Parallel and focal surfaces of wave fronts

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博　士　論　文

Parallel and focal surfaces of wave fronts
波面の平行曲面と焦曲面

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Contents

CHAPTER 1

Introduction

The study of singular surfaces from differential geometric viewpoints is developing with each passing day. In particular, studies of geometry of wave front are remarkable ([**10, 13, 17, 27, 28, 29, 32, 31, 37, 46, 44, 45**]). Wave fronts are due to the following Huygens' principle: Every point on a wave front may be considered a source of secondary spherical wavelets which spread out in the forward direction at the speed of light. The new wave front is the tangential surface to all of these secondary wavelets. V. I. Arnol'd connected singularities of wave fronts and singularities of C^∞ functions and classified generic singularities of wave fronts ([**1, 2**]).

On the other hand, for regular surfaces in the Euclidean 3-space $\boldsymbol{R}^3$, their parallel surfaces and focal surfaces might have singularities in general. Since parallel surfaces of regular surfaces are wave fronts, types of singularities appearing on parallel surfaces are related to Legendrian singularities. Moreover, types of singularities of focal surfaces are related to Lagrangian singularities. In differential geometric viewpoints, these singularities can be characterized by behavior of principal curvatures. In fact, I. R. Porteous studied distance squared functions on surfaces and showed relationships between singularities of focal surfaces and geometric properties of initial regular surfaces. In addition, he found new geometric properties of surfaces called *ridge points* which correspond to cuspidal edges on focal surfaces ([**41, 42**], see also [**4, 11, 12, 20**]). In addition, R. Morris investigated behavior of Gaussian curvatures of focal surfaces and showed relationship between parabolic points on focal surfaces and geometric properties called *sub-parabolic points* on initial surfaces ([**35**]). Both ridge points and sub-parabolic points relate to behavior of principal curvatures. Hence we might obtain new geometric properties of wave fronts investigating singularities of parallel surfaces and focal surfaces.

In this thesis, we consider parallel surfaces and focal surfaces of wave fronts. As in the case of regular surfaces, we might need to consider behavior of principal curvatures for showing relations between types of singularities appearing on parallel surfaces or focal surfaces and geometric properties of initial wave fronts. However, for wave fronts, the following facts are known ([**46**]):

- the Gaussian curvatures of wave fronts are generically unbounded near singular points,
- the mean curvatures of wave fronts are unbounded near singular points.

Thus at least one principal curvature of a wave front might be unbounded near a singular point. In Chapter 3, we show that one principal curvature of a wave front with non-degenerate singular points can be extended as a bounded function on a neighborhood of such a singular point (Theorem 3.6). Moreover, we give a criterion for which principal curvature becomes a bounded even at a non-degenerate singular point by using geometric invariants of fronts obtained in [**32**]. For a bounded principal curvature, we can define a principal vector with respect to it. By using a bounded principal curvature and relative principal vector, we introduce a notion of a *ridge point* for wave fronts. Moreover, we extend the notion of a *line of curvature* and give a condition that singular loci become lines of curvature in terms of geometric invariants (Proposition 3.8). Further, for cuspidal edges, we can define principal vector with respect to unbounded principal curvature by some modifications. Using this principal vector, we define a *sub-parabolic point* for a cuspidal edge, and we give relations among ridge points, sub-parabolic points and known geometric invariants of cuspidal edges (Proposition 3.11).

In Chapter 4, we consider parallel surfaces of wave fronts. It is known that parallel surfaces are also wave fronts. Thus they have singularities in general. We characterize types of singularities appearing on parallel surfaces by ridge points and behavior of a bounded principal curvature (Theorem 4.2). In addition, we consider constant principal curvature (CPC) lines near cuspidal edges. It is known that CPC lines correspond to the set of singular points of parallel surfaces ([**11, 12**]). Using parallel surfaces, we define special points (landmark in the sense of [**42**]) on cuspidal edge as cusps of CPC lines, which seems not to have appeared in the literature (Section 2).

In Chapter 5, we study focal surfaces of wave fronts. We show relations between singularities of a focal surface with respect to the bounded principal curvature and geometric properties of an original front (Theorem 5.6). On the other hand, we consider geometric properties of the another focal surface. This contains the image of the set of singular points of the original front as a curve on it. If the original front has a cuspidal edge, then the focal surface is regular near a cuspidal edge (Proposition 5.7). Thus we can consider the Gaussian curvature and the mean curvature of the focal surface corresponding to the cuspidal edge. We give explicit representations of the Gaussian and the mean curvature of the focal surfaces along the cuspidal edge by using geometric invariants (Theorem 5.8). As an application, we consider a focal surface of the *Beltrami's pseudosphere* which is a negative constant Gaussian curvature surface with singularities. It is known that the screw motion of Beltrami's pseudosphere makes the *Dini's surface*. We also investigate a focal surface of the screw motion of Beltrami's pseudosphere, namely, Dini's surface and give a geometric interpretation of the screw motion of the focal surface (Theorem 5.18).

In Chapter 6, we investigate extended distance squared functions on wave fronts as an application of singularity theory of functions. That function measures contact type between a wave front and a certain sphere. For the case of generic regular surfaces, singularities of extended distance squared functions correspond to types of singularities of parallel surfaces (cf. [**11**, Theorem 3.4]). However, for wave fronts, the same statement does not hold, in fact, different kinds of singularities (D-type) will appear (Theorems 6.3 and 6.4).

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CHAPTER 2

Wave fronts in the Euclidean 3-space

1. Wave fronts

In this section, we recall the notion of wave fronts in the Euclidean 3-space $\boldsymbol{R}^3$. For more details, see [**2, 20, 27, 46**].

Let $f : \Sigma \to \boldsymbol{R}^3$ be a C^∞ map, where $\Sigma \subset \boldsymbol{R}^2$ is a domain with a local coordinate system (u, v). We call this map f a *frontal* if there exists a unit vector field $\nu : \Sigma \to S^2$ along f such that the orthogonality condition

$$\langle df(X_q), \nu(q)\rangle = 0$$

holds for any point $q \in \Sigma$ and any tangent vector $X_q \in T_q\Sigma$ at q, where S^2 denotes the unit sphere and $\langle , \rangle$ means the canonical inner product of $\boldsymbol{R}^3$. The unit vector field ν is called a *unit normal vector* or the *Gauss map* of f. In addition, a frontal f is called a *wave front* or a *front*, for short, if the pair

$$\mathcal{L}_f = (f, \nu) : \Sigma \to T_1\boldsymbol{R}^3 \cong \boldsymbol{R}^3 \times S^2$$

gives an immersion, where $T_1\boldsymbol{R}^3$ is the unit tangent bundle over $\boldsymbol{R}^3$ equipped with the canonical contact structure. This map $\mathcal{L}_f$ is called the *Legendrian immersion* of f. Thus a front can be considered as the image of a projection of a Legendrian immersion. Needless to say, the set of immersions from $\boldsymbol{R}^2$ to $\boldsymbol{R}^3$ is a subset of the set of fronts or frontals.

We fix a frontal f. A point $p \in \Sigma$ is said to be a *singular point* of f if $\operatorname{rank} df_p < 2$ holds. Let $S(f)$ denote the set of singular point of f. We now define a function $\lambda : \Sigma \to \boldsymbol{R}$ by

$$\lambda(u, v) = \det(f_u, f_v, \nu)(u, v), \tag{2.1}$$

where $f_u = \partial f/\partial u$ and $f_v = \partial f/\partial v$. This function λ as in (2.1) is called the *signed area density function* of f. By the definitions of singular points and the signed area density function, the relation $S(f) = \lambda^{-1}(0)$ holds. Let us take a singular point

$p \in S(f)$. Then a singular point p is *non-degenerate* if the exterior derivative of the signed area density function $d\lambda$ does not vanish at p, that is, $(\lambda_u(p), \lambda_v(p)) \neq (0,0)$ holds.

For a non-degenerate singular point p, there exist a neighborhood V of p and a C^∞ regular curve $\gamma : (-\varepsilon, \varepsilon) \ni t \mapsto \gamma(t) \in V(\subset \Sigma)$ through $p = \gamma(0)$ such that $\mathrm{Img}(\gamma) = V \cap S(f)$, where $\varepsilon > 0$ is a sufficiently small real number and $\mathrm{Img}(\gamma)$ means the image of γ. We remark that non-degenerate singular points are (co)rank one singular points of frontal. Thus there exists a never-vanishing vector field η on $V \cap S(f)$ such that for any $q \in V \cap S(f)$, $df_q(\eta_q) = 0$ holds. We call the curve γ, the vector field η and $\hat{\gamma} = f \circ \gamma$ a *singular curve*, a *null vector field* and the *singular locus*, respectively. Moreover, the directions of $\gamma' = d\gamma/dt$ and η a *singular direction* and a *null direction*, respectively.

A non-degenerate singular point p is classified into the following cases. $p = \gamma(0)$ is a non-degenerate singular point of the *first kind* if $\det(\gamma', \eta)(0) \neq 0$. Otherwise, p is of the *second kind* (cf. [**32**]). In addition, we call a non-degenerate singular point of the second kind *admissible* if the singular curve consists of points of the first kind except at p. Otherwise, we call p *non-admissible*.

DEFINITION 2.1. Let f and $g : (\boldsymbol{R}^2, 0) \to (\boldsymbol{R}^3, 0)$ be two C^∞ map-germs. Then f and g are $\mathcal{A}$-*equivalent* if there exist diffeomorphism-germs $\theta : (\boldsymbol{R}^2, 0) \to (\boldsymbol{R}^2, 0)$ on the source and $\Theta : (\boldsymbol{R}^3, 0) \to (\boldsymbol{R}^3, 0)$ on the target such that $\Theta \circ f = g \circ \theta$ holds.

DEFINITION 2.2. Let $f : (\boldsymbol{R}^2, 0) \to (\boldsymbol{R}^3, 0)$ be a C^∞ map-germ at the origin. Then

- f at 0 is *cuspidal edge* if f is $\mathcal{A}$-equivalent to the germ $(u, v) \mapsto (u, v^2, v^3)$ at the origin,
- f at 0 is *swallowtail* if f is $\mathcal{A}$-equivalent to the germ $(u, v) \mapsto (u, 4v^3 + 2uv, 3v^4 + uv^2)$ at the origin,
- f at 0 is *cuspidal beaks* if f is $\mathcal{A}$-equivalent to the germ $(u, v) \mapsto (u, v^3 - u^2 v, 3v^4 - 2u^2 v^2)$ at the origin,
- f at 0 is *cuspidal lips* if f is $\mathcal{A}$-equivalent to the germ $(u, v) \mapsto (u, v^3 + u^2 v, 3v^4 + 2u^2 v^2)$ at the origin,
- f at 0 is *cuspidal butterfly* if f is $\mathcal{A}$-equivalent to the germ $(u, v) \mapsto (u, 5v^4 + 2uv, 4v^5 + uv^2)$ at the origin,

- f at 0 is $D_4^{\pm}$ *singularity* if f is $\mathcal{A}$-equivalent to the germ $(u, v) \mapsto (2uv, \pm u^2 + 3v^2, \pm 2u^2 v + 2v^3)$ at the origin.

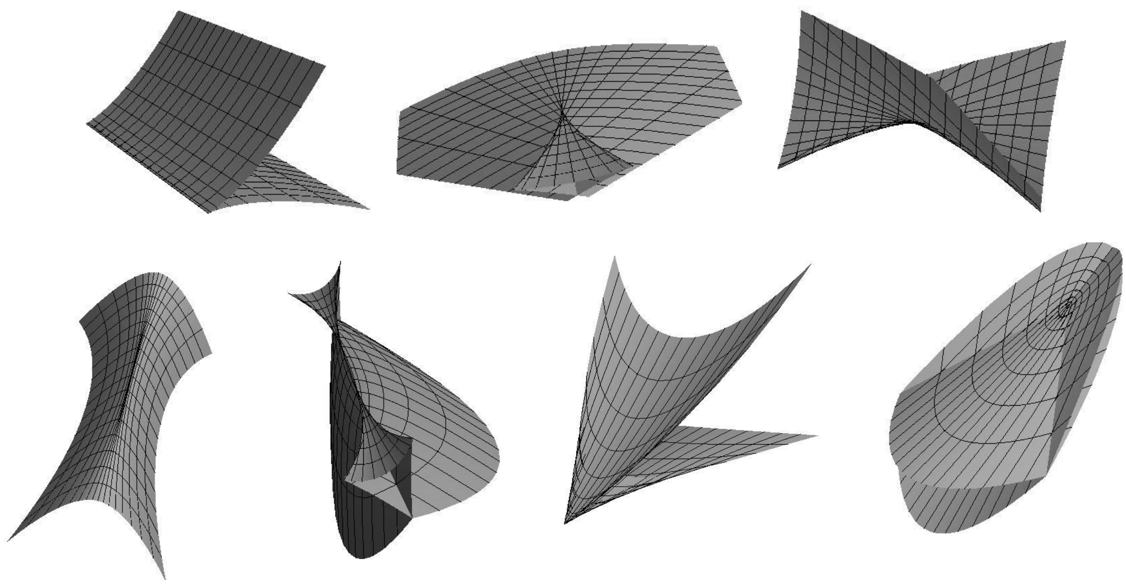


FIGURE 2.1. From top left to bottom right: cuspidal edge, swallowtail, cuspidal butterfly, cuspidal lips, cuspidal beaks, D_4^+ singularity and D_4^- singularity.

REMARK 2.3. Cuspidal edges and swallowtails are generic singularities of wave fronts in $\boldsymbol{R}^3$. Moreover, cuspidal lips, cuspidal beaks, cuspidal butterflies and $D_4^{\pm}$ singularities are singularities of the bifurcations in generic one parameter families of fronts in $\boldsymbol{R}^3$ (see [**2, 20, 52**]). On the other hand, cuspidal edges, swallowtails and cuspidal butterflies are non-degenerate singularities of fronts, and cuspidal beaks and cuspidal lips are degenerate corank one singularities, and $D_4^{\pm}$ singularities are degenerate corank two singularities of fronts (see Figure 2.1).

REMARK 2.4. Cuspidal edges are only non-degenerate singular points of the first kind of fronts, and swallowtails and cuspidal butterflies are non-degenerate singular points of the second kind of fronts. For a frontal but not a front, a particular example of non-degenerate singular point of the first kind is a *cuspidal cross-cap* which is $\mathcal{A}$-equivalent to the map-germ $(u, v) \mapsto (u, v^2, uv^3)$ at the origin (see Figure 2.2). Criteria for cuspidal cross-caps are given in [**10**] and geometric properties of cuspidal cross-caps are studied in [**38**].

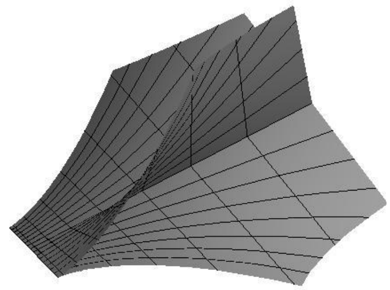


FIGURE 2.2. Cuspidal cross-cap.

It is know that the following criteria for cuspidal edges and swallowtails.

FACT 2.5 ([**27**, Proposition 1.3]). *Let f be a front and p a singular point, with signed area density function λ and null vector field η in a neighborhood of p. Then*
(1) *f at p is a cuspidal edge if and only if $\eta\lambda(p) \neq 0$,*
(2) *f at p is a swallowtail if and only if $\eta\lambda(p) = 0$, $\eta\eta\lambda(p) \neq 0$ and $d\lambda(p) \neq 0$.*
Here $\eta\lambda$ means the directional derivative of λ in the direction η.

We also have the following criteria for cuspidal beaks, cuspidal lips and cuspidal butterflies.

FACT 2.6 ([**22**, Theorem A.1],[**21**, Theorem 8.2]). *Under the same settings as in Lemma 2.5, the following hold.*
(1) *f at p is a cuspidal beaks if and only if $d\lambda(p) = 0$, $\eta\eta\lambda(p) \neq 0$ and $\det \operatorname{Hess} \lambda(p) < 0$.*
(2) *f at p is a cuspidal lips if and only if $d\lambda(p) = 0$ and $\det \operatorname{Hess} \lambda(p) > 0$.*
(3) *f at p is a cuspidal butterfly if and only if $d\lambda(p) \neq 0$, $\eta\lambda(p) = \eta\eta\lambda(p) = 0$ and $\eta\eta\eta\lambda(p) \neq 0$.*
Here $\operatorname{Hess} \lambda(p)$ means the Hessian matrix of λ at p.

There is a criteria for D_4 singularities as well.

FACT 2.7 ([**43**, Theorem 1.1]). *Let f be a front with a unit normal vector ν and the signed area density function λ. A singular point p is a D_4^+ (respectively, D_4^-) singularity if and only if the following conditions hold:*
(1) $\operatorname{rank} df_p = 0$.
(2) $\det \operatorname{Hess} \lambda(p) < 0$ (*respectively,* $\det \operatorname{Hess} \lambda(p) > 0$).

2. Geometric invariants of wave fronts

We focus on geometric invariants of fronts with non-degenerate singular points. See [**46, 32, 31**], for more detail.

2.1. Geometric invariants of cuspidal edges. First, we consider the case of cuspidal edges. Let $f : \Sigma \to \boldsymbol{R}^3$ be a front, $\nu : \Sigma \to S^2$ a unit normal to f and p a cuspidal edge. Then there exist a neighborhood $V(\subset \Sigma)$ of p such that the set of singular points $S(f) \cap V$ on V is a regular curve. Let η be a null vector field. Take a vector field ξ on V such that ξ is tangent to $S(f) \cap V$ and assume that the pair (ξ, η) is positively oriented. This pair of vector fields (ξ, η) is called an *adapted pair of vector fields* introduced by Martins and Saji [**31**]. Then we define the following curvatures by using an adapted pair of vector fields:

$$\kappa_s(u,v) = \operatorname{sgn}(\eta\lambda)\frac{\det(\xi f, \xi\xi f, \nu)}{|\xi f|^3}(u,v), \tag{2.2}$$

$$\kappa_\nu(u,v) = \frac{\langle \xi\xi f, \nu\rangle}{|\xi f|^2}(u,v), \tag{2.3}$$

$$\kappa_c(u,v) = \frac{|\xi f|^{3/2}\det(\xi f, \eta\eta f, \eta\eta\eta f)}{|\xi f \times \eta\eta f|^{5/2}}(u,v), \tag{2.4}$$

$$\kappa_t(u,v) = \frac{\det(\xi f, \eta\eta f, \xi\eta\eta f)}{|\xi f \times \eta\eta f|^2}(u,v) - \frac{\det(\xi f, \eta\eta f, \xi\xi f)\langle \xi f, \eta\eta f\rangle}{|\xi f|^2|\xi f \times \eta\eta f|^2}(u,v), \tag{2.5}$$

$$\kappa_i(u,v) = \frac{\det(\xi f, \eta\eta f, \xi\xi\xi f)}{|\xi f|^3|\xi f \times \eta\eta f|}(u,v) - 3\frac{\det(\xi f, \eta\eta f, \xi\xi f)\langle \xi f, \xi\xi f\rangle}{|\xi f|^5|\xi f \times \eta\eta f|}(u,v), \tag{2.6}$$

where $(u,v) \in S(f) \cap V$. κ_s, κ_ν, κ_c, κ_t and κ_i are called the *singular curvature*, the *limiting normal curvature*, the *cuspidal curvature*, the *cusp-directional torsion* and the *edge inflectional curvature* respectively defined in [**46, 31, 32**]. We note that these invariants as in (2.2), (2.3), (2.4), (2.5) and (2.6) do not depend on the choice of an adapted pair of a vector fields (see [**31**]) and can be defined on frontals with non-degenerate singular points of the first kind. It is known that the singular curvature κ_s is an intrinsic invariant of a cuspidal edge and the cuspidal curvature κ_c does not vanish. κ_s is closely related to the Gaussian curvature K. For example, κ_s is non-positive if the Gaussian curvature is non-negative near the set of singular points (cf. [**46**, Theorem 3.1]). In addition, the product $\kappa_\Pi = \kappa_\nu \kappa_c$ of κ_ν and κ_c is called the *product curvature* along cuspidal edges. We note that κ_Π is an intrinsic invariant of cuspidal edges ([**32**, Theorem 2.9]). For more detailed geometric properties, see [**17, 24, 31, 32, 37**].

On the other hand, one can take a local coordinate system $(U; u, v)$ cantered at p which satisfies the following properties:

(1) $S(f) \cap U = \{v = 0\}$, and
(2) $\eta = \partial_v$ gives a null vector field on $\{v = 0\}$.

We call this coordinate system $(U; u, v)$ an *adapted coordinate system* centered at p. Under an adapted coordinate system $(U; u, v)$, we can write invariants of a cuspidal edge as follows:

$$
\begin{aligned}
\kappa_s(u) &= \operatorname{sgn}(\lambda_v) \frac{\det(f_u, f_{uu}, \nu)}{|f_u|^3}(u, 0), \quad \kappa_\nu(u) = \frac{\langle f_{uu}, \nu \rangle}{|f_u|^2}(u, 0), \\
\kappa_c(u) &= \frac{|f_u|^{3/2} \det(f_u, f_{vv}, f_{vvv})}{|f_u \times f_{vv}|^{5/2}}(u, 0), \\
\kappa_t(u) &= \frac{\det(f_u, f_{vv}, f_{uvv})}{|f_u \times f_{vv}|^2}(u, 0) - \frac{\det(f_u, f_{vv}, f_{uu}) \langle f_u, f_{vv} \rangle}{|f_u|^2 |f_u \times f_{vv}|^2}(u, 0), \\
\kappa_i(u) &= \frac{\det(f_u, f_{vv}, f_{uuu})}{|f_u|^3 |f_u \times f_{vv}|}(u, 0) - 3 \frac{\det(f_u, f_{vv}, f_{uu}) \langle f_u, f_{uu} \rangle}{|f_u|^5 |f_u \times f_{vv}|}(u, 0).
\end{aligned}
$$

2.2. Geometric invariants of non-degenerate singular points of the second kind. Let $f : \Sigma \to \boldsymbol{R}^3$ be a front, $\nu : \Sigma \to S^2$ a unit normal vector to f and $p \in \Sigma$ a non-degenerate singular point of the second kind. Then a local coordinate system $(U; u, v)$ centered at p is called an *adapted* if the following properties hold:

(1) $S(f) \cap U = \{v = 0\}$,
(2) $\eta = \partial_u + \varepsilon(u)\partial_v$ with $\varepsilon(0) = 0$ gives a null vector field on $\{v = 0\}$.

By using this local coordinate system, we define geometric invariants of fronts with non-degenerate singular points of the second kind due to [**32**].

Let $f : \Sigma \to \boldsymbol{R}^3$ be a front, ν a unit normal vector to f and $p \in \Sigma$ a non-degenerate singular point of the second kind. Take an adapted coordinate system $(U; u, v)$ centered at p. Then we define

$$\kappa_\nu(p) = \frac{\langle f_{vv}, \nu \rangle}{|f_v|^2}(p) \tag{2.7}$$

which is called a *limiting normal curvature*. This is a generalization of the limiting normal curvature of cuspidal edges. We note that one can define the limiting normal curvature for fronts with corank one singular points ([**32**]). If $(S(f) \cap U) \setminus \{p\}$ consists of a cuspidal edge, then we can extend κ_ν to $S(f) \cap U$. Next, we define the following

$$\mu_c(p) = \frac{-|f_v|^2 \langle f_{uv}, \nu_u \rangle}{|f_{uv} \times f_v|^2}(p). \tag{2.8}$$

We call $\mu_c(p)$ the *normalized cuspidal curvature* ([**32**]). We note that $\mu_c(p)$ does not vanish if and only if f at p is a front ([**32**, Proposition 3.2]).

We now set $\mu_\Pi = \kappa_\nu \mu_c$ and call it the *normalized product curvature* at p ([**32**]). It is known that μ_Π is an intrinsic invariants ([**32**, Proposition 3.3]).

CHAPTER 3

Behavior of principal curvatures of wave fronts

In this chapter, we consider boundedness of principal curvatures of fronts with non-degenerate singular points. This chapter is based on Section 2 of [**49**] and Section 3 of [**50**].

1. Principal curvatures of fronts

We consider principal curvatures of fronts defined on the set of regular points and give explicit representations in terms of quantities obtained by a frame under an adapted coordinate system.

1.1. Near cuspidal edges. We consider principal curvatures of cuspidal edges. Let $f : \Sigma \to \boldsymbol{R}^3$ be a front and p a cuspidal edge. Then we take an adapted coordinate system $(U; u, v)$ centered at p. By the definition of an adapted coordinate system, there exists a map $h : U \to \boldsymbol{R}^3 \setminus \{0\}$ such that $f_v = vh$ holds by Malgrange preparation theorem ([**14**]). We note that f_u and h are linearly independent. In addition, the pair $\{f_u, h, \nu\}$ gives a frame of f. We set the following functions:

(3.1)

$$\widetilde{E} = \langle f_u, f_u\rangle,\ \widetilde{F} = \langle f_u, h\rangle,\ \widetilde{G} = \langle h, h\rangle,\ \widetilde{L} = -\langle f_u, \nu_u\rangle,\ \widetilde{M} = -\langle h, \nu_u\rangle,\ \widetilde{N} = -\langle h, \nu_v\rangle.$$

We remark that $\widetilde{E}\widetilde{G} - \widetilde{F}^2$ does not vanish near p. Since $\langle \nu, \nu\rangle = 1$, and f_u and h are linearly independent, ν_u and ν_v can be represented by linear combinations of f_u and h. In particular, we have the following.

Lemma 3.1. *It holds that*

$$\nu_u = \frac{\widetilde{F}\widetilde{M} - \widetilde{G}\widetilde{L}}{\widetilde{E}\widetilde{G} - \widetilde{F}^2} f_u + \frac{\widetilde{F}\widetilde{L} - \widetilde{E}\widetilde{M}}{\widetilde{E}\widetilde{G} - \widetilde{F}^2} h,\ \ \nu_v = \frac{\widetilde{F}\widetilde{N} - v\widetilde{G}\widetilde{M}}{\widetilde{E}\widetilde{G} - \widetilde{F}^2} f_u + \frac{v\widetilde{F}\widetilde{M} - \widetilde{E}\widetilde{N}}{\widetilde{E}\widetilde{G} - \widetilde{F}^2} h.$$

Since the pair $\{f_u, h, \nu\}$ gives a frame, f_{uu}, f_{uv} and f_{vv} may be written as linear compositions of f_u, h and ν. We now introduce the following functions:

$$
\begin{aligned}
&\widetilde{\Gamma}^{\,1}_{1\,1} = \frac{\widetilde{G}\widetilde{E}_u - 2\widetilde{F}\widetilde{F}_u + 2\widetilde{F}\langle f_u, h_u\rangle}{2(\widetilde{E}\widetilde{G} - \widetilde{F}^2)}, \qquad &&\widetilde{\Gamma}^{\,2}_{1\,1} = \frac{2\widetilde{E}\widetilde{F}_u - 2\widetilde{E}\langle f_u, h_u\rangle - \widetilde{F}\widetilde{E}_u}{2(\widetilde{E}\widetilde{G} - \widetilde{F}^2)}\\
&\widetilde{\Gamma}^{\,1}_{1\,2} = \frac{2\widetilde{G}\langle f_u, h_u\rangle - \widetilde{F}\widetilde{G}_u}{2(\widetilde{E}\widetilde{G} - \widetilde{F}^2)}, &&\widetilde{\Gamma}^{\,2}_{1\,2} = \frac{\widetilde{E}\widetilde{G}_u - 2\widetilde{F}\langle f_u, h_u\rangle}{2(\widetilde{E}\widetilde{G} - \widetilde{F}^2)}\\
&\widetilde{\Gamma}^{\,1}_{2\,2} = \frac{2\widetilde{G}\widetilde{F}_v - v\widetilde{G}\widetilde{G}_u - \widetilde{F}\widetilde{G}_v}{2(\widetilde{E}\widetilde{G} - \widetilde{F}^2)}, &&\widetilde{\Gamma}^{\,2}_{2\,2} = \frac{\widetilde{E}\widetilde{G}_v - 2\widetilde{F}\widetilde{F}_v + v\widetilde{F}\widetilde{G}_u}{2(\widetilde{E}\widetilde{G} - \widetilde{F}^2)}
\end{aligned}
\tag{3.2}
$$

where the functions $\widetilde{E}, \widetilde{F}$ and $\widetilde{G}$ are defined in (3.1) and $\widetilde{E}_v = 2v\langle f_u, h_u\rangle$ holds. We call functions $\widetilde{\Gamma}^{\,i}_{j\,k}$ as in (3.2) *modified Christoffel symbols.*

Lemma 3.2. *Under the above notations, we have the following:*

$$
\begin{aligned}
f_{uu} &= \widetilde{\Gamma}^{\,1}_{1\,1} f_u + \widetilde{\Gamma}^{\,2}_{1\,1} h + \widetilde{L}\nu,\\
f_{uv} &= v\widetilde{\Gamma}^{\,1}_{1\,2} f_u + v\widetilde{\Gamma}^{\,2}_{1\,2} h + v\widetilde{M}\nu,\\
f_{vv} &= v\widetilde{\Gamma}^{\,1}_{2\,2} f_u + \left(1 + v\widetilde{\Gamma}^{\,2}_{2\,2}\right) h + v\widetilde{N}\nu.
\end{aligned}
$$

Proof. We now set the following:

$$
\begin{aligned}
f_{uu} &= X_1 f_u + X_2 g + X_3 \nu,\\
f_{uv}(= v h_u) &= Y_1 f_u + Y_2 g + Y_3 \nu,\\
f_{vv}(= h + v h_v) &= Z_1 f_u + Z_2 g + Z_3 \nu,
\end{aligned}
$$

where $X_i, Y_i, Z_i : U \to \boldsymbol{R}$ $(i = 1, 2, 3)$ are C^∞ functions.

First we consider f_{uu}. By the definition of $\widetilde{L}$ and $\langle f_u, \nu\rangle = 0$, we have $X_3 = \widetilde{L}$. Let us determine the functions X_1 and X_2. By direct calculations, we have

$$
\langle f_{uu}, f_u\rangle = \widetilde{E}X_1 + \widetilde{F}X_2, \quad \langle f_{uu}, h\rangle = \widetilde{F}X_1 + \widetilde{G}X_2.
$$

Differentiating $\widetilde{E} = \langle f_u, f_u\rangle$ by u, we obtain $\langle f_{uu}, f_u\rangle = \widetilde{E}_u/2$. Moreover, since $\widetilde{F} = \langle f_u, h\rangle$, we have $\langle f_{uu}, h\rangle = \widetilde{F}_u - \langle f_u, h_u\rangle$. Thus the above equations can be rewritten as

$$
\begin{pmatrix} \dfrac{\widetilde{E}_u}{2} \\ \widetilde{F}_u - \langle f_u, h_u\rangle \end{pmatrix} = \begin{pmatrix} \widetilde{E} & \widetilde{F} \\ \widetilde{F} & \widetilde{G} \end{pmatrix} \begin{pmatrix} X_1 \\ X_2 \end{pmatrix}.
$$

Since $\widetilde{E}\widetilde{G}-\widetilde{F}^2>0$, one can solve this equation and get $X_i=\widetilde{\Gamma}_{11}^{\,i}$ $(i=1,2)$.

Next we consider f_{uv}. For Y_3, it follows that $\langle f_{uv},\nu\rangle=v\langle h_u,\nu\rangle=-v\langle h,\nu_u\rangle=v\widetilde{M}=Y_3$ since $\langle h,\nu\rangle=0$. For Y_1 and Y_2, by the similar computations as above, we get the following equation:

$$v\begin{pmatrix}\langle f_u,h_u\rangle\\ \dfrac{\widetilde{G}_u}{2}\end{pmatrix}=\begin{pmatrix}\widetilde{E}&\widetilde{F}\\ \widetilde{F}&\widetilde{G}\end{pmatrix}\begin{pmatrix}Y_1\\ Y_2\end{pmatrix}.$$

Therefore we have $Y_i=v\widetilde{\Gamma}_{12}^{\,i}$ $(i=1,2)$.

Finally we show the case of f_{vv}. f_{vv} can be written as $f_{vv}=h+vh_v$ since $f_v=vh$. Thus the inner product of f_{vv} and ν is calculated as $\langle f_{vv},\nu\rangle=v\langle h_v,\nu\rangle=-v\langle h,\nu_v\rangle=v\widetilde{N}=Y_3$ since $\langle h,\nu\rangle=0$ and $\langle\nu,\nu\rangle=1$. For Z_i $(i=1,2)$, we have the following equation

$$\begin{pmatrix}\widetilde{F}+v\left(\widetilde{F}_v-\dfrac{v\widetilde{G}_u}{2}\right)\\ \widetilde{G}+\dfrac{v\widetilde{G}_v}{2}\end{pmatrix}=\begin{pmatrix}\widetilde{E}&\widetilde{F}\\ \widetilde{F}&\widetilde{G}\end{pmatrix}\begin{pmatrix}Z_1\\ Z_2\end{pmatrix}$$

by the similar calculations as above. Solving this equation, we have $Z_1=v\widetilde{\Gamma}_{22}^1$ and $Z_2=1+v\widetilde{\Gamma}_{22}^2$. □

Using Lemma 3.2, we calculate geometric invariants in our setting.

LEMMA 3.3. *Under the above settings, κ_ν, κ_c, κ_t and κ_i can be expressed as*

$$\kappa_\nu=\frac{\widetilde{L}}{\widetilde{E}},\quad \kappa_c=\pm\frac{2\widetilde{E}^{3/4}\widetilde{N}}{(\widetilde{E}\widetilde{G}-\widetilde{F}^2)^{3/4}},\quad \kappa_t=\pm\frac{\widetilde{E}\widetilde{M}-\widetilde{F}\widetilde{L}}{\widetilde{E}\sqrt{\widetilde{E}\widetilde{G}-\widetilde{F}^2}},\tag{3.3}$$

$$\kappa_i=\pm\left(\frac{(\widetilde{E}\widetilde{M}-\widetilde{F}\widetilde{L})(2\widetilde{F}_u\widetilde{E}-\widetilde{E}\widetilde{E}_{vv}-\widetilde{E}_u\widetilde{F})}{2\widetilde{E}^{5/2}(\widetilde{E}\widetilde{G}-\widetilde{F}^2)}+\frac{\widetilde{E}\widetilde{L}_u-\widetilde{E}_u\widetilde{L}}{\widetilde{E}^{5/2}}\right)\tag{3.4}$$

along the u-axis, where $\pm$ depends on the orientation of the frame $\{f_u,h,\nu\}$.

PROOF. One can check that κ_ν can be expressed as above by (2.3) and definitions of functions. We show κ_c, κ_t and κ_i can be written as above formulas. Since ν is perpendicular to both f_u and h, ν can be written as $\nu=\pm(f_u\times h)/|f_u\times h|$.

First, we show that κ_c can be written as above. We note that $f_{vvv} = 2h_v$ holds on the u-axis. Since $\widetilde{N} = -\langle h, \nu_v\rangle = \langle h_v, \nu\rangle$, κ_c on the u-axis is expressed as

$$\kappa_c(u) = \frac{2\widetilde{E}^{3/4}\det(f_u, h, h_v)}{|f_u \times h|^{5/2}}(u,0) = \pm\frac{2\widetilde{E}^{3/4}\langle \nu, h_v\rangle}{|f_u \times h|^{3/2}}(u,0) = \pm\frac{2\widetilde{E}^{3/4}\widetilde{N}}{(\widetilde{E}\widetilde{G} - \widetilde{F}^2)^{3/4}}(u,0)$$

on the u-axis.

Next, we consider κ_t. Since $f_{uvv} = h_u$ and $\langle h_u, \nu\rangle = -\langle h, \nu_u\rangle = \widetilde{M}$ on the u-axis, we see that

$$\kappa_t(u) = \frac{\det(f_u, h, h_u)}{\widetilde{E}\widetilde{G} - \widetilde{F}^2}(u,0) - \frac{\det(f_u, h, f_{uu})\widetilde{F}}{\widetilde{E}(\widetilde{E}\widetilde{G} - \widetilde{F}^2)}(u,0) = \pm\frac{\widetilde{E}\widetilde{M} - \widetilde{F}\widetilde{L}}{\widetilde{E}\sqrt{\widetilde{E}\widetilde{G} - \widetilde{F}^2}}(u,0).$$

Finally, we consider κ_i. By Lemma 3.2, f_{uuu} is given by

$$f_{uuu} = *_1 f_u + *_2 h + \left(\widetilde{\Gamma}_{11}^1\widetilde{L} + \widetilde{\Gamma}_{11}^2\widetilde{M} + \widetilde{L}_u\right)\nu,$$

where $*_i$ $(i = 1, 2)$ are some functions. Thus it follows that

$$\det(f_u, f_{vv}, f_{uuu}) = \left(\widetilde{\Gamma}_{11}^1\widetilde{L} + \widetilde{\Gamma}_{11}^2\widetilde{M} + \widetilde{L}_u\right)\det(f_u, h, \nu)$$

along the u-axis. Moreover, we have

$$\frac{\det(f_u, f_{vv}, f_{uu})\langle f_u, f_{uu}\rangle}{|f_u \times f_{vv}|} = \pm\frac{\widetilde{L}\widetilde{E}_u}{2}.$$

Therefore we obtain

$$\begin{aligned}\kappa_i(u) &= \frac{\det(f_u, f_{vv}, f_{uuu})}{|f_u|^3|f_u \times f_{vv}|}(u,0) - 3\frac{\det(f_u, f_{vv}, f_{uu})\langle f_u, f_{uu}\rangle}{|f_u|^5|f_u \times f_{vv}|}(u,0)\\ &= \pm\left(\frac{\widetilde{\Gamma}_{11}^1\widetilde{L} + \widetilde{\Gamma}_{11}^2\widetilde{M} + \widetilde{L}_u}{\widetilde{E}^{3/2}} - \frac{3\widetilde{L}\widetilde{E}_u}{2\widetilde{E}^{5/2}}\right)(u,0)\\ &= \pm\left(\frac{(\widetilde{E}\widetilde{M} - \widetilde{F}\widetilde{L})(2\widetilde{F}_u\widetilde{E} - \widetilde{E}\widetilde{E}_{vv} - \widetilde{E}_u\widetilde{F})}{2\widetilde{E}^{5/2}(\widetilde{E}\widetilde{G} - \widetilde{F}^2)} + \frac{\widetilde{E}\widetilde{L}_u - \widetilde{E}_u\widetilde{L}}{\widetilde{E}^{5/2}}\right)(u,0),\end{aligned}$$

where we used the relation $\widetilde{E}_{vv} = 2\langle f_u, h_u\rangle$ along the u-axis and the expressions of $\widetilde{\Gamma}_{11}^i$ $(i = 1, 2)$ as in (3.2). □

We note that the expression

$$\kappa_s(u) = \frac{2\widetilde{F}_u\widetilde{E} - \widetilde{E}\widetilde{E}_{vv} - \widetilde{E}_u\widetilde{F}}{\widetilde{E}^{3/2}\sqrt{\widetilde{E}\widetilde{G} - \widetilde{F}^2}}(u,0) \tag{3.5}$$

holds if $\lambda_v > 0$ along the u-axis ([**46, 37, 17**]).

It is known that $\kappa_c(p)$ does not vanish if p is a cuspidal edge (cf. [**32**, Lemma 2.11]). In particular, $\widetilde{N}$ never vanishes on the u-axis by Lemma 3.3. Take an adapted coordinate system $(U; u, v)$ with $\eta\lambda(u, 0) > 0$. Then $\operatorname{sgn}(\kappa_c) = \operatorname{sgn}(\widetilde{N})$ holds on the u-axis (see Lemma 3.3). If $\eta\lambda(u, 0) < 0$, $\operatorname{sgn}(\kappa_c) = -\operatorname{sgn}(\widetilde{N})$ holds.

We define the following functions on $U \setminus \{v = 0\}$ as

$$\kappa_+ = \frac{2(\widetilde{L}\widetilde{N} - v\widetilde{M}^2)}{\widetilde{A} + \widetilde{B}}, \quad \kappa_- = \frac{2(\widetilde{L}\widetilde{N} - v\widetilde{M}^2)}{\widetilde{A} - \widetilde{B}}, \tag{3.6}$$

where $\widetilde{A} = \widetilde{E}\widetilde{N} - 2v\widetilde{F}\widetilde{M} + v\widetilde{G}\widetilde{L}$, $\widetilde{B} = \sqrt{\widetilde{A}^2 - 4v(\widetilde{E}\widetilde{G} - \widetilde{F}^2)(\widetilde{L}\widetilde{N} - v\widetilde{M}^2)}$. These functions are well-defined on $U \setminus \{v = 0\}$. We remark that κ_+ (resp. κ_-) becomes $-\kappa_-$ (resp. $-\kappa_+$) if we change ν to $-\nu$. Let K and H be the Gaussian and the mean curvature of f defined on $U \setminus \{v = 0\}$. Then $K = \kappa_+\kappa_-$ and $2H = \kappa_+ + \kappa_-$ hold. Thus we may treat κ_+ and κ_- as *principal curvatures* of f defined on $U \setminus \{v = 0\}$. Here K and H can be expressed as

$$K = \frac{\widetilde{L}\widetilde{N} - v\widetilde{M}^2}{v(\widetilde{E}\widetilde{G} - \widetilde{F}^2)}, \quad H = \frac{\widetilde{E}\widetilde{N} - 2v\widetilde{F}\widetilde{M} + v\widetilde{G}\widetilde{L}}{2v(\widetilde{E}\widetilde{G} - \widetilde{F}^2)}$$

on the set of regular points. We note that $\kappa_\pm = H \mp \sqrt{H^2 - K}$ hold on the set of regular points.

1.2. Near singular points of the second kind. Let $f : \Sigma \to \boldsymbol{R}^3$ be a frontal, p a non-degenerate singular point of the second kind and ν a unit normal vector to f. We fix an adapted coordinate system $(U; u, v)$ in the following. Taking a null vector field η, there exists a function $\varepsilon = \varepsilon(u)$ on the u-axis with $\varepsilon(0) = 0$ so that $\eta = \partial_u + \varepsilon(u)\partial_v$. Thus it follows that $df(\eta) = f_u + \varepsilon(u) f_v = 0$ holds along the u-axis. On the other hand, since the u-axis gives the singular curve, there exists a C^∞ function $h : U \to \boldsymbol{R}^3 \setminus \{0\}$ such that $df(\eta) = vh$. Hence we have $f_u = vh - \varepsilon f_v$. We remark that h, f_v and ν are linearly independent since $d\lambda = \det(h, f_v, \nu)dv \neq 0$ holds on the u-axis.

LEMMA 3.4. *Under the adapted coordinate system $(U; u, v)$, ν_u and ν_v on U can be written as*

$$\nu_u = \frac{\widehat{F}(v\widehat{M} - \varepsilon\widehat{N}) - \widehat{G}\widehat{L}}{\widehat{E}\widehat{G} - \widehat{F}^2} h + \frac{\widehat{F}\widehat{L} - \widehat{E}(v\widehat{M} - \varepsilon\widehat{N})}{\widehat{E}\widehat{G} - \widehat{F}^2} f_v,$$
$$\nu_v = \frac{\widehat{F}\widehat{N} - \widehat{G}\widehat{M}}{\widehat{E}\widehat{G} - \widehat{F}^2} h + \frac{\widehat{F}\widehat{M} - \widehat{E}\widehat{N}}{\widehat{E}\widehat{G} - \widehat{F}^2} f_v,$$

where $\widehat{E} = \langle h, h\rangle$, $\widehat{F} = \langle h, f_v\rangle$, $\widehat{G} = \langle f_v, f_v\rangle$, $\widehat{L} = -\langle h, \nu_u\rangle$, $\widehat{M} = -\langle h, \nu_v\rangle$ and $\widehat{N} = -\langle f_v, \nu_v\rangle$.

We now define two C^∞ functions on $U \setminus \{v = 0\}$ by

$$\kappa_+ = \frac{2((\widehat{L} + \varepsilon(u)\widehat{M})\widehat{N} - v\widehat{M}^2)}{\widehat{A} + \widehat{B}}, \quad \kappa_- = \frac{2((\widehat{L} + \varepsilon(u)\widehat{M})\widehat{N} - v\widehat{M}^2)}{\widehat{A} - \widehat{B}}, \tag{3.7}$$

where

$$\widehat{A} = \widehat{G}(\widehat{L} + \varepsilon(u)\widehat{M}) - 2v\widehat{F}\widehat{M} + v\widehat{E}\widehat{N},$$
$$\widehat{B} = \sqrt{\widehat{A}^2 - 4v(\widehat{E}\widehat{G} - \widehat{F}^2)\left((\widehat{L} + \varepsilon(u)\widehat{M})\widehat{N} - v\widehat{M}^2\right)}.$$

Since the Gaussian curvature K and the mean curvature H of f satisfy $K = \kappa_+\kappa_-$ and $2H = \kappa_+ + \kappa_-$, we may regard $\kappa_\pm$ as *principal curvatures* of f on $U \setminus \{v = 0\}$, where K and H are written as

$$K = \frac{(\widehat{L} + \varepsilon(u)\widehat{M})\widehat{N} - v\widehat{M}^2}{v(\widehat{E}\widehat{G} - \widehat{F}^2)}, \quad H = \frac{\widehat{G}(\widehat{L} + \varepsilon(u)\widehat{M}) - 2v\widehat{F}\widehat{M} + v\widehat{E}\widehat{N}}{2v(\widehat{E}\widehat{G} - \widehat{F}^2)}$$

on $U \setminus \{v = 0\}$. We remark that $\kappa_\pm = H \mp \sqrt{H^2 - K}$ hold on the set of regular points.

We put $\widehat{H} = vH$. This is a C^∞ function on U. It follows that

$$2\widehat{H} = \frac{\widehat{G}(\widehat{L} + \varepsilon(u)\widehat{M})}{\widehat{E}\widehat{G} - \widehat{F}^2} \tag{3.8}$$

holds along the u-axis (cf. [**32**]). We note that $\widehat{L} + \varepsilon(u)\widehat{M} = -\langle h, \eta\nu\rangle$ holds. It is known that $2\widehat{H}$ does not vanish on the u-axis if and only if f is a front ([**32**, Proposition 3.2]). In this situation, the normalized cuspidal curvature $\mu_c(p)$ at p can be written as

$$\mu_c(p) = 2\widehat{H}(p) \left(= \frac{\widehat{G}(p)\widehat{L}(p)}{|h(p) \times f_v(p)|^2} \right).$$

By (3.8) and the definition of $\mu_c(p)$, we see that $\operatorname{sgn}(\mu_c(p)) = \operatorname{sgn}(\widehat{L}(p))$ and $\widehat{L}(p) \neq 0$ hold if f is a front.

LEMMA 3.5. *Under the above conditions, the limiting normal curvature κ_ν can be written as $\kappa_\nu = \widehat{N}/\widehat{G}$ at p if p is of the admissible second kind.*

PROOF. By (2.7) (cf. [**32**, Proposition 1.9]), $f_u = vh - \varepsilon(u)f_v$, $f_{uu} = vh_u - \varepsilon'(u)f_v - \varepsilon(u)f_{uv}$ and $f_{uv} = h + vh_v - \varepsilon(u)f_{vv}$, we get the conclusion. □

2. Boundedness of principal curvatures

In this section, we consider boundedness of principal curvatures of fronts by using the above arguments.

THEOREM 3.6. *Let $f : \Sigma \to \boldsymbol{R}^3$ be a front and p a non-degenerate singular point.*

(1) *Let p be a cuspidal edge. If $\eta\lambda(p)\kappa_c(p) > 0$, then the principal curvature κ_+ is a bounded C^∞ function at p. Moreover, $\kappa_+(p) = \kappa_\nu(p)$.*
(2) *Let p be of the second kind. If $\mu_c(p) > 0$, then the principal curvature κ_+ is a bounded C^∞ function at p. Moreover, $\kappa_+(p) = \kappa_\nu(p)$ if p is an admissible.*

Converses are also true. Moreover, if one of $\kappa_\pm$ is bounded at p, then the another is unbounded.

PROOF. We prove the first assertion. Let $f : \Sigma \to \boldsymbol{R}^3$ be a front and p a cuspidal edge. Take an adapted coordinate system $(U; u, v)$ centered at p. We show the case of $\eta\lambda(u, 0) > 0$. In this case, $\operatorname{sgn}(\kappa_c) = \operatorname{sgn}(\widetilde{N})$ holds along the u-axis. For the case of $\eta\lambda(u, 0) < 0$, one can show similarly.

We now assume that $\kappa_c(p) > 0$. Then $\widetilde{N}(p) > 0$ by (3.3). Since $\widetilde{A} \pm \widetilde{B} = \widetilde{E}(\widetilde{N} \pm |\widetilde{N}|)$ and (3.6), we see that κ_+ is a bounded C^∞ function on U and $\kappa_+ = \widetilde{L}/\widetilde{E} = \kappa_\nu$ holds at p. Conversely, we assume that the principal curvature κ_+ is a bounded C^∞ function near p. In this case, it follows that $\widetilde{N} = -\langle h, \eta\nu \rangle$ is positive along the u-axis. This implies that $\eta\lambda \cdot \kappa_c$ is positive along the u-axis by (3.3). Unboundedness of κ_- near p follows from the fact that the mean curvature is unbounded near p.

Next, we prove the second assertion. Take an adapted coordinate system $(U; u, v)$ centered at a non-degenerate singular point of the second kind p. Suppose that $\mu_c(p) = 2\widehat{H}(p) > 0$. It follows that $-\langle h, \eta\nu\rangle > 0$ holds near p from (3.8). Since $\widehat{A} = \widehat{G}(-\langle h, \eta\nu\rangle)$, $\widehat{B} = |\widehat{A}|$ and $-\langle h, \eta\nu\rangle > 0$ along the u-axis, it follows that $\widehat{A} + \widehat{B} = 2\widehat{G}(-\langle h, \eta\nu\rangle) \neq 0$ and $A - B = 0$ hold on the u-axis. Hence by (3.7), we have $\kappa_+ = \widehat{N}/\widehat{G}$ along the u-axis, and κ_+ is a bounded C^∞ function. By Lemma 3.5, we see that $\kappa_+ = \kappa_\nu$ at p if p is admissible. The converse and unboundedness can be shown by using similar arguments to the first assertion. □

Remark 3.7. We assume that κ_+ is bounded near non-degenerate singular point p. Although κ_- is unbounded near p, $\lambda\kappa_-$ is bounded near p. In fact, κ_- can be rewritten as

$$\kappa_- = \begin{cases} \frac{\widetilde{A}+\widetilde{B}}{2v(\widetilde{E}\widetilde{G}-\widetilde{F}^2)} & (p: \text{ cuspidal edge}) \\ \frac{\widehat{A}+\widehat{B}}{2v(\widehat{E}\widehat{G}-\widehat{F}^2)} & (p: \text{ second kind}) \end{cases}$$

on $U \setminus \{v = 0\}$. Thus $\lambda\kappa_-$ is written as

$$\lambda\kappa_- = \begin{cases} \frac{\widetilde{A}+\widetilde{B}}{2\sqrt{\widetilde{E}\widetilde{G}-\widetilde{F}^2}} & (p: \text{ cuspidal edge}) \\ \frac{\widehat{A}+\widehat{B}}{2\sqrt{\widehat{E}\widehat{G}-\widehat{F}^2}} & (p: \text{ second kind}). \end{cases}$$

In particular, $\lambda(p)\kappa_-(p)$ is proportional to $\kappa_c(p)$ when p is a cuspidal edge, and $\lambda(p)\kappa_-(p)$ is proportional to $\mu_c(p)$ when p is of the second kind. Thus $\lambda(p)\kappa_-(p)$ does not vanish.

3. Principal vectors and related properties

3.1. Principal vector for a bounded principal curvature. We consider explicit representations for principal vectors of fronts with respect to a bounded principal curvature.

3.1.1. *Near cuspidal edges.* Let $f : \Sigma \to \boldsymbol{R}^3$ be a front and p a cuspidal edge. Take an adapted coordinate system $(U; u, v)$ satisfying $\eta\lambda(p) > 0$ centered at p. Then we assume that κ_+ as in (3.6) is bounded C^∞ function on U.

If a vector $\boldsymbol{V} = (V_1, V_2)$ is the principal vector with respect to κ_+, then it follows that $II\boldsymbol{V} = \kappa_+ I\boldsymbol{V}$, where

$$I = \begin{pmatrix} \langle f_u, f_u\rangle & \langle f_u, f_v\rangle \\ \langle f_u, f_v\rangle & \langle f_v, f_v\rangle \end{pmatrix}, \quad II = \begin{pmatrix} -\langle f_u, \nu_u\rangle & -\langle f_u, \nu_v\rangle \\ -\langle f_v, \nu_u & -\langle f_v, \nu_v\rangle \end{pmatrix}.$$

By using functions as in (3.1), we can write this equation as

$$\begin{pmatrix} \widetilde{L} & v\widetilde{M} \\ v\widetilde{M} & v\widetilde{N} \end{pmatrix} \begin{pmatrix} V_1 \\ V_2 \end{pmatrix} = \kappa_+ \begin{pmatrix} \widetilde{E} & v\widetilde{F} \\ v\widetilde{F} & v^2\widetilde{G} \end{pmatrix} \begin{pmatrix} V_1 \\ V_2 \end{pmatrix}. \tag{3.9}$$

We can factor out v from (3.9) and obtain

$$\begin{pmatrix} \widetilde{L} - \kappa_+\widetilde{E} & v(\widetilde{M} - \kappa_+\widetilde{F}) \\ v(\widetilde{M} - \kappa_+\widetilde{F}) & v(\widetilde{N} - v\kappa_+\widetilde{G}) \end{pmatrix} \begin{pmatrix} V_1 \\ V_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}. \tag{3.10}$$

Setting

$$\boldsymbol{V} = (V_1, V_2) = (\widetilde{N} - v\kappa_+\widetilde{G}, -\widetilde{M} + \kappa_+\widetilde{F}), \tag{3.11}$$

this satisfies equation (3.10). Since $\hat{N}$ is a non-zero function on the u-axis, $\boldsymbol{V}$ is non-zero on U. This implies that $\boldsymbol{V}$ can be regarded as the principal vector with respect to κ_+.

3.1.2. *Near non-degenerate singular points of the second kind.* We consider principal vectors of fronts with non-degenerate singular points of the second kind with respect to bounded principal curvatures. Let $f : \Sigma \to \boldsymbol{R}^3$ be a front and p a non-degenerate singular point of the second kind. Then we take an adapted coordinate system $(U; u, v)$ centered at p with $\lambda_v(p) > 0$. Suppose that κ_+ is bounded on U.

If a vector $\boldsymbol{V} = (V_1, V_2)$ is the principal vector with respect to κ_+, then as similar in the above discussion, $II\boldsymbol{V} = \kappa_+ I\boldsymbol{V}$ holds. By using functions defined in Lemma 3.4, we have

$$\begin{pmatrix} v\{\widehat{L} - \kappa_+(v\widehat{E} - \varepsilon\widehat{F})\} & v(\widehat{M} - \kappa_+\widehat{F}) \\ v(\widehat{M} - \kappa_+\widehat{F}) - \varepsilon(\widehat{N} - \kappa_+\widehat{G}) & \widehat{N} - \kappa_+\widehat{G} \end{pmatrix} \begin{pmatrix} V_1 \\ V_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}. \tag{3.12}$$

We note that $\widehat{L}$ does not vanish at p. Thus we can take the principal vector $\boldsymbol{V}$ as

$$\boldsymbol{V} = (-\widehat{M} + \kappa_+\widehat{F}, \widehat{L} - \kappa_+(v\widehat{E} - \varepsilon\widehat{F})), \tag{3.13}$$

by factoring out v from (3.12).

3.2. Conditions that singular curves become lines of curvature. By the previous two subsections, we can define one principal vector with respect to a bounded principal curvature of a front locally. Using these results, we can extend the notion of a line of curvature as follows. The singular locus $\hat{\gamma} = f \circ \gamma$ is a *line of curvature* if the principal vector $\boldsymbol{V}$ is tangent to γ.

PROPOSITION 3.8. *Let $f : \Sigma \to \boldsymbol{R}^3$ be a front, p a non-degenerate singular point and γ the singular curve passing through p. Then the following assertions hold*:

(1) *Suppose that p is a cuspidal edge. Then $\hat{\gamma}$ is a line of curvature of f if and only if κ_t vanishes identically along γ.*
(2) *Suppose that p is of the second kind. Then $\hat{\gamma}$ can not be a line of curvature.*

PROOF. First, we consider assertion (1). Take an adapted coordinate system $(U; u, v)$ centered at a cuspidal edge p satisfying $\eta\lambda(u, 0) > 0$. Assume that κ_+ is bounded on U. Then the principal vector $\boldsymbol{V} = (V_1, V_2)$ relative to κ_+ is given by (3.11). Since $\kappa_+ = \widetilde{L}/\widetilde{E}$ on the u-axis, V_2 can be written as

$$V_2 = -\widetilde{M} + \kappa_+\widetilde{F} = -\frac{\widetilde{E}\widetilde{M} - \widetilde{F}\widetilde{L}}{\widetilde{E}} = -\kappa_t\sqrt{\widetilde{E}\widetilde{G} - \widetilde{F}^2}$$

along the u-axis by Lemma 3.3. Thus v_2 vanishes on the u-axis if and only if κ_t vanishes along the u-axis, and we get the conclusion.

Next, we show (2). Take an adapted coordinate system $(U; u, v)$ around p and assume that $\mu_c(p) > 0$ holds. In this case, κ_+ is bounded on U and the principal vector $\boldsymbol{V} = (V_1, V_2)$ of κ_+ is given as (3.13). The second component v_2 is written as

$$V_2 = \widehat{L} + \varepsilon\kappa_+\widehat{F}$$

along the u-axis. Thus we have $V_2 = \widehat{L} \neq 0$ at p. This implies that the u-axis can not be the line of curvature. □

We note that a similar result for cuspidal edges is obtained by Izumiya, Takeuchi and Saji [**24**].

3.3. Ridge points and sub-parabolic points.

3.3.1. *Ridge points.* Using the principal curvature κ_+ and the principal vector $\boldsymbol{V}$ relative to κ_+, we define ridge points for f. Ridge points play important role to study parallel surfaces, focal surfaces and Gauss maps.

DEFINITION 3.9. Under the above settings, a point p is called a *ridge point* if $\boldsymbol{V}\kappa_+(p) = 0$ holds, where $\boldsymbol{V}\kappa_+$ denotes the directional derivative of κ_+ with respect to $\boldsymbol{V}$. Moreover, a point p is called a *k-th order ridge point* if $\boldsymbol{V}^{(m)}\kappa_+(p) = 0$ $(1 \leq m \leq k)$ and $\boldsymbol{V}^{(k+1)}\kappa_+(p) \neq 0$ hold, where $\boldsymbol{V}^{(m)}\kappa_+$ means the m-th directional derivative of κ_+ with respect to $\boldsymbol{V}$.

Ridge points for regular surfaces were first studied deeply by Porteous [**41**]. He showed that ridge points correspond to A_3 singular points, that is, cuspidal edges of caustics. For more details on ridge points, see [**4, 11, 12, 20, 41, 42**].

3.3.2. *Sub-parabolic points on cuspidal edges.* In this subsection we consider sub-parabolic points with respect to a bounded principal curvature at cuspidal edges. Let $f : \Sigma \to \boldsymbol{R}^3$ be a front, ν its unit normal vector and $p \in \Sigma$ a cuspidal edge. Take an adapted coordinate system $(U; u, v)$ centered at p satisfying $\eta\lambda(u, 0) > 0$. Then we assume that κ_+ as in (3.6) is bounded on U.

Let us denote $\widetilde{\boldsymbol{V}} = (\widetilde{V}_1, \widetilde{V}_2)$. If $\widetilde{\boldsymbol{V}}$ is a principal vector with respect to κ_- on $U \setminus \{v = 0\}$, then $\widetilde{\boldsymbol{V}}$ satisfies the relation $(II - \kappa_- I)\widetilde{\boldsymbol{V}} = 0$. By using functions as in (3.1), we can write

$$\begin{pmatrix} \widetilde{L} - \kappa_-\widetilde{E} & v(\widetilde{M} - \kappa_-\widetilde{F}) \\ v(\widetilde{M} - \kappa_-\widetilde{F}) & v(\widetilde{N} - v\kappa_-\widetilde{G}) \end{pmatrix} \begin{pmatrix} \widetilde{V}_1 \\ \widetilde{V}_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}. \tag{3.14}$$

This relation (3.14) is equivalent to the following equations:

$$\begin{cases} (\widetilde{L} - \kappa_-\widetilde{E})\widetilde{V}_1 + v(\widetilde{M} - \kappa_-\widetilde{F})\widetilde{V}_2 = 0 \\ v(\widetilde{M} - \kappa_-\widetilde{F})\widetilde{V}_1 + v(\widetilde{N} - v\kappa_-\widetilde{G})\widetilde{V}_2 = 0. \end{cases}$$

Multiplying λ, we can rewrite the above system as

$$\begin{cases} (\lambda\widetilde{L} - \hat{\kappa}_-\widetilde{E})\widetilde{V}_1 + v(\lambda\widetilde{M} - \hat{\kappa}_-\widetilde{F})\widetilde{V}_2 = 0 \\ (\lambda\widetilde{M} - \hat{\kappa}_-\widetilde{F})\widetilde{V}_1 + (\lambda\widetilde{N} - v\hat{\kappa}_-\widetilde{G})\widetilde{V}_2 = 0, \end{cases}$$

where $\hat{\kappa}_- = \lambda\kappa_-$. Thus we may take $\widetilde{\boldsymbol{V}} = (\widetilde{V}_1, \widetilde{V}_2)$ as

$$\widetilde{\boldsymbol{V}} = (v(\lambda\widetilde{M} - \hat{\kappa}_-\widetilde{F}), -\lambda\widetilde{L} + \hat{\kappa}_-\widetilde{E}) \quad \text{or} \quad \widetilde{\boldsymbol{V}} = (\lambda\widetilde{N} - v\hat{\kappa}_-\widetilde{G}, -\lambda\widetilde{M} + \hat{\kappa}_-\widetilde{F}). \tag{3.15}$$

We note that one of above vectors in (3.15) is well-defined on U since we can write $\widetilde{\boldsymbol{V}}$ as $\widetilde{\boldsymbol{V}} = (0, \hat{\kappa}_-\widetilde{E})$ or $\widetilde{\boldsymbol{V}} = (0, \hat{\kappa}_-\widetilde{F})$ on the u-axis. In particular, $\widetilde{\boldsymbol{V}} = (0, \hat{\kappa}_-\widetilde{E}) \neq (0,0)$ on the u-axis. Thus we can take the (*extended*) *principal vector* $\widetilde{\boldsymbol{V}}$ with respect to κ_- as

$$\widetilde{\boldsymbol{V}} = (v(\lambda\widetilde{M} - \hat{\kappa}_-\widetilde{F}), -\lambda\widetilde{L} + \hat{\kappa}_-\widetilde{E}) \tag{3.16}$$

on U. We now identify the vector $\widetilde{\boldsymbol{V}} = (\widetilde{V}_1, \widetilde{V}_2)$ with the vector field $\widetilde{\boldsymbol{V}} = \widetilde{V}_1\partial_u + \widetilde{V}_2\partial_v$ on U. By (3.16), we note that $\widetilde{\boldsymbol{V}}$ is parallel to the null vector field $\eta = \partial_v$ along the u-axis.

Definition 3.10. Under the above settings, a point p is called the *sub-parabolic point* of f if $\widetilde{\boldsymbol{V}}\kappa_+(p) = 0$ holds, where $\widetilde{\boldsymbol{V}}\kappa_+$ means the directional derivative of κ_+ in the direction $\widetilde{\boldsymbol{V}}$.

For geometric meanings of sub-parabolic points on a regular surface, see [**5, 6, 20, 35**].

3.3.3. *Characterizations of ridge points and sub-parabolic points by geometric invariants.* We consider relationship among ridge points, sub-parabolic points and geometric invariants of cuspidal edges.

Proposition 3.11. *Let $f : \Sigma \to \boldsymbol{R}^3$ be a front and $p \in \Sigma$ a cuspidal edge. Assume that κ_+ (resp. κ_-) is a bounded principal curvature near p. Then*

- *p is a ridge point if and only if $4\kappa_t^3 + \kappa_i\kappa_c^2 = 0$ at p,*
- *p is a sub-parabolic point if and only if $4\kappa_t^2 + \kappa_s\kappa_c^2 = 0$ at p.*

Proof. First, we show the condition for sub-parabolic points. Let $(U; u, v)$ be an adapted coordinate system centered at p satisfying $\eta\lambda(u,0) > 0$. In this case, we may take ν as $\nu = (f_u \times h)/|f_u \times h|$. We assume that κ_+ is a bounded principal curvature of f on U, that is, $\widetilde{N} > 0$ on the u-axis. The directional derivative $\widetilde{\boldsymbol{V}}\kappa_+$ of κ_+ with respect to $\widetilde{\boldsymbol{V}}$ as in (3.16) is $\widetilde{\boldsymbol{V}}\kappa = \widetilde{V}_2(\kappa_+)_v = \widetilde{V}_2\partial\kappa_+/\partial v$ at p. By (3.6), $(\kappa_+)_v$ can be written as

$$(\kappa_+)_v = 2\left(\frac{\widetilde{L}_v\widetilde{N} + \widetilde{L}\widetilde{N}_v - \widetilde{M}^2}{\widetilde{A} + \widetilde{B}} - \frac{\widetilde{L}\widetilde{N}(\widetilde{A}_v + \widetilde{B}_v)}{(\widetilde{A} + \widetilde{B})^2}\right)$$

on the u-axis. We note that $\widetilde{A} + \widetilde{B} = 2\widetilde{E}\widetilde{N}$, $\widetilde{A}_v = \widetilde{E}\widetilde{N}_v - 2\widetilde{F}\widetilde{M} + \widetilde{G}\widetilde{L}$ and $\widetilde{A}_v + \widetilde{B}_v = 2(\widetilde{A}_v - \widetilde{L}|f_u \times h|^2/\widetilde{E})$ hold on the u-axis since $2\widetilde{E}_v(u,0) = \langle f_{uv}, f_u\rangle(u,0) = 0$. Thus it follows that

$$(\kappa_+)_v = \frac{\widetilde{L}_v \widetilde{N} - \kappa_t^2 |f_u \times h|^2}{\widetilde{E}\widetilde{N}}$$

holds along the u-axis by (3.3).

We consider $\widetilde{L}_v$ along the u-axis. Since $\widetilde{L} = \det(f_u, h, f_{uu})/|f_u \times h|$ and $f_{uv} = f_{uuv} = 0$ on the u-axis, we see that

$$\widetilde{L}_v = \frac{\det(f_u, h_v, f_{uu})}{|f_u \times h|} - \frac{|f_u \times h|_v \det(f_u, h, f_{uu})}{|f_u \times h|^2}$$

holds at p. Since h_v and $|f_u \times h|_v$ can be written as

$$h_v = *f_u + \frac{|f_u|^2 \langle h, h_v\rangle - \langle f_u, h\rangle\langle f_u, h_v\rangle}{|f_u \times h|^2} h + \widetilde{N}\nu,$$

$$|f_u \times h|_v = \frac{|f_u|^2 \langle h, h_v\rangle - \langle f_u, h\rangle\langle f_u, h_v\rangle}{|f_u \times h|},$$

we have $\widetilde{L}_v = -\kappa_s \widetilde{N}|f_u|^3/|f_u \times h|$ (see (2.2)). Thus $4\widetilde{L}_v\widetilde{N} = -\kappa_s\kappa_c^2|f_u \times h|^2$ holds on the u-axis by (3.3). Hence we get

$$(\kappa_+)_v = -\frac{1}{2\kappa_c}(4\kappa_t^2 + \kappa_s\kappa_c^2)\left(\frac{|f_u \times h|}{|f_u|}\right)^{1/2} \tag{3.17}$$

at p. Since $2\hat{\kappa}_- \widetilde{E} = \kappa_c(|f_u|^5|f_u \times h|)^{1/2}$ at p, we have the second assertion.

Next, we show the condition for ridge points under the above setting. In this setting, directional derivative $\boldsymbol{V}\kappa_+$ is given by $\boldsymbol{V}\kappa_+ = V_1(\kappa_+)_u + V_2(\kappa_+)_v$, where $\boldsymbol{V}$ is a principal vector with respect to κ_+ as in (3.11). By a direct computation, we have

$$(\kappa_+)_u = \frac{\widetilde{L}_u\widetilde{E} - \widetilde{L}\widetilde{E}_u}{\widetilde{E}^2} \tag{3.18}$$

at p. By Proposition 3.3 and (3.11), it follows that

$$V_1 = \frac{\kappa_c}{2}\left(\frac{\widetilde{E}\widetilde{G} - \widetilde{F}^2}{\widetilde{E}}\right)^{3/4}, \quad V_2 = -\kappa_t(\widetilde{E}\widetilde{G} - \widetilde{F}^2)^{1/2} = -\kappa_t|f_u \times h|$$

at p. Thus we have

$$\boldsymbol{V}\kappa_+ = \frac{\kappa_c(\widetilde{L}_u\widetilde{E} - \widetilde{L}\widetilde{E}_u)}{2\widetilde{E}^2}\left(\frac{\widetilde{E}\widetilde{G} - \widetilde{F}^2}{\widetilde{E}}\right)^{3/4} + \frac{1}{2\kappa_c}(4\kappa_t^3 + \kappa_t\kappa_s\kappa_c^2)\left(\frac{(\widetilde{E}\widetilde{G} - \widetilde{F}^2)^3}{\widetilde{E}}\right)^{1/4}.$$

On the other hand, by Proposition 3.3 and (3.5), it follows that

$$\kappa_i = \kappa_s\kappa_t + \frac{\widetilde{E}\widetilde{L}_u - \widetilde{E}_u\widetilde{L}}{\widetilde{E}^{5/2}}$$

holds when $\lambda_v(u,0) > 0$. Therefore, we see that

$$\begin{aligned}\boldsymbol{V}\kappa_+ &= \frac{\kappa_c(\widetilde{L}_u\widetilde{E} - \widetilde{L}\widetilde{E}_u)}{2\widetilde{E}^2}\left(\frac{\widetilde{E}\widetilde{G} - \widetilde{F}^2}{\widetilde{E}}\right)^{3/4} \\ &\quad + \frac{1}{2\kappa_c}\left(4\kappa_t^3 + \left(\kappa_i - \frac{\widetilde{E}\widetilde{L}_u - \widetilde{E}_u\widetilde{L}}{\widetilde{E}^{5/2}}\right)\kappa_c^2\right)\left(\frac{(\widetilde{E}\widetilde{G} - \widetilde{F}^2)^3}{\widetilde{E}}\right)^{1/4} \\ &= \frac{1}{2\kappa_c}(4\kappa_t^3 + \kappa_i\kappa_c^2)\left(\frac{(\widetilde{E}\widetilde{G} - \widetilde{F}^2)^3}{\widetilde{E}}\right)^{1/4}\end{aligned}$$

holds at p. Hence we obtain the conclusions. □

We will give geometric interpretations of sub-parabolic points.

For cuspidal edges, the following normal form is obtained by [**31**].

FACT 3.12 ([**31**, Theorem 3.1]). *Let* $f : (\boldsymbol{R}^2, 0) \to (\boldsymbol{R}^3, 0)$ *be a map-germ and* 0 *a cuspidal edge. Then there exist a diffeomorphism-germ* $\theta : (\boldsymbol{R}^2, 0) \to (\boldsymbol{R}^2, 0)$ *and an isometry-germ* $\Theta : (\boldsymbol{R}^3, 0) \to (\boldsymbol{R}^3, 0)$ *satisfying*

(3.19)

$$\Theta \circ f \circ \theta(u,v) = \left(u, \frac{a_{20}}{2}u^2 + \frac{a_{30}}{6}u^3 + \frac{v^2}{2}, \frac{b_{20}}{2}u^2 + \frac{b_{30}}{6}u^3 + \frac{b_{12}}{2}uv^2 + \frac{b_{03}}{6}v^3\right) + h(u,v),$$

where $b_{20} \geq 0$, $b_{03} \neq 0$ *and*

$$h(u,v) = (0, u^4h_1(u), u^4h_2(u) + u^2v^2h_3(u) + uv^3h_4(u) + v^4h_5(u,v)),$$

with $h_i(u)$ $(1 \leq i \leq 4)$, $h_5(u,v)$ *smooth functions.*

We note that coefficients in the normal form (3.19) correspond to $\kappa_s(0) = a_{20}$, $\kappa_\nu(0) = b_{20}$, $\kappa_t(0) = b_{12}$ and $\kappa_c(0) = b_{03}$ (see [**31**]).

Using normal form of cuspidal edge (3.19), we have the condition for first order ridge points on cuspidal edges in terms of coefficients of normal form.

PROPOSITION 3.13. *Let* $f : U \to \boldsymbol{R}^3$ *be the normal form* (3.19) *of a cuspidal edge. Assume that* κ_+ *is the principal curvature which extends as a* C^∞ *function* *near* 0 *and* $\boldsymbol{V}$ *is the principal direction corresponding to* κ_+*. Then* 0 *is a first order ridge point if and only if*

$$4b_{12}^3 + b_{30}b_{03}^2 = 0, \quad \textit{and} \tag{3.20}$$

$$\begin{aligned} &-2b_{20}^3 b_{03}^4 - 3b_{20}(4b_{12}^2 + a_{20}b_{03}^2)^2 \\ &\qquad + 24\left(b_{03}^4 h_2(0) + 4b_{12}^2 b_{03}^2 h_3(0) - 8b_{12}^3 b_{03} h_4(0) + 16 b_{12}^4 h_5(0)\right) \neq 0. \end{aligned} \tag{3.21}$$

Proof. Without loss of generality, we assume $\hat{N}$ is positive near 0. Direct computations show $(\kappa_+)_u(0) = b_{30} - a_{20}b_{12}$, $(\kappa_+)_v(0) = -(4b_{12}^2 + a_{20}b_{03}^2)/2b_{03}$, $V_1(0) = b_{03}/2$ and $V_2(0) = -b_{12}$. Hence we get $\boldsymbol{V}\kappa_+(0) = (4b_{12}^3 + b_{30}b_{03}^2)/2b_{03}$, which shows (3.20). Again direct computations shows $(V_1)_u(0) = 3h_4(0), (V_1)_v(0) = -b_{20} + 8h_5(0), (V_2)_u(0) = a_{20}b_{20} - 4h_3(0), (V_2)_v(0) = -3h_4(0)$. Moreover, we have

$$\begin{aligned} (\kappa_+)_{uu} &= -2\left(a_{20}^2 b_{20} + b_{20}^3 + a_{30}b_{12} - 12h_2(0) + 2a_{20}h_3(0)\right), \\ (\kappa_+)_{uv} &= \frac{-1}{2b_{03}^2}\left(a_{30}b_{03}^3 + 8b_{12}(4b_{03}h_3(0) - 3b_{12}h_4(0)) - 2a_{20}b_{03}(4b_{20}b_{12} - 3b_{03}h_4(0))\right), \\ (\kappa_+)_{vv} &= \frac{4}{b_{03}^2}\left(-2b_{20}b_{12}^2 - 6b_{12}b_{03}h_4(0) + 16b_{12}^2 h_5(0) + b_{03}^2(h_3(0) - 2a_{20}h_5(0))\right) \end{aligned}$$

at 0. Since $\boldsymbol{V}^{(2)}\kappa_+ = (V_1(V_1)_u + V_2(V_1)_v)(\kappa_+)_u + (V_1(V_2)_u + V_2(V_2)_v)(\kappa_+)_v + V_1^2(\kappa_+)_{uu} + 2V_1V_2(\kappa_+)_{uv} + V_2^2(\kappa_+)_{vv}$ and $b_{30} = -4b_{12}^3/b_{03}^2$ hold, we have completed the proof. □

This proposition is useful to make examples of cuspidal edges with ridge points.

CHAPTER 4

Parallel surfaces of wave fronts

In this chapter, we consider parallel surfaces of wave fronts. This chapter is based on [**49, 50**].

Throughout this chapter, we assume that κ_+ is bounded near a non-degenerate singular point p of a front $f : \Sigma \to \boldsymbol{R}^3$.

For the case of regular surfaces, principal curvatures relate singularities of parallel surfaces. In this section, we consider singularities of parallel surfaces of fronts and give criteria in terms of principal curvatures and other geometric properties. Here we give criteria for other singularities on parallel surfaces of fronts.

1. Singularities of parallel surfaces of wave fronts

In this subsection, we shall deal with fronts which have singular points of the second kind (swallowtails, for example). Needless to say, the following arguments can be applied to the case of cuspidal edges.

Let $f : \Sigma \to \boldsymbol{R}^3$ be a front, ν a unit normal to f and $p \in \Sigma$ a non-degenerate singular point. Then the *parallel surface* f^t *of* f is defined by $f^t = f + t\nu$, where $t \in \boldsymbol{R} \setminus \{0\}$ is constant. We note that f^t is also a front since ν is a unit normal to f^t.

LEMMA 4.1. *Let $f : \Sigma \to \boldsymbol{R}^3$ be a front, ν its unit normal vector and p a non-degenerate singular point of f. Suppose that κ_+ is a bounded C^∞ function near p and $\kappa_+(p) \neq 0$. Then p is a singular point of f^t if and only if $t = 1/\kappa_+(p)$. Moreover, p is non-degenerate singular point of f^t if and only if p is not a critical point of κ_+.*

PROOF. We show the case that p is of the second kind. Let $(U; u, v)$ be an adapted coordinate system centered at p with the null vector field $\eta = \partial_u + \varepsilon(u)\partial_v$. Then the signed area density function $\lambda^t = \det(f^t_u, f^t_v, \nu)$ of f^t can be written as

$$\lambda^t = \det(f^t_u, f^t_v, \nu) = (1 - t\kappa_+)(\lambda - t\lambda\kappa_-)$$

by Lemma 3.4, where $\lambda = \det(f_u, f_v, \nu)$. Since $\lambda\kappa_-$ does not vanish at p, p is a singular point of f^t if and only if $t = 1/\kappa_+(p)$ holds. Thus we may treat $\widehat{\lambda}^t = \kappa_+(u,v) - \kappa_+(p)$ as the signed area density function of f^t. Non-degeneracy follows $d\widehat{\lambda}^t = (\kappa_+)_u du + (\kappa_+)_v dv$. □

THEOREM 4.2. *Let $f : \Sigma \to \boldsymbol{R}^3$ be a front and p be a non-degenerate singular point. Suppose that the principal curvature κ_+ is a bounded C^∞ function near p and $\kappa_+(p) \neq 0$. Then for the parallel surface f^t with $t = 1/\kappa_+(p)$, the following conditions hold.*

(1) *Assume $d\kappa_+(p) \neq 0$. Then the following hold:*
  (a) *The map-germ f^t at p is $\mathcal{A}$-equivalent to a cuspidal edge if and only if p is not a ridge point of f.*
  (b) *The map-germ f^t at p is $\mathcal{A}$-equivalent to a swallowtail if and only if p is a first order ridge point of f.*
  (c) *The map-germ f^t at p is $\mathcal{A}$-equivalent to a cuspidal butterfly if and only if p is a second order ridge point of f.*

(2) *Assume $d\kappa_+(p) = 0$. Then the following hold:*
  (a) *The map-germ f^t at p is $\mathcal{A}$-equivalent to a cuspidal lips if and only if* $\operatorname{rank}(df^t)_p = 1$ *and* $\det \operatorname{Hess}(\kappa_+(p)) > 0$ *hold.*
  (b) *The map-germ f^t at p is $\mathcal{A}$-equivalent to a cuspidal beaks if and only if p is a first order ridge point of f,* $\operatorname{rank}(df^t)_p = 1$ *and* $\det \operatorname{Hess}(\kappa_+(p)) < 0$ *hold.*

Here $\operatorname{Hess}(\kappa_+(p))$ *is the Hessian matrix of κ_+ at p.*

PROOF. Let $f : \Sigma \to \boldsymbol{R}^3$ be a front, $p \in \Sigma$ a non-degenerate singular point of the second kind and ν a unit normal vector. Then we take an adapted coordinate system $(U; u, v)$ around p. By Lemma 4.1, we can take the signed area density function of parallel surface f^t with $t = 1/\kappa_+(p)$ as $\widehat{\lambda}^t(u,v) = \kappa_+(u,v) - \kappa_+(p)$.

First, we prove the assertion (1). In this case, $(\widehat{\lambda}^t)^{-1}(0)$ is a smooth curve near p and there exists a null vector field η^t of f^t. We set $\eta^t = \eta_1^t \partial_u + \eta_2^t \partial_v$, where η_i^t $(i = 1, 2)$ are functions on U. By Lemma 3.4, $df^t(\eta^t) = 0$ on $S(f^t)$ is equivalent to

$$\begin{pmatrix} \widehat{L} - \kappa_+(v\widehat{E} - \varepsilon\widehat{F}) & \widehat{M} - \kappa_+\widehat{F} \\ v(\widehat{M} - \kappa_+\widehat{F}) - \varepsilon(\widehat{N} - \kappa_+\widehat{G}) & \widehat{N} - \kappa_+\widehat{G} \end{pmatrix} \begin{pmatrix} \eta_1^t \\ \eta_2^t \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

holds on $S(f^t)$. Thus the null vector field η^t can be taken as the principal vector $\boldsymbol{V}$ as in (3.13) with respect to κ_+ restricted to $S(f^t)$. Under these conditions, the equation $(\eta^t)^{(k)}\widehat{\lambda}^t = \boldsymbol{V}^{(k)}\kappa_+$ holds for some natural number k. Thus we have the assertion (1) by Fact 2.5 and Fact 2.6 (3).

Next, we prove (2). In this case, $d\kappa_+$ vanishes at p. We consider the rank of df^t at p. The Jacobian matrix J_{f^t} of f^t is $J_{f^t} = (h, f_v)\mathcal{M}$ at p, where

$$\begin{aligned}\mathcal{M} &= \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} - t \begin{pmatrix} \widehat{E} & \widehat{F} \\ \widehat{F} & \widehat{G} \end{pmatrix}^{-1} \begin{pmatrix} \widehat{L} & \widehat{M} \\ 0 & \widehat{N} \end{pmatrix} \\ &= \frac{1}{\widehat{N}(\widehat{E}\widehat{G} - \widehat{F}^2)} \begin{pmatrix} -\widehat{G}^2\widehat{L} & \widehat{G}(\widehat{F}\widehat{N} - \widehat{G}\widehat{M}) \\ \widehat{F}\widehat{G}\widehat{L} & -\widehat{F}(\widehat{F}\widehat{N} - \widehat{G}\widehat{M}) \end{pmatrix}.\end{aligned} \tag{4.1}$$

Since $\operatorname{rank} \mathcal{M} = 1$, it follows that $\operatorname{rank}\left(J_{f^t}\right)_p = 1$, when $t = 1/\kappa_+(p)$, and it implies that $\operatorname{rank}\left(df^t\right)_p = 1$. Thus there exists a non-zero vector field η^t near p such that if $q \in S(f^t)$ then $df^t(\eta^t) = 0$ holds at q. We can take the principal vector $\boldsymbol{V}$ with respect to κ_+ as η^t, then $\eta^t\eta^t\widehat{\lambda}^t = \boldsymbol{V}^{(2)}\kappa_+$. Moreover, we see that $\widehat{\lambda}^t_{uu} = (\kappa_+)_{uu}$, $\widehat{\lambda}^t_{uv} = (\kappa_+)_{uv}$, $\widehat{\lambda}^t_{vv} = (\kappa_+)_{vv}$. Thus we have $\det \operatorname{Hess}(\widehat{\lambda}^t(p)) = \det \operatorname{Hess}(\kappa_+(p))$. By using Fact 2.6 (1) and (2) and the definition of ridge points, we have the conclusion. □

This theorem implies that the behavior of a bounded principal curvature of fronts determines the types of singularities appearing on parallel surfaces. For regular surfaces and Whitney umbrellas, similar results are obtained in [**11, 12**]. By (4.1) in the proof of Theorem 4.2 and Fact 2.7, we see that a parallel surface f^t does not have D_4 singularity at p.

2. Constant principal curvature lines of cuspidal edges

Let $f : \Sigma \to \boldsymbol{R}^3$ be a front, ν a unit normal vector and p a cuspidal edge. Suppose that κ_+ is bounded at p and $\kappa_+(p) \neq 0$. We set $\widehat{\lambda}^t(u, v) = \kappa_+(u, v) - \kappa_+(p)$. The zero-set of this function gives the singular curve of the parallel surface f^t of f, where $t = 1/\kappa_+(p)$ (Lemma 4.1). We call the curve given by $\widehat{\lambda}^t(u, v) = \kappa_+(u, v) - \kappa_+(p) = 0$ a *constant principal curvature* (*CPC*) *line with the value of* $\kappa_+(p)$ (cf. [**11, 12**]). In this case, the CPC line is a regular curve since $d\widehat{\lambda}^t(p) \neq 0$. In [**11, 12**], CPC lines for regular surfaces and Whitney umbrellas, and relations between singularities of parallel surfaces and the behavior of CPC lines are investigated. For intrinsic properties of Whitney umbrellas, see [**17, 18**].

First, we consider contact of the CPC line with the singular curve.

DEFINITION 4.3. Let $\alpha : I \ni t \mapsto (x(t), y(t)) \in \boldsymbol{R}^2$ be a regular plane curve and let β be another plane curve given as the zero set of a C^∞ function $F : \boldsymbol{R}^2 \to \boldsymbol{R}$, where $I \subset \boldsymbol{R}$ is an open interval. Then α has $(n+1)$-*point contact* at $t_0 \in I$ with β if the function $g(t) = F \circ \alpha(t) = F(x(t), y(t))$ satisfies

$$g(t_0) = g'(t_0) = g''(t_0) = \cdots = g^{(n)}(t_0) = 0 \quad \text{and} \quad g^{(n+1)}(t_0) \neq 0,$$

where $' = d/dt$ and $g^{(m)}$ denotes the m-th order derivative of g.

PROPOSITION 4.4. *Let $f : \Sigma \to \boldsymbol{R}^3$ be a front, p a cuspidal edge and γ a singular curve passing through p. Suppose that κ_+ is bounded near p and $d\kappa_+(p) \neq 0$. Then γ has $(n+1)$-point contact at p with the CPC line if and only if*

$$\kappa_\nu'(p) = \cdots = \kappa_\nu^{(n)}(p) = 0 \quad \textit{and} \quad \kappa_\nu^{(n+1)}(p) \neq 0.$$

PROOF. Let $(U; u, v)$ be an adapted coordinate system. Then $\kappa_+(u, 0) = \kappa_\nu(u)$ holds by Theorem 3.6. Thus the composite function of $\widehat{\lambda}^t$ and γ is given as

$$\widehat{\lambda}^t(u, 0) = \kappa_\nu(u) - \kappa_\nu(p)$$

since $\kappa_+(p) = \kappa_\nu(p)$. Hence we get the conclusion by the definition of contact of two plane curves. □

Next, we consider special points (landmarks in the sense of Porteous [**42**]) on CPC lines of cuspidal edges. Let $f : \Sigma \to \boldsymbol{R}^3$ be a front, $p \in \Sigma$ a cuspidal edge and assume that κ_+ is bounded near p. The condition $\eta\kappa_+ = 0$ at p implies that the CPC line is tangent to the null vector field η of f at p. Moreover, the image $f(S(f^t))$ of the set of singular points of the parallel surface f^t by f is cusped at p. We call such a point an *exactly cusped point for the constant principal curvature* (*CPC*) *line.*

PROPOSITION 4.5. *Let $f : \Sigma \to \boldsymbol{R}^3$ be a front and p a cuspidal edge. Suppose that κ_+ (resp. κ_-) is bounded at p. Then $\eta\kappa_+(p) = 0$ (resp. $\eta\kappa_-(p) = 0$) implies $\kappa_s(p) \leq 0$.*

PROOF. We assume that κ_+ is bounded near p. Let us take an adapted coordinate system $(U; u, v)$ centered at p. Then the null vector field η is given by $\eta = \partial_v$. Thus we have

$$\eta\kappa_+ = (\kappa_+)_v = -\frac{1}{2\kappa_c}(4\kappa_t^2 + \kappa_s\kappa_c^2)\left(\frac{|f_u \times h|}{|f_u|}\right)^{1/2}$$

at p by (3.17). This implies that $\eta\kappa_+ = 0$ at p if and only if

$$\kappa_s = \frac{4\kappa_t^2}{\kappa_c^2} < 0$$

holds at p. Hence we have the assertion.

For the case of κ_- to be bounded, we can show in a similar way. □

Relations between the Gaussian curvature and the singular curvature are stated in [**46**, Theorem 3.1].

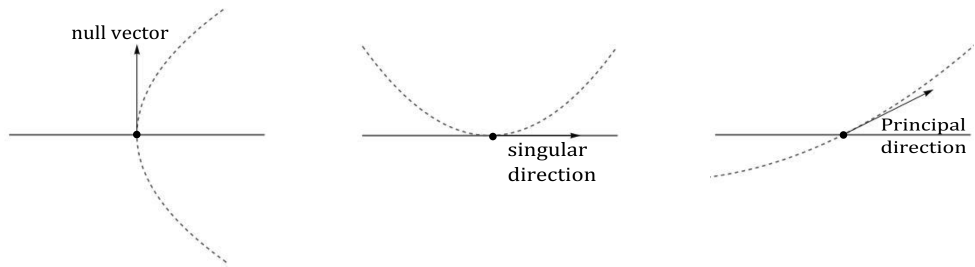
PROPOSITION 4.6. *Let $f : \Sigma \to \boldsymbol{R}^3$ be a front, p a cuspidal edge, γ a singular curve and η a null vector field. Assume that κ_+ is bounded near p, $\kappa_+(p) \neq 0$ and p is not a ridge point of f. Then the cusp-directional torsion κ_t^t of f^t vanishes at p if and only if $\eta\kappa_+$ vanishes at p, namely, p is an exactly cusped point, where $t = 1/\kappa_+(p)$.*

PROOF. Let f be a normal form as in (3.19) and σ be a singular curve of f^t satisfying $\widehat{\lambda}^t(\sigma) = 0$. Note that coefficients in (3.19) satisfy $b_{20} \neq 0$ and $4b_{12}^3 + b_{30}b_{03}^2 \neq 0$ since $\kappa_+(0) = \kappa_\nu(0) = b_{20}$ and 0 is not a ridge point (see Proposition 3.13). We assume that $(\kappa_+)_u(0) \neq 0$. Then we can take $\sigma(v) = (u(v), v)$. Let $\boldsymbol{W} = u'\partial_u + \partial_v$ denote a vector field tangent to σ, where $u' = -(\kappa_+)_v/(\kappa_+)_u$. The pair $(\boldsymbol{W}, \boldsymbol{V})$ gives an adapted pair of vector fields in the sense of [**31**]. Moreover, $\langle \boldsymbol{W}f^t, \boldsymbol{V}\boldsymbol{V}f^t\rangle = 0$ holds at 0. By [**31**, (5.1)], we have

$$\kappa_t^t(0) = \frac{\det(\boldsymbol{W}f^t, \boldsymbol{V}\boldsymbol{V}f^t, \boldsymbol{W}\boldsymbol{V}\boldsymbol{V}f^t)}{|\boldsymbol{W}f^t \times \boldsymbol{V}\boldsymbol{V}f^t|^2}(0) = \frac{b_{20}^2(4b_{12}^2 + a_{20}b_{03}^2)}{4b_{12}^3 + b_{30}b_{03}^2}. \tag{4.2}$$

Comparing $(\kappa_+)_v(0) = -(4b_{12}^2 + a_{20}b_{03}^2)/2b_{03}$ and (4.2), we obtain the result. □

We now consider the case that $(\kappa_+)_u = 0$ at p. Since this is equivalent to $\kappa_\nu' = 0$ at p, we call such a point an *extrema of the limiting normal curvature* κ_ν. Therefore we have three special points on cuspidal edges which have special relations between the singular curve and the CPC line (see Figure 4.1). It seems that exactly cusped points have not appeared in the literature.



exactly cusped point extrema of κ_ν ridge point

FIGURE 4.1. Figures of the singular curve and the CPC line near a cuspidal edge. The solid curve is the singular curve and the dotted one is the CPC line through p.

CHAPTER 5

Focal surfaces of wave fronts

We consider focal surfaces of fronts. Since focal surfaces can be regarded as singular value set of a certain map $\boldsymbol{R}^3 \to \boldsymbol{R}^3$, we consider focal surfaces by using results about Morin singularities of $\boldsymbol{R}^3 \to \boldsymbol{R}^3$. This chapter is based on [**51**].

1. Morin singularities

We recall relations between the A_k-Morin singularities and the A_k-front singularities. The A_k-*Morin singularities* are map germs $f : (\boldsymbol{R}^n, p) \to (\boldsymbol{R}^n, f(p))$ which are $\mathcal{A}$-equivalent to

$$f(x_1, \dots, x_n) = (x_1, \dots, x_{n-1}, x_1 x_n + \cdots + x_{k-1} x_n^{k-1} + x_n^{k+1}) \quad (k \le n)$$

at the origin 0 (see [**34, 14, 45**]). We note that the A_0-Morin singularity is a regular point.

Fact 5.1 ([**45**, Theorem A.1]). *Assume that $k \le n$. Let Ω be a domain of $\boldsymbol{R}^n$, and $f : \Omega \to \boldsymbol{R}^n$ a C^∞ map and p a singular point of f. Assume that p is a corank one singularity. Then f at p is $\mathcal{A}$-equivalent to an A_k-Morin singularity if and only if*

(1) $\Lambda = \Lambda' = \cdots = \Lambda^{(k-1)} = 0$ *and* $\Lambda^{(k)} \neq 0$ *at* p,
(2) $(\Lambda, \Lambda', \dots, \Lambda^{(k-1)}) : \Omega \to \boldsymbol{R}^k$ *is non-singular at* p.

Here, $\Lambda = \det(f_{x_1}, \dots, f_{x_n})$, $(x_1, \dots, x_n)$ *is the canonical coordinate system on* Ω, $\Lambda' = \widetilde{\eta}\Lambda$, $\Lambda^{(i)} = \widetilde{\eta}\Lambda^{(i-1)}$ *and* $\widetilde{\eta}$ *is the extended null vector field of* f.

Fact 5.2 ([**45**, Corollary 2.11]). *Let Ω be a domain of $\boldsymbol{R}^{n+1}$ and $f : \Omega \to \boldsymbol{R}^{n+1}$ a C^∞ map. Suppose that $p \in \Omega$ is a singular point of f such that the exterior derivative of the Jacobian of f does not vanish at p. Then the following are equivalent:*

(1) p *is an* A_k*-Morin singular point of* f,
(2) $f|_{S(f)}$ *is a front, and* p *is an* A_k*-front singularity of* $f|_{S(f)}$.

Here, the A_{k+1}-*front singularity* is a C^∞ map germ defined as

$$X \mapsto \left((k+1)t^{k+2} + \sum_{j=2}^{k} (j-1)t^j x_j, -(k+2)t^{k+1} - \sum_{j=2}^{k} j t^{j-1} x_j, X_1 \right)$$

at 0, where $X = (t, x_2, \ldots, x_n)$ and $X_1 = (x_2, \ldots, x_n)$ (see [**2, 45**]).

Remark 5.3. The image of an A_1-front singularity is a regular point, the image of an A_2-front singularity is a cuspidal edge, and the image of an A_3-front singularity is a swallowtail if the dimension of the source space is two and of the target space is three (see [**2, 4, 45**]).

2. Focal surfaces of wave fronts

Let $f : \Sigma \to \boldsymbol{R}^3$ be a front, ν a unit normal vector to f and $p \in \Sigma$ a non-degenerate singular point. We assume that p is of the second kind, and $(U; u, v)$ is an adapted coordinate system centered at p satisfying $\lambda_v(u, 0) > 0$. Suppose that κ_+ is bounded on U. If p is a cuspidal edge, the following arguments can be applied similarly.

We now consider a map $\mathcal{F} : U \times \boldsymbol{R} \to \boldsymbol{R}^3$ as

$$\mathcal{F}(u, v, w) = f(u, v) + w\nu(u, v) \quad ((u, v) \in U,\ w \in \boldsymbol{R}). \tag{5.1}$$

By direct computations, it follows that

$$\mathcal{F}_u = (v + w\alpha_1)h + (-\varepsilon(u) + w\alpha_2)f_v, \quad \mathcal{F}_v = w\beta_1 h + (1 + w\beta_2)f_v, \quad \mathcal{F}_w = \nu,$$

where

$$\alpha_1 = \frac{\widehat{F}(v\widehat{M} - \varepsilon(u)\widehat{N}) - \widehat{G}\widehat{L}}{|h \times f_v|^2}, \qquad \alpha_2 = \frac{\widehat{F}\widehat{L} - \widehat{E}(v\widehat{M} - \varepsilon(u)\widehat{N})}{|h \times f_v|^2},$$
$$\beta_1 = \frac{\widehat{F}\widehat{N} - \widehat{G}\widehat{M}}{|h \times f_v|^2}, \qquad \beta_2 = \frac{\widehat{F}\widehat{M} - \widehat{E}\widehat{N}}{|h \times f_v|^2}$$

(cf. Lemmas 3.1 and 3.4). From these calculations, the Jacobian of $\mathcal{F}$ can be written by

$$\det(\mathcal{F}_u, \mathcal{F}_v, \mathcal{F}_w) = (1 - w\kappa_+)(\lambda - w\hat{\kappa}_-),$$

where $\hat{\kappa}_- = \lambda\kappa_-$ and $\lambda = \det(f_u, f_v, \nu)$. By Remark 3.7, $\hat{\kappa}$ is a C^∞ function and does not vanish on the u-axis, in particular at p. Thus we see that $\det(\mathcal{F}_u, \mathcal{F}_v, \mathcal{F}_w) = 0$ if and only if $1 - w\kappa_+(u, v) = 0$ or $\lambda(u, v) - w\hat{\kappa}_-(u, v) = 0$. Hence the set of singular points of $\mathcal{F}$ is $S(\mathcal{F}) = S_1(\mathcal{F}) \cup S_2(\mathcal{F})$, where $S_1(\mathcal{F}) = \{(u,v,w) \,|\, w = 1/\kappa_+(u,v)\}$ and $S_2(\mathcal{F}) = \{(u,v,w) \,|\, w = \lambda(u,v)/\hat{\kappa}_-(u,v)\}$. The image of $\mathcal{F}(S(\mathcal{F}))$ is

$$\mathcal{F}(S(\mathcal{F})) = \left\{ f(u,v) + \frac{1}{\kappa_+(u,v)}\nu(u,v) \,\middle|\, (u,v) \in U, w = \frac{1}{\kappa_+(u,v)} \right\} \bigcup \left\{ f(u,v) + \frac{\lambda(u,v)}{\hat{\kappa}_-(u,v)}\nu(u,v) \,\middle|\, (u,v) \in U, w = \frac{\lambda(u,v)}{\hat{\kappa}_-(u,v)} \right\}. \tag{5.2}$$

We set

(5.3)

$$FC_f(u,v) = f(u,v) + \frac{1}{\kappa_+(u,v)}\nu(u,v), \quad \widehat{FC}_f(u,v) = f(u,v) + \frac{\lambda(u,v)}{\hat{\kappa}_-(u,v)}\nu(u,v).$$

These are *focal surfaces* of f (cf. [**9**, pages 231 and 232], see also [**7, 23**]). If f at p is a cuspidal edge, by the similar calculations, we have the same formulae as in (5.3) for focal surfaces of f. We assume that f at p is a cuspidal edge or of the second kind. We note that FC_f can not be defined at p if $\kappa_+(p) = 0$ (such a point is called a *parabolic point*, see next subsection). On the other hand, $\widehat{FC}_f$ can be defined near p even if $\kappa_+(p)$ vanishes. Since the set of singular points $S(f^t)$ of a *parallel surface* $f^t = f + t\nu$, where $t \in \boldsymbol{R} \setminus \{0\}$ is constant, of a front f is given by $S(f^t) = \{q \in U | t = 1/\kappa_+(q)\}$, the union of all the set of singular points of f^t corresponds to the focal surface FC_f if κ_+ never vanishes.

3. Singularities of a focal surface FC_f on a wave front

We consider relations between singularities of FC_f at p and geometric properties of f. We assume that p is not a parabolic point with respect to κ_+.

LEMMA 5.4. *Under the above settings, a singular point $P = (p, w_0 = 1/\kappa_+(p)) \in S_1(\mathcal{F})$ of $\mathcal{F}$ is corank one. Moreover, $S_1(\mathcal{F})$ is a smooth submanifold of $U \times \boldsymbol{R}$ with codimension one near P.*

PROOF. We show the case that p is a non-degenerate singular point of the second kind. For cuspidal edges, one can show in a similar way.

By the above calculations, $\mathcal{F}_w = \nu$ is linearly independent to $\mathcal{F}_u$ and $\mathcal{F}_v$. We note that $\mathcal{F}_u$ and $\mathcal{F}_v$ do not vanish at P simultaneously since $\alpha_1(p) \neq 0$ holds. The cross product of $\mathcal{F}_u$ and $\mathcal{F}_v$ satisfies

$$\mathcal{F}_u \times \mathcal{F}_v = (1 - w\kappa_+)(\lambda - w\hat{\kappa}_-)\nu = 0$$

at $P=(p,w_0)\in S_1(\mathcal{F})$. Thus $\mathcal{F}_u$ and $\mathcal{F}_v$ are linearly dependent at P. This implies that a point $P=(p,w_0)\in S_1(\mathcal{F})$ is corank one.

We show $S_1(\mathcal{F})$ is a smooth submanifold of $U\times\boldsymbol{R}$ near P. By straightforward computations, the Jacobian matrix of $\mathcal{F}$ is rank two at $P=(p,w_0)\in S_1(\mathcal{F})$. We put $\Lambda : U\times\boldsymbol{R}\to\boldsymbol{R}$ as $\Lambda(u,v,w)=1-w\kappa_+(u,v)$. The gradient vector $\mathrm{grad}(\Lambda)$ of Λ is

$$\mathrm{grad}(\Lambda)=\left(-\frac{(\kappa_+)_u}{\kappa_+},-\frac{(\kappa_+)_v}{\kappa},-\kappa_+\right)\neq(0,0,0)$$

at $(p,w_0)\in S_1(\mathcal{F})$ since $\kappa_+(p)\neq 0$, where $(\kappa_+)_u=\partial\kappa_+/\partial u$ and $(\kappa_+)_v=\partial\kappa_+/\partial v$. By the implicit function theorem, we have the conclusion. □

Let $\boldsymbol{V}$ be a principal vector with respect to κ_+. Then $d\mathcal{F}(\boldsymbol{V})=0$ holds on $S_1(\mathcal{F})$ since the definitions of principal curvatures and principal vectors. Therefore $\boldsymbol{V}$ can be considered as the extended null vector field $\widetilde{\eta}$ of $\mathcal{F}$.

LEMMA 5.5. *Under the above conditions, the following assertions hold.*

(1) *$\mathcal{F}$ has an A_1-Morin singularity at $P=(p,w_0)\in S_1(\mathcal{F})$ if and only if p is not a ridge point of f.*

(2) *$\mathcal{F}$ has an A_2-Morin singularity at $P=(p,w_0)\in S_1(\mathcal{F})$ if and only if p is a first order ridge point of f.*

(3) *$\mathcal{F}$ has an A_3-Morin singularity at $P=(p,w_0)\in S_1(\mathcal{F})$ if and only if p is a second order ridge point of f and the ridge line passing through p is a regular curve.*

Here $w_0=1/\kappa_+(p)$.

PROOF. Let $\mathcal{F}: U\times\boldsymbol{R}\to\boldsymbol{R}^3$ be a C^∞ map given by (5.1). By Lemma 5.4, it follows that a singular point $P=(p,w_0)\in S_1(\mathcal{F})$ of $\mathcal{F}$ is corank one. Moreover, the extended null vector field $\widetilde{\eta}$ can be taken as a principal vector $\boldsymbol{V}$ of κ_+ for $\mathcal{F}$, and the function Λ which gives $S_1(\mathcal{F})$ can be taken as $\Lambda=1-w\kappa_+$ in the both cases that f at p is a cuspidal edge and a non-degenerate singular point of the second kind.

First, we show (1). Since $\kappa_+(p)\neq 0$, $d\Lambda(P)\neq 0$. By assumptions, we have $\widetilde{\eta}\Lambda=-\boldsymbol{V}\kappa_+/\kappa_+\neq 0$ at p if and only if a point p is not a ridge point. Thus assertion (1) holds by Fact 5.1.

Next, we prove the assertion (2). We assume that $\widetilde{\eta}\Lambda=-\boldsymbol{V}\kappa_+/\kappa_+=0$ at $P=(p,w_0)$. The second order directional derivative of Λ in the direction $\widetilde{\eta}$ and becomes

$\widetilde{\eta}^{(2)}\Lambda = -\boldsymbol{V}^{(2)}\kappa_+/\kappa_+$ at $P = (p, w_0)$. Moreover, a map $(\Lambda, \widetilde{\eta}\Lambda) : U \times \boldsymbol{R} \to \boldsymbol{R}^2$ is non-singular at $P = (p, w_0)$ if and only if the matrix

$$\begin{pmatrix} -(\kappa_+)_u/\kappa_+ & -(\kappa_+)_v/\kappa_+ & -\kappa_+ \\ -(\boldsymbol{V}\kappa_+)_u/\kappa_+ & -(\boldsymbol{V}\kappa_+)_v/\kappa_+ & 0 \end{pmatrix}$$

has rank two at $P = (p, w_0)$ by Fact 5.1 in that case of $k = 2$. Since $\boldsymbol{V}^{(2)}\kappa_+(p) \neq 0$, $d(\boldsymbol{V}\kappa_+)(p)$ does not vanish, and hence assertion (2) holds.

Finally, we show (3). We assume that $\Lambda = \widetilde{\eta}\Lambda = \widetilde{\eta}^{(2)}\Lambda = 0$ at $P = (p, w_0) \in S_1(\mathcal{F})$, that is, $w_0 = 1/\kappa_+(p)$ and $\boldsymbol{V}\kappa_+(p) = \boldsymbol{V}^{(2)}\kappa_+(p) = 0$ hold. Then $\widetilde{\eta}^{(3)}\Lambda \neq 0$ at P if and only if $\boldsymbol{V}^{(3)}\kappa_+(p) \neq 0$. In addition, a map $(\Lambda, \widetilde{\eta}\Lambda, \widetilde{\eta}^{(2)}\Lambda) : U \times \boldsymbol{R} \to \boldsymbol{R}^3$ is non-singular at P if and only if the matrix

$$\begin{pmatrix} -(\kappa_+)_u/\kappa_+ & -(\kappa_+)_v/\kappa_+ & -\kappa_+ \\ -(\boldsymbol{V}\kappa_+)_u/\kappa_+ & -(\boldsymbol{V}\kappa_+)_v/\kappa_+ & 0 \\ -(\boldsymbol{V}^{(2)}\kappa_+)_u/\kappa_+ & -(\boldsymbol{V}^{(2)}\kappa_+)_v/\kappa_+ & 0 \end{pmatrix}$$

has rank three at P. Since $\boldsymbol{V}^{(3)}\kappa_+(p) \neq 0$, $d(\boldsymbol{V}^{(2)}\kappa_+)$ does not vanish at p. Therefore the above 3×3 matrix has rank three at P if and only if $d(\boldsymbol{V}\kappa_+)$ does not vanish at p. This condition is equivalent to the condition that the ridge line passing through p is a regular curve. Thus we have the assertion by Fact 5.1. □

For the focal surface FC_f, we shall prove the following assertion.

THEOREM 5.6. *Let $f : \Sigma \to \boldsymbol{R}^3$ be a front, $p \in \Sigma$ a non-degenerate singular point. Suppose that κ_+ (resp. κ_-) is C^∞ principal curvature of f near p and FC_f is a focal surface of f with respect to κ_+. Then the following assertions hold.*

(1) *FC_f is non-singular at p if and only if p is not a ridge point of f.*
(2) *FC_f is a cuspidal edge at p if and only if p is a first order ridge point of f.*
(3) *FC_f is a swallowtail at p if and only if p is a second order ridge point of f and the ridge line passing through p is a regular curve.*

PROOF. We prove the case that front has non-degenerate singular point of the second kind. For the case of cuspidal edges, we can show in a similar way.

Let $f : \Sigma \to \boldsymbol{R}^3$ be a front, ν a unit normal vector to f and $p \in \Sigma$ a non-degenerate singular point. Suppose that $(U; u, v)$ is an adapted coordinate system centered at p and a principal curvature κ_+ of f is of class C^∞ on U. We construct a map $\mathcal{F} : U \times \boldsymbol{R} \to \boldsymbol{R}^3$ as in (5.1). Then the image of the set of singular points of $\mathcal{F}$ gives a focal surface of f with respect to κ_+ (see (5.2)). Moreover, a point $P = (p, w_0) \in S_1(\mathcal{F})$ is corank one singular point of $\mathcal{F}$ by Lemma 5.4. Thus we get the conclusions by Lemma 5.5 and Fact 5.2. □

4. Geometric properties of $\widehat{FC}_f$ of cuspidal edges

We consider geometric properties of $\widehat{FC}_f$ as in (5.3) of a front $f : \Sigma \to \boldsymbol{R}^3$ with a cuspidal edge $p \in \Sigma$.

PROPOSITION 5.7. *Let $f : \Sigma \to \boldsymbol{R}^3$ be a front, ν a unit normal vector to f and $p \in \Sigma$ a cuspidal edge. Then the focal surface $\widehat{FC}_f$ is regular at p. Moreover, the limiting tangent plane LT of f at $f(p)$ and the tangent plane TP of $\widehat{FC}_f$ at $\widehat{FC}_f(p)$ intersect orthogonally.*

Here, the *limiting tangent plane of f at $f(p)$* is a plane which is perpendicular to $\nu(p)$.

PROOF. Let us take an adapted coordinate system $(U; u, v)$ around p satisfying $\eta\lambda(u, 0) > 0$. Then the limiting tangent plane LT of f at $f(p)$ is generated by $f_u(p)$ and $h(p)$. Moreover, ν is given by $\nu = (f_u \times h)/|f_u \times h|$.

On the other hand, we consider the tangent plane of the focal surface $\widehat{FC}_f$ at $\widehat{FC}_f(p)$. We note that $\widehat{FC}_f(p) = f(p)$ holds. By direct computations, we have

$$(\widehat{FC}_f)_u = f_u + \left(\frac{\lambda}{\hat{\kappa}_-}\right)_u \nu + \frac{\lambda}{\hat{\kappa}_-}\nu_u, \quad (\widehat{FC}_f)_v = vh + \left(\frac{\lambda}{\hat{\kappa}_-}\right)_v \nu + \frac{\lambda}{\hat{\kappa}_-}\nu_v.$$

Thus $(\widehat{FC}_f)_u(p) = f_u(p)$ and $(\widehat{FC}_f)_v(p) = \lambda_v(p)\nu(p)/\hat{\kappa}_-(p)$ hold, where $\lambda_v = |f_u \times h|$. This implies that $\widehat{FC}_f$ is regular at p, and $f_u(p)$ and $\nu(p)$ are orthogonal basis of the tangent plane TP of $\widehat{FC}_f$ at $\widehat{FC}_f(p)$. A normal vector to $\widehat{FC}_f$ is given as $\widetilde{\boldsymbol{n}} = f_u \times \nu$ along the u-axis. Since $\langle \nu, \widetilde{\boldsymbol{n}} \rangle = 0$ holds on the u-axis, LT and TP intersect orthogonally at $f(p) = \widehat{FC}_f(p)$. □

By Proposition 5.7, we can consider the Gaussian and the mean curvature of $\widehat{FC}_f$ along the singular curve γ of f.

THEOREM 5.8. *The Gaussian curvature $K_{\widehat{FC}_f}$ and the mean curvature $H_{\widehat{FC}_f}$ of the focal surface $\widehat{FC}_f$ are given as*

$$K_{\widehat{FC}_f} = -\frac{1}{4}(4\kappa_t^2 + \kappa_s\kappa_c^2), \quad H_{\widehat{FC}_f} = \pm\frac{1}{8}(\kappa_c^2 - 4\kappa_s)$$

along γ, where the sign $\pm$ of $H_{\widehat{FC}_f}$ depends on the orientation of the unit normal vector to $\widehat{FC}_f$.

Proof. Let us take an adapted coordinate system $(U; u, v)$ centered at p with $\eta\lambda(u,0) = \lambda_v(u,0) > 0$. Then we may take ν as $\nu = (f_u \times h)/|f_u \times h|$, and we have $\lambda_v = \det(f_u, h, \nu) = |f_u \times h|$. Since $(\widehat{FC}_f)_u = f_u$ and $(\widehat{FC}_f)_v = \lambda_v \nu/\hat{\kappa}_-$ on the u-axis, coefficients of the first fundamental form of $\widehat{FC}_f$ are

$$E_{\widehat{FC}_f} = \widetilde{E} = |f_u|^2, \quad F_{\widehat{FC}_f} = 0, \quad G_{\widehat{FC}_f} = \frac{\lambda_v^2}{\hat{\kappa}_-^2}$$

along γ. The second order differentials of $\widehat{FC}_f$ can be written as

$$(\widehat{FC}_f)_{uu} = f_{uu}, \quad (\widehat{FC}_f)_{uv} = \frac{\lambda_v}{\hat{\kappa}_-}\nu_u + *_1\nu, \quad (\widehat{FC}_f)_{vv} = -h + \frac{*_2}{\hat{\kappa}_-}f_u + *_3\nu$$

on γ, where $h : U \to \boldsymbol{R}^3 \setminus \{0\}$ is a C^∞ map satisfying $f_v = vh$ and $*_i$ $(i = 1, 2, 3)$ are some functions. We can take a unit normal vector $\boldsymbol{n}$ to $\widehat{FC}_f$ as $\boldsymbol{n} = \pm(f_u \times \nu)/|f_u|$ along γ. Thus coefficients of the second fundamental form of $\widehat{FC}_f$ are

$$L_{\widehat{FC}_f} = \pm\frac{\det(f_u, \nu, f_{uu})}{|f_u|}, \quad M_{\widehat{FC}_f} = \pm\frac{\lambda_v \det(f_u, \nu, \nu_u)}{\hat{\kappa}_-|f_u|}, \quad N_{\widehat{FC}_f} = \pm\frac{\det(f_u, h, \nu)}{|f_u|}$$

at p. By (2.2), $L_{\widehat{FC}_f} = \mp\kappa_s|f_u|^2$ holds. By Lemma 3.1 and (3.3), ν_u is expressed as

$$\nu_u = \frac{\widetilde{F}\widetilde{M} - \widetilde{G}\widetilde{L}}{|f_u \times h|^2} f_u - \frac{\kappa_t |f_u|^2}{|f_u \times h|} h \tag{5.4}$$

along γ. On the other hand, the following equation holds on γ by (3.3):

$$\frac{\lambda_v}{\hat{\kappa}_-} = \frac{|f_u \times h|^2}{|f_u|^2 \widetilde{N}} = \frac{2}{\kappa_c}\left(\frac{|f_u \times h|}{|f_u|}\right)^{1/2}. \tag{5.5}$$

Hence $M_{\widehat{FC}_f}$ is calculated as

$$M_{\widehat{FC}_f} = \pm\frac{2}{\kappa_c}\left(\frac{|f_u \times h|}{|f_u|}\right)^{1/2} \frac{\kappa_t|f_u|}{|f_u \times h|}\det(f_u, h, \nu) = \pm\frac{2\kappa_t|f_u|^{1/2}}{\kappa_c|f_u \times h|^{1/2}}\det(f_u, h, \nu)$$

on γ by (5.4) and (5.5). Since $\det(f_u, h, \nu) = |f_u \times h|$, we have

$$M_{\widehat{FC}_f} = \pm\frac{2\kappa_t}{\kappa_c}(|f_u||f_u \times h|)^{1/2}, \quad N_{\widehat{FC}_f} = \pm\frac{|f_u \times h|}{|f_u|}$$

and

$$E_{\widehat{FC}_f}G_{\widehat{FC}_f}-F^2_{\widehat{FC}_f}=\frac{4|f_u||f_u\times h|}{\kappa_c^2},\ L_{\widehat{FC}_f}N_{\widehat{FC}_f}-M^2_{\widehat{FC}_f}=-\frac{(4\kappa_t^2+\kappa_s\kappa_c^2)}{\kappa_c^2}|f_u||f_u\times h|,$$

$$E_{\widehat{FC}_f}N_{\widehat{FC}_f}-2F_{\widehat{FC}_f}M_{\widehat{FC}_f}+G_{\widehat{FC}_f}L_{\widehat{FC}_f}=\pm\frac{(\kappa_c^2-4\kappa_s)}{\kappa_c^2}|f_u||f_u\times h|$$

along γ. Thus the assertions hold by the following formulae:

$$K_{\widehat{FC}_f}=\frac{L_{\widehat{FC}_f}N_{\widehat{FC}_f}-M^2_{\widehat{FC}_f}}{E_{\widehat{FC}_f}G_{\widehat{FC}_f}-F^2_{\widehat{FC}_f}},\quad H_{\widehat{FC}_f}=\frac{E_{\widehat{FC}_f}N_{\widehat{FC}_f}-2F_{\widehat{FC}_f}M_{\widehat{FC}_f}+G_{\widehat{FC}_f}L_{\widehat{FC}_f}}{E_{\widehat{FC}_f}G_{\widehat{FC}_f}-F^2_{\widehat{FC}_f}}.$$

□

Comparing Theorem 5.8 and Proposition 3.11, we have the following assertion.

COROLLARY 5.9. *Let f be a front in $\boldsymbol{R}^3$, p a cuspidal edge and $\widehat{FC}_f$ the focal surface. Then the Gaussian curvature $K_{\widehat{FC}_f}$ of $\widehat{FC}_f$ vanishes at p if and only if p is a sub-parabolic point with respect to a bounded principal curvature of f.*

This property is similar as the case of regular surfaces obtained by Morris [**35**] (see also [**20**]). Moreover, we have the following properties immediately.

COROLLARY 5.10. *Let f be a front in $\boldsymbol{R}^3$, p a cuspidal edge and $\widehat{FC}_f$ a focal surface of f.*

(1) *A point p of $\widehat{FC}_f$ is classified as follows:*
- *p is an elliptic point of $\widehat{FC}_f$ if and only if $4\kappa_t^2+\kappa_s\kappa_c^2<0$ at p,*
- *p is a parabolic point of $\widehat{FC}_f$ if and only if $4\kappa_t^2+\kappa_s\kappa_c^2=0$ at p,*
- *p is a hyperbolic point of $\widehat{FC}_f$ if and only if $4\kappa_t^2+\kappa_s\kappa_c^2>0$ at p.*

Moreover, the Gaussian curvature $K_{\widehat{FC}_f}$ is non-negative at p if and only if κ_s is non-positive at p. In particular, if $K_{\widehat{FC}_f}$ is strictly positive along γ, then κ_s is strictly negative.

(2) *If the mean curvature $H_{\widehat{FC}_f}$ vanishes, κ_s is strictly positive along γ.*

Here, a point p is an elliptic, a parabolic or a hyperbolic point of $\widehat{FC}_f$ if $K_{\widehat{FC}_f}>0$, $=0$ or <0 at p, respectively.

The invariant $4\kappa_t^2+\kappa_s\kappa_c^2$ is appeared as the coefficient of v in the Gaussian curvature K of a cuspidal edge (see [**32**]).

Under this setting, since the singular locus $\hat{\gamma}$ is a regular curve on $\widehat{FC}_f$, we can consider the *geodesic curvature* $\hat{\kappa}_g$ and the *normal curvature* $\hat{\kappa}_n$ of $\widehat{FC}_f$ along the singular curve γ.

PROPOSITION 5.11. *Let $f : \Sigma \to \boldsymbol{R}^3$ be a front, $p \in \Sigma$ a cuspidal edge, γ a singular curve passing through p and η a null vector field. Assume that $\eta\lambda > 0$ (resp. $\eta\lambda < 0$) along γ. Then $\hat{\kappa}_g = \kappa_\nu$ and $\hat{\kappa}_n = -\kappa_s$ (resp. $\hat{\kappa}_g = \kappa_\nu$ and $\hat{\kappa}_n = \kappa_s$) hold along γ.*

PROOF. Let us take an adapted coordinate system $(U; u, v)$ centered at p with $\eta\lambda(u, 0) > 0$. Then we take the unit normal vector $\boldsymbol{n}$ to $\widehat{FC}_f$ along γ as $\boldsymbol{n} = (f_u \times \nu)/|f_u|$. The geodesic curvature $\hat{\kappa}_g$ and the normal curvature $\hat{\kappa}_n$ of $\widehat{FC}_f$ along γ are written as

$$\hat{\kappa}_g = \frac{\langle \hat{\gamma}'', \boldsymbol{n} \times \hat{\gamma}' \rangle}{|\hat{\gamma}'|^3} = \frac{\langle f_{uu}, \boldsymbol{n} \times f_u \rangle}{|f_u|^3}, \quad \hat{\kappa}_n = \frac{\langle \hat{\gamma}'', \boldsymbol{n} \rangle}{|\hat{\gamma}'|^2} = \frac{\langle f_{uu}, \boldsymbol{n} \rangle}{|f_u|^2}.$$

By direct calculations, we see that $\boldsymbol{n} \times f_u = |f_u|\nu$ and $\langle f_{uu}, \boldsymbol{n} \rangle = -\det(f_u, f_{uu}, \nu)/|f_u|$ hold. By [**46**, (1.7)] and [**46**, (3.11)], we have the assertions. □

COROLLARY 5.12. *The Gaussian curvature of a cuspidal edge is bounded on a sufficiently small neighborhood of the singular curve γ if and only if $\hat{\gamma}$ is a (pre-) geodesic on $\widehat{FC}_f$.*

Here, a curve on a regular surface is called a *pre-geodesic* if the geodesic curvature vanishes along the curve (cf. [**26**]). In addition, we call a curve on a regular surface a *geodesic* if the curve is pre-geodesic and has unit speed.

PROOF. The Gaussian curvature of a cuspidal edge is bounded if and only if the limiting normal curvature κ_ν vanishes along the singular curve γ ([**46**, Theorem 3.1]). Thus we have the assertion by Proposition 5.11. □

It is known that the singular locus $\hat{\gamma}$ of cuspidal edges is a *line of curvature* if and only if the cusp-directional torsion κ_t vanishes identically on γ ([**50**, Proposition 3.2], see also [**24**]). In this case, we have the following.

PROPOSITION 5.13. *Let f be a front, p a cuspidal edge and γ a singular curve passing through p. Suppose that $\hat{\gamma}$ is a line of curvature on f. Then $\hat{\gamma}$ is also a line of curvature on $\widehat{FC}_f$.*

Proof. It is known that a curve $\sigma(t)$ on a regular surface is a line of curvature if and only if

$$\det(\dot{\sigma}, n, \dot{n}) = 0$$

holds, where $n = n(t)$ is a unit normal to the surface restricted to σ and we denote $\dot{} = d/dt$. We apply this fact to the case of $\widehat{FC}_f$.

We take an adapted coordinate system $(U; u, v)$ centered at p satisfying $\eta\lambda(u, 0) > 0$. Then the unit normal vector to $\widehat{FC}_f$ can be taken as $\boldsymbol{n} = (f_u \times \nu)/|f_u|$ along $\gamma(u) = (u, 0)$. Differentiating $\boldsymbol{n}$, we have

$$\boldsymbol{n}' = \boldsymbol{n}_u = \frac{f_{uu} \times \nu + f_u \times \nu_u}{|f_u|} - |f_u|_u(f_u \times \nu)$$

Thus $\det(\hat{\gamma}', \boldsymbol{n}, \boldsymbol{n}')$ can be written as

$$\det(\hat{\gamma}', \boldsymbol{n}, \boldsymbol{n}') = \frac{1}{\widetilde{E}} \det(f_u, f_u \times \nu, f_{uu} \times \nu + f_u \times \nu_u) = \frac{1}{\widetilde{E}} \det(f_u, f_u \times \nu, f_u \times \nu_u)$$

since $\langle f_u, \nu \rangle = 0$, where we used the relation $\det(\boldsymbol{a} \times \boldsymbol{b}, \boldsymbol{a} \times \boldsymbol{c}, \boldsymbol{d}) = \det(\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c})\langle \boldsymbol{a}, \boldsymbol{d} \rangle$ $(\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c}, \boldsymbol{d} \in \boldsymbol{R}^3)$. By (5.4), ν_u is written as

$$\nu_u = \frac{\widetilde{F}\widetilde{M} - \widetilde{G}\widetilde{L}}{|f_u \times h|^2} f_u - \frac{\kappa_t |f_u|^2}{|f_u \times h|} h.$$

Hence we have

$$\det(\hat{\gamma}', \boldsymbol{n}, \boldsymbol{n}') = \frac{\kappa_t \widetilde{E}}{\sqrt{\widetilde{E}\widetilde{G} - \widetilde{F}^2}} \det(f_u, h, \nu).$$

Since $\lambda_v = \det(f_u, h, \nu)$ does not vanish on γ, we have the assertion. □

In general, a line of curvature on a regular surface does not become a line of curvature on its focal surfaces (cf. [**20**, Proposition 6.19]). Thus Proposition 3.8 gives a characteristic of cuspidal edges.

5. Focal surfaces of Beltrami's pseudosphere

We consider a focal surface of the *Beltrami's pseudosphere* and its screw motion as an example. This surface has constant negative Gaussian curvature. Moreover, one can construct this surface as a surface of revolution of the *tractrix* (c.f. [**8**]).

On the other hand, the Beltrami's pseudosphere is one of solutions of the *sine-Gordon equation*

$$\varphi_{uu} - \varphi_{vv} = \sin \varphi$$

(see [**33**], for example). Although the sine-Gordon equation is a non-linear hyperbolic partial differential equation, it is known that this equation is *integrable*. Thus the Beltrami's pseudosphere is interested in both differential geometry and theory of integrable system. To calculate concretely, we use a parametrization (5.8) of the Beltrami's pseudosphere below in stead of a solution of the sine-Gordon equation above.

5.1. Screw motion of the Beltrami's pseudosphere. Let $\gamma = (\gamma_1, \gamma_2) : I \to \boldsymbol{R}^2$ be a C^∞ map, where $I \subset (\boldsymbol{R}; u)$ is an interval. We assume that $\gamma_1(u) > 0$ for any $u \in I$. Then we consider a surface of revolution $f : I \times \boldsymbol{R}/2\pi\boldsymbol{Z} \to \boldsymbol{R}^3$ whose profile curve is γ given as

$$f(u, v) = (\gamma_1(u)\cos v, \gamma_1(u)\sin v, \gamma_2(u)), \tag{5.6}$$

where $(u, v) \in I \times \boldsymbol{R}/2\pi\boldsymbol{Z}$. We define a *screw motion for a surface of revolution* as follows.

DEFINITION 5.14. Let $f : I \times \boldsymbol{R}/2\pi\boldsymbol{Z} \to \boldsymbol{R}^3$ be a surface of revolution as in (5.6) whose profile curve is $\gamma = (\gamma_1, \gamma_2) : I \to \boldsymbol{R}^2$. Then a *screw motion f^θ of f with screw parameter θ* is a one parameter deformation given by the following form:

$$f^\theta(u, v) = (\cos\theta\gamma_1(u)\cos v, \cos\theta\gamma_1(u)\sin v, \cos\theta\gamma_2(u) + \sin\theta v), \tag{5.7}$$

where $\theta \in [0, \pi/2]$.

By the definition, we see that $f^0 = f$, and $f^{\pi/2}$ degenerates into a line.

We consider the Beltrami's pseudosphere. This surface is a surface of revolution whose profile curve is a tractrix. Let $\gamma : I \to \boldsymbol{R}^2$ be a tractrix given by

$$\gamma(u) = (\sin u, \cos u + \log(\tan(u/2))),$$

where $I = (0, \pi)$. Then we have the Beltrami's pseudosphere f by rotating γ about the y-axis (see Figure 5.1):

$$f(u, v) = (\sin u\cos v, \sin u\sin v, \cos u + \log(\tan(u/2))). \tag{5.8}$$

It is known that f has constant Gaussian curvature -1, and its singular set is $S(f) = \{(\pi/2, v)\}$. We denote by f^θ a screw motion of f with screw parameter $\theta \in [0, \pi/2]$. This surface f^θ is known as the *Dini's surface* (see Figure 5.2). Thus

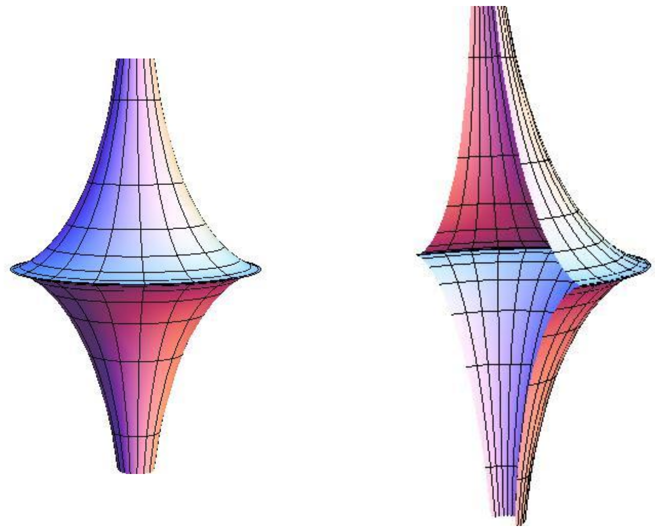


FIGURE 5.1. Beltrami's pseudosphere (left) and its half cut (right).

f^θ also has constant Gaussian curvature -1 for $\theta \in [0, \pi/2)$. We note that the set of singular points $S(f^\theta)$ is same as $S(f) = \{(\pi/2, v)\}$. Moreover, f^θ has only cuspidal edge singularities when $\theta \in [0, \pi/2)$.

We consider geometric properties of f^θ. Let f^θ be the screw motion of the Beltrami's pseudosphere. Then the differentials of f^θ are

$$f_u^\theta = (c(\theta)\cos u\cos v, c(\theta)\cos u\sin v, c(\theta)\cos u\cot u),$$
$$f_v^\theta = (-c(\theta)\sin u\sin v, c(\theta)\sin u\cos v, s(\theta)),$$

where $c(\theta) = \cos\theta$ and $s(\theta) = \sin\theta$. Thus the coefficients of the first fundamental form are

$$E^\theta = c(\theta)^2\cot^2 u,\ F^\theta = c(\theta)s(\theta)\cos u\cot u,\ G^\theta = s(\theta)^2 + c(\theta)^2\sin^2 u. \tag{5.9}$$

We can take unit normal vector ν^θ to f^θ as

$$\nu^\theta = (-c(\theta)\cos u\cos v + s(\theta)\sin v, -s(\theta)\cos v - c(\theta)\cos u\sin v, c(\theta)\sin u). \tag{5.10}$$

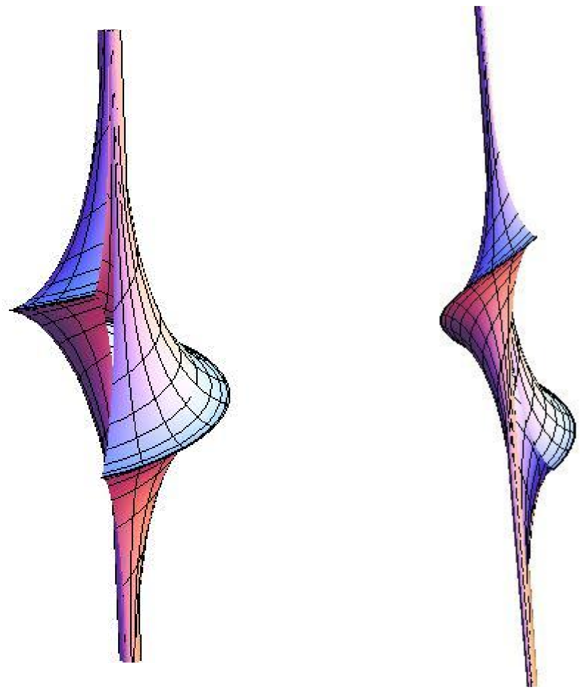


FIGURE 5.2. $f^{\pi/12}$ (left) and $f^{\pi/6}$ (right).

Since the second order differentials of f^θ are

$$\begin{aligned} f^\theta_{uu} &= (-c(\theta)\sin u\cos v, -c(\theta)\sin u\sin v, -c(\theta)(\cos u + \cot u\csc u)),\\ f^\theta_{uv} &= (-c(\theta)\cos u\sin v, c(\theta)\cos u\cos v, 0),\\ f^\theta_{vv} &= (-c(\theta)\sin u\cos v, -c(\theta)\cos u\sin v, 0),\end{aligned}$$

coefficients of the second fundamental form can be given as

$$L^\theta = -c(\theta)^2\cot u,\ M^\theta = -c(\theta)s(\theta)\cos u,\ N^\theta = c(\theta)^2\cos u\sin u. \tag{5.11}$$

Using (5.9) and (5.11), the Weingarten matrix W^θ defined on the set of regular points is given by

$$W^\theta = \begin{pmatrix} -\tan u & t(\theta)\sec(u) \\ 0 & \cot u\end{pmatrix}, \tag{5.12}$$

where $t(\theta) = \tan\theta$. Thus principal curvatures κ_1 and κ_2 of f^θ are $\kappa_1 = \cot u$, $\kappa_2 = -\tan u$. We note that principal curvatures do not depend on the parameter

$\theta \in [0, \pi/2)$. In addition, κ_1 can be extended to the set of singular point on the source and κ_2 becomes unbounded near singular points. On the other hand, if we set principal radius functions $\rho_i = \kappa_i^{-1}$ $(i = 1, 2)$ on the set of regular points, then ρ_1 becomes unbounded near singular points and ρ_2 can be defined as bounded C^∞ function on the source.

5.2. Focal surface of screw motion of the Beltrami's pseudosphere. Let f^θ be a screw motion of the Beltrami's pseudosphere. By the previous section, one principal radius function $\rho = -\cot u$ can be defined on the source even at singular points. By using this function, we define the *focal surface* $\mathfrak{f}^\theta : I \times \boldsymbol{R}/2\pi\boldsymbol{Z} \to \boldsymbol{R}^3$ as

$$
\begin{aligned}
(5.13)\quad \mathfrak{f}^\theta(u, v) &= f^\theta(u, v) + \rho(u, v)\nu^\theta(u, v) \\
&= (c(\theta)\csc u \cos v - s(\theta)\cot u \sin v, s(\theta)\cot u \cos v + c(\theta)\csc u \sin v, \\
&\qquad c(\theta)\log(\tan(u/2)) + s(\theta)v).
\end{aligned}
$$

It is classically known that $\mathfrak{f}^0$ is a catenoid, i.e. a minimal surface of revolution (see Figure 5.3).

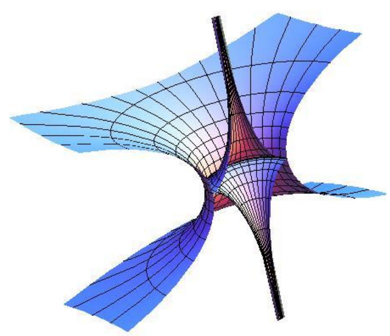


FIGURE 5.3. The Beltrami's pseudosphere and its focal surface (catenoid).

LEMMA 5.15. *The focal surface* $\mathfrak{f}^\theta$ *as in* (5.13) *is a regular surface for any* $\theta \in [0, \pi/2]$.

PROOF. By direct computations, we have

$$\begin{aligned}\mathfrak{f}^\theta_u &= (\csc u(-c(\theta)\cot u\cos v + s(\theta)\csc u\sin v),\\ &\qquad -\csc u(s(\theta)\csc u\cos v + c(\theta)\cot u\sin v), c(\theta)\csc u),\\ \mathfrak{f}^\theta_v &= (-s(\theta)\cot u\cos v - c(\theta)\csc u\sin v, c(\theta)\csc u\cos v - s(\theta)\cot u\sin v, s(\theta)).\end{aligned}$$

The cross product $\mathfrak{f}^\theta_u \times \mathfrak{f}^\theta_v$ is

$$\mathfrak{f}^\theta_u \times \mathfrak{f}^\theta_v = -\csc^2 u(\cos v, \sin v, \cot u) \neq 0$$

for any $(u, v) \in I \times \boldsymbol{R}/2\pi\boldsymbol{Z} = (0, \pi) \times \boldsymbol{R}/2\pi\boldsymbol{Z}$. Thus we have the assertion. □

PROPOSITION 5.16. *Screw motion gives an isometric deformation of* $\mathfrak{f}^\theta$ *as in* (5.13) . *Moreover, for any* $\theta \in [0, \pi/2]$, $\mathfrak{f}^\theta$ *is a minimal surface.*

PROOF. By the proof of the above lemma, we may take a unit normal vector $\mathfrak{n}^\theta$ to $\mathfrak{f}^\theta$ as

$$\mathfrak{n}^\theta(u, v) = (\sin u\cos v, \sin u\sin v, \cos u). \tag{5.14}$$

The differentials of $\mathfrak{n}^\theta$ are

$$\mathfrak{n}^\theta_u = (\cos u\cos v, \cos u\sin v, -\sin u), \quad \mathfrak{n}^\theta_v = (-\sin u\sin v, \sin u\cos v, 0).$$

Thus the coefficients of the first and the second fundamental form are

$$\mathcal{E}^\theta = \csc^4 u,\ \mathcal{F}^\theta = 0,\ \mathcal{G}^\theta = \csc^2 u,\ \mathcal{L}^\theta = c(\theta)\csc^2 u,\ \mathcal{M}^\theta = s(\theta)\csc u,\ \mathcal{N}^\theta = -c(\theta). \tag{5.15}$$

This implies that the first fundamental form does not depend on θ. Hence screw motion gives an isometric deformation of $\mathfrak{f}^\theta$.

Moreover, by (5.15), we see that

$$\mathcal{E}^\theta\mathcal{N}^\theta - 2\mathcal{F}^\theta\mathcal{M}^\theta + \mathcal{G}^\theta\mathcal{L}^\theta \equiv 0,$$

and hence the mean curvature of $\mathfrak{f}^\theta$ vanishes identically. □

PROPOSITION 5.17. *The focal surface* $\mathfrak{f}^\theta$ *is a helicoid when* $\theta = \pi/2$.

PROOF. By (5.13), we have

$$\mathfrak{f}^{\pi/2}(u, v) = (-\cot u\cos v, \cot u\cos v, v).$$

This is a standard parametrization of a catenoid when we set $w = \cot u$. □

Combining Propositions 5.16 and 5.17, we have the following.

THEOREM 5.18. *Let* f^θ *be a screw motion of the Beltrami's pseudosphere* f *as in* (5.8). *Let* $\mathfrak{f}^\theta$ *be the focal surface of* f^θ *given by* (5.13). *Then a screw motion gives an isometric deformation of* $\mathfrak{f}^\theta$ *from catenoid to helicoid* (*see Figure 5.4*).

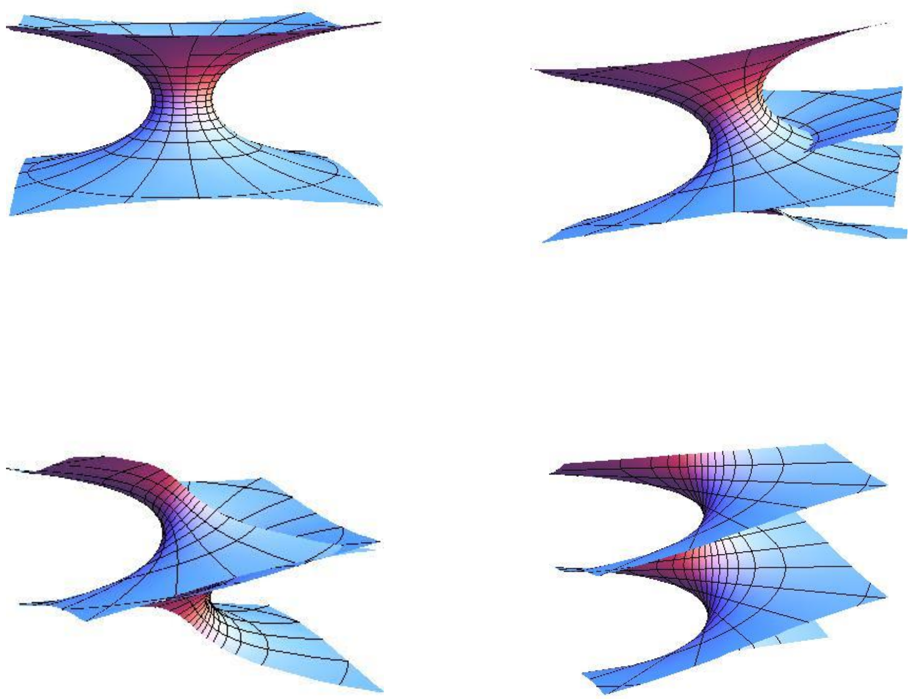


FIGURE 5.4. From top left to bottom right: $\mathfrak{f}^0$, $\mathfrak{f}^{\pi/6}$, $\mathfrak{f}^{\pi/4}$ and $\mathfrak{f}^{\pi/2}$.

CHAPTER 6

Extended distance squared functions on fronts

In this chapter, we consider *extended distance squared functions* on fronts. These functions measure contactness of fronts with certain spheres. This chapter is based on [**49, 50**].

Let $f : \Sigma \to \boldsymbol{R}^3$ be a front, ν its unit normal and p a non-degenerate singular point. Then we define function $\psi : \Sigma \to \boldsymbol{R}$ by

$$\psi(u, v) = -\frac{1}{2}(|\boldsymbol{x}_0 - f(u, v)|^2 - t_0^2), \tag{6.1}$$

where $\boldsymbol{x}_0 \in \boldsymbol{R}^3$ and $t_0 \in \boldsymbol{R} \setminus \{0\}$. We call ψ as in (6.1) the *extended distance squared function with respect to* $\boldsymbol{x}_0$.

Lemma 6.1. *For the function* ψ *as in* (6.1), $\psi(p) = \psi_u(p) = \psi_v(p) = 0$ *if* $\boldsymbol{x}_0 = f(p) + t_0\nu(p)$.

Proof. Let us set $\boldsymbol{x}_0 = f(p)+t_0\nu(p)$. Then $\psi(p) = 0$ holds by the definition of ψ. We now assume that p is of the second kind and take an adapted coordinate system $(U; u, v)$ centered at p. By direct computations, we have $\psi_u = \langle \boldsymbol{x}_0 - f, vh - \varepsilon(u)f_v\rangle$ and $\psi_v = \langle \boldsymbol{x}_0 - f, f_v\rangle$. Since $\langle f_v, \nu\rangle = 0$ and $\varepsilon(0) = 0$, we have $\psi_u(p) = \psi_v(p) = 0$. If p is a cuspidal edge, we can show in a similar way. □

We fix $\boldsymbol{x}_0 = f(p) + t_0\nu(p)$. We are interested in the case of $t_0 = 1/\kappa_+(p)$, because $\boldsymbol{x}_0$ corresponds to the image of a singular point of a parallel surface f^t with $t = 1/\kappa_+(p)$, that is, $\boldsymbol{x}_0$ coincides with a *focal point* of f at p. In such a case, ψ measures contact of f with the *principal curvature sphere* at p (cf. [**20, 30**]).

Proposition 6.2. *If* $\boldsymbol{x}_0 = f(p) + \nu(p)/\kappa_+(p)$ *and* $t_0 = 1/\kappa_+(p)$, *then* $j^2\psi = 0$ *holds, where* $j^2\psi$ *is the* 2*-jet of* ψ *at* p.

Proof. We consider the case that p is of the second kind. Let us take an adapted coordinate system $(U; u, v)$ around p. By Lemma 6.1, we see that $\psi = \psi_u = \psi_v = 0$

at p. By direct calculations, we have

$$\psi_{uu} = -|vh - \varepsilon f_v|^2 + \langle \boldsymbol{x}_0 - f, vh_u - \varepsilon' f_v - \varepsilon f_{uv}\rangle,$$
$$\psi_{uv} = -\langle f_v, vh - \varepsilon f_v\rangle + \langle \boldsymbol{x}_0 - f, h + vh_v - \varepsilon f_{vv}\rangle,$$
$$\psi_{vv} = -|f_v|^2 + \langle \boldsymbol{x}_0 - f, f_{vv}\rangle.$$

Thus $\psi_{uu} = \psi_{uv} = 0$ hold at p since $\langle f_v, \nu\rangle = \langle h, \nu\rangle = 0$. Moreover, it follows that $\psi_{vv} = -\widehat{G} + \langle \nu, f_{vv}\rangle/\kappa_+(p) = -\widehat{G} + \widehat{N}/\kappa_+(p) = 0$ at p since $1/\kappa_+(p) = \widehat{G}(p)/\widehat{N}(p)$. Thus we have the assertion in the case of second kind. For a cuspidal edge, we can show similarly. □

We note that Martins and Nuño-Ballesteros [**30**] investigate singularities of distance squared functions in more general situation. They showed a similar result as Proposition 6.2 by using an *umbilic curvature* κ_u [**30**, Theorem 2.15]. It is known that $|\kappa_\nu(p)| = \kappa_u(p)$ holds when p is a cuspidal edge ([**31**]). Thus Proposition 6.2 might be a special case of [**30**, Theorem 2.15].

Proposition 6.2 implies that ψ may have a D_4 singularity at p if $\boldsymbol{x}_0$ coincides with the focal point of f at p, where a function-germ $h : (\boldsymbol{R}^2, 0) \to (\boldsymbol{R}, 0)$ has a D_4 *singularity* at 0 if h is $\mathcal{R}$-equivalent to the germ $(u, v) \mapsto u^3 \pm uv^2$ at 0 (cf. [**3**, pages 264 and 265]). Therefore we consider the condition that ψ has a D_4 singularity at p in terms of geometric properties of f. (Level sets of these singularities, see Figure 6.1.)

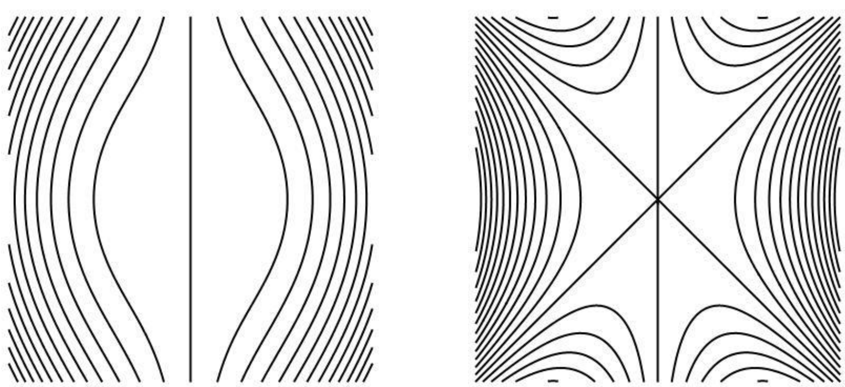


FIGURE 6.1. Level sets of functions $u^3 + uv^2$ (left) and $u^3 - uv^2$ (right).

Let $f : \Sigma \to \boldsymbol{R}^3$ be a front and p a non-degenerate singular point. For a function $\psi : \Sigma \to \boldsymbol{R}$, set

$$\begin{aligned}(6.2)\quad \Delta_\psi = ((\psi_{uuu})^2(\psi_{vvv})^2 - 6\psi_{uuu}\psi_{uuv}\psi_{uvv}\psi_{vvv} \\ - 3(\psi_{uuv})^2(\psi_{uvv})^2 + 4(\psi_{uuv})^3\psi_{vvv} + 4\psi_{uuu}(\psi_{uvv})^3)(p).\end{aligned}$$

It is known that the function ψ is $\mathcal{R}$-equivalent to $u^3 + uv^2$ (resp. $u^3 - uv^2$) if and only if $j^2\psi = 0$ and $\Delta_\psi > 0$ (resp. $\Delta_\psi < 0$) hold (see [**43**, Lemma 3.1], see also [**11**, Theorem 4.2]).

First we consider the case that p is a cuspidal edge. In this case, we have the following assertion.

Theorem 6.3. *Let $f : \Sigma \to \boldsymbol{R}^3$ be a front and p a cuspidal edge. Assume that κ_+ is a bounded C^∞ function near p and $\kappa_+(p) \neq 0$. Then the function ψ as in* (6.1) *with $\boldsymbol{x}_0 = f(p) + \nu(p)/\kappa_+(p)$ and $t_0 = 1/\kappa_+(p)$ has a D_4 singularity at p if and only if $\kappa_i(p) \neq 0$ and p is not a ridge point.*

Proof. Take an adapted coordinate system $(U; u, v)$ centered at p and suppose that $\eta\lambda(u, 0) > 0$. This is equivalent to $\det(f_u, h, \nu)(u, 0) > 0$. By direct computations, the third order differentials of ψ are given by

$$\psi_{uuu}(p) = -\frac{3}{2}\widetilde{E}_u(p) + \frac{\widetilde{E}(p)}{\widetilde{L}(p)}\langle \nu(p), f_{uuu}(p)\rangle, \quad \psi_{uuv}(p) = 0,$$

$$\psi_{uvv}(p) = -\widetilde{F}(p) + \frac{\widetilde{E}(p)\widetilde{M}(p)}{\widetilde{L}(p)} = \frac{\widetilde{E}(p)\widetilde{M}(p) - \widetilde{F}(p)\widetilde{L}(p)}{\widetilde{L}(p)}, \quad \psi_{vvv}(p) = \frac{2\widetilde{N}(p)}{\widetilde{L}(p)},$$

since $\kappa_+(p) = \widetilde{L}(p)/\widetilde{E}(p)$. By Lemma 3.3, $\psi_{uvv}(p)$ and $\psi_{vvv}(p)$ are written as

$$\psi_{uvv}(p) = \kappa_t(p)\frac{\widetilde{E}(p)\sqrt{\widetilde{E}(p)\widetilde{G}(p) - \widetilde{F}(p)^2}}{\widetilde{L}(p)}, \quad \psi_{vvv}(p) = \kappa_c(p)\frac{(\widetilde{E}(p)\widetilde{G}(p) - \widetilde{F}(p)^2)^{3/4}}{\widetilde{E}(p)^{3/4}\widetilde{L}(p)}.$$

By Lemma 3.2, f_{uuu} is given by

$$f_{uuu} = *_1 f_u + *_2 h + \left(\widetilde{\varGamma}^{\,1}_{11}\widetilde{L} + \widetilde{\varGamma}^{\,2}_{11}\widetilde{M} + \widetilde{L}_u\right)\nu,$$

where $*_i$ $(i = 1, 2)$ are some functions. Thus ψ_{uuu} is

$$\psi_{uuu} = -\frac{3}{2}\widetilde{E}_u + \frac{\widetilde{E}}{\widetilde{L}}\left(\widetilde{\varGamma}^{\,1}_{11}\widetilde{L} + \widetilde{\varGamma}^{\,2}_{11}\widetilde{M} + \widetilde{L}_u\right)$$

at p. Hence we have

$$\psi_{uuu}(p) = \frac{(\widetilde{E}\widetilde{M} - \widetilde{F}\widetilde{L})(2\widetilde{F}_u\widetilde{E} - \widetilde{E}\widetilde{E}_{vv} - \widetilde{E}_u\widetilde{F})}{2\widetilde{L}(\widetilde{E}\widetilde{G} - \widetilde{F}^2)}(p) + \frac{\widetilde{E}\widetilde{L}_u - \widetilde{E}_u\widetilde{L}}{\widetilde{L}}(p).$$

Comparing with (3.4), it follows that

$$\psi_{uuu}(p) = \kappa_i(p)\frac{\widetilde{E}^{5/2}}{\widetilde{L}}(p).$$

Since $\psi_{uuv}(p) = 0$, the number Δ_ψ as in (6.2) is written as

$$\Delta_\psi = \psi_{uuu}(p)(4\psi_{uvv}(p)^3 + \psi_{uuu}(p)\psi_{vvv}(p)^2).$$

Summing up the above calculations, we see that

$$\Delta_\psi = \kappa_i(p)(4\kappa_t(p)^3 + \kappa_i(p)\kappa_c(p)^2)\frac{\widetilde{E}(p)^4(\widetilde{E}(p)\widetilde{G}(p) - \widetilde{F}(p)^2)^{3/2}}{\widetilde{L}(p)^4},$$

and hence $\Delta_\psi \neq 0$ if and only if $\kappa_i(p) \neq 0$ and $4\kappa_t(p) + \kappa_i(p)\kappa_c(p)^2 \neq 0$.

On the other hand, the directional derivative $\boldsymbol{V}\kappa_+$ is

$$\boldsymbol{V}\kappa_+(p) = \frac{1}{2\kappa_c(p)}(4\kappa_t(p)^3 + \kappa_i(p)\kappa_c(p)^2)\left(\frac{(\widetilde{E}(p)\widetilde{G}(p) - \widetilde{F}(p)^2)^3}{\widetilde{E}(p)}\right)^{1/4}$$

by Proposition 3.11. Thus we have the assertion. □

Next, we consider the case of the second kind.

THEOREM 6.4. *Let $f : \Sigma \to \boldsymbol{R}^3$ be a front and p a singular point of the second kind. Suppose that κ_+ is bounded near p and $\kappa_+(p) \neq 0$. Then ψ as in* (6.1) *with $\boldsymbol{x}_0 = f(p) + \nu(p)/\kappa_+(p)$ and $t_0 = 1/\kappa_+(p)$ has a D_4 singularity at p if and only if p is not a ridge point of f.*

To prove this theorem, we take a special adapted coordinate system $(U; u, v)$ centered at p called a *strongly adapted coordinate system* which satisfies $\langle f_{uv}, f_v\rangle = 0$ at p (see [**32**, Definition 3.6]). Under this coordinate system, we see that $\widehat{F} = \widehat{G}_u = 0$ at p since $h(p) = f_{uv}(p)$. We prepare a lemma.

LEMMA 6.5. *Under the above conditions, $\Delta_\psi \neq 0$ if and only if*

$$4\psi_{uuv}\psi_{vvv} - 3(\psi_{uvv})^2 = \frac{4\widehat{G}}{\widehat{N}^2}(\widehat{L}(\widehat{G}\widehat{N}_v - \widehat{G}_v\widehat{N}) - \widehat{G}\widehat{M}(\widehat{N}_u + \widehat{M})) \neq 0 \tag{6.3}$$

at p.

PROOF. We take a strongly adapted coordinate system $(U; u, v)$ around p. Direct calculations show that

$$\psi_{uuu} = t_0\langle \nu, f_{uuu}\rangle, \quad \psi_{uuv} = t_0\langle \nu, f_{uuv}\rangle - \langle f_v, f_{uu}\rangle$$

hold at p, where $t_0 = 1/\kappa_+(p) = \widehat{G}(p)/\widehat{N}(p)$. Since $f_{uu} = -\varepsilon' f_v$, $f_{uuu} = -\varepsilon'' f_v - 2\varepsilon' h$ and $f_{uuv} = h_u - \varepsilon' f_{vv}$ at p, it follows that $\psi_{uuu} = 0$ and

$$\psi_{uuv} = t_0\langle \nu, h_u\rangle + \varepsilon'(-t_0\langle \nu, f_{vv}\rangle + |f_v|^2) = t_0\langle \nu, h_u\rangle = \widehat{G}\widehat{L}/\widehat{N} \neq 0 \tag{6.4}$$

hold at p. Thus Δ_ψ as in (6.2) can be written as

$$\Delta_\psi = (\psi_{uuv}(p))^2(4\psi_{uuv}(p)\psi_{vvv}(p) - 3(\psi_{uvv}(p))^2).$$

This implies that $\Delta_\psi \neq 0$ if and only if $4\psi_{uuv}(p)\psi_{vvv}(p) - 3(\psi_{uvv}(p))^2 \neq 0$.

We consider the form of $4\psi_{uuv}(p)\psi_{vvv}(p) - 3(\psi_{uvv}(p))^2 \neq 0$. By direct computations, we have

$$\psi_{uvv} = t_0\langle \nu, f_{uvv}\rangle, \quad \psi_{vvv} = t_0\langle \nu, f_{vvv}\rangle - 3\langle f_v, f_{vv}\rangle$$

at p. Since $f_{uvv} = 2h_v$ at p, it follows that

$$\psi_{uvv}(p) = 2t_0\widehat{M}(p) = \frac{2\widehat{G}(p)\widehat{M}(p)}{\widehat{N}(p)}. \tag{6.5}$$

We now deal with $\psi_{vvv}(p)$. It follows that $\langle \nu, f_v\rangle = 0$ and $\langle \nu, f_{vv}\rangle = -\langle \nu_v, f_v\rangle = \widehat{N}$ on U. So $\langle \nu, f_{vvv}\rangle = \widehat{N}_v - \langle \nu_v, f_{vv}\rangle$ holds. By Lemma 3.4, $\langle \nu_v, f_{vv}\rangle$ is written as

$$\langle \nu_v, f_{vv}\rangle = -\frac{\widehat{M}}{\widehat{E}}\langle h, f_{vv}\rangle - \frac{\widehat{N}}{\widehat{G}}\langle f_v, f_{vv}\rangle$$

at p. On the other hand, $\widehat{N}_u = \langle \nu_u, f_{vv}\rangle + \langle \nu, f_{uvv}\rangle = -\widehat{L}\langle h, f_{vv}\rangle/\widehat{E} + 2\widehat{M}$ at p by Lemma 3.4. Hence we have $\langle h, f_{vv}\rangle = -\widehat{E}(\widehat{N}_u - 2\widehat{M})/\widehat{L}$ and

$$\psi_{vvv} = \frac{\widehat{G}\widehat{N}_v - \widehat{G}_v\widehat{N}}{\widehat{N}} - \frac{\widehat{G}\widehat{M}(\widehat{N}_u - 2\widehat{M})}{\widehat{L}\widehat{N}} \tag{6.6}$$

at p, where we used $2\langle f_v, f_{vv}\rangle = \widehat{G}_v$. By (6.4), (6.5) and (6.6), $4\psi_{uuv}\psi_{vvv} - 3(\psi_{uvv})^2$ can be written as

$$4\psi_{uuv}\psi_{vvv} - 3(\psi_{uvv})^2 = \frac{4\widehat{G}}{\widehat{N}^2}(\widehat{L}(\widehat{G}\widehat{N}_v - \widehat{G}_v\widehat{N}) - \widehat{G}\widehat{M}(\widehat{N}_u + \widehat{M}))$$

at p. Thus we have the assertion. □

Proof of Theorem 6.4. Let us take a strongly adapted coordinate system $(U; u, v)$ centered at p. Then we note that $\widehat{F} = \widehat{G}_u = 0$ holds at p. The differentials $(\kappa_+)_u$ and $(\kappa_+)_v$ are given by

$$(\kappa_+)_u = \frac{\widehat{N}_u}{\widehat{G}}, \quad (\kappa_+)_v = \frac{-\widehat{G}\widehat{M}^2 + \widehat{L}(\widehat{G}\widehat{N}_v - \widehat{G}_v\widehat{N})}{\widehat{G}^2\widehat{L}}$$

at p. Since the principal vector $\boldsymbol{V}$ as in (3.13) is written as $\boldsymbol{V} = (-\widehat{M}, \widehat{L})$ at p, we have

(6.7)

$$\begin{aligned} \boldsymbol{V}\kappa_+(p) &= -\widehat{M}(p)(\kappa_+)_u(p) + \widehat{L}(p)(\kappa_+)_v(p) \\ &= \frac{1}{\widehat{G}(p)^2}(\widehat{L}(p)(\widehat{G}(p)\widehat{N}_v(p) - \widehat{G}_v(p)\widehat{N}(p)) - \widehat{G}(p)\widehat{M}(p)(\widehat{N}_u(p) + \widehat{M}(p))). \end{aligned}$$

Comparing (6.7) and (6.3) in Lemma 6.5, $\boldsymbol{V}\kappa_+(p) \neq 0$, namely, p is not a ridge point of f if and only if $4\psi_{uuv}(p)\psi_{vvv}(p) - 3(\psi_{uvv}(p))^2 \neq 0$. This implies that the number Δ_ψ defined as (6.2) does not vanish by Lemma 6.5. □

We remark that the condition that f is a front in Theorem 6.4 is needed for ψ to have a D_4 singularity at p. In fact, for a frontal $f : \Sigma \to \boldsymbol{R}^3$ with a singular point of the admissible second kind p, we have the following.

Proposition 6.6. *Let $f : \Sigma \to \boldsymbol{R}^3$ be a frontal but not a front and p a singular point of the admissible second kind. Then ψ with $\boldsymbol{x}_0 = f(p) + \nu(p)/\kappa_\nu(p)$ and $t_0 = 1/\kappa_\nu(p)$ does not have a D_4 singularity at p.*

Proof. Let us take an adapted coordinate system $(U; u, v)$ centered at p with the null vector field $\eta = \partial_u + \varepsilon(u)\partial_v$. Since f at p is a frontal but not a front, $\widehat{L}(p) = 0$. Thus $\psi_{uuv}(p) = 0$ by (6.4). By the proof of Lemma 6.5, Δ_ψ vanishes automatically if f is not a front at p. □

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