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Connection coefficients and monodromy representations for a class of Okubo systems of ordinary differential equation

（大久保型常微分方程式系のあるクラスに対する接続係数とモノドロミー表現）

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Connection coefficients and monodromy representations for a class of Okubo systems of ordinary differential equations

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Abstract

Explicit connection coefficients and monodromy representations are constructed for the canonical solution matrices of a class of Okubo systems of ordinary differential equations as an application of the Katz operations.

Introduction

The Okubo systems are one of the most important classes of Fuchsian systems of ordinary differential equations on the Riemann sphere $\mathbb{P}^1$; for the definition of an Okubo system, see Section 1. A characteristic feature of this class is that each Okubo system admits a solution matrix consisting of non-holomorphic solutions around its finite singular points, which we call below *Okubo's canonical solution matrix*. Okubo [7] studied basic properties and in particular established an explicit determinant formula for such a solution matrix (see Theorem 1.1). This determinant formula guarantees that Okubo's canonical solution matrix is in fact a fundamental solution matrix. Also, it is because of this determinant formula that Okubo systems provide with good models for global analysis of Fuchsian systems.

Yokoyama [14] classified the types of irreducible rigid Okubo systems under the condition that the nontrivial local exponents are mutually distinct at each finite singular point; this class consists of eight types I, II, III, IV and I*, II*, III*, IV*, which is usually referred to as *Yokoyama's list.* For each type in Yokoyama's list, Haraoka constructed a canonical form of the Okubo system [1], as well as the corresponding monodromy representation, up to the action of diagonal matrices [2]. The purpose of this doctoral thesis is to propose a method for determining the explicit monodromy representation for Okubo's canonical solution matrix, including the diagonal matrix factor which has not been fixed in [2]. This problem is in fact reduced to determining the connection coefficients among non-holomorphic solutions for the Okubo system. We solve this connection problem by means of the middle convolutions for Schlesinger systems.

It is known by a celebrated work of Katz [3] that any irreducible rigid local system can be constructed by a finite iteration of the so-called Katz operations (additions and middle convolutions) from a local system of rank one. Dettweiler and Reiter [10] reformulated the Katz operations so that they can be applied directly to Schlesinger systems and their monodromy representations. Therefore, any irreducible rigid Schlesinger system, as well as its monodromy representation, can be constructed in principle from the rank one case. In the case of Okubo systems, Yokoyama independently introduced the notions of extending and restricting operations for differential systems and their monodromy matrices [15]. He also proved that any generic rigid Okubo system can be constructed by a finite iteration of his operations. It is clarified by Oshima [12] the Katz operations and the Yokoyama operations are essentially equivalent as far as the Okubo systems are concerned. It is not completely clarified, however, how the Katz (or Yokoyama) operations can be realized on the level of fundamental solution matrices and their monodromy representations.

Our method for solving the connection problem is based on the inductive construction of Okubo systems by middle convolutions. For a given Schlesinger system and a fundamental solution matrix, we give an explicit construction of a fundamental solution matrix of its middle convolution in the sense of [10]. We then apply this procedure for constructing the Okubo systems in Yokoyama's list and their explicit monodromy representations.

This paper is organized as follows. We propose in Section 1 the notion of *Okubo's canonical solution matrix* $\Psi(x)$ for an Okubo system

$$(x-T)\frac{d}{dx}Y = AY, \quad A \in \mathrm{Mat}(n;\mathbb{C}) \tag{0.1}$$

of ordinary differential equations. This solution matrix, introduced by Okubo [7], consists of non-holomorphic solutions around finite singular points, and forms a fundamental solution matrix under a certain genericity condition. We also give a remark on the relation between the connection coefficients among non-holomorphic local solutions and the monodromy matrices for Okubo's canonical solution matrix. The monodromy matrices for $\Psi(x)$ are determined from the connection coefficients through formula (1.17). Our main results are collected in Section 2. We fix a canonical form for each of the Okubo systems of of Yokoyama's list as in Theorems 2.1 to Theorem 2.6, and give the explicit monodromy matrices for the corresponding canonical solution matrix in Theorem 2.7 to Theorem 2.12. Theses results are proved in subsequent sections by iterations of the Katz operations.

Our method for determining the connection coefficients and the monodromy matrices is based on the middle convolution for Schlesinger systems of [10]. The procedure of the middle convolution for a Schlesinger system

$$\frac{d}{dx}y = \sum_{k=1}^{r}\frac{A_k}{x-t_k}Y \tag{0.2}$$

is divided into three steps, which we call the *convolution*, the *K-reduction* and the *L-reduction*. Analyzing these three steps, in Section 3 we clarify how one can construct a fundamental solution matrix for the middle convolution from a given fundamental solution matrix of the system (0.2).

A special combination of Katz operations

$$\mathrm{add}_{(0,\ldots,\rho,\ldots,0)} \circ \mathrm{mc}_{-\rho-c} \circ \mathrm{add}_{(0,\ldots,c,\ldots,0)} \tag{0.3}$$

(middle convolution with additions at a singular points) is used effectively for constructing the Okubo systems in Yokoyama's list. In fact, each Okubo system in Yokoyama's list can be constructed by a finite iteration of such operations from the rank one case. We formulate in Section 4 this type of operations for a general Okubo system and its monodromy representation. We show that the resulting Schlesinger system is also an Okubo system as explicitly given by (4.15). For the monodromy representation, (4.32) provides the moodromy representation of the resulting system specified by (4.15). Furthermore we obtain the connection coefficients for the canonical solution matrix of the resulting system as in Theorem 4.4 by applying these results.

In Section 5 we prove our main theorems for Okubo systems of Yokoyama's list. Our proof is divided into two steps. We first compute a certain part of the connection coefficients for each of those Okubo systems by applying Theorem 4.4 repeatedly. Second, we determine the other connection coefficients by the symmetry of the Okubo system. Our argument is based on the fact that the permutaion of characteristic exponents at a singular point can be realized by the adjoint action of a contant matrix.

We remark that our approach is similar to that of Yokoyama [16] in which recursive relations for connection coefficients are investigated in the framework of extending operations for Okubo systems. It is not clear, however, whether the connection coefficients for individual Okubo systems in Yokoyama's list can be directly determined only from the results of [16]. We also remark that the connection problem for rigid irreducible Fuchsian differential equations of scalar type has been discussed by Oshima [11] as an application of the Katz operations.

The contents of this thesis have been made public as the two papers [5] and [6].

Contents

1 Okubo systems and their canonical solution matrices

In this section we formulate the notion of a *canonical solution matrix* for a system of ordinary differential equations of Okubo type, and investigate fundamental properties of its connection coefficients and monodromy matrices.

1.1 Canonical solution matrix for an Okubo system

Let $t_1, \dots, t_r$ be r distinct points in $\mathbb{C}$. Fixing an r-tuple $(n_1, \dots, n_r)$ of positive integers with $n_1 + \dots + n_r = n$, we denote by

$$T = \operatorname{diag}(t_1 I_{n_1}, t_2 I_{n_2}, \dots, t_r I_{n_r}) = \begin{pmatrix} t_1 I_{n_1} & & & \\ & t_2 I_{n_2} & & \\ & & \ddots & \\ & & & t_r I_{n_r} \end{pmatrix} \tag{1.1}$$

the block diagonal matrix whose diagonal blocks are scalar matrices $t_i I_{n_i}$ $(i = 1, \dots, r)$, where I_k stands for the $k \times k$ identity matrix. A system of ordinary differential equations on $\mathbb{P}^1$ of the form

$$(xI_n - T)\frac{d}{dx}Y = AY \qquad (A \in \mathrm{Mat}(n; \mathbb{C})), \tag{1.2}$$

is called an *Okubo system* of type $(n_1, \dots, n_r)$, where $Y = (y_1, \dots, y_m)^{\mathrm{t}}$ is the column vector of unknown functions. For each $i, j = 1, \dots r$, we denote by A_{ij} the (i, j)-block of the matrix A:

$$A = \begin{pmatrix} A_{11} & \dots & A_{1r} \\ \vdots & & \vdots \\ A_{r1} & \dots & A_{rr} \end{pmatrix}, \qquad A_{ij} \in \mathrm{Mat}(n_i, n_j; \mathbb{C}). \tag{1.3}$$

Then the Okubo system (1.2) can be equivalently rewritten in the Schlesinger form

$$\frac{d}{dx}Y = \left(\sum_{k=1}^{r} \frac{A_k}{x - t_k} \right) Y, \qquad A_k = \begin{pmatrix} & O & \\ A_{k1} & \dots & A_{kr} \\ & O & \end{pmatrix} \quad (k = 1, \dots, r). \tag{1.4}$$

We denoting by $\alpha_1^{(k)}, \dots, \alpha_{n_k}^{(k)}$ the eigenvalues of the diagonal block A_{kk} $(k = 1, \dots, r)$ and by $\rho_1, \dots, \rho_n$ the eigenvalues of $A = A_1 + \dots + A_r$; these eigenvalues are subject to the *Fuchs relation*

$$\sum_{k=1}^{r} \sum_{j=1}^{n_k} \alpha_j^{(k)} = \sum_{j=1}^{n} \rho_j. \tag{1.5}$$

We assume hereafter that

$$\alpha_i^{(k)} - \alpha_j^{(k)} \notin \mathbb{Z} \backslash \{0\} \quad (1 \le i < j \le n_k), \quad \alpha_j^{(k)} \notin \mathbb{Z} \quad (1 \le j \le n_k) \tag{1.6}$$

for $k = 1, \dots, r$.

A characteristic feature of a system of Okubo type is that, if the local exponents are generic, there exists a *canonical* fundamental solution matrix consisting of singular solutions around the finite singular points $x = t_1, \dots, t_r$.

In what follows, we denote by $\mathcal{O}(\widetilde{\mathcal{D}})$ the vector space of all multivalued holomorphic functions on $\mathcal{D} = \mathbb{C} \backslash \{t_1, \dots, t_r\}$ and by $\mathcal{O}_a$ the ring of germs of homomorphic functions at $x = a$. Fixing a base point $p_0 \in D$, we identify a multivalued holomorphic function on D with a germ of holomorphic function at $x = p_0$ which can be continued analytically along any continuous path in D starting from p_0. We denote by γ_k ($k = 1, \dots, r$) and γ_∞ the homotopy classes in $\pi_1(\mathcal{D}, p_0)$ of continuous paths encircling $x = t_k$ and $x = \infty$ in the positive direction, respectively; we choose these paths so that $\gamma_\infty \gamma_1 \cdots \gamma_r = 1$ in $\pi_1(\mathcal{D}, p_0)$ (See Figure 1). In this paper, for two continuous paths α, β, we use the notation $\beta\alpha$ to refer to the path obtained by connecting α and β in this order. Choosing the base point p_0 appropriately, we assume that $\operatorname{Im}(t_i - p_0)/(t_1 - p_0) < 0$ for $i = 2, \dots, r$, and assign the arguments $\theta_k = \arg(p_0 - t_k)$ ($k = 1, 2, \dots, r$) so that

$$\theta_1 > \theta_2 > \cdots > \theta_r > \theta_1 - \pi. \tag{1.7}$$

When we consider the behavior of $f \in \mathcal{O}(\widetilde{\mathcal{D}})$ around $x = t_k$, we use the analytic continuation of $f \in \mathcal{O}_{p_0}$ along the line segment connecting p_0 and t_k.

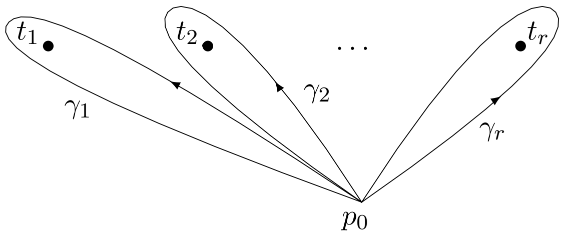

Figure 1:

If condition (1.6) is satisfied, for each $k = 1, \dots, r$, system (1.2) has a unique fundamental solution matrix $\Psi^{(k)}(x) \in \mathrm{Mat}(n; \mathcal{O}(\widetilde{\mathcal{D}}))$ of the form

$$\Psi^{(k)}(x) = F^{(k)}(x)(x - t_k)^{A_k}, \quad F^{(k)}(x) \in \mathrm{Mat}(n; \mathcal{O}_{t_k}), \quad F^{(k)}(t_k) = I_n, \tag{1.8}$$

where we specify the branch of $(x - t_k)^{A_k} = \exp(A_k \log(x - t_k))$ near $x = p_0$ by $\arg(p_0 - t_k) = \theta_k$. We call $\Phi^{(k)}(x)$ the *local canonical solution matrix* of (1.2) at $x = t_k$. We decompose this $\Psi^{(k)}(x)$ as

$$\Psi^{(k)}(x) = (\Psi_1^{(k)}(x), \dots, \Psi_r^{(k)}(x)), \quad \Psi_j^{(k)}(x) \in \mathrm{Mat}(n, n_j; \mathcal{O}(\widetilde{\mathcal{D}})). \tag{1.9}$$

Then the kth block $\Psi_k^{(k)}(x)$ represents a basis of all singular solutions around $x = t_k$, and its local monodromy is given by

$$\gamma_k \cdot \Psi_k^{(k)}(x) = \Psi_k^{(k)}(x) e(A_{kk}), \tag{1.10}$$

where $e(\mu) = \exp(2\pi i \mu)$. Collecting the singular solutions $\Psi_k^{(k)}(x)$ ($k = 1, \dots, r$) together, we define the $n \times n$ solution matrix $\Psi(x)$ by

$$\Psi(x) = (\Psi_1^{(1)}(x), \Psi_2^{(2)}(x), \dots, \Psi_r^{(r)}(x)) \in \mathrm{Mat}(n; \mathcal{O}(\widetilde{\mathcal{D}})). \tag{1.11}$$

We call this $\Psi(x)$ the *canonical solution matrix* for the Okubo system (1.2).

As for the determinant of $\Psi(x)$, the following theorem is known as Okubo's determinant formula (Okubo [7], Kohno [4]).

Theorem 1.1. *Under the assumption* (1.6)*, the determinant of the canonical solution matrix* $\Psi(x)$ *is explicitly given by*

$$\det(\Psi(x)) = \frac{\prod_{k=1}^{r} \prod_{j=1}^{n_k} \Gamma(1 + \alpha_j^{(k)})}{\prod_{i=1}^{n} \Gamma(1 + \rho_i)} \prod_{k=1}^{r} (x - t_k)^{\sum_{j=1}^{n_k} \alpha_j^{(k)}}. \tag{1.12}$$

Corollary 1.2. *Under the assumption* (1.6)*, the canonical solution matrix* $\Psi(x)$ *of the Okubo system* (1.2) *is a fundamental solution matrix if and only if* $\rho_i \notin \mathbb{Z}_{\leq 0}$ *for* $i = 1, \ldots, n$*.*

1.2 Connection coefficients and monodromy matrices

We give some remarks on the relationship between the connection coefficients and the monodromy matrices for the canonical solution matrix $\Psi(x)$ of the Okubo system (1.2).

Let $\Psi(x)$ be the canonical solution matrix of the Okubo system (1.2). The analytic continuation of $\Psi(x)$ by γ_k ($k = 1, \ldots, r$) defines an r-tuple of monodromy matrices $\boldsymbol{M} = (M_1, \ldots, M_r) \in \mathrm{GL}(n;\mathbb{C})^r$ as

$$\gamma_k \cdot \Psi(x) = \Psi(x) M_k \quad (k = 1, \ldots, r). \tag{1.13}$$

Through the analytic continuation along γ_k to a neighborhood of $x = t_k$, the j-th column block $\Psi_j^{(j)}(x)$ ($j = 1, \ldots, r$) is expressed uniquely in the form

$$\Psi_j^{(j)}(x) = \Psi_k^{(k)}(x) C_{kj} + H_j^{(k)}(x), \tag{1.14}$$

where C_{kj} is an $n_k \times n_j$ constant matrix and $H_j^{(k)}(x)$ is an $n \times n_j$ solution matrix which is holomorphic around $x = t_k$. Since $\gamma_k \cdot \Psi_k^{(k)}(x) = \Psi_k^{(k)}(x) e(A_{kk})$, we have

$$\gamma_k \cdot \Psi_j^{(j)}(x) = \Psi_k^{(k)}(x) e(A_{kk}) C_{kj} + H_j^{(k)}(x), \tag{1.15}$$

and hence

$$\gamma_k \cdot \Psi_j^{(j)}(x) = \Psi_k^{(k)}(x)(e(A_{kk}) - 1) C_{kj} + \Psi_j^{(j)}(x) \tag{1.16}$$

eliminating $H_j^{(k)}(x)$ by (1.14). To summarize, the monodromy matrices for the canonical solution matrix $\Psi(x)$ are given as

$$M_k = \begin{pmatrix} 1 & & & & \\ & \ddots & & & \\ (e(A_{kk}) - 1)C_{k1} & \ldots & e(A_{kk}) & \ldots & (e(A_{kk}) - 1)C_{kr} \\ & & & \ddots & \\ & & & & 1 \end{pmatrix} \quad (k = 1, \ldots, r) \tag{1.17}$$

in terms of the connection coefficients as in (1.14). We remark that, in order to compute the monodromy matrices of $\Psi(x)$, we need not explicitly determine the holomorphic part $H_j^{(k)}(x)$ of analytic continuation.

2 Okubo systems in Yokoyama's list

2.1 Canonical forms of the systems

Yokoyama's list is a class of rigid irreducible Okubo systems such that each diagonal block $A_{ii} \in \mathrm{Mat}(n_i;\mathbb{C})$ ($i = 1, \ldots, r$) of A has n_i mutually distinct eigenvalues and that the matrix $A \in \mathrm{Mat}(n;\mathbb{C})$ is diagonalizable. In this class we can assume that the diagonal blocks A_{ii} ($i = 1, \ldots, r$) are diagonal matrices with mutually distinct entries. We assume that A is diagonalized as

$$A \sim \mathrm{diag}(\rho_1 I_{m_1}, \ldots, \rho_q I_{m_q}) \quad (m_1 \geq \cdots \geq m_q) \tag{2.1}$$

with mutually distinct $\rho_1, \dots, \rho_q \in \mathbb{C}$. Then the Okubo systems in Yokoyama's list are characterized by the following eight pairs $(n_1, \dots, n_r)$, $(m_1, \dots, m_q)$ of partitions of n respectively:

$$
\begin{array}{llll}
(\mathrm{I})_n : & \begin{cases} (n_1, n_2) = (n-1, 1) \\ (m_1, \dots, m_n) = (1, \dots, 1) \end{cases} & (\mathrm{I}^*)_n : & \begin{cases} (n_1, \dots, n_n) = (1, \dots, 1) \\ (m_1, m_2) = (n-1, 1) \end{cases} \\
(\mathrm{II})_{2n} : & \begin{cases} (n_1, n_2) = (n, n) \\ (m_1, m_2, m_3) = (n, n-1, 1) \end{cases} & (\mathrm{II}^*)_{2n} : & \begin{cases} (n_1, n_2, n_3) = (n, n-1, 1) \\ (m_1, m_2) = (n, n) \end{cases} \\
(\mathrm{III})_{2n+1} : & \begin{cases} (n_1, n_2) = (n+1, n) \\ (m_1, m_2, m_3) = (n, n, 1) \end{cases} & (\mathrm{III}^*)_{2n+1} : & \begin{cases} (n_1, n_2, n_3) = (n, n, 1) \\ (m_1, m_2) = (n+1, n) \end{cases} \\
(\mathrm{IV})_6 : & \begin{cases} (n_1, n_2) = (4, 2) \\ (m_1, m_2, m_3) = (2, 2, 2) \end{cases} & (\mathrm{IV}^*)_6 : & \begin{cases} (n_1, n_2, n_3) = (2, 2, 2) \\ (m_1, m_2) = (4, 2) \end{cases}
\end{array}
\tag{2.2}
$$

Canonical forms of the Okubo systems in Yokoyama's list are determined by the work of Haraoka [1]. We first specify explicit canonical forms of the Okubo systems of Yokoyama's list as in [1], which we will use throughout this paper.

Case I: We express the Okubo system of type $(\mathrm{I})_n$ in the form

$$
(x - T)\frac{d}{dx}Y = AY, \quad A = \begin{pmatrix} \alpha & A_{12} \\ A_{21} & \beta \end{pmatrix} = \begin{pmatrix} \alpha_1 & & & (A_{12})_1 \\ & \ddots & & \vdots \\ & & \alpha_{n-1} & (A_{12})_{n-1} \\ (A_{21})_1 & \dots & (A_{21})_{n-1} & \alpha_n \end{pmatrix} \sim \mathrm{diag}(\rho_1, \rho_2, \dots, \rho_n)\,,
\tag{2.3}
$$

where α, β and T are diagonal matrices defined by

$$
T = \mathrm{diag}(t_1 I_{n-1}, t_2), \quad \alpha = \mathrm{diag}(\alpha_1, \cdots, \alpha_{n-1}), \quad \beta = (\alpha_n). \tag{2.4}
$$

The Fuchs relation in this case is given by

$$
\sum_{i=1}^{n} \alpha_i = \sum_{i=1}^{n} \rho_i. \tag{2.5}
$$

We assume that the parameters of $(\mathrm{I})_n$ satisfy the following conditions:

$$
\begin{array}{ll}
\alpha_i - \alpha_j \notin \mathbb{Z} \quad (1 \le i < j \le n-1), & \alpha_i \notin \mathbb{Z} \quad (1 \le i \le n), \\
\rho_i - \rho_j \notin \mathbb{Z} \quad (1 \le i < j \le n), & \rho_i \notin \mathbb{Z} \quad (1 \le i \le n).
\end{array}
\tag{2.6}
$$

For the type $(\mathrm{I})_n$, the canonical form can be taken as follows

Theorem 2.1 ([1]). *Canonical form of the Okubo system of type* $(\mathrm{I})_n$ *is given by the matrices* $K = (K_i)_i$ *and* $L = (L_j)_j$ *with the following entries respectively* :

$$
(A_{12})_i = 1, \qquad (A_{21})_j = -\frac{\prod_{k=1}^{n}(\alpha_j - \rho_k)}{\prod_{k \ne j}^{n-1}(\alpha_j - \alpha_k)}, \quad (i, j = 1, \dots, n-1) \tag{2.7}
$$

The symbol $\prod_{k\neq i}^{m}$ stands for the product over k such that $1 \le k \le m$ and $k \neq i$.

Cases II **and** III: We express the Okubo systems of type $(\mathrm{II})_{2n}$ and $(\mathrm{III})_{2n+1}$ in the form

$$(x-T)\frac{d}{dx}Y = AY, \quad A = \begin{pmatrix} \alpha & A_{12} \\ A_{21} & \beta \end{pmatrix} \sim \mathrm{diag}(\rho_1 I_n, \rho_2 I_{m-1}, \rho_3) \ \ (m = n, n+1), \tag{2.8}$$

where α, β and T are diagonal matrices defined by

$$\begin{aligned} (\mathrm{II})_{2n}: \ & T = \mathrm{diag}(t_1 I_n, t_2 I_n), \quad \alpha = \mathrm{diag}(\alpha_1, \cdots, \alpha_n), \quad \beta = \mathrm{diag}(\beta_1, \cdots, \beta_n), \\ (\mathrm{III})_{2n+1}: \ & T = \mathrm{diag}(t_1 I_{n+1}, t_2 I_n), \quad \alpha = \mathrm{diag}(\alpha_1, \cdots, \alpha_{n+1}), \quad \beta = \mathrm{diag}(\beta_1, \cdots, \beta_n), \end{aligned} \tag{2.9}$$

respectively. Note that the Fuchs relations in these cases are given by

$$\sum_{i=1}^{m} \alpha_i + \sum_{i=1}^{n} \beta_i = n\rho_1 + (m-1)\rho_2 + \rho_3. \tag{2.10}$$

We always assume that the parameters satisfy the following conditions:

$$\begin{aligned} &\alpha_i - \alpha_j \notin \mathbb{Z} \quad (1 \le i < j \le m), \qquad \alpha_i \notin \mathbb{Z} \quad (1 \le i \le m), \\ &\beta_i - \beta_j \notin \mathbb{Z} \quad (1 \le i < j \le n), \qquad \beta_i \notin \mathbb{Z} \quad (1 \le i \le n), \\ &\rho_i - \rho_j \notin \mathbb{Z} \quad (1 \le i < j \le 3), \qquad \rho_i \notin \mathbb{Z} \quad (1 \le i \le 3). \end{aligned} \tag{2.11}$$

In the cases of type $(\mathrm{II})_{2n}$ and $(\mathrm{III})_{2n+1}$, the following canonical forms of the Okubo systems are proposed by Haraoka [1].

Theorem 2.2 ([1])**.** *Canonical forms of the Okubo systems of type* $(\mathrm{II})_{2n}$ $(m = n)$ *and* $(\mathrm{III})_{2n+1}$ $(m = n+1)$ *are given by the matrices* $K = (K_{ij})_{ij}$ *and* $L = (L_{ij})_{ij}$ *with the following entries respectively* :

$$(\mathrm{II})_{2n}: \quad \begin{aligned} (A_{12})_{ij} &= (\beta_j - \rho_1)\frac{\prod_{k\neq i}^{n}(\alpha_k + \beta_j - \rho_1 - \rho_2)}{\prod_{k\neq j}^{n}(\beta_j - \beta_k)} \quad (1 \le i, j \le n) \\ (A_{21})_{ij} &= (\alpha_j - \rho_1)\frac{\prod_{k\neq i}^{n}(\alpha_j + \beta_k - \rho_1 - \rho_2)}{\prod_{k\neq j,\,1}^{n}(\alpha_j - \alpha_k)} \quad (1 \le i, j \le n), \end{aligned} \tag{2.12}$$

$$(\mathrm{III})_{2n+1}: \quad \begin{aligned} (A_{12})_{ij} &= \frac{\prod_{k\neq i}^{n+1}(\alpha_k + \beta_j - \rho_1 - \rho_2)}{\prod_{k\neq j}^{n}(\beta_j - \beta_k)} \quad (1 \le i \le n+1;\ 1 \le j \le n) \\ (A_{21})_{ij} &= (\alpha_j - \rho_1)(\alpha_j - \rho_2)\frac{\prod_{k\neq i}^{n}(\alpha_j + \beta_k - \rho_1 - \rho_2)}{\prod_{k\neq j}^{n+1}(\alpha_j - \alpha_k)} \quad (1 \le i \le n;\ 1 \le j \le n+1). \end{aligned} \tag{2.13}$$

To be more precise, the canonical form of Haraoka [1] is the conjugation of our A by the diagonal matrix

$$\mathrm{diag}(a_1^{-1}, \ldots, a_m^{-1}, b_1^{-1}, \ldots, b_n^{-1}); \quad a_i = \prod_{k\neq i}^{m}(\alpha_i - \alpha_k), \quad b_i = \prod_{k\neq i}^{n}(\beta_i - \beta_k). \tag{2.14}$$

Case IV: The system $(\mathrm{IV})_6$ is expressed in the form

$$(x-T)\frac{d}{dx}Y = \begin{pmatrix} \alpha & A_{12} \\ A_{21} & \beta \end{pmatrix} Y \tag{2.15}$$

Then the Fuchs relation is given by

$$\begin{aligned} &\alpha_i - \alpha_j \notin \mathbb{Z} \quad (1 \le i < j \le 4), && \alpha_i \notin \mathbb{Z} \quad (1 \le i \le 4), \\ &\beta_1 - \beta_2 \notin \mathbb{Z}, && \beta_i \notin \mathbb{Z} \quad (1 \le i \le 2), \\ &\rho_i - \rho_j \notin \mathbb{Z}, \quad (1 \le i < j \le 3), && \rho_i \notin \mathbb{Z} \quad (1 \le i \le 3). \end{aligned} \tag{2.16}$$

We assume that the parameters of $(\mathrm{IV})_6$ satisfy the following conditions:

$$\alpha_i \notin \mathbb{Z} \quad (1 \le i \le 4), \quad \rho_1 - \rho_2 \notin \mathbb{Z}, \quad \rho_1, \rho_2 \notin \mathbb{Z}, \tag{2.17}$$

Theorem 2.3 ([1]). *Canonical form of the Okubo system of type* $(\mathrm{IV})_6$ *is given by the matrices* $A = (A_{ij})_{ij}$ *with the following entries respectively:*

$$\begin{aligned} (A_{12})_{ij} &= \frac{\prod_{k \ne i}^{3} (\alpha_k + \alpha_4 + \beta_j - \rho_1 - \rho_2 - \rho_3)}{\prod_{k \ne j}^{2} (\beta_j - \beta_k)} \quad (i = 1, 2, 3, j = 1, 2), \\ (A_{12})_{4j} &= \frac{1}{\prod_{k \ne j}^{2} (\beta_j - \beta_k)} \quad (j = 1, 2), \\ (A_{21})_{ij} &= \prod_{k=1}^{3} (\alpha_j - \rho_k) \frac{\prod_{k \ne i}^{2} (\alpha_j + \alpha_4 + \beta_k - \rho_1 - \rho_2 - \rho_3)}{\prod_{k \ne j}^{4} (\alpha_j - \alpha_k)} \quad (i = 1, 2, j = 1, 2, 3), \\ (A_{21})_{i4} &= \prod_{k=1}^{3} (\alpha_j - \rho_k) \frac{\prod_{k=1}^{3} \prod_{l \ne i}^{2} (\alpha_k + \alpha_4 + \beta_l - \rho_1 - \rho_2 - \rho_3)}{\prod_{k \ne j}^{4} (\alpha_j - \alpha_k)} \quad (i = 1, 2). \end{aligned} \tag{2.18}$$

Case I^*: The Okubo system of type $(\mathrm{I}^*)_n$ is expressed in the form

$$\begin{pmatrix} x - t_1 & & \\ & \ddots & \\ & & x - t_n \end{pmatrix} \frac{d}{dx} Y = \begin{pmatrix} \alpha_1 & a_{12} & \cdots & a_{1n} \\ a_{21} & \alpha_2 & \cdots & a_{2n} \\ \vdots & & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & \alpha_n \end{pmatrix} Y \sim \mathrm{diag}(\rho_1 I_{n-1}, \rho_2). \tag{2.19}$$

Then the Fuchs relation is given by

$$\sum_{k=1}^{n} \alpha_k = (n-1)\rho_1 + \rho_2. \tag{2.20}$$

We assume that the parameters of $(\mathrm{I}^*)_n$ satisfy the following conditions:

$$\alpha_i \notin \mathbb{Z} \quad (1 \le i \le n), \quad \rho_1 - \rho_2 \notin \mathbb{Z}, \quad \rho_1, \rho_2 \notin \mathbb{Z}, \tag{2.21}$$

The canonical forms of the type $(\mathrm{I}^*)_n$ is given by following form:

Theorem 2.4 ([1]). *Canonical form of the Okubo system of type* $(\mathrm{I}^*)_n$ *is given by the components* a_{ij} *with the following entries respectively* :

$$a_{ij} = \alpha_j - \rho_1 \quad (i \ne j, \ i, j = 1, \dots, n) \tag{2.22}$$

Cases II* **and** III*: We express the Okubo systems of type $(\mathrm{II}^*)_{2n}$ and $(\mathrm{III}^*)_{2n+1}$ in the form

$$(x-T)\frac{d}{dx}Y = AY, \quad A = \begin{pmatrix} \alpha & A_{12} & A_{13} \\ A_{21} & \beta & A_{23} \\ A_{31} & A_{32} & \gamma \end{pmatrix} \sim \mathrm{diag}(\rho_1 I_m, \rho_2 I_n) \ \ (m = n, n+1), \tag{2.23}$$

where α, β and T are diagonal matrices defined by

$$\begin{aligned} (\mathrm{II}^*)_{2n}: \ & T = \mathrm{diag}(t_1 I_n, t_2 I_{n-1}, 1), \quad \alpha = \mathrm{diag}(\alpha_1, \cdots, \alpha_n), \quad \beta = \mathrm{diag}(\beta_1, \cdots, \beta_{n-1}), \\ (\mathrm{III}^*)_{2n+1}: \ & T = \mathrm{diag}(t_1 I_n, t_2 I_n, 1), \quad \alpha = \mathrm{diag}(\alpha_1, \cdots, \alpha_n), \quad \beta = \mathrm{diag}(\beta_1, \cdots, \beta_n), \end{aligned} \tag{2.24}$$

respectively. Note that the Fuchs relations in these cases are given by

$$\sum_{i=1}^{n} \alpha_i + \sum_{i=1}^{m-1} \beta_i + \gamma = m\rho_1 + n\rho_2. \tag{2.25}$$

We always assume that the parameters satisfy the following conditions:

$$\begin{aligned} &\alpha_i - \alpha_j \notin \mathbb{Z} \quad (1 \le i < j \le m), && \alpha_i \notin \mathbb{Z} \quad (1 \le i \le m), \\ &\beta_i - \beta_j \notin \mathbb{Z} \quad (1 \le i < j \le n), && \beta_i \notin \mathbb{Z} \quad (1 \le i \le n), \\ &\gamma \notin \mathbb{Z}, \quad \rho_1 - \rho_2 \notin \mathbb{Z}, && \rho_i \notin \mathbb{Z} \quad (1 \le i \le 3). \end{aligned} \tag{2.26}$$

In the cases of type $(\mathrm{II}^*)_{2n}$ and $(\mathrm{III}^*)_{2n+1}$, the following canonical forms of the Okubo systems are proposed by Haraoka [1].

Theorem 2.5 ([1]). *Canonical forms of the Okubo systems of type* $(\mathrm{II}^*)_{2n}$ $(m = n)$ *and* $(\mathrm{III}^*)_{2n+1}$ $(m = n+1)$ *are given by the matrices* $A = (A_{ij})_{ij}$ *with the following entries respectively* :

$$(\mathrm{II}^*)_{2n}: \quad \begin{aligned} (A_{12})_{ij} &= \frac{\prod_{k\neq i}^{n}(\beta_j + \alpha_k - \rho_1 - \rho_2)}{\prod_{k\neq j}^{n-1}(\beta_j - \beta_k)} \quad (i = 1, \dots, n, j = 1, \dots, n-1), \\ (A_{13})_i &= 1 \quad (i = 1, \dots, n), \\ (A_{23})_i &= 1 \quad (i = 1, \dots, n-1), \\ (A_{21})_{ij} &= (\alpha_j - \rho_1)(\alpha_j - \rho_2)\frac{\prod_{k\neq i}^{n-1}(\alpha_j + \beta_k - \rho_1 - \rho_2)}{\prod_{k\neq j}^{n}(\alpha_j - \alpha_k)} \quad (i = 1, \dots, n-1, j = 1, \dots, n), \\ (A_{31})_j &= -(\alpha_j - \rho_1)(\alpha_j - \rho_2)\frac{\prod_{k=1}^{n-1}(\alpha_j + \beta_k - \rho_1 - \rho_2)}{\prod_{k\neq j}^{n}(\alpha_j - \alpha_k)} \quad (j = 1, \dots, n), \\ (A_{32})_j &= -\frac{\prod_{k=1}^{n}(\beta_j + \alpha_k - \rho_1 - \rho_2)}{\prod_{k\neq j}^{n-1}(\beta_j - \beta_k)} \quad (j = 1, \dots, n-1), \end{aligned} \tag{2.27}$$

$$
(\mathrm{III}^*)_{2n+1}: \begin{array}{lll}
(A_{12})_{ij} &= (\beta_j-\rho_1)\dfrac{\prod_{k\neq i}^n(\beta_j+\alpha_k-\rho_1-\rho_2)}{\prod_{k\neq j}^n(\beta_j-\beta_k)} & (i,j=1,\dots,n),\\
(A_{13})_i &= 1 \quad (i=1,\dots,n), &\\
(A_{23})_i &= 1 \quad (i=1,\dots,n), &\\
(A_{21})_{ij} &= (\alpha_j-\rho_1)\dfrac{\prod_{k\neq i}^n(\alpha_j+\beta_k-\rho_1-\rho_2)}{\prod_{k\neq j}^n(\alpha_j-\alpha_k)} & (i,j=1,\dots,n),\\
(A_{31})_j &= -(\alpha_j-\rho_1)\dfrac{\prod_{k=1}^n(\alpha_j+\beta_k-\rho_1-\rho_2)}{\prod_{k\neq j}^n(\alpha_j-\alpha_k)} & (j=1,\dots,n),\\
(A_{32})_j &= -(\beta_j-\rho_1)\dfrac{\prod_{k=1}^n(\beta_j+\alpha_k-\rho_1-\rho_2)}{\prod_{k\neq j}^n(\beta_j-\beta_k)} & (j=1,\dots,n).
\end{array}
\tag{2.28}
$$

Cases IV*: We express the Okubo systems of type $(\mathrm{IV}^*)_6$ in the form

$$
(x-T)\frac{d}{dx}Y = AY, \quad A = \begin{pmatrix} \alpha & A_{12} & A_{13}\\ A_{21} & \beta & A_{23}\\ A_{31} & A_{32} & \gamma \end{pmatrix} \sim \mathrm{diag}(\rho_1 I_4, \rho_2 I_2), \tag{2.29}
$$

where α, β, γ and T are diagonal matrices defined by

$$
T = \mathrm{diag}(t_1 I_n, t_2 I_{n-1}, 1), \quad \alpha = \mathrm{diag}(\alpha_1,\alpha_2), \quad \beta = \mathrm{diag}(\beta_1,\beta_2), \quad \gamma = \mathrm{diag}(\gamma_1,\gamma_2) \tag{2.30}
$$

respectively. Note that the Fuchs relations in these cases are given by

$$
\sum_{k=1}^2 \alpha_k + \sum_{k=1}^2 \beta_+ \sum_{k=1}^2 \gamma_2 = 4\rho_1 + 2\rho_2. \tag{2.31}
$$

We always assume that the parameters satisfy the following conditions:

$$
\begin{array}{llll}
\alpha_1-\alpha_2 \notin \mathbb{Z}, & \alpha_i \notin \mathbb{Z} \quad (1 \le i \le 2), & \beta_i-\beta_j \notin \mathbb{Z}, & \beta_i \notin \mathbb{Z} \quad (1\le i\le 2),\\
\gamma_1-\gamma_2 \notin \mathbb{Z}, & \gamma_i \notin \mathbb{Z} \quad (1 \le i \le 2), & \rho_1-\rho_2 \notin \mathbb{Z}, & \rho_i \notin \mathbb{Z} \quad (1\le i\le 2).
\end{array}
\tag{2.32}
$$

In the cases of type $(\mathrm{IV}^*)_6$, the following canonical forms of the Okubo systems are proposed by Haraoka [1].

Theorem 2.6 ([1])**.** *Canonical form of the Okubo system of type* $(\mathrm{IV}^*)_6$ *is given by the form*

$$
(x-T)\frac{d}{dx}Y = \begin{pmatrix}
\alpha_1 & 0 & h_{212} & h_{222} & 1 & h_{212}h_{222}\\
0 & \alpha_2 & h_{112} & h_{122} & 1 & h_{112}h_{122}\\
h_{122} & h_{222} & \beta_1 & 0 & 1 & -h_{122}h_{222}\\
h_{112} & h_{212} & 0 & \beta_2 & 1 & -h_{112}h_{212}\\
-h_{112}h_{122} & -h_{212}h_{222} & -h_{112}h_{212} & -h_{122}h_{222} & \gamma_1 & 0\\
-1 & -1 & 1 & 1 & 0 & \gamma_2
\end{pmatrix} DY, \tag{2.33}
$$

where the notations

$$
h_{ijk} = \alpha_i + \beta_j + \gamma_k - 2\rho_1 - \rho_2, \tag{2.34}
$$

and $D = \mathrm{diag}(a_1,a_2,b_1,b_2,c_1,c_2)$ *denotes the diagonal matrix defined by*

$$
a_j = \frac{\alpha_j-\rho_1}{\prod_{k\neq j}^2(\alpha_j-\alpha_k)}, \quad bj = \frac{\beta_j-\rho_1}{\prod_{k\neq j}^2(b_j-b_k)}, \quad c_j = \frac{\gamma_j-\rho_1}{\prod_{k\neq j}^2(\gamma_j-\gamma_k)} \quad (j=1,2). \tag{2.35}
$$

2.2 Main theorems

For each Okubo system, the monodromy matrices for the canonical solution matrix are expressed in the form (1.17) by using the connection coefficients. Our main results are the explicit formulas for the connection coefficients and the monodromy matrices for the canonical solution matrices of Yokoyama's list. In what follows, we use the notation $(t_i - t_j)^\alpha = \exp(\alpha \log(t_i - t_j))$ $(\alpha \in \mathbb{C})$ with the convention of arguments such that $\theta_j - \pi < \arg(t_i - t_j) < \theta_j$ for $i < j$, and $\theta_j < \arg(t_i - t_j) < \theta_j + \pi$ for $i > j$, using $\theta_j = \arg(p_0 - t_j)$ as specified in Section 1.1.

Case I: For the Okubo system of type $(\mathrm{I})_n$, the canonical solution matrix

$$\Psi(x) = (\psi_1(x), \dots, \psi_{n-1}(x), \psi_n(x)) \in \mathrm{Mat}(n; \mathcal{O}(\widetilde{\mathcal{D}})) \tag{2.36}$$

is a multivalued holomorphic solution matrix on $\mathcal{D} = \mathbb{P}^1 \backslash \{t_1, t_2, \infty\}$ characterized by the following conditions:

$$\psi_j(x) = (x - t_1)^{\alpha_j}(\boldsymbol{e}_j + O(x - t_1)) \quad (j = 1, \dots, n) \tag{2.37}$$

around $x = t_1$, and

$$\psi_n(x) = (x - t_2)^{\alpha_n}(\boldsymbol{e}_n + O(x - t_2)) \tag{2.38}$$

around $x = t_2$, where $\boldsymbol{e}_j = (\delta_{ij})_{i=1}^n$ denotes the j-th unit column vector.

According to the representation (1.17) of the previous section, the monodromy matrices M_1, M_2 are expressed in the form

$$M_1 = \begin{pmatrix} e_1 & (e_1 - 1)C_{12} \\ 0 & 1 \end{pmatrix}, \quad e_1 = \mathrm{diag}(e(\alpha_1), \dots, e(\alpha_{n-1})),$$
$$M_2 = \begin{pmatrix} I_{n-1} & 0 \\ (e_2 - 1)C_{21} & e_2 \end{pmatrix}, \quad e_2 = e(\alpha_n). \tag{2.39}$$

These connection matrices $C = (C_{12})_i$ and $C_{21} = (D_{21})_j$ are defined as

$$\psi_n(x) = \sum_{i=1}^{n-1} \psi_i(x)(C_{12})_i + h_n(x), \quad h_n(x) \in \mathcal{O}_{t_1} \tag{2.40}$$

around $x = t_1$ and

$$\psi_j(x) = \psi_n(x)(C_{21})_j + h_j(x), \quad h_j(x) \in \mathcal{O}_{t_2} \quad (j = 1, \dots, n-1) \tag{2.41}$$

around $x = t_2$. As to the monodromy around $x = \infty$, we remark that

$$M_\infty^{-1} = M_1 M_2 \sim \mathrm{diag}(f_1, f_2, \dots, f_n), \tag{2.42}$$

where $f_i = e(\rho_i)$ $(i = 1, 2, \dots, n)$. The connection matrices C_{12} and C_{21} are given by the following theorem:

Theorem 2.7. *The monodromy matrices M_1, M_2 for the canonical solution matrix $\Psi(x)$ of type* $(\mathrm{I})_n$ *is expressed as* (2.39) *in terms of the connection matrices $C_{12} = (C_{12})_i$ and $C_{21} = (C_{21})_j$ determined as follows* :

$$\begin{aligned}(C_{12})_i &= (-1)^n\, e(\tfrac{1}{2}(\rho_2-\alpha_i-\alpha_n))\frac{(t_1-t_2)^{\rho_2-\alpha_i}}{(t_2-t_1)^{\rho_2-\alpha_n}}\Gamma(-\alpha_i)\Gamma(\alpha_n+1)\frac{\prod_{k\neq i}^{n-1}\Gamma(1+\alpha_k-\alpha_i)}{\prod_{k=1}^{n}\Gamma(1+\rho_k-\alpha_i)},\\ (C_{21})_j &= e(\tfrac{1}{2}(-\rho_2+\alpha_j+\alpha_n))\frac{(t_2-t_1)^{\rho_2-\alpha_n}}{(t_1-t_2)^{\rho_2-\alpha_j}}\Gamma(1+\alpha_j)\Gamma(-\alpha_n)\frac{\prod_{k\neq j,\,1\leq k\leq n-1}\Gamma(\alpha_j-\alpha_k)}{\prod_{k=1}^{n}\Gamma(\alpha_j-\rho_k)}.\end{aligned} \tag{2.43}$$

Cases II **and** III: For the Okubo system type $(\mathrm{II})_{2n}$ and $(\mathrm{III})_{2n+1}$, the canonical solution matrix

$$\Psi(x) = (\psi_1(x),\dots,\psi_m(x),\psi_{m+1}(x),\dots,\psi_{m+n}(x)) \in \mathrm{Mat}(m+n;\mathcal{O}(\widetilde{\mathcal{D}})) \tag{2.44}$$

is a multivalued holomorphic solution matrix on $\mathcal{D} = \mathbb{P}^1\backslash\{t_1,t_2,\infty\}$ characterized by the following conditions:

$$\psi_j(x) = (x-t_1)^{\alpha_j}(\boldsymbol{e}_j + O(x-t_1)) \quad (j=1,\dots,m) \tag{2.45}$$

around $x = t_1$, and

$$\psi_{m+j}(x) = (x-t_2)^{\beta_j}(\boldsymbol{e}_{n+j} + O(x-t_2)) \quad (j=1,\dots,n) \tag{2.46}$$

around $x = t_2$.

According to the representation (1.17) of the previous section, the monodromy matrices M_1, M_2 are expressed in the form

$$\begin{aligned} M_1 &= \begin{pmatrix} e_1 & (e_1-1)C_{12} \\ 0 & I_n \end{pmatrix}, \quad e_1 = \mathrm{diag}(e(\alpha_1),\dots,e(\alpha_m)),\\ M_2 &= \begin{pmatrix} I_m & 0 \\ (e_2-1)C_{21} & e_2 \end{pmatrix}, \quad e_2 = \mathrm{diag}(e(\beta_1),\dots,e(\beta_n)),\end{aligned} \tag{2.47}$$

These connection matrices $(C_{12}) = (C_{12})_{ij}$ and $C_{21} = (C_{21})_{ij}$ are defined as

$$\psi_{m+j}(x) = \sum_{i=1}^{n}\psi_i(x)(C_{12})_{ij} + h_{m+j}(x), \quad h_{m+j}(x) \in \mathcal{O}_{t_1} \quad (j=1,\dots,m) \tag{2.48}$$

around $x = t_1$ and

$$\psi_j(x) = \sum_{i=1}^{n}\psi_{n+i}(x)(C_{21})_{ij} + h_j(x), \quad h_j(x) \in \mathcal{O}_{t_2} \quad (j=1,\dots,m) \tag{2.49}$$

around $x = t_2$. As to the monodromy around $x = \infty$, we remark that

$$M_\infty^{-1} = M_1M_2 \sim \mathrm{diag}(\overbrace{f_1,\dots,f_1}^{n},\overbrace{f_2,\dots,f_2}^{m-1},f_3), \tag{2.50}$$

where $f_i = e(\rho_i) \quad (i=1,2,3)$.

Theorem 2.8. *The monodromy matrices M_1, M_2 for the canonical solution matrices $\Psi(x)$ of types* $(\mathrm{II})_{2n}$ *and* $(\mathrm{III})_{2n+1}$ *are expressed as* (2.47) *in terms of the connection matrices $C = (C_{12})_{ij}$ and $C_{21} = (C_{21})_{ij}$ determined as follows :*

$(\mathrm{II})_{2n}$:

$$\begin{aligned}(C_{12})_{ij} &= (-1)^{n-1}\frac{(t_1-t_2)^{\rho_3-\alpha_i}}{(t_2-t_1)^{\rho_3-\beta_j}}e(\tfrac{1}{2}(\rho_3-\alpha_i-\beta_j))\frac{\Gamma(\beta_j+1)\Gamma(-\alpha_i)}{\Gamma(1+\rho_1-\alpha_i)\Gamma(\beta_j-\rho_1)}\\ &\frac{\prod_{k\neq i}^{n}\Gamma(1+\alpha_k-\alpha_i)}{\prod_{k\neq j}^{n}\Gamma(1+\rho_1+\rho_2-\alpha_1-\beta_k)}\frac{\prod_{k\neq j}^{n}\Gamma(\beta_j-\beta_k)}{\prod_{k\neq i}^{n}\Gamma(\beta_j+\alpha_k-\rho_1-\rho_2)}\quad(1\le i,j\le n),\\ (C_{21})_{ij} &= (-1)^{n-1}\frac{(t_2-t_1)^{\rho_3-\beta_i}}{(t_1-t_2)^{\rho_3-\alpha_j}}e(\tfrac{1}{2}(\alpha_j+\beta_i-\rho_3))\frac{\Gamma(-\beta_i)\Gamma(\alpha_j+1)}{\Gamma(\alpha_j-\rho_1)\Gamma(1+\rho_1-\beta_i)}\\ &\frac{\prod_{k\neq j}^{n}\Gamma(\alpha_j-\alpha_k)}{\prod_{k\neq i}^{n}\Gamma(\alpha_j+\beta_k-\rho_1-\rho_2)}\frac{\prod_{k\neq i}^{n}\Gamma(1+\beta_k-\beta_i)}{\prod_{k\neq j}^{n}\Gamma(1+\rho_1+\rho_2-\alpha_k-\beta_i)}\quad(1\le i,j\le n),\end{aligned}\tag{2.51}$$

$(\mathrm{III})_{2n+1}$:

$$\begin{aligned}(C_{12})_{ij} &= (-1)^{n}e(\tfrac{1}{2}(\rho_3-\alpha_i-\beta_j))\frac{(t_1-t_2)^{\rho_3-\alpha_i}}{(t_2-t_1)^{\rho_3-\beta_j}}\frac{\Gamma(\beta_j+1)\Gamma(-\alpha_i)}{\Gamma(\alpha_i-\rho_1)\Gamma(\alpha_i-\rho_2)}\\ &\frac{\prod_{k\neq i}^{n+1}\Gamma(1+\alpha_k-\alpha_i)}{\prod_{k\neq j}^{n}\Gamma(1+\rho_1+\rho_2-\alpha_1-\beta_k)}\frac{\prod_{k\neq j}^{n}\Gamma(\beta_j-\beta_k)}{\prod_{k\neq i}^{n+1}\Gamma(\beta_j+\alpha_k-\rho_1-\rho_2)}\quad(1\le i\le n+1;1\le j\le n),\\ (C_{21})_{ij} &= (-1)^{n-1}e(\tfrac{1}{2}(\alpha_j+\beta_i-\rho_3))\frac{(t_2-t_1)^{\rho_3-\beta_i}}{(t_1-t_2)^{\rho_3-\alpha_j}}\frac{\Gamma(-\beta_i)\Gamma(\alpha_j+1)}{\Gamma(1+\rho_1-\alpha_j)\Gamma(1+\rho_2-\alpha_j)}\\ &\frac{\prod_{k\neq j}^{n+1}\Gamma(\alpha_j-\alpha_k)}{\prod_{k\neq i}^{n}\Gamma(\alpha_j+\beta_k-\rho_1-\rho_2)}\frac{\prod_{k\neq i}^{n}\Gamma(1+\beta_k-\beta_i)}{\prod_{k\neq j}^{n+1}\Gamma(1+\rho_1+\rho_2-\alpha_k-\beta_i)}\quad(1\le i\le n;1\le j\le n+1).\end{aligned}\tag{2.52}$$

Case IV: For the Okubo system type $(\mathrm{IV})_6$, the canonical solution matrix

$$\Psi(x) = (\psi_1(x),\ldots,\psi_4(x),\psi_5(x),\psi_6(x)) \in \mathrm{Mat}(6;\mathcal{O}(\widetilde{\mathcal{D}}))\tag{2.53}$$

is a multivalued holomorphic solution matrix on $\mathcal{D} = \mathbb{P}^1\backslash\{t_1,t_2,\infty\}$ characterized by the following conditions:

$$\psi_j(x) = (x-t_1)^{\alpha_j}(\boldsymbol{e}_j + O(x-t_1))\quad(j=1,\ldots,4)\tag{2.54}$$

around $x = t_1$, and

$$\psi_{m+j}(x) = (x-t_2)^{\beta_j}(\boldsymbol{e}_{4+j} + O(x-t_2))\quad(j=1,2)\tag{2.55}$$

around $x = t_2$.

According to the representation (1.17) of the previous section, the monodromy matrices M_1, M_2 are expressed in the form

$$\begin{aligned}M_1 &= \begin{pmatrix} e_1 & (e_1-1)C_{12}\\ 0 & I_n\end{pmatrix},\quad e_1 = \mathrm{diag}(e(\alpha_1),\ldots,e(\alpha_4)),\\ M_2 &= \begin{pmatrix} I_m & 0\\ (e_2-1)C_{21} & e_2\end{pmatrix},\quad e_2 = \mathrm{diag}(e(\beta_1),e(\beta_2)),\end{aligned}\tag{2.56}$$

These connection matrices $C = (C_{12})_{ij}$ and $C_{21} = (C_{21})_{ij}$ are defined as

$$\psi_{4+j}(x) = \sum_{i=1}^{4} \psi_i(x)(C_{12})_{ij} + h_{4+j}(x), \quad h_{4+j}(x) \in \mathcal{O}_{t_1} \quad (j = 1, \ldots, 4) \tag{2.57}$$

around $x = t_1$ and

$$\psi_j(x) = \sum_{i=1}^{2} \psi_{n+i}(x)(C_{21})_{ij} + h_j(x), \quad h_j(x) \in \mathcal{O}_{t_2} \quad (j = 1, 2) \tag{2.58}$$

around $x = t_2$. As to the monodromy around $x = \infty$, we remark that

$$M_\infty^{-1} = M_1 M_2 \sim \mathrm{diag}(f_1, f_1, f_2, f_2, f_3, f_3), \tag{2.59}$$

where $f_i = e(\rho_i) \quad (i = 1, 2, 3)$.

Theorem 2.9. *The monodromy matrices M_1, M_2 for the canonical solution matrix $\Psi(x)$ of types* $(\mathrm{IV})_6$ *are expressed as* (2.56) *in terms of the connection matrices $C_{12} = (C_{12})_{ij}$ and $C_{21} = (C_{21})_{ij}$ determined as follows:*

$$(C_{12})_{ij} = e(\tfrac{1}{2}(-\alpha_i - \beta_j)) \frac{(t_2 - t_1)^{\rho_3 - \alpha_i}}{(t_1 - t_2)^{\rho_3 - \beta_j}} \frac{\Gamma(-\alpha_i)\Gamma(\beta_j + 1)}{\prod_{k=1}^{3} \Gamma(1 + \rho_k - \alpha_i)}$$
$$\frac{\prod_{k\neq j}^{2} \Gamma(\beta_j - \beta_k) \prod_{k\neq i,4}^{4} \Gamma(1 + \alpha_k - \alpha_i)}{\prod_{k\neq i}^{3} \Gamma(\beta_j + \alpha_k + \alpha_4 - \rho_1 - \rho_2 - \rho_3) \prod_{k\neq j}^{2} \Gamma(1 + \rho_1 + \rho_2 + \rho_3 - \alpha_i - \alpha_4 - \beta_k)} \quad (i = 1, 2, 3; j = 1, 2). \tag{2.60}$$

$$(C_{12})_{4j} = \frac{e(\frac{1}{2}(-\alpha_4 - \beta_j))}{\prod_{k\neq j}^{2} \prod_{k\neq 1}^{3} (\alpha_1 + \alpha_k + \beta_j - \rho_1 - \rho_2 - \rho_3)} \frac{(t_2 - t_1)^{\rho_3 - \alpha_4}}{(t_1 - t_2)^{\rho_3 - \beta_j}} \frac{\Gamma(-\alpha_4)\Gamma(\beta_j + 1)}{\prod_{k=1}^{3} \Gamma(1 + \rho_k - \alpha_4)}$$
$$\frac{\prod_{k\neq j}^{2} \Gamma(\beta_j - \beta_k) \prod_{k\neq 4}^{4} \Gamma(1 + \alpha_k - \alpha_i)}{\prod_{k\neq 1,4}^{4} \Gamma(\beta_j + \alpha_k + \alpha_1 - \rho_1 - \rho_2 - \rho_3) \prod_{k\neq j}^{2} \Gamma(1 + \rho_1 + \rho_2 + \rho_3 - \alpha_1 - \alpha_4 - \beta_k)} \quad (j = 1, 2). \tag{2.61}$$

$$(C_{21})_{ij} = -e(\tfrac{1}{2}(\alpha_j + \alpha_4 + \beta_i - 2\rho_3)) \frac{(t_2 - t_1)^{\alpha_4 - \beta_i}}{(t_1 - t_2)^{\alpha_4 - \alpha_j}} \frac{\Gamma(-\beta_i)\Gamma(\alpha_j + 1)}{\Gamma(\alpha_j - \rho_1)\Gamma(\alpha_j - \rho_2)\Gamma(\alpha_j - \rho_3)}$$
$$\frac{\prod_{k\neq j}^{4} \Gamma(\alpha_j - \alpha_k) \prod_{k\neq i}^{2} \Gamma(1 + \beta_k - \beta_i)}{\prod_{k\neq i}^{2} \Gamma(\alpha_j + \alpha_4 + \beta_k - \rho_1 - \rho_2 - \rho_3) \prod_{k\neq j,4}^{4} \Gamma(1 + \rho_1 + \rho_2 + \rho_3 - \alpha_k - \beta_i - \alpha_4)} \quad (i = 1, 2; j = 1, 2, 3). \tag{2.62}$$

$$(C_{21})_{i4} = \prod_{k\neq 1}^{3} (\alpha_1 + \alpha_k + \beta_j - \rho_1 - \rho_2 - \rho_3) e(\tfrac{1}{2}(\alpha_1 + \alpha_4 + \beta_i - 2\rho_3)) \frac{(t_2 - t_1)^{\alpha_4 - \beta_i}}{(t_1 - t_2)^{\alpha_4 - \alpha_j}} \frac{\Gamma(-\beta_i)\Gamma(\alpha_4 + 1)}{\Gamma(\alpha_4 - \rho_1)\Gamma(\alpha_4 - \rho_2)\Gamma(\alpha_4 - \rho_3)}$$
$$\frac{\prod_{k\neq j}^{4} \Gamma(\alpha_j - \alpha_k) \prod_{k\neq i}^{2} \Gamma(1 + \beta_k - \beta_i)}{\prod_{k\neq i}^{2} \Gamma(\alpha_1 + \alpha_4 + \beta_k - \rho_1 - \rho_2 - \rho_3) \prod_{k\neq 1,4}^{4} \Gamma(1 + \rho_1 + \rho_2 + \rho_3 - \alpha_k - \beta_i - \alpha_1)} \quad (i = 1, 2). \tag{2.63}$$

Case I*: For the Okubo system of type $(\mathrm{I}^*)_n$, the canonical solution matrix

$$\Psi(x) = (\psi_1(x), \psi_2(x), \ldots, \psi_n(x)) \in \mathrm{Mat}(n; \mathcal{O}(\widetilde{\mathcal{D}})) \tag{2.64}$$

is a multivalued holomorphic solution matrix on $\mathcal{D} = \mathbb{P}^1 \backslash \{t_1, t_2, \ldots, t_n \infty\}$ characterized by the following conditions:

$$\psi_j(x) = (x - t_j)^{\alpha_j} (\boldsymbol{e}_j + O(x - t_j)) \quad (j = 1, \ldots, n) \tag{2.65}$$

around $x = t_j$.

The monodromy matrices $M_1, M_2, \ldots, M_n$ are expressed in the form

$$M_i = \begin{pmatrix} 1 & & O & & \\ (e_i - 1)C_{i1} & \cdots & e_i & \cdots & (e_i - 1)C_{in} \\ & & O & & 1 \end{pmatrix} \quad (i = 1, \ldots, n). \tag{2.66}$$

These connection matrices C_{ij} are defined as

$$\psi_j(x) = \psi_i(x) C_{ij} + h_{ij}(x), \quad h_{ij}(x) \in \mathcal{O}_{t_i} \quad (j = 1, \ldots, n). \tag{2.67}$$

Theorem 2.10. *The monodromy matrices $M_1, M_2, \ldots, M_n$ for the canonical solution matrix $\Psi(x)$ of type $(\mathrm{I}^*)_n$ is expressed as (2.66) in terms of the connection matrices C_{ij} determined as follows :*

$$C_{ij} = \begin{cases} e(\frac{-\rho_1}{2}) \dfrac{\prod_{k \neq i}^{n} (t_i - t_k)^{\alpha_k - \rho_1}}{\prod_{k \neq j}^{n} (t_j - t_k)^{\alpha_k - \rho_1}} \dfrac{\Gamma(-\alpha_i)\Gamma(\alpha_j + 1)}{\Gamma(\alpha_j - \rho_1)\Gamma(1 + \rho_1 - \alpha_i)} & (i < j), \\ e(\frac{\rho_1}{2}) \dfrac{\prod_{k \neq i}^{n} (t_i - t_k)^{\alpha_k - \rho_1}}{\prod_{k \neq j}^{n} (t_j - t_k)^{\alpha_k - \rho_1}} \dfrac{\Gamma(-\alpha_i)\Gamma(\alpha_j + 1)}{\Gamma(\alpha_j - \rho_1)\Gamma(1 + \rho_1 - \alpha_i)} & (i > j). \end{cases} \tag{2.68}$$

Cases II* **and** III*: For the Okubo system type $(\mathrm{II}^*)_{2n}$ and $(\mathrm{III}^*)_{2n+1}$, the canonical solution matrix

$$\Psi(x) = (\psi_1(x), \ldots, \psi_n(x), \psi_{m-1}(x), \ldots, \psi_{m+n-1}(x), \psi_{m+n}(x)) \in \mathrm{Mat}(m + n; \mathcal{O}(\widetilde{\mathcal{D}})) \tag{2.69}$$

is a multivalued holomorphic solution matrix on $\mathcal{D} = \mathbb{P}^1 \backslash \{t_1, t_2, t_3, \infty\}$ characterized by the following conditions:

$$\begin{aligned} \psi_j(x) &= (x - t_1)^{\alpha_j} (\boldsymbol{e}_j + O(x - t_1)) \quad (j = 1, \ldots, n), \\ \psi_{m+j}(x) &= (x - t_2)^{\beta_j} (\boldsymbol{e}_{n+j} + O(x - t_2)) \quad (j = 1, \ldots, m - 1), \\ \psi_{m+n}(x) &= (x - t_3)^{\gamma} (\boldsymbol{e}_{m+n} + O(x - t_3)) \end{aligned} \tag{2.70}$$

around $x = t_1$, $x = t_2$ and $x = t_3$ respectively.

According to the representation (1.17) of the previous section, the monodromy matrices M_1, M_2, M_3 are expressed in the form

$$\begin{aligned} M_1 &= \begin{pmatrix} e_1 & (e_1 - 1)C_{12} & (e_1 - 1)C_{13} \\ 0 & I_{m-1} & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad e_1 = \mathrm{diag}(e(\alpha_1), \ldots, e(\alpha_n)), \\ M_2 &= \begin{pmatrix} I_n & 0 & 0 \\ (e_2 - 1)C_{21} & e_2 & (e_2 - 1)C_{23} \\ 0 & 0 & 1 \end{pmatrix}, \quad e_2 = \mathrm{diag}(e(\beta_1), \ldots, e(\beta_{m-1})), \\ M_3 &= \begin{pmatrix} I_n & 0 & 0 \\ 0 & I_{m-1} & 0 \\ (e_3 - 1)C_{31} & e_3 C_{32} & e_3 \end{pmatrix}, \quad e_3 = e(\gamma). \end{aligned} \tag{2.71}$$

These connection matrices $C = (C_{ij})_{ij}$ are defined as

$$\psi_{n+j}(x) = \sum_{i=1}^{n} \psi_i(x)(C_{12})_{ij} + h_{n+j}^{(12)}(x) \quad (j = 1, \ldots, m-1), \quad \psi_{m+n}(x) = \sum_{i=1}^{n} \psi_i(x)(C_{13})_i + h_{m+n}^{(13)}(x),$$

around $x = t_1$,

$$\psi_j(x) = \sum_{i=1}^{m-1} \psi_{n+i}(x)(C_{21})_{ij} + h_j^{(21)}(x) \quad (j = 1, \ldots, n), \quad \psi_{m+n}(x) = \sum_{i=1}^{m-1} \psi_{n+i}(x)(C_{23})_i + h_{m+n}^{(23)}(x),$$

around $x = t_2$ and

$$\psi_j(x) = \psi_{m+n}(x)(C_{31})_j + h_j^{(31)}(x) \quad (j = 1, \ldots, n), \quad \psi_{m+n}(x) = \sum_{i=1}^{m-1} \psi_{n+i}(x)(C_{32})_i + h_{m+n}^{(32)}(x),$$

around $x = t_3$ where $h_l^{(ij)}(x) \in \mathcal{O}_{t_i}$ $(i = 1, 2, 3)$. As to the monodromy around $x = \infty$, we remark that

$$M_\infty^{-1} = M_1 M_2 M_3 \sim \mathrm{diag}(\overbrace{f_1, \ldots, f_1}^{m}, \overbrace{f_2, \ldots, f_2}^{n}), \tag{2.72}$$

where $f_i = e(\rho_i) \quad (i = 1, 2)$.

Theorem 2.11. *The monodromy matrices M_1, M_2 and M_3 for the canonical solution matrices $\Psi(x)$ of types $(\mathrm{II}^*)_{2n}$ and $(\mathrm{III}^*)_{2n+1}$ are expressed as (2.71) in terms of the connection matrices determined as follows:*

$(\mathrm{II}^*)_{2n}$

$$\begin{aligned}(C_{12}^{(2n)})_{ij} &= (-1)^{n-1} e(\tfrac{1}{2}(\rho_1 + \rho_2 - \alpha_i - \beta_j - \gamma) \frac{(t_1 - t_2)^{\rho_1+\rho_2-\alpha_i-\gamma}}{(t_2 - t_1)^{\rho_1+\rho_2-\beta_j-\gamma}} \left(\frac{t_1 - t_3}{t_2 - t_3}\right)^{\rho_1+\rho_2-\alpha_i-\beta_j} \\ &\quad \frac{\Gamma(-\alpha_i)\Gamma(\beta_j + 1)}{\Gamma(1 + \rho_1 - \alpha_i)\Gamma(1 + \rho_2 - \alpha_i)} \frac{\prod_{k \neq j}^{n-1} \Gamma(\beta_j - \beta_k)}{\prod_{k \neq i}^{n} \Gamma(\beta_j + \alpha_k - \rho_1 - \rho_2)} \frac{\prod_{k \neq i}^{n} \Gamma(1 + \alpha_k - \alpha_i)}{\prod_{k \neq j}^{n-1} \Gamma(1 + \rho_1 + \rho_2 - \alpha_i - \beta_k)}\end{aligned} \tag{2.73}$$

$$\begin{aligned}(C_{13}^{(2n)})_i &= (-1)^n e(\tfrac{1}{2}(\rho_1 + \rho_2 - \alpha_i - \beta_1 - \gamma) \frac{(t_1 - t_3)^{\rho_1+\rho_2-\alpha_i-\beta_1}}{(t_3 - t_1)^{\rho_1+\rho_2-\beta_1-\gamma}} \left(\frac{t_1 - t_2}{t_3 - t_2}\right)^{\rho_1+\rho_2-\alpha_i-\gamma} \\ &\quad (\rho_1 + \rho_2 - \alpha_i - \beta_1)^{-1} \frac{\Gamma(\gamma + 1)\Gamma(-\alpha_i)}{\Gamma(1 + \rho_1 - \alpha_i)\Gamma(1 + \rho_2 - \alpha_i)} \frac{\prod_{k \neq i}^{n} \Gamma(1 + \alpha_k - \alpha_i)}{\prod_{k=1}^{n-1} \Gamma(1 + \rho_1 + \rho_2 - \alpha_i - \beta_k)}\end{aligned} \tag{2.74}$$

$$\begin{aligned}(C_{23}^{(2n)})_i &= (-1)^n e(\tfrac{1}{2}(\rho_1 + \rho_2 - \alpha_1 - \beta_i - \gamma)) \frac{(t_2 - t_3)^{\rho_1+\rho_2-\alpha_1-\beta_i}}{(t_3 - t_2)^{\rho_1+\rho_2-\alpha_1-\gamma}} \left(\frac{t_2 - t_1}{t_3 - t_1}\right)^{\rho_1+\rho_2-\beta_i-\gamma} \\ &\quad (\rho_1 + \rho_2 - \alpha_1 - \beta_i)^{-1} \Gamma(\gamma + 1)\Gamma(-\beta_i) \frac{\prod_{k \neq i}^{n-1} \Gamma(1 + \beta_k - \beta_i)}{\prod_{k=1}^{n} \Gamma(1 + \rho_1 + \rho_2 - \alpha_k - \beta_i)}\end{aligned} \tag{2.75}$$

$$(C_{21}^{(2n)})_{ij} = (-1)^n e(\tfrac{-1}{2}(\alpha_j + \beta_i + \gamma - \rho_1 - \rho_2) \frac{(t_2 - t_1)^{\rho_1+\rho_2-\beta_i-\gamma}}{(t_1 - t_2)^{\rho_1+\rho_2-\alpha_j-\gamma}} \left(\frac{t_2 - t_3}{t_1 - t_3}\right)^{\rho_1+\rho_2-\alpha_i-\beta_j}$$
$$\frac{\Gamma(\alpha_j + 1)\Gamma(-\beta_i)}{\Gamma(1 + \rho_1 - \beta_j)\Gamma(1 + \rho_2 - \beta_j)} \frac{\prod_{k\neq i}^{n-1} \Gamma(\beta_k - \beta_i + 1)}{\prod_{k\neq j}^{n} \Gamma(1 + \rho_1 + \rho_2 - \beta_i - \alpha_k)} \frac{\prod_{k\neq j}^{n} \Gamma(\alpha_j - \alpha_k))}{\prod_{k\neq i}^{n-1} \Gamma(\alpha_j + \beta_k - \rho_1 - \rho_2)} \tag{2.76}$$

$$(C_{31}^{(2n)})_j = (-1)^n e(\tfrac{1}{2}(\rho_1 + \rho_2 - \alpha_j - \beta_1 - \gamma) \frac{(t_3 - t_1)^{\rho_1+\rho_2-\alpha_j-\beta_1}}{(t_1 - t_3)^{\rho_1+\rho_2-\beta_1-\gamma}} \left(\frac{t_3 - t_2}{t_1 - t_2}\right)^{\rho_1+\rho_2-\alpha_j-\gamma}$$
$$(\rho_1 + \rho_2 - \alpha_i - \beta_1) \frac{\Gamma(-\gamma)\Gamma(\alpha_j + 1)}{\Gamma(\alpha_j - \rho_1)\Gamma(\alpha_j - \rho_2)} \frac{\prod_{k\neq j}^{n} \Gamma(\alpha_j - \alpha_k)}{\prod_{k=1}^{n-1} \Gamma(\alpha_j + \beta_k - \rho_1 - \rho_2)} \tag{2.77}$$

$$(C_{32}^{(2n)})_j = (-1)^n e(\tfrac{1}{2}(\rho_1 + \rho_2 - \alpha_1 - \beta_j - \gamma)) \frac{(t_3 - t_2)^{\rho_1+\rho_2-\alpha_1-\gamma}}{(t_2 - t_3)^{\rho_1+\rho_2-\alpha_1-\beta_j}} \left(\frac{t_3 - t_1}{t_2 - t_1}\right)^{\rho_1+\rho_2-\beta_j-\gamma}$$
$$(\rho_1 + \rho_2 - \alpha_1 - \beta_j)\Gamma(-\gamma)\Gamma(\beta_j + 1) \frac{\prod_{k\neq j}^{n-1} \Gamma(\beta_j - \beta_k)}{\prod_{k=1}^{n} \Gamma(\alpha_k + \beta_j - \rho_1 - \rho_2)} \tag{2.78}$$

$(\mathrm{III}^*)_{2n+1}$

$$(C_{12}^{(2n+1)})_{ij} = (-1)^n e(\tfrac{1}{2}(\rho_1 + \rho_2 - \alpha_i - \beta_j - \gamma) \frac{(t_1 - t_2)^{\rho_1+\rho_2-\alpha_i-\gamma}}{(t_2 - t_1)^{\rho_1+\rho_2-\beta_j-\gamma}} \left(\frac{t_1 - t_3}{t_2 - t_3}\right)^{\rho_1+\rho_2-\alpha_i-\beta_j}$$
$$\frac{\Gamma(-\alpha_i)\Gamma(\beta_j + 1)}{\Gamma(1 + \rho_1 - \alpha_i)\Gamma(\beta_j - \rho_1)} \frac{\prod_{k\neq j}^{n} \Gamma(\beta_j - \beta_k)}{\prod_{k\neq i}^{n} \Gamma(\beta_j + \alpha_k - \rho_1 - \rho_2)} \frac{\prod_{k\neq i}^{n} \Gamma(1 + \alpha_k - \alpha_i)}{\prod_{k\neq j}^{n} \Gamma(1 + \rho_1 + \rho_2 - \alpha_i - \beta_k)} \tag{2.79}$$

$$(C_{13}^{(2n+1)})_i = (-1)^n e(\tfrac{1}{2}(\rho_1 + \rho_2 - \alpha_i - \beta_1 - \gamma) \frac{(t_1 - t_3)^{\rho_1+\rho_2-\alpha_i-\beta_1}}{(t_3 - t_1)^{\rho_1+\rho_2-\beta_1-\gamma}} \left(\frac{t_1 - t_2}{t_3 - t_2}\right)^{\rho_1+\rho_2-\alpha_i-\gamma}$$
$$\frac{\Gamma(\gamma + 1)\Gamma(-\alpha_i)}{\Gamma(1 + \rho_1 - \alpha_i)} \frac{\prod_{k\neq i}^{n} \Gamma(1 + \alpha_k - \alpha_i)}{\prod_{k=1}^{n} \Gamma(1 + \rho_1 + \rho_2 - \alpha_i - \beta_k)} \tag{2.80}$$

$$(C_{23}^{(2n+1)})_i = (-1)^n e(\tfrac{1}{2}(\rho_1 + \rho_2 - \alpha_1 - \beta_i - \gamma)) \frac{(t_2 - t_3)^{\rho_1+\rho_2-\alpha_1-\beta_i}}{(t_3 - t_2)^{\rho_1+\rho_2-\alpha_1-\gamma}} \left(\frac{t_2 - t_1}{t_3 - t_1}\right)^{\rho_1+\rho_2-\beta_i-\gamma}$$
$$\frac{\Gamma(\gamma + 1)\Gamma(-\beta_i)}{\Gamma(1 + \rho_1 - \beta_i)} \frac{\prod_{k\neq i}^{n} \Gamma(1 + \beta_k - \beta_i)}{\prod_{k=1}^{n} \Gamma(1 + \rho_1 + \rho_2 - \alpha_k - \beta_i)} \tag{2.81}$$

$$(C_{21}^{(2n+1)})_{ij} = (-1)^n e(\tfrac{-1}{2}(\rho_1 + \rho_2 - \alpha_j - \beta_i - \gamma) \frac{(t_2 - t_1)^{\beta_i+\gamma-\rho_1-\rho_2}}{(t_1 - t_3)^{\rho_1+\rho_2-\alpha_i-\beta_j}} \left(\frac{t_2 - t_3}{t_1 - t2}\right)^{\rho_1+\rho_2-\alpha_j-\gamma}$$
$$\frac{\Gamma(\alpha_j + 1)\Gamma(-\beta_i)}{\Gamma(\alpha_j - \rho_1)\Gamma(1 + \rho_1 - \beta_i)} \frac{\prod_{k\neq i}^{n} \Gamma(1 + \beta_k - \beta_j)}{\prod_{k\neq j}^{n} \Gamma(1 + \rho_1 + \rho_2 - \beta_i - \alpha_k)} \frac{\prod_{k\neq j}^{n} \Gamma(\alpha_j - \alpha_k)}{\prod_{k\neq i}^{n} \Gamma(\alpha_j + \beta_k - \rho_1 - \rho_2)} \tag{2.82}$$

$$(C_{31}^{(2n+1)})_j = (-1)^n e(\tfrac{1}{2}(\rho_1+\rho_2-\alpha_j-\beta_1-\gamma)\frac{(t_3-t_1)^{\rho_1+\rho_2-\alpha_j-\beta_1}}{(t_1-t_3)^{\rho_1+\rho_2-\beta_1-\gamma}}\left(\frac{t_3-t_2}{t_1-t_2}\right)^{\rho_1+\rho_2-\alpha_j-\gamma}$$
$$\frac{\Gamma(-\gamma)\Gamma(\alpha_j+1)}{\Gamma(\alpha_j-\rho_1)}\frac{\prod_{k\neq j}^{n}\Gamma(\alpha_j-\alpha_k)}{\prod_{k=1}^{n}\Gamma(\alpha_j+\beta_k-\rho_1-\rho_2)} \tag{2.83}$$

$$(C_{32}^{(2n+1)})_j = (-1)^n e(\tfrac{1}{2}(\rho_1+\rho_2-\alpha_1-\beta_i-\gamma))\frac{(t_3-t_2)^{\rho_1+\rho_2-\alpha_1-\gamma}}{(t_2-t_3)^{\rho_1+\rho_2-\alpha_1-\beta_i}}\left(\frac{t_3-t_1}{t_2-t_1}\right)^{\rho_1+\rho_2-\beta_i-\gamma}$$
$$\frac{\Gamma(-\gamma)\Gamma(\beta_j+1)}{\Gamma(\beta_j-\rho_1)}\frac{\prod_{k\neq j}^{n}\Gamma(\beta_j-\beta_k)}{\prod_{k=1}^{n}\Gamma(\alpha_k+\beta_j-\rho_1-\rho_2)} \tag{2.84}$$

Case IV*: For the Okubo system type $(\mathrm{IV}^*)_6$, the canonical solution matrix

$$\Psi(x) = (\psi_1(x),\ldots,\psi_6(x)) \in \mathrm{Mat}(6;\mathcal{O}(\widetilde{\mathcal{D}})) \tag{2.85}$$

is a multivalued holomorphic solution matrix on $\mathcal{D}=\mathbb{P}^1\backslash\{t_1,t_2,t_3,\infty\}$ characterized by the following conditions:

$$\begin{aligned}\psi_j(x) &= (x-t_1)^{\alpha_j}(\boldsymbol{e}_j+O(x-t_1)) \quad (j=1,2),\\ \psi_{2+j}(x) &= (x-t_2)^{\beta_j}(\boldsymbol{e}_{2+j}+O(x-t_2)) \quad (j=1,2),\\ \psi_{4+j}(x) &= (x-t_3)^{\gamma}(\boldsymbol{e}_{4+j}+O(x-t_3)) \quad (j=1,2)\end{aligned} \tag{2.86}$$

around $x=t_1$, $x=t_2$ and $x=t_3$ respectively.

According to the representation (1.17) of the previous section, the monodromy matrices M_1, M_2, M_3 are expressed in the form

$$\begin{aligned}M_1 &= \begin{pmatrix} e_1 & (e_1-1)C_{12} & (e_1-1)C_{13}\\ 0 & I_2 & 0\\ 0 & 0 & I_2\end{pmatrix}, \quad e_1=\mathrm{diag}(e(\alpha_1),e(\alpha_2)),\\ M_2 &= \begin{pmatrix} I_n & 0 & 0\\ (e_2-1)C_{21} & e_2 & (e_2-1)C_{23}\\ 0 & 0 & I_2\end{pmatrix}, \quad e_2=\mathrm{diag}(e(\beta_1),e(\beta_2)),\\ M_3 &= \begin{pmatrix} I_2 & 0 & 0\\ 0 & I_2 & 0\\ (e_3-1)C_{31} & e_3C_{32} & e_3\end{pmatrix}, \quad e_3=\mathrm{diag}(e(\gamma_1),e(\gamma_2)).\end{aligned} \tag{2.87}$$

These connection matrices $C=(C_{ij})_{ij}$ are defined as

$$\psi_{2+j}(x) = \sum_{i=1}^{2}\psi_i(x)(C_{12})_{ij}+h_{2+j}^{(12)}(x) \quad (j=1,2), \quad \psi_{4+j}(x) = \sum_{i=1}^{2}\psi_i(x)(C_{13})_{ij}+h_{4+j}^{(13)}(x) \quad (j=1,2),$$

around $x=t_1$,

$$\psi_j(x) = \sum_{i=1}^{2}\psi_{4+2}(x)(C_{21})_{ij}+h_j^{(21)}(x) \quad (j=1,2), \quad \psi_{4+j}(x) = \sum_{i=1}^{2}\psi_{2+i}(x)(C_{23})_{ij}+h_{4+j}^{(23)}(x) \quad (j=1,2),$$

around $x = t_2$ and

$$\psi_j(x) = \sum_{i=1}^{2} \psi_{4+i}(x)(C_{31})_{ij} + h_j(x)^{(31)} \quad (j = 1, 2), \psi_{2+j}(x) = \sum_{i=1}^{2} \psi_{4+i}(x)(C_{32})_{ij} + h_{2+j}^{(32)}(x) \quad (j = 1, 2),$$

around $x = t_3$ where $h_l^{(ij)}(x) \in \mathcal{O}_{t_i}$ $(i = 1, 2, 3)$. As to the monodromy around $x = \infty$, we remark that

$$M_\infty^{-1} = M_1 M_2 M_3 \sim \mathrm{diag}(\overbrace{f_1, \dots, f_1}^{4}, \overbrace{f_2, \dots, f_2}^{2}), \tag{2.88}$$

where $f_i = e(\rho_i) \quad (i = 1, 2)$.

Theorem 2.12. *The monodromy matrices M_1, M_2 and M_3 for the canonical solution matrix $\Psi(x)$ of types* $(\mathrm{IV}^*)_6$ *are expressed as* (2.87) *in terms of the connection matrices determined as follows:*

$$(C_{12})_{ij} = e(\tfrac{1}{2}(2\rho_1 + \rho_2 - \alpha_i - \beta_j - \gamma_1 - \gamma_2)) \frac{(t_1 - t_2)^{\rho_1+\rho_2-\alpha_i-\gamma_1}}{(t_2 - t_1)^{\rho_1+\rho_2-\beta_j-\gamma_1}} \left(\frac{t_1 - t_3}{t_2 - t_3}\right)^{\rho_1+\rho_2-\alpha_i-\beta_j}$$
$$\frac{\Gamma(-\alpha_i)\Gamma(\beta_j + 1)}{\Gamma(1 + \rho_1 - \alpha_i)\Gamma(\beta_j - \rho_1)} \frac{\prod_{k\neq j}^2 \Gamma(\beta_j - \beta_k) \prod_{k\neq i}^2 \Gamma(1 + \alpha_k - \alpha_i)}{\prod_{k\neq i}^2 \Gamma(\alpha_k + \beta_j + \gamma_2 - 2\rho_1 - \rho_2) \prod_{k\neq j}^2 \Gamma(1 + 2\rho_1 + \rho_2 - \alpha_i - \beta_k - \gamma_2)} \tag{2.89}$$

$$(C_{13})_{ij} = (\prod_{k\neq i}^{2} h_{k2j} \prod_{k\neq j}^{2} h_{i2k})^{\delta_{2j}} e(\tfrac{1}{2}(2\rho_1 + \rho_2 - \alpha_i - \beta_1 - \gamma_1 - \gamma_2) \frac{(t_1 - t_3)^{\rho_1+\rho_2-\alpha_i-\beta_1}}{(t_3 - t_1)^{\rho_1+\rho_2-\beta_1-\gamma_j}} \left(\frac{t_1 - t_2}{t_3 - t_2}\right)^{\rho_1+\rho_2-\alpha_i-\gamma_j}$$
$$\frac{\Gamma(\gamma_j + 1)\Gamma(-\alpha_i)}{\Gamma(1 + \rho_1 - \alpha_i)\Gamma(\gamma_j - \rho_1)} \frac{\prod_{k\neq j}^2 \Gamma(\gamma_j - \gamma_k) \prod_{k\neq i}^2 \Gamma(1 + \alpha_k - \alpha_i)}{\prod_{k\neq i}^2 \Gamma(1 + \alpha_k + \beta_2 + \gamma_j - 2\rho_1 - \rho_2) \prod_{k\neq j}^2 \Gamma(1 + 2\rho_1 + \rho_2 - \alpha_i - \beta_2 - \gamma_k)} \tag{2.90}$$

$$(C_{23})_{ij} = (\prod_{k\neq i}^{2} h_{2kj} \prod_{k\neq j}^{2} h_{2ik})^{\delta_{2j}} e(\tfrac{1}{2}(\rho_1 + \rho_2 - \alpha_1 - \beta_i - \gamma_j)) \frac{(t_2 - t_3)^{\rho_1+\rho_2-\alpha_1-\beta_i}}{(t_3 - t_2)^{\rho_1+\rho_2-\alpha_1-\gamma_j}} \left(\frac{t_2 - t_1}{t_3 - t_1}\right)^{\rho_1+\rho_2-\beta_i-\gamma_j}$$
$$\frac{\Gamma(\gamma_j + 1)\Gamma(-\beta_i)}{\Gamma(1 + \rho_1 - \beta_i)\Gamma(\gamma_j - \rho_1)} \frac{\prod_{k\neq j}^2 \Gamma(\gamma_j - \gamma_k) \prod_{k\neq i}^2 \Gamma(1 + \beta_k - \beta_i)}{\prod_{k\neq i}^2 \Gamma(1 + \alpha_2 + \beta_k + \gamma_j - 2\rho_1 - \rho_2) \prod_{k\neq j}^2 \Gamma(1 + 2\rho_1 + \rho_2 - \alpha_2 - \beta_i - \gamma_k)} \tag{2.91}$$

$$(C_{21})_{ij} = e(\tfrac{1}{2}(2\rho_1 + \rho_2 - \alpha_i - \beta_j - \gamma_1 - \gamma_2)) \frac{(t_1 - t_2)^{\rho_1+\rho_2-\alpha_j-\gamma_1}}{(t_2 - t_1)^{\rho_1+\rho_2-\beta_i-\gamma_1}} \left(\frac{t_1 - t_3}{t_2 - t_3}\right)^{\rho_1+\rho_2-\alpha_i-\beta_j}$$
$$\frac{\Gamma(\alpha_j + 1)\Gamma(-\beta_i)}{\Gamma(1 + \rho_1 - \beta_i)\Gamma(\alpha_j - \rho_1)} \frac{\prod_{k\neq j}^2 \Gamma(\alpha_j - \alpha_k) \prod_{k\neq i}^2 \Gamma(1 + \beta_k - \beta_i)}{\prod_{k\neq i}^2 \Gamma(\alpha_j + \beta_k + \gamma_2 - 2\rho_1 - \rho_2) \prod_{k\neq j} \Gamma(1 + 2\rho_1 + \rho_2 - \alpha_k - \beta_i - \gamma_2)} \tag{2.92}$$

$$(C_{31})_{ij} = \frac{e(\frac{1}{2}(2\rho_1+\rho_2-\alpha_j-\beta_1-\gamma_i-\gamma_2))}{(\prod_{k\neq i}^2 h_{j2k}\prod_{k\neq j}^2 h_{k2i})^{\delta_{i2}}} \frac{(t_1-t_3)^{\rho_1+\rho_2-\alpha_j-\beta_1}}{(t_3-t_1)^{\beta_1+\gamma_i-\rho_1-\rho_2}} \left(\frac{t_1-t_2}{t_3-t_2}\right)^{\rho_1+\rho_2-\alpha_j-\gamma_i}$$
$$\frac{\Gamma(-\gamma_i)\Gamma(1+\alpha_j)}{\Gamma(1+\rho_1-\gamma_i)\Gamma(\alpha_j-\rho_1)} \frac{\prod_{k\neq j}\Gamma(\alpha_j-\alpha_k)\prod_{k\neq i}\Gamma(1+\gamma_i-\gamma_k)}{\prod_{k\neq i}\Gamma(\alpha_j+\beta_2+\gamma_k-2\rho_1-\rho_2)\prod_{k\neq j}^2\Gamma(2\rho_1+\rho_2-\alpha_k-\beta_2-\gamma_i)} \tag{2.93}$$

$$(C_{32})_{ij} = \frac{e(\frac{1}{2}(2\rho_1+\rho_2-\alpha_1-\beta_j-\gamma_1-\gamma_2))}{(\prod_{k\neq i}^2 h_{2jk}\prod_{k\neq i}^2 h_{2ki})^{\delta_{i2}}} \frac{(t_2-t_3)^{\rho_1+\rho_2-\alpha_1-\beta_j}}{(t_3-t_2)^{\alpha_1+\gamma_i-\rho_1-\rho_2}} \left(\frac{t_2-t_1}{t_3-t_1}\right)^{\rho_1+\rho_2-\beta_j-\gamma_i}$$
$$\frac{\Gamma(-\gamma_i)\Gamma(1+\beta_j)}{\Gamma(1+\rho_1-\gamma_i)\Gamma(\beta_j-\rho_1)} \frac{\prod_{k\neq j}\Gamma(\beta_j-\beta_k)\prod_{k\neq i}\Gamma(1+\gamma_i-\gamma_k)}{\prod_{k\neq i}\Gamma(\alpha_2+\beta_j+\gamma_k-2\rho_1-\rho_2)\prod_{k\neq j}\Gamma(\rho_1+\rho_2-\alpha_2-\beta_k-\gamma_i)} \tag{2.94}$$

In the following sections, we prove these theorems by the method of middle convolutions.

3 Middle convolutions for Schlesinger systems

In this section, we briefly recall the definitions of the Katz operations (the addition and the middle convolution) for a Schlesinger system and its monodromy representation ([10]).

3.1 Addition

For each r-tuple of matrices $\boldsymbol{A} = (A_1, \dots, A_r) \in \mathrm{Mat}(n;\mathbb{C})^r$, we consider the Schlesinger system

$$\frac{d}{dx}Y = \left(\sum_{k=1}^{r}\frac{A_k}{x-t_k}\right)Y, \quad A_k \in \mathrm{Mat}(n;\mathbb{C}) \ \ (k=1,\dots,r), \tag{3.1}$$

of ordinary differential equations on $\mathcal{D} = \mathbb{P}^1\backslash\{t_1,\dots,t_r,\infty\}$. We denote by $A_\infty = -A_1 - \cdots - A_r$ the residue matrix at $x=\infty$. For an r-tuple $\boldsymbol{a} = (a_1,\dots,a_r) \in \mathbb{C}^r$, the addition $\mathrm{add}_{\boldsymbol{a}}(\boldsymbol{A})$ of $\boldsymbol{A}$ by $\boldsymbol{a}$ is defined simply as

$$\mathrm{add}_{\boldsymbol{a}}(\boldsymbol{A}) = (A_1+a_1,\dots,A_r+a_r). \tag{3.2}$$

The corresponding Schlesinger system is given by

$$\frac{d}{dx}Z = \left(\sum_{k=1}^{r}\frac{A_k+a_k}{x-t_k}\right)Z, \quad A_k \in \mathrm{Mat}(n;\mathbb{C}) \ \ (k=1,\dots,r). \tag{3.3}$$

Let $Y(x)$ be a fundamental solution matrix of the Schlesinger system (3.1) associated with an r-tuple of matrices $\boldsymbol{A} \in \mathrm{Mat}(n;\mathbb{C})^r$. Then the monodromy matrices M_k $(k=1,\dots,r)$ defined as

$$\gamma_k \cdot Y(x) = Y(x)M_k \tag{3.4}$$

give rise to an r-tuple $\boldsymbol{M} = (M_1,\dots,M_r) \in \mathrm{GL}(n;\mathbb{C})^r$ of invertible matrices. The addition for an r-tuple $\boldsymbol{M} = (M_1,\dots,M_r)$ is defined by

$$\mathrm{Add}_{\boldsymbol{\lambda}}(\boldsymbol{M}) = (\lambda_1 M_1,\dots,\lambda_r M_r) \tag{3.5}$$

for each $\boldsymbol{\lambda} = (\lambda_1, \dots, \lambda_r) \in (\mathbb{C}^*)^r$, where $\mathbb{C}^* = \mathbb{C}\backslash\{0\}$.

We remark that the Schlesinger system (3.3) is obtained from (3.1) by the operation

$$Z = \prod_{k=1}^{r} (x - t_k)^{a_k} Y. \tag{3.6}$$

Then the monodromy matrices for the fundamental solution matrix $Z(x) = \prod_{k=1}^{r}(x - t_k)^{a_k} Y(x)$ of (3.3) is given by (3.5) with $\lambda_k = e(a_k)$ $(k = 1, \dots, r)$.

3.2 Middle convolution of a Schlesinger system

The middle convolution $\mathrm{mc}_\mu(\boldsymbol{A})$ of $\boldsymbol{A}$ with parameter $\mu \in \mathbb{C}^*$ is defined through three steps ([10]).

Step 1 (Convolution): We first define the *convolution* $\mathrm{c}_\mu(\boldsymbol{A})$ of $\boldsymbol{A}$ with parameter μ as the r-tuple of matrices $\boldsymbol{B} = (B_1, \dots, B_r) \in \mathrm{Mat}(nr; \mathbb{C})$ by setting

$$B_k = \begin{pmatrix} 0 & & \dots & & 0 \\ & \ddots & & & \\ A_1 & \dots & A_k + \mu & \dots & A_r \\ & & & \ddots & \\ 0 & & \dots & & 0 \end{pmatrix} \in \mathrm{Mat}(nr; \mathbb{C}) \qquad (k = 1, \dots, r) \tag{3.7}$$

where B_k is zero outside the k-th block row. The Schlesinger system associated with $\boldsymbol{B}$ can be written in the form of an Okubo system

$$(x - T)\frac{d}{dx} Z = BZ, \tag{3.8}$$

where $T = \mathrm{diag}(t_1 I_n, \dots, t_r I_n)$ and $B = B_1 + \cdots + B_r$.

Step 2 (K-Reduction): Setting $n_k = \mathrm{rank}\, A_k$, we decompose A_k $(k = 1, \dots, r)$ as $A_k = P_k Q_k$ with two matrices $P_k \in \mathrm{Mat}(n, n_k; \mathbb{C})$, $Q_k \in \mathrm{Mat}(n_k, n; \mathbb{C})$, and take the block matrix

$$Q = \begin{pmatrix} Q_1 & & \\ & \ddots & \\ & & Q_r \end{pmatrix} \in \mathrm{Mat}(\widetilde{n}, nr; \mathbb{C}) \qquad (\widetilde{n} = \sum_{k=1}^{r} n_k). \tag{3.9}$$

Since the (i, j)-block of B is given by $B_{ij} = A_j + \mu\delta_{ij} = P_j Q_j + \mu\delta_{ij}$, we have $Q_i B_{ij} = (Q_i P_j + \mu\delta_{ij})Q_j$, namely

$$QB = \widetilde{B}Q, \quad \widetilde{B} = \Big(Q_i P_j + \mu\delta_{ij}\Big)_{i,j=1}^{r}. \tag{3.10}$$

Hence, the system (3.8) gives rise to the Okubo system

$$(x - T)\frac{d}{dx}\widetilde{Z} = \widetilde{B}\widetilde{Z} \tag{3.11}$$

for $\widetilde{Z} = QZ$. Equivalently, we obtain the Schlesinger system

$$\frac{d}{dx}\widetilde{Z} = \left(\sum_{k=1}^{r} \frac{\widetilde{B}_k}{x - t_k}\right)\widetilde{Z} \tag{3.12}$$

corresponding to $\widetilde{\boldsymbol{B}} = (\widetilde{B}_1, \dots, \widetilde{B}_r)$, where

$$\widetilde{B}_k = \begin{pmatrix} 0 & & \dots & & 0 \\ & \ddots & & & \\ Q_kP_1 & \dots & Q_kP_k + \mu & \dots & Q_kP_r \\ & & & \ddots & \\ 0 & & \dots & & 0 \end{pmatrix} \quad (k = 1, \dots, r). \tag{3.13}$$

Step 3 (L-Reduction): Setting $m = \operatorname{rank} \widetilde{B}$, we further decompose $\widetilde{B}$ as $\widetilde{B} = P_0Q_0$ by two matrices $P_0 \in \mathrm{Mat}(\widetilde{n}, m; \mathbb{C})$, $Q_0 \in \mathrm{Mat}(m, \widetilde{n}; \mathbb{C})$. Since $\widetilde{B}_k$ is expressed as $\widetilde{B}_k = E_k\widetilde{B}$, $E_k = \mathrm{diag}(0.\dots, I_{n_k}, \dots, 0)$, we have

$$Q_0\widetilde{B}_k = Q_0E_k\widetilde{B} = Q_0E_kP_0Q_0. \tag{3.14}$$

Hence, multiplying (3.12) by Q_0 we obtain

$$\frac{d}{dx}Q_0\widetilde{Z} = \left(\sum_{k=1}^{r} \frac{Q_0E_kP_0}{x - t_k}\right) Q_0\widetilde{Z}. \tag{3.15}$$

Setting $\widehat{Z} = Q_0\widetilde{Z}$, we obtain the Schlesinger system

$$\frac{d}{dx}\widehat{Z} = \left(\sum_{k=1}^{r} \frac{\widehat{B}_k}{x - t_k}\right) \widehat{Z}, \quad \widehat{B}_k = Q_0E_kP_0 \quad (k = 1, \dots, r) \tag{3.16}$$

associated with $\mathrm{mc}_\mu(\boldsymbol{A}) = (\widehat{B}_1, \dots, \widehat{B}_r)$. We call the system (3.16) the *middle convolution* of (3.1) with parameter μ. We note that these matrices $\widehat{B}_i$ $(i = 1, \dots, r)$ are realizations of the linear transformations on $\mathbb{C}^{nr}/\big((\bigoplus_{k=1}^{r} \operatorname{Ker} A_k) \oplus \operatorname{Ker} B\big)$ induced from B_i.

3.3 Middle convolution of a monodromy representation

We next recall the middle convolution of monodromy matrices.

Let $Y(x)$ be a fundamental solution matrix of the Schlesinger system (3.1) associated with an r-tuple of matrices $\boldsymbol{A} \in \mathrm{Mat}(n; \mathbb{C})^r$. Then the monodromy matrices M_k $(k = 1, \dots, r)$ defined as

$$\gamma_k \cdot Y(x) = Y(x)M_k \tag{3.17}$$

give rise to an r-tuple $\boldsymbol{M} = (M_1, \dots, M_r) \in \mathrm{GL}(n; \mathbb{C})^r$ of invertible matrices. The multiplicative middle convolution $\mathrm{MC}_\lambda(\boldsymbol{M})$ we are going to explain below provides a way to construct the monodromy matrices for a certain fundamental solution matrix of the Schlesinger system associated with the middle convolution $\mathrm{mc}_\mu(\boldsymbol{A})$ with $\lambda = e(\mu)$.

Let $\boldsymbol{M} = (M_1, \dots, M_r) \in \mathrm{GL}(n; \mathbb{C})^r$ be an arbitrary r-tuple of invertible matrices. The multiplicative middle convolution $\mathrm{MC}_\lambda(\boldsymbol{M})$ of $\boldsymbol{M}$ with parameter $\lambda \in \mathbb{C}^*$ is constructed through three steps.

Step 1 (Convolution): We define an r-tuple of invertible matrices $\boldsymbol{N} = (N_1, \ldots, N_r) \in \mathrm{GL}(nr;\mathbb{C})^r$ as follows:

$$N_k = \begin{pmatrix} 1 & & \ldots & & 0 \\ & \ddots & & & \\ \lambda(M_1 - 1) & \ldots & \lambda M_k & \ldots & M_r - 1 \\ & & & \ddots & \\ 0 & & \ldots & & 1 \end{pmatrix} \in \mathrm{GL}(nr;\mathbb{C}). \tag{3.18}$$

Then we call $\boldsymbol{N}$ the *convolution* of $\boldsymbol{M}$ with parameter λ, and denote it by $\mathrm{C}_\lambda(\boldsymbol{M})$. As we will see later, $\mathrm{C}_\lambda(\boldsymbol{M})$ represents the r-tuple of monodromy matrices for a fundamental solution matrix of the Schlesinger system $\mathrm{c}_\mu(\boldsymbol{A})$.

Step 2 ($\mathcal{K}$-Reduction): Setting $n_k = \mathrm{rank}\,(M_k - 1)$, we decompose $M_k - 1\,(k = 1, \ldots, r)$ as $M_k - 1 = \mathcal{P}_k \mathcal{Q}_k$ with two matrices $\mathcal{P}_k \in \mathrm{Mat}(n, n_k;\mathbb{C})$, $\mathcal{Q}_k \in \mathrm{Mat}(n_k, n;\mathbb{C})$. Taking a right inverse $\mathcal{S}_k \in \mathrm{Mat}(n, n_k;\mathbb{C})$ of $\mathcal{Q}_k$ for each $k = 1, \ldots, r$ so that $\mathcal{Q}_k \mathcal{S}_k = I_{n_k}$, we define the block matrices $\mathcal{Q}$ and $\mathcal{S}$ by

$$\mathcal{Q} = \begin{pmatrix} \mathcal{Q}_1 & & \\ & \ddots & \\ & & \mathcal{Q}_r \end{pmatrix} \in \mathrm{Mat}(\widetilde{n}, nr;\mathbb{C}), \quad \mathcal{S} = \begin{pmatrix} \mathcal{S}_1 & & \\ & \ddots & \\ & & \mathcal{S}_r \end{pmatrix} \in \mathrm{Mat}(nr, \widetilde{n};\mathbb{C}). \tag{3.19}$$

where $\widetilde{n} = \sum_{i=1}^r n_i$. Since the (k, j)-block of $N_k - 1$ is given as

$$(N_k - 1)_{kj} = \begin{cases} \lambda(M_j - 1) = \lambda \mathcal{P}_k \mathcal{Q}_k & (1 \le j < k) \\ \lambda M_k - 1 = \lambda \mathcal{P}_k \mathcal{Q}_k + (\lambda - 1) & (j = k) \\ M_j - 1 = \mathcal{P}_k \mathcal{Q}_k & (k < j \le r), \end{cases} \tag{3.20}$$

by $\mathcal{Q}_k \mathcal{S}_k = 1$ we obtain $\mathcal{Q}(N_k - 1)\mathcal{S} = \widetilde{N}_k - 1$, i.e. $\mathcal{Q} N_k \mathcal{S} = \widetilde{N}_k$, where

$$\widetilde{N}_k = \begin{pmatrix} 1 & & \ldots & & 0 \\ & \ddots & & & \\ \lambda \mathcal{Q}_k \mathcal{P}_1 & \ldots & \lambda(\mathcal{Q}_k \mathcal{P}_k + 1) & \ldots & \mathcal{Q}_k \mathcal{P}_r \\ & & & \ddots & \\ 0 & & \ldots & & 1 \end{pmatrix} \in \mathrm{GL}(\widetilde{n};\mathbb{C}), \tag{3.21}$$

which gives a $\mathcal{K}$-reduction of the middle convolution $\widetilde{\boldsymbol{N}} = (\widetilde{N}_1, \ldots, \widetilde{N}_r)$.

Step 3 ($\mathcal{L}$-reduction): To construct monodromy matrices corresponding to the middle convolution $\mathrm{MC}_\lambda(\boldsymbol{M})$, we set $\widetilde{N}_0 = \widetilde{N}_1 \cdots \widetilde{N}_r$, and decompose $\widetilde{N}_0 - 1$, as $\widetilde{N}_0 - 1 = \mathcal{P}_0 \mathcal{Q}_0$ by two matrices $\mathcal{P}_0 \in \mathrm{Mat}(\widetilde{n}, m;\mathbb{C})$, $\mathcal{Q}_0 \in \mathrm{Mat}(m, \widetilde{n};\mathbb{C})$ with $m = \mathrm{rank}\,(\widetilde{N}_0 - 1)$, and take a right inverse $\mathcal{S}_0 \in \mathrm{Mat}(\widetilde{n}, m;\mathbb{C})$ so that $\mathcal{Q}_0 \mathcal{S}_0 = 1$. Then we obtain an r-tuple of monodromy matrices $\widehat{\boldsymbol{N}}(\widehat{N}_1, \ldots, \widehat{N}_r)$ as

$$\widehat{N}_k = \mathcal{Q}_0 \widetilde{N}_k \mathcal{S}_0 \qquad (k = 1, \ldots, r). \tag{3.22}$$

We call the r-tuple $\widehat{\boldsymbol{N}}$ the *middle convolution* of $\boldsymbol{M}$ with parameter λ, and denote by $\mathrm{MC}_\lambda(\boldsymbol{M})$. We remark that these matrices $\widehat{N}_i$ $(i = 1, \ldots, r)$ are realizations of the linear transformations on $\mathbb{C}^{nr} / \big((\bigoplus_{k=1}^r \mathrm{Ker}\,(M_k - 1)) \oplus \mathrm{Ker}\,(N_0 - 1) \big)$ induced from N_i, where $N_0 = N_1 \cdots N_r$.

3.4 Fundamental solution matrices

Let $Y(x)$ be a fundamental solution matrix of the Schlesinger system of (1.4) and $\boldsymbol{M} = (M_1, \ldots, M_r)$ the r-tuple of invertible matrices defined by the monodromy representation as in (3.17). We summarize below how one can obtain fundamental solution matrices whose monodromy representations correspond to the convolution $\mathrm{C}_\lambda(\boldsymbol{M})$ and the middle convolution $\mathrm{MC}_\lambda(\boldsymbol{M})$ of $\boldsymbol{M}$.

Following the construction by [10], for a complex parameter $\mu \in \mathbb{C}$ we consider the $nr \times n$ block matrices

$$I^{\mu}_{L_k} Y(x) = \left(\int_{L_k} (x-u)^{\mu} Y(u) \frac{du}{u - t_i} \right)_{i=1}^{r} \qquad (k = 1, \ldots, r), \tag{3.23}$$

called the *Euler transforms* of $Y(x)$. Here L_k denotes the double loop in the u-plane $\mathcal{D}_x = \mathbb{C}\backslash\{t_1, \ldots, t_r, x\}$ encircling $u = t_k$ and $u = x$ for each $k = 1, \ldots, r$. We use the symbols $\alpha_\infty, \alpha_1, \ldots, \alpha_r$ and α_x for the generators of $\pi_1(\mathcal{D}_x, p_0)$ encircling $u = \infty, t_1, \ldots, t_r$ and $u = x$ in the positive direction such that $\alpha_\infty \alpha_1 \cdots \alpha_r \alpha_x = 1$. Then the double loops L_k $(k = 1, \ldots, r)$ are expressed as $L_k = L[t_k, x] = \alpha_k^{-1} \alpha_x^{-1} \alpha_k \alpha_x$ (See Figure 2). We also set $L_\infty = L[\infty, x] = \alpha_\infty^{-1} \alpha_x^{-1} \alpha_\infty \alpha_x$.

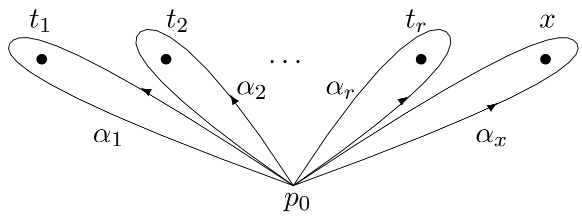

Figure 2:

We define the $nr \times nr$ block matrix $I^{\mu} Y(x)$ by

$$\begin{aligned} I^{\mu} Y(x) &= \left(I^{\mu}_{L_1} Y(x), \ldots, I^{\mu}_{L_r} Y(x) \right) \\ &= \left(\int_{L_j} (x-u)^{\mu} Y(u) \frac{du}{u - t_i} \right)_{1 \le i,j \le r} \in \mathrm{Mat}(nr; \mathcal{O}(\widetilde{D})). \end{aligned} \tag{3.24}$$

It is known by [10] that $I^{\mu} Y(x)$ is a solution matrix of the Okubo system (3.8) associated with $\mathrm{c}_\mu(\boldsymbol{A})$, and that the monodromy of $I^{\mu} Y(x)$ is described as

$$\gamma_k \cdot I^{\mu} Y(x) = I^{\mu} Y(x)\, N_k \qquad (k = 1, \ldots, r) \tag{3.25}$$

in terms of the convolution $\mathrm{C}_\lambda(\boldsymbol{M}) = (N_1, \ldots, N_r)$ of $\boldsymbol{M}$ with $\lambda = e(\mu)$.

Theorem 3.1. *Suppose that the following three conditions are satisfied.*

(i) $\mu \in \mathbb{C}\backslash\mathbb{Z}$.

(ii) *For each* $k = 1, \ldots, r$ *and* ∞*, any pair of distinct eigenvalues of* A_k *has no integer difference.*

(iii) A_k $(k = 1, \ldots, r)$ *and* $A_\infty - \mu$ *have no eigenvalue in* $\mathbb{Z}_{>0}$.

Then the $nr \times nr$ *block matrix* $Z(x) = I^{\mu} Y(x)$ *is a fundamental solution matrix of the Okubo system* (3.8) *associated with* $\mathrm{c}_\mu(\boldsymbol{A})$.

We prove that $I^\mu Y(x)$ is a fundamental solution matrix. The fundamental solution matrix $Y(x)$ of (3.1) is written of the form

$$Y(x) = F^{(k)}(x)\,(x-t_k)^{A_k} C^{(k)}, \quad F^{(k)}(x) \in \mathrm{Mat}(n;\mathcal{O}_{t_k}), \quad F^{(k)}(t_k) = I_n \tag{3.26}$$

around $x = t_k$ for each $k = 1, \dots, r$, and

$$Y(x) = F^{(\infty)}(x)\, x^{-A_\infty} C^{(\infty)}, \quad F^{(\infty)}(x) \in \mathrm{Mat}(n;\mathcal{O}_\infty), \quad F^{(\infty)}(\infty) = I_n \tag{3.27}$$

around $x = \infty$. We investigate the local behavior of

$$I^\mu Y(x)_{ij} = \int_{L_j} \frac{(x-u)^\mu}{u-t_i} Y(u)du = \int_{L_j} \frac{(x-u)^\mu}{u-t_i} F^{(j)}(u)(u-t_j)^{A_j} C^{(j)} du \tag{3.28}$$

at $x = t_j$. Changing the integration variable by $u = (x-t_j)v + t_j$, for $j \neq i$ we have

$$\begin{aligned}
\int_{L_j} \frac{(x-u)^\mu}{u-t_i} Y(u)du &= \int_{L[0,1]} \frac{(1-v)^\mu}{(x-t_j)v - (t_i - t_j)} F^{(j)}((x-t_j)v + t_j) v^{A_j} (x-t_j)^{A_j+\mu+1} C^{(j)} dv \\
&= \int_{L[0,1]} \frac{(1-v)^\mu}{t_j - t_i} F^{(j)}(t_j) v^{A_j} dv\, (x-t_j)^{A_j+\mu+1} C^{(j)} \big(1 + O(x-t_j)\big) \\
&= \frac{\widetilde{B}(A_j+1, \mu+1)}{t_j - t_i} (x-t_j)^{A_j+\mu+1} C^{(j)} \big(1 + O(x-t_j)\big),
\end{aligned} \tag{3.29}$$

where

$$\begin{aligned}
\widetilde{B}(A,\beta) &= \int_{L[0,1]} u^{A-1}(1-u)^{\beta-1} du \\
&= (e(A)-1)(e(\beta)-1)B(A,\beta) \qquad (A \in \mathrm{Mat}(n;\mathbb{C}),\ \beta \in \mathbb{C})
\end{aligned} \tag{3.30}$$

denotes the regularized beta function with a matrix argument. In this integral, as the base point we take a point in the interval $(0,1)$ with $\arg u = \arg(1-u) = 0$. Similarly we have

$$\begin{aligned}
&\int_{L_i} \frac{(x-u)^\mu}{u-t_i} Y(u)du = \widetilde{B}(A_i, \mu+1)(x-t_i)^{A_i+\mu} C^{(i)} \big(1 + O(x-t_i)\big) \quad (i = 1, \cdots, r), \\
&\int_{L_\infty} \frac{(x-u)^\mu}{u-t_i} Y(u)du = -e^{-\pi i \mu} \widetilde{B}(A_\infty - \mu, \mu+1) x^{-A_\infty+\mu} C^{(\infty)} \big(1 + O(x^{-1})\big).
\end{aligned} \tag{3.31}$$

Since A_i has no eigenvalue in $\mathbb{Z}_{>0}$ by the assumption, we have $\det \widetilde{B}(A_i, \mu+1) \neq 0$, and hence $\det(I^\mu Y(x)_{ii}) \neq 0$ for $i = 1, \dots, r$. Similarly, $\det(I^\mu_{L_\infty} Y(x)_i) \neq 0$ for $i = 1, \dots, r$. Consider the vector space

$$\begin{aligned}
\mathcal{L} &= \bigcap_{i=1}^{r} \mathrm{Ker}(N_i - 1) = \mathrm{Ker}(N_\infty - 1) \\
&= \{\boldsymbol{v} = (M_2 \cdots M_r v, \dots, M_r v, v)^t \in \mathbb{C}^{nr} \mid v \in \mathrm{Ker}(\lambda M_1 \cdots M_r - 1)\},
\end{aligned} \tag{3.32}$$

where we regard square matrices as linear transformations acting on column vectors.

Lemma 3.2. *Under the assumption of Theorem 3.1, we have* $\mathrm{Ker}(I^\mu Y(x)) \subseteq \mathcal{L}$.

Proof. Assume that a column vector $\boldsymbol{v} = (v_1, \dots, v_r)^t \in \mathbb{C}^{nr}$ belongs to $\mathrm{Ker}(I^\mu Y(x))$, namely, $I^\mu Y(x)\boldsymbol{v} = 0$. Then we have $(\gamma_k - 1) \cdot I^\mu Y(x)\boldsymbol{v} = 0$ for $k = 1, \dots, r$. This implies that the k-th row of $I^\mu Y(x)$ satisfies

$$I^\mu Y(x)_{kk}\big(\sum_{1\le i<k} e(\mu)(M_i - 1)v_i + (e(\mu)M_k - 1)v_k + \sum_{k<i\le r} (M_i - 1)v_i\big) = 0 \quad (k = 1, \dots, r). \tag{3.33}$$

Since $\det(I^\mu Y(x)_{kk}) \neq 0$, we have

$$\sum_{1\le i<k} e(\mu)(M_i - 1)v_i + (e(\mu)M_k - 1)v_k + \sum_{k<i\le r} (M_i - 1)v_i = 0 \quad (k = 1, \dots, r). \tag{3.34}$$

Hence we find that $\boldsymbol{v} \in \mathrm{Ker}(N_k - 1) \quad (k = 1, \dots, r)$. □

To complete the proof, we assume $\boldsymbol{v} \in \mathrm{Ker}(I^\mu Y(x))$. Then by Lemma 3.2, $\boldsymbol{v}$ is expressed as $\boldsymbol{v} = (M_2 \cdots M_r v, \dots, M_r v, v)$ for some $v \in \mathrm{Ker}(\lambda M_1 \cdots M_r - 1)$. In the following we set

$$I^\mu_\alpha Y(x) = \left(\int_\alpha (x-u)^\mu\, Y(u) \frac{du}{u - t_i} \right)_{i=1}^r \tag{3.35}$$

for each $\alpha \in \pi(\mathcal{D}_x, p_0)$. Note that by $L_k = \alpha_k^{-1}\alpha_x^{-1}\alpha_k\alpha_x$ we have

$$\begin{aligned} I^\mu_{L_k} Y(x) &= I^\mu_{\alpha_k} Y(x)(\lambda - 1) - I^\mu_{\alpha_x} Y(x)(M_k - 1) \quad (k = 1, \dots, r), \\ I^\mu_{L_\infty} Y(x) &= I^\mu_{\alpha_\infty} Y(x)(\lambda - 1) - I^\mu_{\alpha_x} Y(x)((\lambda M_1 \cdots M_r)^{-1} - 1). \end{aligned} \tag{3.36}$$

In this notation $I^\mu Y(x)\boldsymbol{v}$ is computed as follows:

$$\begin{aligned} I^\mu Y(x)\boldsymbol{v} &= \sum_{k=1}^r I^\mu_{L_k} Y(x) M_{k+1} \cdots M_r v \\ &= \sum_{k=1}^r \big(I^\mu_{\alpha_k} Y(x)(\lambda - 1) - I^\mu_{\alpha_x} Y(x)(M_k - 1)\big) M_{k+1} \cdots M_r v \\ &= (\lambda - 1) \sum_{k=1}^r I^\mu_{\alpha_k} Y(x) M_{k+1} \cdots M_r v - \sum_{k=1}^r I^\mu_{\alpha_x} Y(x)(M_k - 1) M_{k+1} \cdots M_r v \\ &= (\lambda - 1) I^\mu_{\alpha_1\cdots\alpha_r} Y(x) v - I^\mu_{\alpha_x} Y(x)(M_1 \cdots M_r - 1)v \end{aligned} \tag{3.37}$$

By

$$\begin{aligned} I^\mu_{\alpha_1\cdots\alpha_r} Y(x) = I^\mu_{\alpha_\infty^{-1}\alpha_x^{-1}} Y(x) &= I^\mu_{\alpha_\infty^{-1}} Y(x)\lambda^{-1} + I^\mu_{\alpha_x^{-1}} Y(x) \\ &= -I^\mu_{\alpha_\infty} Y(x) M_1 \cdots M_r - I^\mu_{\alpha_x} Y(x)\lambda^{-1} \end{aligned} \tag{3.38}$$

we obtain

$$\begin{aligned} I^\mu Y(x)\boldsymbol{v} &= -(\lambda - 1) I^\mu_{\alpha_\infty} Y(x) M_1 \cdots M_r v - I^\mu_{\alpha_x} Y(x)(M_1 \cdots M_r - \lambda^{-1})v \\ &= \big(-(\lambda - 1) I^\mu_{\alpha_\infty} Y(x) + I^\mu_{\alpha_x} Y(x)((\lambda M_1 \cdots M_r)^{-1} - 1)\big) M_1 \cdots M_r v \\ &= -I^\mu_{L_\infty} Y(x) M_1 \cdots M_r v. \end{aligned} \tag{3.39}$$

Since $\det(I^\mu_{L_\infty} Y(x)_i) \neq 0$ $(i = 1, \dots, r)$, $I^\mu Y(x)\boldsymbol{v} = 0$ implies $v = 0$, and hence $\boldsymbol{v} = 0$. This completes the proof of Theorem 3.1.

We set $n_k = \mathrm{rank}\, A_k$ for $k = 1, \dots, r$, and $\widetilde{n} = \sum_{i=1}^r n_i$.

Theorem 3.3. *In addition to the conditions* (i), (ii), (iii) *of Theorem 3.1, suppose that the following condition is satisfied.*

(iv) *For each $k = 1, \ldots, r$, A_k has non-integer eigenvalues $\alpha_1^{(k)}, \ldots, \alpha_{n_k}^{(k)}$ where $n_k = \operatorname{rank} A_k$.*

Using the constant matrices Q and $\mathcal{S}$ as in (3.9) *and* (3.19) *respectively, we set $\widetilde{Z}(x) = QZ(x)\mathcal{S}$, $Z(x) = I^\mu Y(x)$. Then the $\widetilde{n} \times \widetilde{n}$ matrix $\widetilde{Z}(x)$ is a fundamental solution matrix of the K-reduction* (3.12) *of the convolution, and the monodromy matrices for $\widetilde{Z}(x)$ are given by the $\mathcal{K}$-reduction $\widetilde{\boldsymbol{N}}$ of $\mathrm{C}_\lambda(\boldsymbol{M}) = \boldsymbol{N}$.*

Theorem 3.4. *In addition to the conditions* (i), (ii), (iii), (iv) *above, suppose that the following condition is satisfied.*

(v) *$A_\infty - \mu$ has non-integer eigenvalues $\alpha_1^{(\infty)} - \mu, \ldots, \alpha_l^{(\infty)} - \mu$ where $l = \operatorname{rank}(A_\infty - \mu)$.*

Using Q_0 and $\mathcal{S}_0$ as in (3.14) *and* (3.22) *respectively, set $\widehat{Z}(x) = Q_0\widetilde{Z}(x)\mathcal{S}_0$. Then $\widehat{Z}(x)$ is a fundamental solution matrix of the Schlesinger system corresponding to the middle convolution* (3.16)*, and the monodromy matrices for $\widehat{Z}(x)$ are given by the middle convolution $\mathrm{MC}_\lambda(\boldsymbol{M}) = \widehat{\boldsymbol{N}}$.*

As we already verified, $QZ(x)$ is a solution matrix of the K-reduction (3.12). Also, under the assumption (iv), we have $\operatorname{rank} A_k = \operatorname{rank}(M_k - 1)$ for $k = 1, \ldots, r$. For $\mathcal{Q}_k$, $\mathcal{S}_k$ in (3.19), we take $\mathcal{Q}'_k$, $\mathcal{S}'_k$ such that

$$\begin{pmatrix} \mathcal{Q}_k \\ \mathcal{Q}'_k \end{pmatrix} \begin{pmatrix} \mathcal{S}_k & \mathcal{S}'_k \end{pmatrix} = I_n. \tag{3.40}$$

Since $M_k - 1 = \mathcal{P}_k\mathcal{Q}_k$, we have

$$(M_k - 1)\begin{pmatrix} \mathcal{S}_k & \mathcal{S}'_k \end{pmatrix} = (\mathcal{P}_k, 0). \tag{3.41}$$

We consider solution matrix $QZ(x)\widetilde{\mathcal{S}}$ of (3.12) defined by

$$\widetilde{S} = \begin{pmatrix} \mathcal{S} & \mathcal{S}' \end{pmatrix} = \begin{pmatrix} \mathcal{S}_1 & & & \mathcal{S}'_1 & & \\ & \ddots & & & \ddots & \\ & & \mathcal{S}_r & & & \mathcal{S}'_r \end{pmatrix}. \tag{3.42}$$

Noting that

$$Z(x)\widetilde{\mathcal{S}} = I^\mu Y(x)\widetilde{\mathcal{S}} = (I^\mu_{L_1} Y(x)\mathcal{S}_1, \ldots I^\mu_{L_r} Y(x)\mathcal{S}_r, I^\mu_{L_1} Y(x)\mathcal{S}'_1, \ldots I^\mu_{L_r} Y(x)\mathcal{S}'_r) \tag{3.43}$$

we compute the analytic continuation by γ_k as

$$\gamma_k \cdot I^\mu_{L_j} Y(x)S'_j = \begin{cases} I^\mu_{L_j} Y(x)S'_j & (j \neq k) \\ I^\mu_{L_k} Y(x)e(\mu)S'_k & (j = k) \end{cases} \tag{3.44}$$

by $(M_j - 1)\mathcal{S}'_j = 0$. Hence we have

$$\gamma_k \cdot QI^\mu_{L_k} Y(x)\mathcal{S}'_k = QI^\mu_{L_k} Y(x)\mathcal{S}'_k\, e(\mu) \quad (k = 1, \ldots, r). \tag{3.45}$$

However, under the assumption (iv), $e(\mu)$ is *not* an eigenvalue of γ_k on the solution space of the K-reduction (3.12). This means that $QI^\mu_{L_k} Y(x)\mathcal{S}'_k = 0$ $(k = 1, \ldots, r)$, namely,

$$QZ(x)\widetilde{\mathcal{S}} = (QZ(x)\mathcal{S}, 0). \tag{3.46}$$

Since $\operatorname{rank} Q = \widetilde{n}$, the matrix $\operatorname{rank} QZ(x)\widetilde{\mathcal{S}} = \widetilde{n}$ for any regular point $x \in \mathcal{D}$. Hence we have $\operatorname{rank} QZ(x)\mathcal{S} = \widetilde{n}$. This completes the proof of Theorem 3.3. Theorem 3.4 can be proved in a similar way.

4 Construction of Okubo systems by middle convolutions

4.1 Middle convolution for an Okubo system with additions at a singular point

We consider the operation

$$\mathrm{add}_{(0,\dots,\rho,\dots,0)} \circ \mathrm{mc}_{-\rho-c} \circ \mathrm{add}_{(0,\dots,c,\dots,0)}(\boldsymbol{A}) \qquad (c, \rho \in \mathbb{C}) \tag{4.1}$$

for an Okubo system (1.2) with residue matrices $\boldsymbol{A} = (A_1, \dots, A_r)$, where two additions are applied at $x = t_k$ $(k = 1, \dots, r)$. The Okubo systems of types I, II and III are included in the class of the differential systems obtained from $(\mathrm{II})_2$ by a finite iteration of operations in the form (4.1). We show that the Schlesinger system corresponding to (4.1) is an Okubo system if the parameters c and ρ are generic in the sense that $\mathrm{Ker}(A_k + c) = 0$ and $\mathrm{Ker}(A_{kk} - \rho) = 0$.

With the notations of (1.3) and (1.4), we first decompose the components of $\mathrm{add}_{0,\dots,c,\dots,0}(\boldsymbol{A}) = (A_1, \dots, A_k + c, \dots, A_r)$ as

$$\begin{aligned} &A_i = P_i Q_i; \quad P_i = \begin{pmatrix} 0 \\ I_{n_i} \\ 0 \end{pmatrix}, \quad Q_i = \begin{pmatrix} A_{i1} & \cdots & A_{ir} \end{pmatrix}, \quad (i \neq k) \\ &A_k + c = P_k Q_k; \quad P_k = I_n, \quad Q_k = A_k + c. \end{aligned} \tag{4.2}$$

For the decomposition of $A - \rho$, we use the following lemma.

Lemma 4.1. *Let $X = (X_{ij})_{i,j=1}^r \in \mathrm{Mat}(n; \mathbb{C})$ be an $n \times n$ block matrix of type $(n_1, \dots, n_r)$, $n_1 + \cdots + n_r = n$, where $X_{ij} \in \mathrm{Mat}(n_i, n_j; \mathbb{C})$ $(i, j = 1, \dots, r)$. Suppose that* $\mathrm{rank}\, X_{kk} = n_k$ *and* $\mathrm{rank}\, X \geq n_k$ *for an index $k = 1, \dots, r$ and set $l = \mathrm{rank}\, X - n_k$. Then there exist matrices*

$$\xi_i \in \mathrm{Mat}(n_i, l; \mathbb{C}) \quad (1 \leq i \leq r,\ i \neq k), \quad \eta_j \in \mathrm{Mat}(l, n_j; \mathbb{C}) \quad (1 \leq j \leq r,\ i \neq k) \tag{4.3}$$

such that

$$X_{ij} = X_{ik} X_{kk}^{-1} X_{kj} + \xi_i \eta_j \quad (1 \leq i, j \leq r;\ i, j \neq k). \tag{4.4}$$

Furthermore, the $(n - n_k) \times l$ matrix $\xi = (\xi_i)_{1 \leq i \leq r; i \neq k}$ and the $l \times (n - n_k)$ matrix $\eta = (\eta_j)_{1 \leq j \leq r; i \neq k}$ are of maximal rank.

Proof. Without loss of generality we assume $k = r$ and set

$$A = (X_{ij})_{i,j=1}^{r-1}, \quad B = (X_{ir})_i^{r-1}, C = (X_{rj})_{j=1}^{r-1}, \quad D = X_{rr}. \tag{4.5}$$

Noting that D is invertible, we decompose this matrix into the form

$$X = \begin{pmatrix} A & B \\ C & D \end{pmatrix} = \begin{pmatrix} 1 & BD^{-1} \\ 0 & 1 \end{pmatrix} \begin{pmatrix} A - BD^{-1}C & 0 \\ 0 & D \end{pmatrix} \begin{pmatrix} 1 & 0 \\ D^{-1}C & 1 \end{pmatrix} \tag{4.6}$$

Since $\mathrm{rank}(A - BD^{-1}C) = \mathrm{rank}\ X - n_r = l$, there exist matrices $\xi \in \mathrm{Mat}(n - n_r, l; \mathbb{C})$ and $\eta \in \mathrm{Mat}(l, n - n_r; \mathbb{C})$ of rank l such that $A - BD^{-1}C = \xi\eta$. From $A = BD^{-1}C + \xi\eta$ we obtain

$$X_{ij} = X_{ir} X_{rr}^{-1} X_{rj} + \xi_i \eta_j \qquad (i, j = 1, \dots, r - 1) \tag{4.7}$$

where $\xi = (\xi_i)_{i=1}^{r-1}$ and $\eta = (\eta_j)_{j=1}^{r-1}$. □

If we apply Lemma 4.1 to the matrix $A-\rho$ with the invertible block $A_{kk}-\rho$, then formula (4.4) implies the decomposition $A-\rho=P_0Q_0$ of $A-\rho$ where

$$P_0=\begin{pmatrix} A_{1k}(A_{kk}-\rho)^{-1} & \xi_1 \\ \vdots & \vdots \\ A_{k-1,k}(A_{kk}-\rho)^{-1} & \xi_{k-1} \\ 1 & 0 \\ A_{k+1,k}(A_{kk}-\rho)^{-1} & \xi_{k+1} \\ \vdots & \vdots \\ A_{rk}(A_{kk}-\rho)^{-1} & \xi_r \end{pmatrix}, \quad Q_0=\begin{pmatrix} A_{k1} & \dots & A_{k,k-1} & A_{kk}-\rho & A_{k,k+1} & \dots & A_{kr} \\ \eta_1 & \dots & \eta_{k-1} & 0 & \eta_{k+1} & \dots & \eta_r \end{pmatrix}. \tag{4.8}$$

In this case, the linear mapping

$$\mathbb{C}^{nr}\to\mathbb{C}^{nr}: \quad (v_1,\dots,v_r)\mapsto(v_1-v_k,\dots,v_{k-1}-v_k,v_k,v_{k+1}-v_k,\dots,v_r-v_k) \tag{4.9}$$

induces the isomorphism from the quotient space $\mathbb{C}^{nr}/\big((\bigoplus_{p\neq k}\mathrm{Ker}A_p\oplus\mathrm{Ker}(A_k+c))\oplus\mathrm{Ker}B\big)$ for the middle convolution to

$$(\mathbb{C}^n/\mathrm{Ker}A_1)\oplus\cdots\oplus(\mathbb{C}^n/\mathrm{Ker}A_{k-1})\oplus(\mathbb{C}^n/\mathrm{Ker}(A-\rho))\oplus(\mathbb{C}^n/\mathrm{Ker}A_{k+1})\oplus\cdots\oplus(\mathbb{C}^n/\mathrm{Ker}A_r). \tag{4.10}$$

Therefore we redefine Q as

$$Q=\begin{pmatrix} Q_1 & & -Q_1 & & \\ & \ddots & \vdots & & \\ & & Q_0 & & \\ & & \vdots & \ddots & \\ & & -Q_r & & Q_r \end{pmatrix}. \tag{4.11}$$

Then, from the convolution

$$(x-T)\frac{d}{dx}Z=BZ; \quad B=(B_{ij})_{i,j=1}^r, \quad B_{ij}=A_j+\delta_{jk}c-\delta_{ij}(\rho+c), \tag{4.12}$$

by the transformation $\widehat{Z}=QZ$ we obtain the Schlesinger system

$$\frac{d}{dx}\widehat{Z}=\sum_{l=1}^r\frac{\widehat{B}_l+\rho\delta_{kl}}{x-t_l}\widehat{Z}, \qquad \widehat{B}_lQ=QB_l \quad (l=1,\dots,r) \tag{4.13}$$

corresponding to the operation (4.1), where

$$\widehat{B}_l=\begin{pmatrix} & O & \\ A_{l1} & \cdots & Q_lP_0 & \cdots & A_{lr} \\ & O & \end{pmatrix}-(\rho+c)E_{ll} \quad (l\neq k), \quad \widehat{B}_k=\begin{pmatrix} -Q_1 \\ \vdots \\ Q_0 \\ \vdots \\ -Q_r \end{pmatrix}\begin{pmatrix} P_1 & \cdots P_0 & \cdots & P_r \end{pmatrix} \tag{4.14}$$

with Q_lP_0 in the (l,k) block of $\widehat{B}_l$.

Then the Schlesinger system (4.13) is transformed into the Okubo system

$$(x-T)\frac{d}{dx}W=A^{mc}W, \qquad A^{mc}=\mathrm{Ad}(G)(\widehat{B}_1+\cdots+\widehat{B}_r+\rho) \tag{4.15}$$

for $W = G\widehat{Z}$, where

$$A^{mc} = \begin{pmatrix} & & & A_{1k}(A_{kk}+c)(A_{kk}-\rho)^{-1} & (\rho+c)\xi_1 & & & \\ & A_{ij}-(\rho+c)\delta_{ij} & & \vdots & \vdots & & A_{ij} & \\ & & & A_{k-1,k}(A_{kk}+c)(A_{kk}-\rho)^{-1} & (\rho+c)\xi_{k-1} & & & \\ A_{k1} & \dots & A_{k,k-1} & A_{kk} & 0 & A_{k,k+1} & \dots & A_{kr} \\ \eta_1 & \dots & \eta_{k-1} & 0 & \rho & \eta_{k+1} & \dots & \eta_r \\ & & & A_{k+1,k}(A_{kk}+c)(A_{kk}-\rho)^{-1} & (\rho+c)\xi_{k+1} & & & \\ & A_{ij} & & \vdots & \vdots & & A_{ij}-(\rho+c)\delta_{ij} & \\ & & & A_{rk}(A_{kk}+c)(A_{kk}-\rho)^{-1} & (\rho+c)\xi_r & & & \end{pmatrix} \tag{4.16}$$

and

$$G = \begin{pmatrix} 1 & & & A_{1k}(A_{kk}-\rho)^{-1} & \xi_1 & & & \\ & \ddots & & \vdots & \vdots & & & \\ & & 1 & A_{k-1k}(A_{kk}-\rho)^{-1} & \xi_{k-1} & & & \\ & & & 1 & 0 & & & \\ & & & 0 & 1 & & & \\ & & & A_{k+1k}(A_{kk}-\rho)^{-1} & \xi_{k+1} & 1 & & \\ & & & \vdots & \vdots & & \ddots & \\ & & & A_{rk}(A_{kk}-\rho)^{-1} & \xi_r & & & 1 \end{pmatrix}. \tag{4.17}$$

We remark that the transformation from the convolution (4.12) to the Okubo system (4.15) is given by $W = GQZ$:

$$GQ = \begin{pmatrix} Q_1 & & & (-\rho I_{n_1},\,0) & & & \\ & \ddots & & \vdots & & & \\ & & Q_{k-1} & (0,\,-\rho I_{n_{k-1}},\,0) & & & \\ & & & Q_0 & & & \\ & & & (0,\,-\rho I_{n_{k+1}},\,0) & Q_{k+1} & & \\ & & & \vdots & & \ddots & \\ & & & (0,\,-\rho I_{n_r}) & & & Q_r \end{pmatrix}. \tag{4.18}$$

Lemma 4.2. *If the Okubo system* (1.2) *satisfies the conditions* $\mathrm{Ker}(A_k+c)=0$ *and* $\mathrm{Ker}(A_{kk}-\rho)=0$ *for an index* $k=1,\dots,r$*, then the Schlesinger system with residue matrices* (4.1) *is equivalent to the Okubo system* (4.15)*.*

4.2 Middle convolution for monodromy matrices of Okubo type

Let $\Psi(x)$ be the canonical solution matrix of the Okubo system (1.2). The analytic continuation $\Psi(x)$ by γ_k $(k=1,\dots,r)$ gives an r-tuple of monodromy matrices $\boldsymbol{M}$ as defined in Section 1.2. Then $\boldsymbol{M}$ is *of Okubo type* in the sense that M_i are expressed as

$$M_i = \begin{pmatrix} 1 & & & & \\ & \ddots & & & \\ M_{i1} & \cdots & M_{ii} & \cdots & M_{ir} \\ & & & \ddots & \\ & & & & 1 \end{pmatrix} \qquad (i=1,\dots,r) \tag{4.19}$$

in terms of matices $M_{ij} \in \mathrm{Mat}(n_i, n_j; \mathbb{C})$ $(1 \le i, j \le r)$. We show that the operation

$$\mathrm{Add}_{(1,\dots,\lambda^{-1},\dots,1)} \circ \mathrm{MC}_{\lambda s^{-1}} \circ \mathrm{Add}_{(1,\dots,s,\dots,1)}(\boldsymbol{M}) \tag{4.20}$$

gives rises to monodromy matrices of Okubo type as in the case of Okubo system (1.2). We decompose $M_i - 1$, $(i = 1, \dots, r)$ and take $\mathcal{S}_i$ as

$$M_i - 1 = \mathcal{P}_i \mathcal{Q}_i = \begin{pmatrix} 0 \\ I_{n_i} \\ 0 \end{pmatrix} \begin{pmatrix} M_{i1} \cdots M_{ii} - 1 \cdots M_{ir} \end{pmatrix}, \quad \mathcal{S}_i = \begin{pmatrix} 0 \\ (M_{ii} - 1)^{-1} \\ 0 \end{pmatrix}. \tag{4.21}$$

Also, setting $M_0 = \lambda M_1 \cdots M_r$, we take the decomposition

$$\begin{aligned} M_0^{(k)} - 1 &= \mathcal{P}_0^{(k)} \mathcal{Q}_0^{(k)} \\ &= \begin{pmatrix} M_{1k}^{(k)}(M_{kk}^{(k)} - 1)^{-1} & \xi_1 \\ \vdots & \vdots \\ M_{k-1,k}^{(k)}(M_{kk}^{(k)} - 1)^{-1} & \xi_{k-1} \\ 1 & 0 \\ M_{k+1,k}^{(k)}(M_{kk}^{(k)} - 1)^{-1} & \xi_{k+1} \\ \vdots & \vdots \\ M_{rk}^{(k)}(M_{kk}^{(k)} - 1)^{-1} & \xi_r \end{pmatrix} \begin{pmatrix} M_{k1}^{(k)} & \cdots & M_{k,k-1}^{(k)} & M_{kk}^{(k)} - 1 & M_{k,k+1}^{(k)} & \cdots & M_{k,r}^{(k)} \\ \eta_1 & \cdots & \eta_{k-1} & 0 & \eta_{k+1} & \cdots & \eta_r \end{pmatrix} \end{aligned} \tag{4.22}$$

for

$$M_0^{(k)} = \mathrm{Ad}(M_{k+1} \cdots M_r)(M_0) = M_{k+1} \cdots M_r M_1 \cdots M_k = (M_{ij}^{(k)})_{i,j=1}^r \tag{4.23}$$

as in Lemma 4.1, and define $\mathcal{P}_0 = (M_{k+1} \cdots M_r)^{-1} \mathcal{P}_0^{(k)}$, $\mathcal{Q}_0 = \mathcal{Q}_0^{(k)}(M_{k+1} \cdots M_r)$, $\mathcal{S}_0$ so that $M_0 - 1 = \mathcal{P}_0 \mathcal{Q}_0$, $\mathcal{Q}_0 \mathcal{S}_0 = I_{n_0}$. As in the case of the differential system, the linear mapping $\mathbb{C}^{nr} \to \mathbb{C}^{nr}$:

$$(v_1, \dots, v_r) \mapsto (v_1 - sM_2 \cdots v_k, \dots, v_{k-1} - sM_k v_k, v_k, v_{k+1} - M_{k+1}^{-1} v_k, \dots, v_r - (M_{k+1} \cdots M_r)^{-1} v_k) \tag{4.24}$$

induces the isomorphism from the quotient space $\mathbb{C}^{nr} / \big((\bigoplus_{p \ne k} \mathrm{Ker}(M_p - 1) \oplus \mathrm{Ker}(sM_k - 1)) \oplus \mathrm{Ker}(N_0 - 1) \big)$ for the middle convolution to

$$\begin{aligned} &(\mathbb{C}^n / \mathrm{Ker}(M_1 - 1)) \oplus \cdots \oplus (\mathbb{C}^n / \mathrm{Ker}(M_{k-1} - 1)) \\ &\oplus (\mathbb{C}^n / \mathrm{Ker}(M_0^{(k)} - 1)) \oplus (\mathbb{C}^n / \mathrm{Ker}(M_{k+1} - 1)) \oplus \cdots \oplus (\mathbb{C}^n / \mathrm{Ker}(M_r - 1)). \end{aligned} \tag{4.25}$$

Therefore, taking block matrices

$$\mathcal{Q} = \begin{pmatrix} \mathcal{Q}_1 & & & -s\mathcal{Q}_1 M_2 \cdots M_k & & & \\ & \ddots & & \vdots & & & \\ & & \mathcal{Q}_{k-1} & s\mathcal{Q}_{k-1} M_k & & & \\ & & & \mathcal{Q}_0 (M_{k+1} \cdots M_r)^{-1} & & & \\ & & & \mathcal{Q}_{k+1} M_{k+1}^{-1} & \mathcal{Q}_{k+1} & & \\ & & & \vdots & & \ddots & \\ & & & \mathcal{Q}_r (M_{k+1} \cdots M_r)^{-1} & & & \mathcal{Q}_r \end{pmatrix}, \quad \mathcal{S} = \begin{pmatrix} \mathcal{S}_1 & & & sM_2 \cdots M_r \mathcal{S}_0 & & & \\ & \ddots & & \vdots & & & \\ & & \mathcal{S}_{k-1} & sM_k \cdots M_r \mathcal{S}_0 & & & \\ & & & M_{k+1} \cdots M_r \mathcal{S}_0 & & & \\ & & & M_{k+2} \cdots M_r \mathcal{S}_0 & \mathcal{S}_{k+1} & & \\ & & & \vdots & & \ddots & \\ & & & \mathcal{S}_0 & & & \mathcal{S}_r \end{pmatrix}, \tag{4.26}$$

we obtain the monodromy matrices $(\widehat{N}_1,\dots,\widehat{N}_r)=(\mathcal{Q}N_1\mathcal{S},\dots,\mathcal{Q}N_r\mathcal{S})$ corresponding to $\mathrm{MC}_{\lambda s^{-1}}\circ\mathrm{Add}_{(1,\dots,s,\dots,1)}(\boldsymbol{M})$ as

$$\widehat{N}_i=\begin{pmatrix}1&&&&\\&\ddots&&&\\ \lambda s^{-1}M_{i1}&\cdots&\lambda s^{-1}M_{ii}&\cdots&M_{ir}\\&&&\ddots&\\&&&&1\end{pmatrix}\quad(i\neq k)\tag{4.27}$$

with the (i,k)-block replaced by $\mathcal{Q}_i\mathcal{P}_0$, and

$$\widehat{N}_k-1=\begin{pmatrix}-s\mathcal{Q}_1M_2\cdots M_k\\ \vdots\\ -s\mathcal{Q}_{k-1}M_k\\ \mathcal{Q}_0(M_{k+1}\cdots M_r)^{-1}\\ -\mathcal{Q}_{k+1}M_{k+1}^{-1}\\ \vdots\\ -\mathcal{Q}_r(M_{k+1}\cdots M_r)^{-1}\end{pmatrix}\begin{pmatrix}\lambda s^{-1}\mathcal{P}_1&\cdots&\mathcal{P}_0&\cdots&\mathcal{P}_r\end{pmatrix}.\tag{4.28}$$

We remark here that the i-th rows of $M_0^{(k)}$ and $(M_0^{(k)})^{-1}$ are given respectively as follows:

$$\begin{aligned}(M_0^{(k)})_i&=\begin{cases}(\lambda M_i\cdots M_rM_1\cdots M_k)_i&(k+1\le i\le r)\\(\lambda M_i\cdots M_k)_i&(1\le i\le k),\end{cases}\\(M_0^{(k)})_i^{-1}&=\begin{cases}(\lambda^{-1}M_i^{-1}\cdots M_{k+1}^{-1})_i&(k+1\le i\le r)\\(\lambda^{-1}M_i^{-1}\cdots M_1^{-1}M_r^{-1}\cdots M_{k+1}^{-1})_i&(1\le i\le k).\end{cases}\end{aligned}\tag{4.29}$$

Using these relations (4.29), we rewrite $\widehat{N}_k-1$ as

$$\widehat{N}_k-1=\begin{pmatrix}-s\big(\lambda^{-1}M_{ij}^{(k)}-\delta_{ij}\big)_{\substack{i=1,\dots,k-1\\ j=1,\dots,r}}\\ \mathcal{Q}_0^{(k)}\\ -\big(\delta_{ij}-\lambda\widetilde{M}_{ij}^{(k)}\big)_{\substack{i=k+1,\dots,r\\ j=1\dots r}}\end{pmatrix}\begin{pmatrix}\lambda s^{-1}\mathcal{P}_1&\cdots&\mathcal{P}_0&\cdots&\mathcal{P}_r\end{pmatrix},\tag{4.30}$$

where $(M_0^{(k)})^{-1}=(\widetilde{M}_{ij}^{(k)})_{i,j=1}^r$. Noting that $M_{ij}^{(k)}-\delta_{ij}-M_{ik}^{(k)}(M_{kk}^{(k)}-1)^{-1}M_{kj}^{(k)}=\xi_i\eta_j$, we set

$$\mathcal{G}=\begin{pmatrix}1&&&\lambda^{-1}s(\mathcal{P}_0^{(k)})_1&&&\\&\ddots&&\vdots&&&\\&&1&\lambda^{-1}s(\mathcal{P}_0^{(k)})_{k-1}&&&\\&&&1&&&\\&&&\lambda(M_0^{(k)})_{k+1}^{-1}\mathcal{P}_0^{(k)}&1&&\\&&&\vdots&&\ddots&\\&&&\lambda(M_0^{(k)})_r^{-1}\mathcal{P}_0^{(k)}&&&1\end{pmatrix},\tag{4.31}$$

where $(\mathcal{P}_0^{(k)})_i$ stands for the i-th row of $\mathcal{P}_0^{(k)}$. Using this matrix $\mathcal{G}$ for the conjugation, from $\widehat{\boldsymbol{N}} = (\widehat{N}_1, \ldots, \widehat{N}_r)$ we obtain the following monodromy matrices of Okubo type $\boldsymbol{M}^{mc} = (M_1^{mc}, \ldots, M_r^{mc}) = \mathrm{Add}_{(1,\ldots,\lambda^{-1},\ldots,1)}(\mathrm{Ad}(\mathcal{G})\widehat{\boldsymbol{N}})$:

$$M_i^{mc} = \begin{pmatrix} 1 & & & & & & \\ & \ddots & & & & & \\ \lambda s^{-1} M_{i1} & \cdots & \lambda s^{-1} M_{ii} & \cdots & \begin{pmatrix} M_{i(k1)}^{mc} & M_{i(k2)}^{mc} \end{pmatrix} & \cdots & M_{ir} \\ & & & \ddots & & & \\ & & & & & & 1 \end{pmatrix} \quad (i < k)$$

$$M_i^{mc} = \begin{pmatrix} 1 & & & & & & \\ & & & \ddots & & & \\ \lambda s^{-1} M_{i1} \cdots & \begin{pmatrix} M_{i(k1)}^{mc} & M_{i(k2)}^{mc} \end{pmatrix} & \cdots & \lambda s^{-1} M_{ii} & \cdots & M_{ir} \\ & & & & \ddots & \\ & & & & & 1 \end{pmatrix} \quad (k < i) \tag{4.32}$$

$$M_k^{mc} = \begin{pmatrix} 1 & & & & & & \\ & \ddots & & & & & \\ \lambda s^{-1} M_{k1} & \cdots & M_{kk} & 0 & \cdots & M_{kr} \\ s^{-1}\eta_1 & \cdots & 0 & \lambda^{-1} & \cdots & \lambda^{-1}\eta_r \\ & & & & \ddots & \\ & & & & & 1 \end{pmatrix}$$

where $M_{i(k1)}^{mc}$ and $M_{i(k2)}^{mc}$ are computed as follows:

$$\begin{aligned} M_{i(k1)}^{mc} &= M_{ik} + (1-\lambda^{-1}s)\Big(\sum_{j=i+1}^{k-1} M_{ij} M_{jr}^{(k)} (M_{kk}^{(k)} - 1)^{-1} - M_{ir}^{(k)} (M_{kk}^{(k)} - 1)^{-1}\Big) \\ &= M_{ik} - (1-\lambda^{-1}s) M_{ik} M_{kk}^{(k)} (M_{kk}^{(k)} - 1)^{-1} \\ &= \lambda^{-1} s M_{ik} (M_{kk}^{(k)} - \lambda s^{-1})(M_{kk}^{(k)} - 1)^{-1}, \\ M_{i(k2)}^{mc} &= (1-\lambda^{-1}s)\Big(\sum_{j=i+1}^{k-1} M_{ij}\xi_j - \xi_i\Big) \end{aligned} \tag{4.33}$$

for $i < k$, and

$$\begin{aligned} M_{i(k1)}^{mc} &= M_{ik} - \lambda(1-\lambda s^{-1})\Big(\sum_{j=k+1}^{i} M_{ij} \widetilde{M}_{jr}^{(k)} (M_{kk}^{(k)} - 1)^{-1}\Big) \\ &= M_{ik} + \lambda(1-\lambda s^{-1}) M_{ik} (M_{kk}^{(k)} - 1)^{-1} \\ &= M_{ik} (M_{kk}^{(k)} - \lambda s^{-1})(M_{kk}^{(k)} - 1)^{-1}, \\ M_{i(k2)}^{mc} &= \lambda(1-\lambda s^{-1}) \sum_{j=k+1}^{i} M_{ij}\Big(\sum_{p=1, p\neq k}^{r} \widetilde{M}_{jp}^{(k)} \xi_p\Big). \end{aligned} \tag{4.34}$$

for $i > k$. Note that $\boldsymbol{M}^{mc}$ is the r-tuple of monodromy matrices for the fundamental solution matrix $Y^{mc}(x) = GQ(x-t_k)^{\rho} Z(x) \mathcal{S}\mathcal{G}^{-1}$, where $Z(x) = I^{-\rho-c}((x-t_k)^c \Psi(x))$. Therefore we need to specify the right inverse $\mathcal{S}\mathcal{G}^{-1}$ of $\mathcal{G}\mathcal{Q}$ to solve the connection problem. In this case the matrices $\mathcal{G}\mathcal{Q}$ and $\mathcal{S}\mathcal{G}^{-1}$ are given as

$$\mathcal{G}\mathcal{Q} = \begin{pmatrix} (\mathcal{Q}_i \delta_{ij})_{i,j=1}^{k-1} & (s(1-\lambda^{-1}) I_{n_i} \delta_{ij})_{\substack{i=1,\dots,k-1 \\ j=1,\dots,r}} & 0 \\ 0 & Q_0^{(k)} & 0 \\ 0 & ((\lambda-1)\delta_{ij})_{\substack{i=k+1,\dots,r \\ j=1,\dots,r}} & (\mathcal{Q}_i \delta_{ij})_{i,j=k+1}^{r} \end{pmatrix}, \tag{4.35}$$

$$\mathcal{S}\mathcal{G}^{-1} = \begin{pmatrix} \mathcal{S}_1 & & (s(\lambda^{-1}-1)\mathcal{S}_1, 0)\mathcal{S}_0^{(k)} & & \\ & \ddots & \vdots & & \\ & & \mathcal{S}_{k-1} \; (0, s(\lambda^{-1}-1)\mathcal{S}_{k-1}, 0)\mathcal{S}_0^{(k)} & & \\ & & \mathcal{S}_0^{(k)} & & \\ & & (0, (1-\lambda)\mathcal{S}_{k+1}, 0)\mathcal{S}_0^{(k)} & \mathcal{S}_{k+1} & \\ & & \vdots & & \ddots \\ & & (0, (1-\lambda)\mathcal{S}_r)\mathcal{S}_0^{(k)} & & & \mathcal{S}_r \end{pmatrix}. \tag{4.36}$$

We remark that one can take $\mathcal{S}_0^{(k)}$ in the form

$$\mathcal{S}_0^{(k)} = \begin{pmatrix} 0 & \widetilde{\eta}_1 \\ \vdots & \vdots \\ 0 & \widetilde{\eta}_{k-1} \\ (M_{kk}^{(k)} - 1)^{-1} & 0 \\ 0 & \widetilde{\eta}_{k+1} \\ \vdots & \vdots \\ 0 & \widetilde{\eta}_r \end{pmatrix}. \tag{4.37}$$

Summarizing these arguments, we conclude:

Proposition 4.3. *The operation* (4.20) *gives rise to the r-tuple of monodromy matrices of Okubo type $\boldsymbol{M}^{mc}$ as in* (4.32), (4.33)*. Furthermore $\boldsymbol{M}^{mc}$ is realized as the monodromy matrices for the fundamental solution matrix $Y^{mc}(x) = GQ(x-t_k)^{\rho} Z(x) \mathcal{S}\mathcal{G}^{-1}$, where $Z(x) = I^{-\rho-c}((x-t_k)^c \Psi(x))$.*

4.3 Recurrence relations for the connection coefficients

We consider the Okubo system (4.15) obtained from the Okubo system (1.2) by the operation (4.1). Let $\Psi^{mc}(x) = (\Psi_1^{mc}(x), \dots, \Psi_{(k1)}^{mc}(x), \Psi_{(k2)}^{mc}(x), \dots, \Psi_r^{mc}(x))$ be the canonical solution matrix of (4.15). Then the monodromy matrices Γ_i^{mc} $(i = 1, \dots, r)$ corresponding to (1.17) are represented as

$$\Gamma_i^{mc} = \begin{pmatrix} 1 & & & & \\ & \ddots & & & \\ (e(A_{ii}-\rho-c)-1)C_{i1}^{mc} & \cdots & e(A_{ii}-\rho-c) & \cdots & (e(A_{ii}-\rho-c)-1)C_{ir}^{mc} \\ & & & \ddots & \\ & & & & 1 \end{pmatrix} \quad (i \neq k) \tag{4.38}$$

where $C_{ik}^{mc} = (C_{i(k1)}^{mc}, C_{i(k2)}^{mc})$, and

$$\Gamma_k^{mc} = \begin{pmatrix} 1 & & & & & & \\ & \ddots & & & & & \\ (e(A_{kk})-1)C_{(k1)1}^{mc} & \cdots & e(A_{kk}) & 0 & \cdots & (e(A_{kk})-1)C_{(k1)r}^{mc} & \\ (e(\rho)-1)C_{(k2)1}^{mc} & \cdots & 0 & e(\rho) & \cdots & (e(\rho)-1)C_{(k2)r}^{mc} & \\ & & & & \ddots & & \\ & & & & & 1 & \end{pmatrix}. \tag{4.39}$$

We now investigate the relation between C_{ij} and $C_{ij}^{mc}, C_{(k1)j}^{mc}$. From the forms of (4.32), the fundamental solution matrix $Y^{mc}(x) = GQ(x-t_k)^\rho Z(x)\mathcal{S}\mathcal{G}^{-1}$ is transformed into $\Psi^{mc}(x) = Y^{mc}(x)R^{-1}$ by some block diagonal matrix $R = (R_1, \ldots, R_{(k1)}, R_{(k2)}, \ldots, R_r)$. Therefore we have recurrence relations

$$\begin{aligned} C_{ij}^{mc} &= \begin{cases} (e(A_{ii}-\rho-c)-1)^{-1}R_i(e(A_{ii})-1)\lambda s^{-1}C_{ij}R_j^{-1} & (j<i;\, i,j\neq k) \\ (e(A_{ii}-\rho-c)-1)^{-1}R_i(e(A_{ii})-1)C_{ij}R_j^{-1} & (i<j;\, i,j\neq k) \end{cases} \\ C_{i(k1)}^{mc} &= \begin{cases} (e(A_{ii}-\rho-c)-1)^{-1}R_i(e(A_{ii})-1)C_{ik}(e(A_{kk}+c)-1)(e(A_{kk}-\rho)-1)^{-1}R_{(k1)}^{-1} & (i<k) \\ (e(A_{ii}-\rho-c)-1)^{-1}R_i(e(A_{ii})-1)\lambda s^{-1}C_{ik}(e(A_{kk}+c)-1)(e(A_{kk}-\rho)-1)R_{(k1)}^{-1} & (k<i) \end{cases} \\ C_{(k1)j}^{mc} &= \begin{cases} R_{(k1)}\lambda s^{-1}C_{kj}R_j^{-1} & (j<k) \\ R_{(k1)}C_{kj}R_j^{-1} & (k<j). \end{cases} \end{aligned} \tag{4.40}$$

The new connection coefficients $C_{(k2)j}, C_{i(k2)}$ are also given as

$$\begin{aligned} C_{(k2)j}^{mc} &= \begin{cases} R_{(k2)}s^{-1}\eta_j R_j^{-1} & (j<k) \\ R_{(k2)}\lambda^{-1}\eta_j R_j^{-1} & (k<j) \end{cases} \\ C_{i(k2)}^{mc} &= \begin{cases} (e(A_{ii}-\rho-c)-1)^{-1}R_i(1-e(\rho+c))(\sum_{j=i+1}^{k-1}C_{ij}\xi_j - \xi_i)R_{(k2)}^{-1} & (i<k) \\ (e(A_{ii}-\rho-c)-1)^{-1}R_i e(-\rho)(1-e(-\rho-c))(\sum_{j=k+1}^{r}C_{ij}\sum_{p=1,p\neq k}^{r}\widetilde{C}_{jp}^{(k)}\xi_p)R_{(k2)}^{-1} & (k<i). \end{cases} \end{aligned} \tag{4.41}$$

Looking at the diagonal blocks of the solution matrix $Z(x)$ and $GQ, \mathcal{S}\mathcal{G}^{-1}$, we can compute R_i as follows:

$$\begin{aligned} R_i &= A_{ii}\lim_{x\to t_i}(x-t_i)^{-A_{ii}+\rho+c}(x-t_k)^\rho\int_{L_i}(x-u)^{-\rho-c}(u-t_k)^c\Psi_i(u)\frac{du}{u-t_i}(e(A_{ii})-1)^{-1} \\ &= A_{ii}(t_i-t_k)^{\rho+c}\widetilde{B}(A_{ii},-\rho-c+1)(e(A_{ii})-1)^{-1} \\ R_{(k1)} &= (A_{kk}-\rho)\lim_{x\to t_k}(x-t_k)^{-A_{kk}+\rho}\int_{L_i}(x-u)^{-\rho-c}(u-t_k)^c\Psi_k(u)\frac{du}{u-t_k}(e(A_{kk}-\rho)-1)^{-1} \\ &= (A_{kk}-\rho)\widetilde{B}(A_{kk}+c,-\rho-c+1)(e(A_{kk}-\rho)-1)^{-1} \end{aligned}. \tag{4.42}$$

Using these formulas, the recurrence relations for the connection coefficients are rewritten as

$$\begin{aligned} C^{mc}_{ij} &= (e(A_{ii}-\rho-c)-1)^{-1}A_{ii}(t_i-t_k)^{\rho+c}\widetilde{B}(A_{ii},-\rho-c+1)(e(-\rho-c))^{\delta(i<j)}C_{ij} \\ &\quad\cdot(e(A_{jj})-1)(\widetilde{B}(A_{jj},-\rho-c+1))^{-1}(t_j-t_k)^{-c-\rho}A_{jj}^{-1} \\ &= \begin{cases} \left(\dfrac{t_i-t_k}{t_j-t_k}\right)^{\rho+c}\dfrac{e(\frac{-1}{2}(\rho+c))\Gamma(\rho+c-A_{ii})}{\Gamma(-A_{ii})}C_{ij}\dfrac{\Gamma(A_{jj}-\rho-c+1)}{\Gamma(A_{jj}+1)} & (j<i;\, i,j\neq k) \\ \left(\dfrac{t_i-t_k}{t_j-t_k}\right)^{\rho+c}\dfrac{e(\frac{1}{2}(\rho+c))\Gamma(\rho+c-A_{ii})}{\Gamma(-A_{ii})}C_{ij}\dfrac{\Gamma(A_{jj}-\rho-c+1)}{\Gamma(A_{jj}+1)} & (i<j;\, i,j\neq k), \end{cases} \end{aligned} \tag{4.43}$$

where $\delta(i<j)=1$ if $i<j$ and $\delta(i<j)=0$ otherwise. Similarly,

$$C^{mc}_{i(k1)} = \begin{cases} (t_i-t_k)^{\rho+c}e(\frac{1}{2}(\rho+c))\dfrac{\Gamma(\rho+c-A_{ii})}{\Gamma(-A_{ii})}C_{ik}\dfrac{\Gamma(A_{kk}-\rho)}{\Gamma(A_{kk}+c)} & (i<k) \\ (t_i-t_k)^{\rho+c}e(\frac{-1}{2}(\rho+c))\dfrac{\Gamma(\rho+c-A_{ii})}{\Gamma(-A_{ii})}C_{ik}\dfrac{\Gamma(A_{kk}-\rho)}{\Gamma(A_{kk}+c)} & (k<i) \end{cases} \tag{4.44}$$

$$C^{mc}_{(k1)j} = \begin{cases} -e(\frac{-1}{2}(\rho+c))(t_j-t_k)^{-\rho-c}\dfrac{\Gamma(1+\rho-A_{kk})}{\Gamma(1-A_{kk}-c)}C_{kj}\dfrac{\Gamma(A_{jj}-\rho-c+1)}{\Gamma(A_{jj}+1)} & (j<k) \\ -e(\frac{1}{2}(\rho+c))(t_j-t_k)^{-\rho-c}\dfrac{\Gamma(1+\rho-A_{kk})}{\Gamma(1-A_{kk}-c)}C_{kj}\dfrac{\Gamma(A_{jj}-\rho-c+1)}{\Gamma(A_{jj}+1)} & (k<j) \end{cases} \tag{4.45}$$

Theorem 4.4. *The connection coefficients C^{mc}_{ij} and $C^{mc}_{i(k1)}$, $C^{mc}_{(k1)j}$ $(i,j\neq k)$ for the canonical solution matrix $\Psi^{mc}(x)$ are determined from the connection coefficients C_{ij} for $\Psi(x)$ by the recurrence formulas* (4.43) *and* (4.44), (4.45) *respectively.*

It seems that the matrices $R_{(k2)}$ and $C^{mc}_{(k2)}$ cannot be written as products of gamma functions in general, while they can be described as the limits as $x\to t_k$ of certain explicit integrals. In this paper we do not go into the detail of the description of $R_{(k2)}$ and $C^{mc}_{(k2)}$, since they can be determined by symmetry arguments in the context of the Okubo systems in Yokoyama's list that will be discussed below.

5 Connection coefficients for the Okubo systems of Yokoyama's list

5.1 Construction of canonical forms of the Okubo systems

In what follows, we use the symbols $(\mathrm{I})_n$, $(\mathrm{II})_{2n}$, $(\mathrm{III})_{2n+1}$ $(\mathrm{IV})_6$, and $(\mathrm{I}^*)_n$, $(\mathrm{II}^*)_{2n}$, $(\mathrm{III}^*)_{2n+1}$, $(\mathrm{IV}^*)_6$ to refer to the corresponding tuples $\boldsymbol{A}$ of residue matrices as in (1.4). As for the nontrivial eigenvalues, we also use the notations $\alpha^{(l)}_i=\alpha_i$, $\beta^{(l)}_i=\beta_i$, $\rho^{(l)}_i=\rho_i$ in order to specify the rank $l=n,n,2n,2n+1$ of the Okubo system.

Case I: We first construct $(\mathrm{I})_2=(\mathrm{II})_2$ by $\mathrm{mc}_{\mu_1}(\alpha^{(1)}_1,\beta^{(1)}_1)$ from the differential equation

$$\frac{dy}{dx}=\left(\frac{\alpha^{(1)}_1}{x-t_1}+\frac{\beta^{(1)}_1}{x-t_2}\right)y \tag{5.1}$$

of rank 1. The resulting system is given by

$$(x-T)\frac{d}{dx}Y = AY;\quad A = \begin{pmatrix} \alpha_1^{(1)}+\mu_1 & \beta_1^{(1)} \\ \alpha_1^{(1)} & \beta_1^{(1)}+\mu_1 \end{pmatrix} = \begin{pmatrix} \alpha_1^{(2)} & \beta_1^{(2)}-\rho_1^{(2)} \\ \alpha_1^{(2)}-\rho_1^{(2)} & \beta_1^{(2)} \end{pmatrix}. \tag{5.2}$$

The canonical form of type $(\mathrm{I})_2$ is obtained as the conjugation of A by the matrix

$$G = \begin{pmatrix} 1 & 0 \\ 0 & \beta_1^{(2)}-\rho_1^{(2)} \end{pmatrix} \tag{5.3}$$

Starting with $(\mathrm{I})_2$, the Okubo systems $(\mathrm{I})_n$ can be constructed inductively by the Katz operations

$$(\mathrm{I})_{n+1} = \mathrm{add}_{(\rho,0)} \circ \mathrm{mc}_{-c-\rho} \circ \mathrm{add}_{(c,0)} (\mathrm{I})_n \quad (n \geq 2). \tag{5.4}$$

Lemma 5.1. *For the system* (2.7) *of type* $(\mathrm{I})_n$, $\alpha_n - \rho - A_{21}(\alpha-\rho)^{-1}A_{12}$ *is a matrix of rank 1. It is expressed as*

$$\alpha_n - \rho_2 - A_{21}(\alpha-\rho)^{-1}A_{12} = \xi\eta \tag{5.5}$$

with ξ *and* η *defined by*

$$\xi = -\frac{\prod_{k=1}^{n}(\rho-\rho_k)}{\prod_{k=1}^{n-1}(\rho-\alpha_k)}, \quad \eta = 1 \tag{5.6}$$

Proof. A direct computation of $\alpha_n - \rho_2 - A_{21}(\alpha-\rho_2)^{-1}A_{12} = \xi\eta$ gives the relation

$$\xi\eta = \alpha_n - \rho + \sum_{k=1}^{n-1} \frac{1}{\alpha_k - \rho} \frac{\prod_{l=1}^{n}(\alpha_k-\rho_l)}{\prod_{l\neq k}^{n-1}(\alpha_k-\alpha_l)}. \tag{5.7}$$

By the partial fraction expansion of the function $\frac{\prod_{k=1}^{n}(x-\rho_k)}{\prod_{k=1}^{n-1}(x-\alpha_k)}$ and the Fuchs relation, we have

$$\xi\eta = \alpha_n + \sum_{k=1}^{n-1}\alpha_k - \sum_{k=1}^{n}\rho_k - \frac{\prod_{k=1}^{n}(\rho-\rho_k)}{\prod_{k=1}^{n-1}(\rho-\alpha_k)} = -\frac{\prod_{k=1}^{n}(\rho-\rho_k)}{\prod_{k=1}^{n-1}(\rho-\alpha_k)}. \tag{5.8}$$

□

We show how the canonical form (2.7) of the Okubo system $(\mathrm{I})_{n+1}$ is obtained from the canonical form (2.7) of $(\mathrm{I})_n$ by the operation of (5.4). By Lemma 4.2 the Katz operation (5.4) for $(\mathrm{I})_n$ gives rise to the Okubo system $(x-T)\frac{d}{dx}W = A^{mc}W$ where

$$A^{mc} = \begin{pmatrix} \alpha & 0 & K \\ 0 & \rho & \eta \\ L(\alpha+c)(\alpha-\rho)^{-1} & (\rho+c)\xi & \alpha_n \end{pmatrix}. \tag{5.9}$$

This matrix A^{mc} is precisely the canonical form of Okubo system $(\mathrm{I})_{n+1}$ with characteristic exponents $(\alpha_i^{(n+1)})_{i=1}^{n}$, $(\rho_i^{(n+1)})_{i=1}^{n}$ specified by

$$\begin{cases} \alpha_i^{(n+1)} = \alpha_i \quad (1 \leq i \leq n-1), \quad \alpha_n^{(n+1)} = \rho \\ \rho_i^{(n+1)} = \rho_i, \quad \rho_n^{(n+1)} = -c. \end{cases} \tag{5.10}$$

Cases II **and** III: The Okubo system $(\mathrm{III})_3$ as in (2.13) is obtained from the system (5.2) by $\mathrm{add}_{(\rho,0)} \circ \mathrm{mc}_{-a_1-\rho} \circ \mathrm{add}_{(c,0)}$ with

$$\begin{cases} \alpha_1^{(3)} = \alpha_1^{(2)} - a_1 - \rho, \quad \alpha_2^{(3)} = \rho \\ \beta_1^{(3)} = \beta_1^{(2)} - \rho - a_1, \\ \rho_1^{(3)} = -a_1, \quad \rho_2^{(3)} = \rho_1^{(2)}, \quad \rho_3^{(3)} = \alpha_1^{(2)} + \beta_1^{(2)} - \rho_1^{(2)}. \end{cases} \tag{5.11}$$

The Okubo systems $(\mathrm{II})_{2n}$ and $(\mathrm{III})_{2n+1}$ can be constructed inductively by the following Katz operations:

$$\begin{aligned} (\mathrm{II})_{2n} &= \mathrm{add}_{(0,\rho_2^{(2n-1)})} \circ \mathrm{mc}_{-b_n-\rho_2^{(2n-1)}} \circ \mathrm{add}_{(0,b_{n-1})} (\mathrm{III})_{2n-1} \quad (n \geq 2). \\ (\mathrm{III})_{2n+1} &= \mathrm{add}_{(\rho_2^{(2n)},0)} \circ \mathrm{mc}_{-a_n-\rho_2^{(2n)}} \circ \mathrm{add}_{(a_n,0)} (\mathrm{II})_{2n}, \quad (n \geq 2) \end{aligned} \tag{5.12}$$

Lemma 5.2. (1) *For the system* (2.12) *of type* $(\mathrm{II})_{2n}$, $\beta - \rho_2 - A_{21}(\alpha - \rho_2)^{-1}A_{12}$ *is a matrix of rank 1. It is expressed as*

$$\beta - \rho_2 - A_{21}(\alpha - \rho_2)^{-1}A_{12} = \xi\eta \tag{5.13}$$

with a column vector ξ *and a row vector* η *defined by*

$$\xi_i = (\rho_2 - \rho_1)\frac{\prod_{k\neq i\, 1\leq k\leq n}(\beta_k - \rho_1)}{\prod_{k=1}^n(\rho_2 - \alpha_k)}, \quad \eta_j = \frac{\prod_{k=1}^n(\beta_j + \alpha_k - \rho_1 - \rho_2)}{\prod_{k\neq j\, 1\leq k\leq n}(\beta_j - \beta_k)}. \tag{5.14}$$

(2) *For the system* (2.13) *of type* $(\mathrm{III})_{2n+1}$, $\alpha - \rho_2 - A_{12}(\beta - \rho_2)^{-1}A_{21}$ *is a matrix of rank 1. It is expressed as*

$$\alpha - \rho_2 - A_{12}(\beta - \rho_2)^{-1}A_{21} = \xi\eta \tag{5.15}$$

with a column vector ξ *and a row vector* η *defined by*

$$\xi_i = \frac{\prod_{k\neq i,\, 1\leq k\leq n+1}(\alpha_k - \rho_1)}{\prod_{k=1}^n(\rho_2 - \beta_k)}, \quad \eta_j = (\alpha_j - \rho_2)\frac{\prod_{1\leq k\leq n}(\beta_k + \alpha_j - \rho_1 - \rho_2)}{\prod_{k\neq j,\, 1\leq k\leq n+1}(\alpha_j - \alpha_k)}. \tag{5.16}$$

Proof. For the canonical form of the Okubo system type $(\mathrm{II})_{2n}$, the values of $\xi_i\eta_j$ are also computed as follows by partial fraction expansions:

$$\begin{aligned} \xi_i\eta_j &= (\beta_i - \rho_2)\delta_{ij} - \sum_{k=1}^n \frac{\alpha_k - \rho_1}{\alpha_k - \rho_2}\frac{\prod_{p\neq i}^n(\alpha_k + \beta_p - \rho_1 - \rho_2)}{\prod_{p\neq k}^n(\alpha_k - \alpha_p)}(\beta_j - \rho_1)\frac{\prod_{p\neq k}^n(\beta_j + \alpha_p - \rho_1 - \rho_2)}{\prod_{p\neq j}^n(\beta_j - \beta_p)} \\ &= (\beta_i - \rho_2)\delta_{ij} - \frac{(\beta_j - \rho_1)}{\prod_{k\neq j}^n(\beta_j - \beta_k)}\sum_{k=1}^n \frac{\alpha_k - \rho_1}{\alpha_k - \rho_2}\frac{\prod_{p\neq i}^n(\alpha_k + \beta_p - \rho_1 - \rho_2)}{\prod_{p\neq k}^n(\alpha_k - \alpha_p)}\prod_{p\neq k}^n(\beta_j + \alpha_p - \rho_1 - \rho_2) \\ &= (\beta_i - \rho_2)\delta_{ij} - \frac{(\beta_j - \rho_1)\prod_{k=1}^n(\beta_j + \alpha_k - \rho_1 - \rho_2)}{\prod_{k\neq j}^n(\beta_j - \beta_k)}\sum_{k=1}^n \frac{\alpha_k - \rho_1}{\alpha_k - \rho_2}\frac{\prod_{p\neq i,j}^n(\alpha_k + \beta_p - \rho_1 - \rho_2)}{\prod_{p\neq k}^n(\alpha_k - \alpha_p)} \\ &= (\beta_i - \rho_2)\delta_{ij} - \frac{(\beta_j - \rho_1)\prod_{k=1}^n(\beta_j + \alpha_k - \rho_1 - \rho_2)}{\prod_{k\neq j}^n(\beta_j - \beta_k)}\sum_{k=1}^n \frac{\prod_{p\neq i,j}^{n+1}(\alpha'_k + \beta'_p - \rho'_1 - \rho'_2)}{\prod_{p\neq k}^{n+1}(\alpha'_k - \alpha'_p)} \\ &= \frac{(\beta_j - \rho_1)\prod_{k=1}^n(\beta_j + \alpha_k - \rho_1 - \rho_2)}{\prod_{k\neq j}^n(\beta_j - \beta_k)}\frac{\prod_{k\neq i,j}^{n+1}(\alpha'_{n+1} + \beta'_k - \rho'_1 - \rho'_2)}{\prod_{k\neq n+1}^{n+1}(\alpha'_{n+1} - \alpha'_k)} \\ &= (\rho_2 - \rho_1)\frac{(\beta_j - \rho_1)\prod_{k=1}^n(\beta_j + \alpha_k - \rho_1 - \rho_2)}{\prod_{k\neq j}^n(\beta_j - \beta_k)}\frac{\prod_{k\neq i,j}^n(\beta_k - \rho_1)}{\prod_{k=1}^n(\rho_2 - \alpha_k)} \end{aligned} \tag{5.17}$$

where $\alpha_i = \alpha'_i$, $\beta_i = \beta'_i + \alpha'_{n+1} - \beta'_{n+1}$, $\rho_1 = \rho'_1 + \rho'_2 - \beta'_{n+1}$, $\rho_2 = \alpha'_{n+1}$ Therefore we can choose ξ_i and η_j as

$$\xi_i = (\rho_2 - \rho_1)\frac{\prod_{k\neq i}^{n-1}(\beta_k - \rho_1)}{\prod_{k=1}^{n-1}(\rho_2 - \alpha_k)}, \quad \eta_j = \frac{\prod_{k=1}^{n-1}(\beta_j + \alpha_k - \rho_1 - \rho_2)}{\prod_{k\neq j}^{n-1}(\beta_j - \beta_k)}, \tag{5.18}$$

For the canonical form of type $(\mathrm{III})_{2n+1}$, the values of ξ_i and η_j are computed similarly. □

From $(\mathrm{III})_{2n-1}$ to $(\mathrm{II})_{2n}$

We show how the canonical form (2.12) of the Okubo system $(\mathrm{II})_{2n}$ is obtained from the canonical form (2.13) of $(\mathrm{III})_{2n-1}$ by the operation of (5.12). By Lemma 4.2 the Katz operation (5.12) for $(\mathrm{III})_{2n-1}$ gives rise to the Okubo system $(x-T)\frac{d}{dx}W = A^{mc}W$ where

$$A^{mc} = \begin{pmatrix} \alpha - \rho_2 - b_n & K(\beta + b_{n-1})(\beta - \rho_2)^{-1} & (\rho_2 + b_{n-1})\xi \\ L & \beta & 0 \\ \eta & 0 & \rho_2 \end{pmatrix}. \tag{5.19}$$

This matrix A^{mc} is precisely the canonical form of Okubo system $(\mathrm{II})_{2n}$ with characteristic exponents $(\alpha_i^{(2n)})_{i=1}^n$, $(\beta_i^{(2n)})_{i=1}^n$, $(\rho_i^{(2n)})_{i=1}^3$ specified by

$$\begin{cases} \alpha_i^{(2n)} = \alpha_i - \rho_2 - b_{n-1} \quad (1 \le i \le n), \\ \beta_i^{(2n)} = \beta_i \quad (1 \le i \le n), \quad \beta_{n+1}^{(2n)} = \rho_2, \\ \rho_1^{(2n)} = -b_{n-1}, \quad \rho_2^{(2n+1)} = \rho_1, \quad \rho_3^{(2n+1)} = \rho_3. \end{cases} \tag{5.20}$$

From $(\mathrm{II})_{2n}$ to $(\mathrm{III})_{2n+1}$

Similarly one can construct the canonical form (2.12) of the Okubo system $(\mathrm{II})_{2n+2}$ from $(\mathrm{III})_{2n+1}$ by the middle convolution. From Lemma 4.2, the Schlesinger system of (5.12) for $(\mathrm{III})_{2n+1}$ is given by the Okubo system $(x-T)\frac{d}{dx}W = A^{mc}W$ of type $(\mathrm{II})_{2n+2}$, where

$$A^{mc} = \begin{pmatrix} \alpha & 0 & K \\ 0 & \rho_2 & \eta \\ L(\alpha + a_n)(\alpha - \rho_2)^{-1} & (\rho_2 + a_n)\xi & \beta - a_n - \rho_2 \end{pmatrix}. \tag{5.21}$$

The characteristic exponents $(\alpha_i^{(2n+1)})_{i=1}^{n+1}$, $(\beta_i^{(2n+1)})_{i=1}^n$, $(\rho_i^{(2n+1)})_{i=1}^3$ are determined as

$$\begin{cases} \alpha_i^{(2n+1)} = \alpha_i \quad (1 \le i \le n), \quad \alpha_{n+1}^{(2n+1)} = \rho_2, \\ \beta_i^{(2n+1)} = \beta_i - \rho_2 - a_n \quad (1 \le i \le n), \\ \rho_1^{(2n+1)} = -a_n, \quad \rho_2^{(2n+1)} = \rho_1, \quad \rho_3^{(2n+1)} = \rho_3. \end{cases} \tag{5.22}$$

Note that the passage from $(\mathrm{II})_2$ to $(\mathrm{III})_3$ is also included in this procedure with ρ_2 replaced by ρ.

Case I^*: The system of type $(\mathrm{I}^*)_n$ is obtained by mc_μ with a generic parameter μ from the Schlesinger system

$$\frac{d}{dx}y = \sum_{k=1}^n \frac{\alpha_k}{x - t_k} y \tag{5.23}$$

of rank 1, where $\alpha_k \in \mathbb{C}^*$ $(k = 1, \dots, n)$. The resulting system is given by

$$(x - T)\frac{d}{dx}Z = \begin{pmatrix} \alpha_1 + \mu & \cdots & \alpha_n \\ \vdots & \ddots & \vdots \\ \alpha_1 & \cdots & \alpha_n + \mu \end{pmatrix} Z \tag{5.24}$$

Then characteristic exponents $(\alpha_i^{(n)})_{i=1}^n$ and $\rho_1^{(n)}, \rho_2^{(n)}$ are determined as

$$\alpha_i^{(n)} = \alpha_i + \mu, \quad \rho_1^{(n)} = \mu, \quad \rho_2^{(n)} = \sum_{k=1}^n \alpha_k + \mu. \tag{5.25}$$

Cases II* **and** III*: Firstly, the system $(\mathrm{III}^*)_3$ is obtained by mc_{μ_1} from the differential equation

$$\frac{dy}{dx} = \left(\frac{\alpha_1^{(1)}}{x - t_1} + \frac{\beta_1^{(1)}}{x - t_2} + \frac{\gamma_1^{(1)}}{x - t_3} \right) y. \tag{5.26}$$

The resulting system is given by

$$(x - T)\frac{d}{dx}Y = \begin{pmatrix} \alpha_1 & \beta_1 - \rho_1 & \rho_1 + \rho_2 - \alpha_1 - \beta_1 \\ \alpha_1 - \rho_1 & \beta_1 & \rho_1 + \rho_2 - \alpha_1 - \beta_1 \\ \alpha_1 - \rho_1 & \beta_1 - \rho_1 & \gamma \end{pmatrix} Y \tag{5.27}$$

with characteristic exponents specified by

$$\begin{cases} \alpha_1 = \alpha_1^{(1)} + \mu_1, \quad \beta_1 = \beta_1^{(1)} + \mu_1, \quad \gamma = \gamma_1^{(1)} + \mu_1, \\ \rho_1 = \mu_1, \quad \rho_2 = \alpha_1^{(1)} + \beta_1^{(1)} + \gamma_1^{(1)} + \mu_1. \end{cases} \tag{5.28}$$

It can be transformed into the canonical form

$$(\mathrm{III}^*)_3 = \begin{pmatrix} \alpha_1 & \beta_1 - \rho_1 & 1 \\ \alpha_1 - \rho_1 & \beta_1 & 1 \\ \frac{\alpha_1 - \rho_1}{\rho_1 + \rho_2 - \alpha_1 - \beta_1} & \frac{\beta_1 - \rho_1}{\rho_1 + \rho_2 - \alpha_1 - \beta_1} & \gamma \end{pmatrix} \tag{5.29}$$

by conjugation with the diagonal matrix

$$\mathrm{diag}(1,\, 1,\, \rho_1 + \rho_2 - \alpha_1 - \beta_1). \tag{5.30}$$

Starting from the $(\mathrm{III}^*)_3$, the Okubo systems $(\mathrm{II}^*)_{2n}$ and $(\mathrm{III}^*)_{2n+1}$ can be constructed inductively by the following Katz operations:

$$\begin{aligned} (\mathrm{II}^*)_{2n} &= \mathrm{add}_{(\rho_2,0,0)} \circ \mathrm{mc}_{-a_{n-1}-\rho_2} \circ \mathrm{add}_{(a_{n-1},0,0)} (\mathrm{III}^*)_{2n-1}, \\ (\mathrm{III}^*)_{2n+1} &= \mathrm{add}_{(0,\rho_2,0)} \circ \mathrm{mc}_{-b_n-\rho_2} \circ \mathrm{add}_{(0,b_n,0)} (\mathrm{II}^*)_{2n}. \end{aligned} \tag{5.31}$$

Therefore, by Lemma 4.2 we obtain canonical forms of the Okubo system types $(\mathrm{II}^*)_{2n}$ and $(\mathrm{III}^*)_{2n+1}$ inductively.

Lemma 5.3. (1) *For the system* (2.27) *of type* $(\mathrm{II}^*)_{2n}$*,* $\alpha-\rho_2-A_{12}(\beta-\rho_2)^{-1}A_{21}$ *and* $\gamma-\rho_2-A_{32}(\beta-\rho_2)A_{23}$ *are matrices of rank* 1*. They are expressed as*

$$\alpha-\rho_2-A_{12}(\beta-\rho_2)^{-1}A_{21}=\xi_1\eta_1,\quad \gamma-\rho_2-A_{32}(\beta-\rho_2)A_{23} \tag{5.32}$$

with column vectors ξ_1,ξ_3 *and row vectors* η_1,η_3 *defined by*

$$\begin{aligned}(\xi_1)_i&=\frac{\prod_{k\neq i}^{n}(\alpha_k-\rho_1)}{\prod_{k=1}^{n-1}(\rho_2-\beta_k)},\quad (\eta_1)_j=(\alpha_j-\rho_1)\frac{\prod_{k=1}^{n-1}(\alpha_j+\beta_k-\rho_1-\rho_2)}{\prod_{k\neq j}^{n}(\alpha_j-\alpha_k)},\\ \xi_3&=-\frac{\prod_{k=1}^{n}(\alpha_k-\rho_1)}{\prod_{k=1}^{n-1}(\rho_2-\beta_k)},\quad \eta_3=1\end{aligned} \tag{5.33}$$

(2) *For the system* (2.28) *of type* $(\mathrm{III}^*)_{2n+1}$*,* $\beta-\rho_2-A_{21}(\alpha-\rho_2)^{-1}A_{12}$ *and* $\gamma-\rho_2-A_{31}(\alpha-\rho_2)A_{13}$ *are matrices of rank* 1*. They are expressed as*

$$\beta-\rho_2-A_{21}(\alpha-\rho_2)^{-1}A_{12}=\xi_2\eta_2,\quad \gamma-\rho_2-A_{31}(\alpha-\rho_2)A_{13}=\xi_3\eta_3 \tag{5.34}$$

with column vectors ξ_2,ξ_3 *and row vectors* η_2,η_3 *defined by*

$$\begin{aligned}(\xi_2)_i&=(\rho_2-\rho_1)\frac{\prod_{k\neq i}^{n}(\beta_k-\rho_1)}{\prod_{k=1}^{n}(\rho_2-\alpha_k)},\quad (\eta_2)_j=\frac{\prod_{k=1}^{n}(\beta_j+\alpha_k-\rho_1-\rho_2)}{\prod_{k\neq j}^{n}(\beta_j-\beta_k)},\\ \xi_3&=\frac{\prod_{k=1}^{n}(\beta_k-\rho_1)}{\prod_{k=1}^{n}(\rho_2-\alpha_k)},\quad \eta_3=1\end{aligned} \tag{5.35}$$

This lemma is proved using partial fraction expansions as the case of type $(\mathrm{I})_n$ and types $(\mathrm{II})_{2n}$, $(\mathrm{III})_{2n+1}$. Assume that the system $(\mathrm{III}^*)_{2n-1}$ is given as in our canonical form of Theorem 2.5. By Theorem 4.2, the Katz operation (5.31) for $(\mathrm{III}^*)_{2n-1}$ gives rise to the Okubo system $(x-T)\frac{d}{dx}W=A^{mc}W$ where

$$A^{mc}=\begin{pmatrix}\alpha & 0 & A_{12} & A_{13}\\ 0 & \rho_2 & \eta_2 & \eta_3\\ A_{21}(\alpha+a_{n-1})(\alpha-\rho_2)^{-1} & (\rho_2+a_{n-1})\xi_2 & \beta-a_{n-1}-\rho_2 & A_{23}\\ A_{31}(\alpha+a_{n-1})(\alpha-\rho_2)^{-1} & (\rho_2+a_{n-1})\xi_3 & A_{32} & \gamma-a_{n-1}-\rho_2\end{pmatrix}. \tag{5.36}$$

Renaming the characteristic exponents of (5.36) as

$$\begin{cases}\alpha_i^{(2n)}=\alpha_i\quad(1\le i\le n-1),\quad \alpha_n^{(2n)}=\rho_2,\\ \beta_i^{(2n)}=\beta_i-\rho_2-a_{n-1}\quad(1\le i\le n),\\ \gamma^{(2n)}=\gamma-a_{n-1}-\rho_2,\quad \rho_1^{(2n)}=-a_{n-1},\quad \rho_2^{(2n)}=\rho_1,\end{cases} \tag{5.37}$$

we see that A^{mc} is in the canonical form of type $(\mathrm{II}^*)_{2n}$. Similarly, applying the Katz operation to the canonical form of $(\mathrm{II}^*)_{2n}$, we obtain the canonical form of type $(\mathrm{III}^*)_{2n+1}$.

Case IV: The Okubo system of type $(\mathrm{IV})_6$ is constructed by

$$(\mathrm{IV})_6=\mathrm{add}_{(\rho_3,0)}\circ\mathrm{mc}_{-c-\rho_3}\circ\mathrm{add}_{(c,0)}(\mathrm{III})_5. \tag{5.38}$$

One can directly verify that the the canonical form of type $(\mathrm{IV})_6$ is obtained from the canonical form of type $(\mathrm{III})_5$ of (2.18) by choosing ξ and η appropriately.

Case IV*: The Okubo system of type $(\mathrm{IV}^*)_6$ is constructed by

$$(\mathrm{IV}^*)_6=\mathrm{add}_{(0,0,\rho_1)}\circ\mathrm{mc}_{-c-\rho_1}\circ\mathrm{add}_{(0,0,c)}(\mathrm{III}^*)_5. \tag{5.39}$$

One can directly verify that the the canonical form of type $(\mathrm{IV}^*)_6$ is obtained from the canonical form of type $(\mathrm{III}^*)_5$ of (2.33) by choosing ξ and η appropriately.

5.2 Recurrence relations for the connection coefficients of Yokoyama's list

In what follows we prove our main theorems in Section 2 using recurrence relations for the connection coefficients. Before the proof of the main theorems, we give a simple remark concerning the symmetry of Okubo systems.

Lemma 5.4. *Assume that the Okubo system* (1.2) *is obtained by conjugation from an Okubo system*

$$(x-T)\frac{d}{dx}Y = BY \tag{5.40}$$

as

$$A = \mathrm{Ad}(D)B = DBD^{-1}, \quad D = \mathrm{diag}(D_1, \dots, D_r), \quad D_i \in \mathrm{GL}(n_i, \mathbb{C}) \quad (i = 1, \dots, r). \tag{5.41}$$

Let C^A_{ij} and C^B_{ij} $(1 \le i, j \le r)$ be the connection matrices for the canonical solution matrices of (1.2) *and* (5.40)*, respectively. Then we have*

$$C^A_{ij} = D_i C^B_{ij} D_j^{-1} \quad (i, j = 1, \dots, r). \tag{5.42}$$

In the context of the Okubo system in Yokoyama's list, we apply this lemma to analyze how the connection coefficients transform under the permutation of characteristic exponents.

Case I: We determine the connection coefficients $(C_{12})_i = (C^{(n)}_{12})_i$ and $(C_{21})_j = (C^{(n)}_{21})_j$ for the canonical solution matrix $\Psi^{(n)}(x)$ of the Okubo system $(\mathrm{I})_n$ in the canonical form (2.7).

Recall that the Okubo system of type $(\mathrm{I})_n$ is constructed inductively by the Katz operation (5.4). Hence, by Proposition 4.3 the canonical solution matrix $\Psi^{(n)}(x)$ of type $(\mathrm{I})_n$ is obtained inductively in the form

$$\Psi^{(n+1)}(x) = GQ(x-t_1)^{\rho} I^{-a_n-\rho}\big((x-t_1)^{a_n}\Psi^{(n)}(x)\big)\mathcal{S}\mathcal{G}R^{-1}, \tag{5.43}$$

where $R = (R_{(11)}, R_{(12)}, R_2)$. Also, by Theorem 4.4 we obtain the recurrence relations for the connection coefficients $C^{(n+1)}_i$ and $D^{(n+1)}_i$ $(i = 1, \dots, n-1)$ as follows:

$$\begin{aligned}
(C^{(n+1)}_{12})_i &= (C^{mc}_{(11)2})_i = -e(\tfrac{1}{2}(\rho + a_n))(t_2 - t_1)^{-\rho - a_n}\frac{\Gamma(1+\rho-\alpha^{(n)}_i)}{\Gamma(1-\alpha^{(n)}_i - a_n)}C^{(n)}_i\frac{\Gamma(\alpha^{(n)}_n - \rho - a_n + 1)}{\Gamma(\alpha^{(n)}_n + 1)}\\
&= -e(\tfrac{1}{2}(\alpha^{(n+1)}_n - \rho^{(n+1)}_{n+1}))(t_2-t_1)^{\rho^{(n+1)}_{n+1} - \alpha^{(n+1)}_n}\frac{\Gamma(1+\alpha^{(n+1)}_n - \alpha^{(n+1)}_i)}{\Gamma(1+\rho^{(n+1)}_{n+1} - \alpha^{(n+1)}_i)}\frac{\Gamma(\alpha^{(n+1)}_{n+1} + 1)}{\Gamma(\alpha^{(n)}_n + 1)}(C^{(n)}_{12})_i\\
(C^{(n+1)}_{21})_j &= (C^{mc}_{2(11)})_j = e(\tfrac{-1}{2}(\rho + a_n))(t_2 - t_1)^{\rho + a_n}\frac{\Gamma(\rho + a_n - \alpha^{(n)}_n)}{\Gamma(-\alpha^{(n)}_n)}(C^{(n)}_{21})_j\frac{\Gamma(\alpha^{(n)}_j - \rho)}{\Gamma(\alpha^{(n)}_j + a_n)}\\
&= e(\tfrac{-1}{2}(\alpha^{(n+1)}_n - \rho^{(n+1)}_{n+1}))(t_2-t_1)^{\alpha^{(n+1)}_n - \rho^{(n+1)}_{n+1}}\frac{\Gamma(-\alpha^{(n+1)}_{n+1})}{\Gamma(-\alpha^{(n+1)}_n)}(C^{(n)}_{21})_j\frac{\Gamma(\alpha^{(n+1)}_j - \alpha^{(n+1)}_n)}{\Gamma(\alpha^{(n+1)}_j - \rho^{(n+1)}_{n+1})}
\end{aligned} \tag{5.44}$$

Therefore, using these relations we can completely determine $(C^{(n)}_{12})_1$ and $(C^{(n)}_{21})_1$ in terms of the gamma function, if we know the initial data $(C^{(2)}_{12})_1$ and $(C^{(2)}_{21})_1$. The other connection coefficients

$(C_{12}^{(n)})_i$ and $(C_{21}^{(n)})_i$ are obtained from $(C_{12}^{(n)})_1$ and $(C_{21}^{(n)})_1$ by using symmetry of the Okubo system and Lemma 5.4. In the following we denote by σ_{ij}^{α} (resp. σ_{ij}^{β}) the operation which exchanges the parameters α_i and α_j (resp. β_i and β_j). Then the matrix A of the canonical form of type $(\mathrm{I})_n$ satisfies

$$A = \mathrm{Ad}(D)(\sigma_{ij}^{\alpha} A), \quad D = \mathrm{diag}(S_{ij}, 1) \tag{5.45}$$

where S_{ij} is the permutation matrix corresponding the transposition (ij) of matrix indices. From (5.75), applying Lemma 5.4 to $B = \sigma_{1i}^{\alpha}(A)$ we obtain

$$C_{12} = S_{1i}\sigma_{1i}^{\alpha}(C_{12})S_{1j}, \quad C_{21} = \sigma_{1i}^{\alpha}(C_{21})S_{1i}. \tag{5.46}$$

Therefore we obtain the other connection coefficients as

$$(C_{12}^{(n)})_i = \sigma_{1i}^{\alpha}(C_{12}^{(n)})_1, \quad (C_{21}^{(n)})_i = \sigma_{1i}^{\alpha}(C_{21}^{(n)})_1, \quad (i = 1, \dots, n-1). \tag{5.47}$$

The connection coefficients $(C_{12}^{(2)})_1$ and $(C_{21}^{(2)})_1$ for the Okubo system of type $(\mathrm{I})_2$ are computed as follows. Firstly, the equation and the solution of the rank 1 case are given by

$$\frac{d}{dx}Y = \left(\frac{\alpha_1}{x - t_1} + \frac{\beta_1}{x - t_2}\right)Y, \quad Y(x) = (x - t_1)^{\alpha_1}(x - t_2)^{\beta_1}. \tag{5.48}$$

Then the canonical form of Okubo system $(\mathrm{I})_2$ is given as a conjugation of (5.2). The monodromy matrices $\mathrm{MC}_\lambda(e(\alpha_1), e(\beta_1))$ are given by

$$M_1 = \begin{pmatrix} e(\alpha_1) & e(\alpha_2 - \rho_1) - 1 \\ 0 & 1 \end{pmatrix}, \quad M_2 = \begin{pmatrix} 1 & 0 \\ e(\rho_1)(e(\alpha_1 - \rho_1) - 1) & e(\alpha_2) \end{pmatrix}. \tag{5.49}$$

Then the coefficients $r_1^{(2)}$ and $r_2^{(2)}$ are computed as

$$r_1^{(2)} = (t_1 - t_2)^{\alpha_2 - \rho_1}\widetilde{B}(\alpha_1 - \rho_1, \rho_1 + 1), \quad r_2^{(2)} = (\alpha_2 - \rho_1)(t_2 - t_1)^{\alpha_1 - \rho_1}\widetilde{B}(\alpha_2 - \rho_1, \rho_1 + 1) \tag{5.50}$$

From $r_1^{(2)}$and $r_2^{(2)}$, we obtain the connection coefficients $(C_{12}^{(2)})_1$ and $(C_{21}^{(2)})_1$:

$$\begin{aligned}
(C_{12}^{(2)})_1 &= \frac{(e(\alpha_2 - \rho_1) - 1)}{(e(\alpha_1) - 1)}\frac{r_1^{(2)}}{r_2^{(2)}} \\
&= \frac{1}{\alpha_2 - \rho_1}\frac{e(\alpha_1 - \rho_1) - 1}{e(\alpha_1) - 1}\frac{(t_1 - t_2)^{\alpha_2 - \rho_1}}{(t_2 - t_1)^{\alpha_1 - \rho_1}}\frac{\Gamma(\alpha_1 - \rho_1)\Gamma(\alpha_2 + 1)}{\Gamma(\alpha_2 - \rho_1)\Gamma(\alpha_1 + 1)} \\
&= e(\tfrac{-\rho_1}{2})\frac{(t_1 - t_2)^{\rho_2 - \alpha_1}}{(t_2 - t_1)^{\alpha_1 - \rho_1}}\frac{\Gamma(-\alpha_1)\Gamma(\alpha_2 + 1)}{\Gamma(1 + \rho_2 - \alpha_1)\Gamma(1 + \rho_1 - \alpha_1)} \\
(C_{21}^{(2)})_1 &= \frac{r_2^{(2)}}{r_1^{(2)}}\frac{e(\rho_1)(e(\alpha_2 - \rho_1) - 1)}{e(\alpha_2) - 1} \\
&= e(\tfrac{\rho_1}{2})\frac{(t_2 - t_1)^{\alpha_1 - \rho_1}}{(t_1 - t_2)^{\alpha_2 - \rho_1}}\frac{\Gamma(-\alpha_2)\Gamma(\alpha_1 + 1)}{\Gamma(\alpha_1 - \rho_1)\Gamma(\alpha_1 - \rho_2)}.
\end{aligned} \tag{5.51}$$

Therefore we can obtain $(C_{12}^{(n)})_1$ from $(C_{21}^{(n)})_1$ as follows:

$$\begin{aligned}(C_{12}^{(n)})_1 =&(-1)^{n-2}\prod_{k=2}^{n-1} e(\tfrac{1}{2}(\alpha_k^{(k+1)}-\rho_{k+1}^{(k+1)}))(t_2-t_1)^{\rho_{k+1}^{(k+1)}-\alpha_k^{(k+1)}}\frac{\Gamma(1+\alpha_k^{(k+1)}-\alpha_1^{(k+1)})}{\Gamma(1+\rho_{k+1}^{(k+1)}-\alpha_1^{(k+1)})}\frac{\Gamma(\alpha_{k+1}^{(k+1)}+1)}{\Gamma(\alpha_k^{(k)}+1)}C_1^{(2)}\\ =&(-1)^n e(\tfrac{1}{2}(\rho_2^{(n)}-\alpha_1^{(n)}-\alpha_n^{(n)}))\frac{(t_1-t_2)^{\rho_2^{(n)}-\alpha_1^{(n)}}}{(t_2-t_1)^{\rho_2^{(n)}-\alpha_n^{(n)}}}\Gamma(-\alpha_1^{(n)})\Gamma(\alpha_n^{(n)}+1)\frac{\prod_{k=2}^{n-1}\Gamma(1+\alpha_k^{(n)}-\alpha_1^{(n)})}{\prod_{k=1}^{n}\Gamma(1+\rho_k^{(n)}-\alpha_1^{(n)})}\end{aligned} \tag{5.52}$$

The connection coefficient $(C_{21}^{(n)})_1$ is obtained in the same way. Furthermore, exchanging $\alpha_1^{(n)}$, $\alpha_i^{(n)}$, we obtain $(C_{12}^{(n)})$ and $(C_{21}^{(n)})_1$. This completes the proof of Theorem 2.7.

Cases II **and** III: Keeping the notation of Section 2, we consider the Okubo system of type $(\mathrm{II})_{2n}$ or $(\mathrm{III})_{2n+1}$, and set $m=n$ or $m=n+1$ respectively. We denote by $\Psi^{(m+n)}(x)=(\psi_1(x),\dots,\psi_{m+n}(x))$ the canonical solution matrix of rank $m+n$.

We determine the connection coefficients $(C_{12})_{ij}=(C_{12}^{(m+n)})_{ij}$ and $D_{ij}=D_{ij}^{(n+m)}$ for $\Psi^{(m+n)}(x)$. From Theorem 4.4, we obtain the recurrence relations

$$\begin{aligned}C_{ij}^{(2n)} &= (C_{1(21)}^{mc})_{ij} = e(\tfrac{1}{2}(\rho_2+b_n))(t_1-t_2)^{\rho_2+b_{n-1}}\frac{\Gamma(\rho_2+b_{n-1}-\alpha_i)}{\Gamma(-\alpha_i)}C_{ij}^{(2n)}\frac{\Gamma(\beta_j-\rho_2)}{\Gamma(\beta_j+c)}\\ &= e(\tfrac{1}{2}(\beta_n^{(2n)}-\rho_1^{(2n)}))(t_1-t_2)^{\beta_n^{(2n)}-\rho_1^{(2n)}}\frac{\Gamma(-\alpha_i^{(2n-1)})}{\Gamma(-\alpha_i^{(2n)})}C_{ij}^{(2n-1)}\frac{\Gamma(\beta_j^{(2n)}-\beta_n^{(2n)})}{\Gamma(\beta_j^{(2n)}-\rho_1^{(2n)})},\\ D_{ij}^{(2n)} &= (C_{(21)1}^{mc})_{ij} = -e(\tfrac{-1}{2}(\rho_2+b_{n-1}))(t_1-t_2)^{-\rho_2-b_{n-1}}\frac{\Gamma(1+\rho_2-\beta_i)}{\Gamma(1-\beta_i-b_{n-1})}D_{ij}^{(2n)}\frac{\Gamma(\alpha_j-\rho-b_{n-1}+1)}{\Gamma(\alpha_j+1)}\\ &= -e(\tfrac{-1}{2}(\beta_n^{(2n)}-\rho_1^{(2n)}))(t_1-t_2)^{\rho_1^{(2n)}-\beta_n^{(2n)}}\frac{\Gamma(1+\beta_n^{(2n)}-\beta_i^{(2n)})}{\Gamma(1+\rho_1^{(2n)}-\beta_i^{(2n)})}D_{ij}^{(2n-1)}\frac{\Gamma(\alpha_j^{(2n)}+1)}{\Gamma(\alpha_j^{(2n-1)}+1)},\end{aligned} \tag{5.53}$$

$$\begin{aligned}C_{ij}^{(2n+1)} &= -e(\tfrac{1}{2}(\alpha_{n+1}^{(2n+1)}-\rho_1^{(2n+1)}))(t_2-t_1)^{-\alpha_{n+1}^{(2n+1)}+\rho_1^{(2n+1)}}\frac{\Gamma(1+\alpha_{n+1}^{(2n+1)}-\alpha_i^{(2n+1)})}{\Gamma(1+\rho_1^{(2n+1)}-\alpha_i^{(2n+1)})}C_{ij}^{(2n)}\frac{\Gamma(\beta_j^{(2n+1)}+1)}{\Gamma(\beta_j^{(2n)}+1)},\\ D_{ij}^{(2n+1)} &= e(\tfrac{-1}{2}(\alpha_{n+1}^{(2n+1)}-\rho_1^{(2n+1)}))(t_2-t_1)^{\alpha_{n+1}^{(2n+1)}-\rho_1^{(2n+1)}}\frac{\Gamma(-\beta_i^{(2n+1)})}{\Gamma(-\beta_i^{(2n)})}D_{ij}^{(2n)}\frac{\Gamma(\alpha_j^{(2n+1)}-\alpha_{n+1}^{(2n+1)})}{\Gamma(\alpha_j^{(2n+1)}-\rho_1^{(2n+1)})}.\end{aligned} \tag{5.54}$$

Therefore using these relations, we can completely determine $C_{11}^{(m+n)}$ and $D_{11}^{(m+n)}$ in terms of the gamma function, if we know the initial data $C_{11}^{(2)}$ and $D_{11}^{(2)}$. The connection coefficients $C_{11}^{(2)}$ and $D_{11}^{(2)}$ for the Okubo system (5.2) of type $(\mathrm{II})_2$ are given as follows.

$$\begin{aligned}C_{11}^{(2)} &= -e(\tfrac{-\rho_1}{2})\frac{(t_1-t_2)^{\beta_1-\rho_1}}{(t_2-t_1)^{\alpha_1-\rho_1}}\frac{\Gamma(-\alpha_1)\Gamma(\beta_1+1)}{\Gamma(\beta_1-\rho_1)\Gamma(1-\alpha_1+\rho_1)}\\ D_{11}^{(2)} &= -e(\tfrac{\rho_1}{2})\frac{(t_2-t_1)^{\alpha_1-\rho_1}}{(t_1-t_2)^{\beta_1-\rho_1}}\frac{\Gamma(-\beta_1)\Gamma(\alpha_1+1)}{\Gamma(\alpha_1-\rho_1)\Gamma(1-\beta_1-\rho_1)}.\end{aligned} \tag{5.55}$$

Therefore we can obtain $C_{11}^{(m+n)}$ from $C_{11}^{(2)}$as follows:

$$\begin{aligned}
C_{11}^{(2n)} &= (-1)^{n-1}\prod_{k=2}^{n} e(\tfrac{1}{2}(\alpha_k^{(2k-1)}-\rho_1^{(2k-1)}+\beta_k^{(2k)}-\rho_1^{(2k)}))\frac{(t_1-t_2)^{\beta_k^{(2k)}-\rho_1^{(2k)-1}}}{(t_2-t_1)^{\alpha_k^{(2k-1)}-\rho_1^{(2k-1)}}}\\
&\prod_{k=2}^{n}\frac{\Gamma(-\alpha_1^{(2k)})}{\Gamma(-\alpha_1^{(2k-1)})}\frac{\Gamma(\beta_1^{(2k)}-\beta_k^{(2k)})}{\Gamma(\beta_1^{(2k)}-\rho_1^{(2k)})}\frac{\Gamma(1+\alpha_k^{(2k-1)}-\alpha_1^{(2k-1)})}{\Gamma(1+\rho_1^{(2k-1)}-\alpha_1^{(2k-1)})}\frac{\Gamma(\beta_1^{(2k-1)}+1)}{\Gamma(\beta_1^{(2k-2)}+1)}C_{11}^{(2)}\\
&= (-1)^{n-1}e(\tfrac{\alpha_2^{(3)}+\beta_2^{(4)}-\rho_1^{(2n)}-\rho_2^{(2n)}}{2})\frac{(t_1-t_2)^{\beta_2^{(4)}-\rho_2^{(2n)}}}{(t_2-t_1)^{\alpha_2^{(3)}-\rho_1^{(2n)}}}\frac{\Gamma(\beta_1^{(2n)}+1)\Gamma(-\alpha_1^{(2n)})}{\Gamma(\beta_1^{(4)}+1)\Gamma(-\alpha_1^{(3)})}\\
&\prod_{k=2}^{n}\frac{\Gamma(\beta_1^{(2k)}-\beta_k^{(2k)})}{\Gamma(\beta_1^{(2k)}-\rho_1^{(2k)})}\frac{\Gamma(1+\alpha_k^{(2k-1)}-\alpha_1^{(2k-1)})}{\Gamma(1+\rho_1^{(2k-1)}-\alpha_1^{(2k-1)})}C_{11}^{(2)}\\
&= (-1)^{n}e(\tfrac{1}{2}(\rho_3^{(2n)}-\alpha_1^{(2n)}-\beta_1^{(2n)}))\frac{(t_1-t_2)^{\rho_3^{(2n)}-\alpha_1^{(2n)}}}{(t_2-t_1)^{\rho_3^{(2n)}-\beta_1^{(2n)}}}\frac{\Gamma(\beta_1^{(2n)}+1)\Gamma(-\alpha_1^{(2n)})}{\Gamma(1+\rho_1^{(2n)}-\alpha_1^{(2n)})\Gamma(\beta_1^{(2n)}-\rho_1^{(2n)})}\\
&\prod_{k=2}^{n}\frac{\Gamma(1+\alpha_k^{(2n)}-\alpha_1^{(2n)})}{\Gamma(1+\rho_1^{(2n)}+\rho_2^{(2n)}-\alpha_1^{(2n)}-\beta_k^{(2n)})}\frac{\Gamma(\beta_1^{(2n)}-\beta_k^{(2n)})}{\Gamma(\beta_1^{(2n)}+\alpha_k^{(2n)}-\rho_1^{(2n)}-\rho_2^{(2n)})}
\end{aligned} \tag{5.56}$$

The connection coefficients $D_{11}^{(2n)}, C_{11}^{(2n+1)}$, and $D_{11}^{(2n+1)}$ are obtained in the same way. Furthermore, the other connection coefficients $C_{ij}^{(m+n)}$, and $D_{ij}^{(m+n)}$ are derived by Lemma 5.4 and operations σ_{ij}^{α}, σ_{ij}^{β}. For σ_{1i}^{α}, σ_{1j}^{β}, the canonical form of type $(\mathrm{II})_{2n}$ and $(\mathrm{III})_{2n+1}$ satisfies the relations

$$A = \mathrm{Ad}(D)\sigma_{1i}^{\alpha}\sigma_{1j}^{\beta}(A) \quad D = \mathrm{diag}(S_{1i}, S_{1j}). \tag{5.57}$$

By taking account of Lemma 5.4 and (5.57), the connection coefficients $(C_{12}^{(m+n)})_{ij}$ and $(C_{21}^{(m+n)})_{ij}$are given by

$$(C_{12}^{(m+n)})_{ij} = \sigma_{1i}^{\alpha}\sigma_{1j}^{\beta}(C_{12})_{11}, \quad (C_{21}^{(m+n)})_{ij} = \sigma_{1j}^{\alpha}\sigma_{1i}^{\beta}(C_{12})_{11}. \tag{5.58}$$

This completes the proof of Theorem 2.8.

Case IV: The canonical forms of type IV is given by the operation

$$(\mathrm{IV})_6 = \mathrm{add}_{(\rho_3,0)} \circ \mathrm{mc}_{-\rho_3-c} \circ \mathrm{add}_{(c,0)}(\mathrm{III}_5). \tag{5.59}$$

Using the connection coefficient of type $(\mathrm{III})_5$, we can completely determine $(C_{12}^{(6)})_{11}$ for (2.18) by the following computation:

$$
\begin{aligned}
(C^{(6)}_{12})_{11} &= (C^{mc}_{(11)2})_{11} = -e(\tfrac{1}{2}(\rho_3+c))(t_2-t_1)^{-\rho_3-c}\frac{\Gamma(1+\rho_3-\alpha_1)}{\Gamma(1-\alpha_1-c)}(C^{(5)}_{12})_{11}\frac{\Gamma(\beta_1-\rho_3-c+1)}{\Gamma(\beta_1+1)} \\
&= -e(\tfrac{1}{2}(\alpha_4-\rho_3))(t_2-t_1)^{\rho_3-\alpha_4}\frac{\Gamma(1+\alpha_4-\alpha_1)}{\Gamma(1+\rho_3-\alpha_1)}(C_{12})_{11}\frac{\Gamma(\beta^{(6)}_1+1)}{\Gamma(\beta^{(5)}_1+1)} \\
&= e(\tfrac{1}{2}(\alpha_4-\rho_3))(t_2-t_1)^{\rho_3-\alpha_4}\frac{\Gamma(1+\alpha_4-\alpha_1)}{\Gamma(1+\rho_3-\alpha_1)}\frac{\Gamma(\beta^{(6)}_1+1)}{\Gamma(\beta^{(5)}_1+1)}\frac{(t_2-t_1)^{\alpha_4-\alpha_i}}{(t_1-t_2)^{\rho_3-\beta_j}}e(\tfrac{1}{2}(\rho_3-\alpha_1-\beta_1-\alpha_4)) \\
&\quad \frac{\Gamma(-\alpha_1)\Gamma(\beta^{(5)}_1+1)}{\Gamma(1+\rho_1-\alpha_1)\Gamma(1+\rho_2-\alpha_1)}\frac{\prod_{k\neq 1}^{2}\Gamma(\beta_1-\beta_k)}{\prod_{k\neq 1}^{3}\Gamma(\beta_1+\alpha_k+\alpha_4-\rho_1-\rho_2-\rho_3)}\frac{\prod_{k\neq 1}^{3}\Gamma(1+\alpha_k-\alpha_1)}{\prod_{k\neq 1}^{2}\Gamma(1+\rho_1+\rho_2+\rho_3-\beta_k-\alpha_1-\alpha_4)} \\
&= e(\tfrac{1}{2}(-\alpha_1-\beta_1))\frac{(t_2-t_1)^{\rho_3-\alpha_1}}{(t_1-t_2)^{\rho_3-\beta_j}}\frac{\Gamma(-\alpha_1)\Gamma(\beta_1+1)}{\Gamma(1+\rho_1-\alpha_1)\Gamma(1+\rho_2-\alpha_1)\Gamma(1+\rho_3-\alpha_1)} \\
&\quad \frac{\prod_{k\neq 1}^{2}\Gamma(\beta_1-\beta_k)}{\prod_{k\neq 1}^{3}\Gamma(\beta_1+\alpha_k+\alpha_4-\rho_1-\rho_2-\rho_3)}\frac{\prod_{k\neq 1}^{4}\Gamma(1+\alpha_k-\alpha_1)}{\prod_{k\neq 1}^{2}\Gamma(1+\rho_1+\rho_2+\rho_3-\beta_k-\alpha_1-\alpha_4)}.
\end{aligned}
\tag{5.60}
$$

As in the cases of types II and III, we determine the other connection coefficients by Lemma 5.4 and operations σ^{α}_{ij}, σ^{β}_{ij}. For σ^{α}_{1i}, σ^{β}_{12}, the Okubo system (2.18) satisfies the relations

$$A = \mathrm{Ad}(D)\sigma^{\alpha}_{1i}\sigma^{\beta}_{12}(A) \quad D = \mathrm{diag}(S_{1i}, S_{12}), \quad (i = 1, 2, 3). \tag{5.61}$$

By taking account of Lemma 5.4 and (5.61), the connection coefficients $(C_{12})_{ij}$ $(i = 1, 2, 3; j = 1, 2)$ are given by

$$(C_{12})_{ij} = \sigma^{\alpha}_{1i}\sigma^{\beta}_{1j}(C_{12})_{11}. \tag{5.62}$$

Although the system (2.18) is *not* symmetric with respect to (α_1, α_4), for the operation σ^{α}_{14} it satisfies the relation

$$A = \mathrm{Ad}(D^{\alpha}_{14})\sigma^{\alpha}_{14}(A) \tag{5.63}$$

where $D^{\alpha}_{14} = \mathrm{diag}(d_1, \dots, d_6)\mathrm{diag}(S_{14},\ I_2)$ is the matrix defined by

$$
\begin{aligned}
&d_1 = \prod_{k=1}^{2}(\alpha_1+\alpha_2+\beta_k-\rho_1-\rho_2-\rho_3)(\alpha_1+\alpha_3+\beta_k-\rho_1-\rho_2-\rho_3), \\
&d_2 = -\prod_{k=1}^{2}(\alpha_1+\alpha_3+\beta_k-\rho_1-\rho_2-\rho_3), \quad d_3 = -\prod_{k=1}^{2}(\alpha_1+\alpha_2+\beta_k-\rho_1-\rho_2-\rho_3), \quad d_4 = 1, \\
&d_5 = \prod_{k\neq 1}^{3}(\alpha_1+\alpha_k+\beta_1-\rho_1-\rho_2-\rho_3), \quad d_6 = \prod_{k\neq 1}^{3}(\alpha_1+\alpha_k+\beta_2-\rho_1-\rho_2-\rho_3).
\end{aligned}
\tag{5.64}
$$

Therefore the connection coefficient $(C_{12})_{41}$ is derived by

$$(C_{12})_{41} = d_4\sigma^{\alpha}_{14}(C_{12})_{11}d_6^{-1}. \tag{5.65}$$

Since $C_{12} = \sigma^{\beta}_{12}(C_{12})S_{12}$, we obtain

$$(C_{12})_{4j} = \sigma^{\beta}_{12}(d_4\sigma^{\alpha}_{14}(C_{12})_{11}d_6^{-1}). \tag{5.66}$$

The other connection coefficients $(C_{21})_{ij}$ of type IV are computed similarly.

Case I*: We consider the Schlesinger system of rank 1 and its solution defined by

$$\frac{d}{dx}Y = \sum_{k=1}^{n} \frac{\alpha_k}{x-t_k}Y, \quad Y(x) = \prod_{k=1}^{n}(x-t_k)^{\alpha_k}. \tag{5.67}$$

Then the Okubo system of type $(\mathrm{I}^*)_n$ is constructed by mc_μ, and its canonical solution matrix is expressed in the form $I^\mu(Y(x))R^{-1}$, where $R = \mathrm{diag}(r_1, \ldots, r_n)$. The components of the diagonal matrix R are computed as

$$\begin{aligned} r_i &= \lim_{x\to t_i}(x-t_i)^{-\alpha_i-\mu}\int_{L_i}(x-u)^{\mu}\prod_{k=1}^{n}(u-t_k)^{\alpha_k}\frac{du}{u-t_i} \\ &= \lim_{x\to t_i}(x-t_i)^{-\alpha_i^{(n)}}\int_{L_i}(x-u)^{\rho_1^{(n)}}\prod_{k=1}^{n}(u-t_k)^{\alpha_k^{(n)}-\rho_1^{(n)}}\frac{du}{u-t_i} \\ &= \prod_{k\neq i, 1\le k\le n}(t_i-t_k)^{\alpha_k^{(n)}-\rho_1^{(n)}}\widetilde{B}(\alpha_i^{(n)}-\rho_1^{(n)}, \rho_1^{(n)}+1). \end{aligned} \tag{5.68}$$

Therefore the connection coefficients C_{ij} are obtained as follows:

$$\begin{aligned} C_{ij} &= \frac{e(\rho_1^{\delta(i>j)})(e(\alpha_j-\rho_1)-1)}{e(\alpha_i)-1}\frac{r_i}{r_j} \\ &= \begin{cases} e(\frac{\rho_1}{2})\dfrac{\prod_{k\neq i,1\le k\le n}(t_i-t_k)^{\alpha_k-\rho_1}}{\prod_{k\neq j,1\le k\le n}(t_j-t_k)^{\alpha_k-\rho_1}}\dfrac{\Gamma(-\alpha_i)\Gamma(\alpha_j+1)}{\Gamma(\alpha_j-\rho_1)\Gamma(1+\rho_1-\alpha_i)} & (i<j) \\ e(\frac{-\rho_1}{2})\dfrac{\prod_{k\neq i,1\le k\le n}(t_i-t_k)^{\alpha_k-\rho_1}}{\prod_{k\neq j,1\le k\le n}(t_j-t_k)^{\alpha_k-\rho_1}}\dfrac{\Gamma(-\alpha_i)\Gamma(\alpha_j+1)}{\Gamma(\alpha_j-\rho_1)\Gamma(1+\rho_1-\alpha_i)} & (i>j) \end{cases} \end{aligned} \tag{5.69}$$

This completes the proof of Theorem 2.10 .

Case II*, III*: Recall that the canonical form of the system $(\mathrm{III}^*)_3$ is constructed by the middle convolution mc_μ for the differential system

$$\frac{d}{dx}Y = \left(\frac{\alpha}{x-t_1}+\frac{\beta}{x-t_2}+\frac{\gamma}{x-t_3}\right)Y, \quad Y = (x-t_1)^{\alpha}(x-t_2)^{\beta}(x-t_3)^{\gamma}. \tag{5.70}$$

Since the Okubo system of type $(\mathrm{III}^*)_3$ is equivalent to the Okubo system of type $(\mathrm{I}^*)_3$ up to conjugation, the connection coefficients in this case are directly computed. The connection coefficients $(C_{12}^{(3)})_{11}$ and $(C_{13}^{(3)})_1$ are in fact given as

$$\begin{aligned} (C_{12}^{(3)})_{11} = -e(\tfrac{1}{2}(\rho_1+\rho_2-\alpha_1-\beta_1-\gamma))\frac{(t_1-t_2)^{\rho_1+\rho_2-\alpha_1-\gamma}}{(t_2-t_1)^{\beta_1+\gamma-\rho_1-\rho_2}} \\ \left(\frac{t_1-t_3}{t_2-t_3}\right)^{\rho_1+\rho_2-\alpha_1-\beta_1}\frac{\Gamma(-\alpha_1)\Gamma(\beta_1+1)}{\Gamma(1+\rho_1-\alpha_1)\Gamma(\beta_1-\rho_1)} \end{aligned} \tag{5.71}$$

$$\begin{aligned} (C_{13}^{(3)})_1 = -(\gamma-\rho_1)^{-1}e(\tfrac{1}{2}(\rho_1+\rho_2-\alpha_1-\beta_1-\gamma))\frac{(t_1-t_3)^{\rho_1+\rho_2-\alpha_1-\beta_1}}{(t_3-t_1)^{\beta_1+\gamma-\rho_1-\rho_2}} \\ \left(\frac{t_1-t_2}{t_3-t_2}\right)^{\rho_1+\rho_2-\alpha_1-\gamma}\frac{\Gamma(-\alpha_i)\Gamma(\gamma+1)}{\Gamma(1+\rho_1-\alpha_1)\Gamma(\gamma-\rho_1)} \end{aligned} \tag{5.72}$$

Using Theorem 4.4, the connection coefficients $(C_{12}^{(2n+1)})_{11}$ and $(C_{13}^{(2n+1)})_1$ of $(\mathrm{III}^*)_{2n+1}$ are computed as follows:

$$
\begin{aligned}
(C_{12}^{(2n+1)})_{11} &= (-1)^{n-1}\prod_{k=2}^{n} e(\tfrac{1}{2}(\beta_k^{(2k+1)}-\rho_1^{(2k+1)}))(t_1-t_2)^{\beta_k^{(2k+1)}-\rho_1^{(2k+1)}} e(\tfrac{1}{2}(\alpha_k^{(2k)}-\rho_1^{(2k)}))(t_2-t_1)^{\rho_1^{(2k)}-\alpha_k^{(2k)}} \\
&\prod_{k=2}^{n}\frac{\Gamma(-\alpha_1^{(2k+1)})}{\Gamma(-\alpha_1^{(2k)})}\frac{\Gamma(\beta_1^{(2k+1)}-\beta_k^{(2k+1)})}{\Gamma(\beta_1^{(2k+1)}-\rho_1^{(2k+1)})}\frac{\Gamma(1+\alpha_k^{(2k)}-\alpha_1^{(2k)})}{\Gamma(1+\rho_1^{(2k)}-\alpha_1^{(2k)})}\frac{\Gamma(\beta_1^{(2k)}+1)}{\Gamma(\beta_1^{(2k-1)}+1)}(C_{12}^{(3)})_{11} \\
&= (-1)^{n-1}e(\tfrac{1}{2}(-\rho_1^{(2n+1)}+\beta_2^{(5)}))e(\tfrac{1}{2}(-\rho_1^{(2n)}+\alpha_2^{(4)}))(t_1-t_2)^{-\rho_1^{(2n+1)}+\beta_2^{(5)}}(t_2-t_1)^{\rho_1^{(2n)}-\alpha_2^{(4)}} \\
&\frac{\Gamma(-\alpha_1^{(2n+1)})\Gamma(\beta_1^{(2n)}+1)}{\Gamma(-\alpha_1^{(3)})\Gamma(\beta_1^{(3)}+1)}\prod_{k=2}^{n}\frac{\Gamma(\beta_1^{(2k+1)}-\beta_k^{(2k+1)})}{\Gamma(\beta_1^{(2k+1)}-\rho_1^{(2k+1)})}\frac{\Gamma(1+\alpha_k^{(2k)}-\alpha_1^{(2k)})}{\Gamma(1+\rho_1^{(2k)}-\alpha_1^{(2k)})}(C_{12}^{(3)})_{11} \\
&= (-1)^{n}e(\tfrac{1}{2}(-\rho_1^{(2n+1)}-\rho_1^{(2n)}+\beta_2^{(5)}+\alpha_2^{(4)}-\rho_1^{(3)}))\frac{(t_1-t_2)^{-\rho_1^{(2n+1)}+\beta_1^{(3)}}}{(t_2-t_1)^{\rho_1^{(2n)}-\alpha_2^{(4)}+\rho_1^{(3)}-\alpha_1^{(3)}}} \\
&\left(\frac{t_1-t_3}{t_2-t_3}\right)^{\rho_1^{(2n+1)}+\rho_2^{(2n+1)}-\alpha_1^{(2n+1)}-\beta_1^{(2n+1)}}\frac{\Gamma(-\alpha_1^{(2n+1)})\Gamma(\beta_1^{(2n+1)}+1)}{\Gamma(1+\rho_1^{(2n+1)}-\alpha_1^{(2n+1)})\Gamma(\beta_1^{(2n+1)}-\rho_1^{(2n+1)})} \\
&\prod_{k=2}^{n}\frac{\Gamma(\beta_1^{(2n+1)}-\beta_k^{(2n+1)})}{\Gamma(\beta_1^{(2n+1)}+\alpha_k^{(2n+1)}-\rho_1^{(2n+1)}-\rho_2^{(2n+1)})}\frac{\Gamma(1+\alpha_k^{(2n+1)}-\alpha_1^{(2n+1)})}{\Gamma(1+\rho_1^{(2n+1)}+\rho_2^{(2n+1)}-\alpha_1^{(2n+1)}-\beta_k^{(2n+1)})} \\
&= (-1)^{n}e(\tfrac{1}{2}(\rho_1+\rho_2-\alpha_1-\beta_1-\gamma)\frac{(t_1-t_2)^{\rho_1+\rho_2-\alpha_1-\gamma}}{(t_2-t_1)^{\beta_1+\gamma-\rho_1-\rho_2}}\left(\frac{t_1-t_3}{t_2-t_3}\right)^{\rho_1+\rho_2-\alpha_1-\beta_1} \\
&\frac{\Gamma(-\alpha_1)\Gamma(\beta_1+1)}{\Gamma(1+\rho_1-\alpha_1)\Gamma(\beta_1-\rho_1)}\prod_{k=2}^{n}\frac{\Gamma(\beta_1-\beta_k)}{\Gamma(\beta_1+\alpha_k-\rho_1-\rho_2)}\frac{\Gamma(1+\alpha_k-\alpha_1)}{\Gamma(1+\rho_1+\rho_2-\alpha_1-\beta_k)}
\end{aligned}
\tag{5.73}
$$

$$
\begin{aligned}
(C_{13}^{(2n+1)})_{1} &= (-1)^{n-1}\prod_{k=2}^{n}\left(\frac{t_1-t_2}{t_3-t_2}\right)^{\beta_k^{(2k+1)}-\rho_1^{(2k+1)}}\frac{e(\frac{1}{2}(\beta_k^{(2k+1)}-\rho_1^{(2k+1)}))\Gamma(-\alpha_1^{(2k+1)})}{\Gamma(-\alpha_1^{(2k)})}\frac{\Gamma(\gamma^{(2k+1)}+1)}{\Gamma(\gamma^{(2k)}+1)} \\
&e(\tfrac{1}{2}(\alpha_k^{(2k)}-\rho_1^{(2k)}))(t_3-t_1)^{\rho_1^{(2k)}-\alpha_k^{(2k)}}\frac{\Gamma(1+\alpha_k^{(2k)}-\alpha_1^{(2k)})}{\Gamma(1+\rho_1^{(2k)}-\alpha_1^{(2k)})}\frac{\Gamma(\gamma^{(2k)}+1)}{\Gamma(\gamma^{(2k-1)}+1)}(C_{13}^{(3)})_{11} \\
&= (-1)^{n-1}e(\frac{1}{2}(-\rho_1^{(2n+1)}-\rho_2^{(2n+1)}+\beta_2^{(5)}+\alpha_2^{(2)}))(t_3-t_1)^{\rho_2^{(2n+1)}-\alpha_2^{(4)}}\left(\frac{t_1-t_2}{t_3-t_2}\right)^{-\rho_1^{(2n+1)}+\beta_2^{(5)}} \\
&\frac{\Gamma(\gamma^{(2n+1)}+1)}{\Gamma(\gamma^{(3)}+1)}\frac{\Gamma(-\alpha_1^{(2n+1)})}{\Gamma(-\alpha_1^{(4)})}\prod_{k=2}^{n}\frac{\Gamma(1+\alpha_k^{(2n+1)}-\alpha_1^{(2n+1)})}{\Gamma(1+\rho_1^{(2n+1)}+\rho_2^{(2n+1)}-\alpha_1^{(2n+1)}-\beta_k^{(2n+1)})}(C_{13}^{(3)})_{11} \\
&= (-1)^{n}e(\tfrac{1}{2}(\rho_1+\rho_2-\alpha_1-\beta_1-\gamma))(t_1-t_3)^{\rho_1+\rho_2-\alpha_1-\beta_1}(t_3-t_1)^{\beta_1+\gamma-\rho_1-\rho_2} \\
&\left(\frac{t_1-t_2}{t_3-t_2}\right)^{\rho_1+\rho_2-\alpha_1-\gamma}\frac{\Gamma(\gamma+1)\Gamma(-\alpha_1)}{\Gamma(1+\rho_1-\alpha_1)}\frac{\prod_{k=2}^{n}\Gamma(1+\alpha_k-\alpha_1)}{\prod_{k=1}^{n}\Gamma(1+\rho_1+\rho_2-\alpha_1-\beta_k)}
\end{aligned}
\tag{5.74}
$$

The connection coefficients $(C_{12})_{ij}$ and $(C_{13})_i$ for other i,j are derived by Lemma 5.4. Then the matrix A of the canonical form of type $(\mathrm{III}^*)_{2n+1}$ satisfies

$$A=\mathrm{Ad}(D)\sigma_{1i}^{\alpha}\sigma_{1j}^{\beta}(A),\quad D=\mathrm{diag}(S_{1i},S_{1j},1) \tag{5.75}$$

From (5.75), applying Lemma 5.4 to $B = \sigma_{1i}^{\alpha}\sigma_{1j}^{\beta}(A)$ we obtain

$$C_{12} = S_{1i}\sigma_{1i}^{\alpha}\sigma_{1j}^{\beta}(C_{12})S_{1j}, \quad C_{13} = S_{1i}\sigma_{1i}^{\alpha}\sigma_{1j}^{\beta}(C_{13}). \tag{5.76}$$

Therefore the connection coefficients $(C_{12})_{ij}$ and $(C_{13})_i$ are given by

$$(C_{12})_{ij} = \sigma_{1i}^{\alpha}\sigma_{1j}^{\beta}(C_{12})_{11}, \quad (C_{13})_i = \sigma_{1i}^{\alpha}(C_{13})_1. \tag{5.77}$$

The other connection coefficients of type III* and connection coefficients of type II* are computed similarly.

Case IV*: The canonical form of type IV* is constructed by

$$(\mathrm{IV}^*)_6 = \mathrm{add}_{(0,0,\rho_1)} \circ \mathrm{mc}_{-c-\rho_1} \circ \mathrm{add}_{(0,0,c)}(\mathrm{III}^*)_5. \tag{5.78}$$

The connection coefficients $(C_{12})_{11}, (C_{13})_{11}, (C_{23})_{11}$ are directly computed from the connection coefficients $(C_{ij})_{11}^{(5)}$ of type $(\mathrm{III}^*)_5$ by Theorem 4.4. The other connection coefficients can be determined by combining Lemma 5.4 and the operations σ_{12}^{α}, σ_{12}^{β}, σ_{12}^{γ}. For σ_{12}^{α}, σ_{12}^{β}, the Okubo system (2.33) satisfies the relation

$$A = \mathrm{Ad}(D)\sigma_{12}^{\alpha}\sigma_{12}^{\beta}(A), \quad D = \mathrm{diag}(S_{12}, S_{12}, I_2). \tag{5.79}$$

For σ_{12}^{γ}, the Okubo system (2.33) satisfies

$$A = \mathrm{Ad}(D^{\gamma})\sigma_{12}^{\gamma}(A), \tag{5.80}$$

where D^{γ} is the matrix given by

$$D^{\gamma} = \mathrm{diag}(h_{111}h_{121},\, h_{211}h_{221},\, -h_{111}h_{211},\, -h_{121}h_{221},\, h_{111}h_{121}h_{211}h_{221},\, 1)\mathrm{diag}(I_2,\, I_2,\, S_{12}). \tag{5.81}$$

With these relations, the other connection matrices are determined by Lemma 5.4 as follows:

$$\begin{aligned} &C_{12} = S_{1i}\sigma_{1i}^{\alpha}\sigma_{1j}^{\beta}(C_{12})S_{1j}, \quad C_{13} = S_{1i}\sigma_{1i}^{\alpha}(C_{13}), \quad C_{23} = S_{1i}\sigma_{1i}^{\beta}(C_{23}), \\ &C_{13} = \begin{pmatrix} h_{111}h_{121} & 0 \\ 0 & h_{211}h_{221} \end{pmatrix} \sigma_{12}^{\gamma}(C_{13})S_{12} \begin{pmatrix} h_{111}h_{121}h_{211}h_{221} & 0 \\ 0 & 1 \end{pmatrix}^{-1}, \\ &C_{23} = \begin{pmatrix} -h_{111}h_{211} & 0 \\ 0 & -h_{121}h_{221} \end{pmatrix} \sigma_{12}^{\gamma}(C_{23})S_{12} \begin{pmatrix} h_{111}h_{121}h_{211}h_{221} & 0 \\ 0 & 1 \end{pmatrix}^{-1}. \end{aligned} \tag{5.82}$$

This completes the proof of the main theorems.

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