have constant sectional curvature 1 (resp. -1).

Let M^3 be a 3-dimensional spaceform one of $\mathbb{R}^3$, $\mathbb{R}^{2,1}$, $\mathbb{S}^3$, $\mathbb{H}^3$, $\mathbb{S}^{2,1}$ or $\mathbb{H}^{2,1}$. Let $f : U \to M^3$ be a (spacelike) immersion, where $U \subset (\mathbb{R}^2; u, v)$ is an open set. Then we consider the *unit normal vector* ν to f. If $M^3 = \mathbb{R}^3$ or $\mathbb{R}^{2,1}$, ν is defined as

$$\nu = \frac{f_u \wedge f_v}{|f_u \wedge f_v|} \quad (f_u = \partial f/\partial u,\ f_v = \partial f/\partial v),$$

where $|\boldsymbol{x}| = \sqrt{|\langle \boldsymbol{x}, \boldsymbol{x}\rangle|}$. If M^3 is one of the other spaceforms, ν can be taken as

$$\nu = \frac{f \wedge f_u \wedge f_v}{|f \wedge f_u \wedge f_v|}.$$

In these cases, we use $\langle \cdot, \cdot \rangle$ as the induced metric from the ambient space $\mathbb{R}^{4-k,k}$ $(k = 0, 1, 2)$.

6.1.2 CMC surface theory

In this section, we explain some basical notations, as in [10] and [17]. Let M^3 be one of $\mathbb{R}^3$, $\mathbb{R}^{2,1}$, $\mathbb{S}^3$, $\mathbb{H}^3$, $\mathbb{S}^{2,1}$ or $\mathbb{H}^{2,1}$. Let $f : U \to M^3$ be a conformally (spacelike) immersion, where U is a simply-connected domain in $\mathbb{C}$ with usual complex coordinate $z = u + iv$ $(i = \sqrt{-1})$. By using the map $\mathbb{C} \ni z = u + iv \mapsto (u, v) \in \mathbb{R}^2$, we can identify $\mathbb{C}$ with $\mathbb{R}^2$. We note that $\partial_z = (\partial_u - i\partial_v)/2$ and $\partial_{\bar z} = (\partial_u + i\partial_v)/2$. Then $\langle f_z, f_z\rangle = \langle f_{\bar z}, f_{\bar z}\rangle = 0$ and $\langle f_z, f_{\bar z}\rangle = 2g^2$ hold for some function $g : U \to \mathbb{R}$, and in particular, the first fundamental form of f is given as

$$ds^2 = 4g^2(du^2 + dv^2).$$

Take the unit normal vector field ν. Then the mean curvature H and Hopf differential factor Q are given by

$$H = \frac{1}{2g^2}\langle f_{z\bar z}, \nu\rangle, \quad Q = \langle f_{zz}, \nu\rangle. \tag{6.1.1}$$

By (6.1.1), one can check that H and Q change to $-H$ and $-Q$, respectively, when we change ν to $-\nu$. We assume that H is constant. It is known that the Codazzi equation implies that Q is holomorphic. Moreover, the extrinsic Gaussian curvature K is written as

$$K = -\frac{1}{4g^4}Q\bar{Q} + H^2. \tag{6.1.2}$$

We now define the *parallel transforms* $\hat{f}^t$ *and* $\check{f}^t$ of f. If $M^3 = \mathbb{R}^3$ or $\mathbb{R}^{2,1}$,

$$\hat{f}^t = f + t\nu \tag{6.1.3}$$

for some constant $t \in \mathbb{R}$. In this case, ν is also a unit normal vector to $\hat{f}^t$. If $M^3 = \mathbb{S}^3$ or $\mathbb{H}^{2,1}$, $\hat{f}^t$ and $\hat{\nu}^t$ are

$$\hat{f}^t = (\cos t)f + (\sin t)\nu, \quad \hat{\nu}^t = -(\sin t)f + (\cos t)\nu \tag{6.1.4}$$

for some constant $t \in \mathbb{R}$. If $M^3 = \mathbb{H}^3$ or $\mathbb{S}^{2,1}$, we define

$$\hat{f}^t = (\cosh t)f + (\sinh t)\nu, \quad \check{f}^t = \hat{\nu}^t = (\sinh t)f + (\cosh t)\nu \tag{6.1.5}$$

for some constant $t \in \mathbb{R}$ (cf. [20] and [24]).

6.1.3 Wave fronts

We recall some notions of wave fronts. For details, see [3, 32, 51, 52, 83].

Let $f : U \to (M^3, h)$ be a C^∞ map, where U is a simply-connected domain in $\mathbb{R}^2$ and (M^3, h) is an oriented Riemannian or semi-Riemannian manifold with metric h. We call f a *wave front* or *front* if for each point $p \in U$ there exists a unit normal vector field ν along f and the map $L = (f, \nu) : U \to T_1M^3$ gives an immersion, where T_1M^3 is the unit tangent bundle over M^3. A point $p \in U$ is called a *singular point* if f is not an immersion at p. Let $S(f)$ denote the set of singular points of f. We set a function λ on U as

$$f_u \wedge f_v = \lambda\nu \tag{6.1.6}$$

when $M^3 = \mathbb{R}^3$ or $\mathbb{R}^{2,1}$, and

$$f \wedge f_u \wedge f_v = \lambda\nu \tag{6.1.7}$$

when $M^3 = \mathbb{S}^3$, $\mathbb{H}^3$, $\mathbb{S}^{2,1}$ or $\mathbb{H}^{2,1}$, where $\wedge$ denotes the vector product with respect to the metric h. We call this function λ the *signed area density function*. By definition, $\lambda^{-1}(0) = S(f)$ holds. A singular point p is called *non-degenerate* if the exterior derivative $d\lambda$ does not vanish at p. On a neighborhood of a non-degenerate singular point, there exists a smooth regular curve $\gamma(t)$ satisfying $\gamma(0) = p$ such that $\gamma(t)$ parametrizes the set of singular points. We call this curve γ a *singular curve* and the direction of $\gamma' = d\gamma/dt$ a *singular direction*. The dimension of the kernel $\operatorname{Ker} df_{\gamma(t)}$ of the differential map $df_{\gamma(t)}$ is 1 and there exists a never-vanishing vector field $\eta(t)$ such that $\langle\eta(t)\rangle_{\mathbb{R}} = \operatorname{Ker} df_{\gamma(t)}$. We call $\eta(t)$ a *null vector field* and the direction of η a *null direction*.

Definition 6.1.1. Let $f : (U, p) \to (\mathbb{R}^3, f(p))$ be a map-germ around p. Then f has a *cuspidal edge* at p if the map-germ f at p is $\mathcal{A}$-equivalent to the map-germ $(u, v) \mapsto (u, v^2, v^3)$ at $\mathbf{0}$, and f has a *swallowtail* at p if the map-germ f at p is $\mathcal{A}$-equivalent to the map-germ $(u, v) \mapsto (u, 3v^4 + uv^2, 4v^3 + 2uv)$ at $\mathbf{0}$, and f has a $D_4^\pm$*-singularity* at p if the map-germ f at p is $\mathcal{A}$-equivalent to the map-germ $(u, v) \mapsto (2uv, \pm u^2 + 3v^2, \pm 2u^2v + 2v^3)$ at $\mathbf{0}$, where the two map-germs $f, g : (\mathbb{R}^2, \mathbf{0}) \to (\mathbb{R}^3, \mathbf{0})$ are $\mathcal{A}$*-equivalent* if there exist diffeomorphism-germs $\theta : (\mathbb{R}^2, \mathbf{0}) \to (\mathbb{R}^2, \mathbf{0})$ on the source and $\Theta : (\mathbb{R}^3, \mathbf{0}) \to (\mathbb{R}^3, \mathbf{0})$ on the target such that $\Theta \circ f = g \circ \theta$ holds.

We note that cuspidal edges and swallowtails are non-degenerate singular points of fronts. On the other hand, D_4-singularities are degenerate singular points with corank two. There are well-known criteria for cuspidal edges and swallowtails (see [57, Proposition 1.3]). There is a criterion for $D_4^\pm$-singularities as well.

Fact 6.1.1 ([82, Theorem 1.1])**.** Let f be a front and λ the signed area density function. A singular point p is a D_4^+-singularity (resp. D_4^--singularity) if and only if the following conditions hold:

1. $\operatorname{rank} df_p = 0$.

2. $\det \operatorname{Hess} \lambda < 0$ (respectively, $\det \operatorname{Hess} \lambda > 0$) at p.

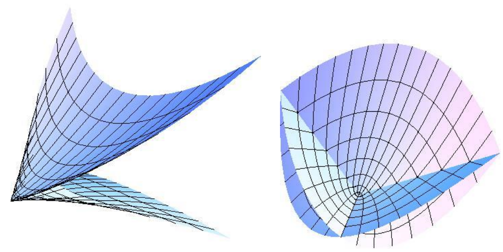


Fig. 6.1: The left hand side is a D_4^+-singularity and the right hand side is a D_4^-- singularity of a wave front.

6.2 Constant mean curvature surfaces with D_4-singularities

6.2.1 Surfaces with non-zero constant mean curvature

In this section, we consider cases such that

$$\begin{cases} H \neq 0 \text{ if } M^3 = \mathbb{R}^3,\ \mathbb{R}^{2,1},\ \mathbb{S}^3 \text{ or } \mathbb{H}^{2,1}, \\ H \neq 0, 1 \text{ if } M^3 = \mathbb{H}^3 \text{ or } \mathbb{S}^{2,1}. \end{cases} \tag{6.2.1}$$

Lemma 6.2.1. *Let $f : (U, z) \to M^3$ be a conformally CMC H immersion, ν a unit normal vector to f and p an umbilic point.*

(1) *Suppose that $M^3 = \mathbb{S}^3$ or $\mathbb{H}^{2,1}$ and $H > 0$. Then p is a corank two singular point of $\hat{f}^t$ if and only if $t = \operatorname{arccot}(H)$.*

(2) *Suppose that $M^3 = \mathbb{H}^3$ or $\mathbb{S}^{2,1}$ and $H > 1$. Then p is a corank two singular point of $\hat{f}^t$ if and only if $t = \operatorname{arccoth}(H)$.*

(3) *Suppose that $M^3 = \mathbb{H}^3$ or $\mathbb{S}^{2,1}$ and $0 < H < 1$. Then p is a corank two singular point of $\check{f}^t$ if and only if $t = \operatorname{arctanh}(H)$.*

Proof. We show (2) and (3) in the case of $M^3 = \mathbb{H}^3$. For the case of $M^3 = \mathbb{S}^{2,1}$ and (1), one can show the result in a similar way.

Let $f : U \to M^3 = \mathbb{H}^3$ be a conformally CMC H immersion. Suppose that $H > 1$. Then we consider $\hat{f}^t$ as in (6.1.5). Since $\nu_z = (-2Hf_z - Qg^{-2}f_{\bar{z}})/2$ and $\nu_{\bar{z}} = (-\bar{Q}g^{-2}f_z - 2Hf_{\bar{z}})/2$ by (6.1.1), we have

$$\hat{f}^t_z = (\cosh t - H \sinh t) f_z - \frac{Q}{2g^2}(\sinh t) f_{\bar{z}},$$
$$\hat{f}^t_{\bar{z}} = -\frac{\bar{Q}}{2g^2}(\sinh t) f_z + (\cosh t - H \sinh t) f_{\bar{z}}.$$

Since p is an umbilic point, $Q(p) = \bar{Q}(p) = 0$. Thus $\hat{f}^t_z = \hat{f}^t_{\bar{z}} = \mathbf{0}$ at p if and only if $t = \operatorname{arccoth}(H)$. Therefore we have the assertion (2).

Next we show (3). Assume that $0 < H < 1$. By direct computations, we see that

$$\check{f}^t_z = (\sinh t - H\cosh t)f_z - \frac{Q}{2g^2}(\cosh t)f_{\bar{z}},$$
$$\check{f}^t_{\bar{z}} = -\frac{\bar{Q}}{2g^2}(\cosh t)f_z + (\sinh t - H\cosh t)f_{\bar{z}}.$$

Hence $\check{f}^t_z = \check{f}^t_{\bar{z}} = \mathbf{0}$ at p if and only if $t = \operatorname{arctanh}(H)$. □

Theorem 6.2.1. *Let f be a CMC surface in M^3 with mean curvature H with* (6.2.1)*, Hopf differential factor $Q(z)$ and p a corank two singular point. Then f has a D_4^--singularity at p if and only if $Q_z(p) \neq 0$. Moreover, f does not have D_4^+-singularities.*

Proof. By Facts 0.0.2 and 0.0.3, if $f : U \to M^3$ is a CMC H immersion, then $\hat{f}^t$ and $\check{f}^t$ are CMC $-H$ immersion on the set of regular points for suitable distance t. Therefore we consider the CMC immersion f with an umbilic point p and singularities of its parallel transform $\hat{f}^t$ at p. Since one can prove this similarly by using Lemma 6.2.1 in other cases, we consider just the case of $M^3 = \mathbb{R}^3$.

Let $f : U \to \mathbb{R}^3$ be a CMC immersion and p an umbilic point, where $U \subset \mathbb{C}$ is a simply-connected domain with conformal coordinate $z = u + iv$. By the previous section, $\langle f_z, f_z\rangle = \langle f_{\bar{z}}, f_{\bar{z}}\rangle = 0$ and $\langle f_z, f_{\bar{z}}\rangle = 2g^2$ for some function g on U. We now consider the parallel transformation of f given by $\hat{f}^t = f + t\nu$, where $t \in \mathbb{R}$ is constant. In this case, we can take a unit normal vector $\hat{\nu}^t$ of $\hat{f}^t$ as ν. By Fact 0.0.2, $\operatorname{rank} d\hat{f}^t(p) = 0$ if and only if $t = 1/H$.

We fix $t = 1/H$. The signed area density function of $\hat{f}^t$ is given by $\hat{\lambda}^t = \langle \hat{f}^t_u \wedge \hat{f}^t_v, \nu\rangle = -2i\langle \hat{f}^t_z \wedge \hat{f}^t_{\bar{z}}, \nu\rangle$. Using $\nu_z = (-2Hf_z - Qg^{-2}f_{\bar{z}})/2$, $\nu_{\bar{z}} = (-2Hf_{\bar{z}} - \bar{Q}g^{-2}f_z)/2$ and (6.1.2), the signed area density function $\hat{\lambda}^t$ is rewritten as

$$\hat{\lambda}^t = (1 - 2tH + t^2K)\langle f_z \wedge f_{\bar{z}}, \nu\rangle.$$

Since $\langle f_z \wedge f_{\bar{z}}, \nu\rangle \neq 0$, we may regard

$$\tilde{\lambda}^t = 1 - 2tH + t^2K \tag{6.2.2}$$

as the signed area density function of $\hat{f}^t$. By direct computations, $\tilde{\lambda}^t_z$ and $\tilde{\lambda}^t_{\bar{z}}$ are

$$\tilde{\lambda}^t_z = -t^2\frac{(Q_z\bar{Q} + Q\bar{Q}_z)g - 4Q\bar{Q}g_z}{4g^5}, \tilde{\lambda}^t_{\bar{z}} = -t^2\frac{(Q_{\bar{z}}\bar{Q} + Q\bar{Q}_{\bar{z}})g - 4Q\bar{Q}g_{\bar{z}}}{4g^5}.$$

Since $Q(p) = \bar{Q}(p) = 0$, $\tilde{\lambda}^t_z(p) = \tilde{\lambda}^t_{\bar{z}}(p) = 0$, that is, $d\tilde{\lambda}^t(p) = 0$ holds. We consider the Hessian of $\tilde{\lambda}^t$. The second derivative $\tilde{\lambda}^t_{zz}$ becomes

$$\begin{aligned}\tilde{\lambda}^t_{zz} = -\frac{t^2}{4g^5}\{&(Q_{zz}\bar{Q} + 2Q_z\bar{Q}_z + Q\bar{Q}_{zz})g + (Q_z\bar{Q} + Q\bar{Q}_z)g \\ &- 4(Q_z\bar{Q}g_z + Q\bar{Q}_zg_z + Q\bar{Q}g_{zz})\} \\ &- t^2\frac{5\{(Q_z\bar{Q} + Q\bar{Q}_z)g - 4Q\bar{Q}g_z\}g_z}{4g}.\end{aligned} \tag{6.2.3}$$

Thus $\tilde{\lambda}^t_{zz}(p) = 0$ holds. Similarly, we see that $\tilde{\lambda}^t_{\bar{z}\bar{z}}(p) = 0$ holds. By direct calculation, we have

$$\begin{aligned}\tilde{\lambda}^t_{z\bar{z}} = -\frac{t^2}{4g^5}\{(Q_{z\bar{z}}\bar{Q} + Q_z\bar{Q}_{\bar{z}} + Q_{\bar{z}}\bar{Q}_z + Q\bar{Q}_{z\bar{z}})g + (Q_z\bar{Q} + Q\bar{Q}_z)g_{\bar{z}} \\ - 4(Q_{\bar{z}}\bar{Q}g_z + Q\bar{Q}_{\bar{z}}g_z + Q\bar{Q}g_{z\bar{z}})\} \\ - t^2\frac{5\{(Q_z\bar{Q} + Q\bar{Q}_z)g - 4Q\bar{Q}g_z\}g_{\bar{z}}}{4g}. \quad (6.2.4)\end{aligned}$$

By this equation, $\tilde{\lambda}^t_{z\bar{z}} = -t^2 Q_z\bar{Q}_{\bar{z}}/4g^4$ holds at p. Identifying $\mathbb{C}$ with $\mathbb{R}^2$, we have

$$\tilde{\lambda}^t_{uu} = \tilde{\lambda}^t_{zz} + 2\tilde{\lambda}^t_{z\bar{z}} + \tilde{\lambda}^t_{\bar{z}\bar{z}}, \quad \tilde{\lambda}^t_{uv} = i(\tilde{\lambda}^t_{zz} - \tilde{\lambda}^t_{\bar{z}\bar{z}}), \quad \tilde{\lambda}^t_{vv} = -(\tilde{\lambda}^t_{zz} - 2\tilde{\lambda}^t_{z\bar{z}} + \tilde{\lambda}^t_{\bar{z}\bar{z}}).$$

By the above computations, it follows that

$$\tilde{\lambda}^t_{uu} = \tilde{\lambda}^t_{vv} = 2\tilde{\lambda}^t_{z\bar{z}} = -t^2\frac{Q_z\bar{Q}_{\bar{z}}}{2g^4}, \quad \tilde{\lambda}^t_{uv} = 0$$

hold at p. Thus we have

$$\det \mathrm{Hess}_{(u,v)}(\tilde{\lambda}^t)_p = \tilde{\lambda}^t_{uu}(p)\tilde{\lambda}^t_{vv}(p) - \tilde{\lambda}^t_{uv}(p)^2 = t^4\frac{(Q_z\bar{Q}_{\bar{z}})^2}{4g^8} > 0.$$

This completes the proof of the case $M^3 = \mathbb{R}^3$, by Fact 6.1.1.

If $M^3 = \mathbb{R}^{2,1}$, we can take $\tilde{\lambda}^t$ as same as the case of $\mathbb{R}^3$. If $M^3 = \mathbb{S}^3$ or $\mathbb{H}^{2,1}$, we have the assertion by using the signed area density for $\hat{f}^t$ as in (6.1.4)

$$\tilde{\lambda}^t = \cos^2 t - 2\cos t \sin tH + \sin^2 tK.$$

If $M^3 = \mathbb{H}^3$ or $\mathbb{S}^{2,1}$, we show by using

$$\tilde{\lambda}^t = \cosh^2 t - 2\cosh t \sinh tH + \sinh^2 tK$$

for $\hat{f}^t$ and

$$\tilde{\lambda}^t = \sinh^2 t - 2\cosh t \sinh tH + \cosh^2 tK$$

for $\check{f}^t$. □

Examples: Here we construct CMC surfaces with D_4^--singularities in $\mathbb{H}^3$. By Theorem 6.2.1, we need to choose the Hopf differential factor so that $Q_z(p) \neq 0$ at a point p. Now we fix $Q = -z$ for CMC $H > 1$ or $0 \leq H < 1$ surfaces and we have the 3-legged Smyth-type surfaces as in [8], [70], [71] and [87]. Applying Theorems 6.2.1, we get the following figures with a D_4^--singularity at the origin $z = 0$:

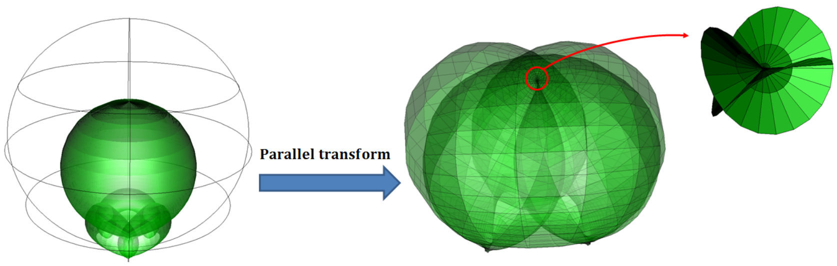


Fig. 6.2: 3-legged Smyth surface with $H > 1$ in $\mathbb{H}^3$ and its parallel transform with D_4^--singularity.

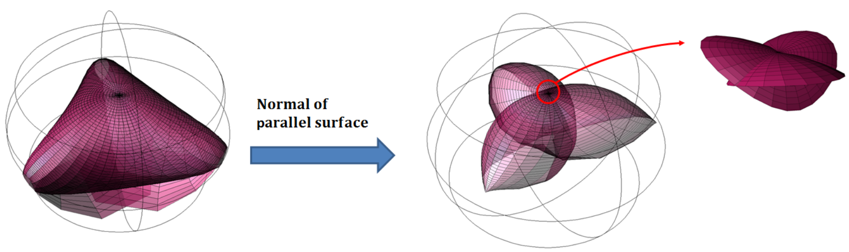


Fig. 6.3: 3-legged Smyth surface with $0 < H < 1$ in $\mathbb{H}^3$ and the normal vector of its parallel transform into $\mathbb{S}^{2,1}$ with D_4^--singularity.

6.2.2 Minimal surfaces with D_4-singularities

We now consider the condition that minimal surfaces (resp. maximal surfaces) in $\mathbb{R}^3$ (resp. $\mathbb{R}^{2,1}$) have D_4-singularities. For minimal surfaces, the following representation formula is known.

Fact 6.2.1. Any simply-connected minimal surface $f : U(\subset \mathbb{C}) \to \mathbb{R}^3$ can be parametrized as

$$f = \mathrm{Re} \int (1 - g^2, i(1 + g^2), 2g)\omega, \tag{6.2.5}$$

where $g : U \to \mathbb{C}$ is a meromorphic function and $\omega = \hat{\omega} dz$ is a holomorphic 1-form.

We call the pair (g, ω) the *Weierstrass data*. On the other hand, the representation formula for maximal surfaces is also known.

Fact 6.2.2 ([55])**.** Any simply-connected maximal surface $f : U(\subset \mathbb{C}) \to \mathbb{R}^{2,1}$ can be parametrized as

$$f = \mathrm{Re} \int (1 + g^2, i(1 - g^2), -2g)\omega, \tag{6.2.6}$$

where $g : U \to \mathbb{C}$ is a meromorphic function and $\omega = \hat{\omega} dz$ is a holomorphic 1-form

We also call the pair (g, ω) the *Weierstrass data*. We should remark that there are several studies for maximal surfaces (see [28, 32, 59, 92], for example).

Lemma 6.2.2. *Let $f : U \to \mathbb{R}^3$ (resp. $\mathbb{R}^{2,1}$) be a minimal surface (resp. a maximal surface) constructed by (6.2.5) (resp. (6.2.6)). Then $p \in U$ is a corank two singular point of f if and only if $g(p)$ is of finite value (i.e., p is not a pole of g) and $\hat{\omega}(p) = 0$. Moreover, f is a front at p if and only if $g_z(p) \neq 0$.*

Proof. Let $f : U \to \mathbb{R}^3$ be a minimal surface with the Weierstrass data $(g, \omega = \hat{\omega} dz^2)$. The differentials of f are

$$f_z = \frac{1}{2}(1 - g^2, i(1 + g^2), 2g)\hat{\omega}, \quad f_{\bar{z}} = \frac{1}{2}(1 - \bar{g}^2, -i(1 + \bar{g}^2), 2\bar{g})\bar{\hat{\omega}}.$$

Thus we have the first assertion.

Next, we show the condition of f to be a front at p. Let p be a corank two singular point of f. Then f is a front at p if and only if its unit normal vector $\nu : U \to \mathbb{S}^2$ gives an immersion at p. Under the above settings, the unit normal vector ν to f is given by

$$\nu = \left(\frac{g + \bar{g}}{|g|^2 + 1}, i\frac{\bar{g} - g}{|g|^2 + 1}, \frac{|g|^2 - 1}{|g|^2 + 1} \right).$$

Since g is holomorphic at p, $g_{\bar{z}}(p) = 0$ holds. Differentiating ν, we have

$$\nu_z = g_z \left(\frac{1 - \bar{g}^2}{(1 + |g|^2)^2}, i\frac{-1 + \bar{g}^2}{(1 + |g|^2)^2}, \frac{2\bar{g}}{(1 + |g|^2)^2} \right),$$
$$\nu_{\bar{z}} = \overline{g_z} \left(\frac{1 - g^2}{(1 + |g|^2)^2}, i\frac{1 - g^2}{(1 + |g|^2)^2}, \frac{2g}{(1 + |g|^2)^2} \right)$$

at p. Thus we have the conclusion.

For maximal surfaces, one can show this similarly by identifying $\mathbb{R}^{2,1}$ with $\mathbb{R}^3$ and using the Euclidean unit normal vector $\boldsymbol{n}_E$ given as

$$\boldsymbol{n}_E = \frac{1}{\sqrt{(1 + |g|^2)^2 + 4|g|^2}}(g + \bar{g}, i(\bar{g} - g), 1 + |g|^2)$$

(see [92]). □

Theorem 6.2.2. *Let $f : U \subset (\mathbb{C}, z) \to \mathbb{R}^3$ (resp. $f : U \subset (\mathbb{C}, z) \to \mathbb{R}^{2,1}$) be a minimal surface (resp. a maximal surface) given by the Weierstrass data $(g, \omega = \hat{\omega} dz)$. Then a point $p \in U$ is a D_4^--singularity of f if and only if $\hat{\omega}(p) = 0$ and $Q_z(p) \neq 0$ (resp. $\hat{\omega}(p) = 0$, $Q_z(p) \neq 0$ and $|g(p)| \neq 1$). Here $Q = g_z \hat{\omega}$ is the Hopf differential factor. Moreover, f does not have D_4^+-singularities.*

Proof. First, we show the case of a minimal surface. The signed area density function λ of f can be given as

$$\lambda = -2i\langle f_z \wedge f_{\bar{z}}, \nu \rangle = (1 + |g|^2)^2 |\hat{\omega}|^2.$$

Since $1 + |g|^2 \neq 0$, we may treat $\hat{\lambda} = |\hat{\omega}|^2 = \hat{\omega}\bar{\hat{\omega}}$ as the signed area density function. Moreover, since $f_z(p) = f_{\bar{z}}(p) = \mathbf{0}$, p is a corank two singular point. The differentials of $\hat{\lambda}$ in $z, \bar{z} \in U$ are

$$\tilde{\lambda}_z = 0,\ \tilde{\lambda}_{\bar{z}} = 0,\ \tilde{\lambda}_{zz} = 0,\ \tilde{\lambda}_{z\bar{z}} = \hat{\omega}_z \bar{\hat{\omega}}_{\bar{z}},\ \tilde{\lambda}_{\bar{z}\bar{z}} = 0$$

at p, since $\hat{\omega}(p) = \bar{\hat{\omega}}(p) = \hat{\omega}_{\bar{z}}(p) = \bar{\hat{\omega}}_z(p) = 0$. Identifying $z = u + iv \in \mathbb{C}$ and $(u, v) \in \mathbb{R}^2$, we see that $\tilde{\lambda}_{uu}(p) = \tilde{\lambda}_{vv}(p) = 2\hat{\omega}_z(p)\bar{\hat{\omega}}_{\bar{z}}(p)$ and $\tilde{\lambda}_{uv}(p) = 0$. Hence the Hessian of $\tilde{\lambda}$ is

$$\det \mathrm{Hess}_{(u,v)}(\tilde{\lambda})_p = 4|\hat{\omega}_z(p)|^4 \geq 0.$$

By Fact 6.1.1 and Lemma 6.2.2, f has a D_4^--singularity at p if and only if $g_z(p) \neq 0$ and $\hat{\omega}_z(p) \neq 0$. On the other hand, the derivative of the Hopf differential factor Q is

$$Q_z(p) = g_{zz}(p)\hat{\omega}(p) + g_z(p)\hat{\omega}_z(p) = g_z(p)\hat{\omega}_z(p),$$

since g_{zz} is finite at p. Thus we have the assertion.

Next, we consider a maximal surface that is not a maxface. The signed area density function is

$$\lambda = -2i\langle f_z \wedge f_{\bar{z}}, \nu\rangle = (|g|^2 - 1)^2|\hat{\omega}|^2$$

where $\wedge$ means the vector product with respect to the metric of $\mathbb{R}^{2,1}$. From Lemmas 6.2.2, we may take $\tilde{\lambda} = \hat{\omega}\bar{\hat{\omega}}$. By using similar arguments, we obtain the assertion. $\square$

By the Lawson correspondence, the first fundamental forms of (spacelike) CMC 1 surfaces in $\mathbb{H}^3$ (resp. $\mathbb{S}^{2,1}$) are equal to the first fundamental forms of corresponding minimal surfaces in $\mathbb{R}^3$ (resp. maximal surfaces in $\mathbb{R}^{2,1}$). This means that they have the same signed area density functions. Thus we obtain the condition that (spacelike) CMC 1 surfaces have D_4^--singularities similarly.

On the other hand, when $f : U \to M^3 = \mathbb{S}^3,\ \mathbb{H}^3,\ \mathbb{S}^{2,1}$ or $\mathbb{H}^{2,1}$ is a minimal immersion, there is no representation formula. However, if $p \in U$ is an umbilic point for f, then its unit normal vector ν has a corank two singularity at p. By using similar calculations as in the proof of Theorem 6.2.1, we see that ν has a D_4^--singularity at p if and only if $Q_z(p) \neq 0$, where Q is the Hopf differential factor of f.

Example: Here we construct CMC 1 surfaces with D_4^--singularities in $\mathbb{H}^3$. By Theorems 6.2.2, we fix the Weierstrass data $(g, \omega) = (\cot(z-1), (e^z - 1)dz)$ for CMC 1 surface in $\mathbb{H}^3$. Applying Theorem 6.2.2, we get the following figure with a D_4^--singularity at the origin $z = 0$:

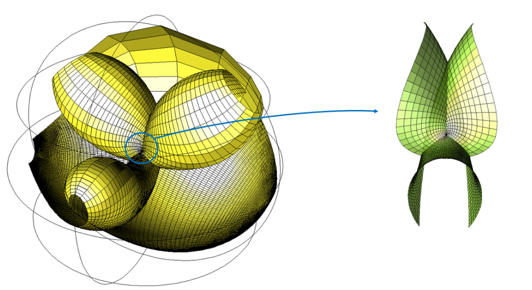


Fig. 6.4: CMC 1 surface with D_4^--singularity in $\mathbb{H}^3$.

6.3 Curvatures of unit normal vector fields to CMC surfaces

In this section, M^3 denotes one of $\mathbb{S}^3$, $\mathbb{H}^3$, $\mathbb{S}^{2,1}$ or $\mathbb{H}^{2,1}$. We consider relations between a CMC immersion $f : U \to M^3$ and a unit normal vector ν to f.

Proposition 6.3.1. *Let $f : U \to M^3$ be a (spacelike) CMC H immersion and ν be a unit normal vector to f. Let K denote the extrinsic Gaussian curvature of f. Then the extrinsic Gaussian curvature K_ν and the mean curvature H_ν of ν are*

$$K_\nu = \frac{1}{K}, \quad H_\nu = \frac{H}{K},$$

Moreover, the unit normal vector ν has a constant harmonic mean curvature $1/2H$.

Here the *harmonic mean curvature HMC* is given by

$$HMC = \frac{K}{2H}.$$

Proof. We consider the case $f : U \to \mathbb{H}^3$. One can show other cases similarly. Let (u, v) be a conformal coordinate system on U. Then the determinant of the first fundamental matrix I_ν is given by

$$\det I_\nu = E_\nu G_\nu - F_\nu^2 = K^2 E^2,$$

where

$$I_\nu = \begin{pmatrix} \langle \nu_u, \nu_u \rangle & \langle \nu_u, \nu_v \rangle \\ \langle \nu_v, \nu_u \rangle & \langle \nu_v, \nu_v \rangle \end{pmatrix} = \begin{pmatrix} E_\nu & F_\nu \\ F_\nu & G_\nu \end{pmatrix},$$

and $E = \langle f_u, f_u \rangle = \langle f_v, f_v \rangle$. It follows that the coefficients of the second fundamental form of ν are the same as of f by definition. Thus we have the assertions by straightforward calculations. $\square$

For $M^3 = \mathbb{S}^3$, $\mathbb{H}^{2,1}$ and $H > 0$, or $M^3 = \mathbb{H}^3$, $\mathbb{S}^{2,1}$ and $H > 1$, the parallel transformations $\hat{f}^t$ and $\hat{\nu}^t$ are defined as in (6.1.4) and (6.1.5). If $M^3 = \mathbb{H}^3$, $\mathbb{S}^{2,1}$ and $0 < H < 1$, the parallel transforms of f and ν are $\check{f}^t = \hat{\nu}^t$, $\check{\nu}^t = \hat{f}^t$, respectively. Thus it is sufficient to consider $\hat{f}^t$ and $\hat{\nu}^t$.

Lemma 6.3.1. *Under the above settings, the extrinsic Gaussian curvature K^t and K_ν^t, and the mean curvatures H^t and H_ν^t for $\hat{f}^t$ and $\hat{\nu}^t$ are given by the following:*

$$K^t = \begin{cases} \dfrac{\sin^2 t + 2H \cos t \sin t + K \cos^2 t}{\cos^2 t - 2H \cos t \sin t + K \sin^2 t}, \\ \dfrac{\sinh^2 t + 2H \cosh t \sinh t + K \cosh^2 t}{\cosh^2 t - 2H \cosh t \sinh t + K \sinh^2 t}, \end{cases}$$

$$K_\nu^t = \begin{cases} \dfrac{\cos^2 t - 2H \cos t \sin t + K \sin^2 t}{\sin^2 t + 2H \cos t \sin t + K \cos^2 t}, \\ \dfrac{\cosh^2 t - 2H \cosh t \sinh t + K \sinh^2 t}{\sinh^2 t + 2H \cosh t \sinh t + K \cosh^2 t}, \end{cases}$$

$$H^t = \begin{cases} \dfrac{(1-K) \cos t \sin t + H(\cos^2 t - \sin^2 t)}{\cos^2 t - 2H \cos t \sin t + K \sin^2 t}, \\ -\dfrac{(1+K) \cosh t \sinh t - H(\cosh^2 t + \sinh^2 t)}{\cosh^2 t - 2H \cosh t \sinh t + K \sinh^2 t}, \end{cases}$$

$$H_\nu^t = \begin{cases} \dfrac{(1-K) \cos t \sin t + H(\cos^2 t - \sin^2 t)}{\cos^2 t + 2H \cos t \sin t + K \sin^2 t}, \\ -\dfrac{(1+K) \cosh t \sinh t - H(\cosh^2 t + \sinh^2 t)}{\cosh^2 t + 2H \cosh t \sinh t + K \sinh^2 t}. \end{cases}$$

Proof. One can show the above formulas directly by applying the same computations as in [56]. □

By Proposition 6.3.1 and Lemma 6.3.1, we immediately have the following:

Proposition 6.3.2. *Let $f : U \to M^3 = \mathbb{S}^3,\ \mathbb{H}^3,\ \mathbb{S}^{2,1},\ \mathbb{H}^{2,1}$ be a CMC $H \neq 0$ immersion and ν its unit normal vector. A point p is an umbilic point for f if and only if p is a non-flat umbilic point for ν. Moreover, if $\hat{f}^t$ (resp. $\check{f}^t$) has a D_4^--singularity at p, p is a flat umbilic point of $\hat{\nu}^t$ (resp. $\check{\nu}^t$) (see Figure 6.5, 6.6).*

$$\begin{tikzcd} f : \text{CMC with umbilic point at } p \arrow[r, dotted, "\text{Normal}"] \arrow[d, leftrightarrow, "\text{Parallel transform}"'] & \nu : \text{CHMC with non-flat umbilict } p \arrow[d, leftrightarrow, "\text{Parallel transform}"] \\ \hat{f} : \text{CMC with } D_4^-\text{-singularity at } p \arrow[r, dotted, "\text{Normal}"] & \hat{\nu} : \text{CHMC with flat umbilic } p \end{tikzcd}$$

Fig. 6.5: The case that f is a (spacelike) CMC surface in $\mathbb{S}^3$, $\mathbb{H}^{2,1}$ (resp. $\mathbb{H}^3$, $\mathbb{S}^{2,1}$) with $H > 0$ (resp. $H > 1$)

$$\begin{tikzcd} f : \text{CMC with umbilic point at } p \arrow[r, dotted, "\text{Normal}"] \arrow[d, leftrightarrow, "\text{Normal of parall. surf.}"'] & \nu : \text{CHMC with non-flat umbilic } p \arrow[d, leftrightarrow, "\text{Normal of parall. surf.}"] \\ \check{f} : \text{CMC with } D_4^-\text{-singularity at } p \arrow[r, dotted, "\text{Normal}"] & \check{\nu} : \text{CHMC with flat umbilic } p \end{tikzcd}$$

Fig. 6.6: The case that f is a (spacelike) CMC surface in $\mathbb{H}^3$, $\mathbb{S}^{2,1}$ with $0 < H < 1$

Chapter 7

Deformation of minimal surfaces with planar curvature lines

7.1 Minimal surfaces with planar curvature lines in $\mathbb{R}^3$

We wish to classify minimal surfaces with planar curvature lines and obtain their parametrizations by calculating the Weierstrass data. However, applying the planar curvature line condition directly to the Weierstrass data involves heavy calculation in a complex-analysis setting. Therefore, from the zero mean curvature condition and planar curvature line condition, we obtain a system of partial differential equations for the metric function. Then by applying the method analogous to the one employed by Abresch in [1], we solve the system of partial differential equations by transforming it into a system of ordinary differential equations. Finally, using explicit solutions for the metric function, we recover the Weierstrass data and the parametrization by calculating the unit normal vector, through an analogous approach to the one taken by Walter in [93].

7.1.1 Minimal surface theory

Let $\Sigma \subset \mathbb{R}^2$ be a simply-connected domain with coordinates (u, v), and let $X : \Sigma \to \mathbb{R}^3$ be a conformally immersed surface. Since $X(u, v)$ is conformal,

$$\mathrm{d}s^2 = e^{2\omega}(\mathrm{d}u^2 + \mathrm{d}v^2)$$

for some $\omega : \Sigma \to \mathbb{R}$. We choose the unit normal vector field $N : \Sigma \longrightarrow \mathbb{S}^2$ of X, and then the mean curvature H and Hopf differential Q are

$$H := \frac{1}{2e^{2\omega}}\langle X_{uu} + X_{vv}, N\rangle \quad \text{and} \quad Q := \frac{1}{4}\langle X_{uu} - 2iX_{uv} - X_{vv}, N\rangle.$$

Since we are interested in minimal surfaces, we let $H = 0$ and calculate the Gauss-Weingarten equations,

$$\begin{cases} X_{uu} = \omega_u X_u - \omega_v X_v + (Q + \bar{Q})N \\ X_{vv} = -\omega_u X_u + \omega_v X_v - (Q + \bar{Q})N \\ X_{uv} = \omega_v X_u + \omega_u X_v + i(Q - \bar{Q})N \\ N_u = -e^{-2\omega}(Q + \bar{Q})X_u - ie^{-2\omega}(Q - \bar{Q})X_v \\ N_v = -ie^{-2\omega}(Q - \bar{Q})X_u + e^{-2\omega}(Q + \bar{Q})X_v, \end{cases}$$

and the Gauss-Codazzi equation,

$$\Delta\omega - 4Q\bar{Q}e^{-2\omega} = 0 \quad \text{and} \quad Q_{\bar{z}} = 0$$

for $z := u + iv$. Note that the Gauss-Codazzi equation is equivalent to the Hopf differential factor Q being holomorphic. Moreover, the Gauss-Codazzi equation is invariant under the deformation $Q \mapsto \lambda^{-2}Q$ for $\lambda \in \mathbb{S}^1 \subset \mathbb{C}$. In fact, when $X(u, v)$ is a minimal surface in $\mathbb{R}^3$, $\lambda \in \mathbb{S}^1$ allows us to create a single-parameter family of minimal surfaces $X^\lambda(u, v)$ associated to $X(u, v)$, called the *associated family*. In particular if $\lambda^{-2} = i$, then the new surface is called the *conjugate* surface of X.

Since the families of curvature lines of a plane are trivially planar, we may assume that $X(u, v)$ is not totally umbilic, and that (u, v) are conformal curvature line (or isothermic) coordinates. Then we can normalize the Hopf differential such that $Q = -\frac{1}{2}$, and the relevant Gauss-Weingarten equations become

$$\begin{cases} X_{uu} = \omega_u X_u - \omega_v X_v - N \\ X_{vv} = -\omega_u X_u + \omega_v X_v + N \\ X_{uv} = \omega_v X_u + \omega_u X_v \\ N_u = e^{-2\omega} X_u \\ N_v = -e^{-2\omega} X_v \end{cases} \tag{7.1.1}$$

where $k_1 = -e^{-2\omega}$ and $k_2 = e^{-2\omega}$ are the principal curvatures of X. Furthermore, the Gauss equation becomes the following Liouville equation:

$$\Delta\omega - e^{-2\omega} = 0.$$

On the other hand, as the following lemma shows, we may attain an additional partial differential equation regarding ω from the planar curvature line condition, allowing us to solve for ω.

Lemma 7.1.1. *For non-planar umbilic-free minimal surfaces with isothermic coordinates (u, v), the following statements are equivalent:*

1. *u-curvature lines are planar.*
2. *v-curvature lines are planar.*
3. *$\omega_{uv} + \omega_u \omega_v = 0$.*

Proof. Since (u, v) is an isothermic coordinate, u-curvature lines are planar if and only if

$$\det(X_u, X_{uu}, X_{uuu}) = 0.$$

However, from (7.1.1),

$$X_{uuu} = (\omega_{uu} + \omega_u^2 - \omega_v^2 - e^{-2\omega})X_u + (-2\omega_u\omega_v - \omega_{uv})X_v - \omega_u N.$$

Therefore, u-curvature lines are planar if and only if

$$0 = \det(X_u, X_{uu}, X_{uuu}) = -e^{2\omega}(\omega_{uv} + \omega_u\omega_v).$$

Similarly, (7.1.1) implies that v-curvature lines are planar if and only if $\omega_{uv} + \omega_u\omega_v = 0$. □

Remark 7.1.1. It should be noted that the condition $\omega_{uv}+\omega_u\omega_v = 0$ is equivalent to the condition $(e^\omega)_{uv} = 0$ as found in [5] or [6]. Since the curvature lines of the original minimal surface correspond to asymptotic lines of its conjugate surface, the above equivalence shows that the conjugate of the minimal surface with planar curvature lines is an affine minimal surface ([5], [6], [85], [88]).

Hence, finding non-planar umbilic-free minimal surfaces with planar curvature lines is equivalent to finding solutions to the following system of partial differential equations:

$$\begin{cases} \Delta\omega - e^{-2\omega} = 0 & \text{(minimality condition)} \qquad (7.1.2a) \\ \omega_{uv} + \omega_u\omega_v = 0 & \text{(planar curvature line condition).} \qquad (7.1.2b) \end{cases}$$

Generally, solving systems of partial differential equations may prove to be difficult. However, the next lemma shows that (7.1.2) can be reduced to a system of ordinary differential equations.

Lemma 7.1.2. *The solution $\omega : \Sigma \to \mathbb{R}$ of* (7.1.2) *is precisely given by*

$$e^{\omega(u,v)} = \frac{1 + f(u)^2 + g(v)^2}{f_u(u) + g_v(v)} \tag{7.1.3}$$

where $f(u)$ and $g(v)$ are real-valued meromorphic functions satisfying the following system of ordinary differential equations:

$$\begin{cases} (f_u(u))^2 = (c-d)f(u)^2 + c & (7.1.4a) \\ f_{uu}(u) = (c-d)f(u) & (7.1.4b) \\ (g_v(v))^2 = (d-c)g(v)^2 + d & (7.1.4c) \\ g_{vv}(v) = (d-c)g(v). & (7.1.4d) \end{cases}$$

for some real constants c and d such that $c^2 + d^2 \neq 0$. Moreover, f and g can be recovered from ω by

$$\begin{cases} \omega_u = e^{-\omega} f(u) \\ \omega_v = e^{-\omega} g(v) \end{cases} . \tag{7.1.5}$$

Proof. Integrating (7.1.2b) with respect to u and v gives (7.1.5) for some constants of integration $f(u)$ and $g(v)$. Using these definitions of f and g, it is straightforward to check that (7.1.3) holds.

Now, from the fact that $\omega_u e^{-\omega} = e^{-2\omega} f$,

$$\frac{f_{uu}}{1 + f^2 + g^2} - \frac{f((f_u)^2 - (g_v)^2)}{(1 + f^2 + g^2)^2} = 0, \tag{7.1.6}$$

and multiplying both sides by $2f_u$ and integrating with respect to u tells us that

$$\frac{(f_u)^2}{1+f^2+g^2} = \frac{(g_v)^2}{1+f^2+g^2} + D(v)$$

for some constant of integration $D(v)$, implying that

$$(f_u)^2 = (g_v)^2 + D(v)(1+f^2+g^2). \tag{7.1.7}$$

Substituting (7.1.7) into (7.1.6), we get $f_{uu}(u) = D(v)f(u)$ implying that $D(v) = \tilde{c}$ for some constant $\tilde{c}$. Hence,

$$f_{uu} = \tilde{c}f,$$

and again multiplying both sides by $2f_u$ and integrating with respect to u implies that

$$(f_u)^2 = \tilde{c}f^2 + c$$

for some constant c.

Similarly, from the fact that $\omega_v e^{-\omega} = e^{-2\omega}g$, we can show that

$$\begin{cases} g_{vv} = \tilde{d}g \\ (g_v)^2 = \tilde{d}g^2 + d \end{cases}$$

for some constants d and $\tilde{d}$. Substituting these differential equations into (7.1.6) shows that $-\tilde{d} = \tilde{c} = c - d$. □

To find the explicit solution for f, we must consider the initial conditions of $f(u)$ and $g(v)$ satisfying (7.1.4). We would like to assume $f(0) = g(0) = 0$ for simplicity; therefore, we first identify the conditions for $f(u)$ and $g(v)$ having a zero and prove that both $f(u)$ and $g(v)$ has a zero, using the next pair of lemmas.

Lemma 7.1.3. *$f(u)$ (resp. $g(v)$) satisfying Lemma 7.1.2 has a zero if and only if $c \geq 0$ (resp. $d \geq 0$).*

Proof. First, we note that by (7.1.5), $f(u)$ satisfying Lemma 7.1.2 must be real-valued since $\omega(u, v)$ is real-valued.

Now, if $c = d$, then the statement is a result of direct computation; hence, we assume $c \neq d$. First, to see that $c \geq 0$ implies that $f(u)$ has a zero, from (7.1.4a) and (7.1.4b), we get

$$f(u) = C_1 e^{\sqrt{c-d}\,u} + C_2 e^{-\sqrt{c-d}\,u}$$

for some complex constants C_1 and C_2 such that $c = -4C_1C_2(c-d)$. If $c = 0$, then either $f(u) \equiv 0$ or (7.1.4a) implies $d < 0$, a contradiction to (7.1.4c). Now assume $c > 0$. Then, C_1 and C_2 are non-zero, and we may let $C_1 = \frac{1}{2}\sqrt{\frac{c}{c-d}} = -C_2$ so that f is real-valued and $f(0) = 0$.

On the other hand, if $f(u_0) = 0$ for some u_0, then $(f_u(u_0))^2 = c$, implying that $c \geq 0$. Therefore, $f(u)$ has a zero if and only if $c \geq 0$. The case for $g(v)$ is proven similarly using (7.1.4d). □

Lemma 7.1.4. *Let $f(u)$ and $g(v)$ be functions satisfying* (7.1.4). *Then both $f(u)$ and $g(v)$ have a zero.*

Proof. Suppose by way of contradiction that f does not have a zero. Then by the previous lemma, $c < 0$. From (7.1.4a), we see that $c < 0$ implies $c-d > 0$. However, (7.1.4c) implies that if $c-d > 0$, then $d > 0$, a contradiction since $c < 0$ and $c > d$. Therefore, f must always have a zero. Similarly, g must have a zero, from (7.1.4b) and (7.1.4d). $\square$

By shifting parameters u and v, we may assume $f(0) = g(0) = 0$. Using these initial conditions, we may solve (7.1.4) to get the following:

$$\begin{aligned} f(u) &= \begin{cases} \pm\dfrac{\alpha}{\sqrt{\alpha^2-\beta^2}}\sinh\left(\sqrt{\alpha^2-\beta^2}\,u\right), & \text{if } \alpha \neq \beta \\ \pm\alpha u, & \text{if } \alpha = \beta \end{cases} \\ g(v) &= \begin{cases} \pm\dfrac{\beta}{\sqrt{\beta^2-\alpha^2}}\sinh\left(\sqrt{\beta^2-\alpha^2}\,v\right), & \text{if } \alpha \neq \beta \\ \pm\beta v, & \text{if } \alpha = \beta \end{cases} \end{aligned} \tag{7.1.8}$$

where $\alpha^2 = c$ and $\beta^2 = d$. It should be noted that by letting $u \mapsto -u$ and $v \mapsto -v$, we may drop the plus or minus condition of (7.1.8). Finally, we arrive at the following result.

Proposition 7.1.1. *For a non-planar minimal surface $X(u, v)$ with planar curvature lines, the real-analytic solution $\omega : \mathbb{R}^2 \to \mathbb{R}$ of (7.1.2) is precisely given by*

$$e^{\omega(u,v)} = \frac{1 + f(u)^2 + g(v)^2}{f_u(u) + g_v(v)} \tag{7.1.9}$$

with

$$\begin{aligned} f(u) &= \begin{cases} \dfrac{\alpha}{\sqrt{\alpha^2-\beta^2}}\sinh\left(\sqrt{\alpha^2-\beta^2}\,u\right), & \text{if } \alpha \neq \beta \\ \alpha u, & \text{if } \alpha = \beta \end{cases} \\ g(v) &= \begin{cases} \dfrac{\beta}{\sqrt{\beta^2-\alpha^2}}\sinh\left(\sqrt{\beta^2-\alpha^2}\,v\right), & \text{if } \alpha \neq \beta \\ \beta v, & \text{if } \alpha = \beta \end{cases} \end{aligned} \tag{7.1.10}$$

where $\alpha + \beta > 0$.

Proof. To see that the ω in (7.1.9) with $f(u)$ and $g(v)$ as in (7.1.10) is real, we only need to show that $f_u(u) + g_v(v) > 0$ for any $(u, v) \in \Sigma$. If $\alpha = \beta$, then $f_u + g_v = \alpha + \beta > 0$. Without loss of generality, assume $\alpha > \beta$; since $\alpha + \beta > 0$, $\alpha > |\beta|$. From (7.1.10),

$$\begin{aligned} f_u(u) &= \alpha\cosh(\sqrt{\alpha^2-\beta^2}\,u) \geq \alpha \\ g_v(v) &= \beta\cos(\sqrt{\alpha^2-\beta^2}\,v) \geq -|\beta| \end{aligned}$$

implying that $f_u + g_v \geq \alpha - |\beta| > 0$. The case for $\alpha < \beta$ can be proved similarly. Finally, the real-analyticity of $f(u)$ and $g(v)$ tells us that the domain of $\omega(u, v)$ can be extended to $\mathbb{R}^2$ globally. $\square$

Since the u-direction and v-direction of $\omega(u, v)$ depend only on $f(u)$ and $g(v)$ respectively, by choosing different values for α and β, we may analytically understand how the surfaces behave in either direction. The following theorem and figure explains the relationship between different values of α and β and the surface generated by the corresponding $\omega(u, v)$. Note that in the figure, the subscript $u \leftrightarrow v$ denotes that the role of u and v are switched.

Theorem 7.1.1. *Let $X(u,v)$ be a non-planar minimal surface in $\mathbb{R}^3$ with isothermic coordinates (u,v) such that $\mathrm{d}s^2 = e^{2\omega}(\mathrm{d}u^2 + \mathrm{d}v^2)$. Then X has planar curvature lines if and only if $\omega(u,v)$ satisfies Proposition 7.1.1. Furthermore, for different values of α and β, the curvature lines of $X(u,v)$ have the following properties, based on Figure 7.1:*

- *①, ①′ are not periodic in the u-direction but periodic in the v-direction.*
- *② is not periodic in the u-direction but constant in the v-direction.*
- *③ is not periodic in both the u-direction and v-direction.*

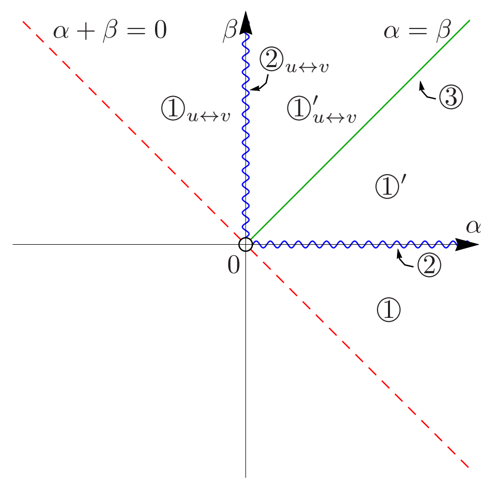


Fig. 7.1: Classification diagram for non-planar minimal surfaces with planar curvature lines.

7.1.2 Axial directions and normal vector

Through Theorem 7.1.1, we were able to identify the non-planar minimal surfaces with planar curvature lines. In fact, we may even understand that the surfaces represented by ①, ②, or ③ in Figure 7.1 are surfaces in the Bonnet family, catenoid, or Enneper surface, respectively. However, we would like to find their parametrizations, allowing us to visualize these surfaces and obtain a deformation between them. To do this, we utilize the Weierstrass representation theorem as follows: find the axial directions of the non-planar minimal surfaces with planar curvature lines by referring to the method developed by Walter in [93], calculate the unit normal vector, and recover the Weierstrass data from the metric function $e^{2\omega(u,v)}$ in Proposition 7.1.1. First, we show the existence of axial directions for non-planar minimal surfaces with planar curvature lines.

Proposition 7.1.2. *If $f(u)$ (resp. $g(v)$) is not identically equal to zero, then there is a unique constant direction $\vec{v}_1$ (resp. $\vec{v}_2$) such that*

$$\langle m(u,v), \vec{v}_1\rangle = \langle m_v(u,v), \vec{v}_1\rangle = 0 \tag{7.1.11}$$

$$(\textit{resp. } \langle n(u,v), \vec{v}_2\rangle = \langle n_u(u,v), \vec{v}_2\rangle = 0)$$

where $m = e^{-2\omega}(X_u \times X_{uu})$ *(resp.* $n = e^{-2\omega}(X_v \times X_{vv})$*) and*

$$\vec{v}_1 = \omega_{uu} X_u - \omega_{uv} X_v + \omega_u N$$

$$(\textit{resp. } \vec{v}_2 = \omega_{uv} X_u - \omega_{vv} X_v + \omega_v N).$$

Furthermore, if $\vec{v}_1$ *and* $\vec{v}_2$ *both exist, then* $\vec{v}_1$ *is orthogonal to* $\vec{v}_2$*. We call* $\vec{v}_1$ *and* $\vec{v}_2$ *the* axial directions *of the surface.*

Proof. We will only prove the statement regarding $\vec{v}_1$. From (7.1.1),

$$m = e^{-2\omega} X_v - \omega_v N \quad \text{and} \quad m_v = -\omega_u e^{-2\omega} X_u + \omega_{uu} N.$$

Since $f(u)$ is not identically equal to zero, let $f(u_0) \neq 0$. Since $\omega_u(u_0, v) \neq 0$, m and m_v are linearly independent at (u_0, v). Therefore, the following definition

$$\vec{v}_1 := e^{-2\omega}(m \times m_v) = \omega_{uu} X_u - \omega_{uv} X_v + \omega_u N$$

is well-defined on (u_0, v). Furthermore, by calculation,

$$(\vec{v}_1)_u = (\vec{v}_1)_v = 0$$

for all (u, v), implying that $\vec{v}_1$ is constant. It is easy to check that (7.1.11) holds, and checking orthogonality is a straight-forward calculation using the definitions of $\vec{v}_1$ and $\vec{v}_2$. □

By normalizing these vectors, we may calculate the unit normal vector of the surface as follows.

Proposition 7.1.3. *Let* $f(u)$ *and* $g(v)$ *be as in Proposition 7.1.1. If* $\alpha\beta \neq 0$*, then the unit normal vector* $N(u, v)$ *is given by the following:*

$$N(u, v) = \left(\frac{1}{\alpha}\,\omega_u, \frac{1}{\beta}\,\omega_v, \sqrt{1 - \frac{1}{\alpha^2}\,\omega_u^2 - \frac{1}{\beta^2}\,\omega_v^2} \right).$$

Proof. First we normalize the axial direction such that $\vec{v}_1$ is parallel to $\mathbf{e}_1$, the unit vector in the x_1-direction. Then, from $\langle m, \vec{v}_1 \rangle = 0$, we get

$$-\omega_v N_1 - (N_1)_v = 0, \tag{7.1.12}$$

where $N = (N_1, N_2, N_3)$. Now, using (7.1.2b) and integrating with respect to v, we obtain

$$N_1 = B_1(u) \cdot \omega_u$$

for some function $B_1(u)$. Similarly, from $\langle m_v, \vec{v}_1 \rangle = 0$, we get

$$N_1 = B_2(v) \cdot \omega_u$$

for some function $B_2(v)$. Therefore, $B_1(u) = B_2(v) = B$ for some constant B, and

$$N_1 = B \cdot \omega_u.$$

To compute B, first note that since $\omega_u(0, v) = e^{\omega(0,v)} f(0) = 0$, we note that $\langle m, X_u \rangle = \langle m_v, X_u \rangle = 0$ on $(0, v)$. Therefore, $X_u(0, v) \in \operatorname{span}\{\mathbf{e}_1\}$, and

$$e^{2\omega} = \|X_u(0, v)\|^2 = ((X_1(0, v))_v)^2 = e^{2\omega} B^2 \alpha^2$$

where $X = (X_1, X_2, X_3)$. Hence,

$$B = \frac{1}{\alpha} \quad \text{and} \quad N_1 = \frac{1}{\alpha}\,\omega_u.$$

Similarly, by letting $\vec{v}_2$ be parallel to $\mathbf{e}_2$, we get

$$N_2 = \frac{1}{\beta}\,\omega_v.$$

Finally, using the fact that N is a unit normal vector gives us the desired result. □

Using the normal vector, we may now calculate the Weierstrass data. Since the meromorphic function h is the normal vector function under stereographic projection,

$$h^{(\alpha,\beta)}(u,v) = \frac{1}{1-N_3}\,(N_1 + iN_2) = \frac{\sqrt{\alpha^2-\beta^2}}{\alpha-\beta}\tanh\left(\frac{\sqrt{\alpha^2-\beta^2}}{2}\,(u+iv)\right),$$

while since $Q = -\frac{1}{2}(h_u + ih_v)\eta = -\frac{1}{2}$, we also have

$$\eta^{(\alpha,\beta)}(u,v) = \frac{1}{h_u + ih_v} = \frac{1}{\alpha+\beta}\cosh^2\left(\frac{\sqrt{\alpha^2-\beta^2}}{2}(u+iv)\right)$$

for $\alpha+\beta > 0$. If we let $\alpha = r\cos\theta$ and $\beta = r\sin\theta$, then it is easy to see that r is a homothety factor. Therefore, we may assume $r = 1$, and rewrite $h^{(\alpha,\beta)}(u,v)$ and $\eta^{(\alpha,\beta)}(u,v)$ as follows:

$$\begin{aligned}
h^{\theta}(u,v) &= \begin{cases} \dfrac{\sqrt{\cos(2\theta)}}{\cos\theta-\sin\theta}\tanh\left(\dfrac{\sqrt{\cos(2\theta)}}{2}(u+iv)\right), & \text{if } \theta \neq \frac{\pi}{4} \\ \dfrac{u+iv}{\sqrt{2}}, & \text{if } \theta = \frac{\pi}{4} \end{cases} \\
\eta^{\theta}(u,v) &= \begin{cases} \dfrac{1}{\cos\theta+\sin\theta}\cosh^2\left(\dfrac{\sqrt{\cos(2\theta)}}{2}(u+iv)\right), & \text{if } \theta \neq \frac{\pi}{4} \\ \dfrac{1}{\sqrt{2}}, & \text{if } \theta = \frac{\pi}{4} \end{cases}
\end{aligned} \tag{7.1.13}$$

where $\theta \in \left(-\frac{\pi}{4}, \frac{3\pi}{4}\right)$. Since h^θ is meromorphic, and η^θ is holomorphic such that $(h^\theta)^2\eta^\theta$ is holomorphic, we may use the Weierstrass representation theorem to obtain the following parametrizations for minimal surfaces with planar curvature lines.

Proposition 7.1.4. *Let $X(u,v)$ be a non-planar minimal surface with planar curvature lines in*

$\mathbb{R}^3$. Then X must have the following parametrization

$$X^\theta(u,v) = \begin{cases} \begin{pmatrix} \dfrac{u\cos\theta\sqrt{\cos 2\theta} - \sin\theta \sinh\left(u\sqrt{\cos 2\theta}\right)\cos\left(v\sqrt{\cos 2\theta}\right)}{(\cos 2\theta)^{3/2}} \\ \dfrac{v\sin\theta\sqrt{\cos 2\theta} - \cos\theta \cosh\left(u\sqrt{\cos 2\theta}\right)\sin\left(v\sqrt{\cos 2\theta}\right)}{(\cos 2\theta)^{3/2}} \\ \dfrac{\cosh\left(u\sqrt{\cos 2\theta}\right)\cos\left(v\sqrt{\cos 2\theta}\right) - 1}{\cos 2\theta} \end{pmatrix}^t, & \text{if } \theta \neq \frac{\pi}{4} \\ \left(-\dfrac{u(-6+u^2-3v^2)}{6\sqrt{2}}, \dfrac{v(-6-3u^2+v^2)}{6\sqrt{2}}, \dfrac{u^2-v^2}{2}\right), & \text{if } \theta = \frac{\pi}{4} \end{cases} \tag{7.1.14}$$

for some $\theta \in \left(-\frac{\pi}{4}, \frac{3\pi}{4}\right)$ on its domain up to isometries and homotheties of $\mathbb{R}^3$.

7.2 Continuous deformation of minimal surfaces with planar curvature lines

In the previous section, we obtained the Weierstrass data and the parametrizations of non-planar minimal surfaces in $\mathbb{R}^3$ with planar curvature lines; in fact, the Weierstrass data and the parametrizations of such surfaces depended on a single parameter θ. In this section, we show that this parameter defines a locally continuous deformation between non-planar minimal surfaces preserving the planar curvature line condition. Furthermore, we show that by introducing a suitable homothety factor depending on θ, we may also extend the deformation to include the plane.

First, we show that the deformation of minimal surfaces in $\mathbb{R}^3$ with planar curvature lines we obtain by using the parameter θ is continuous. By "continuous", we mean that the deformation converges uniformally over compact subdomains component-wise. To show this, it is enough to show that each component function in the parametrization is continuous for all θ at any point (u,v) in the domain. The continuity is self-evident at any $\theta \neq \frac{\pi}{4}$; hence, we only need to check the for the case $\theta = \frac{\pi}{4}$.

However, for the Weierstrass data of minimal surfaces with planar curvature lines as stated in (7.1.13), it is easy to check that at any point (u,v),

$$\lim_{\theta \to \frac{\pi}{4}} h^\theta(u,v) = h^{\frac{\pi}{4}}(u,v) \quad \text{and} \quad \lim_{\theta \to \frac{\pi}{4}} \eta^\theta(u,v) = \eta^{\frac{\pi}{4}}(u,v).$$

In addition, each component of the parametrization in (7.1.14) is also continuous at $\theta = \frac{\pi}{4}$ at any point (u,v), as

$$\lim_{\theta \to \frac{\pi}{4}} X^\theta(u,v) = X^{\frac{\pi}{4}}(u,v).$$

Therefore, $X^\theta(u,v)$ is a continuous deformation.

Now, we would like to extend $X^\theta(u,v)$ to include the plane. To do so, we define the homotethy factor R^θ as follows:

$$R^\theta = \left(1 - \sin\left(\theta + \frac{\pi}{4}\right)\right)|\cos 2\theta| + \sin\left(\theta + \frac{\pi}{4}\right)$$

for $\theta \in [-\frac{\pi}{4}, \frac{3\pi}{4}]$. Note that $R^\theta \geq 0$, and $R^\theta = 0$ if and only if θ is at an endpoint of its domain.

If we consider $\tilde{X}^\theta(u,v) = R^\theta X^\theta(u,v)$, then

$$\lim_{\theta \searrow -\frac{\pi}{4}} \tilde{X}^\theta(u,v) = \frac{3}{\sqrt{2}}(u,-v,0) = \lim_{\theta \nearrow \frac{3\pi}{4}} \tilde{X}^\theta(u,v).$$

Therefore, the extension of $X^\theta(u,v)$ to $\tilde{X}^\theta(u,v)$ defined as

$$\tilde{X}^\theta(u,v) = \begin{cases} R^\theta X^\theta(u,v), & \text{if } \theta \in \left(-\frac{\pi}{4}, \frac{3\pi}{4}\right) \\ \frac{3}{\sqrt{2}}(u,-v,0), & \text{if } \theta = -\frac{\pi}{4}, \frac{3\pi}{4} \end{cases}$$

is again, a continuous deformation for $\theta \in [-\frac{\pi}{4}, \frac{3\pi}{4}]$.

In conclusion, we obtain the following classification and deformation of minimal surfaces with planar curvature lines.

Theorem 7.2.1. *If $\tilde{X}(u,v)$ is a minimal surface with planar curvature lines in $\mathbb{R}^3$, then the surface is given by the following parametrization on its domain*

$$\tilde{X}^\theta(u,v) = \begin{cases} R^\theta \begin{pmatrix} \dfrac{u\cos\theta\sqrt{\cos 2\theta} - \sin\theta \sinh\left(u\sqrt{\cos 2\theta}\right)\cos\left(v\sqrt{\cos 2\theta}\right)}{(\cos 2\theta)^{3/2}} \\ \dfrac{v\sin\theta\sqrt{\cos 2\theta} - \cos\theta \cosh\left(u\sqrt{\cos 2\theta}\right)\sin\left(v\sqrt{\cos 2\theta}\right)}{(\cos 2\theta)^{3/2}} \\ \dfrac{\cosh\left(u\sqrt{\cos 2\theta}\right)\cos\left(v\sqrt{\cos 2\theta}\right) - 1}{\cos 2\theta} \end{pmatrix}^t, & \\ \hfill \textit{if } \theta \neq -\frac{\pi}{4}, \frac{\pi}{4}, \frac{3\pi}{4} & \\ \left(-\dfrac{u(-6+u^2-3v^2)}{6\sqrt{2}}, \dfrac{v(-6-3u^2+v^2)}{6\sqrt{2}}, \dfrac{u^2-v^2}{2}\right), & \textit{if } \theta = \frac{\pi}{4} \\ \dfrac{3}{\sqrt{2}}(u,-v,0), & \textit{if } \theta = -\frac{\pi}{4}, \frac{3\pi}{4} \end{cases}$$

up to isometries and homotheties of $\mathbb{R}^3$ for some $\theta \in \left[-\frac{\pi}{4}, \frac{3\pi}{4}\right]$, where $R^\theta = \left(1 - \sin\left(\theta + \frac{\pi}{4}\right)\right)|\cos 2\theta| + \sin\left(\theta + \frac{\pi}{4}\right)$. In fact, it must be a piece of one, and only one, of the following:

- *plane ($\theta = -\frac{\pi}{4}, \frac{3\pi}{4}$),*
- *catenoid ($\theta = 0, \frac{\pi}{2}$),*
- *Enneper surface ($\theta = \frac{\pi}{4}$), or*
- *a surface in the Bonnet family ($\theta \in (-\frac{\pi}{4}, \frac{3\pi}{4}) \setminus \{0, \frac{\pi}{4}, \frac{\pi}{2}\}$).*

Moreover, the deformation $\tilde{X}^\theta(u,v)$ depending on the parameter θ is continuous.

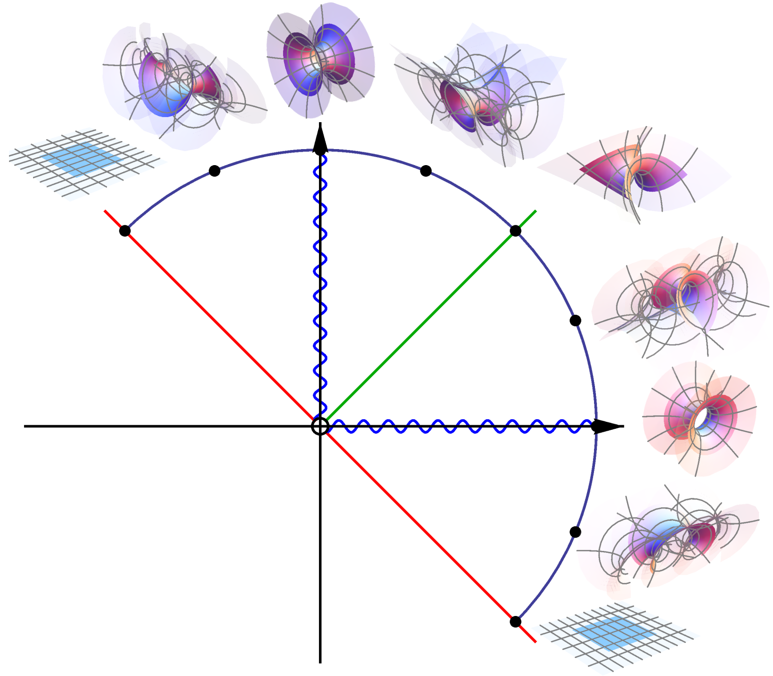


Fig. 7.2: Deformation of minimal surfaces with planar curvature lines with parametrization as in Theorem 7.2.1.

Furthermore, by considering the conjugate of minimal surfaces with planar curvature lines, we get the following classification and deformation of minimal surfaces that are also affine minimal.

Corollary 7.2.1. *If $\hat{X}(u,v)$ is a minimal surface that is also an affine minimal surface in $\mathbb{R}^3$, then the surface is given by the following parametrization on its domain*

$$\hat{X}^\theta(u,v) = \begin{cases} R^\theta \begin{pmatrix} \dfrac{u\cos\theta\sqrt{\cos 2\theta} - \sin\theta\cosh\left(u\sqrt{\cos 2\theta}\right)\sin\left(v\sqrt{\cos 2\theta}\right)}{-(\cos 2\theta)^{3/2}} \\ \dfrac{v\sin\theta\sqrt{\cos 2\theta} - \cos\theta\sinh\left(u\sqrt{\cos 2\theta}\right)\cos\left(v\sqrt{\cos 2\theta}\right)}{(\cos 2\theta)^{3/2}} \\ \dfrac{-\sinh\left(u\sqrt{\cos 2\theta}\right)\sin\left(v\sqrt{\cos 2\theta}\right)}{\cos 2\theta} \end{pmatrix}^t, & \text{if } \theta \neq -\frac{\pi}{4}, \frac{\pi}{4}, \frac{3\pi}{4} \\ \left(-\dfrac{v(6-3u^2+v^2)}{6\sqrt{2}}, \dfrac{u(6+u^2-3v^2)}{6\sqrt{2}}, -uv\right), & \text{if } \theta = \frac{\pi}{4} \\ \dfrac{3}{\sqrt{2}}(-v,-u,0), & \text{if } \theta = -\frac{\pi}{4}, \frac{3\pi}{4} \end{cases}$$

up to isometries and homotheties of $\mathbb{R}^3$ *for some* $\theta \in \left[-\frac{\pi}{4}, \frac{3\pi}{4}\right]$. *In fact, it must be a piece of one, and only one, of the following:*

- *plane* ($\theta = -\frac{\pi}{4}, \frac{3\pi}{4}$),
- *helicoid* ($\theta = 0, \frac{\pi}{2}$),
- *Enneper surface* ($\theta = \frac{\pi}{4}$), *or*
- *a surface in the Thomsen family* ($\theta \in (-\frac{\pi}{4}, \frac{3\pi}{4}) \setminus \{0, \frac{\pi}{4}, \frac{\pi}{2}\}$).

Moreover, the deformation $\hat{X}^\theta(u, v)$ *depending on the parameter* θ *is continuous.*

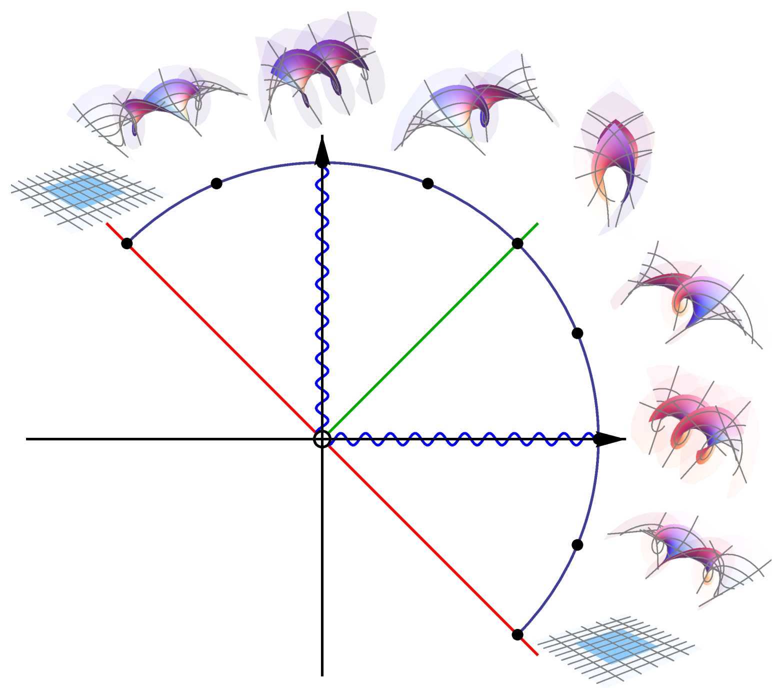


Fig. 7.3: Deformation of minimal surfaces that are also affine minimal with parametrization as in the corollary to Theorem 7.2.1.

7.3 Remarks for our future works

With appropriate modifications, the analytic method in conjunction with axial directions introduced in this paper can be applied to study maximal surfaces with planar curvature lines in Lorentz-Minkowski space $\mathbb{R}^{2,1}$. The complete classification is already given by Leite in [62], where she developed and used the orthogonal systems of cycles on the hyperbolic plane. The analytic method can also be used to fully classify such surfaces; however, using the analytic method further allows us to see that all maximal surfaces with planar curvature lines can be joined by a single deformation. In addition, using this method allows us to understand that maximal Bonnet-type surfaces can be classified into three general types analytically, and five specific types by means of singularity theory. We will defer a complete discussion of these properties to our subsequent work [21].

Chapter 8

Discrete DPW method

8.1 The DPW method for smooth CMC surfaces

First we introduce construction of smooth CMC $H \neq 0$ surfaces in $\mathbb{R}^3$, $\mathbb{S}^3$ and $\mathbb{H}^3$ introduced in [25] (see also [30]), which is now called the DPW method. Throughout this paper, $I := \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$, $\sigma_1 := \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, $\sigma_2 := \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$, $\sigma_3 := \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$, $\mathbb{S}^3 := \{Y \in \mathbb{R}^4 | \langle Y, Y \rangle = 1\}$, $\mathbb{R}^{3,1}$ denotes Minkowski 4-space with signature $(+++-)$, and $\mathbb{H}^3 := \{Y \in \mathbb{R}^{3,1} | \langle Y, Y \rangle = -1\}$.

Let Σ be in a simply connected domain in the complex plane $\mathbb{C}$ with the usual complex coodinate $z = x+iy$, let $f : \Sigma \to \mathbb{R}^3$ be a conformal immersion satisfying $\langle f_u, f_u \rangle = \langle f_v, f_v \rangle = 4e^{2u}$, $\langle f_u, f_v \rangle = 0$ for some scalar function $u : \Sigma \to \mathbb{R}$, and let $N : \Sigma \to \mathbb{S}$ be its unit normal vector field. In this paper we identify $\mathbb{R}^4$ (resp. $\mathbb{R}^{3,1}$) with the unitary group $\left\{ X \in M_{2\times 2} \left| X \cdot \bar{X}^t = I \right. \right\}$ (resp. another matrix group) as follows:

$$\mathbb{R}^4 \text{ (or, } \mathbb{R}^{3,1}) \ni x = (x_1, x_2, x_3, x_4) \longmapsto \begin{pmatrix} x_4 + \nu \cdot x_3 & x_1 - i x_2 \\ -\varepsilon \cdot (x_1 + i x_2) & x_4 - \nu \cdot x_3 \end{pmatrix}, \tag{8.1.1}$$

with $\nu = i, \varepsilon = 1$ for $\mathbb{R}^4$ (resp. $\nu = 1, \varepsilon = -1$ for $\mathbb{R}^{3,1}$). The metric becomes, under this identification,

$$\langle X, Y \rangle = \varepsilon \cdot \frac{1}{2} \text{trace}(X \sigma_2 Y^t \sigma_2).$$

In particular, $\langle X, X \rangle = \varepsilon \cdot \det(X)$, and we can identify $\mathbb{S}^3$ (resp. $\mathbb{H}^3$), with SU_2 respectively, with the self-adjoint matrices ($X = \bar{X}^t$), via

$$\mathbb{H}^3 \ni x = (x_1, x_2, x_3, x_4) \longmapsto \begin{pmatrix} x_4 + x_3 & x_1 - i x_2 \\ x_1 + i x_2 & x_4 - x_3 \end{pmatrix} \in \left\{ F \bar{F}^t \left| F \in \mathrm{SL}_2(\mathbb{C}) \right. \right\}.$$

Here we give a description of the DPW method by using loop groups and algebras. First, we define the holomorphic potential ξ as in Definition 8.1.1:

Definition 8.1.1. Let Σ be a simply-connected domain. Then the $1-$form

$$\xi := A dz, \quad A = A(z, \lambda) = \sum_{j=-1}^{\infty} A_j(z) \lambda^j \text{ for } z \in \Sigma \text{ and } \lambda \in \mathbb{C} \tag{8.1.2}$$

is called a *holomorphic potential*, where each $A_j(z)$ is a 2×2 matrix that is independent of λ, is holomorphic in $z \in \Sigma$, is traceless, is a diagonal (resp. off-diagonal) matrix when j is even (resp. odd), and the upper-right entry of $A_{-1}(z)$ is never zero.

Given a holomorphic potential ξ, we then solve the equation

$$d\varphi = \varphi\xi, \quad \varphi(z_*) = I \quad \text{for} \quad \varphi \in \Lambda\mathrm{SL}_2(\mathbb{C}), \tag{8.1.3}$$

where $\Lambda\mathrm{SL}_2(\mathbb{C}) = \left\{\varphi(\lambda) \in M_{2\times 2}\middle|\varphi : \mathbb{S}^1 \xrightarrow{C^\infty} \mathrm{SL}_2(\mathbb{C}), \varphi(-\lambda) = \sigma_3\varphi(\lambda)\sigma_3\right\}$ for some choice of initial point $z_* \in \Sigma$. We will use the following "SU_2-Iwasawa splitting" of this loop group $\Lambda\mathrm{SL}_2(\mathbb{C})$. The following proposition was proven in [25].

Proposition 8.1.1. *For all φ, there exist unique loops F and B such that*

$$\begin{aligned} &\varphi = F\cdot B, \ \textit{where} \quad F \in \Lambda\mathrm{SU}_2, \ B \in \Lambda^+_{\mathbb{R}}\mathrm{SL}_2(\mathbb{C}), \\ &\Lambda\mathrm{SU}_2 = \left\{\varphi(\lambda) \in M_{2\times 2}\middle|\varphi : \mathbb{S}^1 \xrightarrow{C^\infty} \mathrm{SU}_2, \varphi(-\lambda) = \sigma_3\varphi(\lambda)\sigma_3\right\}, \\ &\Lambda^+_{\mathbb{R}}\mathrm{SL}_2(\mathbb{C}) = \left\{ B_+(\lambda) \in \Lambda\mathrm{SL}_2(\mathbb{C}) \,\middle|\, \begin{array}{c} B_+ \textit{ extends holomorphically to } \mathbb{D}, \\ B_+(0) = \begin{pmatrix} \rho & 0 \\ 0 & \rho^{-1}\end{pmatrix} \textit{ for some } \rho > 0. \end{array}\right\}. \end{aligned} \tag{8.1.4}$$

Using the SU_2-Iwasawa splitting, we have the following recipe for smooth CMC surfaces in $\mathbb{R}^3$, $\mathbb{S}^3$ and $\mathbb{H}^3$, which was originally derived in [25] (see also [30]):

Proposition 8.1.2. *Any conformal CMC $H \neq 0$ (resp. $\cot(-2\gamma_1)$, $\coth(-\gamma)$) surface in $\mathbb{R}^3$ (resp. $\mathbb{S}^3$, $\mathbb{H}^3$) can be constructed by the following steps:*

1. *Set a holomorphic potential ξ, and solve $d\varphi = \varphi\xi$ for some initial data $\varphi|_{z_*\in\Sigma} \in \Lambda\mathrm{SL}_2\mathbb{C}$.*
2. *For the φ obtained in (1), apply SU_2-Iwasawa splitting $\varphi = F\cdot B$ with $F \in \Lambda\mathrm{SU}_2$, $B \in \Lambda^+_{\mathbb{R}}\mathrm{SL}_2\mathbb{C}$.*

$\mathbb{R}^3$ **case:** *Substitute the F in (2) into*

$$f = \frac{1}{2H}\left[\frac{-1}{2}Fi\sigma_3F^{-1} - i\lambda(\partial_\lambda F)F^{-1}\right]\Bigg|_{\lambda=1} \tag{8.1.5}$$

with unit normal vector field $N = \frac{1}{2}\left[Fi\sigma_3F^{-1}\right]\Big|_{\lambda=1}$.

$\mathbb{S}^3$ **case:** *Set $F_1 = F|_{\lambda=e^{i\gamma_1}}$ and $F_2 = F|_{\lambda=e^{-i\gamma_1}}$ for $\gamma_1 \in \mathbb{R}$ and $2\gamma_1 \neq n\pi$ ($n \in \mathbb{Z}$), and substitute the F in (2) into*

$$f = F_1\begin{pmatrix} e^{i\gamma_1} & 0 \\ 0 & e^{-i\gamma_1}\end{pmatrix}{F_2}^{-1} \tag{8.1.6}$$

with normal vector field $N = iF_1\begin{pmatrix} e^{i\gamma_1} & 0 \\ 0 & -e^{-i\gamma_1}\end{pmatrix}{F_2}^{-1}$.

$\mathbb{H}^3$ **case:** *Set $F_0 = F|_{\lambda=e^{\gamma/2}}$ for $\gamma \in \mathbb{R}\setminus\{0\}$, and substitute the F in (2) into*

$$f = F_0\begin{pmatrix} e^{\gamma/2} & 0 \\ 0 & e^{-\gamma/2}\end{pmatrix}\overline{F_0}^t \tag{8.1.7}$$

with normal vector field $N = F_0 \begin{pmatrix} e^{\gamma/2} & 0 \\ 0 & -e^{-\gamma/2} \end{pmatrix} \overline{F_0}^t$.

In particular, choosing $\xi = \lambda^{-1} \begin{pmatrix} 0 & h(z) \\ h(z)^{-1} & 0 \end{pmatrix} dz$*, where* $h(z)$ *is a meromorphic function with respect to* z*, we have any smooth isothermic CMC surface in each* 3*-space.*

8.2 Notations in discrete surface theory

In this section we define some notations of discrete differential geometry, as in [9], [12], [19], [43]. In order to define discrete isothermic CMC surfaces, we need to introduce the cross ratio of a quadrilateral. Henceforth, each element in $\mathbb{S}^3$ and $\mathbb{H}^3$ is identified with the matrix form as in Equation (8.1.1).

Definition 8.2.1. Let X_1, X_2, X_3 and X_4 be points in $\mathbb{S}^3$ or $\mathbb{H}^3$. Then

$$cr(X_1, X_2, X_3, X_4) := (X_1 - X_2)(X_2 - X_3)^{-1}(X_3 - X_4)(X_4 - X_1)^{-1}$$

is called the *cross ratio* of X_1, X_2, X_3 and X_4. Note that, the cross ratio $cr(X_1, X_2, X_3, X_4)$ is a scalar function multiple of the identity matrix if and only if the four points X_1, X_2, X_3 and X_4 lie in a circle.

Like in $\mathbb{R}^3$, we define discrete isothermic surfaces in $\mathbb{S}^3$ and $\mathbb{H}^3$, and their dual durfaces. As will be seen later, existence of the dual surfaces characterizes discrete CMC surfaces in Riemannian spaceforms (see [11]).

Definition 8.2.2. A *discrete isothermic surface* is a map $f : \mathbb{Z}^2 \longrightarrow \mathbb{S}^3$ or $\mathbb{H}^3$ for which all elementary quadrilaterals satisfy $cr(f_p, f_q, f_r, f_s) = -\frac{\alpha_{pq}}{\alpha_{ps}} I$ for all

$$p = (m, n),\ q = (m+1, n),\ r = (m+1, n+1),\ s = (m, n+1)\ ((m, n) \in \mathbb{Z}^2),$$

where α_{pq} (resp. α_{ps}) is a positive scalar function depending only on horizontal edges (resp. vertical edges). Then α_{pq}, α_{ps} are called the *cross-ratio factorizing functions.* In particular, a discrete isothermic surface $\mathcal{Z} : \mathbb{Z}^2 \to \mathbb{R}^2 \cong \mathbb{C}$ is called a *discrete holomorphic map.*

Let $f : \mathbb{Z}^2 \longrightarrow \mathbb{S}^3$ or $\mathbb{H}^3$ be a discrete isothermic surface. Then a discrete isothermic surface f^* in $\mathbb{R}^4$ or $\mathbb{R}^{3,1}$ with the same cross-ratio as f solving

$$f^*_q - f^*_p = \alpha_{pq} \frac{f_q - f_p}{\|f_q - f_p\|^2},\ f^*_s - f^*_p = -\alpha_{ps} \frac{f_s - f_p}{\|f_s - f_p\|^2}$$

is called a *dual surface* of f, where $\|\cdot\|^2 := \langle \cdot, \cdot \rangle$. Note that, the dual surface f^* of f is uniquely determined, up to scaling and translation, and f^* is also a discrete isothermic surface.

Remark 8.2.1. In general, a dual surface of a discrete isothermic surface f does not lie in the same spaceform, that is, when f is discrete isothermic in $\mathbb{S}^3$ (resp. $\mathbb{H}^3$), f^* does not necessarily lie in $\mathbb{S}^3$ (resp. $\mathbb{H}^3$). On the other hand, as will be seen in Proposition 8.2.1, a dual surface f^* of a discrete CMC H (resp. H with $|H| > 1$) surface f in $\mathbb{S}^3$ (resp. $\mathbb{H}^3$) can be chosen so that $f^* \in \mathbb{S}^3$ (resp. $\mathbb{H}^3$).

Let $\mathbb{L}^4$ be a 4-dimensional light cone in Minkowski 5-space $\mathbb{R}^{4,1}$ with signature $(++++-)$. Then $M_\kappa := \{X \in \mathbb{L}^4; \langle X, \mathfrak{q}\rangle_{\mathbb{R}^{4,1}} = -1\}$ for some constant vector $\mathfrak{q} \in \mathbb{R}^{4,1}$. Then M_κ has constant sectional curvature $\kappa = -\langle \mathfrak{q}, \mathfrak{q}\rangle$. In particular, take $\mathfrak{q} = (0,0,0,0,1)^t$ (resp. $\mathfrak{q} = (0,0,0,1,0)^t$), and we have $M_1 \cong \mathbb{S}^3$ (resp. $M_{-1} \cong \mathbb{H}^3$). First we define several notations here.

Definition 8.2.3. Let $\mathfrak{f} : \mathbb{Z}^2 \to \mathbb{R}^{4,1}$ be a discrete surface with each quadrilateral $(\mathfrak{f}_p, \mathfrak{f}_q, \mathfrak{f}_r, \mathfrak{f}_s,)$ planar. Then $\mathfrak{f}$ is called a *discrete conjugate net*. And two discrete conjugate surfaces $\mathfrak{f}$ and $\mathfrak{f}^*$ are *Königs dual* to each other if they are discrete surfaces with corresponding edges parallel and with opposite diagonal edges parallel, meaning $\mathfrak{f}_{12} - \mathfrak{f} \parallel \mathfrak{f}^*_2 - \mathfrak{f}^*_1,\ \mathfrak{f}_2 - \mathfrak{f}_1 \parallel \mathfrak{f}^*_{12} - \mathfrak{f}^*$.

As already introduced in [11] (see also [18]), the Gaussian and mean curvatures are defined for the projection of discrete conjugate nets in $\mathbb{L}^4$ to M_κ, that is, discrete conjugate nets in M_κ. In particular, the lift $\mathfrak{f}$ of a discrete isothermic surface f in $\mathbb{S}^3$ or $\mathbb{H}^3$ to $\mathbb{L}^4$ is also a conjugate net and it has a Königs dual partner. In fact, considering the following particular lifts

$$\mathbb{S}^3 \ni f \mapsto \mathfrak{f} := (f, 1)^t \in \mathbb{L}^4 \text{ (resp. } \mathbb{H}^3 \ni (f_1, f_2, f_3, f_4)^t \mapsto \mathfrak{f} := (f_1, f_2, f_3, -1, f_4)^t \in \mathbb{L}^4),$$

one can confirm that the lift $\mathfrak{f}$ of a discrete isothermic surface f in $\mathbb{S}^3$ (resp. $\mathbb{H}^3$) has a Königs dual net $\mathfrak{f}^*$. The $\mathfrak{f}^*$ can be explicitly written as

$$\mathfrak{f}^* = (f^*, 1)^t \quad (\text{resp. } (f^*_1, f^*_2, f^*_3, -1, f^*_4)^t),$$

where $f^* := (f^*_1, f^*_2, f^*_3, f^*_4)$ is a dual surface of f. The Königs duality of $\mathfrak{f}$*and* $\mathfrak{f}^*$ follows from the non-corresponding diagonal parallel condition of discrete isothermic surfaces f and f^* (see [13]). In this paper, restricting the class of $\mathfrak{f}$ to the lift of discrete isothermic surfaces in $\mathbb{S}^3$ or $\mathbb{H}^3$, we define the Gaussian and mean curvatures of discrete isothermic surfaces as follows:

Definition 8.2.4. Let $f : \mathbb{Z}^2 \to \mathbb{S}^3$ (resp. $\mathbb{H}^3$) be a discrete isothermic surface and let $\mathfrak{f}$ be its lift to $\mathbb{L}^4$. Then a discrete surface $n : \mathbb{Z}^2 \to \mathbb{S}^3$ (resp. $\mathbb{S}^{2,1} := \{X \in \mathbb{R}^{3,1} | \langle X, X\rangle = 1\}$) is called the *Gauss map* of f if $\langle f, n\rangle = 0$ and $dn_{pq} + \kappa_{pq} f_{pq} = 0$ and $dn_{ps} + \kappa_{ps} f_{ps} = 0$, where

$$df_{pq} := f_q - f_p,\ df_{ps} := f_s - f_p,\ dn_{pq} := n_q - n_p,\ dn_{ps} := n_s - n_p$$

and κ_{pq}, κ_{ps} are real-valued functions defined on horizontal edges or vertical edges, respectively. Then κ_{pq}, κ_{ps} are called the *principal curvatures* of f. Consider a particular lift

$$\begin{aligned} &\mathbb{S}^3 \ni n \mapsto \mathfrak{n} := (n, 0) \in \mathbb{S}^{3,1} \\ &(\text{resp. } \mathbb{S}^{2,1} \ni (n_1, n_2, n_3, n_4) \mapsto \mathfrak{n} := (n_1, n_2, n_3, 0, n_4) \in \mathbb{S}^{3,1}), \end{aligned} \tag{8.2.1}$$

where $\mathbb{S}^{n,1} := \{X \in \mathbb{R}^{n+1,1} | \langle X, X\rangle_{\mathbb{R}^{n+1,1}} = 1\}$. Then $\mathfrak{n}$ is called the lift of n (or, $\mathfrak{n}$ is called the Gauss map of $\mathfrak{f}$). Note that, $\mathfrak{n}$ satisfies $\langle \mathfrak{f}, \mathfrak{n}\rangle = 0$, $d\mathfrak{n}_{pq} + \kappa_{pq}\mathfrak{f}_{pq} = 0$ and $d\mathfrak{n}_{ps} + \kappa_{ps}\mathfrak{f}_{ps} = 0$ with κ_{pq} and κ_{ps} the same as those of f. The *Gaussian curvature* K and the *mean curvature* H of f are real values satisfying

$$A(\mathfrak{n}, \mathfrak{n}) = KA(\mathfrak{f}, \mathfrak{f}), \quad A(\mathfrak{f}, \mathfrak{n}) = -HA(\mathfrak{f}, \mathfrak{f}),$$

where $A(\mathfrak{g}, \mathfrak{g}) := \dfrac{1}{2}(\delta\mathfrak{g}_{pr} \wedge \delta\mathfrak{g}_{qs})$, $A(\mathfrak{g}, \mathfrak{h}) := \dfrac{1}{4}(\delta\mathfrak{g}_{pr} \wedge \delta\mathfrak{h}_{qs} + \delta\mathfrak{h}_{pr} \wedge \delta\mathfrak{g}_{qs})$ for some discrete conjugate nets $\mathfrak{g}, \mathfrak{h} \in \mathbb{R}^{4,1}$ with $\delta\mathfrak{g}_{pr} := \mathfrak{g}_r - \mathfrak{g}_p$ and $\delta\mathfrak{g}_{qs} := \mathfrak{g}_s - \mathfrak{g}_q$ (regarding the definition of wedge product, see [80], for example).

As already mentioned in Lemma & Def 2.3 in [11], a necessary and sufficient condition for a discrete conjugate net $\mathfrak{f}$ and $\mathfrak{f}^*$ to be Königs dual to each other is $A(\mathfrak{f},\mathfrak{f}^*)=0$. Using that, discrete CMC H (resp. H with $|H|>1$) in $\mathbb{S}^3$ (resp. $\mathbb{H}^3$) can be characterized as follows.

Proposition 8.2.1. *Let f be a discrete isothermic surface in $\mathbb{S}^3$ (resp. $\mathbb{H}^3$), let n be the Gauss map of f, and let $\mathfrak{f}$, $\mathfrak{n}$ be their lifts. Then f is a discrete CMC H (resp. H with $|H|>1$) surface in $\mathbb{S}^3$ (resp. $\mathbb{H}^3$) if and only if its dual surface f^* can be chosen as $f^*=f^{\theta_1}\in\mathbb{S}^3$ (resp. $\mathbb{H}^3$) with θ_1 satisfying $\theta_1=\cot^{-1}H$ (resp. $\coth^{-1}H$), where $f^\theta:=\cos\theta\cdot f+\sin\theta\cdot n$ (resp. $\cosh\theta\cdot f+\sinh\theta\cdot n$). Moreover, f^{θ_2} is a constant positive Gaussian curvature $(|H|+\sqrt{H^2+1})^2$ (resp. $(|H|+\sqrt{H^2-1})^2$) surface in $\mathbb{S}^3$ (resp. $\mathbb{H}^3$), where $\theta_2=\dfrac{1}{2}\cot^{-1}H$ (resp. $\dfrac{1}{2}\coth^{-1}H$).*

Proof. First we consider the case in $\mathbb{S}^3$. By definition, the following two equations

$$A(\mathfrak{n},\mathfrak{n})=KA(\mathfrak{f},\mathfrak{f}),\quad A(\mathfrak{f},\mathfrak{n})=-HA(\mathfrak{f},\mathfrak{f})$$

hold. Assume that H is constant. Then the Königs dual net $\mathfrak{f}^*$ of $\mathfrak{f}$ can be chosen as $\mathfrak{f}^*=H\mathfrak{f}+\mathfrak{n}=\begin{pmatrix}Hf+n\\ H\end{pmatrix}$. By translations and homotheties of $\mathbb{R}^{4,1}$, $\mathfrak{f}^*$ can be chosen as $\mathfrak{f}^*=\begin{pmatrix}\frac{H}{\sqrt{H^2+1}}f+\frac{1}{\sqrt{H^2+1}}n\\ 1\end{pmatrix}\in\mathbb{L}^4$, implying that the projection f^* of $\mathfrak{f}^*$ can be chosen as $f^*=\cos\theta_1\cdot f+\sin\theta_1\cdot n$ ($\theta_1=\cot^{-1}H$).

In this proof K^θ denotes the Gaussian curvature of a parallel surface $f^\theta=\cos\theta\cdot f+\sin\theta\cdot n$ of f at distance θ. Then the unit normal vector field n^θ of f^θ is $n^\theta=-\sin\theta\cdot f+\cos\theta\cdot n$. Let $\mathfrak{f}^\theta$ be a lift of f^θ and let $\mathfrak{n}^\theta$ be a lift of n^θ. Then we have

$$\begin{aligned}&A(\mathfrak{n}^\theta,\mathfrak{n}^\theta)=K^\theta A(\mathfrak{f}^\theta,\mathfrak{f}^\theta)=K^\theta\{\cos^2\theta A(\mathfrak{f},\mathfrak{f})+2\cos\theta\sin\theta A(\mathfrak{f},\mathfrak{n})+\sin^2\theta A(\mathfrak{n},\mathfrak{n})\}\\ \Leftrightarrow\ &\{K\cos^2\theta+H\sin(2\theta)+\sin^2\theta\}A(\mathfrak{f},\mathfrak{f})=K^\theta\{\cos^2\theta-H\sin(2\theta)+K\sin^2\theta\}A(\mathfrak{f},\mathfrak{f}).\end{aligned}$$

Thus we have $K^\theta=\dfrac{K\cos^2\theta+H\sin(2\theta)+\sin^2\theta}{\cos^2\theta-H\sin(2\theta)+K\sin^2\theta}$. Taking $\theta=\theta_2=\dfrac{1}{2}\cot^{-1}H$, we have $K^{\theta_2}=(|H|+\sqrt{H^2+1})^2$.

Similarly, take a discrete isothermic surface f in $\mathbb{H}^3$, then f is a discrete CMC H ($|H|>1$) surface if and only if f^* can be chosen as f^{θ_1} with $\theta_1=\coth^{-1}H$, where $f^\theta:=\cosh\theta\cdot f+\sinh\theta\cdot n$. And the Gaussian curvature K^θ of f^θ is $K^\theta=\dfrac{K\cosh^2\theta-H\sinh(2\theta)+\sinh^2\theta}{\cosh^2\theta-H\sinh(2\theta)+K\sinh^2\theta}$. In particular, when f is a discrete CMC H surface in $\mathbb{H}^3$, the Gaussian curvature K^{θ_2} of f^{θ_2} is $(|H|+\sqrt{H^2-1})^2$ by taking $\theta_2=\dfrac{1}{2}\coth^{-1}H$, proving the proposition. □

8.2.1 The discrete Lax pair

Away from umbilic points, we can reparametrize the conformal coordinates to isothermic ones. Then isothermic surfaces have real constant Hopf differentials Q (see [12]). Then, as a natural choice, we can define the discrete Lax pair as follows:

$$F_q=F_pU_{pq},\quad F_s=F_pV_{ps},\ \text{where}\tag{8.2.2}$$

$$U_{pq}=\begin{pmatrix}a_{pq}&\lambda b_{pq}+\frac{1}{\lambda b_{pq}}\\ -\frac{\bar{b}_{pq}}{\lambda}-\frac{\lambda}{\bar{b}_{pq}}&\bar{a}_{pq}\end{pmatrix},\quad V_{ps}=\begin{pmatrix}d_{ps}&\lambda e_{ps}+\frac{1}{\lambda e_{ps}}\\ -\frac{\bar{e}_{ps}}{\lambda}-\frac{\lambda}{\bar{e}_{ps}}&\bar{d}_{ps}\end{pmatrix},\tag{8.2.3}$$

where $a_{pq}, b_{pq}, d_{ps}, e_{ps} \in \mathbb{C}$. The compatibility condition of F is

$$V_{ps}U_{sr} = U_{pq}V_{qr}, \tag{8.2.4}$$

and taking the determinant of Equation (8.2.4), we have

$$\det(V_{ps}) \cdot \det(U_{sr}) = \det(U_{pq}) \cdot \det(V_{qr}).$$

We now assume that $\det(U_{pq})$ (resp. $\det(V_{ps})$) is independent of vertical edges (resp. horizontal edges). Our goal is to find such U_{pq} and V_{ps} to obtain a discrete CMC surface. Note that, the compatibility condition implies a discrete version of the sinh-Gordon equation (see [76]).

Remark 8.2.2. Our formulation of Lax pairs for discrete CMC surfaces in $\mathbb{R}^3$ is slightly different from the original one in [12]. However, we can again construct the discrete non-zero CMC surfaces in $\mathbb{R}^3$ using the solution F of Equations (8.2.2) and (8.2.3) (see Proposition 8.4.1 here).

8.3 Discrete versions of the matrix splitting theorems

Here we recall the definition of the projectivization of loop groups and matrix-splitting theorems for them, which were given in [43], and play important roles in our construction of discrete CMC surfaces in Riemannian spaceforms.

Definition 8.3.1. For $\lambda_0 \in \mathbb{R}\backslash\{\pm 1\}$ and $\lambda_1 \in i\mathbb{R}\backslash\{\pm i\}$, the set $\mathrm{P}\Lambda\mathrm{SL}_2(\mathbb{C})$, the projectivization of $\Lambda\mathrm{SL}_2(\mathbb{C})$, is defined by the following conditions: $C(\lambda) \in \mathrm{P}\Lambda\mathrm{SL}_2(\mathbb{C})$ if

1. $C(\lambda)$ is a Laurent polynomial in λ.
2. $C(\lambda)$ is twisted: $C(-\lambda) = \sigma_3 C(\lambda)\sigma_3$.
3. $\det(C(\lambda)) = (1-(\lambda_0/\lambda)^2)^i(1-(\lambda_1/\lambda)^2)^j(1-\lambda_0^2\lambda^2)^k(1-\lambda_1^2\lambda^2)^l$ for some $i, j, k, l \in \mathbb{N}$.

Remark 8.3.1. We can identify $F \in \mathrm{P}\Lambda\mathrm{SL}_2(\mathbb{C})$ with $\tilde{F} \in \mathrm{SL}_2(\mathbb{C})$ projectively up to some scalar function $\beta : \mathbb{S}^1 \longrightarrow \mathbb{C}$, i.e. we can write $\tilde{F} = \beta(\lambda)F$.

We introduce the *discrete Birkhoff splitting* in the following proposition, as shown in [43]. In Proposition 8.3.1, X lies in the set corresponding to what would be the minus group, and $\tilde{C}$ lies in the set corresponding to what would be the plus group, in the smooth case. We denote these groups by $\mathrm{P}\Lambda^-\mathrm{SL}_2(\mathbb{C})$ and $\mathrm{P}\Lambda^+\mathrm{SL}_2(\mathbb{C})$.

Proposition 8.3.1. *Let $C \in \mathrm{P}\Lambda\mathrm{SL}_2(\mathbb{C})$. Then there exist matrices X, $\tilde{C} \in \mathrm{P}\Lambda\mathrm{SL}_2(\mathbb{C})$ such that*

$$C = X\tilde{C},\ X(\lambda) \overset{\lambda\to\infty}{\longrightarrow} I,\ \det(\tilde{C}(\lambda)) = \frac{\det(C(\lambda))}{1-(\lambda_l/\lambda)^2},$$

where $l = 0$ or 1.

Proof. We can write $C(\lambda_l) = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ $(l = 0, 1)$ with $\det(C(\lambda_l)) = 0$ but $C(\lambda_l) \neq \mathbf{0}$, by taking an appropriate scalar multiple of C. So we show the only case when $ab \neq 0$ or $cd \neq 0$ here. This case is the one that always appears in the discrete DPW method here.

Solve $\tilde{X}(\lambda_l)\cdot C(\lambda_l)=\mathbf{0}$, and we obtain the unique solution $\tilde{X}=\begin{pmatrix}1 & -\frac{x_l}{\lambda}\\ -\frac{\lambda_l^2}{x_l\lambda} & 1\end{pmatrix}$, where $x_l=\lambda_l\frac{a}{c}=\lambda_l\frac{b}{d}$, and $\tilde{X}\in \mathrm{P\Lambda SL_2}(\mathbb{C})$. We set $X:=\begin{pmatrix}1 & \frac{x_l}{\lambda}\\ \frac{\lambda_l^2}{x_l\lambda} & 1\end{pmatrix}$ and $\tilde{C}:=X^{-1}C$, then we have $X,\tilde{C}\in \mathrm{P\Lambda SL_2}(\mathbb{C})$, $X(\lambda)\overset{\lambda\to\infty}{\longrightarrow} I$ and $\det(\tilde{C}(\lambda))=\frac{\det(C(\lambda))}{1-(\lambda_l/\lambda)^2}$. □

Next we introduce the *discrete* SU_2*-Iwasawa splitting* for $\mathrm{P\Lambda^-SL_2}(\mathbb{C})$ as follows:

Proposition 8.3.2 ([43])**.** *Let $L_l^-\in \mathrm{P\Lambda^-SL_2}(\mathbb{C})$ be of the form*

$$L_l^-(\lambda)=\begin{pmatrix}1 & \frac{x_l}{\lambda}\\ \frac{\lambda_l^2}{x_l\lambda} & 1\end{pmatrix}\qquad (l=0,1).$$

Then, there exist a matrix $L_l(\lambda)$ of the form

$$L_l(\lambda)=\begin{pmatrix}a_l & \lambda b_l+\frac{1}{\lambda b_l}\\ -\frac{\bar{b}_l}{\lambda}-\frac{\lambda}{\bar{b}_l} & \bar{a}_l\end{pmatrix}$$

with $\left(\sigma_3 L_l^-(\lambda_l)\sigma_3\right)\cdot L_l(\lambda_l)=\mathbf{0}$ and $\det(L_l(\lambda))=|\lambda_l|^{-2}\left(1-(\lambda_l/\lambda)^2\right)\left(1-\lambda_l^2\lambda^2\right)$, and this $L_l^-(\lambda)$ is unique up to sign.

In Proposition 8.3.2, $L_l(\lambda)$ lies in the set corresponding to what would be SU_2, and $(L_l)^{-1}\cdot L_l^-$ lies in the set corresponding to what would be the plus group, in the smooth case. We denote these groups by $\mathrm{P\Lambda SU_2}$ and $\mathrm{P\Lambda^+SL_2}(\mathbb{C})$.

Proof. Solve

$$\begin{pmatrix}1 & -\frac{x_l}{\lambda_l}\\ -\frac{\lambda_l}{x_l} & 1\end{pmatrix}\begin{pmatrix}a_l & \lambda_l b_l+\frac{1}{\lambda_l b_l}\\ -\frac{\bar{b}_l}{\lambda_l}-\frac{\lambda_l}{\bar{b}_l} & \bar{a}_l\end{pmatrix}=\mathbf{0}$$
$$\iff\quad a_l+\frac{x_l}{\lambda_l^2}\bar{b}_l+\frac{x_l}{\bar{b}_l}=0,\ \bar{a}_l-\frac{\lambda_l^2}{x_l}b_l-\frac{1}{x_l b_l}=0.$$

These implies that $b_l=\pm\sqrt{\dfrac{-\lambda_l^2(1+|x_l|^2)}{|\lambda_l|^4+|x_l|^2}}$, and we obtain that a_l and b_l are defined up to sign. By direct computation, we get $\det(L_l(\lambda))=|\lambda_l|^{-2}\left(1-(\lambda_l/\lambda)^2\right)\left(1-\lambda_l^2\lambda^2\right)$. □

8.4 Discrete CMC surfaces via the discrete DPW method

In this section, first we introduce the discrete DPW method for discrete CMC $H\neq 0$ surfaces in $\mathbb{R}^3$, studied in [43]. After that, we show the main result here, which is a discrete analogue of the DPW method for CMC surfaces in $\mathbb{S}^3$ with any mean curvature H and CMC surfaces in $\mathbb{H}^3$ with $|H|>1$, by using discrete holomorphic potentials.

Proposition 8.4.1 ([43])**.** *Any discrete isothermic CMC $H\neq 0$ surface $f:\mathbb{Z}^2\longrightarrow\mathbb{R}^3$ can be constructed by the following Steps 1-4:*

<u>Step 1</u> *Let $\mathcal{Z}:\mathbb{Z}^2 \longrightarrow \mathbb{C}$ be a discrete holomorphic function with*

$$cr(\mathcal{Z}_p,\mathcal{Z}_q,\mathcal{Z}_r,\mathcal{Z}_s)=\lambda_0^2/\lambda_1^2<0$$

satisfying $|\lambda_0|,|\lambda_1|\neq 1$ and set

$$L_{pq}^- = \begin{pmatrix} 1 & \frac{x_{pq}}{\lambda} \\ \frac{\lambda_0^2}{\lambda x_{pq}} & 1 \end{pmatrix}, \quad M_{ps}^- = \begin{pmatrix} 1 & \frac{y_{ps}}{\lambda} \\ \frac{\lambda_1^2}{\lambda y_{ps}} & 1 \end{pmatrix} \tag{8.4.1}$$

for $x_{pq}:=\mathcal{Z}_q-\mathcal{Z}_p$ and $y_{ps}:=\mathcal{Z}_s-\mathcal{Z}_p$.

<u>Step 2</u> *Solve*

$$\varphi_q=\varphi_p L_{pq}^-,\ \varphi_s=\varphi_p M_{ps}^- \tag{8.4.2}$$

with the initial condition $\varphi_{0,0}=I$.

<u>Step 3</u> *Split*

$$\varphi_p=F_pB_p \quad (F_p\in \mathrm{P}\Lambda\mathrm{SU}_2, B_p\in \mathrm{P}\Lambda^+\mathrm{SL}_2(\mathbb{C})). \tag{8.4.3}$$

<u>Step 4</u> *Substitute F_p to the Sym-Bobenko formula*

$$f_p=\mathrm{Sym}_{\mathbb{R}^3}(F_p)=\mathrm{Im}\left[\frac{1}{2H}\left(\frac{-1}{2}F_pi\sigma_3F_p^{-1}-i\lambda(\partial_\lambda F_p)F_p^{-1}\right)\right]\Bigg|_{\lambda=1} \tag{8.4.4}$$

with unit normal vector field $n_p=\dfrac{1}{2}\left[F_pi\sigma_3F_p^{-1}\right]\Big|_{\lambda=1}$, where

$$\mathrm{Im}:\begin{pmatrix} a+ib & c+id \\ -c+id & a-ib\end{pmatrix}\longmapsto\begin{pmatrix} ib & c+id \\ -c+id & -ib\end{pmatrix}.$$

Remark 8.4.1. We have two remarks here.

- We do not have to choose λ_0 or λ_1 as in $\mathbb{R}\setminus\{0,\pm1\}$ or $\mathbb{R}\setminus\{0,\pm i\}$. In fact, even when we choose $\lambda_0=\sqrt{\alpha_{pq}}$ and $\lambda_1=i\sqrt{\alpha_{ps}}$ ($|\lambda_0|,|\lambda_1|\neq 1$) with α_{pq} and α_{ps} the cross ratio factorizing functions of a discrete holomorphic function $\mathcal{Z}$, we can still construct discrete CMC surfaces. In order to simplify the argument, we assume that λ_0 and λ_1 are constant.

- As already mentioned in Remark 8.2.2, because of the difference of the formulation of Lax pairs for discrete isothermic CMC surfaces in $\mathbb{R}^3$ from the one in [12], the Sym-Bobenko formula in Step 4 in Proposition 8.4.1 is different from [12]. On the other hand, we can see that any discrete surface f defined in Step 4 is a discrete isothermic surface and the dual surface f^* of f can be chosen as $f^*=f+H^{-1}n$, i.e., f is a discrete isothermic CMC H surface in $\mathbb{R}^3$. The proof of this fact is similar to the one in [12], so here we omit this proof.

Here we introduce our first main result. We can construct any discrete CMC H (resp. H with $|H|>1$) in $\mathbb{S}^3$ (resp. $\mathbb{H}^3$) via the following recipe, which is a generalization of a result by Hoffmann [43].

Theorem 8.4.1. *Any discrete isothermic CMC surface $f : \mathbb{Z}^2 \longrightarrow \mathbb{S}^3$ (resp. $\mathbb{H}^3$) with CMC H (resp. H with $|H| > 1$) can be constructed by the same Steps 1-3 in Proposition 8.4.1 and the following Step $4'$ or $4''$, respectively:*

<u>*Step* $4'$</u> *Set $F_p^1 = F_p|_{\lambda=t_1:=e^{i\gamma_1}}$ and $F_p^2 = F_p|_{\lambda=t_1^{-1}}$ for $\gamma_1 \in \mathbb{R}$ and $2\gamma_1 \neq n\pi$ $(n \in \mathbb{Z})$. We define the following Sym-Bobenko type formulas*

$$f_p = \mathrm{Sym}_{\mathbb{S}^3}(F_p) = F_p^1 \begin{pmatrix} t_1 & 0 \\ 0 & t_1^{-1} \end{pmatrix} (F_p^2)^{-1} \tag{8.4.5}$$

with its normal vector field $n_p = iF_p^1 \begin{pmatrix} t_1 & 0 \\ 0 & -t_1^{-1} \end{pmatrix} (F_p^2)^{-1}$. Then, f is a discrete CMC $H = \cot(-2\gamma_1)$ surface in $\mathbb{S}^3$ with normal n.

<u>*Step* $4''$</u> *Set $F_p^0 = F|_{\lambda=t_0:=\mathrm{e}^{\gamma/2}}$ for some constant $\gamma \in \mathbb{R} \setminus \{0\}$ satisfying*

$$\min\left(\lambda_0^2, \lambda_0^{-2}\right) < \mathrm{e}^{\gamma} < \max\left(\lambda_0^2, \lambda_0^{-2}\right).$$

We define the following Sym-Bobenko type formulas

$$f_p = \mathrm{Sym}_{\mathbb{H}^3}(F_p) := \frac{1}{\det F_p^0} F_p^0 \begin{pmatrix} t_0 & 0 \\ 0 & t_0^{-1} \end{pmatrix} \overline{F_p^0}^t \tag{8.4.6}$$

with normal vector field $n_p = \dfrac{1}{\det F_p^0} F_p^0 \begin{pmatrix} t_0 & 0 \\ 0 & -t_0^{-1} \end{pmatrix} \overline{F_p^0}^t$. Then, f is a discrete CMC $H = \coth(-\gamma)$ surface in $\mathbb{H}^3$ with normal n.

We have some lemmas for these steps as follows, which implies that φ_p are well-defined for all p:

Lemma 8.4.1. *In Step 1, L_{pq}^- and M_{ps}^- satisfy $M_{ps}^- L_{sr}^- = L_{pq}^- M_{qr}^-$.*

Proof. By direct computations, we get

$$\begin{aligned} & M_{ps}^- L_{sr}^- = L_{pq}^- M_{qr}^- \\ \iff & \begin{cases} 1 + \frac{\lambda_0^2 y_{ps}}{\lambda^2 x_{sr}} = 1 + \frac{\lambda_1^2 x_{pq}}{\lambda^2 y_{qr}}, & \frac{x_{pq}}{\lambda} + \frac{y_{ps}}{\lambda} = \frac{y_{qr}}{\lambda} + \frac{x_{pq}}{\lambda}, \\ \frac{\lambda_1^2}{\lambda y_{ps}} + \frac{\lambda_0^2}{\lambda x_{sr}} = \frac{\lambda_0^2}{\lambda x_{pq}} + \frac{\lambda_1^2}{\lambda y_{qr}}, & \frac{\lambda_1^2 x_{sr}}{\lambda^2 y_{ps}} + 1 = \frac{\lambda_0^2 y_{qr}}{\lambda^2 x_{pq}} + 1 \end{cases}. \end{aligned} \tag{8.4.7}$$

Here, by the definition of x_{pq} and y_{ps}, we have

$$x_{sr} + y_{ps} = y_{qr} + x_{pq} \ (= \mathcal{Z}_r - \mathcal{Z}_p), \tag{8.4.8}$$

and by $cr(\mathcal{Z}_p, \mathcal{Z}_q, \mathcal{Z}_r, \mathcal{Z}_s) = \lambda_0^2/\lambda_1^2$, we also have

$$\frac{x_{pq} x_{sr}}{y_{ps} y_{qr}} = \frac{\lambda_0^2}{\lambda_1^2}. \tag{8.4.9}$$

Thus, by using Equations (8.4.8), (8.4.9), Equation (8.4.7) are clear. □

In the next step, we show that the solution of Equation (8.4.2) can be split like in Step 3 of Proposition 8.4.1, which we call SU_2-Iwasawa splitting for φ.

Lemma 8.4.2. *As in Step 3, for all φ_p we can split*

$$\varphi_p = F_p B_p \quad (F_p \in \mathrm{P\Lambda SU}_2, B_p \in \mathrm{P\Lambda^+ SL}_2(\mathbb{C})).$$

Proof. By Step 2, we have the following solution φ:

$$\varphi_p = L^-_{0,0} L^-_{1,0} L^-_{2,0} \cdots L^-_{m-1,0} \cdot M^-_{m,0} M^-_{m,1} M^-_{m,2} \cdots M^-_{m,n-1}.$$

In the above equation, $L^-_{u,0} := L^-_{(u,0)(u+1,0)}$, $M^-_{m,v} := M^-_{(m,v)(m,v+1)}$ for $u, v \in \mathbb{Z}_{\geq 0}$ Applying Proposition 8.3.2 to $L^-_{0,0}$, we obtain

$$\varphi_p = U_{0,0} A^+_{0,0} L^-_{1,0} L^-_{2,0} \cdots L^-_{m-1,0} \cdot M^-_{m,0} M^-_{m,1} M^-_{m,2} \cdots M^-_{m,n-1}$$

for $A_{0,0} \in \mathrm{P\Lambda SU}_2$ and $A^+_{0,0} \in \mathrm{P\Lambda^+ SL}_2(\mathbb{C})$. Next we name $U^g_{1,0} = A^+_{0,0} L^-_{1,0} \in \mathrm{P\Lambda SL}_2(\mathbb{C})$ and split $U^g_{1,0} = U^-_{1,0} A^{++}_{1,0}$ by the discrete Birkhoff splitting in Proposition 8.3.1:

$$\varphi_p = U_{0,0} U^-_{1,0} A^{++}_{1,0} L^-_{2,0} \cdots L^-_{m-1,0} \cdot M^-_{m,0} M^-_{m,1} M^-_{m,2} \cdots M^-_{m,n-1}.$$

We again use Proposition 8.3.2 for $U^-_{1,0}$:

$$\varphi_p = U_{0,0} U_{1,0} A^+_{1,0} A^{++}_{1,0} L^-_{2,0} \cdots L^-_{m-1,0} \cdot M^-_{m,0} M^-_{m,1} M^-_{m,2} \cdots M^-_{m,n-1}$$

for $U_{1,0} \in \mathrm{P\Lambda SU}_2$ and $A^+_{1,0} \in \mathrm{P\Lambda^+ SL}_2(\mathbb{C})$. We can repeat these splittings and we get

$$\varphi_p = U_{0,0} U_{1,0} \cdots U_{m-1,0} \cdot V_{m,0} V_{m,1} \cdots V_{m,n-1} A^+_{m-1,n-1} A^{++}_{m-1,n-1}.$$

Taking $F_p = U_{0,0} U_{1,0} \cdots U_{m-1,0} \cdot V_{m,0} V_{m,1} \cdots V_{m,n-1}$ and $B_p = A^+_{m-1,n-1} A^{++}_{m-1,n-1}$, the lemma is proven. $\square$

If U_{pq} and V_{ps} obtained by the above process satisfy the compatibility condition $V_{ps} U_{sr} = U_{pq} V_{qr}$, we have the solotion F_p of Equations (8.2.2), (8.2.3). In fact, the compatibility condition follows from Lemma 8.4.3.

Lemma 8.4.3. *Let U_{pq} and V_{ps} be matrices obtained by Lemma 8.4.2. Then*

$$V_{ps} U_{sr} = U_{pq} V_{qr} \Longleftrightarrow M^-_{ps} L^-_{sr} = L^-_{pq} M^-_{qs}.$$

Proof. By Steps 2 and 3, we have $F_q B_q = F_p B_p L^-_{pq}$ and $F_s B_s = F_p B_p M^-_{ps}$. These imply that $U_{pq} = B_p L^-_{pq} B_q^{-1}$ and $V_{ps} = B_p M^-_{ps} B_s^{-1}$. Substituting them into $V_{ps} U_{sr} = U_{pq} V_{sr}$, we can prove the lemma. $\square$

By the proof of Proposition 8.3.2 and by $\lambda_0 \in \mathbb{R} \setminus \{\pm 1\}$, $\lambda_1 \in i\mathbb{R} \setminus \{\pm i\}$, we can choose the components $b_{pq} \in i\mathbb{R} \setminus \{0\}$ and $e_{ps} \in \mathbb{R} \setminus \{0\}$ of U_{pq} and V_{ps} in Equation (8.2.3). Replacing b_{pq} with ib_{pq} ($b_{pq} \in \mathbb{R} \setminus \{0\}$) in Equation (8.2.3), we have the following expression:

$$U_{pq} = \begin{pmatrix} a_{pq} & \lambda i b_{pq} - \lambda^{-1} i b_{pq}^{-1} \\ \lambda^{-1} i b_{pq} - \lambda i b_{pq}^{-1} & \bar{a}_{pq} \end{pmatrix}, \; V_{ps} = \begin{pmatrix} d_{ps} & \lambda e_{ps} + \lambda^{-1} e_{ps}^{-1} \\ -\lambda^{-1} e_{ps} - \lambda e_{ps}^{-1} & \bar{d}_{ps} \end{pmatrix}$$

for $a_{pq}, d_{ps} \in \mathbb{C} \setminus \{0\}$ and $b_{pq}, e_{ps} \in \mathbb{R} \setminus \{0\}$. Finally, we have the following lemma.

Lemma 8.4.4. *In Step $4'$ (resp. $4''$), f is a discrete isothermic CMC $\cot(-2\gamma_1)$ (resp. $\coth(-q)$) surface in $\mathbb{S}^3$ (resp. $\mathbb{H}^3$).*

Proof. Defining $\hat{F}_p := F \begin{pmatrix} \sqrt{\lambda} & 0 \\ 0 & \frac{1}{\sqrt{\lambda}} \end{pmatrix}$, we have

$$f_p = \mathrm{Sym}_{\mathbb{S}^3}(F_p) = \mathrm{Sym}_{\mathbb{S}^3}\left(\hat{F}_p \begin{pmatrix} \frac{1}{\sqrt{\lambda}} & 0 \\ 0 & \sqrt{\lambda} \end{pmatrix}\right) = \hat{F}_p^1(\hat{F}_p^2),$$

where $\hat{F}_p^1 := \hat{F}_p|_{\lambda=t_1},\ \hat{F}_p^2 := \hat{F}_p|_{\lambda=t_1^{-1}}$. Then,

$$f_q - f_p = -\hat{F}_p^1 \cdot \frac{t_1^2 - t_1^{-2}}{\alpha_{pq}^2(t_1)} \begin{pmatrix} b_{pq}^{-2}t_1^{-2} - 1 & -ia_{pq}b_{pq}^{-1} \\ i\bar{a}_{pq}b_{pq}^{-1} & 1 - b_{pq}^{-2}t_1^2 \end{pmatrix} \cdot (\hat{F}_p^2)^{-1},$$
$$f_s - f_p = -\hat{F}_p^1 \cdot \frac{t_1^2 - t_1^{-2}}{\alpha_{ps}^2(t_1)} \begin{pmatrix} 1 + e_{ps}^{-2}t_1^{-2} & d_{ps}e_{ps}^{-1} \\ \bar{d}_{ps}e_{ps}^{-1} & -1 - e_{ps}^{-2}t_1^2 \end{pmatrix} \cdot (\hat{F}_p^2)^{-1},$$

where $\alpha_{pq}(\lambda) := \sqrt{\det U_{pq}}$ and $\alpha_{ps}(\lambda) := \sqrt{\det V_{ps}}$. Furthermore, we have the following expressions:

$$\hat{F}_q = \hat{F}_p \begin{pmatrix} a_{pq} & i\left(b_{pq} - \frac{1}{\lambda^2 b_{pq}}\right) \\ i\left(b_{pq} - \frac{\lambda^2}{b_{pq}}\right) & a_{pq} \end{pmatrix},\ \hat{F}_s = \hat{F}_p \begin{pmatrix} d_{ps} & e_{ps} + \frac{1}{\lambda^2 e_{ps}} \\ -\left(e_{ps} + \frac{\lambda^2}{e_{ps}}\right) & d_{ps} \end{pmatrix}.$$

Setting

$$\hat{U}_{pq} := \frac{1}{\alpha_{pq}(\lambda)} \begin{pmatrix} a_{pq} & i\left(b_{pq} - \frac{1}{\lambda^2 b_{pq}}\right) \\ i\left(b_{pq} - \frac{\lambda^2}{b_{pq}}\right) & a_{pq} \end{pmatrix},\ \hat{V}_{ps} := \frac{1}{\alpha_{ps}(\lambda)} \begin{pmatrix} d_{ps} & e_{ps} + \frac{1}{\lambda^2 e_{ps}} \\ -\left(e_{ps} + \frac{\lambda^2}{e_{ps}}\right) & d_{ps} \end{pmatrix},$$

$\hat{U}_{pq}^1 := \hat{U}_{pq}|_{\lambda=t_1}$ and $\hat{V}_{ps}^1 := \hat{V}_{ps}|_{\lambda=t_1}$, we can express the above equations as

$$f_q - f_p = -\hat{F}_p^1 \cdot \frac{t_1^2 - t_1^{-2}}{\alpha_{pq}(t_1)} b_{pq}^{-1}\hat{U}_{pq}^1 \cdot \sigma_2 \cdot (\hat{F}_p^2)^{-1},$$
$$f_s - f_p = -\hat{F}_p^1 \cdot \frac{t_1^2 - t_1^{-2}}{\alpha_{ps}(t_1)} e_{ps}^{-1}\hat{V}_{pq}^1 \cdot \sigma_1 \cdot (\hat{F}_p^2)^{-1}.$$

Similarly, by using $\hat{U}_{pq}^2 := \hat{U}_{pq}|_{\lambda=t_1^{-1}}$ and $\hat{V}_{ps}^2 := \hat{V}_{ps}|_{\lambda=t_1^{-1}}$, we also have

$$f_r - f_q = \hat{F}_p^1\hat{U}_{pq}^1 \cdot \frac{t_1^2 - t_1^{-2}}{\alpha_{ps}(t_1)} e_{qr}^{-1}\hat{V}_{qr}^1 \cdot \sigma_1 \cdot (\hat{U}_{pq}^2)^{-1}(\hat{F}_p^2)^{-1},$$
$$f_r - f_s = -\hat{F}_p^1\hat{V}_{ps}^1 \cdot \frac{t_1^2 - t_1^{-2}}{\alpha_{pq}(t_1)} b_{sr}^{-1}\hat{U}_{sr}^1 \cdot \sigma_2 \cdot (\hat{V}_{ps}^2)^{-1}(\hat{F}_p^2)^{-1}.$$

Using these expressions and the compatibility condition $\hat{U}_{pq}\hat{V}_{qr} = \hat{V}_{ps}\hat{U}_{sr}$, we have

$$cr(f_p, f_q, f_r, f_s) = -\frac{\alpha_{ps}^2(t_1)}{\alpha_{pq}^2(t_1)}.$$

Thus f is a discrete isothermic surface in $\mathbb{S}^3$. And by a direct computation, we can check that the dual surface f^* of f can be chosen as $f^{-2\gamma_1} = \cos(-2\gamma_1)\cdot f + \sin(-2\gamma_1)\cdot n$, implying that f is a discrete isohermic CMC $\cot(-2\gamma_1)$ surface in $\mathbb{S}^3$.

In the same vein, but a tedious computation, we have that the f in Step $4''$ is a discrete isothermic CMC $\coth(-q)$ surface in $\mathbb{H}^3$ with cross ration $-\dfrac{\alpha_{ps}^2(t_0)}{\alpha_{pq}^2(t_0)}$. □

Combining Lemmas 8.4.1, 8.4.2, 8.4.3, 8.4.4, we have proven Theorem 8.4.1.

8.5 Examples

In this section, we introduce three examples which are round cylinders, Smyth surfaces and Delaunay surfaces. In the following we will take $\lambda_1 = i\lambda_0$ for simplicity.

Example 8.5.1 (Round Cylinders and 2-legged Smyth surfaces)**.** Taking $\mathcal{Z}_{m,n} = c\lambda_0(m+in)$ ($c \in \mathbb{R}\setminus\{0\}$). Then we have discrete CMC surfaces in $\mathbb{R}^3$, $\mathbb{S}^3$ and $\mathbb{H}^3$ as in Figure 8.1. When $c = 1$, the resulting surfaces are round cylinders in $\mathbb{R}^3$, $\mathbb{S}^3$ and $\mathbb{H}^3$, and when $c \neq 1$, we have discrete 2-legged Smyth surfaces, as in the lower column in Figure 8.1. In order to visualize the surfaces in $\mathbb{S}^3$ and $\mathbb{H}^3$, we use stereographic projections.

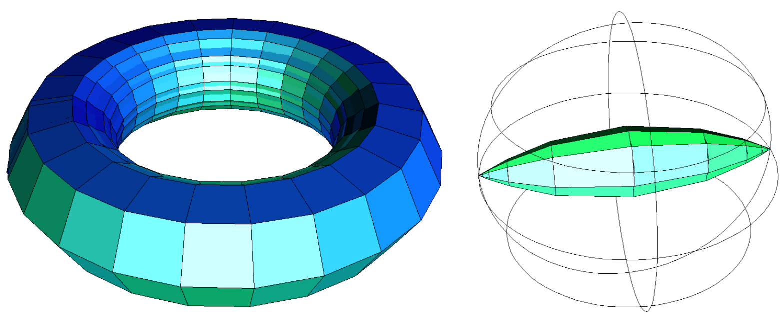


Fig. 8.1: Round cylinders in $\mathbb{S}^3$ and $\mathbb{H}^3$ (left to right).

Example 8.5.2 (k-legged Smyth surfaces)**.** Let $\mathcal{Z}_{m,n}$ be a dicrete power function $z_{m,n}$ defined as follows (see [12] for detail):

$$\gamma\cdot z_{m,n} = 2m\frac{(z_{m+1,n}-z_{m,n})(z_{m-1,n}-z_{m,n})}{z_{m+1,n}-z_{m-1,n}} + 2n\cdot\frac{(z_{m,n+1}-z_{m,n})(z_{m,n}-z_{m,n-1})}{z_{m,n+1}-z_{m,n-1}},$$
$$z_{0,0}=0,\ z_{1,0}=1\ z_{0,1}=i^\gamma.$$

This discrete power function is the discrete analogue of the holomorphic function w^γ for $w \in \mathbb{C}$. In order to construct discrete k-legged Smyth surfaces, we set $\gamma = \frac{2}{k+2}$. See Figure 8.2.

Example 8.5.3 (Delaunay surfaces)**.** Here we construct discrete Delaunay surfaces with constant cross ratios. Although discrete Delaunay surfaces in $\mathbb{R}^3$ were already constructed and the construction is explained in [43], here we introduce more detailed explanations.

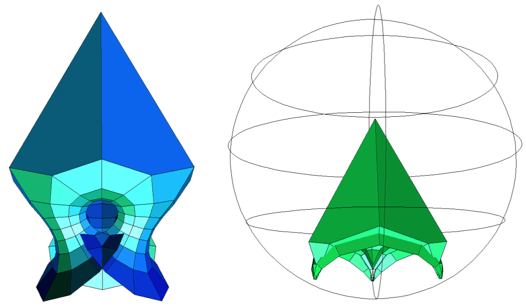


Fig. 8.2: 3-legged Smyth surfaces in $\mathbb{S}^3$ and $\mathbb{H}^3$ (left to right).

At first, assume that we have a discrete CMC surface of revolution in $\mathbb{R}^3$. Take a parametrization of the discrete surface of revolution $f_{m,n}$ in $\mathbb{R}^3$ as $f_{m,n} = (X_n \cos\theta_m, X_n \sin\theta_m, Y_n)^t$, where X_n and Y_n are real-valued functions depending on only n, and θ_m is a real-valued function depending on only m. Then, by a direct computation, we can show that $f_{m,n}$ is discrete isothermic. In particular, if $cr(f_p, f_q, f_r, f_s)$ is constant, θ_m must be chosen as $\theta_m = c_1 m$, where c_1 is a real cosntant.

This implies that the frame F_p as in Equation (8.2.2) along the horizontal direction (i.e. rotational direction) is the multiple of scalar constant matrices U_{pq}. Here we rename the scalar matrix U_{pq} as U. Starting from the data, first we recover the holomorphic data $L_{(m,0)(m+1,0)}$ as in Equation (8.4.1) (symbolic notation is the same as in the proof of Lemma 8.4.2). Then, splitting U into $U = L^-_{0,0}L^+_{0,0}$ by applying Proosition 8.3.1, we have

$$\begin{aligned} F_{m,0} &= \underbrace{UUU\cdots U}_{m \text{ times}} \\ &= L^-_{0,0}L^+_{0,0}L^-_{0,0}L^+_{0,0}L^-_{0,0}L^+_{0,0}\cdots L^-_{0,0}L^+_{0,0}. \end{aligned}$$

Applying Proposition 8.3.1 to the first $L^+_{0,0}L^-_{0,0}$ from the left in the above equation, we have $L^+_{0,0}L^-_{0,0} = L^-_{1,0}L^+_{1,0}$. Then

$$F_{m,0} = L^-_{0,0}L^-_{1,0}L^+_{1,0}L^+_{0,0}L^-_{0,0}L^+_{0,0}\cdots L^-_{0,0}L^+_{0,0}$$

Applying Proposition 8.3.1 to the next $L^+_{0,0}L^-_{0,0}$, we can again split $L^+_{0,0}L^-_{0,0}$ into $L^-_{1,0}L^+_{1,0}$. Then we have $L^+_{1,0}L^-_{1,0}$ in the above equation. We can split $L^+_{1,0}L^-_{1,0}$ into $L^-_{2,0}L^+_{2,0}$ i.e.

$$\begin{aligned} F_{m,0} &= L^-_{0,0}L^+_{0,0}L^-_{0,0}L^+_{0,0}L^-_{0,0}L^+_{0,0}\cdots L^-_{0,0}L^+_{0,0} \\ &= L^-_{0,0}L^-_{1,0}L^+_{1,0}L^+_{0,0}L^-_{0,0}L^+_{0,0}\cdots L^-_{0,0}L^+_{0,0} \\ &= L^-_{0,0}L^-_{1,0}L^+_{1,0}L^-_{1,0}L^+_{1,0}L^+_{0,0}\cdots L^-_{0,0}L^+_{0,0} \\ &= L^-_{0,0}L^-_{1,0}L^-_{2,0}L^+_{2,0}L^+_{1,0}L^+_{0,0}\cdots L^-_{0,0}L^+_{0,0}. \end{aligned}$$

Iterating the splitting into

$$L^+_{k,0}L^-_{k,0} = L^-_{k+1,0}L^+_{k+1,0} \ (k \in \mathbb{N} \cup \{0\}), \tag{8.5.1}$$

we can recover the L^-_{pq} as in Equation (8.4.1) by reduction.

Here we assume that U is constant, so $\det U = \lambda_0^{-2}(1-\lambda_0^2/\lambda^2)(1-\lambda_0^2\lambda^2)$ is also constant. In particular, $\det U^+_{m,0} = \lambda_0^{-2}(1-\lambda_0^2\lambda^2)$ and $U^+_{m,0}$ is of the form $\begin{pmatrix} S_{pq} & \lambda b_{pq} \\ \frac{\lambda}{b_{pq}} & \frac{\lambda_0^2}{S_{pq}} \end{pmatrix}$.

Conversely, we derive the condition for discrete holomorphic function $\mathcal{Z}_{m,n}$ from Equation (8.5.1). Equation (8.5.1) implies

$$t := \sqrt{\frac{-\lambda_0^2(1+|x_{(0,0)(1,0)}|^2)}{\lambda_0^4+|x_{(0,0)(1,0)}|^2}}, \quad S_{(0,0)(1,0)} = \frac{t x_{(0,0)(1,0)}}{\lambda_0^2},\ x_{jk} = \frac{S_{ij}+\frac{\lambda_0^2 t}{x_{ij}}}{\frac{1}{\lambda_0^2 x_{ij}S_{ij}}+\frac{1}{t}},\ S_{jk} = \frac{x_{jk}}{\lambda_0^2 x_{ij}S_{ij}},$$

where $i=(m-1,0), j=(m,0), k=(m+1,0)$. So we can define a discrete function $\mathcal{Z}_{m,0}$ as the solution of $x_{(m,0)(m+1,0)} = \mathcal{Z}_{m+1,0}-\mathcal{Z}_{m,0}$. In particular, here we assume that $x_{(0,0)(1,0)} \in i\mathbb{R}$.

For given initial data $y_{(0,n)(0,n+1)} \in \mathbb{C}$ ($n \in \mathbb{N}\cup\{0\}$), defining $\mathcal{Z}_{0,n}$ as the solution of $y_{(0,n+1)(0,n)} = \mathcal{Z}_{0,n+1}-\mathcal{Z}_{0,n}$, we can describe the discrete holomorphic function $\mathcal{Z}_{m,n}$ as the solution of cross ratio condition $cr(\mathcal{Z}_p,\mathcal{Z}_q,\mathcal{Z}_r,\mathcal{Z}_s) = \dfrac{\lambda_0^2}{\lambda_1^2} = -1$.

On the other hand, by a symmetry condition for the meridian curve of a discrete surface of revolution, we must choose the initial data $y_{(0,0)(0,1)}$ so that $V_{(0,0)(0,1)} = V_{(1,0)(1,1)}$. By Proposition 8.3.1, we can factorize $V_{(0,0)(0,1)}$ into $V_{(0,0)(0,1)} = M^-_{(0,0)(0,1)}M^+_{(0,0)(0,1)}$. Moreover, by definition of $V_{(1,0)(1,1)}$ and a construction of $V_{(1,0)(1,1)}$, we have

$$V_{(1,0)(1,1)} = (F_{1,0})^{-1}F_{1,1} = B_{(0,0)(1,0)}\varphi^{-1}_{(0,0)(1,0)}\varphi_{(0,0)(1,0)}B^{-1}_{(1,0)(1,1)} = L^+_{(0,0)(1,0)}M^-_{(1,0)(1,1)}B^{-1}_{(1,0)(1,1)}.$$

Again, splitting $L^+_{(0,0)(1,0)}M^-_{(1,0)(1,1)}$ into $L^+_{(0,0)(1,0)}M^-_{(1,0)(1,1)} = U^-U^+$, we have $M^-_{(0,0)(0,1)} = U^-$. This condition, the compatibility condition $x_{(0,0)(1,0)}+y_{(1,0)(1,1)} = y_{(0,0)(0,1)}+x_{(0,1)(1,1)}$, the assumption of $x_{(0,0)(1,0)}$, and the cross ratio condition $\dfrac{x_{(0,0)(1,0)}x_{(0,1)(1,1)}}{y_{(0,0)(0,1)}y_{(1,0)(1,1)}} = -1$ imply

$$y_{(0,0)(0,1)} = \sqrt{-\frac{2\lambda_0^4 - x^2_{(0,0)(1,0)}\lambda_0^4 - x^2_{(0,0)(1,0)}}{-1+2x^2_{(0,0)(1,0)}-\lambda_0^4}}.$$

Thus we have the initial condition for $y_{(0,n)(0,n+1)} = \mathcal{Z}_{0,n+1}-\mathcal{Z}_{0,n}$. In a similar way as for $x_{(m,0)(m+1,0)}$, solving

$$T_{(0,0)(0,1)} = \frac{t y_{(0,0)(0,1)}}{\lambda_1^2}, \quad y_{jk} = \frac{T_{ij}+\frac{\lambda_1^2 t}{y_{ij}}}{\frac{1}{\lambda_1^2 y_{ij}T_{ij}}+\frac{1}{t}}, \quad T_{jk} = \frac{y_{jk}}{\lambda_1^2 y_{ij}T_{ij}},$$

where $i=(0,n-1), j=(0,n), k=(0,n+1)$, we have $\mathcal{Z}_{0,n}$ and the discrete holomorphic function $\mathcal{Z}_{m,n}$. This data determines a CMC surface of revolution (see Figures 8.3, 8.4).

8.6 Singularities of discrete positive constant Gaussian curvature surfaces

As mentioned in Section 8.2, we can obtain discrete positive constant Gaussian curvature (p-CGC, for short) surfaces in $\mathbb{S}^3$ and $\mathbb{H}^3$. In the smooth case, it is known that p-CGC surfaces generally have singularities, so it is natural to expect that discrete p-CGC surfaces also generally have singularities.

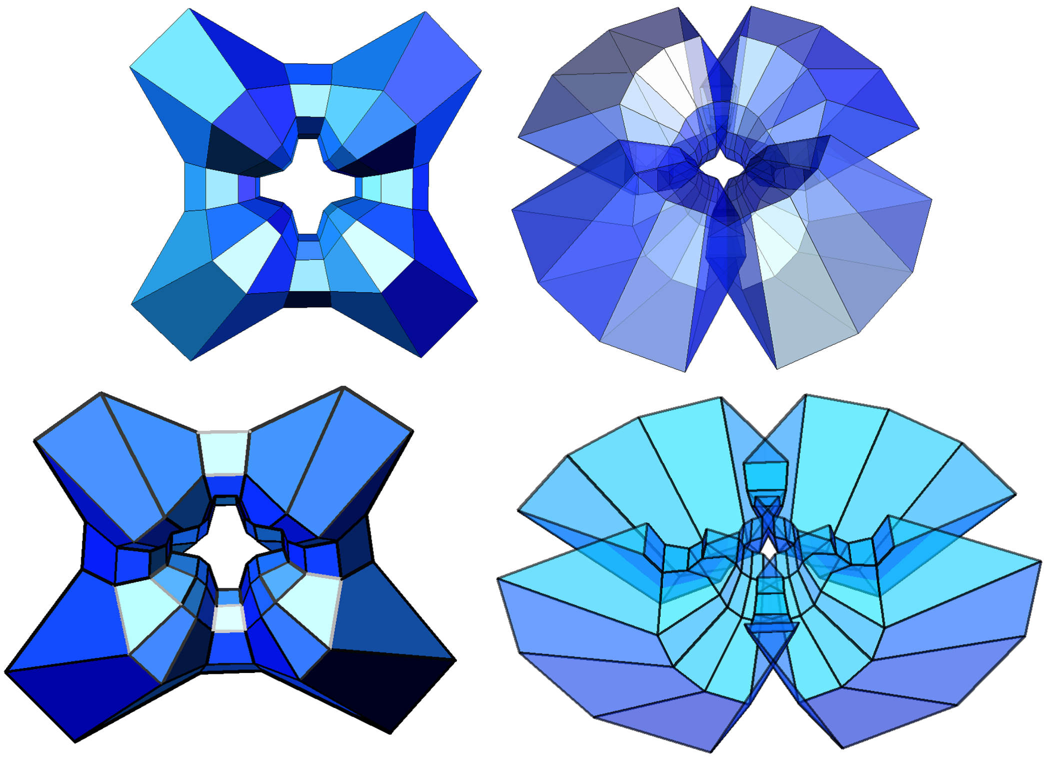


Fig. 8.3: Discrete Delaunay surfaces in $\mathbb{S}^3$. Like in the smooth case, there exist discrete closed isothermic CMC surfaces of revolution. In the upper row, a discrete unduloid and a discrete nodoid with four lobs are shown. In the lower row, the half cuts of the pictures above are shown.

Hoffmann, Rossman, Sasaki, Yoshida [44] analyzed discrete flat surfaces (discrete circular surfaces with Gaussian curvature in the sense of Definition 8.2.4 identically 1) in $\mathbb{H}^3$, and Rossman and the second author [81] analyzed singularities of discrete linear Weingarten surfaces of Bryant (resp. Bianchi) type in $\mathbb{H}^3$ (resp. $\mathbb{S}^{2,1}$). Here we introduce vertices defined in [81] that are potentially singularities of discrete surfaces.

Definition 8.6.1. Let i, j, k be three consective points in the horizontal or vertical direction in $\mathbb{Z}^2$, let f be a discrete circular surface in $\mathbb{S}^3$ (resp. $\mathbb{H}^3$) and let n its Gauss map. Then f_j is called a *flat, parabolic, singular vertex* (FPS vertex, for short) if $\kappa_{ij} \cdot \kappa_{jk} < 0$.

FPS vertices possibly become singularities of discrete surfaces. On the other hand, in order to define singularities of discrete surfaces, it is unsufficient to see only signatures of principal curvatures, which was explained in [81]. In this paper we define and analyze singularities (i.e. singular vertices) of discrete p-CGC surfaces in $\mathbb{S}^3$. For other 3-spaces ($\mathbb{R}^3$ or $\mathbb{H}^3$), we can also discuss singularities of discrete constant positive Gaussian curvature surfaces in the same way.

Before discussing singularities of discrete p-CGC surfaces in $\mathbb{S}^3$, we see smooth p-CGC surfaces in $\mathbb{S}^3$. At a singular point of a front, at least one of the principal curvatures of a front diverges (see

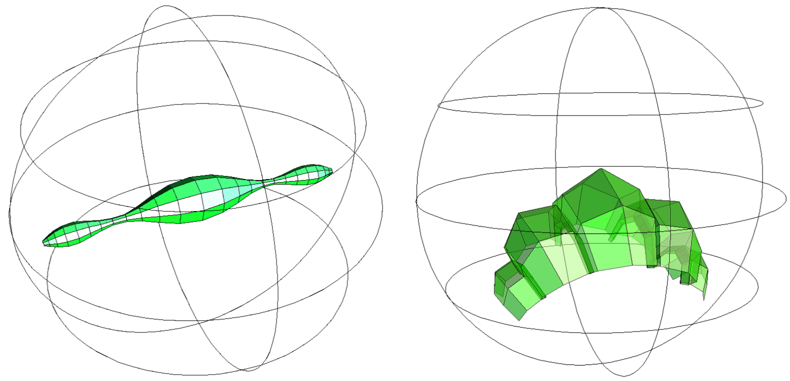


Fig. 8.4: Discrete Delaunay surfaces in $\mathbb{H}^3$

[67]), which we would like to reflect in the discrete case. First we compute the principal curvatures of smooth p-CGC surfaces.

Let f be a conformal CMC $\cot(-2\gamma_1)$ surface in $\mathbb{S}^3$ described in Proposition 8.1.2 and let f^θ be its parallel surface at distance θ. In particular, we assume that f is isothermic. In order to simplify the argument, we only consider that $\gamma_1 \in (0, \pi/4]$. Then $f^{-\gamma_1}$ becomes a p-CGC $\cot^2\gamma_1$ surface in $\mathbb{S}^3$ and the two principal curvatures κ_1, κ_2 become $\kappa_1 = \cot(-2\gamma_1) + e^{-2u}\csc(-2\gamma_1)$, $\kappa_2 = \cot(-2\gamma_1) - e^{-2u}\csc(-2\gamma_1)$, implying that the principal curvatures $\kappa_1^{-\gamma_1}$, $\kappa_2^{-\gamma_1}$ of a smooth p-CGC surface $f^{-\gamma_1}$ are

$$\kappa_1^{-\gamma_1} = \cot\gamma_1 \cdot \frac{e^{-2u}+1}{e^{-2u}-1}, \quad \kappa_2^{-\gamma_1} = \cot\gamma_1 \cdot \frac{e^{-2u}-1}{e^{-2u}+1}.$$

Thus we know that $|\kappa_1^{-\gamma_1}| > \cot\gamma_1$ and $|\kappa_2^{-\gamma_1}| < \cot\gamma_1$ at a regular point, and that $\kappa_1^{-\gamma_1}$ diverges and $\kappa_2^{-\gamma_1}$ converges to 0 at a singular point.

Henceforth, we consider the discrete case. Let f be a discrete CMC $\cot(-2\gamma_1)$ ($\gamma_1 \in (0, \pi/4]$) surface in $\mathbb{S}^3$ described in Theorem 8.4.1 and let f^θ be its parallel surface at distance θ. Like in the smooth case, using Proposition 8.2.1, $f^{-\gamma_1}$ is a discrete p-CGC $\cot^2\gamma_1$ surface in $\mathbb{S}^3$. One can compute that the principal curvatures of f are $\kappa_{pq} = \dfrac{\cos(-2\gamma_1) - b_{pq}^2}{\sin(-2\gamma_1)}$, $\kappa_{ps} = \dfrac{\cos(-2\gamma_1) + e_{ps}^2}{\sin(-2\gamma_1)}$, and the principal curvatures $\kappa_{pq}^\theta, \kappa_{ps}^\theta$ of f^θ are $\kappa_{pq}^\theta = \dfrac{1+\kappa_{pq}\cot\theta}{\cot\theta - \kappa_{pq}}$, $\kappa_{ps}^\theta = \dfrac{1+\kappa_{ps}\cot\theta}{\cot\theta - \kappa_{ps}}$. Using that, we have

$$\kappa_{pq}^{-\gamma_1} = \cot\gamma_1 \cdot \frac{b_{pq}^2 - 1}{b_{pq}^2+1}, \quad \kappa_{ps}^{-\gamma_1} = \cot\gamma_1 \cdot \frac{e_{ps}^2+1}{e_{ps}^2-1}.$$

Like in the smooth case, we have $|\kappa_{pq}^{-\gamma_1}| < \cot\gamma_1$, $|\kappa_{ps}^{-\gamma_1}| > \cot\gamma_1$. Mimicking the smooth case, this justifies that there is no singular vertex in the $m-$direction but there exists a singular vertex in the $n-$direction. In conclusion, we precisely define singular vertices of discrete p-CGC surfaces in $\mathbb{S}^3$ as follows:

Definition 8.6.2. Let $f^{-\gamma}$ be a discrete p-CGC $\cot^2\gamma$ surface obtained by taking a parallel surface of a discrete isothermic CMC $\cot(-2\gamma_1)$ surface described in Theorem 8.4.1 and let i, j, k be three consective three points in the vertical direction. Then the FPS vertex $f_j^{-\gamma}$ is a *singular vertex.*

Remark 8.6.1. By a similar observation, we can define singular vertices of discrete p-CGC surfaces in $\mathbb{R}^3$ or $\mathbb{H}^3$ as follows: Let $\tilde{f}$ be a discrete p-CGC K surface obtained by taking a parallel surface of a discrete isothermic CMC surface described in Proposition 8.4.1 or Theorem 8.4.1, and let i, j, k be three consective three points in the horizontal or vertical direction. Then the FPS vertex $\tilde{f}_j$ is a *singular vertex* if $|\tilde{\kappa}_{ij}|,\ |\tilde{\kappa}_{jk}| > \sqrt{K}$, where $\tilde{\kappa}$ are the principal curvatures of $\tilde{f}$.

Finally we conclude this paper by examining examples of discrete p-CGC surfaces in $\mathbb{S}^3$. On the left-hand side of Figure 8.5, an example of discrete p-CGC surfaces in $\mathbb{S}^3$ with FPS vertices is shown, and on the right-hand of Figure 8.5, the same discrete p-CGC surface in $\mathbb{S}^3$ with singular vertices is shown. The singular vertices seem to capture the characteristic of singularities of discrete p-CGC surfaces much more than the FPS vertices.

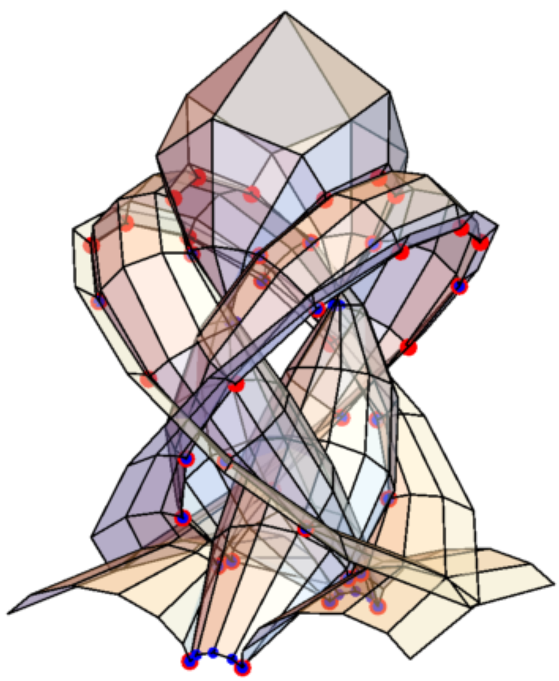


Fig. 8.5: A discrete p-CGC surface in $\mathbb{S}^3$ obtained by taking a parallel surface of a discrete 2-legged Smyth surface with FPS vertices (left-hand side) or singular vertices (right-hand side).

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