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# 博　士　論　文

# Relations among Alexander-Conway polynomials of Turk's head links
# （タークセット絡み目のアレクサンダー・コンウェイ多項式間の関係）

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# RELATIONS AMONG ALEXANDER-CONWAY POLYNOMIALS OF TURK'S HEAD LINKS

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Abstract. The $(m,n)$-Turk's head link is presented by the alternating diagram which is obtained from the standard diagram of the $(m,n)$-torus link by crossing changes. In this paper, we show the two properties. First, we show that for any integers $m \geq 1$ and $n = 2, 3$, the coefficients of $z^i$ in the Conway polynomials of the $(m,n)$- and $(n,m)$-Turk's head links coincide for $i \equiv 0, 1 \pmod 4$ and differ by sign for $i \equiv 2, 3 \pmod 4$. We conjecture that this property holds for any $n$. Second, we show that for any positive integers $a, b, m, n$, the Alexander polynomial of the $(am, bn)$-Turk's head link is divisible by that of the $(m,n)$-Turk's head link.

## 1. Introduction

The $(m,n)$-Turk's head link $TH(m,n)$ is presented by the alternating diagram which is obtained from the standard diagram of the $(m,n)$-torus link by crossing changes. There are several studies on Turk's head links (cf. [3, 4, 6, 10]).

It is well-known that the $(m,n)$- and $(n,m)$-torus links have the same link type, and hence, their invariants are the same. However, the $(m,n)$- and $(n,m)$-Turk's head links have distinct link types ([10]) and their invariants are not the same generally.

The Jones polynomials $V_{TH(m,n)}(t)$ and the Alexander polynomials $\Delta_{TH(m,n)}(t)$ for $\{m,n\} = \{6,2\}$ and $\{5,3\}$ are given by the following.

$$\begin{cases} V_{TH(6,2)}(t) = -t^{-\frac{7}{2}} + 3t^{-\frac{5}{2}} - 6t^{-\frac{3}{2}} + 9t^{-\frac{1}{2}} \\ \qquad\qquad -11t^{\frac{1}{2}} + 12t^{\frac{3}{2}} - 11t^{\frac{5}{2}} + 8t^{\frac{7}{2}} - 6t^{\frac{9}{2}} + 2t^{\frac{11}{2}} - t^{\frac{13}{2}}, \\ V_{TH(2,6)}(t) = -t^{\frac{5}{2}} - t^{\frac{9}{2}} + t^{\frac{11}{2}} - t^{\frac{13}{2}} + t^{\frac{15}{2}} - t^{\frac{17}{2}}, \end{cases}$$

$$\begin{cases} V_{TH(5,3)}(t) = t^{-6} - 6t^{-5} + 16t^{-4} - 30t^{-3} + 44t^{-2} - 54t^{-1} \\ \qquad\qquad +59 - 54t + 44t^2 - 30t^3 + 16t^4 - 6t^5 + t^6, \\ V_{TH(3,5)}(t) = -t^{-5} + 5t^{-4} - 10t^{-3} + 15t^{-2} - 19t^{-1} \\ \qquad\qquad +21 - 19t + 15t^2 - 10t^3 + 5t^4 - t^5, \end{cases}$$

$$\begin{cases} \Delta_{TH(6,2)}(t) = t^{-\frac{5}{2}} - 9t^{-\frac{3}{2}} + 25t^{-\frac{1}{2}} - 25t^{\frac{1}{2}} + 9t^{\frac{3}{2}} - t^{\frac{5}{2}}, \\ \Delta_{TH(2,6)}(t) = t^{-\frac{5}{2}} - t^{-\frac{3}{2}} + t^{-\frac{1}{2}} - t^{\frac{1}{2}} + t^{\frac{3}{2}} - t^{\frac{5}{2}}, \end{cases}$$

$$\begin{cases} \Delta_{TH(5,3)}(t) = t^{-4} - 10t^{-3} + 39t^{-2} - 80t^{-1} + 101 - 80t + 39t^2 - 10t^3 + t^4, \text{ and} \\ \Delta_{TH(3,5)}(t) = t^{-4} - 6t^{-3} + 15t^{-2} - 24t^{-1} + 29 - 24t + 15t^2 - 6t^3 + t^4. \end{cases}$$

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The Jones polynomials of the $(m,n)$- and $(n,m)$-Turk's head links are quite different. Although the Alexander polynomials also look different, their Conway polynomials are similar as follows.

$$\begin{cases} \nabla_{TH(6,2)}(z) = -3z + 4z^3 - z^5, \\ \nabla_{TH(2,6)}(z) = -3z - 4z^3 - z^5, \end{cases} \begin{cases} \nabla_{TH(5,3)}(z) = 1 + 2z^2 - z^4 - 2z^6 + z^8, \quad \text{and} \\ \nabla_{TH(3,5)}(z) = 1 - 2z^2 - z^4 + 2z^6 + z^8. \end{cases}$$

We observe that the coefficients of $z^i$ in $\nabla_{TH(m,n)}(z)$ and $\nabla_{TH(n,m)}(z)$ coincide or differ by sign. In addition, we calculate the Conway polynomials of Turk's head links for $n = 2, 3$ as follows.

$$\begin{cases} \nabla_{TH(3,2)}(z) = 1 - z^2, \\ \nabla_{TH(2,3)}(z) = 1 + z^2, \end{cases}$$
$$\begin{cases} \nabla_{TH(4,2)}(z) = -2z + z^3, \\ \nabla_{TH(2,4)}(z) = -2z - z^3, \end{cases}$$
$$\begin{cases} \nabla_{TH(5,2)}(z) = 1 - 3z^2 + z^4, \\ \nabla_{TH(2,5)}(z) = 1 + 3z^2 + z^4, \end{cases}$$
$$\begin{cases} \nabla_{TH(10,2)}(z) = -5z + 20z^3 - 21z^5 + 8z^7 - z^9, \\ \nabla_{TH(2,10)}(z) = -5z - 20z^3 - 21z^5 - 8z^7 - z^9, \end{cases}$$
$$\begin{cases} \nabla_{TH(4,3)}(z) = 1 - z^2 - z^4 + z^6, \\ \nabla_{TH(3,4)}(z) = 1 + z^2 - z^4 - z^6, \end{cases}$$
$$\begin{cases} \nabla_{TH(6,3)}(z) = 4z^4 - 3z^8 + z^{10}, \\ \nabla_{TH(3,6)}(z) = 4z^4 - 3z^8 - z^{10}, \end{cases}$$
$$\begin{cases} \nabla_{TH(10,3)}(z) = 1 - 3z^2 - 6z^4 + 18z^6 + 11z^8 - 29z^{10} + 2z^{12} + 14z^{14} - 7z^{16} + z^{18}, \\ \nabla_{TH(3,10)}(z) = 1 + 3z^2 - 6z^4 - 18z^6 + 11z^8 + 29z^{10} + 2z^{12} - 14z^{14} - 7z^{16} - z^{18}, \end{cases}$$
$$\nabla_{TH(2,2)}(z) = -z, \quad \text{and}$$
$$\nabla_{TH(3,3)}(z) = z^4.$$

The first aim of this paper is to generalize this property as follows.

**Theorem 1.1.** *For any integers $m \geq 1$ and $n = 2, 3$, the Conway polynomials*

$$\nabla_{TH(m,n)}(z) = \sum_{i=0}^{\infty} a_i z^i \quad \textit{and} \quad \nabla_{TH(n,m)}(z) = \sum_{i=0}^{\infty} b_i z^i$$

*of the $(m,n)$- and $(n,m)$-Turk's head links satisfy*

$$\begin{cases} a_i = b_i & \text{for } i \equiv 0, 1 \pmod 4, \quad \textit{and} \\ a_i = -b_i & \text{for } i \equiv 2, 3 \pmod 4. \end{cases}$$

We can not prove this Theorem for $n \geq 4$ yet. To prove this Theorem for $n \geq 4$, Theorem 1.3 may be useful.

It is known that the Alexander polynomial $\Delta_{T(m,n)}(t)$ of the $(m,n)$-torus link $T(m,n)$ is given by the following.

**Theorem 1.2** (cf. [8]). *For any positive integers $m$ and $n$, we have*

$$\Delta_{T(m,n)}(t) = \frac{(t-1)(t^{\frac{mn}{\mu}}-1)^{\mu}}{(t^m-1)(t^n-1)},$$

*where $\mu$ is the greatest common divisor of $m$ and $n$.*

By this theorem, we can easily see that $\Delta_{T(am,bn)}(t)$ is divisible by $\Delta_{T(m,n)}(t)$ for any positive integers $a, b, m, n$.

In this paper, we study the Alexander polynomial $\Delta_{TH(m,n)}(t)$ of the $(m,n)$-Turk's head link $TH(m,n)$. For example, we have

$$\begin{cases} \Delta_{TH(2,3)}(t) = t^{-1} - 1 + t, \\ \Delta_{TH(4,3)}(t) = \Delta_{TH(2,3)}(t) \cdot (t^{-1} - 3 + t)^2, \\ \Delta_{TH(2,6)}(t) = \Delta_{TH(2,3)}(t) \cdot (t^{-\frac{1}{2}} - t^{\frac{1}{2}})(t^{-1} + 1 + t), \quad \text{and} \\ \Delta_{TH(4,6)}(t) = \Delta_{TH(2,3)}(t) \cdot (t^{-\frac{1}{2}} - t^{\frac{1}{2}})(t^{-1} + 1 + t)(t^{-1} - 3 + t)^2 \\ \qquad\qquad\qquad \cdot (t^{-1} - 1 + t)^2 (t^{-1} - 4 + t). \end{cases}$$

In particular, $\Delta_{TH(4,3)}(t)$, $\Delta_{TH(2,6)}(t)$, and $\Delta_{TH(4,6)}(t)$ are divisible by $\Delta_{TH(2,3)}(t)$. The second aim of this paper is to generalize this property as follows.

**Theorem 1.3.** *For any positive integers $a, b, m, n$, $\Delta_{TH(am,bn)}(t)$ is divisible by $\Delta_{TH(m,n)}(t)$.*

Theorem 1.3 for $a = 1$ can be proved by a property of periodic links in [9, 11]. We remark that the $(m, bn)$-Turk's head link is a periodic link of order $b$, and the quotient is the $(m,n)$-Turk's head link. In this paper, we give an alternative proof from another viewpoint.

This paper is organized as follows. In Section 2, we review braids, Turk's head links, and the Conway polynomial. In Sections 3 and 4, we prove Theorem 1.1 for $n = 2$ and 3, respectively. In Section 5, we give supporting computational evidence for the conjecture that Theorem 1.1 holds for any $n \geq 2$ by the program "knotGTK" ([12]), which is the Windows version of the program "KNOT" ([5]). In Section 6, we give a Seifert matrix for $TH(m,n)$. In Section 7, we give a formula of $\Delta_{TH(m,n)}(t)$ by the determinant of a certain matrix of size $n-1$ (Theorem 7.1). In Section 8, we prove that $\Delta_{TH(cm,n)}(t)$ is divisible by $\Delta_{TH(m,n)}(t)$. In Section 9, we prove that $\Delta_{TH(m,cn)}(t)$ is divisible by $\Delta_{TH(m,n)}(t)$. The combination of these results implies Theorem 1.3 immediately.

## 2. Definitions

A *braid* is a collection of $n$ parallel strands such that adjacent strands are allowed to cross over or under one another (cf. [1, 2]). Two braids on the same number of strands can be composed by placing them end to end. The *braid group* on $n$ strands has a presentation with generators $\sigma_1, \sigma_2, \ldots, \sigma_{n-1}$ and the relations

$$\sigma_i\sigma_{i+1}\sigma_i = \sigma_{i+1}\sigma_i\sigma_{i+1}, \quad \text{and} \quad \sigma_i\sigma_j = \sigma_j\sigma_i \text{ for } |i-j| \geq 2.$$

Here $\sigma_i$ is the braid as shown in Figure 1. In this paper every braid is oriented from top to bottom.

Given a braid $\alpha$, the *closure* of $\alpha$ is the oriented link obtained by connecting the top and bottom of $\alpha$ simply as shown in Figure 2. We denote it by $Cl(\alpha)$.

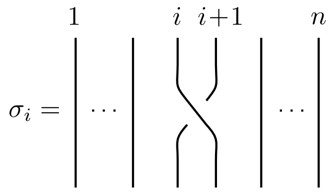


FIGURE 1

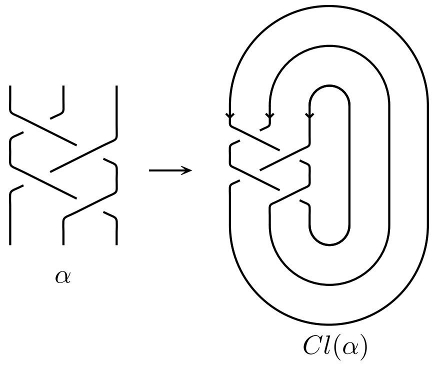


FIGURE 2

A *Markov move of type* 1 takes an $n$-strand braid to another $n$-strand braid via conjugation by $\sigma_i$ for some $i \in \{1, 2, \ldots, n-1\}$. A *Markov move of type* 2 takes an $n$-strand braid to an $(n+1)$-strand braid by adding $\sigma_n$ or $\sigma_n^{-1}$ to the end. In other words, an $n$-strand braid $\alpha$ becomes $\alpha\sigma_n$ or $\alpha\sigma_n^{-1}$.

**Theorem 2.1** ([7])**.** *The closures of two braids present the same knot or link if and only if one braid can be deformed into the other by a finite number of Markov moves or their inverses.* □

We denote by $A_m$ and $A_m^*$ the $m$-strand braids as shown in Figure 3.

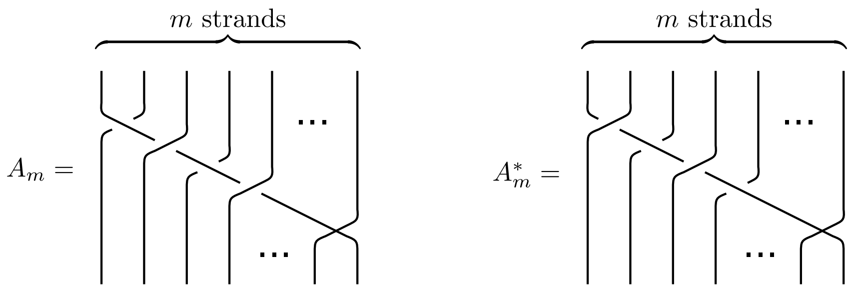


FIGURE 3

**Definition 2.2** ([10])**.** For any integrs $m, n \geq 2$, the $(m, n)$-*Turk's head link* is the closure of the $m$-strand braid $(A_m)^n$. We denote it by $TH(m, n)$.

We remark that the number of components of $TH(m,n)$ is the greatest common divisor $\mathrm{GCD}(m,n)$.

The *Conway polynomial* $\nabla_L(z)$ of an oriented link $L$ is a polynomial on $z$, which is computed by the following recursive formulas:

$$\begin{cases} \nabla_{L_+}(z) - \nabla_{L_-}(z) = z\nabla_{L_0}(z), \quad \text{and} \\ \nabla_{\bigcirc}(z) = 1, \end{cases}$$

where $\bigcirc$ is the trivial knot and $(L_+, L_-, L_0)$ is a skein triple of oriented knots or links that are identical except in a crossing neighborhood where they look as in Figure 4. We often abbreviate $\nabla_L(z)$ to $\nabla_L$.

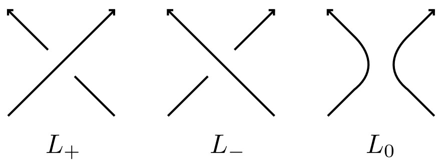


FIGURE 4

We denote by $L^*$ the mirror image of a link $L$. The Conway polynomial $\nabla_{L^*}$ satisfies

$$\nabla_{L^*} = \begin{cases} \nabla_L & \text{if the number of components of } L \text{ is odd,} \\ -\nabla_L & \text{if the number of components of } L \text{ is even.} \end{cases}$$

## 3. THE CONWAY POLYNOMIALS OF $TH(m,2)$ AND $TH(2,m)$

In this section, we prove Theorem 1.1 for $n = 2$.

**Lemma 3.1.** *The Conway polynomial of $TH(m,2)$ satisfies*

$$\begin{cases} \nabla_{TH(1,2)} = 1, \\ \nabla_{TH(2,2)} = -z, \quad \textit{and} \\ \nabla_{TH(m,2)} = \nabla_{TH(m-2,2)} - (-1)^m z \nabla_{TH(m-1,2)} \ (m \geq 3). \end{cases}$$

*Proof.* Since $TH(1,2)$ is the trivial knot, we have $\nabla_{TH(1,2)} = 1$. By the skein relation, it holds that

$$\begin{aligned} \nabla_{TH(2,2)} &= \nabla_{Cl(\sigma_1\sigma_1)} \\ &= \nabla_{Cl(\sigma_1^{-1}\sigma_1)} - z\nabla_{Cl(\sigma_1)} \\ &= -z. \end{aligned}$$

By the skein relation as shown in Figure 5, where a crossing in the skein relation is marked by a dot, we have

$$\nabla_{TH(m,2)} = \nabla_{TH(m-2,2)} - z\nabla_{TH^*(m-1,2)}$$

for $m \geq 3$. Since the number of components of $TH(m-1,2)$ is $\mathrm{GCD}(m-1,2)$, we have

$$\nabla_{TH^*(m-1,2)} = (-1)^m \nabla_{TH(m-1,2)}.$$

□

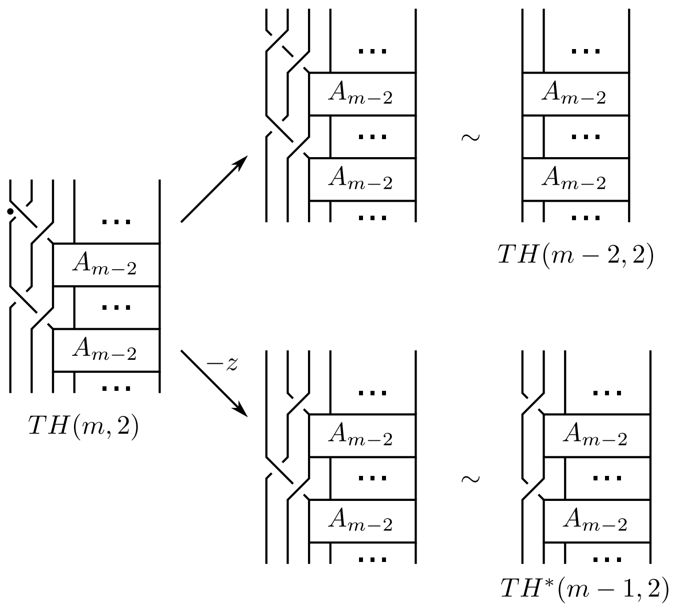


FIGURE 5

**Lemma 3.2.** *The Conway polynomial of $TH(2,m)$ satisfies*

$$\begin{cases} \nabla_{TH(2,1)} = 1, \\ \nabla_{TH(2,2)} = -z, \quad \text{and} \\ \nabla_{TH(2,m)} = \nabla_{TH(2,m-2)} - z\nabla_{TH(2,m-1)} \ (m \geq 3). \end{cases}$$

*Proof.* Since $TH(2,1)$ is the trivial knot, we have $\nabla_{TH(2,1)} = 1$. The second equation is given in Lemma 3.1. For $m \geq 3$, we have

$$\begin{aligned} \nabla_{TH(2,m)} =& \nabla_{Cl(\sigma_1^m)} \\ =& \nabla_{Cl(\sigma_1^{m-2})} - z\nabla_{Cl(\sigma_1^{m-1})} \\ =& \nabla_{TH(2,m-2)} - z\nabla_{TH(2,m-1)}. \end{aligned}$$

□

**Theorem 3.3.** *For any integer $m \geq 1$, the Conway polynomials*

$$\nabla_{TH(m,2)} = \sum_{i=0}^{\infty} a_i z^i \text{ and } \nabla_{TH(2,m)} = \sum_{i=0}^{\infty} b_i z^i$$

*satisfy*

$$\begin{cases} a_i = b_i & \text{for } i \equiv 0,1 \pmod 4, \text{ and} \\ a_i = -b_i & \text{for } i \equiv 2,3 \pmod 4. \end{cases}$$

*Proof.* We prove the theorem by induction on $m$. For $m = 1$, we have

$$\nabla_{TH(1,2)} = \nabla_{TH(2,1)} = 1.$$

Hence it holds that $a_0 = b_0 = 1$ and $a_i = b_i = 0$ for $i \geq 1$. For $m = 2$, we have

$$\nabla_{TH(2,2)} = -z.$$

Hence it holds that $a_1 = b_1 = -1$ and $a_i = b_i = 0$ for $i \neq 1$.

Assume that the theorem holds for $m = k-2$ and $k-1$ with $k \geq 3$. In other words, there are polynomials $f_i$ and $g_i \in \mathbb{Z}[z^4]$ $(i = 0, 1, 2, 3)$ such that

$$\begin{cases} \nabla_{TH(k-2,2)} = f_0 + zf_1 + z^2 f_2 + z^3 f_3, \\ \nabla_{TH(2,k-2)} = f_0 + zf_1 - z^2 f_2 - z^3 f_3, \\ \nabla_{TH(k-1,2)} = g_0 + zg_1 + z^2 g_2 + z^3 g_3, \quad \text{and} \\ \nabla_{TH(2,k-1)} = g_0 + zg_1 - z^2 g_2 - z^3 g_3. \end{cases}$$

By Lemmas 3.1 and 3.2, we have

$$\begin{cases} \nabla_{TH(k,2)} = \nabla_{TH(k-2,2)} - (-1)^k z \nabla_{TH(k-1,2)} \\ \qquad = (f_0 + (-1)^{k-1} z^4 g_3) + z(f_1 + (-1)^{k-1} g_0) \\ \qquad +z^2(f_2 + (-1)^{k-1} g_1) + z^3(f_3 + (-1)^{k-1} g_2), \quad \text{and} \\ \nabla_{TH(2,k)} = \nabla_{TH(2,k-2)} - z\nabla_{TH(2,k-1)} \\ \qquad = (f_0 + z^4 g_3) + z(f_1 - g_0) - z^2(f_2 + g_1) - z^3(f_3 - g_2). \end{cases}$$

(i) Assume that $k$ is odd. The number of components of $TH(k-2, 2)$ and $TH(k-1, 2)$ are one and two, respectively. Hence we have

$$\begin{cases} f_1 = f_3 = g_0 = g_2 = 0, \\ \nabla_{TH(k,2)} = (f_0 + z^4 g_3) + z^2(f_2 + g_1), \quad \text{and} \\ \nabla_{TH(2,k)} = (f_0 + z^4 g_3) - z^2(f_2 + g_1). \end{cases}$$

Therefore the theorem holds for $m = k$.

(ii) Assume that $k$ is even. The number of components of $TH(k-2, 2)$ and $TH(k-1, 2)$ are two and one, respectively. Hence we have

$$\begin{cases} f_0 = f_2 = g_1 = g_3 = 0, \\ \nabla_{TH(k,2)} = z(f_1 - g_0) + z^3(f_3 - g_2), \quad \text{and} \\ \nabla_{TH(2,k)} = z(f_1 - g_0) - z^3(f_3 - g_2). \end{cases}$$

Therefore the theorem holds for $m = k$. □

## 4. The Conway polynomials of $TH(m, 3)$ and $TH(3, m)$

In this section, we prove Theorem 1.1 for $n = 3$.

**Lemma 4.1.** *The Conway polynomial of $TH(m, 3)$ satisfies*

$$\begin{cases} \nabla_{TH(1,3)} = 1, \\ \nabla_{TH(2,3)} = 1 + z^2, \quad \textit{and} \\ \nabla_{TH(m,3)} = (-1 + z^2)\nabla_{TH(m-1,3)} - \nabla_{TH(m-2,3)} + 2 \ (m \geq 3). \end{cases}$$

*Proof.* Since $TH(1,3)$ is the trivial knot, we have $\nabla_{TH(1,3)} = 1$. By the skein relation, it holds that

$$\begin{aligned}\nabla_{TH(2,3)} &= \nabla_{Cl(\sigma_1^3)} \\ &= \nabla_{Cl(\sigma_1)} - z\nabla_{Cl(\sigma_1^2)} \\ &= 1 + z^2.\end{aligned}$$

For $m = 3$, we have

$$\begin{aligned}\nabla_{TH(3,3)} &= \nabla_{\text{a split link}} - z(\nabla_{TH(2,2)} + z\nabla_{TH(3,2)}) \\ &= -z(-z + z(\nabla_{\bigcirc} + z\nabla_{TH(2,2)})) \\ &= z^2 - z^2 + z^4 \\ &= z^4\end{aligned}$$

as shown in Figure 6. Then it holds that

$$\begin{aligned}(-1+z^2)\nabla_{TH(2,3)} - \nabla_{TH(1,3)} + 2 =&(-1+z^2)(1+z^2) - 1 + 2 \\ =& -1 + z^4 - 1 + 2 \\ =& z^4 \\ =& \nabla_{TH(3,3)}.\end{aligned}$$

Let $P(m)$ be the link $Cl(\sigma_1^{-1}A_m^3)$. By the skein relations as shown in Figures 7

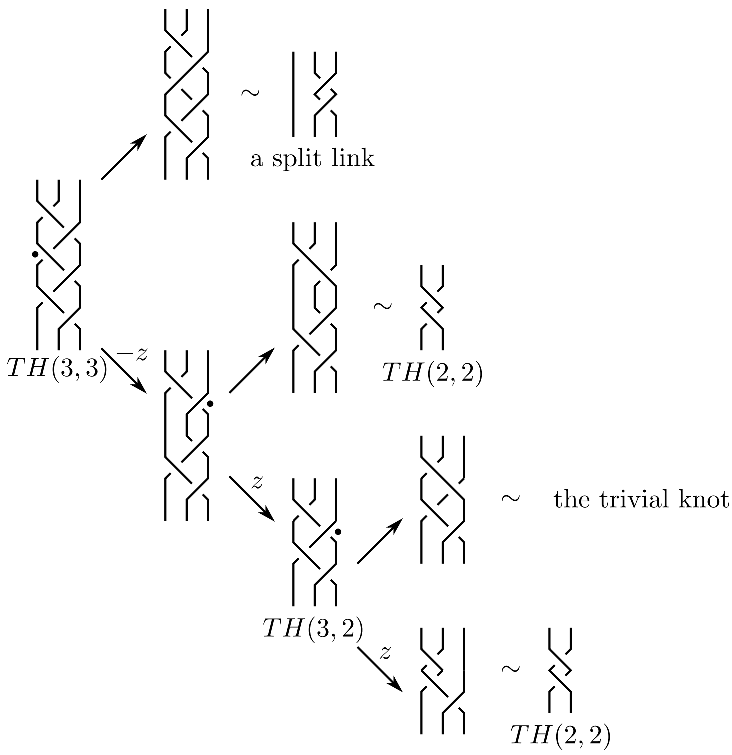


FIGURE 6

and 8, we have

$$\begin{cases} \nabla_{TH(m,3)} = \nabla_{TH^*(m-3,3)} + z\nabla_{P(m-2)} - z\nabla_{P(m)} \quad \text{for} \quad m \geq 4, \quad \text{and} \\ \nabla_{P(m)} = \nabla_{P^*(m-1)} - z\nabla_{TH^*(m-1,3)} \quad \text{for} \quad m \geq 3. \end{cases}$$

We remark that the numbers of components of $TH(m,3)$ and $P(m)$ are odd and even, respectively. Then it holds that

$$\begin{aligned} \nabla_{TH(m,3)} =& \nabla_{TH^*(m-3,3)} + z\nabla_{P(m-2)} - z\nabla_{P(m)} \\ =& \nabla_{TH^*(m-3,3)} + z\nabla_{P(m-2)} - z(\nabla_{P^*(m-1)} - z\nabla_{TH^*(m-1,3)}) \\ =& \nabla_{TH^*(m-3,3)} + z\nabla_{P(m-2)} \\ & - z((-\nabla_{P^*(m-2)} + z\nabla_{TH^*(m-2,3)}) - z\nabla_{TH^*(m-1,3)}) \\ =& \nabla_{TH^*(m-3,3)} + z\nabla_{P(m-2)} \\ & + z\nabla_{P^*(m-2)} - z^2\nabla_{TH^*(m-2,3)} + z^2\nabla_{TH^*(m-1,3)} \\ =& \nabla_{TH(m-3,3)} + z\nabla_{P(m-2)} \\ & - z\nabla_{P(m-2)} - z^2\nabla_{TH(m-2,3)} + z^2\nabla_{TH(m-1,3)} \\ =& \nabla_{TH(m-3,3)} - z^2\nabla_{TH(m-2,3)} + z^2\nabla_{TH(m-1,3)}. \end{aligned}$$

Hence we have

$$\begin{aligned} \nabla_{TH(m,3)} + (1-z^2)\nabla_{TH(m-1,3)} + \nabla_{TH(m-2,3)} & \\ &= \nabla_{TH(m-1,3)} + (1-z^2)\nabla_{TH(m-2,3)} + \nabla_{TH(m-3,3)} \\ &= \nabla_{TH(m-2,3)} + (1-z^2)\nabla_{TH(m-3,3)} + \nabla_{TH(m-4,3)} \\ &= \nabla_{TH(3,3)} + (1-z^2)\nabla_{TH(2,3)} + \nabla_{TH(1,3)}. \end{aligned}$$

Then we have

$$\begin{aligned} \nabla_{TH(m,3)} + (1-z^2)\nabla_{TH(m-1,3)} + \nabla_{TH(m-2,3)} & \\ &= \nabla_{TH(3,3)} + (1-z^2)\nabla_{TH(2,3)} + \nabla_{TH(1,3)} \\ &= z^4 + (1-z^2)(1+z^2) + 1 \\ &= 2. \end{aligned}$$

□

**Lemma 4.2.** *The Conway polynomial of $TH(3,m)$ satisfies*

$$\begin{cases} \nabla_{TH(3,1)} = 1, \\ \nabla_{TH(3,2)} = 1 - z^2, \quad \textit{and} \\ \nabla_{TH(3,m)} = (-1-z^2)\nabla_{TH(3,m-1)} - \nabla_{TH(3,m-2)} + 2 \ (m \geq 3). \end{cases}$$

*Proof.* Since $TH(3,1)$ is the trivial knot, we have $\nabla_{TH(3,1)} = 1$. By the skein relation as shown in a lower part of Figure 6, it holds that

$$\begin{aligned} \nabla_{TH(3,2)} &= \nabla_{\bigcirc} + z\nabla_{TH(2,2)} \\ &= 1 - z^2. \end{aligned}$$

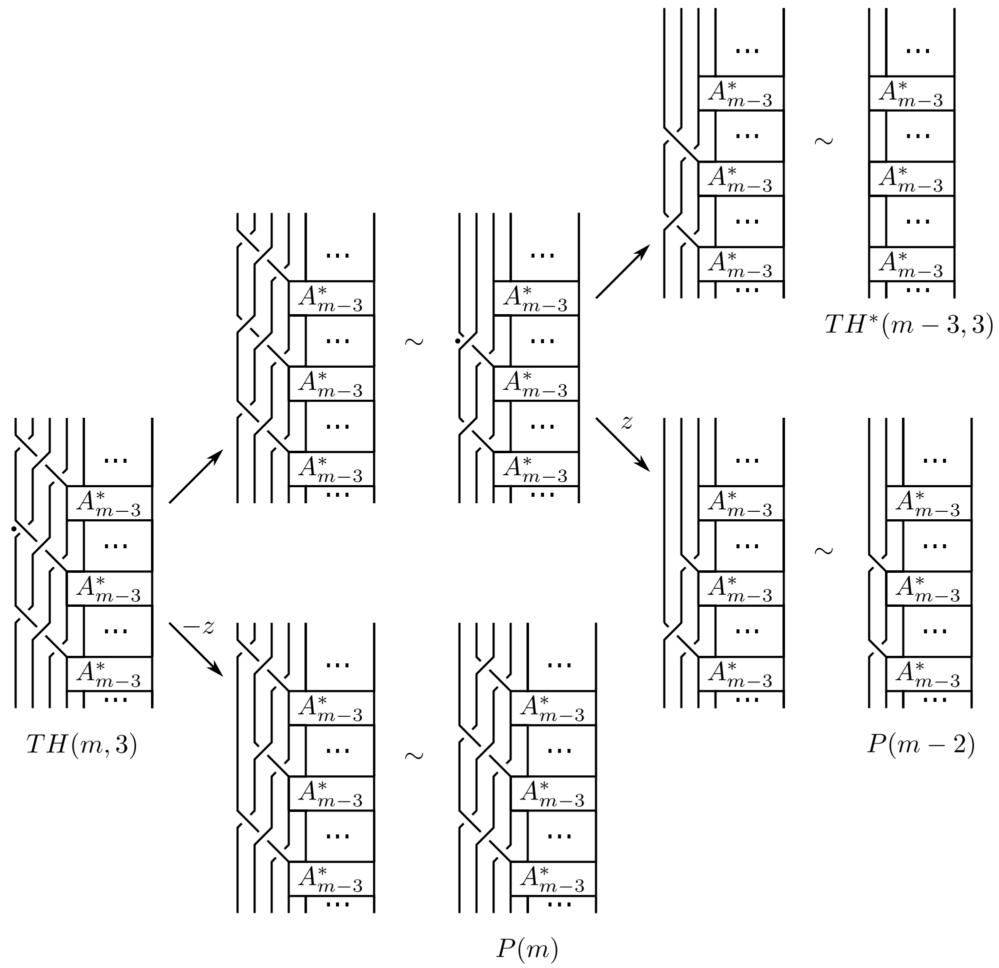


FIGURE 7

For $m = 3$, it holds that

$$\begin{aligned}(-1 - z^2)\nabla_{TH(3,2)} - \nabla_{TH(3,1)} + 2 =& (-1 - z^2)(1 - z^2) - 1 + 2\\ =& z^4\\ =& \nabla_{TH(3,3)}.\end{aligned}$$

Let $Q(m)$ and $R(m)$ be the links $Cl(\sigma_1\sigma_1\sigma_2 A_3^m)$ and $Cl(\sigma_1\sigma_1 A_3^m)$, respectively. By the skein relations as shown in Figures 9 and 10, we have

$$\begin{cases} \nabla_{TH(3,m)} = \nabla_{R(m-3)} + z(\nabla_{Q(m-2)} + z\nabla_{R(m-2)}) \\ \qquad\qquad = \nabla_{R(m-3)} + z\nabla_{Q(m-2)} + z^2\nabla_{R(m-2)}, \quad \text{and} \\ \nabla_{R(m)} = \nabla_{TH(3,m)} - z(\nabla_{Q(m-1)} + z\nabla_{R(m-1)}) \\ \qquad\quad = \nabla_{TH(3,m)} - z\nabla_{Q(m-1)} - z^2\nabla_{R(m-1)} \end{cases}$$

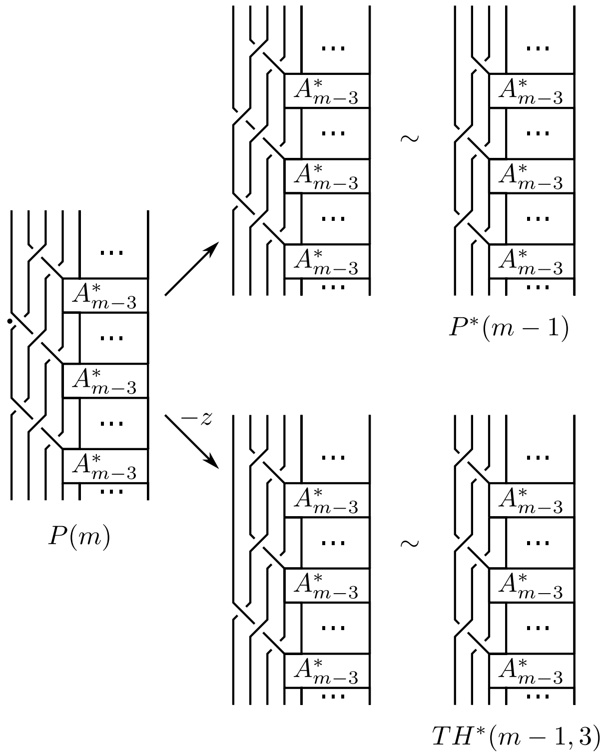


FIGURE 8

for $m \geq 4$. By Figure 11, we have

$$\nabla_{Q(m)} = \nabla_{Q(m-2)} = \nabla_{TH(2,2)} = -z.$$

Then it holds that

$$\begin{cases} \nabla_{TH(3,m)} = \nabla_{R(m-3)} - z^2 + z^2\nabla_{R(m-2)} \quad \text{for} \quad m \geq 4, \quad \text{and} \\ \nabla_{R(m)} = \nabla_{TH(3,m)} + z^2 - z^2\nabla_{R(m-1)} \quad \text{for} \quad m \geq 2. \end{cases}$$

By these equations, we have

$$\begin{cases} \nabla_{R(m-2)} = \dfrac{1}{z^4-1}(z^2\nabla_{TH(3,m)} - \nabla_{TH(3,m-2)} + z^4 - z^2), \quad \text{and} \\ \nabla_{R(m-3)} = \dfrac{1}{z^4-1}(-\nabla_{TH(3,m)} + z^2\nabla_{TH(3,m-2)} + z^4 - z^2). \end{cases}$$

Therefore we obtain

$$z^2\nabla_{TH(3,m)} - \nabla_{TH(3,m-2)} = -\nabla_{TH(3,m+1)} + z^2\nabla_{TH(3,m-1)} \quad \text{for} \quad m \geq 4$$

and hence

$$\begin{aligned} \nabla_{TH(3,m+1)} &+ (1+z^2)\nabla_{TH(3,m)} + \nabla_{TH(3,m-1)} \\ &= \nabla_{TH(3,m)} + (1+z^2)\nabla_{TH(3,m-1)} + \nabla_{TH(3,m-2)} \\ &= \nabla_{TH(3,3)} + (1+z^2)\nabla_{TH(3,2)} + \nabla_{TH(3,1)}. \end{aligned}$$

Then we have

$$\begin{aligned}\nabla_{TH(3,m)} + (1+z^2)\nabla_{TH(3,m-1)} + \nabla_{TH(3,m-2)} \\ &= \nabla_{TH(3,3)} + (1+z^2)\nabla_{TH(3,2)} + \nabla_{TH(3,1)} \\ &= z^4 + (1+z^2)(1-z^2) + 1 \\ &= 2.\end{aligned}$$

□

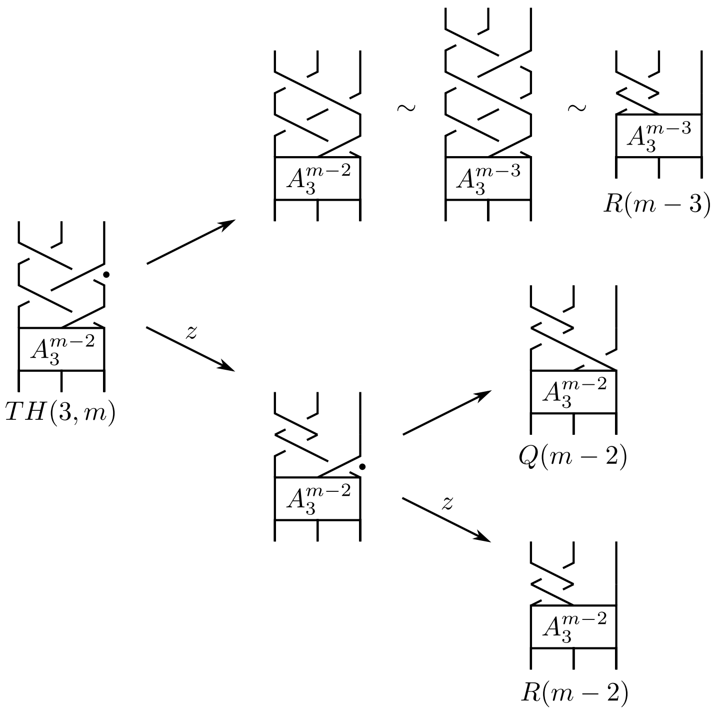


FIGURE 9

**Theorem 4.3.** *For any integer $m \geq 1$, the Conway polynomials*

$$\nabla_{TH(m,3)} = \sum_{i=0}^{\infty} a_i z^i \quad \text{and} \quad \nabla_{TH(3,m)} = \sum_{i=0}^{\infty} b_i z^i$$

*satisfy*

$$\begin{cases} a_i = b_i & \text{for } i \equiv 0,1 \pmod 4, \quad \text{and} \\ a_i = -b_i & \text{for } i \equiv 2,3 \pmod 4. \end{cases}$$

*Proof.* We prove the theorem by induction on $m$. For $m = 1$, we have

$$\nabla_{TH(1,3)} = \nabla_{TH(3,1)} = 1.$$

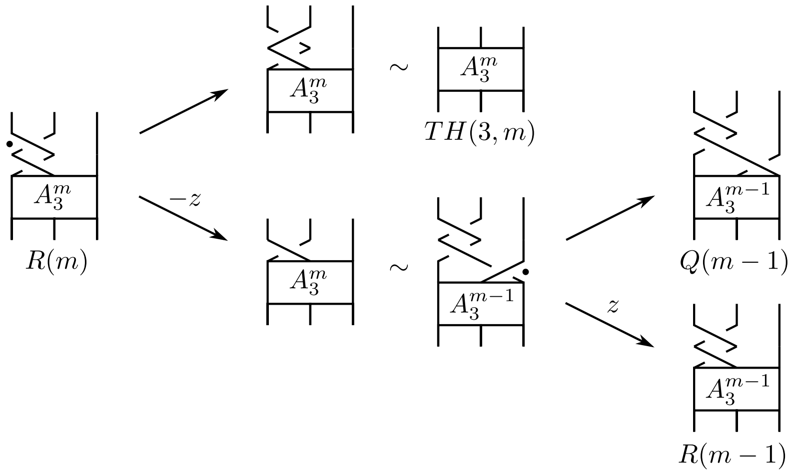


FIGURE 10

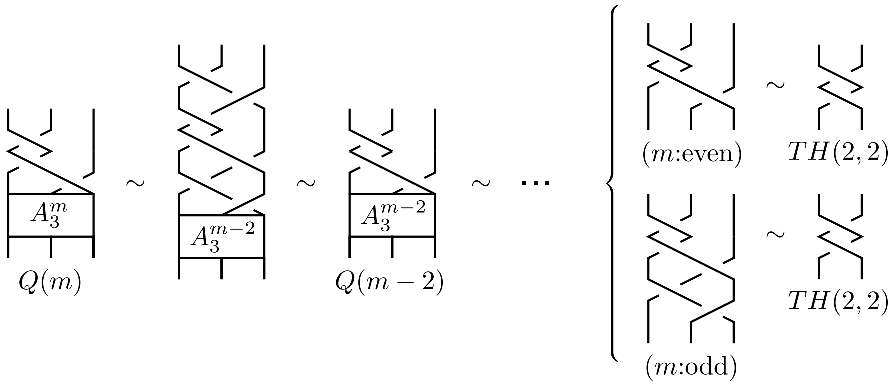


FIGURE 11

Hence it holds that $a_0 = b_0 = -1$ and $a_i = b_i = 0$ for $i \geq 1$. For $m = 2$, we have

$$\begin{cases} \nabla_{TH(2,3)} = 1 + z^2, & \text{and} \\ \nabla_{TH(3,2)} = 1 - z^2. \end{cases}$$

Hence it holds that $a_0 = b_0 = 1$, $a_2 = -b_2 = 1$ and $a_i = b_i = 0$ for $i \neq 0, 2$.

Assume that the theorem holds for $m = k - 2$ and $k - 1$ with $k \geq 3$. In other words, there are polynomials $f_i$ and $g_i \in \mathbb{Z}[z^4]$ $(i = 0, 2)$ such that

$$\begin{cases} \nabla_{TH(k-2,3)} = f_0 + z^2 f_2, \\ \nabla_{TH(3,k-2)} = f_0 - z^2 f_2, \\ \nabla_{TH(k-1,3)} = g_0 + z^2 g_2, & \text{and} \\ \nabla_{TH(3,k-1)} = g_0 - z^2 g_2. \end{cases}$$

By Lemmas 4.1 and 4.2, we have

$$\begin{cases} \nabla_{TH(k,3)} = (-1+z^2)\nabla_{TH(k-1,3)} - \nabla_{TH(k-2,3)} + 2 \\ \qquad\qquad = (-f_0 - g_0 + z^4 g_2 + 2) + z^2(-f_2 + g_0 - g_2), \quad \text{and} \\ \nabla_{TH(3,k)} = (-1-z^2)\nabla_{TH(3,k-1)} - \nabla_{TH(3,k-2)} + 2 \\ \qquad\qquad = (-f_0 - g_0 + z^4 g_2 + 2) - z^2(-f_2 + g_0 - g_2). \end{cases}$$

Therefore the theorem holds for $m = k$. □

## 5. Conjecture

By computer calculations, we have

$$\begin{cases} \nabla_{TH(4,5)} = 1 + 5z^2 + 6z^4 - 3z^6 - 6z^8 + z^{10} + z^{12}, \\ \nabla_{TH(5,4)} = 1 - 5z^2 + 6z^4 + 3z^6 - 6z^8 - z^{10} + z^{12}, \end{cases}$$

$$\begin{cases} \nabla_{TH(4,6)} = -6z - 5z^3 + 14z^5 + 11z^7 - 10z^9 - 7z^{11} + 2z^{13} + z^{15}, \\ \nabla_{TH(6,4)} = -6z + 5z^3 + 14z^5 - 11z^7 - 10z^9 + 7z^{11} + 2z^{13} - z^{15}, \end{cases}$$

$$\begin{cases} \nabla_{TH(5,7)} = 1 - 8z^2 - 2z^4 + 82z^6 + 57z^8 - 156z^{10} \\ \qquad -113z^{12} + 106z^{14} + 72z^{16} - 26z^{18} - 15z^{20} + 2z^{22} + z^{24}, \\ \nabla_{TH(7,5)} = 1 + 8z^2 - 2z^4 - 82z^6 + 57z^8 + 156z^{10} \\ \qquad -113z^{12} - 106z^{14} + 72z^{16} + 26z^{18} - 15z^{20} - 2z^{22} + z^{24}, \end{cases}$$

$$\nabla_{TH(4,4)} = -4z^5 + z^9,$$
$$\nabla_{TH(5,5)} = 25z^8 - 10z^{12} + z^{16}, \quad \text{and}$$
$$\nabla_{TH(6,6)} = -144z^9 + 232z^{13} - 105z^{17} + 18z^{21} - z^{25}.$$

By these equations, we conjecture the following.

**Conjecture 5.1.** *For any integers $m \geq 1$ and $n \geq 2$, the Conway polynomials*

$$\nabla_{TH(m,n)} = \sum_{i=0}^{\infty} a_i z^i \quad \textit{and} \quad \nabla_{TH(n,m)} = \sum_{i=0}^{\infty} b_i z^i$$

*satisfy*

$$\begin{cases} a_i = b_i & \textit{for} \quad i \equiv 0, 1 \pmod 4, \quad \textit{and} \\ a_i = -b_i & \textit{for} \quad i \equiv 2, 3 \pmod 4. \end{cases}$$

## 6. A Seifert matrix of $TH(m,n)$

Let $L$ be a link in $S^3$, $F$ a Seifert surface for $L$, and $V$ a Seifert matrix associated with $F$. The Alexander polynomial $\Delta_L(t)$ of $L$ is defined by $\Delta_L(t) \doteq \left| t^{\frac{1}{2}} V - t^{-\frac{1}{2}} V^T \right|$. Here, $f(t) \doteq g(t)$ means that $f(t) = \pm t^c g(t)$ for some integer $c$.

A Seifert matrix of $TH(m,n)$ is obtained as follows. Let $M_n$ be the square matrix of size $n-1$ of the following form.

$$M_n = \begin{pmatrix} 1 & -1 & & \text{\Large 0} \\ & \ddots & \ddots & \\ & & \ddots & -1 \\ \text{\Large 0} & & & 1 \end{pmatrix}.$$

Let $V_{mn}$ be the square matrix of size $(m-1)(n-1)$ of the following form.

$$V_{mn} = \begin{cases} \begin{pmatrix} M_n & & & & & \text{\Large 0} \\ -M_n & -M_n^T & & & & \\ & -M_n & M_n & & & \\ & & \ddots & \ddots & & \\ & & & -M_n & -M_n^T & \\ \text{\Large 0} & & & & -M_n & M_n \end{pmatrix} & (m : \text{even}), \\ \begin{pmatrix} M_n & & & & & \text{\Large 0} \\ -M_n & -M_n^T & & & & \\ & -M_n & M_n & & & \\ & & \ddots & \ddots & & \\ & & & -M_n & M_n & \\ \text{\Large 0} & & & & -M_n & -M_n^T \end{pmatrix} & (m : \text{odd}), \end{cases}$$

where $M_n^T$ denotes the transposed matrix of $M_n$.

**Lemma 6.1.** *$V_{mn}$ is a Seifert matrix of $TH(m,n)$.*

*Proof.* Let $F$ and $\{a_{ij} | 1 \le i \le m-1, 1 \le j \le n-1\}$ be the Seifert surface for $TH(m,n)$ and the basis of $H_1(F;\mathbb{Z})$ as shown in Figure 12, respectively. Then we see that $V_{mn}$ is a Seifert matrix of $TH(m,n)$ associated with the basis. $\square$

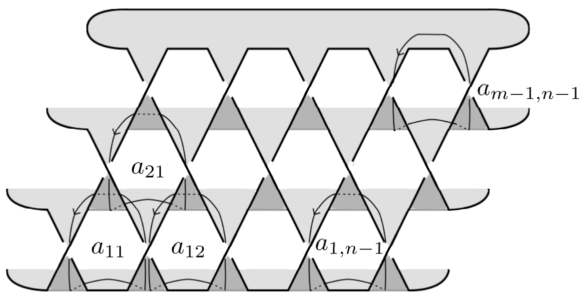


FIGURE 12

## 7. THE ALEXANDER POLYNOMIAL OF $TH(m,n)$

For any integer $n \ge 2$, let $B_n$ be the square matrix of size $n-1$ defined by

$$B_2 = (1) \text{ and } B_n = \begin{pmatrix} 0 & \cdots & 0 & 1 \\ -1 & & \text{\Large 0} & \vdots \\ & \ddots & & 1 \\ \text{\Large 0} & & -1 & 1 \end{pmatrix} \quad (n \ge 3),$$

We remark that $|B_n| = 1$. Let $X_n = X_n(t)$ and $Y_n = Y_n(t)$ be the matrices defined by

$$X_n(t) = t^{\frac{1}{2}} E_{n-1} - t^{-\frac{1}{2}} B_n \text{ and } Y_n(t) = X_n(t^{-1}),$$

respectively. Here, $E_{n-1}$ is the unit matrix of size $n-1$.

We define the square matrix $P_{mn} = P_{mn}(t)$ of size $n-1$ inductively as follows. For $m = 2, 3$, we have

$$P_{2n} = X_n, \quad P_{3n} = t^{-\frac{1}{2}} X_n Y_n + t^{-\frac{1}{2}} B_n.$$

for $m \geq 4$, we have

$$P_{mn} = \begin{cases} t^{-\frac{1}{2}} X_n P_{m-1,n} + t^{-1} B_n P_{m-2,n} & (m : \text{even}), \\ t^{-\frac{1}{2}} Y_n P_{m-1,n} + t^{-1} B_n P_{m-2,n} & (m : \text{odd}). \end{cases}$$

Then the Alexander polynomial of $TH(m,n)$ is obtained as follows.

**Theorem 7.1.** *For any integers $m, n \geq 2$, we have*

$$\Delta_{TH(m,n)}(t) \doteq |P_{mn}(t)|.$$

*Proof.* Let $X'_n = X'_n(t)$ and $Y'_n = Y'_n(t)$ be the matrices defined by

$$X'_n(t) = t^{\frac{1}{2}} M_n - t^{-\frac{1}{2}} M_n^T \quad \text{and} \quad Y'_n(t) = X'_n(t^{-1}).$$

First, we concider the case when $m$ is odd.

$$\Delta_{TH(m,n)} \doteq \left| t^{\frac{1}{2}} V_{mn} - t^{-\frac{1}{2}} V_{mn}^T \right|$$

$$= \begin{vmatrix} X'_n & t^{-\frac{1}{2}} M_n^T & & & & 0 \\ -t^{\frac{1}{2}} M_n & Y'_n & t^{-\frac{1}{2}} M_n^T & & & \\ & -t^{\frac{1}{2}} M_n & X'_n & \ddots & & \\ & & \ddots & \ddots & t^{-\frac{1}{2}} M_n^T & \\ & & & -t^{\frac{1}{2}} M_n & X'_n & t^{-\frac{1}{2}} M_n^T \\ 0 & & & & -t^{\frac{1}{2}} M_n & Y'_n \end{vmatrix}.$$

For each $0 \leq i \leq m-2$, we perform the fundamental deformations on this determinant as follows.

- Add the sum of $i(n-1)+j$th rows for $2 \leq j \leq n-1$ to the $i(n-1)+1$st row.
- Add the sum of $i(n-1)+j$th rows for $3 \leq j \leq n-1$ to the $i(n-1)+2$nd row.
- ...
- Add the $i(n-1)+(n-1)$st row to the $i(n-1)+(n-2)$nd row.

Then we have

$$\Delta_{TH(m,n)} \doteq \begin{vmatrix} X_n & t^{-\frac{1}{2}}B_n & & & & 0 \\ -t^{\frac{1}{2}}E_{n-1} & Y_n & t^{-\frac{1}{2}}B_n & & & \\ & -t^{\frac{1}{2}}E_{n-1} & X_n & \ddots & & \\ & & \ddots & \ddots & t^{-\frac{1}{2}}B_n & \\ & & & -t^{\frac{1}{2}}E_{n-1} & X_n & t^{-\frac{1}{2}}B_n \\ 0 & & & & -t^{\frac{1}{2}}E_{n-1} & Y_n \end{vmatrix}.$$

This matrix is divided into $(m-1)^2$ blocks of size $n-1$. We perform the fundamental deformation on this determinant as follows.

- Multiply the 2nd row of blocks by $t^{-\frac{1}{2}}X_n = t^{-\frac{1}{2}}P_{2n}$ from the left, and add them to the 1st row of blocks.
- Multiply the 3rd row of blocks by $t^{-\frac{1}{2}}P_{3n}$ from the left, and add them to the 1st row of blocks.
- ...
- Multiply the $m-1$st row of blocks by $t^{-\frac{1}{2}}P_{m-1,n}$ from the left, and add them to the 1st row of blocks.

By the definiton of $P_{mn}$, we have

$$\Delta_{TH(m,n)} \doteq \begin{vmatrix} 0 & & \cdots & & 0 & P_{mn} \\ -t^{\frac{1}{2}}E_{n-1} & Y_n & t^{-\frac{1}{2}}B_n & & & 0 \\ & -t^{\frac{1}{2}}E_{n-1} & X_n & \ddots & & \\ & & \ddots & \ddots & t^{-\frac{1}{2}}B_n & \\ & & & -t^{\frac{1}{2}}E_{n-1} & X_n & t^{-\frac{1}{2}}B_n \\ 0 & & & & -t^{\frac{1}{2}}E_{n-1} & Y_n \end{vmatrix}$$
$$= \left((-1)^n(-t^{\frac{1}{2}})\right)^{(m-2)(n-1)} |P_{mn}| \doteq |P_{mn}|.$$

□

There is an explicit formula for $P_{mn}$ as follows.

**Lemma 7.2.** *For any $m, n \geq 2$, it holds that*

$$P_{mn} = \begin{cases} t^{-\frac{m-2}{2}} X_n B_n^{\frac{m-2}{2}} \displaystyle\sum_{k=0}^{\frac{1}{2}(m-2)} \binom{\frac{m}{2}+k}{\frac{m-2}{2}-k} X_n^k Y_n^k B_n^{-k} & (m : \textit{even}), \\ t^{-\frac{m-2}{2}} B_n^{\frac{m-1}{2}} \displaystyle\sum_{k=0}^{\frac{1}{2}(m-1)} \binom{\frac{m-1}{2}+k}{\frac{m-1}{2}-k} X_n^k Y_n^k B_n^{-k} & (m : \textit{odd}). \end{cases}$$

*Proof.* The lemma can be proved by the induction on $m$. □

## 8. First divisibility on $\Delta_{TH(m,n)}(t)$

In this section, we prove the following.

**Theorem 8.1.** *For any positive integers $c, m, n$, $\Delta_{TH(cm,n)}(t)$ is divisible by $\Delta_{TH(m,n)}(t)$.*

For any integer $m \geq 0$, let $f_m = f_m(x)$ be the polynomial defined as follows. For $m = 0, 1, 2$, we have

$$f_0(x) = 0, \quad f_1(x) = 1, \quad f_2(x) = 1,$$

For $m \geq 3$, we have

$$f_m(x) = \begin{cases} \displaystyle\sum_{k=0}^{\frac{m-2}{2}} \binom{\frac{m}{2}+k}{\frac{m-2}{2}-k} x^k & (m : \text{even}), \\ \displaystyle\sum_{k=0}^{\frac{m-1}{2}} \binom{\frac{m-1}{2}+k}{\frac{m-1}{2}-k} x^k & (m : \text{odd}). \end{cases}$$

The sequence of polynomials $\{f_m(x) | m \geq 0\}$ satisfies the Fibonacci property as follows. The proof is straightforward, and we omit it.

**Lemma 8.2.** *For any integers $m, n \geq 0$ and $c \geq 2$, $\{f_m(x)\}$ satisfies the following.*

(i) $f_{m+2} = \begin{cases} f_{m+1} + f_m & (m \equiv 0 \pmod 2) \text{ and} \\ x f_{m+1} + f_m & (m \equiv 1 \pmod 2). \end{cases}$

(ii) $f_{m+n+1} = \begin{cases} f_m f_n + f_{m+1} f_{n+1} & (m \not\equiv n \pmod 2), \\ f_m f_n + x f_{m+1} f_{n+1} & (m \equiv n \equiv 1 \pmod 2) \text{ and} \\ x f_m f_n + f_{m+1} f_{n+1} & (m \equiv n \equiv 0 \pmod 2). \end{cases}$

(iii) *$f_{cm}$ is divisible by $f_m$.* □

*Proof of Theorem 8.1.* By Theorem 7.1 and Lemma 7.2, we have

$$\Delta_{TH(m,n)}(t) \doteq \begin{cases} |X_n| \left| f_m(X_n Y_n B_n^{-1}) \right| & (m : \text{even}), \\ \left| f_m(X_n Y_n B_n^{-1}) \right| & (m : \text{odd}). \end{cases}$$

Hence we have

$$\frac{\Delta_{TH(cm,n)}(t)}{\Delta_{TH(m,n)}(t)} \doteq \begin{cases} \dfrac{|X_n|\,\left|f_{cm}(X_nY_nB_n^{-1})\right|}{\left|f_m(X_nY_nB_n^{-1})\right|} & (m : \text{odd and } c : \text{even}), \\ \\ \dfrac{\left|f_{cm}(X_nY_nB_n^{-1})\right|}{\left|f_m(X_nY_nB_n^{-1})\right|} & (\text{otherwise}). \end{cases}$$

By Lemma 8.2, we have $f_{cm}(x) = g(x)f_m(x)$ for some polynomial $g(x)$ and

$$f_{cm}(X_nY_nB_n^{-1}) = g(X_nY_nB_n^{-1})f_m(X_nY_nB_n^{-1}).$$

Then we have

$$\frac{\Delta_{TH(cm,n)}(t)}{\Delta_{TH(m,n)}(t)} \doteq \begin{cases} |X_n|\,\left|g(X_nY_nB_n^{-1})\right| & (m : \text{odd and } c : \text{even}), \\ \\ \left|g(X_nY_nB_n^{-1})\right| & (\text{otherwise}). \end{cases}$$

Therefore, $\Delta_{TH(cm,n)}(t)$ is divisible by $\Delta_{TH(m.n)}(t)$. □

## 9. Second divisibility on $\Delta_{TH(m,n)}(t)$.

In this section, we prove the following.

**Theorem 9.1.** *For any positive integers $c, m, n$, $\Delta_{TH(m,cn)}(t)$ is divisible by $\Delta_{TH(m,n)}(t)$.*

For the proof of this theorem, we use the following lemma.

**Lemma 9.2.** $$|B_n + \alpha E_{n-1}| = \sum_{k=0}^{n-1} \alpha^k.$$

*Proof.* We have

$$|B_n + \alpha E_{n-1}| = \begin{vmatrix} \alpha & & & 0 & 1 \\ -1 & \alpha & & & 1 \\ & \ddots & \ddots & & \vdots \\ & & -1 & \alpha & 1 \\ 0 & & & -1 & 1+\alpha \end{vmatrix}$$

$$= \begin{vmatrix} \alpha & & & 0 & 1 \\ -1 & \alpha & & & 1 \\ & \ddots & \ddots & & \vdots \\ & & -1 & \alpha & 1 \\ 0 & & & -1 & 1 \end{vmatrix} + \begin{vmatrix} \alpha & & & 0 & 1 \\ -1 & \alpha & & & 1 \\ & \ddots & \ddots & & \vdots \\ & & -1 & \alpha & 1 \\ 0 & & & & \alpha \end{vmatrix}$$

$$= \sum_{k=0}^{n-2} \alpha^k + \alpha^{n-1} = \sum_{k=0}^{n-1} \alpha^k.$$

□

*Proof of Theorem 9.1.* By Theorem 7.1, Lemmas 7.2 and 9.2, we have

$$\Delta_{TH(m,n)}(t) \doteq \begin{cases} \displaystyle\sum_{k=0}^{n-1}(-t)^k \\ \displaystyle\times\left|\sum_{k=0}^{\frac{1}{2}(m-2)} \binom{\frac{m}{2}+k}{\frac{m-2}{2}-k}(B_n^2-(t^{-1}+t)B_n+E_{n-1})^k B_n^{\frac{m-2}{2}-k}\right| & (m:\text{even}), \\ \displaystyle\left|\sum_{k=0}^{\frac{1}{2}(m-1)} \binom{\frac{m-1}{2}+k}{\frac{m-1}{2}-k}(B_n^2-(t^{-1}+t)B_n+E_{n-1})^k B_n^{\frac{m-1}{2}-k}\right| & (m:\text{odd}). \end{cases}$$

Then we represent $\Delta_{TH(m,n)}(t)$ as follows.

$$\Delta_{TH(m,n)}(t) \doteq \begin{cases} \displaystyle\sum_{k=0}^{n-1}(-t)^k\left|\sum_{k=0}^{m-2} a_{mk}B_n^k\right| & (m:\text{even}), \\ \displaystyle\left|\sum_{k=0}^{m-1} a_{mk}B_n^k\right| & (m:\text{odd}) \end{cases}$$

for some $a_{m0}, a_{m1}, \ldots, a_{m,m-1} \in \mathbb{Z}[t,t^{-1}]$. Here $a_{m,m-2} = 1$ for even $m$ and $a_{m,m-1} = 1$ for odd $m$. Hence we have

$$\frac{\Delta_{TH(m,cn)}(t)}{\Delta_{TH(m,n)}(t)} \doteq \begin{cases} \displaystyle\sum_{k=0}^{c-1}(-t)^{kn}\frac{\left|\sum_{k=0}^{m-2} a_{mk}B_{cn}^k\right|}{\left|\sum_{k=0}^{m-2} a_{mk}B_n^k\right|} & (m:\text{even}), \\ \displaystyle\frac{\left|\sum_{k=0}^{m-1} a_{mk}B_{cn}^k\right|}{\left|\sum_{k=0}^{m-1} a_{mk}B_n^k\right|} & (m:\text{odd}). \end{cases}$$

Hence we have the conclusion by Lemma 9.3 below. □

**Lemma 9.3.** *For any $\ell$ Laurent polynomials $h_0, h_1, \ldots, h_{\ell-1} \in \mathbb{Z}[t,t^{-1}]$ and $h_\ell = 1$, $\left|\sum_{k=0}^{\ell} h_k B_{cn}^k\right|$ is divisible by $\left|\sum_{k=0}^{\ell} h_k B_n^k\right|$.*

*Proof.* We consider a factorization of the polynomial $\sum_{k=0}^{\ell} h_k x^k$ such that

$$\sum_{k=0}^{\ell} h_k x^k = \prod_{k=1}^{\ell}(x+\alpha_k),$$

where $\alpha_1, \alpha_2, \ldots, \alpha_\ell$ are elements in some finite extension field of a quotient field of $\mathbb{Z}[t,t^{-1}]$. We remark that any symmetric polynomial in $\alpha_1, \alpha_2, \ldots, \alpha_\ell$ can be expressed as a polynomial in $h_0, h_1, \ldots, h_{\ell-1}$.

By substituting the matrix $B_n$ for $x$, we have

$$\left|\sum_{k=0}^{\ell} h_k B_n^k\right| = \prod_{k=1}^{\ell} |B_n + \alpha_k E_{n-1}| = \prod_{k=1}^{\ell} \sum_{i=0}^{n-1} \alpha_k^i$$

by Lemma 9.2. Similarly by substituting $B_{cn}$ for $x$, we have

$$\frac{\left|\sum_{k=0}^{\ell} h_k B_{cn}^k\right|}{\left|\sum_{k=0}^{\ell} h_k B_n^k\right|} = \prod_{k=1}^{\ell} \frac{\sum_{i=0}^{cn-1} \alpha_k^i}{\sum_{i=0}^{n-1} \alpha_k^i} = \prod_{k=1}^{\ell} \sum_{i=0}^{c-1} \alpha_k^{in}.$$

Since $\prod_{k=1}^{\ell} \sum_{i=0}^{c-1} \alpha_k^{in}$ is a symmetric polynomial in $\alpha_1, \alpha_2, \ldots, \alpha_\ell$, it is expressed as a polynomial in $h_0, h_1, \ldots, h_{\ell-1}$ and hence in $\mathbb{Z}[t, t^{-1}]$. Therefore $\left|\sum_{k=0}^{\ell} h_k B_{cn}^k\right|$ is divisible by $\left|\sum_{k=0}^{\ell} h_k B_n^k\right|$. □

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