



Chiral fermions and mass hierarchy from extra dimensions with boundaries

Tatsumi, Kentaro

(Degree)

博士 (理学)

(Date of Degree)

2018-03-25

(Date of Publication)

2019-03-01

(Resource Type)

doctoral thesis

(Report Number)

甲第7119号

(URL)

<https://hdl.handle.net/20.500.14094/D1007119>

※ 当コンテンツは神戸大学の学術成果です。無断複製・不正使用等を禁じます。著作権法で認められている範囲内で、適切にご利用ください。



Doctoral Dissertation

Chiral fermions and mass hierarchy from extra dimensions with boundaries

(境界をもつ余剰次元によるカイラルフェルミオンと質量階層性の生成)

Kentaro Tatsumi

巽 健太郎

January 2018



Kobe University
Department of Physics, Graduate School of Science
1-1, Rokkodai, Nada, Kobe
Hyogo, 657-8501
Japan

Abstract

In this thesis, we reveal two classes of the hidden quantum mechanical supersymmetry in the higher dimensional free Dirac field, the $\mathcal{N} = 2$ and the $\mathcal{N}$ -extended supersymmetries.

The standard model is one of the most successful theories which describe the nature. On the other hand, there are still reminded mysteries. The extra-dimensional theory is one of the great candidates of the beyond the standard model. In order to reproduce the standard model, it should be revealed that what symmetry controls the spectral degeneracy of the theory and the profile of the mode function.

We show that there exist the two hidden quantum mechanical supersymmetries, the $\mathcal{N} = 2$ and the $\mathcal{N}$ -extended supersymmetries, in the KK decomposition of the higher dimensional Dirac field. The supersymmetries control the degeneracy and the profile. The first one, the $\mathcal{N} = 2$ supersymmetry ensures the existence of the chiral partners in the 4D spectrum. That hidden symmetry always exists because its existence is equivalent to the existence of the higher dimensional theory. On the other hand, there typically exist additional isospectral pairs in the 4D spectrum of the higher dimensional free Dirac field. The second one, the $\mathcal{N}$ -extended supersymmetry represents the additional isospectral pairs in the 4D spectrum. We stress that the two hidden symmetry control the degeneracy of the spectrum and the profile of the mode functions. However, in contrast to the $\mathcal{N} = 2$ supersymmetry, the existence of the $\mathcal{N}$ -extended supersymmetry deeply depends on the boundary condition of the theory. We will demonstrate the correspondence between the hidden supersymmetry and the isospectrality on the 4D spectrum in several explicit models after discussing the boundary condition consistent with the theory.

Acknowledgements

I would like to thank my supervisor professor M. Sakamoto for may advice. Without his suggestions and discussions, the thesis cannot be produced. I also thank I. Ueba, T. Fujimoto, K. Hasegawa, and K. Nishiwaki. Their valuable comments and discussions made great progress in this research. Without discussion with T. Houri and K. Tomoda, the research could not have been achieved. I would like to thank them also.

Publication List

- [1] T. Houri, M. Sakamoto and **Kentaro Tatsumi**, *Spectral intertwining relations in exactly solvable quantum mechanical systems*, [PTEP **2017** \(2017\) 063A01](#), arXiv: [1701.04307](#).
- [2] Y. Fujimoto, K. Hasegawa, K. Nishiwaki, M. Sakamoto and **Kentaro Tatsumi**, *Supersymmetry in 6d Dirac Action*, [PTEP **2017** \(2017\) 073B03](#), arXiv: [1609.04565](#).
- [3] Y. Fujimoto, K. Hasegawa, K. Nishiwaki, M. Sakamoto and **Kentaro Tatsumi**, *6d Dirac fermion on a rectangle; scrutinizing boundary conditions, mode functions and spectrum*, [NPB **922** \(2017\) 186](#), arXiv: [1609.01413](#).
- [4] Y. Fujimoto, K. Hasegawa, T. Nagasawa, K. Nishiwaki, M. Sakamoto and **Kentaro Tatsumi**, *Active Dirac Neutrinos via $SU(2)_L$ Doublets in 5d*, [JHEP **2016** \(2016\) 178](#), arXiv: [1601.05265](#).

Contents

1	Introduction	1
1.1	Extra-dimensional theory as the beyond standard model	1
1.2	Kaluza–Klein decomposition	2
1.3	Higher dimensional Higgs mechanism	4
1.4	Boundary condition and its aspects	6
1.5	Outline	7
2	Higher dimensional Dirac action	9
2.1	Brief review of 4D Dirac action	9
2.2	Higher dimensional gamma matrices	12
2.3	Higher dimensional Dirac action and the KK decomposition	16
2.4	The five dimensional case	19
2.5	Summary: Higher dimensional Dirac action	19
3	Supersymmetric Quantum Mechanics	21
3.1	Brief review of supersymmetric quantum mechanics	21
3.2	Higher dimensional $\mathcal{N} = 2$ SQM realization	26
3.3	Hidden $\mathcal{N}$ -extended SQM	28
3.4	Summary: Supersymmetric Quantum Mechanics	31
4	Boundary condition	33
4.1	Hermiticity and the non-local boundary condition	34
4.2	BC at each point on the boundary	36
4.3	BC for all points on the boundary	38
4.4	Summary: Boundary condition	39
5	Models	43
5.1	Hyperrectangular internal space	43
5.2	Toroidal internal space	47
5.3	Two-dimensional cylindrical internal space	51
5.4	One-dimensional interval internal space	56
5.5	Summary: Models	60
6	Summary	63

A	Higher dimensional Dirac action	67
A.1	Two component spinor	67
A.2	Higher dimensional gamma matrices	69
B	Supersymmetric quantum mechanics	77
B.1	$\mathcal{N} = 2$ SQM supercharges	77
C	Boundary condition	79
C.1	Derivation of the surface integral term	79
D	Models	83
D.1	One-dimensional interval internal space	83
E	Mathematical Supplement	85
E.1	Solution of the screened Poisson equation	85
E.2	Direct sum and tensor product	86

Chapter 1

Introduction

1.1 Extra-dimensional theory as the beyond standard model

By discovering the Higgs boson in 2012 [1, 2], the standard model (SM) that is a four-dimensional (4D) non-Abelian gauge theory established as a successful model describing the nature below the TeV scale. The high energy experiments have increased the plausibility of the SM. For example, the experimental value of the electron's g -factor matches to the theoretically predicted value with the extraordinarily high accuracy of $\sim 10^{-12}$ [3].

In spite of the great successes, there still remain mysteries in the SM and the particle physics. We focus on the three mysteries.

The first mystery is the generation problem. The SM is a non-Abelian gauge theory with three generations of quarks and leptons. That is, there are three copies of quarks and leptons in the theory which have exactly the same interaction with the gauge fields. In order to realize experimentally verified CP violation, there should be three generations at least [4–6]. However, the SM could not answer the question; why is not the high generation but the minimal generation realized? This is called the generation problem.

The second mystery is the fermion mass hierarchy problem. There is a huge hierarchy in the quark' masses even though they have exactly the same interaction with the gauge fields. Especially, the ratio of the masses between the top and up quark m_t/m_u is approximately 10^5 [3]. The fermions obtain the masses by the Higgs mechanism (we briefly review in [Section 1.3](#)). Therefore, such the huge mass hierarchy comes from the huge hierarchy of the Yukawa coupling constants which are dimensionless coupling constants. The ratio m_t/m_u is almost the same as the ratio of the masses between a mosquito and an elephant. It is unnatural that there exist such the huge hierarchy in the theory. In the SM, the Yukawa coupling constants are the free parameters of the theory. Hence, the reason is not explained in the SM. This is called the fermion mass hierarchy problem.

The last mystery is the chiral asymmetric problem. Namely, the left-handed and right-handed quarks and leptons interact with the $SU(2)_L$ gauge fields differently.¹ The

¹ The left-handed quarks and leptons belong to the fundamental representation of the $SU(2)_L$ gauge fields, but the right-handed ones belong to trivial representation. Therefore, the right-handed quarks and leptons do

reason is, also, not explained in the SM.

In addition to the problems, many problems remain in the standard model. A lot of models have been proposed so far to solve them.

The models [7, 8] with two hidden spatial directions with the solitonic background will generate chiral 4D theory effectively. If the magnetic flux pierces through the T^2 and S^2 which are the hidden direction, the three generations can appear [9–11]. If the background space-time equips the exponentially decaying metric (warped metric), the theory will naturally obtain huge hierarchy [12]. The orbifold space-time can generate the chiral 4D theory effectively also [13, 14].

We should stress that these models aiming to solve the problems are defined on the higher dimensional space-time. Such the theory is called the *extra-dimensional theory*. In the extra-dimensional theory, the extra-dimensions are compactified, and the standard model appears as the low energy effective theory. We could not verify the extra-dimension directly because it is compactified with a tiny scale.

The super string theory which is a great candidate for the “Theory of Everything” predicts the higher dimensional space-time [15, 16]. Therefore, such the extra-dimensional theory is interesting from the bottom-up approach for the “Theory of Everything”.

1.2 Kaluza–Klein decomposition

The extra-dimensional theory is defined on the higher dimensional space-time more than 4D. The field appearing in the higher dimensional theory depends on the extra-dimensions which we refer to the *internal space*. On the other hand, the SM is the 4D theory, and we recognize only the four dimensions (including time and space). Therefore, we need to remove the internal space dependency from the higher dimensional fields to obtain the theory depending on only the 4D space-time. We note that it is not enough restricting the field on the 4D hypersurface because field is profiled globally on the internal space in contrast to point particle.

In this section, we briefly review the Kaluza–Klein decomposition which is the method to obtain the 4D effective theory from a higher dimensional theory. We would like to point out that it is important to reveal the hidden symmetry in the Kaluza–Klein decomposition.

The Kaluza–Klein (KK) decomposition is a method of removing the dependency of the higher dimensional field on the internal space. Roughly speaking, the KK decomposition is the Fourier expansion of the higher dimensional theory. Here, we demonstrate the KK decomposition of the d -dimensional scalar field.²

Given a scalar field $\Phi(x, y)$ on a space-time $M^4 \times \Omega$ where M^4 is the 4D Minkowski space-time and the Ω is a d -dimensional internal space. We denote x^μ ($\mu = 0, 1, 2, 3$) as the coordinates of the 4D Minkowski space-time M^4 and $y = y_1, \dots, y_d$ as the coordinates of the internal space Ω . The scalar field satisfies the d -dimensional Klein–Gordon equation,

$$(-\partial_\mu^2 - \partial_y^2 + M^2)\Phi(x, y) = 0,$$

not interact to the $SU(2)_L$ gauge fields.

² We will strictly define the symbols in after chapters.

where M is mass of the scalar field $\Phi(x, y)$. Supposing that the internal space is compact, there exist harmonic functions $\mathbf{h}^{(n)}(y)$ ($n = 0, 1, \dots$) which satisfy,

$$(-\partial_y^2 + M^2)\mathbf{h}^{(n)}(y) = m_n^2\mathbf{h}^{(n)}(y)$$

where m_n^2 ($0 \leq m_0 \leq m_1 \leq \dots$ where $n = 0, 1, \dots$) are the eigenvalues of the equation. Supposing that the harmonic functions are complete, one can expand the higher dimensional scalar field $\Phi(x, y)$ as

$$\Phi(x, y) = \sum_n \phi^{(n)}(x)\mathbf{h}^{(n)}(y)$$

where $\phi^{(n)}(x)$ are the expansion coefficients depending only on the 4D coordinate x^μ . We obtain the 4D equation of the coefficients as

$$(-\partial_\mu^2 + m_n^2)\phi^{(n)}(x) = 0$$

because of the harmonic functions satisfying the eigenvalue equation. It implies that the expansion by the complete harmonic functions reduces the higher dimensional scalar field $\Phi(x, y)$ into the infinite 4D scalar fields $\phi^{(n)}(x)$ with mass m_n^2 .

If appropriate harmonic functions exist, we can remove the dependence on the internal space from the higher dimensional field in the same method. The method is called the KK decomposition. The harmonic functions are called the mode functions. And, we label the mode functions by n . We call the modes with nonzero mass ($m_n \neq 0$) as massive KK modes and the modes with zero mass ($m_n = 0$) as massless KK modes.

The eigenvalues m_n of the KK modes may be understood as the 4D masses. The mass' scale is approximately the same as the inverse of the scale of the internal space. We note that the scale of the internal space is taken to be much small. Therefore, the KK modes typically have huge masses except for the massless KK modes. Thus, the massive KK modes will be decoupled from the massless KK modes. Namely, after the KK decomposition, naively, the 4D theory is constructed by the massless KK modes effectively.

It implies that the degeneracy of the massless KK modes results in the generation in the SM. We emphasize that there exists massless KK mode(s) even if the higher dimensional field has nonzero mass (e.g. scalar field $\Phi(x, y)$ with mass M) because the massless KK mode(s) corresponds to the “bound state” of the eigenvalue equation. That is, the high dimensional theory has the possibility of solving the generation problem as the degeneracy of the spectrum.³

Surprisingly, they pointed out that there exists a hidden $\mathcal{N} = 2$ quantum mechanical supersymmetry in the KK decomposition of the higher dimensional gravity and gauge field in [17].⁴ It is also known that there exists a hidden $\mathcal{N} = 2$ quantum mechanical supersymmetry in the KK decomposition of the five- and six-dimensional Dirac field on the simple shaped

³ We note that the bound state corresponds to the “geometry” or “topological quantity” of the theory, e.g., the de Rham cohomology. Such the higher dimensional theory gives an interpretation of the generation problems as a “geometric” or “topological” problem.

⁴ The symbol $\mathcal{N}$ implies the number of the supercharges, e.g., $\mathcal{N} = 2$ supersymmetry contains two supercharges generating supersymmetric transformation.

internal space [18–20]. We call it a hidden symmetry because supersymmetry does not appear explicitly in the action. Nonetheless, it should be emphasized that supersymmetry controls 4D spectrum.

We should discuss symmetry to solve the generation problem because spectral degeneracy is generally controlled symmetry. For example, the symmetry group $SO(3)$ (isometry of S^2) characterizes degeneracy of the spherical harmonic functions. It is natural that the degeneracy of the 4D spectrum can be controlled by (hidden) symmetry.

The previous studies imply the importance of revealing the hidden symmetry, especially in the Dirac field. We note that the study on the Dirac field had been achieved in a heuristic way. And, the symmetry hidden in the general dimensional model is not known. For the above reasons, in this thesis, we focus on the hidden symmetry in the KK decomposition of the higher dimensional Dirac field.

1.3 Higher dimensional Higgs mechanism

In the extra-dimensional theory, the Higgs mechanisms will be modified a bit. In this section, we briefly review the higher dimensional Higgs mechanism.

Before discussing the higher dimensional Higgs mechanism, we review the ordinary Higgs mechanism.

In the SM, quarks and leptons are introduced as Weyl fermions without mass terms in the Lagrangian. That is, quarks and leptons are massless in principle. And, they acquire the masses through the Higgs mechanism with the Yukawa coupling terms. We briefly review the mechanism in a simple model.

Roughly speaking, the Higgs mechanism generates a mass term of the fermion with mass $m = gv$ where g is a Yukawa coupling constant and v is a vacuum expectation value (VEV) of a scalar field. Given a 4D Dirac field $\psi(x)$ and a 4D real scalar field $\phi(x)$. We consider the action with a Yukawa coupling term,

$$\int_{M^4} dx \bar{\psi}(x) i\gamma^\mu \partial_\mu \psi(x) + g\phi(x) \bar{\psi}(x) \psi(x),$$

where g is the Yukawa coupling constant. We emphasize that the Dirac field is massless (there is no mass term).

We suppose that the scalar field $\phi(x)$ has a nonzero VEV,

$$\langle \phi(x) \rangle = v,$$

and, we expand the scalar field around the VEV,

$$\phi(x) = v + \dots$$

Substituting the expression into the action, one finds that the Dirac field acquires the mass term,

$$\int_{M^4} dx \bar{\psi}(x) i\gamma^\mu \partial_\mu \psi(x) + gv \bar{\psi}(x) \psi(x) + \dots$$

We note that the scalar field called the Higgs field acquires the nonzero VEV through the spontaneous symmetry breaking in the SM.

In the higher dimensional theory, there are two mechanism acquiring masses. The first one is the KK decomposition. As we mentioned in the previous section, the n -th KK level acquires masses whose scale is approximately the inverse of the internal space' scale. The second one is the higher dimensional Higgs mechanism which is slightly different from the 4D Higgs mechanism. We explain it with an example model.

Given two higher dimensional Dirac fields $\Psi_k(x, y)$ ($k = 1, 2$) and a higher dimensional scalar field $\Phi(x, y)$ on the higher dimensional space-time $M^4 \times \Omega$. For simplicity, we consider the higher dimensional Yukawa coupling term only,

$$\int_{M_4} dx \int_{\Omega} dy g \Phi(x, y) \bar{\Psi}_1(x, y) \Psi_2(x, y).$$

We suppose that the higher dimensional fields are expanded as,

$$\begin{aligned} \Phi(x, y) &= \phi(x) \mathbf{h}(y) + \text{higher modes}, \\ \Psi_k(x, y) &= \psi_k(x) \mathbf{f}_k(y) + \text{higher modes}, \end{aligned}$$

where $\mathbf{f}_k(y)$ ($k = 1, 2$) and $\mathbf{h}(y)$ are the mode functions. The coefficients $\psi_k(x)$ ($k = 1, 2$) and $\phi(x)$ correspond to the 4D fields. Let us substitute these expressions into the higher dimensional Yukawa term,

$$\int_{M_4} dx \int_{\Omega} dy g \Phi(x, y) \bar{\Psi}_1(x, y) \Psi_2(x, y) = \int_{M_4} dx g_{4D} \phi(x) \bar{\psi}_1(x) \psi_2(x) + \text{higher modes}.$$

One finds that the higher dimensional Yukawa coupling term becomes into the mixing 4D Yukawa coupling term with a Yukawa coupling constant g_{4D} after integral out dependence on the internal space. We emphasize that the 4D Yukawa coupling constant g_{4D} is proportional to overlap integral (or convolution) of the mode functions,

$$g_{4D} = g \int_{\Omega} dy \mathbf{h}(y) \mathbf{f}_2^\dagger(y) \mathbf{f}_1(y).$$

Therefore, the 4D mass not only depends on the value of the VEV of the scalar field but also depends on the profile of the mode functions in the higher dimensional theory. We call this mechanism as a higher dimensional Higgs mechanism, that is, the mass acquiring mechanism through the Yukawa term with profile depending Yukawa coupling constant.

The 4D Yukawa coupling constant is almost the overlapped area of the mode functions. If the overlap between the functions increases (decreases), the 4D mass increases (decreases) (see [Figure 1.1](#)). Especially, the localized profile can generate the huge hierarchy on the 4D Yukawa coupling. It implies that the higher dimensional theory has a potential to solve the fermion mass hierarchy problem because the theory can generate the mass hierarchy from the theory that there is no hierarchy in the Yukawa coupling constant(s).

We note that the symmetry also controls the profiles. For example, the symmetry group $SO(3)$ (isometry of S^2) completely determines the profiles of the spherical harmonic functions. It implies the importance of revealing the (hidden) symmetry for solving the fermion mass hierarchy problem.

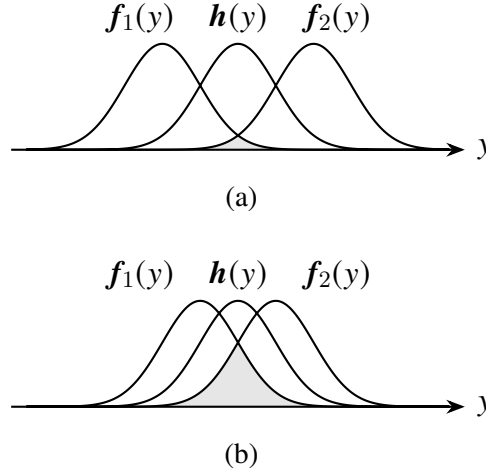


Figure 1.1: The schismatic illustration of the overlapping integral of localized mode functions. The overlapped area of the mode functions $f_k(x)$ ($k = 1, 2$) and $h(y)$ is colored by gray. The area is approximately equal to the 4D Yukawa coupling constant. If the overlap between the functions increases (decreases), the mass of the four dimensions will increases (decreases).

1.4 Boundary condition and its aspects

In general, the nontrivial boundary condition can generate the localized profile. The higher dimensional theories with boundary have been studied in the context of the orbifold models. There exist equivalent models which have internal spaces with boundaries to the orbifold models. For example, it is well-known that the S/Z_2 orbifold model is equivalent to the model which has the interval internal space. Moreover, the boundary condition generates the 4D chiral asymmetry in the massless KK modes of the higher dimensional Dirac field [18]. We show examples in Chapter 5.

Following the previous examples, we note the simple example which provides the localized modes. Given scalar function $h(y)$ defined on the interval $[0, L]$. We assume that the scalar function is a solution of the equation

$$(-\partial_y^2 + M^2)h(y) = 0$$

with the boundary condition,

$$\begin{aligned} h(y) &= 0 & \text{at } y = 0, \\ (a + b\partial_y)h(y) &= 0 & \text{at } y = L. \end{aligned}$$

We should stress that the equation and the boundary condition will be found in later chapters. Then, the scalar function is the hyperbolic sine function,

$$h(y) \propto \sinh(My),$$

if the following condition holds,

$$a \sinh(ML) + bM \cosh(ML) = 0.$$

The function $h(y)$ is obviously localized.

From the viewpoint of studying hidden symmetry, it is also important to clarify possible classes of boundary conditions. Since the spectrum of the theory deeply depends on the boundary condition, it is natural that the hidden symmetry deeply depends on the boundary condition. Therefore, boundary conditions consistent with theory should be revealed with hidden symmetry.

1.5 Outline

As we mentioned in the above, it is important to reveal the hidden symmetry in the KK decomposition of the higher dimensional field to solve the generation problem, the fermion mass hierarchy problem, and the chiral asymmetry problem. In this thesis, we focus on revealing the hidden symmetry of the higher dimensional free Dirac field which has very high degrees of freedom from the viewpoint of the 4D theory.

This thesis organized as follow; in the next chapter, we clarify the properties that the mode function of the higher dimensional Dirac field should satisfy. We note that, due to the high degrees of freedom of the higher dimensional Dirac field, the KK decomposition does not always provide the 4D mass eigenstates in general. By summarizing only the most important properties that the mode function should satisfy, we facilitate the discussion of the hidden symmetry. In prior, we briefly review the 4D Dirac field and the 4D Weyl representation. The 4D Weyl representation clarifies the discussion and the hidden symmetry. In our analysis, the 4D Weyl representation has an important role. In [Chapter 3](#), we reveal the hidden symmetry is the two classes of the quantum mechanical supersymmetry, the $\mathcal{N} = 2$ and the $\mathcal{N}$ -extended quantum mechanical supersymmetry. The $\mathcal{N} = 2$ supersymmetric quantum mechanics (SQM) is introduced by Witten [21]. One of the most significant properties of the supersymmetric quantum mechanics is isospectrality of two quantum mechanical systems. The $\mathcal{N} = 2$ SQM supercharges ensures the existence of the 4D chiral partners of the 4D theory. There, typically, exist additional isospectral pairs in the 4D spectrum. We reveal the $\mathcal{N}$ -extended SQM hidden in the KK decomposition which generates the additional isospectral pairs. In general, it is not trivial that the supersymmetry is well-defined even if the supercharges satisfy the supersymmetric algebra. We should pay attention whether the supercharges are well-defined. In [Chapter 4](#), we discuss the boundary condition consistent with the theory in prior introducing the explicit models. We show that the $\mathcal{N} = 2$ SQM is always well-defined with any boundary condition which is derived from the variation principle of the action. And, we partially classify the locally given boundary condition which is compatible with the $\mathcal{N} = 2$ SQM. We then show there exists the huge parameter space of the boundary condition at each point on the boundary. In [Chapter 5](#), we consider various models to show how the hidden symmetry generates the isospectrality and the complicated dependency on the boundary condition of the supercharges. We consider four models which have internal spaces such as hyperrectangle, torus, cylinder, and interval. We devote [Chapter 6](#) as a summary.

Chapter 2

Higher dimensional Dirac action

Roughly speaking, the Kaluza–Klein (KK) decomposition is a variable separation of the higher dimensional field similar to Fourier series expansion. If dependency on the internal space is completely separated from the 4D space-time, the 4D theory is obtained by integrating out the internal space from the higher dimensional theory. We note that, in order to construct the 4D theory with the mass eigenstates, one should take appropriate mode functions.

It is well-known that the higher dimensional Dirac field has higher degrees of freedom more than 4D one. Because of that, in fact, the naive KK decomposition does not provide the 4D mass eigenstates in the higher dimensional Dirac field. A question arises *what mode functions should be taken to provide the 4D action with mass eigenstates?* Summarizing the properties also advances the discussion of the hidden symmetry. In this chapter, we clarify the properties which the mode functions should satisfy.

This chapter is organized as follows: first, we briefly review the 4D Dirac field and the 4D Weyl representation in [Section 2.1](#). The 4D Weyl representation helps to reveal the properties. Second, we introduce the higher dimensional gamma matrices and analyze the properties from the viewpoint of 4D theory in [Section 2.2](#). Finally, we introduce the higher dimensional free Dirac action and reveal the properties which the mode functions should satisfy in [Section 2.3](#).

2.1 Brief review of 4D Dirac action

In this section, we briefly review the free 4D Dirac action and introduce the Weyl representation of the 4D gamma matrices [\[22–24\]](#).

The 4D Lorentz group $SO(1, 3)$ is a part of the isometry group of the 4D Minkowski space-time M^4 . The Dirac field $\psi(x)$ is a spinor of the 4D Lorentz group. The action of the free 4D Dirac field $\psi(x)$ with mass m on the 4D Minkowski space-time is given by

$$\int_{M_4} d^4x \bar{\psi}(x)(i\gamma^\mu \partial_\mu - m)\psi(x). \quad (2.1)$$

We take the 4D Minkowski metric as

$$\eta_{\mu\nu} = \eta^{\mu\nu} = \text{diag}(-1, +1, +1, +1). \quad (2.2)$$

The 4D Dirac conjugate is given by $\bar{\psi}(x) = \psi^\dagger(x)\gamma^0$. The symbols x^μ ($\mu = 0, 1, 2, 3$) denote the coordinates of the 4D Minkowski space-time. The 4D Dirac field has four-component and the 4D gamma matrices γ^μ ($\mu = 0, 1, 2, 3$) are 4-by-4 matrices and satisfy the Clifford algebra and ordinary Hermiticity,

$$\{\gamma^\mu, \gamma^\nu\} = -2\eta^{\mu\nu}\mathbf{1}_4, \quad \gamma^0\gamma^\mu\gamma^0 = (\gamma^\mu)^\dagger. \quad (2.3)$$

We denote the commutation $[A, B]$ and anti-commutation $\{A, B\}$ by,

$$[A, B] = AB - BA, \quad \{A, B\} = AB + BA. \quad (2.4)$$

We denote $\mathbf{1}_n$ by the n -by- n identity matrix. We omit the subscript n when no confusions arise.

The spinor representation of the 4D Lorentz group which acts on the 4D Dirac field is generated by the quadratic of gamma matrices,

$$\gamma^{\mu\nu} = \frac{i}{4}[\gamma^\mu, \gamma^\nu], \quad (2.5)$$

for $\mu, \nu = 0, 1, 2, 3$. The generators $\gamma^{\mu\nu}$, off cause, satisfy the 4D Lorentz algebra $\mathfrak{so}(1, 3)$,

$$[\gamma^{\mu\nu}, \gamma^{\rho\sigma}] = -i(\eta^{\mu\rho}\gamma^{\nu\sigma} - \eta^{\nu\rho}\gamma^{\mu\sigma} - \eta^{\mu\sigma}\gamma^{\nu\rho} + \eta^{\nu\sigma}\gamma^{\mu\rho}), \quad (2.6)$$

for $\mu, \nu, \rho, \sigma = 0, 1, 2, 3$.

This representation of the 4D Lorentz group $SO(1, 3)$ which is given by the gamma matrices is reducible. We can find the matrix distinguishing two different representation of the 4D Lorentz group which is called the chiral matrix. The 4D chiral matrix γ^5 is defined as

$$\gamma^5 = i\gamma^0\gamma^1\gamma^2\gamma^3. \quad (2.7)$$

The chiral matrix has the properties,

$$(\gamma^5)^\dagger = \gamma^5, \quad (\gamma^5)^2 = \mathbf{1}, \quad \{\gamma^\mu, \gamma^5\} = 0, \quad [\gamma^{\mu\nu}, \gamma^5] = 0, \quad (2.8)$$

for $\mu, \nu = 0, 1, 2, 3$. The second property implies the eigenvalues (called chirality) of γ^5 are ± 1 . The last property implies that the two eigenstates of the γ^5 independently response under the 4D Lorentz group. Hence, we can decompose the 4D Dirac field into two fields which have different eigenvalues of γ^5 by the projection operators $(1 \pm \gamma^5)/2$. The eigenstate with eigenvalues $+1$ (-1) is the 4D chiral right-handed (left-handed) field. It is known that the chiral fields belong the different irreducible representation of the 4D Lorentz group. That is, the 4D Dirac field is the doublet of two individual chiral fields, the 4D left-handed and right-handed field.

We emphasize that the mass term in the action (2.1) is made by coupling of the left-handed and right-handed field. Such the term is called as the Dirac mass term. We can find it by decomposing the action (2.1) by the chirality,

$$\int_{M_4} d^4x \bar{\psi}_L(x) i\gamma^\mu \partial_\mu \psi_L(x) + \bar{\psi}_R(x) i\gamma^\mu \partial_\mu \psi_R(x) - m(\bar{\psi}_R(x)\psi_L(x) + \bar{\psi}_L(x)\psi_R(x)), \quad (2.9)$$

where

$$\psi_L(x) = \frac{1 - \gamma^5}{2} \psi(x), \quad \psi_R(x) = \frac{1 + \gamma^5}{2} \psi(x). \quad (2.10)$$

Such the representation of the Lorentz group is convenient to discuss the higher dimensional theory because the representation and the Dirac fields exist in any dimension.¹ However, from the viewpoint of the 4D theory, the Dirac fields obscure the degrees of freedom of the theory.² The degrees of freedom characterizes the physical models. Therefore, we reformulate the 4D Dirac field by the spinor of an irreducible representation of the 4D Lorentz group to clarify the degrees of freedom.

Such the representation of the gamma matrices is called the Weyl representation. The 4D gamma matrices in the 4D Weyl representation are given as

$$\gamma^0 = \sigma^1 \otimes \mathbf{1}_2, \quad \gamma^k = i\sigma^2 \otimes \sigma^k, \quad (2.11)$$

for $k = 1, 2, 3$. The symbol σ^k ($k = 1, 2, 3$) are the Pauli matrices. The generators of the 4D Lorentz group and the chiral matrix are given by, in the 4D Weyl representation,

$$\gamma^{0j} = -\frac{i}{2} \sigma^3 \otimes \sigma^k, \quad \gamma^{ij} = \frac{1}{2} \sum_k \varepsilon^{ijk} \mathbf{1}_2 \otimes \sigma^k, \quad \gamma^5 = -\sigma^3 \otimes \mathbf{1}_2, \quad (2.12)$$

for $\mu, \nu = 0, 1, 2, 3$. The symbol ε^{ijk} is the Levi-Civita symbol normalized as $\varepsilon^{123} = 1$. Thus, we decompose the 4D Dirac field into two 4D chiral fields,

$$\psi(x) = \mathbf{e}_L \otimes \bar{\phi}(x) + \mathbf{e}_R \otimes \chi(x), \quad (2.13)$$

where $\mathbf{e}_L = (1, 0)^T$, $\mathbf{e}_R = (0, 1)^T$ and $\bar{\phi}(x)$ ($\chi(x)$) is a two-component 4D left-handed (right-handed) field.³ Substituting the expansion into the 4D action (2.1), we find the expression of the action by the two-component fields by,

$$\int_{M_4} d^4x \phi(x) i \bar{\sigma}^\mu \partial_\mu \bar{\phi}(x) + \bar{\chi}(x) i \sigma^\mu \partial_\mu \chi(x) - m(\bar{\phi}(x) \bar{\chi}(x) + \chi(x) \phi(x)), \quad (2.14)$$

where $\sigma^\mu = (\mathbf{1}_2, \sigma^k)$ and $\bar{\sigma}^\mu = (\mathbf{1}_2, -\sigma^k)$.

We note that, by the nesting rule,⁴ the gamma matrices and the 4D chiral matrix in the 4D Weyl representation are given by the 4-by-4 matrix form,⁵

$$\gamma^\mu = \begin{pmatrix} & \sigma^\mu \\ \bar{\sigma}^\mu & \end{pmatrix}, \quad \gamma^5 = \begin{pmatrix} -\mathbf{1}_2 & \\ & \mathbf{1}_2 \end{pmatrix}. \quad (2.15)$$

¹ We prove the existence in the [Appendix A](#).

² The higher dimensional Dirac field has the higher degrees of freedom more than 4D one. We discuss in the next section.

³ These are called the Weyl spinor. We note the convention in [Appendix A](#).

⁴ We take the nesting rule as $A \otimes B = \begin{pmatrix} a_{11}B & \cdots \\ \vdots & \ddots \end{pmatrix}$ where A and B are matrices and a_{11} is the $(1, 1)$ component of A . We note the definition in [Appendix E](#).

⁵ For simplicity, we omit writing 0 in the empty components.

The generators are block diagonalized and the component are explicit anti-symmetric,

$$\gamma^{\mu\nu} = \frac{i}{4} \begin{pmatrix} \sigma^\mu \bar{\sigma}^\nu - \sigma^\nu \bar{\sigma}^\mu & \\ & \bar{\sigma}^\mu \sigma^\nu - \bar{\sigma}^\nu \sigma^\mu \end{pmatrix}. \quad (2.16)$$

Thus, in this notation, the upper-half (lower-half) component of the 4D Dirac field is the left-handed (right-handed) field,

$$\psi(x) = \begin{pmatrix} \bar{\phi}(x) \\ \chi(x) \end{pmatrix}. \quad (2.17)$$

2.2 Higher dimensional gamma matrices

The Dirac field is characterized by the properties of the gamma matrices. In this section, we introduce the higher dimensional gamma matrices and discuss about its property [16, 25].

2.2.1 The definition and basic properties

We can define the higher dimensional gamma matrices and discuss about the properties in the same way as the 4D one.

We define the $2^{\lfloor (4+d)/2 \rfloor}$ -by- $2^{\lfloor (4+d)/2 \rfloor}$ $(4+d)$ -dimensional gamma matrices Γ^N satisfying the Clifford algebra and ordinary Hermiticity,

$$\{\Gamma^N, \Gamma^M\} = -2\eta^{NM}\mathbf{1}, \quad \Gamma^0 \Gamma^N \Gamma^0 = (\Gamma^N)^\dagger, \quad (2.18)$$

where $N = \mu, y$ for $\mu = 0, 1, 2, 3$ and $y = y_1, \dots, y_d$. The $\lfloor (4+d)/2 \rfloor$ is the largest integer less than or equal to $(4+d)/2$. We take the $4+d$ dimensional metric as,⁶

$$\eta_{NM} = \eta^{NM} = \text{diag}(-1, +1, \dots, +1). \quad (2.19)$$

The quadratics of the higher dimensional gamma matrices,

$$\Gamma^{NM} = \frac{i}{4} [\Gamma^N, \Gamma^M], \quad (2.20)$$

for $N, M = \mu, y$, generate the higher dimensional Lorentz group $SO(1, 3+d)$ because the quadratics Γ^{NM} satisfy the higher dimensional Lorentz algebra $\mathfrak{so}(1, 3+d)$,

$$[\Gamma^{NM}, \Gamma^{N'M'}] = -i(\eta^{NN'} \Gamma^{MM'} - \eta^{MN'} \Gamma^{NM'} - \eta^{NM'} \Gamma^{MM'} + \eta^{MM'} \Gamma^{NN'}). \quad (2.21)$$

Such the representation of the higher dimensional Lorentz group acts on the higher dimensional Dirac field.

In even dimensions, such the representation of the higher dimensional Lorentz group is reducible. For even dimensions, in the same manner as the 4D one, the higher dimensional chiral matrix Γ is defined by,

$$\Gamma = i^{d/2-1} \Gamma^0 \dots \Gamma^3 \Gamma^{y_1} \dots \Gamma^{y_d}. \quad (2.22)$$

⁶ There exist such matrices for any dimensional space-time. We show it by a formal proof and explicitly construct by the Pauli matrices in [Appendix A](#).

The chiral matrix Γ has the properties,

$$(\Gamma)^\dagger = \Gamma, \quad (\Gamma)^2 = \mathbf{1}, \quad \{\Gamma^N, \Gamma\} = 0, \quad [\Gamma^{NM}, \Gamma] = 0. \quad (2.23)$$

The higher dimensional Dirac field is decomposed into two higher dimensional chiral fields. The two chiral fields are the eigenstates of the higher dimensional chiral matrix Γ with the eigenvalue ± 1 and belong to the different irreducible representations.

For odd dimensions, any nontrivial chiral matrix such as (2.38) does not exist because the product $\Gamma^0 \dots \Gamma^3 \Gamma^{y_1} \dots \Gamma^{y_d}$ is promotional to the identity matrix. This reflects the fact that the chiral field does not exist and such the representation is irreducible [16, 25–27].

2.2.2 Generators of the $SO(1, 3) \times SO(d)$ subgroup

To provide the 4D mass eigenstates, we separate the dependence of the higher dimensional Dirac field on the 4D Minkowski and the internal space. Thus, we focus on the $SO(1, 3) \times SO(d)$ subgroup of the higher dimensional Lorentz group $SO(1, 3 + d)$.

The $SO(1, 3) \times SO(d)$ subgroup of the higher dimensional Lorentz group is generated by the following generators,

$$\Gamma^{\mu\nu} = \frac{i}{4}[\Gamma^\mu, \Gamma^\nu], \quad \Gamma^{yy'} = \frac{i}{4}[\Gamma^y, \Gamma^{y'}], \quad (2.24)$$

where $\mu, \nu = 0, 1, 2, 3$ and $y, y' = y_1, \dots, y_d$. These generators $\Gamma^{\mu\nu}$ and $\Gamma^{yy'}$ satisfy the 4D Lorentz algebra $\mathfrak{so}(1, 3)$ and $\mathfrak{so}(d)$ algebra respectively,

$$[\Gamma^{\mu\nu}, \Gamma^{\rho\sigma}] = -i(\eta^{\mu\rho}\Gamma^{\nu\sigma} - \eta^{\nu\rho}\Gamma^{\mu\sigma} - \eta^{\mu\sigma}\Gamma^{\nu\rho} + \eta^{\nu\sigma}\Gamma^{\mu\rho}), \quad (2.25)$$

$$[\Gamma^{yy'}, \Gamma^{y''y'''}] = -i(\eta^{yy''}\Gamma^{y'y'''} - \eta^{y'y''}\Gamma^{yy'''} - \eta^{yy'''}\Gamma^{y'y''} + \eta^{y'y'''}\Gamma^{yy''}), \quad (2.26)$$

where the 4D Minkowski metric $\eta^{\mu\nu}$ and the d dimensional internal space metric $\eta^{yy'}$ are given by,⁷

$$\eta^{\mu\nu} = \eta_{\mu\nu} = \text{diag}(-1, +1, +1, +1), \quad (2.27)$$

$$\eta^{yy'} = \eta_{yy'} = \text{diag}(+1, \dots, +1). \quad (2.28)$$

The whole metric η_{NM} of the space-time $M^4 \times \Omega$ can be written as

$$\eta_{NM} = \begin{pmatrix} \eta_{\mu\nu} & \\ & \eta_{yy'} \end{pmatrix}. \quad (2.29)$$

According to the previous discussion, we define the 4D chiral matrix Γ_{4D} and the internal space chiral matrix Γ_{in} (for even dimensions) as,

$$\Gamma_{4D} = i\Gamma^0 \dots \Gamma^3, \quad \Gamma_{in} = i^{d/2-2}\Gamma^{y_1} \dots \Gamma^{y_d}, \quad (2.30)$$

⁷ The 4D Minkowski metric is the same as the metric in the [Section 2.1](#).

which satisfy

$$(\Gamma_{4D})^\dagger = \Gamma_{4D}, \quad (\Gamma_{4D})^2 = \mathbf{1}, \quad (2.31)$$

$$(\Gamma_{in})^\dagger = \Gamma_{in}, \quad (\Gamma_{in})^2 = \mathbf{1}. \quad (2.32)$$

Thus, the eigenvalues of Γ_{4D} and Γ_{in} are ± 1 . We note that the product of these chiral matrices are the higher dimensional gamma matrix Γ ,

$$\Gamma = \Gamma_{4D}\Gamma_{in} = \Gamma_{in}\Gamma_{4D}. \quad (2.33)$$

In any dimensions, the 4D chiral matrix Γ_{4D} commutes with the generators of the $SO(1, 3) \times SO(d)$ subgroup,

$$[\Gamma_{4D}, \Gamma^{\mu\nu}] = 0, \quad [\Gamma_{4D}, \Gamma^{yy'}] = 0, \quad (2.34)$$

and

$$\{\Gamma_{4D}, \Gamma^\mu\} = 0, \quad [\Gamma_{4D}, \Gamma^y] = 0, \quad (2.35)$$

for $\mu, \nu = 0, 1, 2, 3$ and $y, y' = y_1, \dots, y_d$. Therefore, there are the two individual fields in the higher dimensional Dirac field characterized by the eigenvalues of Γ_{4D} .

In even dimensions, the internal space chiral matrix Γ_{in} also commutes with the generators of the subgroup $SO(1, 3) \times SO(d)$,

$$[\Gamma_{in}, \Gamma^{\mu\nu}] = 0, \quad [\Gamma_{in}, \Gamma^{yy'}] = 0, \quad (2.36)$$

and

$$[\Gamma_{in}, \Gamma^\mu] = 0, \quad \{\Gamma_{in}, \Gamma^y\} = 0, \quad (2.37)$$

for $\mu, \nu = 0, 1, 2, 3$ and $y, y' = y_1, \dots, y_d$. Hence, in even dimensions, we can decompose the higher dimensional Dirac field which are characterized by the eigenvalues of the 4D chiral matrix Γ_{4D} and the internal space chiral matrix Γ_{in} . These fields are differently transformed by the action of the subgroup $SO(1, 3 + d)$.

In odd dimensions, there is no internal space chiral matrix because higher dimensional chiral fields do not exist. Thus, we can only decompose the higher dimensional Dirac field respecting the subgroup into the two 4D chiral Dirac fields.

2.2.3 The degeneracy of the 4D Dirac fields

From the viewpoint of 4D theory, the higher dimensional Dirac field has the large degrees of freedom. One can find that such the degrees of freedom appear in the KK decomposition. In this section, we discuss one of the origins of the large degrees of freedom by taking the representation of the higher dimensional gamma matrices respecting the $SO(1, 3) \times SO(d)$ subgroup.

We take the representation of the higher dimensional gamma matrices respecting the $SO(1, 3) \times SO(d)$ subgroup as,

$$\Gamma^\mu = \mathbf{1}_{2^{[(d-4)/2]}} \otimes \gamma^\mu, \quad \Gamma^y = \gamma^y \otimes \gamma^5. \quad (2.38)$$

We note the definition of the 4D gamma matrices again.⁸ The 4-by-4 4D gamma matrices γ^μ is given as,

$$\{\gamma^\mu, \gamma^\nu\} = -2\eta^{\mu\nu} \mathbf{1}_4, \quad \gamma^0 \gamma^\mu \gamma^0 = (\gamma^\mu)^\dagger, \quad (2.39)$$

for all $\mu = 0, 1, 2, 3$. The chiral matrix γ^5 is given as,

$$\gamma^5 = i\gamma^0 \gamma^1 \gamma^2 \gamma^3, \quad \{\gamma^\mu, \gamma^5\} = 0, \quad (2.40)$$

The $2^{\lfloor d/2 \rfloor}$ -by- $2^{\lfloor d/2 \rfloor}$ internal space gamma matrices γ^y satisfy the Clifford algebra and are anti-Hermitian,

$$\{\gamma^y, \gamma^{y'}\} = -2\eta^{yy'} \mathbf{1}_{2^{\lfloor d/2 \rfloor}}, \quad (\gamma^y)^\dagger = -\gamma^y. \quad (2.41)$$

The internal space chiral matrix γ_{in} is defined by,

$$\gamma_{\text{in}} = i^{(4+d)/2-2} \gamma^{y_1} \dots \gamma^{y_d}. \quad (2.42)$$

The internal space chiral matrix γ_{in} has the properties,

$$(\gamma_{\text{in}})^\dagger = \gamma_{\text{in}}, \quad (\gamma_{\text{in}})^2 = \mathbf{1}_{2^{\lfloor d/2 \rfloor}}, \quad \{\gamma^y, \gamma_{\text{in}}\} = 0, \quad [\gamma^{yy'}, \gamma_{\text{in}}] = 0. \quad (2.43)$$

In odd dimensions, there is no such matrix which has the properties.

The generators of the subgroup $\Gamma^{\mu\nu}$ and $\Gamma^{yy'}$ can be written by the generators $\gamma^{\mu\nu}$ and $\gamma^{yy'}$ as,

$$\Gamma^{\mu\nu} = \mathbf{1}_{2^{\lfloor (d-4)/2 \rfloor}} \otimes \gamma^{\mu\nu}, \quad \gamma^{\mu\nu} = \frac{i}{4} [\gamma^\mu, \gamma^\nu], \quad (2.44)$$

$$\Gamma^{yy'} = \gamma^{yy'} \otimes \mathbf{1}_4, \quad \gamma^{yy'} = \frac{i}{4} [\gamma^y, \gamma^{y'}]. \quad (2.45)$$

The generators $\gamma^{\mu\nu}$ and $\gamma^{yy'}$ satisfy the 4D Lorentz algebra $\mathfrak{so}(1, 3)$ and the $\mathfrak{so}(d)$ algebra respectively,

$$[\gamma^{\mu\nu}, \gamma^{\rho\sigma}] = -i(\eta^{\mu\rho} \gamma^{\nu\sigma} - \eta^{\nu\rho} \gamma^{\mu\sigma} - \eta^{\mu\sigma} \gamma^{\nu\rho} + \eta^{\nu\sigma} \gamma^{\mu\rho}), \quad (2.46)$$

$$[\gamma^{yy'}, \gamma^{y''y'''}] = -i(\eta^{yy''} \gamma^{y'y'''} - \eta^{y'y''} \gamma^{yy'''} - \eta^{yy'''} \gamma^{y'y''} + \eta^{y'y'''} \gamma^{yy''}). \quad (2.47)$$

For even dimensions, the three types of the chiral matrices which are given by the tensor product of the 4D chiral matrix γ^5 and the internal space chiral matrix γ_{in} ,

$$\Gamma = \gamma^{\text{in}} \otimes \gamma^5, \quad \Gamma_{4\text{D}} = \mathbf{1}_{2^{\lfloor (d-4)/2 \rfloor}} \otimes \gamma^5, \quad \Gamma_{\text{in}} = \gamma_{\text{in}} \otimes \mathbf{1}_4. \quad (2.48)$$

We should emphasize that the 4D Lorentz generators of the $SO(1, 3) \times SO(d)$ subgroup can be identified with the $2^{\lfloor d/2 \rfloor}$ direct sum of the generators $\gamma^{\mu\nu}$,

$$\Gamma^{\mu\nu} = \mathbf{1}_{2^{\lfloor d/2 \rfloor}} \otimes \gamma^{\mu\nu} = \underbrace{\gamma^{\mu\nu} \oplus \dots \oplus \gamma^{\mu\nu}}_{2^{\lfloor d/2 \rfloor} \text{ times}} = \begin{pmatrix} \gamma^{\mu\nu} & & \\ & \ddots & \\ & & \gamma^{\mu\nu} \end{pmatrix}. \quad (2.49)$$

Eq. (2.49) implies that the higher dimensional Dirac field contains multiple 4D Dirac fields. This is one of the most significant properties of the higher dimensional Dirac field. The origin of such the higher degrees of freedom is the algebraic properties of the spinor representation (see (2.44)).⁹

⁸ We introduce 4D gamma matrices γ^μ in Section 2.1.

⁹ We discuss the gauge (vector) field in Section A.2 to compare with the Dirac field.

2.3 Higher dimensional Dirac action and the KK decomposition

As we showed in the previous section, from the viewpoint of the 4D theory, the higher dimensional Dirac field has higher degrees of freedom more than the 4D one. Because of that, if one naively decompose the higher dimensional Dirac field, the KK modes are not 4D mass eigenstates in general. The KK modes will have mixing mass term. In this section, we introduce the higher dimensional Dirac action and revile that the properties of the mode function to provide mass eigenstates.

2.3.1 The higher dimensional Dirac action

Let us consider the action of a free Dirac field on the Cartesian product of the four dimensional (4D) Minkowski space-time M^4 and the flat d dimensional internal space Ω given as

$$S[\Psi, \bar{\Psi}] = \int_{M^4} d^4x \int_{\Omega} d^d y \bar{\Psi}(x, y) (i\Gamma^\mu \partial_\mu + i\Gamma^y \partial_y - M) \Psi(x, y). \quad (2.50)$$

The symbol $\Psi(x, y)$ is the d dimensional Dirac field with mass M which has the $2^{\lfloor (4+d)/2 \rfloor}$ components. The Dirac conjugate is defined by $\bar{\Psi}(x, y) = \Psi^\dagger(x, y) \Gamma^0$. The symbol x^μ ($\mu = 0, 1, 2, 3$) are the coordinates of the Minkowski space-time M^4 and $y = y_1, \dots, y_d$ are the coordinates of the internal space Ω . We take the metric as

$$\eta_{NM} = \eta^{NM} = \text{diag}(-1, +1, \dots, +1) \quad (2.51)$$

where $N, M = \mu, y$. The differential operator $\Gamma^y \partial_y$ is defined as $\Gamma^y \partial_y = \sum_{k=1}^d \partial_{y_k} \Gamma^{y_k}$. We note the definition of the higher dimensional gamma matrices Γ^N ($N = \mu, y$) again; the $2^{\lfloor (4+d)/2 \rfloor}$ -by- $2^{\lfloor (4+d)/2 \rfloor}$ $(4+d)$ -dimensional gamma matrices Γ^N satisfy the Clifford algebra and ordinary Hermitically,

$$\{\Gamma^N, \Gamma^M\} = -2\eta^{NM} \mathbf{1}, \quad \Gamma^0 \Gamma^N \Gamma^0 = (\Gamma^N)^\dagger, \quad (2.52)$$

where $N = \mu, y$ for $\mu = 0, 1, 2, 3$ and $y = y_1, \dots, y_d$.¹⁰

2.3.2 Representation of gamma matrices

The KK decomposition is a method of separating dependency on the internal space of the higher dimensional field. Before decomposing the higher dimensional Dirac field, we introduce a representation of the higher dimensional gamma matrices which is appropriate for the KK decomposition. The representation separates the dependency on the 4d space-time and the internal space well and clarifies mass eigenstates.

¹⁰ We introduce the higher dimensional gamma matrices Γ^N ($N = \mu, y$) in [Section 2.2](#).

We take the representation of the higher dimensional gamma matrices based on 4D Weyl representation as,

$$\Gamma^0 = \sigma^1 \otimes \mathbf{1}_{2[d/2]} \otimes \mathbf{1}_2, \quad \Gamma^k = i\sigma^2 \otimes \mathbf{1}_{2[d/2]} \otimes \sigma^k, \quad \Gamma^y = -\sigma^3 \otimes \gamma^y \otimes \mathbf{1}_2, \quad (2.53)$$

where $k = 1, 2, 3$. The internal space gamma matrices γ^y ($y = y_1, \dots, y_d$) are defined in Eq. (2.38). In this representation, the dependency on the 4D Minkowski space-time and the internal space of the higher dimensional Dirac operator $i\Gamma^\mu \partial_\mu + i\Gamma^y \partial_y - M$ is separated as,

$$i\Gamma^\mu \partial_\mu + i\Gamma^y \partial_y - M = \begin{pmatrix} (i\gamma^y \partial_y - M) \otimes \mathbf{1}_2 & \mathbf{1}_{2[d/2]} \otimes i\sigma^\mu \partial_\mu \\ \mathbf{1}_{2[d/2]} \otimes i\bar{\sigma}^\mu \partial_\mu & (-i\gamma^y \partial_y - M) \otimes \mathbf{1}_2 \end{pmatrix}. \quad (2.54)$$

The chiral matrices (if it exist) are expressed by σ^3 and γ_{in} as,

$$\Gamma = -\sigma^3 \otimes \gamma_{\text{in}} \otimes \mathbf{1}_2, \quad \Gamma_{4D} = -\sigma^3 \otimes \mathbf{1}_{2[d/2]} \otimes \mathbf{1}_2, \quad \Gamma_{\text{in}} = \mathbf{1}_2 \otimes \gamma_{\text{in}} \otimes \mathbf{1}_2. \quad (2.55)$$

We note the 4D chiral projection operator is presented as,

$$\frac{1 - \Gamma_{4D}}{2} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \otimes \mathbf{1}_{2[d/2]} \otimes \mathbf{1}_2, \quad \frac{1 + \Gamma_{4D}}{2} = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \otimes \mathbf{1}_{2[d/2]} \otimes \mathbf{1}_2. \quad (2.56)$$

Thus, in the matrix form, the upper-half (lower-half) components of the higher dimensional Dirac field have the -1 ($+1$) eigenvalue of Γ_{4D} . Hence, after the KK decomposition, the upper-half (lower-half) components will be the 4D left-handed (right-handed) chiral fields.

We should stress that, like 4D, the 4D chiral matrix Γ_{4D} is (block) diagonalized and the left-most part of the tensor product characterizes the 4D chirality. In addition, the dependency on the internal space is well separated. Therefore, by taking this representation, it will be very clear how the 4D left-handed and the right-handed field couple and how the KK mass appears in the KK decomposition.

2.3.3 The KK decomposition

Supposing the completeness of the mode functions, we can decompose the $(4 + d)$ -dimensional Dirac field into the 4D two component chiral fields $\phi(x)$ and $\bar{\chi}(x)$ where $\phi(x)$ is left-handed and $\bar{\chi}(x)$ is right-handed. The requirement that the 4D fields are mass eigenstates restrict the mode functions. We show the properties which the mode functions satisfy to provide the 4D mass eigenstates after the KK decomposition.

Using the mode functions $\mathbf{f}_\alpha^{(n)}(y)$ and $\mathbf{g}_\alpha^{(n)}(y)$, the KK decomposition of the $(4 + d)$ -dimensional Dirac field $\Psi(x, y)$ will be given by

$$\Psi(x, y) = \sum_n \sum_\alpha \mathbf{e}_L \otimes \mathbf{f}_\alpha^{(n)}(y) \otimes \bar{\phi}_\alpha(x) + \mathbf{e}_R \otimes \mathbf{g}_\alpha^{(n)}(y) \otimes \chi_\alpha(x) \quad (2.57)$$

$$= \sum_n \sum_\alpha \begin{pmatrix} \mathbf{f}_\alpha^{(n)}(y) \otimes \bar{\phi}_\alpha(x) \\ \mathbf{g}_\alpha^{(n)}(y) \otimes \chi_\alpha(x) \end{pmatrix}, \quad (2.58)$$

where $\mathbf{e}_L = (1, 0)^\top$ and $\mathbf{e}_R = (0, 1)^\top$ characterize the 4D chirality. In the second line, we note the expansion with nesting the first tensor product. The index n represents the n -th level of

the KK modes and α denotes the index that distinguishes the degeneracy of the n -th KK modes (if exist). The mode functions $f_\alpha^{(n)}(y)$ and $g_\alpha^{(n)}(y)$ have $2^{\lfloor d/2 \rfloor}$ components and are assumed to form a complete set of the $C^{2\lfloor d/2 \rfloor} \otimes L^2(\Omega)$ where $L^2(\Omega)$ is the square integrable Hilbert space on the internal space Ω . The Hilbert space $C^{2\lfloor d/2 \rfloor} \otimes L^2(\Omega)$ respects the degrees of freedom of the internal space spinor.

Substituting this expansion (2.57) into the action (2.50), we find the dependency on the internal space is completely removed from the action to the scalar factor $\langle X|Y \rangle$,

$$S = \int_{M_4} d^4x \sum_{\alpha, \beta} \sum_{n, m} \langle f_\alpha^{(n)} | f_\beta^{(m)} \rangle \phi_\alpha^{(n)}(x) i \bar{\sigma}^\mu \partial_\mu \bar{\phi}_\beta^{(m)}(x) + \langle g_\alpha^{(n)} | g_\beta^{(m)} \rangle \bar{\chi}_\alpha^{(n)}(x) i \sigma^\mu \partial_\mu \chi_\beta^{(m)}(x) \\ + \langle f_\alpha^{(n)} | A g_\beta^{(m)} \rangle \bar{\phi}_\alpha^{(n)}(x) \bar{\chi}_\beta^{(m)}(x) + \langle g_\alpha^{(n)} | A^\dagger f_\beta^{(m)} \rangle \chi_\alpha^{(n)}(x) \phi_\beta^{(m)}(x). \quad (2.59)$$

The factor $\langle X|Y \rangle = \int_\Omega dy X^\dagger(y) Y(y)$ is the inner product of the Hilbert space $C^{2\lfloor d/2 \rfloor} \otimes L^2(\Omega)$. The operators A and $A^\dagger$ are the Dirac operator on the internal space given as

$$A = +i\gamma^y \partial_y - M, \quad A^\dagger = -i\gamma^y \partial_y - M. \quad (2.60)$$

The $A^\dagger$ is the Hermitian conjugate operator of A .¹¹ We note that the A and $A^\dagger$ are the off-diagonal components of the operator $\Gamma^0(i\gamma^y \partial_y - M)$ written as

$$\Gamma^0(i\gamma^y \partial_y - M) = \begin{pmatrix} & A \\ A^\dagger & \end{pmatrix} \otimes \mathbf{1}_2 = \begin{pmatrix} & i\gamma^y \partial_y - M \\ -i\gamma^y \partial_y - M & \end{pmatrix} \otimes \mathbf{1}_2. \quad (2.61)$$

By multiplying the higher dimensional gamma matrix Γ^0 , the A and $A^\dagger$ move into the off-diagonal components.

We require that $\bar{\phi}_\alpha^{(n)}(x)$ and $\chi_\alpha^{(n)}(x)$ are mass eigenstates of the 4D spectrum and that the action should be written into the form

$$S = \int_{M^4} dx \sum_\alpha \sum_n \bar{\psi}_\alpha^{(n)}(x) (i\gamma^\mu \partial_\mu + m_n) \psi_\alpha^{(n)}(x) \\ + \sum_\alpha \bar{\phi}_\alpha^{(0)}(x) i \bar{\sigma}^\mu \partial_\mu \bar{\phi}_\alpha^{(0)}(x) + \bar{\chi}_\alpha^{(0)}(x) i \sigma^\mu \partial_\mu \chi_\alpha^{(0)}(x), \quad (2.62)$$

where $\psi_\alpha^{(n)}(x) = (\bar{\phi}_\alpha^{(n)}(x), \chi_\alpha^{(n)}(x))^T$ are the n -th KK 4D Dirac fields with a KK mass m_n and the $\bar{\phi}_\alpha^{(0)}(x)$ and $(\chi_\alpha^{(0)}(x))$ in second line are massless right-handed (left-handed) 4D chiral fields.

This can be realized, if the the mode functions $f_\alpha^{(n)}(y)$ and $g_\alpha^{(n)}(y)$ satisfy the following relations;

$$\langle f_\alpha^{(n)} | f_\beta^{(m)} \rangle = \langle g_\alpha^{(n)} | g_\beta^{(m)} \rangle = \delta_{\alpha\beta} \delta^{nm}, \quad (2.63)$$

$$\langle f_\alpha^{(n)} | A g_\beta^{(m)} \rangle = \langle g_\alpha^{(n)} | A^\dagger f_\beta^{(m)} \rangle = m_n \delta_{\alpha\beta} \delta^{nm}. \quad (2.64)$$

¹¹ One might think that it is not trivial if the internal space has boundaries. In fact, the variation principle ensures this conjugation. We discuss this in Chapter 4.

The first relation (2.63) implies that the two sets of the mode functions $f_\alpha^{(n)}(y)$ and $g_\alpha^{(n)}(y)$ are the complete normalized orthogonal system of the Hilbert space $\mathcal{C}^{2^{[d/2]}} \otimes L^2(\Omega)$. The second relation (2.64) ensures that the chiral partners of $f_\alpha^{(n)}(y)$ ($g_\alpha^{(n)}(y)$) is given by $g_\alpha^{(n)}(y)$ ($f_\alpha^{(n)}(y)$). Let us summarize the properties of $f_\alpha^{(n)}(y)$ and $g_\alpha^{(n)}(y)$. The mode functions $f_\alpha^{(n)}(y)$ and $g_\alpha^{(n)}(y)$ should have the properties i) Completeness, ii) Orthogonality, iii) Normalizability and iv) Degeneracy between the left-handed and the right-handed modes.

The plane waves which are usually taken for the mode functions accidentally satisfy the relation (2.63) and (2.64) on the torus. However, it is not appropriate in the general shape of the internal space.

2.4 The five dimensional case

We should mention about the KK decomposition of the five-dimensional Dirac field. Due to less degrees of freedom of internal space, the KK decomposition of the five-dimensional Dirac field becomes exceptionally simple.

In five dimensions, we can always take the higher dimensional gamma matrices Γ^N ($N = 0, 1, 2, 3, y_1$) as the same size of the 4D gamma matrices γ^μ ($\mu = 0, 1, 2, 3$) as

$$\Gamma^\mu = \gamma^\mu, \quad \Gamma^{y_1} = i\gamma^5, \quad (2.65)$$

where γ^5 is defined in (2.7). Hence, the five-dimensional Dirac field is the same size as the 4D one. The KK decomposition of the five-dimensional Dirac field is driven by the “scalar” mode functions $f_\alpha^{(n)}(y)$ and $g_\alpha^{(n)}(y)$ as

$$\Psi(x, y) = \sum_\alpha \sum_n \mathbf{e}_L \otimes \bar{\phi}_\alpha^{(n)}(x) f_\alpha^{(n)}(y) + \mathbf{e}_R \otimes \chi_\alpha^{(n)}(x) g_\alpha^{(n)}(y) \quad (2.66)$$

$$= \sum_\alpha \sum_n \begin{pmatrix} \bar{\phi}_\alpha^{(n)}(x) f_\alpha^{(n)}(y) \\ \chi_\alpha^{(n)}(x) g_\alpha^{(n)}(y) \end{pmatrix}. \quad (2.67)$$

The action after the KK decomposition has the same form as (2.62). There is no difference in the requirement for the mode functions except the definition of the inner product and the operators, A and $A^\dagger$,

$$\langle X|Y \rangle = \int_\Omega dy X^\dagger(y) Y(y), \quad A = +\partial_y + M, \quad A^\dagger = -\partial_y + M. \quad (2.68)$$

We emphasize that the operator A and $A^\dagger$ are, actuarially, the first degree differential operators. The mode functions $f_\alpha^{(n)}(y)$ and $g_\alpha^{(n)}(y)$ form the orthogonal normalized complete system of the square integrable Hilbert space $L^2(\Omega)$ over the internal space Ω .

2.5 Summary: Higher dimensional Dirac action

In this chapter, we showed the properties which the mode functions should have with the natural three assumptions,

- i) there exists the higher dimensional free Dirac field (which has a non-trivial profile)
- ii) there exists the complete system which is able to expand any higher dimensional free Dirac field¹²
- iii) there exists the 4D Lorentz invariant 4D spectrum with 4D mass eigenstates.

The properties are summarized in the relations (2.63) and (2.64) which imply;

- i) the mode functions should be *complete* to expand the higher dimensional Dirac field
- ii) the mode functions should be *orthogonal* to provide the canonical 4D action
- iii) the mode functions should be *normalized* to provide the canonical 4D action
- iv) the mode functions should be *degenerate* to provide the 4D mass eigenstates with the 4D Dirac mass term
- v) the massless KK modes are exception of degeneracy.

These properties are not special (e.g. the plane waves have the properties). However, we didn't discuss the existence and the construction of such the mode functions. In the next chapter, we show a hidden supersymmetric quantum mechanical structure and there exist the mode functions which have the properties (and/or satisfy the relation (2.63) and (2.64) with completeness).

We note that the representation of the higher dimensional gamma matrices (2.55) made the discussion clear and simple because, in the representation, dependency on the 4D space-time and the internal space are well separated. As see in the next chapter, this representation makes the hidden SQM clear also.

¹² We assume the on-shell condition, also.

Chapter 3

Supersymmetric Quantum Mechanics

The supersymmetry is known as the symmetry between bosons and fermions in the field theory [28]. The supersymmetric quantum mechanics (SQM) is introduced by Witten as a toy model to show the spontaneous supersymmetry breaking [21]. The supersymmetry of the quantum mechanical system is the non-trivial duality between the different quantum mechanical systems which are isospectral except zero-energy modes [29, 30].¹ Surprisingly, almost all of well-known exactly solvable one-dimensional quantum mechanical systems, every model what one can imagine, are classified in this model [30].

In this chapter, as one of the applications of the supersymmetric quantum mechanics, we introduce the SQM relation naturally provides the properties based on the hidden higher dimensional $\mathcal{N} = 2$ supersymmetric quantum mechanical structure. In other words, the $\mathcal{N} = 2$ SQM supercharges generate chiral partners of the 4D fields.

However, there are additional isospectral pairs in the 4D spectrum of the higher dimensional Dirac field. We show that there exists the hidden symmetry, *the $\mathcal{N}$ -extended quantum mechanical supersymmetry*, providing the additional isospectral pairs in the KK decomposition of the higher dimensional Dirac field. We introduce them in this chapter.

This chapter is organized as follows: in Section 3.1, we briefly review the $\mathcal{N} = 2$ supersymmetric quantum mechanics which was introduced by Witten. In Section 3.2, we introduce the d dimensional realization of the $\mathcal{N} = 2$ SQM and that the structure naturally provides the properties which the mode functions satisfy. In Section 3.3, we reveal the $\mathcal{N}$ -extended SQM.

3.1 Brief review of supersymmetric quantum mechanics

In this section, we briefly review $\mathcal{N} = 2$ supersymmetric quantum mechanics before introducing the SQM relation in this section. The $\mathcal{N} = 2$ supersymmetric quantum mechanical system is introduced by Witten as a toy model to show spontaneous supersymmetry breaking [21]. He also introduced an index called Witten index which characterizes the spontaneous supersymmetry breaking.

¹ The systems are isospectral if they have the same spectrum.

3.1.1 Definition and general properties

The $\mathcal{N} = 2$ SQM is a Hermitian operator quadruplet $\{Q_\pm, H, (-1)^F\}$ satisfying,

$$\{Q_i, Q_j\} = 2H\delta_{ij}, \quad \{Q_i, (-1)^F\} = 0, \quad (-1)^F(-1)^F = 1, \quad (3.1)$$

for $i, j = \pm$ where H is the quantum mechanical Hamiltonian, Q_i ($i = \pm$) are the supercharges and $(-1)^F$ is a “fermion number” operator.² The supercharge Q_- is not “independent” from the supercharge Q_+ and the “fermion number” operator $(-1)^F$. The supercharge Q_- is formed as a product of them. We fix the phase of the supercharge Q_- as $Q_- = i(-1)^F Q_+$ for convenient.³

This definition implies that the supercharges and the fermion number operator are conserved,

$$[Q_i, H] = [(-1)^F, H] = 0, \quad (3.2)$$

for $i = \pm$. Thus, we label every energy eigenstate by the eigenvalues ± 1 of $(-1)^F$. We denote $|n, \pm\rangle$ by such the states satisfying

$$H|n, \pm\rangle = E_n|n, \pm\rangle, \quad (-1)^F|n, \pm\rangle = \pm|n, \pm\rangle, \quad (3.3)$$

for $n = 0, 1, \dots$

The Hamiltonian H is positive definite, for all state $|\psi\rangle$, $\langle\psi|H|\psi\rangle > 0$, because there exist the Hermitian square roots of the Hamiltonian, $Q_\pm$,

$$\langle\psi|H|\psi\rangle = \langle\psi|Q_i Q_i|\psi\rangle = \langle Q_i \psi | Q_i \psi \rangle \geq 0, \quad (3.4)$$

for all state $|\psi\rangle$ and $i = \pm$.

The spectrum on $\{|n, +\rangle\}$ and $\{|n, -\rangle\}$ are isospectral except for the zero energy state(s). That is, the spectrum of the Hamiltonian completely (at least) doubly degenerates except for the zero energy state(s) because the supercharges Q_i ($i = \pm$) flip the signature of the eigenvalues of $(-1)^F$,

$$(-1)^F Q_i |n, \pm\rangle = -Q_i (-1)^F |n, \pm\rangle = \mp Q_i |n, \pm\rangle, \quad (3.5)$$

for all states $|n, \pm\rangle$ with $0 < E_n$ and $i = \pm$. Moreover, the states $Q_i |n, \pm\rangle$ are still at the same energy level as $|n, \pm\rangle$,

$$H Q_i |n, \pm\rangle = Q_i H |n, \pm\rangle = E_n Q_i |n, \pm\rangle, \quad (3.6)$$

for all non zero energy state $|n, \pm\rangle$. It is easy to verify that the proportionality constants m_n are real, e.g. $Q_+ |n, \pm\rangle = m_n |n, \mp\rangle$. However, the RHS of (3.6) vanishes for the zero energy mode(s) $E_n = 0$. Thus, the zero energy mode(s) does not necessarily degenerate. The existence of the zero energy mode(s) depends on the models.

² Usually, the supercharges are labeled by $Q_+ = Q_1$ and $Q_- = Q_2$.

³ It is difficult to characterize the zero energy modes without the “fermion number” operator $(-1)^F$ because the zero energy modes belong to the kernel of the supercharges. The “fermion number” operator is indispensable.

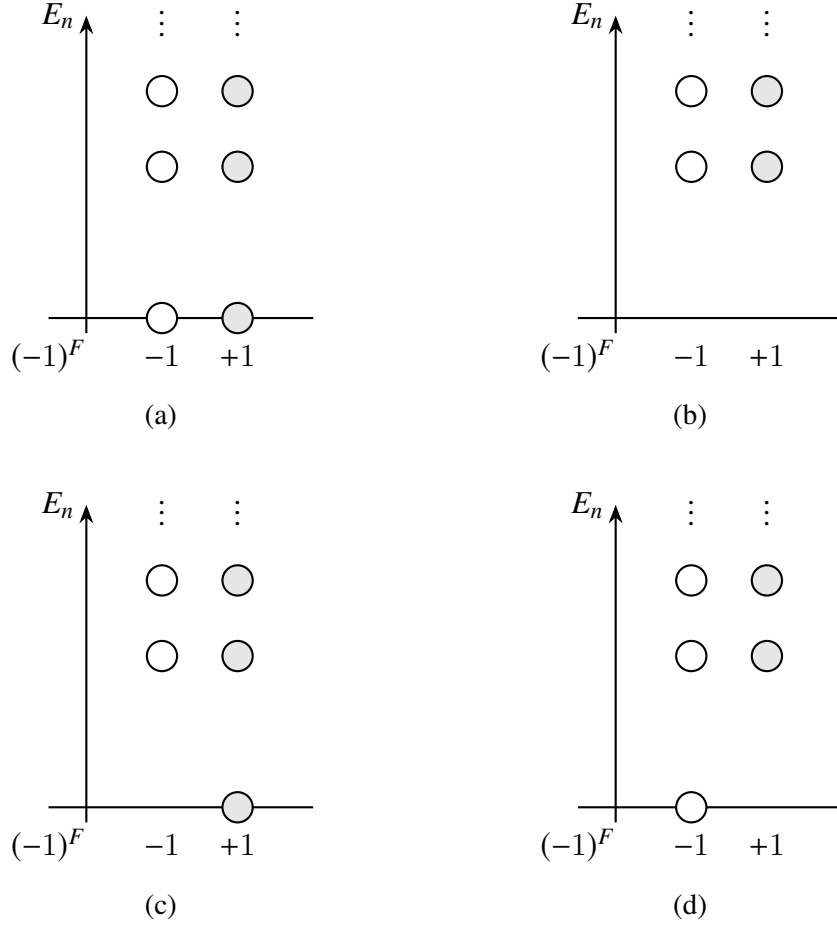


Figure 3.1: We illustrate typical spectrums of the $\mathcal{N} = 2$ supersymmetric quantum mechanical system. There is at least double degeneracy in every spectrum. In the figure (a), there are two zero energy modes $|0, \pm\rangle$. In figure (b), there is no zero energy mode. First two cases are completely degenerated spectrums. In the figure (c), there is only the zero mode $|0, +\rangle$ and, in the figure (d), there is only the zero mode $|0, -\rangle$. The corresponding Witten index is listed in Table 3.1.

One find the the eigenvectors of the the supercharges $Q_{\pm}$ with the eigenvalues m_n (real constants) by the linear combination of the same energy modes $|n\rangle_{\pm} = |n, +\rangle \pm |n, -\rangle$ respectively,

$$Q_{\pm} |n\rangle_{\pm} = \pm m_n |n\rangle_{\pm}, \quad Q_{\pm} |n\rangle_{\mp} = \mp m_n |n\rangle_{\mp}. \quad (3.7)$$

The supersymmetry holds if there exists a zero energy state $|0, \pm\rangle$,

$$Q_i |0, \pm\rangle = 0, \quad (3.8)$$

for $i = \pm$. As the same manner of the quantum field theory, we can define the spontaneous supersymmetry breaking in quantum mechanics. The supersymmetry spontaneously breaks if there does not exist the zero energy state $|0, \pm\rangle$,

$$Q_i |0, \pm\rangle \neq 0. \quad (3.9)$$

In contrast to the quantum field theory, the supercharges are well-defined even if the supersymmetry is spontaneously broken.

Witten introduced the Witten index Δ_W to characterize the spontaneous supersymmetry breaking. The Witten index is the difference between the number of zero energy $(-1)^F = +1$ states and the number of zero energy $(-1)^F = -1$ states [21]. In the $\mathcal{N} = 2$ SQM, such the index has the analytical definition,

$$\Delta_W = \dim \ker_{|n,+\rangle} Q - \dim \ker_{|n,-\rangle} Q. \quad (3.10)$$

where Q is the supercharges $Q_{\pm}$ of the $\mathcal{N} = 2$ SQM.⁴ It is obvious that if the Witten index is non zero, there exists a zero energy mode(s) and the supersymmetry holds,

$$\Delta_W \neq 0 \quad \implies \quad \text{supersymmetry holds.} \quad (3.11)$$

By taking contraposition, the Witten index values zero, only if supersymmetry spontaneously breaks,

$$\Delta_W = 0 \quad \Longleftarrow \quad \text{spontaneous supersymmetry breaking.} \quad (3.12)$$

3.1.2 The Witten model: a minimal realization of the $\mathcal{N} = 2$ SQM

The Witten index Δ_W is related to “topological” quantities.⁵ We show it on the Witten model named after Witten.

The Witten model is a minimal realization of $\mathcal{N} = 2$ SQM as a one dimensional quantum mechanical system. The elements of the quadruplet are given by 2-by-2 matrices,

$$Q_+ = \begin{pmatrix} & A \\ A^\dagger & \end{pmatrix}, \quad Q_- = -i \begin{pmatrix} & A \\ -A^\dagger & \end{pmatrix}, \quad (-1)^F = \begin{pmatrix} 1 & \\ & -1 \end{pmatrix}, \quad (3.13)$$

where A and $A^\dagger$ are the first degree differential operators with a smooth real valued function $W'(x)$ given by

$$A = +\frac{d}{dx} + W'(x), \quad A^\dagger = -\frac{d}{dx} + W'(x), \quad W'(x) = \frac{dW(x)}{dx}. \quad (3.14)$$

The function $W(x)$ is called a superpotential (or prepotential). The supercharges have only the off-diagonal components in the Witten model. Then, the Hamiltonian has only diagonal components,

$$H = \begin{pmatrix} AA^\dagger & \\ & A^\dagger A \end{pmatrix} = \begin{pmatrix} H_+ & \\ & H_- \end{pmatrix}, \quad (3.15)$$

⁴ Given an operator $X : D \rightarrow H$ where D and H are vector spaces. The kernel of D is a set $\{v \mid v \in D, Xv = 0\}$. We denote it as $\ker_D X$ to emphasize the domain D of the operator X .

⁵ We do not mention about what is the “topological” here. One can find the relation between the Witten index and truly topological quantities in the supersymmetric non-linear sigma model area (see [31–33]).

where H_+ and H_- are factorized one dimensional quantum mechanical Hamiltonians,

$$H_{\pm} = -\frac{d^2}{dx^2} + W'(x)^2 \pm W''(x) = \left(\pm \frac{\partial}{\partial x} + W'(x) \right) \left(\mp \frac{\partial}{\partial x} + W'(x) \right). \quad (3.16)$$

Almost all of well-known exactly solvable one-dimensional quantum mechanical systems, e.g. the radial part of the three-dimensional harmonic oscillator, the radial part of the three-dimensional hydrogen-like atom and the free particle etc., are classified in this model.

By definition, isospectrality of the non-zero energy modes is obvious. We focus on the zero energy mode(s). The zero energy mode equation $H|0\rangle = 0$ is equivalent to the two equations $H_+|0, +\rangle = 0$ and $H_-|0, -\rangle = 0$ because of Eq. (3.15). The zero energy mode(s) is a solution of the first-degree differential equation, $A^\dagger|0, +\rangle = 0$ and $A|0, -\rangle = 0$ because of the two factorized Hamiltonians. Thus, one immediately find the two candidates for the zero energy modes,

$$|0, \pm\rangle \propto \exp[\pm W(x)]. \quad (3.17)$$

However, square integrability of the solutions depends on the asymptotic behavior (or boundary values) of the superpotential $W(x)$ because the norm of the zero energy modes $\langle 0, \pm|0, \pm\rangle$ is integral of $\exp[\pm 2W(x)]$,

$$\langle 0, \pm|0, \pm\rangle \propto \int dx \exp[\pm 2W(x)]. \quad (3.18)$$

For simplicity, we assume that the system is defined on $\mathbf{R}$ and the absolute value of the superpotential $|W(x)|$ behaves asymptotically infinite for $x \rightarrow \pm\infty$ rapidly enough, e.g. $W(x)$ is a finite polynomial. In this case, the zero energy mode $|0, +\rangle$ ($|0, -\rangle$) is square integrable if and only if $W(\pm\infty) = \infty$ ($W(\pm\infty) = -\infty$) holds. Hence, the two zero energy solutions cannot be square integrable simultaneously. At most, the one zero energy state exists.

In Witten model, the Witten index can take the values 0 and ± 1 depending on the asymptotic behavior of the superpotential. We can classify the system by the asymptotic behavior of the superpotential.

- i) If ends of the potential tend to ∞ and $-\infty$, there is no zero energy mode. Thus, the Witten index is zero.
- ii) If both ends of the potential tend to ∞ , there is a one zero energy mode with eigenvalue -1 of $(-1)^F$. Thus, the Witten index is -1 .
- iii) If both ends of the potential tend to $-\infty$, there is a one zero energy mode with eigenvalue $+1$ of $(-1)^F$. Thus, the Witten index is $+1$.

The Witten index does not depend on the local shape of the potential but only asymptotical behavior. That is, “topological” property of the model characterizes the Witten index. We illustrate the four examples in Figure 3.2 and summarize in Table 3.1. The spectra of the examples are illustrated in Figure 3.1.

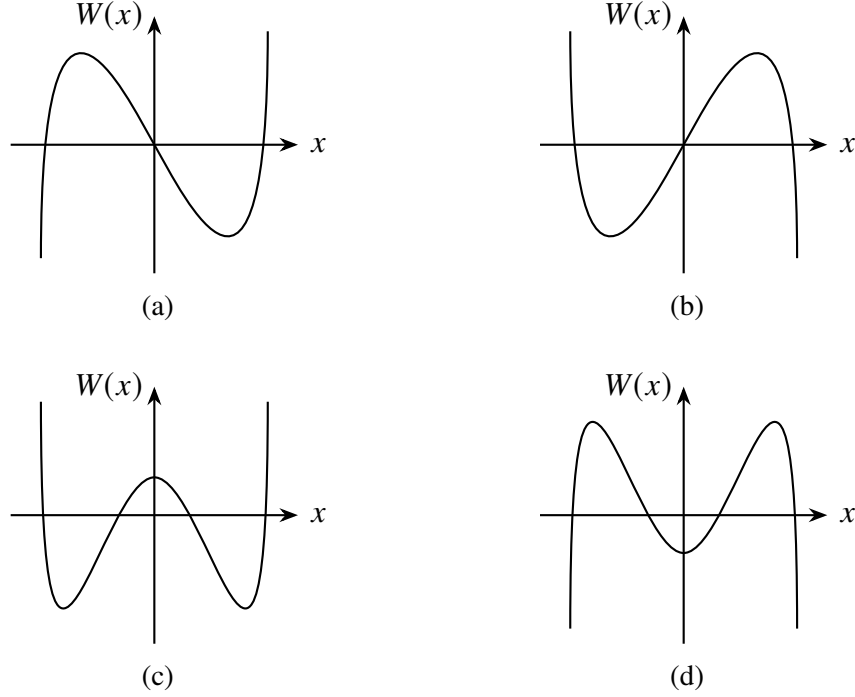


Figure 3.2: The illustration of the four examples of the superpotential $W(x)$ divergent rapidly enough. The figure (a) illustrates the case $W(\pm\infty) = \pm\infty$. The figure (b) illustrates the case $W(\pm\infty) = \mp\infty$. The figure (c) illustrates $W(\pm\infty) = +\infty$ and the figure (d) illustrates $W(\pm\infty) = -\infty$. The corresponding spectrum is given in Figure 3.1 and the Witten index is listed in Table 3.1.

3.2 Higher dimensional $\mathcal{N} = 2$ SQM realization

Now, we introduce the SQM relation naturally providing the mode functions which have the five properties (and/or (2.63) and (2.64) with completeness).

Based on the $\mathcal{N} = 2$ SQM, the SQM relation with (real constant) mass m_n is given as

$$Q_+ \begin{pmatrix} f_\alpha^{(n)}(y) \\ g_\alpha^{(n)}(y) \end{pmatrix} = \begin{pmatrix} A & \\ A^\dagger & \end{pmatrix} \begin{pmatrix} f_\alpha^{(n)}(y) \\ g_\alpha^{(n)}(y) \end{pmatrix} = m_n \begin{pmatrix} f_\alpha^{(n)}(y) \\ g_\alpha^{(n)}(y) \end{pmatrix}, \quad (3.19)$$

for all n and α . The operator Q_+ in (3.19) is a supercharge of the d dimensional realization of the $\mathcal{N} = 2$ SQM quadruplet given by,

$$Q_+ = \begin{pmatrix} A & \\ A^\dagger & \end{pmatrix}, \quad Q_- = -i \begin{pmatrix} A & \\ -A^\dagger & \end{pmatrix}, \quad (3.20a)$$

$$H = -\partial_y^2 + M^2, \quad (-1)^F = \begin{pmatrix} -1 & \\ & 1 \end{pmatrix}, \quad (3.20b)$$

where $A = i\gamma^y \partial_y - M$ and $A^\dagger = -i\gamma^y \partial_y - M$. It is easy to verify that the quadruplet satisfies the $\mathcal{N} = 2$ SQM algebra. Moreover, in next chapter, we show the fact that the variation

$W(-\infty)$	$W(+\infty)$	$ 0, +\rangle$	$ 0, -\rangle$	Δ_W
$-\infty$	$+\infty$	-	-	0
$+\infty$	$-\infty$	-	-	0
$+\infty$	$+\infty$	-	square integrable	-1
$-\infty$	$-\infty$	square integrable	-	+1

Table 3.1: The values of the Witten index Δ_W in the Witten model with the superpotential behaving asymptotically infinite rapidly enough. The value depends on only the asymptotic behavior of the superpotential $W(x)$.

principle ensures that the operators of the quadruplet are Hermitian even if boundaries exist. Therefore, the realization of the $\mathcal{N} = 2$ SQM is always well-defined.

In particular, in our representation, the supercharge Q_+ is naturally embedded in the action, and the operator $(-1)^F$ is naturally embedded in the 4D chiral matrix,

$$\Gamma^0(i\Gamma^y\partial_y - M) = Q_+ \otimes \mathbf{1}_2, \quad \Gamma_{4D} = (-1)^F \otimes \mathbf{1}_2. \quad (3.21)$$

Therefore, it is obvious that Hermiticity of supercharge Q_+ is equivalent to Hermiticity of the action ensured by the variation principle. We emphasize that, because of this embedding, it is natural that there exists such the supersymmetric quantum mechanical structure in the KK decomposition of the higher dimensional Dirac field.

It is easy to verify that the five properties (and/or (2.63) and (2.64)) are realized from the properties of the $\mathcal{N} = 2$ SQM. That is, the $\mathcal{N} = 2$ SQM supercharges generate the chiral partners of the 4D fields.

Let us introduce the abstract notation for respecting $\mathcal{N} = 2$ supersymmetric quantum mechanics. In this realization, the mode function multiples are energy eigenstates of the supersymmetric quantum mechanical Hamiltonian H . We denote $|n, \pm; \alpha\rangle$ as the energy eigenstates by

$$|n, +; \alpha\rangle = \begin{pmatrix} 0 \\ \mathbf{g}_\alpha^{(n)}(y) \end{pmatrix}, \quad |n, -; \alpha\rangle = \begin{pmatrix} \mathbf{f}_\alpha^{(n)}(y) \\ 0 \end{pmatrix}. \quad (3.22)$$

It is easy to verify the states $|n, \pm; \alpha\rangle$ are the eigenvectors of the “fermion number” operator $(-1)^F$,

$$H |n, \pm; \alpha\rangle = m_n^2 |n, \pm; \alpha\rangle, \quad (-1)^F |n, \pm; \alpha\rangle = \pm |n, \pm; \alpha\rangle. \quad (3.23)$$

The label α indicates the additional degeneracy which would not be mentioned by the $\mathcal{N} = 2$ SQM. We also introduce the eigenvectors $|n; \alpha\rangle_\pm$ as

$$|n; \alpha\rangle_+ = |n, +; \alpha\rangle + |n, -; \alpha\rangle = \begin{pmatrix} \mathbf{f}_\alpha^{(n)}(y) \\ \mathbf{g}_\alpha^{(n)}(y) \end{pmatrix}, \quad (3.24)$$

$$|n; \alpha\rangle_- = |n, +; \alpha\rangle - |n, -; \alpha\rangle = \begin{pmatrix} -\mathbf{f}_\alpha^{(n)}(y) \\ \mathbf{g}_\alpha^{(n)}(y) \end{pmatrix}. \quad (3.25)$$

One finds the SQM relation is the eigenvalue equation of the supercharge Q_+ and the mode function multiplets $(f_\alpha^{(n)}(y), g_\alpha^{(n)}(y))$ are the eigenvectors,⁶

$$Q_+ |n; \alpha\rangle_+ = m_n |n; \alpha\rangle_+ . \quad (3.26)$$

We note that the component form of the SQM relation (3.19),

$$A^\dagger f_\alpha^{(n)}(y) = m_n g_\alpha^{(n)}(y), \quad A g_\alpha^{(n)}(y) = m_n f_\alpha^{(n)}(y) . \quad (3.27)$$

Especially, the massless mode functions follow the first-degree differential equation, if such modes exist,

$$A^\dagger f_\alpha^{(0)}(y) = 0, \quad A g_\alpha^{(0)}(y) = 0 . \quad (3.28)$$

These mode functions are the solutions of the screened Poisson equations [34],

$$(-\partial_y^2 + M^2) f_\alpha^{(n)}(y) = m_n f_\alpha^{(n)}(y), \quad (-\partial_y^2 + M^2) g_\alpha^{(n)}(y) = m_n g_\alpha^{(n)}(y) . \quad (3.29)$$

We stress that the SQM relation (3.19) (or (3.27) and (3.28)) is the equation determining the profiles of the mode function and the zero mode equations (3.28) are the Dirac equation with mass M on the d dimensional space.

We should stress that, in this realization, the Witten index Δ_W characterizes to the chiral asymmetry of the massless KK modes,

$$\Delta_W = \# \{ \text{right-handed massless modes} \} - \# \{ \text{left-handed massless modes} \} . \quad (3.30)$$

We emphasize again that it is natural that there exists the $\mathcal{N} = 2$ SQM in the KK decomposition of the higher dimensional Dirac field. After the decomposition, the 4D left-handed and right-handed fields which are mass eigenstates should obtain the same mass to construct mass term at each KK level which referred as KK tower. Therefore, there exists at least one isospectral pair in the KK decomposition which provides the mass eigenstates and there should exist some mechanism ensuring isospectrality such as the $\mathcal{N} = 2$ SQM.

Although the $\mathcal{N} = 2$ supersymmetry assures an isospectral pair in the spectrum (we schematically illustrate in Figure 3.4), there are, in general, more isospectral pairs than expected by the $\mathcal{N} = 2$ supersymmetry. This fact suggests that there should exist some symmetries in addition to the $\mathcal{N} = 2$ supersymmetry. To reveal such hidden symmetries which provide the isospectral pairs is the purpose of the next section.

3.3 Hidden $\mathcal{N}$ -extended SQM

In general, there are additional isospectral pairs which are distinguished by label α at each KK level. One of the origins of such additional isospectral pairs is the geometrical symmetry of the internal space. For example, if the internal space is a d -dimensional torus, there appear the isospectral pairs of left-moving and right-moving modes.

⁶ We note that the $|n; \alpha\rangle_-$ has the eigenvalues $-m_n$, $Q_+ |n; \alpha\rangle_- = -m_n |n; \alpha\rangle_-$.

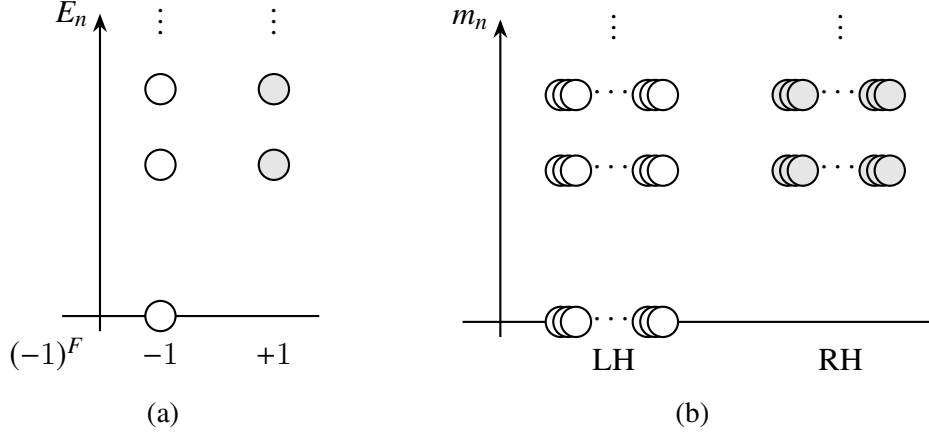


Figure 3.3: The figures of the typical spectrums of $\mathcal{N} = 2$ SQM (figure (a)) and the 4D spectrum of higher dimensional Dirac field (figure (b)). There are at least double degeneracy at each KK level except the lowest level. One finds that there is additional degeneracy in the 4D spectrum.

Moreover, in the Dirac field, there is another origin, because the high degrees of freedom from the viewpoint of a 4D theory that we mention in [Section 2.2](#). It is natural to expect that there is a hidden symmetry which ensures the isospectral pairs corresponding to the high degrees of freedom.

In this section, we reveal that the hidden symmetry is a $\mathcal{N}$ -extended supersymmetry [35] which is one of an extension of the $\mathcal{N} = 2$ supersymmetry. The supercharges of the $\mathcal{N}$ -extended supersymmetry generate the whole mode functions corresponding to such high degrees of freedom.

The $\mathcal{N}$ -extended supersymmetry is the set of the $d + 2$ ($d + 1$) supercharges for even (odd) dimensions. The every supercharge formally satisfies the $\mathcal{N}$ -extended SQM algebra,

$$\{Q_i, Q_j\} = 2H\delta_{ij}, \quad [Q_i, H] = 0. \quad (3.31)$$

We list the supercharges of the $\mathcal{N}$ -extended SQM. The symbol R_k ($k = 1, \dots, d$) denote the reflection operator for the y_k direction; i.e. $R_k : y_k \rightarrow -y_k$, and $P = \prod_{k=1}^d R_k$ denotes the point reflection operator on the internal space (also known as the parity operator). For even dimensions, the $\mathcal{N}$ -extended SQM supercharges are given as

$$Q_k = \begin{pmatrix} A^\dagger \gamma_{\text{in}} \gamma^{y_k} R_k & -\gamma_{\text{in}} \gamma^{y_k} R_k A \end{pmatrix}, \quad (3.32a)$$

$$Q_{d+1} = \begin{pmatrix} A^\dagger \gamma_{\text{in}} P & -\gamma_{\text{in}} P A \end{pmatrix}, \quad (3.32b)$$

$$Q_{d+2} = i(-1)^F Q_+, \quad (3.32c)$$

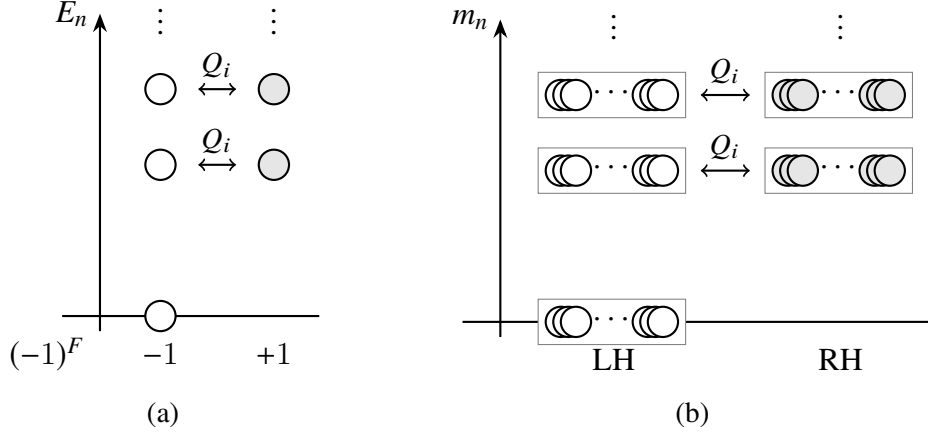


Figure 3.4: The figures of action of the $\mathcal{N} = 2$ SQM supercharges. The $\mathcal{N} = 2$ SQM supercharges only flips the eigenvalues of $(-1)^F$. The $\mathcal{N} = 2$ SQM does not explain about such additional isospectral pairs in the 4D spectrum.

where $k = 1, \dots, d$. For odd dimensions, the supercharges are given as

$$Q_k = \begin{pmatrix} & -\gamma^{y_d} \gamma^{y_k} R_d R_k A \\ A^\dagger \gamma^{y_d} \gamma^{y_k} R_d R_k & \end{pmatrix}, \quad (3.33a)$$

$$Q_d = \begin{pmatrix} & -\gamma^{y_d} R_d P A \\ A^\dagger \gamma^{y_d} R_d P & \end{pmatrix}, \quad (3.33b)$$

$$Q_{d+1} = i(-1)^F Q_+, \quad (3.33c)$$

where $k = 1, \dots, d - 1$.

It is easy to verify that the supercharges satisfy the $\mathcal{N}$ -extended SQM algebra. However, if there exist boundaries, it is not trivial that the $\mathcal{N}$ -extended SQM supercharges are well-defined, that is, Hermitian and consistent with the boundary condition. To explain the additional isospectral pairs, the supercharges should be well-defined. Further, the supercharges should be consistent with the boundary condition, that is, the image of the supercharges should belong to the same class of the boundary condition.

Once given the well-defined supercharges, the supercharges will generate such the additional isospectral pairs. On each n -th KK level with $m_n \neq 0$, the $\mathcal{N}$ -extended SQM algebra will be the Clifford algebra itself,

$$\{Q_i, Q_j\} = 2H\delta_{ij}|_{H=m_n} \implies \{Q_i, Q_j\} = 2m_n\delta_{ij} \quad (3.34)$$

The supercharges generate $2 \times 2^{\lfloor d/2 \rfloor}$ mode functions. Hence, one might expect that the $\mathcal{N}$ -extended SQM will bring the whole isospectral pairs at each KK level. And, if the several supercharges are not well-defined, the degrees of pairs will decrease. However, there is models which have additional isospectral pairs but not the $\mathcal{N}$ -extended SQM. We will show such examples in [Chapter 5](#).

Before closing this section, we note that the relation between the $\mathcal{N} = 2$ SQM supercharge and the $\mathcal{N}$ -extended SQM supercharges. There exist unitary transformations

U_k which relate the $\mathcal{N} = 2$ SQM supercharge Q_+ to the $\mathcal{N}$ -extended SQM supercharges Q_k for each k . The unitary transformations U_k are given by each the $\mathcal{N}$ -extended SQMs supercharges Q_i ($i = 1, \dots, d$ or $i = 1, \dots, d + 1$ formally) as,

$$Q'_i = U_i Q_+ U_i^{-1}, \quad U_i U_i^\dagger = U_i^\dagger U_i = \mathbf{1}, \quad U_i = \begin{pmatrix} X_i & \\ & \mathbf{1} \end{pmatrix}, \quad (3.35)$$

where X_i ($i = 1, \dots, d$ or $i = 1, \dots, d + 1$ formally) are also the unitary transformations. For even dimensions, the X

$$X_k = \gamma_{\text{in}} \gamma^{y_k} R_k, \quad X_{d+1} = \gamma_{\text{in}} P, \quad X_{d+2} = i\mathbf{1}, \quad (3.36)$$

where $k = 1, \dots, d$, For odd dimensions,

$$X_k = \gamma^{y_d} \gamma^{y_k} R_d R_k, \quad X_d = \gamma^{y_d} R_d P, \quad X_{d+1} = i\mathbf{1}, \quad (3.37)$$

where $k = 1, \dots, d - 1$.

3.4 Summary: Supersymmetric Quantum Mechanics

In this chapter, we showed the two kinds of a supersymmetric quantum mechanics hidden in the KK decomposition of the higher dimensional Dirac field.

We introduced the SQM relation (3.19) based on the $\mathcal{N} = 2$ supersymmetric quantum mechanics in Section 3.2. The supersymmetric quantum mechanics is one of the generalizations of the quantum mechanics. The SQM relation is the eigenvalue equation of the supercharge. It naturally provides the mode functions which have the five properties given in Chapter 2, that is, it naturally provides the mass eigenstates in the 4D spectrum. In other words, the $\mathcal{N} = 2$ SQM supercharges generate the chiral partners of the 4D fields. The result is compatible to previous papers [18–20].

On the other hand, there are, typically, additional isospectral pairs in the 4D spectrum which are not described by the SQM relation. A question naturally arises; is there any hidden symmetry in the KK decomposition of the higher dimensional Dirac field?

We showed that the hidden symmetry is the $\mathcal{N}$ -extended quantum mechanical supersymmetry which has $d + 2$ ($d + 1$) supercharges for even (odd) dimensions in Section 3.3. The well-defined $\mathcal{N}$ -extended SQM supercharges formally generates the whole isospectral pairs because it satisfies the Clifford algebra at each KK level formally.

Although we introduced the SQMs, we should mention whether the supercharges are well-defined, i.e., Hermitian and consistent with the boundary condition. In the next chapter, we discuss the boundary condition to discuss well-definedness.

We find the $\mathcal{N}$ -extended SQM supercharges in a heuristic way. Therefore, a question has still remained; why and when does such the supersymmetry exist?

Chapter 4

Boundary condition

We show various hidden supersymmetric quantum mechanical structures in the KK decomposition of the higher dimensional Dirac field. Such the hidden supersymmetric structures describe the isospectral pairs in the 4D spectrum. Namely, the supercharges generates the isospectral pairs *if the supercharges are well-defined*.

The supercharges are well-defined if the supercharges are i) Hermitian and ii) consistent with the boundary condition. The supercharges are Hermitian if the surface term vanishes by the boundary condition. The supercharges are consistent with the boundary condition if the image of the supercharges are compatible with the boundary condition. Consistency is necessary for generating the isospectral pairs.¹ If the internal space has the boundary, obviously, it is not trivial that the supercharges are well-defined.² Therefore, it is important to classify the boundary condition ensuring that the supercharges are well-defined.

The extra-dimensional theory with boundary can be understood as a kind of a generalization of orbifold models. In contrast to orbifold models, there is a hidden parameter space of the boundary condition which does not appear explicitly in the action. The complete classification of the boundary condition is also interesting from the point of a phenomenological view because of a possibility to represent the experiments by fitting such the parameter space.

It is well-known that the most general boundary condition of the scalar field for each point on the boundary is the Robin boundary condition [36]. However, it is not known how to derive the boundary condition in an easy and simple applicable method for the higher spin fields. Furthermore, even in the case of the higher dimensional Klein-Gordon equations, the parameter space which provides nontrivial solutions has not been classified.

We completely classified the boundary condition of the five-dimensional Dirac field in [18] and of the six-dimensional one in [19, 20] with a restriction of the shape of the internal space. The internal space of the five-dimensional space-time is an interval and of

¹ We note that any physical operator which defined on the Hilbert space is closed. However, in general, the operators are not closed on the function space, e.g. a set of the smooth functions satisfying some boundary conditions. As a specific example, the momentum operator $\hat{p}$ of the one-dimensional infinite-well model is not closed ($\hat{p} \sin(x)$ does not satisfy the Dirichlet boundary condition, therefore $\hat{p}$ is not closed on such the function space).

² In fact, even if there is no boundary in the internal space it is not trivial. We show examples in next chapter.

the six-dimensional one is a rectangle. In our classification, we require that the boundary condition has the properties that

- i) the boundary condition is derived by the variation principle of the action
- ii) the boundary condition ensures the 4D Lorentz invariance
- iii) the boundary condition is a local linear equation of the mode functions

Especially, in five-dimensions, we have succeeded in classifying the boundary condition based on the 4D spectrum completely.³ However, in six-dimensions, we obtained the 4D spectrum in only specific boundary condition because of difficulty analyzing the spectrum in general boundary condition.

In this chapter, we try to classify the general boundary condition which is applicable for the internal space with boundary. However, we will skip classification of the boundary condition which provides the nontrivial solution of the higher dimensional Dirac field. The results given in this chapter will be reported.

This chapter is organized as follows: we introduce the non-local boundary condition in [Section 4.1](#). We can capture various boundary conditions as the non-local boundary condition. One finds that the $\mathcal{N} = 2$ SQM is always well-defined with the non-local boundary condition. We focus on the locally given boundary condition, namely the boundary condition at each point on boundary, in [Section 4.2](#). In [Section 4.3](#), we show the special classes of the boundary condition at all point on boundary.

4.1 Hermiticity and the non-local boundary condition

It is obvious that the realization (3.20) satisfies the $\mathcal{N} = 2$ SQM algebra except for Hermiticity of the supercharges. In fact, the variation principle of the action, $\delta S = 0$, implies that the supercharges are Hermitian even if the boundary exists. Thus, the $\mathcal{N} = 2$ SQM holds always. In this section, we discuss the boundary condition derived from the variation principle of the action.

The variation principle and the Stokes' theorem imply that the mode functions should vanish the $d - 1$ dimensional surface integral on the boundary $\partial\Omega$,

$$\oint_{\partial\Omega} d^{d-1}y \begin{pmatrix} \tilde{f}_{\tilde{\alpha}}^{(\tilde{n})}(y) \\ \tilde{g}_{\tilde{\beta}}^{(\tilde{n})}(y) \end{pmatrix}^{\dagger} \begin{pmatrix} -in_y\gamma^y & in_y\gamma^y \end{pmatrix} \begin{pmatrix} f_{\alpha}^{(n)}(y) \\ g_{\beta}^{(n)}(y) \end{pmatrix} = 0, \quad (4.1)$$

for all $(f, g)^T$ and $(\tilde{f}, \tilde{g})^T$ satisfying the same BC. The symbol n_y is a normal vector at each point on the boundary. If the surface integral vanishes, the action will be Hermitian and the $U(1)$ current will conserve.⁴

The equation is not the first-degree equation and is *non-local*. We can capture the various boundary conditions because of these properties. For example, the well-known boundary conditions,

³ We note in [Section 5.4](#).

⁴ We show in [Appendix C](#).

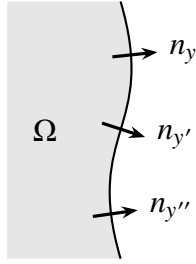


Figure 4.1: The normal vector n_y is vertical to the boundary and is defined at each point on the boundary.

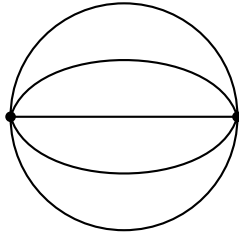


Figure 4.2: The quantum graph is the one-dimensional quantum mechanical system on the one-dimensional graph. The nodes are “singular” point, and there is the boundary condition to conserve the currents. This figure is an example of the one-dimensional graph. There are two nodes on each side.

- i) The Dirichlet boundary condition
- ii) The Neumann boundary condition
- iii) The Robin boundary condition
- iv) The periodic boundary condition
- v) The twisted boundary condition

satisfy the Eq. (4.1) and the $\mathcal{N} = 2$ supercharges will be Hermitian. Thus, we call the surface integral as a *non-local boundary condition*. The non-local boundary condition can also capture interesting examples, e.g., quantum graph, patch worked space [37], etc., with a small modification.⁵

We stress again that it is natural that there exists the $\mathcal{N} = 2$ SQM structure in the KK decomposition of the higher dimensional Dirac field. There should exist some mechanism which ensures the isospectral pairs such as the $\mathcal{N} = 2$ supersymmetry to provide mass eigenstates in the 4D spectrum.

It is clear that if the non-local boundary condition holds, the variation principle holds. That is, the existence of the $\mathcal{N} = 2$ SQM is equivalent to the existence of the higher

⁵ It may be not conformable to call the surface integral as a boundary condition. In fact, the condition that the surface integral vanishes (4.1) is one of the most generalized boundary conditions. The non-local boundary condition (the surface integral vanishing) indicates the following case: even if a current is flowing out at some point on the boundary, the total current is conserved with flowing in from another point. Indeed, as we will show in Appendix C, the non-local boundary condition can be derived from the conservation law (the continuity equation).

dimensional Dirac field theory.

However, the Hermiticity of the $\mathcal{N}$ -extended SQM is non-trivial because of the $\mathcal{N}$ -extended SQM supercharges contains the gamma matrices γ^{y_k} and the reflection operator R_k ($k = 1, \dots, d$). We classify the locally given boundary condition prior because the $\mathcal{N} = 2$ SQM is always well-defined. In the next chapter, we will discuss the well-definedness of the $\mathcal{N}$ -extended SQM in explicit models. The results shown in following two section are the compatible to the previous papers [18–20].

4.2 BC at each point on the boundary

It is not easy to treat the non-local boundary condition due to huge variation of the boundary condition. In this section, for simplicity, we focus on and classify the boundary condition which has properties that

- i) the boundary condition ensures Hermiticity of the supercharges
- ii) the boundary condition ensures the $\mathcal{N} = 2$ supersymmetry
- iii) the boundary condition is a local linear equation of the mode functions

The boundary condition is sufficiently classified by considering the following equation at each point on the boundary,

$$\begin{pmatrix} f_\alpha^{(n)}(y) \\ g_\alpha^{(n)}(y) \end{pmatrix}^\dagger \begin{pmatrix} & N \\ -N & \end{pmatrix} \begin{pmatrix} f_\alpha^{(n)}(y) \\ g_\alpha^{(n)}(y) \end{pmatrix} = 0 \quad \text{at each point on boundary} \quad (4.2)$$

for all n and α . The symbol $N = in_y \gamma^y$ satisfies $N^\dagger = N$ and $N^2 = \mathbf{1}$. If the $(f_\alpha^{(n)}(y), g_\alpha^{(n)}(y))^\top$ and $(\tilde{f}_\alpha^{(n)}(y), \tilde{g}_\alpha^{(n)}(y))^\top$ satisfy this equation, the integrand of the non-local boundary condition (4.1) vanishes at each point on the boundary. Thus, the integrand of the non-local boundary condition (4.1) vanishes at each point on the boundary if and only if this equation hold at each point on boundary.

One can reformulate the Eq. (4.2) into the first-degree equation by applying the polarization identity,⁶

$$u^\dagger v = \frac{1}{4} (\|u + v\|^2 - \|u - v\|^2) - \frac{i}{4} (\|u + iv\|^2 - \|u - iv\|^2), \quad (4.3)$$

where $\|v\|^2 = v^\dagger v$. The real part (the first two terms) trivially vanishes, and the complex part (the last two terms) remains. By $N^\dagger = N$ and $N^2 = \mathbf{1}$, one finds that the complex part will reduce to the simple equation,

$$\|N f_\alpha^{(n)}(y) + i g_\alpha^{(n)}(y)\|^2 = \|N f_\alpha^{(n)}(y) - i g_\alpha^{(n)}(y)\|^2. \quad (4.4)$$

⁶ Although we use the polarization identity to rewrite the inner product into the norm, the important aspect of the polarization identity is the formula inducing inner product by the norm. It goes without saying that inner product induces norm.

The first-degree solution of this equation is given by,

$$N\mathbf{f}_\alpha^{(n)}(y) + i\mathbf{g}_\alpha^{(n)}(y) = U(N\mathbf{f}_\alpha^{(n)}(y) - i\mathbf{g}_\alpha^{(n)}(y)), \quad (4.5)$$

where U is an arbitrary $2^{\lfloor d/2 \rfloor}$ -by- $2^{\lfloor d/2 \rfloor}$ unitary matrix. Finally, one obtains the equation,

$$(\mathbf{1} - U)N\mathbf{f}_\alpha^{(n)}(y) = -i(\mathbf{1} + U)\mathbf{g}_\alpha^{(n)}(y). \quad (4.6)$$

If the boundary condition is the linear combination of the mode functions $\mathbf{f}_\alpha^{(n)}(y)$ and $\mathbf{g}_\alpha^{(n)}(y)$, the “fermion number” operator $(-1)^F$ do not commute with the SQM Hamiltonian H . The operator $(-1)^F$ corresponds to the 4D chiral matrix and should conserve. Thus, each side of Eq. (4.6) vanishes respectively,

$$(\mathbf{1} - NUN)\mathbf{f}_\alpha^{(n)}(y) = 0, \quad (\mathbf{1} + U)\mathbf{g}_\alpha^{(n)}(y) = 0, \quad (4.7)$$

where $N^2 = \mathbf{1}$ is used. For both sides to vanish, the $\mathbf{1} \pm U$ should be projection operators. Such U is given by,

$$U = V^{-1}DV, \quad (4.8)$$

where D is a diagonal matrix of any number of ± 1 (e.x. $D = \text{diag}(+1, \dots, +1, -1, \dots, -1)$) and V is any $2^{\lfloor d/2 \rfloor}$ -by- $2^{\lfloor d/2 \rfloor}$ unitary matrix. The diagonal matrix D determine the combination of the mode functions which vanish the boundary. The unitary matrix V is a weigh (or coefficients) of the linear combination. These will be clear by denoting the boundary condition by $V^{-1}DV$ instead of U ,

$$NV^{-1}(\mathbf{1} - D)VN\mathbf{f}_\alpha^{(n)}(y) = 0, \quad V^{-1}(\mathbf{1} + D)V\mathbf{g}_\alpha^{(n)}(y) = 0. \quad (4.9)$$

There exists the parameter space BC_y ($y \in \partial\Omega$) at each point on the boundary, like a tangent space of the boundary $\partial\Omega$. The essential parameter space BC_y is the disjoint union of the complex Grassmannian which is the coset of the unitary group,

$$BC_y = \bigsqcup_{k=1}^{2^{\lfloor d/2 \rfloor}} \frac{SU(2^{\lfloor d/2 \rfloor})}{SU(2^{\lfloor d/2 \rfloor} - k) \times SU(k)}. \quad (4.10)$$

The coset is compact and simply connected because the group $SU(N)$ is compact and simply connected. Therefore, its fundamental group is trivial [38–40].

Hence, we find the BC satisfying the three condition i)-iii) is characterized by the projection operator P ,

$$P \begin{pmatrix} \mathbf{f}_\alpha^{(n)}(y) \\ \mathbf{g}_\alpha^{(n)}(y) \end{pmatrix} = 0, \quad P = \frac{1}{2} \begin{pmatrix} \mathbf{1} - NUN & \\ & \mathbf{1} + U \end{pmatrix}, \quad (4.11)$$

at each point on the boundary, for all n and α . The projection operator P and its orthogonal projection operator $P_\perp$ satisfies,

$$\mathbf{1} = P + P_\perp, \quad PP_\perp = P_\perp P = 0, \quad P^2 = P, \quad P_\perp^2 = P_\perp, \quad (4.12)$$

where the orthogonal projection operator $P_\perp$ is defined by

$$P_\perp = \frac{1}{2} \begin{pmatrix} \mathbf{1} + NUN & \\ & \mathbf{1} - U \end{pmatrix}. \quad (4.13)$$

The projection operator P is defined at each point on the boundary and has a parameter space as the disjoint union of the complex Grassmannian at each point on the boundary.⁷

It should be emphasized that the projection operators P and $P_\perp$ exchange each other by multiplying $n_y \Gamma^0 \Gamma^y$,

$$(P \otimes \mathbf{1}_2) n_y \Gamma^0 \Gamma^y = n_y \Gamma^0 \Gamma^y (P_\perp \otimes \mathbf{1}_2), \quad (4.14)$$

at each point on the boundary.⁸

Since we focus on the local boundary condition, there still exists a huge parameter space. Therefore, we derive more simple boundary conditions with natural restriction.

4.3 BC for all points on the boundary

The previous class of the boundary condition allows the different boundary condition at each point on the boundary. In this section, we focus on and classify special classes, i.e. the same BC at all points on the boundary.

Such the boundary conditions are solutions of the following equation, there exists U , for all point on the boundary,⁹

$$(P \otimes \mathbf{1}_2) n_y \Gamma^0 \Gamma^y = n_y \Gamma^0 \Gamma^y (P_\perp \otimes \mathbf{1}_2), \quad (4.15)$$

or, equivalently,

$$NUN = U. \quad (4.16)$$

We find that there are only two solutions, $U = \pm \mathbf{1}$ and $U = \pm \gamma_{\text{in}}$. We list the two classes below.¹⁰

The 4D chiral BC In the case of $U = \pm \mathbf{1}$, the projection operators are the 4D chiral projection operators,

$$P = \frac{\mathbf{1}_2 \otimes \mathbf{1} \pm (-1)^F}{2} = \frac{1}{2} \begin{pmatrix} \mathbf{1} \mp \mathbf{1} & \\ & \mathbf{1} \pm \mathbf{1} \end{pmatrix}, \quad \frac{\mathbf{1} \pm \Gamma_{4\text{D}}}{2} = P \otimes \mathbf{1}_2. \quad (4.17)$$

Hence, we call this class as the *4D chiral BC*.

⁷ It seems to be difficult to decompose the action by this projection operator because of the $y \in \partial\Omega$ dependency.

⁸ $\forall y \in \partial\Omega \exists n_y \Gamma^0 \Gamma^y \exists P, P_\perp \quad \text{s.t.} \quad (P \otimes \mathbf{1}_2) n_y \Gamma^0 \Gamma^y = n_y \Gamma^0 \Gamma^y (P_\perp \otimes \mathbf{1}_2).$

⁹ $\exists P, P_\perp \forall y \in \partial\Omega \exists n_y \Gamma^0 \Gamma^y \quad \text{s.t.} \quad (P \otimes \mathbf{1}_2) n_y \Gamma^0 \Gamma^y = n_y \Gamma^0 \Gamma^y (P_\perp \otimes \mathbf{1}_2).$

¹⁰ They are a simple extension of the results given in [18–20].

The internal space chiral BC In the case of $U = \pm\gamma_{\text{in}}$, the projection operators are the internal space chiral projection operators,

$$P = \frac{1}{2} \begin{pmatrix} \mathbf{1} \pm \gamma_{\text{in}} & \\ & \mathbf{1} \pm \gamma_{\text{in}} \end{pmatrix}, \quad \frac{\mathbf{1} \pm \Gamma_{\text{in}}}{2} = P \otimes \mathbf{1}_2. \quad (4.18)$$

Hence, we call this class as the inner space chiral BC. In contrast to the 4D chiral BC, *this class exists for only even dimensions.*

We note that the action can be decomposed by the projections operators. By the 4D chiral projection operators, the action decomposed as

$$S = \int_{M_4} d^4x \int_{\Omega} d^d y \bar{\Psi}_L(x, y) i\Gamma^\mu \partial_\mu \Psi_L(x, y) + \bar{\Psi}_R(x, y) i\Gamma^\mu \partial_\mu \Psi_R(x, y) \\ + \bar{\Psi}_R(x, y) (i\Gamma^y \partial_y - M) \Psi_L(x, y) + \bar{\Psi}_L(x, y) (i\Gamma^y \partial_y - M) \Psi_R(x, y). \quad (4.19)$$

For even dimensions, the action can alternatively be decomposed by the internal space chiral projection operators as follow:

$$S = \int_{M_4} d^4x \int_{\Omega} d^d y \bar{\Psi}_+(x, y) (i\Gamma^\mu \partial_\mu - M) \Psi_+(x, y) + \bar{\Psi}_-(x, y) (i\Gamma^\mu \partial_\mu - M) \Psi_-(x, y) \\ + \bar{\Psi}_-(x, y) i\Gamma^y \partial_y \Psi_+(x, y) + \bar{\Psi}_+(x, y) i\Gamma^y \partial_y \Psi_-(x, y). \quad (4.20)$$

The fields $\Psi_{L/R}(x, y)$ and $\Psi_{\pm}(x, y)$ are the 4D chiral fields and the internal space chiral fields respectively defined by

$$\Psi_{L/R}(x, y) = \frac{\mathbf{1} \mp \Gamma_{4\text{D}}}{2} \Psi(x, y), \quad \Psi_{\pm}(x, y) = \frac{\mathbf{1} \pm \Gamma_{\text{in}}}{2} \Psi(x, y). \quad (4.21)$$

Dependency on the 4D Minkowski space-time and the internal space is clearly separated by these projections.

4.4 Summary: Boundary condition

In this chapter, we have classified the boundary condition to discuss the well-definedness of the supercharges.

First, we introduced the non-local boundary condition (4.1) which is derived by the variation principle of the action. The non-local boundary condition ensures that the d dimensional realization of the $\mathcal{N} = 2$ SQM (3.20) is always well-defined. We note that the non-local boundary condition is the condition that the surface integral vanishes. Although we can capture various the boundary condition, there is technical difficulty due to the huge variety of the boundary condition.

Therefore, we focus on the locally defined boundary condition at each point on the boundary.

Theorem. *Given d dimensional realization of the $\mathcal{N} = 2$ SQM such as (3.20), the boundary condition which has properties that*

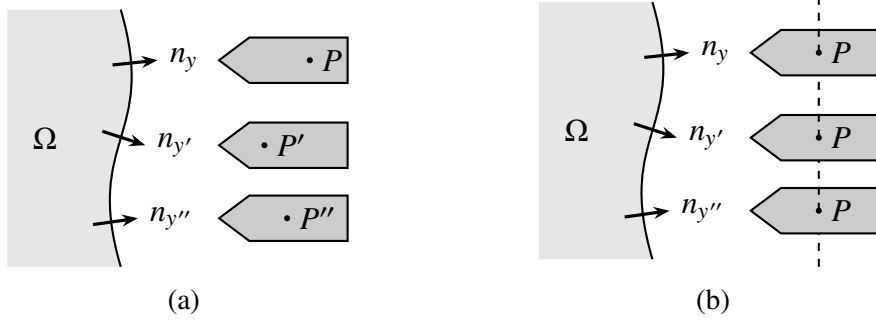


Figure 4.3: The figures of a schematic image of the parameter space of the boundary condition. The projection operator P is defined at each point in the boundary and depends on the normal vector n_y . Also, there is a parameter space of the boundary condition BC_y at each point on the boundary. We illustrate the parameter space BC_y as diamond-like shape. In general, the projection operator is different at each point on the boundary illustrated in figure (a). There are special classes that the BC for all point illustrated in figure (b).

- i) the BC ensures Hamiticity of the supercharges
- ii) the BC ensures the $N = 2$ supersymmetry
- iii) the BC is a local first-ordered equation

are characterized by the projection operator P as

$$P \begin{pmatrix} f_\alpha^{(n)}(y) \\ g_\alpha^{(n)}(y) \end{pmatrix} = 0, \quad \text{where } P = \frac{1}{2} \begin{pmatrix} \mathbf{1} - NUN & \\ & \mathbf{1} + U \end{pmatrix}, \quad (4.22)$$

at each point on the boundary.

Such the boundary has the huge parameter space BC_y at each point in the boundary. The parameter space BC_y is given by the disjoint union of Grassmannians (4.10) and is the same for all points on the boundary.

We focused on the two special classes of the boundary condition.

Theorem. *Given d dimensional realization of the $N = 2$ SQM such as (3.20), there are only two boundary condition which has the properties i)-iii) for all point on the boundary.*

The 4D chiral BC *The projection operators are the 4D chiral projection operators.*

The internal space chiral BC *The projection operators are the inner space chiral projection operators.*

The internal space chiral BC exists for only even dimensions.

We stress that the variation principle of the action derives the property i). Moreover, the property ii) is equivalent to the 4D Lorentz covariance of the boundary condition. That is, we also find the boundary condition of the $(4 + d)$ -dimensional Dirac field $\Psi(x, y)$;¹¹

¹¹ The proof is given in [Appendix C](#).

Theorem. *Given a higher $(4 + d)$ -dimensional Dirac field $\Psi(x, y)$ on the flat space, the boundary condition of the Dirac field which has properties that*

- i) the BC is consistent with the variation principle of the action*
- ii) the BC keeps the 4D Lorentz covariance*
- iii) the BC is a local primary equation*

are characterized by the projection operator P as

$$(P \otimes \mathbf{1}_2)\Psi(x, y) = 0 \quad (4.23)$$

at each point on the boundary, where P is given in (4.22). We take the representation between the higher dimensional gamma matrices as (2.3.2).

We should emphasize that we only focus on the consistency of the boundary condition and the supersymmetric quantum mechanical structure. Therefore, there are trivial classes in this clarification which have no nontrivial solution. However, it is important to classify the boundary condition which provides the nontrivial solution of the mode functions and also the higher dimensional Dirac field. We skip the discussion in this chapter.

We show the parameter space of the boundary condition of the higher dimensional Dirac field is compact. Therefore, it will be expected that the *geometric phase* (or the *Pancharatnam–Berry phase*) appears [41]. Given the parameter space is restricted on some manifold, the curvature of the manifold might appear as a phase of the states by adiabatic evolution along the closed path on the parameter space. The phase is called geometric phase or Pancharatnam–Berry phase. In [42], it is proved that the geometric phase appears from the parameter space of the boundary condition. In this chapter, we show that there is the parameter space of the boundary condition restricted on the Grassmannian. Therefore, the nontrivial geometric phase will appear respecting the parameter space of the boundary condition by the adiabatic evolution.

Chapter 5

Models

As we mentioned in [Chapter 3](#), the $\mathcal{N}$ -extended SQM supercharges are not always well defined even if it satisfies the $\mathcal{N}$ -extended SQM algebra. If the supercharges are not well-defined, it cannot generate the additional isospectral pairs. In this chapter, we construct explicit models with simple shaped internal spaces and the simple boundary conditions. There are examples that the supercharges are not generate the whole isospectral pairs even there exists the additional isospectral pairs.

This chapter is organized as follows: we construct a model with a hyperrectangular internal space and the 4D chiral BC in the next section. We show that we can naturally construct the mode functions respecting chiral partners by the $\mathcal{N} = 2$ SQM supercharges. The construction is highly applicable to various models, and we will use this construction. The $\mathcal{N}$ -extended SQM generates the whole additional isospectral pairs in this model. In [Section 5.2](#), we construct another simplest model which has the toroidal internal space. We will solve the mode functions with the periodic and twisted boundary condition. If the twisted boundary condition is taken, the $\mathcal{N}$ -extended SQM supercharges will be not well-defined. Therefore, the additional isospectral pairs cannot be generated by the $\mathcal{N}$ -extended SQM supercharges if the twisted boundary condition is taken. It indicates that the $\mathcal{N}$ -extended SQM are sensitive for the geometric symmetry of the theory. In [Section 5.3](#), we construct the model with a cylindrical internal space to show the complicated dependence of the $\mathcal{N}$ -extended SQM on the boundary condition. The cylinder has an unconnected boundary, i.e., an up-side and a down-side. We solve the mode functions with taking different boundary condition for each side. In any model which we introduce in this section, the $\mathcal{N}$ -extended SQM generates the additional isospectral pairs if additional degeneracy exists. In [Section 5.4](#), we consider the five-dimensional model with intervally interval space that we discussed in [\[18\]](#). In this model, the $\mathcal{N}$ -extended SQM does not exist. However, it is worth to discuss because of the 4D spectrum with the completely determined profiles of the mode functions. The results given in this chapter will be reported in [\[35\]](#).

5.1 Hyperrectangular internal space

First, we construct a model with the d dimensional hyperrectangular internal space with the 4D chiral BC. This model holds the maximal $\mathcal{N}$ -extended SQM even if the boundary exists.

In addition, it is obvious how the geometric supercharges generate the additional isospectral pairs. We will construct the mode functions explicitly and show how the supercharges generate them. The construction method of the mode functions does not depend on the shape of the internal space and is highly applicable to other models.¹

Shape Hyperrectangular internal space (even dimensions for simplicity)

BC The Dirichlet boundary condition (the 4D right-handed component vanishes at all points on the boundary)

Properties

- The additional isospectral pairs exist
- The maximal $\mathcal{N}$ -extended SQM holds
- The massless KK modes are chiral (only the 4D left-handed fields exist)
- The multiple massless KK modes exist
- The massless KK modes are localized on the extra-dimensions
- It seems to be difficult to determine the massless KK mode profiles completely

5.1.1 Set-up

We consider the $4 + d$ dimensional space-time with the d dimensional hyperrectangular internal space Ω ,

$$\Omega = [0, \pi L_1] \times \cdots \times [0, \pi L_d], \quad (5.1)$$

where L_k ($k = 1, \dots, d$) are the positive real constants. For simplicity, we assume even dimensions. We take the 4D chiral BC, the 4D right-handed fields vanishing, for all points on boundary,

$$A^\dagger f_\alpha^{(n)}(y) = 0, \quad g_\alpha^{(n)}(y) = 0, \quad (5.2)$$

for all n and α . The boundary condition of the left-handed mode functions is derived from the SQM relation (3.19). We note that the equivalent boundary condition of the higher dimensional Dirac field is the 4D chiral Dirichlet boundary condition,

$$\Psi_R(x, y) = 0, \quad (5.3)$$

at all points on the boundary.

¹ We will use this construction method for the posterior models.

5.1.2 Massive KK modes

We construct the massive KK mode functions explicitly. The n -th massive KK mode functions $f_\alpha^{(n)}(y)$ and $g_\alpha^{(n)}(y)$ with KK mass $m_n^2 > 0$ are given as

$$f_\alpha^{(n)}(y) = \frac{1}{m_n} A g_\alpha^{(n)}(y), \quad g_\alpha^{(n)}(y) = h^{(n)}(y) e_{s_1 \dots s_{d/2}}, \quad (5.4)$$

where $\alpha = s_1 \dots s_{d/2}$ and $s_m = \pm$ (for $m = 1, \dots, d/2$).² The symbol $e_{s_1 \dots s_{d/2}}$ are the eigenvectors of the internal space gamma matrices,

$$\gamma_{(m)} e_{s_1 \dots \pm \dots s_{d/2}}^{m\text{-th}} = \pm e_{s_1 \dots \pm \dots s_{d/2}}^{m\text{-th}}, \quad \gamma_{(m)} = i \gamma^{y_{2m-1}} \gamma^{y_{2m}}, \quad (5.5)$$

for $m = 1, \dots, d/2$. The scalar functions $h^{(n)}(y)$ are the eigenfunctions which has the eigenvalues m_n^2 of the screened Poisson equation

$$(-\partial_y^2 + M^2) h^{(n)}(y) = m_n^2 h^{(n)}(y) \quad \forall y \in \Omega, \quad (5.6)$$

with the Dirichlet boundary condition,

$$h^{(n)}(y) = 0 \quad \forall y \in \partial\Omega. \quad (5.7)$$

The scalar functions $h^{(n)}(y)$ are explicitly given by the sine functions, and the n -th KK masses are shifted by the square of mass,

$$h^{(n)}(y) = \prod_{k=1}^d \sqrt{\frac{2}{\pi L_k}} \sin\left(\frac{n_k}{L_k} y_k\right), \quad m_n^2 = M^2 + \sum_{k=1}^d \frac{n_k^2}{L_k^2}. \quad (5.8)$$

where $n_k = 1, 2, \dots$ ($k = 1, \dots, d$).

We can show that every supercharges are Hermitian and consistent with the BC. Thus, *the N -extended SQMs maximally hold* [35]. The supercharges generates the mode functions at each KK level with flipping L/R and s_j . The diagram of the action of the N -extended supercharges on the ten-dimensional mode functions is schematically illustrated in Figure 5.1.

We note that we will use this construction of the mode functions later.

5.1.3 Massive KK modes

There are no 4D right-handed massless KK modes in this model due to the boundary condition. Thus, *there exist only the 4D left-handed massless KK modes*. However, it seems to be difficult to completely determine the profiles of the massless KK mode functions.

We explain it by explicitly constructing the massless KK mode functions. In fact, one can construct the left-handed massless KK mode functions $f_\alpha^{(0)}(y)$ by

$$f_\alpha^{(0)}(y) = A^\dagger G(y) e_{s_1 \dots s_{d/2}}, \quad (5.9)$$

² We should note that the index α of the right-handed mode functions $g_\alpha^{(n)}(y)$ indicate the position of the component (scalar function $h(y)$) but the left-handed mode functions $f_\alpha^{(n)}(y)$ does not. The index describes as the partners of the right-handed mode functions $g_\alpha^{(n)}(y)$ satisfy the relation (2.64).

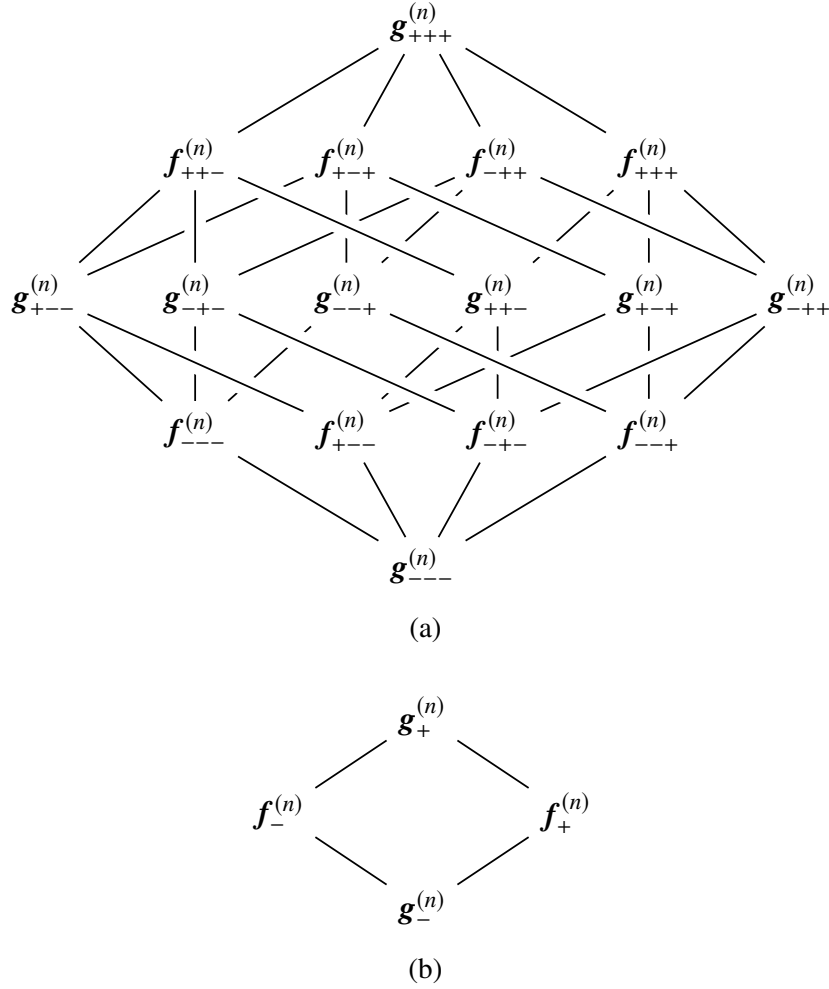


Figure 5.1: Figure (a) and figure (a) are the diagrams of the action of the $\mathcal{N}$ -extended supercharges on the ten-dimensional and six-dimensional mode functions respectively. The lines denote the action of the supercharges of the $\mathcal{N}$ -extended SQM. The $\mathcal{N}$ -extended SQM generates the whole diagram.

where $G(y)$ is any solution of the d dimensional screened Poisson equation defined on the internal space Ω ,

$$(-\partial_y^2 + M^2)G(y) = 0 \quad y \in \Omega. \quad (5.10)$$

We should emphasize that there is any more condition for the mode function. Thus, any solution of the screened Poisson equation is allowed, and there exist infinitely many free parameters characterizing its profile. For example, one can give such the solution as a series of the modified Bessel functions I_n and K_n and the hyperspherical harmonics Y_m^ℓ on S^{d-1} ,

$$G(y) = \sum_{\ell=1}^{\infty} \sum_m [c_{\ell m}^1 I_{(d+2\ell-2)/2}(|y-a|) + c_{\ell m}^2 K_{(d+2\ell-2)/2}(|y-a|)] |y-a|^{1-d/2} Y_m^\ell(\vec{\theta}), \quad (5.11)$$

where $c_{\ell m}^k$ ($k = 1, 2$) are complex constants and $a \in \mathbf{R}^d$.³ These constants are the free parameter of this theory.

This is a consequence of the equivalency between the equation which determines the profile and the boundary condition. We note that the equation which determines the profile of the massless modes is given by

$$A f_{\alpha}^{(0)}(y) = 0, \quad (5.12)$$

on the internal space Ω , and the boundary condition of the massless modes $f_{\alpha}^{(0)}(y)$ is given by

$$A f_{\alpha}^{(0)}(y) = 0, \quad (5.13)$$

for all points on the boundary. These are clearly identical. Therefore, one cannot determine the profile completely without any further condition. In [20], we take Ansatz for the massless KK modes to determine its profile. This difficulty arises in any model with any shape of the internal space if one takes the 4D chiral BC or internal space chiral BC for all point on the boundary.

5.2 Toroidal internal space

In this section, we show models that the $\mathcal{N}$ -extended SQM maximally holds even if there is no boundary.⁴

First, we consider the toroidal internal space with periodic boundary condition. In this model, there are accidental isospectral pairs in the 4D spectrum, that is, the left-moving and the right-moving degeneracy. In spite of it, the $\mathcal{N} = 2$ SQM and the $\mathcal{N}$ -extended SQM generate the whole isospectral pairs. Of course, the $\mathcal{N}$ -extended SQM maximally holds.

Shape Toroidal internal space (even dimensions for simplicity)

BC Periodic boundary condition

Properties

- The additional and accidental isospectral pairs exist
- The maximal $\mathcal{N}$ -extended SQM holds
- The massless KK modes do not exist with nonzero mass M
- If $M = 0$, there exist massless KK modes but they are not 4D chiral

Subsequently, we consider the toroidal internal space with twisted boundary condition as a generalization of the model. In this model, there are additional isospectral pairs but not accidental isospectral pairs. However, the $\mathcal{N}$ -extended SQM is completely broken. This model indicates the complicated dependency of the $\mathcal{N}$ -extended SQM on the boundary condition.

³ The hyperspherical harmonics Y_m^{ℓ} is introduced in [Section E.1](#).

⁴ We emphasize again that the $\mathcal{N} = 2$ SQM holds always.

Shape Toroidal internal space (even dimensions for simplicity)

BC Twisted boundary condition (not including periodic boundary condition)

Properties

- The additional isospectral pairs exist
- The maximal $\mathcal{N}$ -extended SQM is broken
- The massless KK modes do not exist even if mass is zero, $M = 0$

5.2.1 Set-up: periodic boundary condition

We consider the d dimensional toroidal internal space with the periodic boundary condition. Such the internal space is given by the hyperrectangle,

$$\Omega = [0, 2\pi R_1) \times \cdots \times [0, 2\pi R_d), \quad (5.14)$$

with the periodic boundary condition for each y_k direction ($k = 1, \dots, d$),

$$f_\alpha^{(n)}(y_1, \dots, y_k, \dots, y_d) = f_\alpha^{(n)}(y_1, \dots, y_k + 2\pi R_k, \dots, y_d) \quad (5.15)$$

$$g_\alpha^{(n)}(y_1, \dots, y_k, \dots, y_d) = g_\alpha^{(n)}(y_1, \dots, y_k + 2\pi R_k, \dots, y_d) \quad (5.16)$$

for all n and α . The R_k ($k = 1, \dots, d$) are the positive real constants. We note that the equivalent boundary condition of the higher dimensional Dirac field $\Psi(x, y)$ is explicitly given as,

$$\Psi(x, y_1, \dots, y_k, \dots, y_{d-1}) = \Psi(x, y_1, \dots, y_k + 2\pi R_k, \dots, y_{d-1}), \quad (5.17)$$

for each y_k direction.

5.2.2 Massive KK modes

The mode functions are proportional to the plane wave because the system is periodic. We can take the massive mode functions as,

$$f_\alpha^{(n)}(y) = \frac{1}{m_n} A g_\alpha^{(n)}(y), \quad (5.18)$$

$$g_\alpha^{(n)}(y) = \prod_{k=1}^d \sqrt{\frac{1}{2\pi R_k}} e^{s_1 \dots s_{d/2}} \exp \left[i \sum_{k=1}^d \frac{n_k}{R_k} y_k \right], \quad (5.19)$$

without loss generality. We note that at least one of n_k ($k = 1, \dots, d$) is not zero to be massive. It is easy to verify that the n -th KK mass is shifted by a square of mass M^2 ,

$$m_n^2 = M^2 + \sum_{k=1}^d \frac{n_k^2}{R_k^2}. \quad (5.20)$$

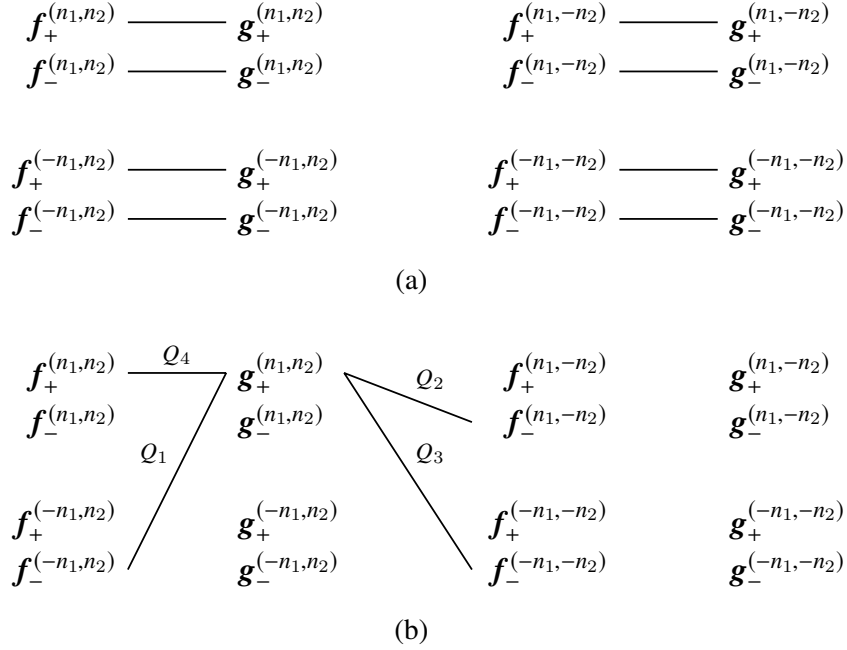


Figure 5.2: The lines indicate the action of the supercharges of the six-dimensional space-time with toroidal internal space. Figure (a) illustrates the action of the $\mathcal{N} = 2$ SQM supercharges which generate the chiral partners. Figure (b) illustrates the action of the $\mathcal{N}$ -extended SQM supercharges acting on the mode function $\mathbf{g}_+^{(n_1, n_2)}(y)$. Since given the mode functions $\mathbf{f}_\pm^{(n_1, n_2)}(y)$ and $\mathbf{g}_\pm^{(n_1, n_2)}(y)$, the whole mode functions are generated by the $\mathcal{N} = 2$ SQM and the $\mathcal{N}$ -extended SQM.

There is the accidental degeneracy between the $\pm n_k$ modes for each y_k direction ($k = 1, \dots, d$) except massless KK modes.⁵ Thus, the massive KK modes degenerates in $2 \times 2^{\lfloor d/2 \rfloor} \times 2^d$ degrees and the massless KK modes degenerates in $2 \times 2^{\lfloor d/2 \rfloor}$ degree in total.

We emphasize that the SQM supercharges have a different action in this model.

The $\mathcal{N} = 2$ SQM The supercharges of the $\mathcal{N} = 2$ SQM generate the chiral partners.

The $\mathcal{N}$ -extended SQM The $\mathcal{N}$ -extended SQM supercharges generates the *accidental isospectral pairs* at each KK level. However, the $\mathcal{N}$ -extended SQM supercharges cannot generate the isospectral pairs only in the left-moving or right-moving modes because the $\mathcal{N}$ -extended SQM supercharges are not closed on the left-moving (right-moving) mode functions.

Given mode function $\mathbf{f}_\alpha^{(n)}(y)$ and $\mathbf{g}_\alpha^{(n)}(y)$ with fixing α at each KK level, the whole isospectral pairs is generated by the $\mathcal{N} = 2$ SQM and the $\mathcal{N}$ -extended SQM. We illustrate the six-dimensional example in Figure 5.2.

⁵ The degeneracy is given by that between the left-moving and the right-moving modes.

5.2.3 Massless KK modes

There is a constant profile on the torus. However, the profile does not provide the massless KK modes and not provide the massless 4D fields because the constant profile has the eigenvalue M^2 of the screen Poisson equation (3.29). These modes are not 4D chiral. As the result, there are $2^{\lfloor d/2 \rfloor} \times d$ 4D Dirac fields with mass M in the 4D spectrum. If one takes $M = 0$, these modes will be massless but not be chirally asymmetric.

5.2.4 Set-up: twisted boundary condition

We generalize the model to show the $\mathcal{N}$ -extended SQM supercharges will be not well-defined if the reflection symmetry is violated.

We consider the d dimensional toroidal internal space with the twisted boundary condition. Such internal space is given by the hyperrectangle,

$$\Omega = [0, 2\pi R_1) \times \cdots \times [0, 2\pi R_d), \quad (5.21)$$

with the twisted boundary condition for each y_k direction ($k = 1, \dots, d$),

$$f_\alpha^{(n)}(y_1, \dots, y_k, \dots, y_d) = e^{i2\pi\theta_k} f_\alpha^{(n)}(y_1, \dots, y_k + 2\pi R_k, \dots, y_d) \quad (5.22)$$

$$g_\alpha^{(n)}(y_1, \dots, y_k, \dots, y_d) = e^{i2\pi\theta_k} g_\alpha^{(n)}(y_1, \dots, y_k + 2\pi R_k, \dots, y_d) \quad (5.23)$$

where $0 < \theta_k < 1$ ($k = 1, \dots, d$). We note the equivalent boundary condition of the higher dimensional Dirac field $\Psi(x, y)$ is explicitly given as

$$\Psi(x, y_1, \dots, y_k, \dots, y_d) = e^{i2\pi\theta_k} \Psi(x, y_1, \dots, y_k + 2\pi R_k, \dots, y_d), \quad (5.24)$$

for each y_k direction. For simplicity, we assume $\theta_k \neq \theta_h$ for all $k, h = 1, \dots, d$.

5.2.5 Massive KK modes

We can show that the mode functions can be expanded by Fourier series even one takes the twisted boundary condition. Therefore, the mode functions are given as

$$f_\alpha^{(n)}(y) = \frac{1}{m_n} A g_\alpha^{(n)}(y), \quad (5.25)$$

$$g_\alpha^{(n)}(y) = \prod_{k=1}^d \sqrt{\frac{1}{2\pi R_k}} e^{s_1 \dots s_{d/2}} \exp \left[i \sum_{k=1}^d \frac{n_k + \theta_k}{R_k} y_k \right]. \quad (5.26)$$

where the quantum number n_k is shifted by θ_k ($k = 1, \dots, d$). The n -th KK mass is also shifted by θ_k ,

$$m_n^2 = M^2 + \sum_{k=1}^d \frac{(n_k + \theta_k)^2}{R_k^2}. \quad (5.27)$$

It should be stressed that *the $\mathcal{N}$ -extended SQM is completely broken* except the $\mathcal{N} = 2$ SQM sector. The twisted boundary condition violates Hermiticity of the reflection operators R_k ($k = 1, \dots, d$). As the result, the transformations (3.35) are not unitary and the $\mathcal{N}$ -extended SQM is broken.

5.2.6 Massless KK modes

There is no constant profile in this model because the twisted boundary condition prohibits the constant profile. The minimum of the momentum is θ_k or $\theta_k - 1$ for each y_k direction ($k = 1, \dots, d$). Therefore, the KK mass of the lowest KK mode is given as,

$$m_{\text{lowest}}^2 = M^2 + \sum_{k=1}^d \frac{\tilde{\theta}_k^2}{R_k^2} \quad \text{where } \tilde{\theta}_k = \begin{cases} \theta_k & \text{for } 0 < \theta_k \leq 1/2 \\ \theta_k - 1 & \text{for } 1/2 < \theta_k < 1 \end{cases}. \quad (5.28)$$

The degrees of degeneracy depend on the value of θ_k ($k = 1, \dots, d$). If $\theta_k \neq 1/2$ for all k , there are $2^{\lfloor d/2 \rfloor} \times d$ isospectral pairs in the lowest KK modes. The degrees of the degeneracy are the same as the rectangular internal space model.

5.3 Two-dimensional cylindrical internal space

In the previous section, the $\mathcal{N}$ -extended SQM is completely broken by violating the reflection symmetry. However, in general, if the reflection symmetry is partially broken, the $\mathcal{N}$ -extended SQM is not necessarily completely broken. In this section, we consider the two-dimensional cylindrical internal space model to show complicated dependence on the boundary condition of the $\mathcal{N}$ -extended SQM.

Shape Two-dimensional cylindrical internal space

BC Twisted boundary condition with the 4D chiral BC or the internal space chiral BC

Properties

- The additional isospectral pairs exist in some cases
- The $\mathcal{N}$ -extended SQM partially holds
- The massless KK modes do not exist (in the mentioned models)

5.3.1 Set-up

The two-dimensional cylindrical inner space is given as the open rectangular inner space,

$$\Omega = [0, \pi L] \times [0, 2\pi R], \quad (5.29)$$

with the twisted boundary condition for the y_2 direction,

$$f_{\alpha}^{(n)}(y_1, y_2) = e^{i2\pi\theta} f_{\alpha}^{(n)}(y_1, y_2 + 2\pi R), \quad (5.30)$$

$$g_{\alpha}^{(n)}(y_1, y_2) = e^{i2\pi\theta} g_{\alpha}^{(n)}(y_1, y_2 + 2\pi R), \quad (5.31)$$

for all n and α . The L and R are the non-zero real constants. We take θ to be $0 < \theta < 1$. In the cylinder, there are two boundaries which are not connected. As we showed, the boundary condition has the huge parameter space at each point on the boundary. For

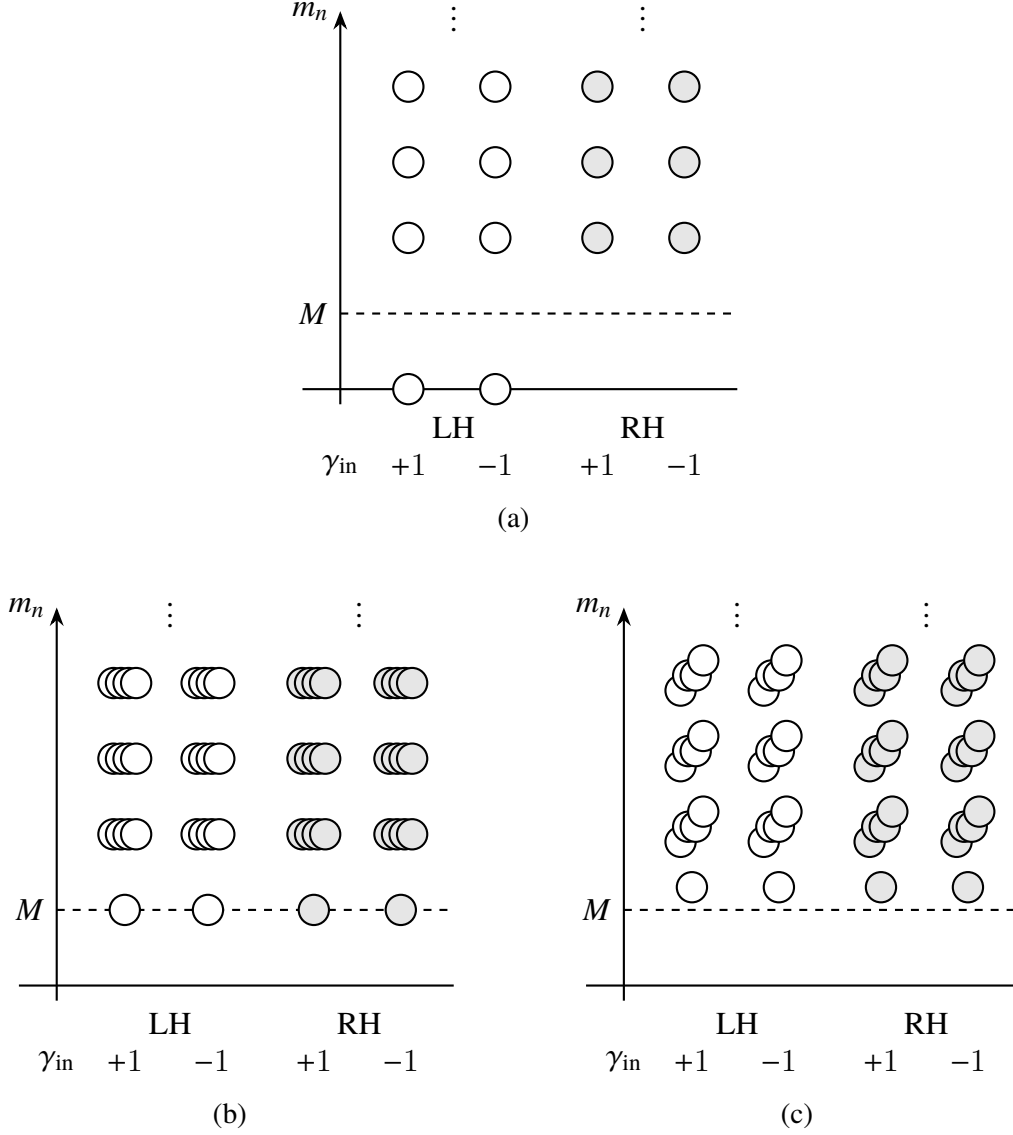


Figure 5.3: The schematic illustration of the 4D spectrum of the six-dimensional theory presented in [Section 5.1](#) and [Section 5.2](#). Figure (a) illustrates the rectangular internal space mode in [Section 5.1](#). There are two massless KK modes which are the 4D left-handed. Figure (b) and (c) illustrate the toroidal internal space mode in [Section 5.2](#). There is no massless KK mode due to mass M and the lower bound of the KK mass m_n is mass M . Figure (b) is the periodic boundary condition and figure (c) is the twisted boundary condition. The twisting parameters θ_k solve the accidental degeneracy of the massive KK modes in general (figure (c)).

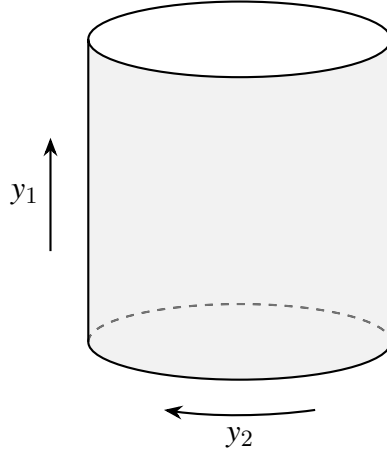


Figure 5.4: There are two boundaries in the cylinder which are not connected.

simplicity, we focus on the BC for all point in the boundary, i.e., the 4D chiral BC and the inner chiral BC.

We note the equivalent boundary condition of the higher dimensional Dirac field $\Psi(x, y)$ is explicitly given as

$$\Psi(x, y_1, y_2) = e^{i2\pi\theta}\Psi(x, y_1, y_2 + 2\pi R). \quad (5.32)$$

5.3.2 Massive KK modes

We discuss the three practical models with combinations of the boundary condition which breaks the $\mathcal{N}$ -extended SQM. One can obtain the remained combinations by exchanging the boundary condition of each side.⁶

In any model, the n -th KK mass is shifted by a square of mass M^2 ,

$$m_n = M^2 + k^2 + p_n^2, \quad (5.33)$$

where k is the momentum for the y_1 direction and p_n is the y_2 direction. The momentum p_n values,

$$p_n = \frac{n + \theta}{R}, \quad n \in \mathbf{Z}. \quad (5.34)$$

For simplicity, we fix the momentum p_n .

In this section, we denote $\mathbf{f}_{\pm}^{(n)}(y)$ and $\mathbf{g}_{\pm}^{(n)}(y)$ by the eigenstates of the γ_{in} with $\gamma_{\text{in}}\mathbf{f}_{\pm}^{(n)}(y) = \pm\mathbf{f}_{\pm}^{(n)}(y)$ and $\gamma_{\text{in}}\mathbf{g}_{\pm}^{(n)}(y) = \pm\mathbf{g}_{\pm}^{(n)}(y)$. We should emphasize that the eigenstates $\mathbf{f}_{\pm}^{(n)}(y)$ (or $\mathbf{g}_{\pm}^{(n)}(y)$) are generally related by the boundary conditions. That is, the two mode functions $\mathbf{f}_{\pm}^{(n)}(y)$ are not linearly independent in general. Therefore, we should pay

⁶ We note that there are no degeneracy between the the right-going and the left-going modes with $\theta \in (0, 1)$.

attention whether the mode functions are linearly independent due to complicated boundary condition.

Before listing the models, we comment about the total supercharges of the cylindrical internal space models. Formally, there are four $\mathcal{N}$ -extended supercharges in this models,

$$Q_1 = \begin{pmatrix} A^\dagger \gamma_{\text{in}} \gamma^{y_1} R_1 & -\gamma_{\text{in}} \gamma^{y_1} R_1 A \end{pmatrix}, \quad Q_2 = \begin{pmatrix} A^\dagger \gamma_{\text{in}} \gamma^{y_2} R_2 & -\gamma_{\text{in}} \gamma^{y_2} R_2 A \end{pmatrix}, \quad (5.35)$$

$$Q_3 = \begin{pmatrix} A^\dagger \gamma_{\text{in}} P & -\gamma_{\text{in}} P A \end{pmatrix}, \quad Q_4 = i(-1)^F Q_+. \quad (5.36)$$

We note that the supercharges Q_2 and Q_3 are not well-defined because of the twisted boundary condition for the y_2 direction. And, the supercharge Q_4 is always well-defined because it is a $\mathcal{N} = 2$ SQM supercharge Q_- itself. Therefore, we focus on whether the supercharge Q_1 is well-defined.

The internal space chiral and the 4D chiral BC

First, we show the spectrum of the combination of the inner space chiral BC and the 4D chiral BC,

$$f_+^{(n)}(y) = 0, \quad f_-^{(n)}(y) = 0, \quad \text{at } y_1 = 0, \quad (5.37)$$

$$f_+^{(n)}(y) = 0, \quad g_+^{(n)}(y) = 0, \quad \text{at } y_1 = L. \quad (5.38)$$

The mode functions $f_+^{(n)}(y)$ are a sine function vanishing at each boundary. Thus, the momentum k of the mode functions $f_+^{(n)}(y)$ and its 4D chiral partners is promotional to the positive integer,

$$k = \frac{n'}{L}, \quad n' = 1, 2, \dots \quad (5.39)$$

The mode functions $f_-^{(n)}(y)$ are also given by the sine function. The momentum k should be chosen to be consistent with the boundary conditions at $y_1 = L$. That is, the momentum k of the mode functions $f_-^{(n)}(y)$ and 4D chiral partners is a solution of the non-linear equation,

$$k = -p \tan(\pi k L). \quad (5.40)$$

It implies that the mode functions $f_\pm^{(n)}(y)$ have different KK masses and cannot be an isospectral pair. The well-defined supercharges generates isospectral pairs. If there is no isospectral pair, the supercharges are not well-defined. Therefore, the supercharge Q_1 is *not* well-defined in this model. We note that the combination leads to

- i) the two massive KK mode functions $f_\pm^{(n)}$ (and also $g_\pm^{(n)}$) are linearly independent
- ii) there is *no additional degeneracy*
- iii) only the supercharges Q_4 is well-defined for the $\mathcal{N}$ -extended SQM

The 4D chiral and the 4D chiral BC

The 4D chiral BC and the 4D chiral BC combination is given by

$$f_+^{(n)}(y) = 0, \quad f_-^{(n)}(y) = 0, \quad \text{at } y_1 = 0, \quad (5.41)$$

$$g_+^{(n)}(y) = 0, \quad g_-^{(n)}(y) = 0, \quad \text{at } y_1 = L. \quad (5.42)$$

The mode functions are given by the sine functions. With this boundary condition, the momentum k is a solution of the non-linear equation,

$$k = \sqrt{M^2 + p^2} \tan(\pi k L). \quad (5.43)$$

The coefficient is shifted by square of mass M^2 due to the boundary condition. At $y_1 = L$, the boundary condition implies that the two left-handed mode functions $f_\pm^{(n)}(y)$ are mixed at $y_1 = L$. That is, *the two massive KK mode functions $f_\pm^{(n)}(y)$ are not linearly independent*. Therefore, the supercharge Q_1 is *not* well-defined. We note that the combination leads to

- i) *the two massive KK mode functions $f_\pm^{(n)}$ (and also $g_\pm^{(n)}$) are not linearly independent*
- ii) *there is no additional degeneracy*
- iii) *only the supercharges Q_4 is well-defined for the $\mathcal{N}$ -extended SQM*

The internal space chiral and the internal space chiral BC

The internal space chiral BC and the internal space chiral BC combination is given by

$$f_-^{(n)}(y) = 0, \quad g_-^{(n)}(y) = 0, \quad \text{at } y_1 = 0, \quad (5.44)$$

$$f_+^{(n)}(y) = 0, \quad g_+^{(n)}(y) = 0, \quad \text{at } y_1 = L. \quad (5.45)$$

The mode functions are also given by the sine functions. The momentum k is a solution of the non-linear equation,

$$k = -p \tan(\pi k L). \quad (5.46)$$

The two eigenstates $f_\pm^{(n)}(y)$ (and $g_\pm^{(n)}(y)$) has the same momentum k and linearly independent. Hence, there are additional isospectral pairs and *the supercharge Q_1 is well-defined*. We note that the combination leads to

- i) *the two massive KK mode functions $f_\pm^{(n)}$ (and also $g_\pm^{(n)}$) are linearly independent*
- ii) *there is an additional degeneracy*
- iii) *only two supercharges Q_1 and Q_4 of the $\mathcal{N}$ -extended SQM supercharges are well-defined*

BC	Solutions	Total degeneracy	Well-defined supercharges
$(L, +)$	4	2	Q_4
(L, R)	2	2	Q_4
$(-, +)$	2	4	Q_1 and Q_4

Table 5.1: A brief summary of the [Section 5.3](#). There are the additional isospectral pairs in the combination $(-, +)$. If additional isospectral pairs exist, the part of the $\mathcal{N}$ -extended SQM remains and generates the whole degeneracy.

In the first two cases, the $\mathcal{N}$ -extended SQM completely is broken except the $\mathcal{N} = 2$ SQM sector. We note that, in the first case, there are two linearly independent solutions $f_{\pm}^{(n)}(y)$ (or $g_{\pm}^{(n)}(y)$). However, the individual modes have different KK masses. Therefore, only the supercharge Q_4 is well-defined for the $\mathcal{N}$ -extended SQM. In second case, in contrast to the first case, *there are no two independent solutions*. Therefore, the $\mathcal{N}$ -extended SQM completely is broken except the $\mathcal{N} = 2$ SQM sector.

In these two cases, the $\mathcal{N}$ -extended SQM is broken because *there is no independent solutions which can be the additional isospectral pairs*.

In the last case, the $\mathcal{N}$ -extended SQM is partially broken *even though the additional isospectral pairs exist*. On the other hand, the well-defined supercharges Q_1 and Q_4 of the $\mathcal{N}$ -extended SQM can generate the whole mode functions (see [Figure 5.1\(b\)](#)). Thus, the $\mathcal{N} = 2$ SQM and the partial $\mathcal{N}$ -extended SQM still generate the whole degeneracy and provides the additional isospectral pairs in the third case.

We also note that the reflection symmetry of the models. In previous two cases, the reflection symmetry for the y_1 direction is violated by the boundary condition. Hence, the reflection operator R_1 is not Hermitian and the supercharge Q_1 cannot be Hermitian. On the other hand, in the last model, it is not violated. Therefore, the supercharge Q_1 can be Hermitian.

We summarize these cases in [Table 5.1](#).

5.3.3 Massless KK modes

The cases which we mentioned have no massless KK modes.

5.4 One-dimensional interval internal space

Finally, we show the most simple example; the interval internal space model [[18](#)]. This system is, exceptionally, not complicated. For example, the $\mathcal{N}$ -extended SQM and additional degeneracy do not exist.⁷ However, it is still worth to discuss because we can completely classify the 4D spectrum. We will solve the whole mode functions explicitly with any boundary condition in this section.

Shape One dimensional interval internal space

⁷ We discussed the reason in [Section 2.4](#).

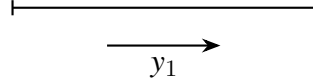


Figure 5.5: The illustration of the interval internal space.

BC 4D chiral BC

Properties

- The $\mathcal{N}$ -extended supersymmetry does not exist
- The massless KK mode is 4D chiral
- The massless KK mode is localized
- The multiple localized tiny KK mass modes exist

5.4.1 Set-up

The one-dimensional interval inner space is given as

$$\Omega = [0, \pi L] . \quad (5.47)$$

There are two boundary (two points) at $y = 0, \pi L$.

In five-dimensions, the Dirac field has the same components as the 4D one. Therefore, there are only two possible the boundary conditions, i.e., the left-handed modes vanish,

$$f^{(n)}(y) = 0, \quad A^\dagger g^{(n)}(y) = 0, \quad (5.48)$$

or the right-handed modes vanish,

$$A f^{(n)}(y) = 0, \quad g^{(n)}(y) = 0, \quad (5.49)$$

at each point on the boundary. Thus, there are four combinations of boundary condition in total. We note that the equivalent boundary condition of the higher dimensional Dirac field $\Psi(x, y)$ is given by

$$\Psi_L(x, y) = 0 \quad \text{or} \quad \Psi_R(x, y) = 0, \quad (5.50)$$

at each point on the boundary.

5.4.2 Massive KK modes

It is easy to solve the equation (3.19) which determines the profile of the mode functions. We note the mode functions and the momentum p .⁸

The n -th KK mass of the massive KK modes is simply given as

$$m_n^2 = M^2 + p^2 . \quad (5.51)$$

⁸ We give every mode functions in [Appendix D](#).

RH and RH Let us consider the boundary condition that the right-handed modes vanish at each boundary,

$$A\mathbf{f}^{(n)}(y) = 0, \quad \mathbf{g}^{(n)}(y) = 0, \quad \text{at } y = 0, \quad (5.52)$$

$$A\mathbf{f}^{(n)}(y) = 0, \quad \mathbf{g}^{(n)}(y) = 0, \quad \text{at } y = \pi L. \quad (5.53)$$

The system for the right-handed modes is the infinite-well quantum mechanical system itself. We immediately find the normalized mode functions as

$$\mathbf{f}^{(n)}(y) = \frac{1}{m_n} A^\dagger \mathbf{g}^{(n)}(y), \quad (5.54)$$

$$\mathbf{g}^{(n)}(y) = \sqrt{\frac{2}{\pi L}} \sin(py), \quad (5.55)$$

where the momentum p takes the values

$$p = \frac{n}{L}, \quad n = 1, 2, \dots \quad (5.56)$$

RH and LH Let us consider the another combination of boundary condition; the right-handed modes vanish at $y = 0$ and the left-handed modes vanish at $y = \pi L$,

$$A\mathbf{f}^{(n)}(y) = 0, \quad \mathbf{g}^{(n)}(y) = 0, \quad \text{at } y = 0, \quad (5.57)$$

$$\mathbf{f}^{(n)}(y) = 0, \quad A^\dagger \mathbf{g}^{(n)}(y) = 0, \quad \text{at } y = \pi L. \quad (5.58)$$

One can find that the normalized mode functions can be given by

$$\mathbf{f}^{(n)}(y) = \frac{1}{m_n} A^\dagger \mathbf{g}^{(n)}(y), \quad (5.59)$$

$$\mathbf{g}^{(n)}(y) = \sqrt{\frac{4p}{2p\pi L - \sin(2p\pi L)}} \sin(py), \quad (5.60)$$

where the momentum p is given by the solutions to the equation

$$p = M \tan(p\pi L). \quad (5.61)$$

The remained combinations of the boundary condition are obtained by merely exchanging the boundary condition at each boundary. Therefore, we obtained the 4D spectrum of every class of the boundary condition. It is obvious that the mode function has n nodes (the zero points) for each n -th KK level.

It is interesting that, in contrast to the first case whose mass M only shifts the 4D spectrum, the spectrum totally depends on the mass M in the second case. Such dependence results from the boundary condition which is the feature of these models.

5.4.3 Massless KK modes

It is also easy to solve the massless mode (and localized mode). We note that the massless mode function (and localized mode) is the solution of the first-degree equation in this model.

RH and RH With this boundary condition, the right-handed massless mode does not exist, but the left-handed exists. Therefore, the massless KK mode is 4D chiral. The massless left-handed normalized mode function is given by,

$$f^{(0)}(y) = \sqrt{\frac{2M}{1 - e^{-2\pi LM}}} \exp(-My). \quad (5.62)$$

We stress that this mode is localized. The sign of the mass M determines which side the mode function is localized on.⁹ One can easily show the Witten index Δ_W is equal to -1 , and the supersymmetry is *not* spontaneously broken.

RH and LH One can find the localized modes with this boundary condition. If $1 < \pi LM$, there are bound states which are given by a hyperbolic sine function as

$$f^{(0)}(y) = \sqrt{\frac{4\kappa}{2\kappa\pi L - \sinh(2\kappa\pi L)}} \sinh(\kappa(y - L)), \quad (5.63)$$

$$g^{(0)}(y) = \sqrt{\frac{4\kappa}{2\kappa\pi L - \sinh(2\kappa\pi L)}} \sinh(\kappa y). \quad (5.64)$$

where the momentum κ satisfies

$$\kappa = M \tanh(\pi L M). \quad (5.65)$$

We stress that these modes are localized on one side and correspond to bound states. However, *they are not 4D chiral and the modes are not massless modes*. That is, the KK mass of the states is given by the subtraction of the momentum κ^2 from mass square M^2 ,

$$m_{\text{lowest}}^2 = M^2 - \kappa^2. \quad (5.66)$$

The momentum κ never takes the value M . Therefore, these modes cannot be massless even though these are localized. If $1 > \pi LM$, there are no localized modes because there is no non-trivial solution of (5.65). The Witten index Δ_W is 0 and there are no massless KK modes. Hence, the supersymmetry is spontaneously broken.

We note that the massless KK mode in the first case is 4D chiral and the lowest KK modes in the second case are not 4D chiral. These mode functions have similar localized profiles but have different KK masses. The first case has zero KK mass. The second case has non zero but tiny KK mass. For example, by taking the two parameters as the grand unification theory scale, $L = 2.0 \times 10^{25}$ GeV and $M = 1.3 \times 10^{26}$ GeV, the lowest modes' mass m_{lowest} approximately has the value 8.7×10^{-1} eV which is the neutrino mass scale. It implies that the model provides the neutrino mass scale 10^{-1} eV from the grand unification theory scale 10^{25} GeV. We constructed a phenomenological model based on the mechanism in [18].

⁹ The ambiguity of the origin is removed by the normalization condition.

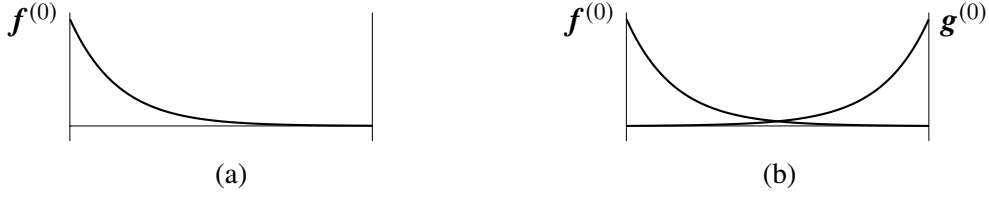


Figure 5.6: The illustration of the localized modes. Figure (a) illustrates the massless KK mode of the RH and RH combination. Figure (b) illustrates the localized but tiny massive mode of the RH and RL combination. These have almost the same profile, but the KK mass is decisively different.

5.5 Summary: Models

In this chapter, we have considered various models to show how the $\mathcal{N} = 2$ SQM generates the 4D chiral partners and the $\mathcal{N}$ -extended SQM generate the additional isospectral pairs.

First, we constructed one of the simplest models with boundary in [Section 5.1](#). In this model, the $\mathcal{N}$ -extended SQM maximally holds. By giving the mode functions explicitly, we showed that the supercharges generate the whole 4D spectrum in this model. We constructed the mode functions explicitly in a highly applicable way and used this construction in later models. On the other hand, it seems to be difficult to determine the profiles of the massless KK modes because, for the massless KK modes, the boundary condition (5.13) and the equation (5.12) which determine the profile of the massless KK modes have the same form. This difficulty arises in any model with any shape of the internal space if one takes the 4D chiral BC or internal space chiral BC for all points on the boundary.

In [Section 5.2](#), we introduced the two models with no boundary. In the first model, a toroidal internal space with the periodic boundary condition, there are accidental isospectral pairs, i.e., the left-moving and the right-moving modes. The supercharges of the $\mathcal{N} = 2$ SQM generate the 4D chiral partners as usual. In addition, the supercharges of the $\mathcal{N}$ -extended SQM generates the mode functions interfering the left-moving and right-moving modes. Therefore, the whole 4D spectrum is generated by the combination of the $\mathcal{N} = 2$ SQM and the $\mathcal{N}$ -extended SQM the supercharges. In the second model, a toroidal internal space with the twisted boundary condition, the $\mathcal{N}$ -extended SQM is broken because the reflection symmetry completely breaks in general. This model indicates the sensitivity of the $\mathcal{N}$ -extended SQM on the boundary condition. In these models, all of the mode functions are completely determined in contrast to the model with a hyperrectangular internal space. However, there is no 4D chiral mode.

In [Section 5.3](#), we introduced the model with a cylindrical internal space to emphasize the complicated dependency of the $\mathcal{N}$ -extended SQM on the boundary condition. For simplicity, we take the 4D chiral boundary condition and the internal space chiral boundary condition for the two unconnected boundaries. We show three models breaking the $\mathcal{N}$ -extended SQM with combinations of the boundary condition. The $\mathcal{N}$ -extended SQM breaking is caused by the different mechanisms; a) there are no additional isospectral pairs in the 4D spectrum, b) there are no linearly independent mode functions which can be an isospectral pairs and c) the boundary condition violates the reflection symmetry of the

mode functions.¹⁰Table 5.1 is a summary that the combination breaks the $\mathcal{N}$ -extended SQM.

In Section 5.4, we introduce the most simple mode which has an interval internal space to discuss the lowest KK modes. We show that there are essentially two different models from the viewpoint of the lowest KK modes. The first model has only one localized massless KK mode which has an exponential profile. The partner mode is prohibited by the Dirichlet boundary condition. Therefore, the massless KK mode is 4D chiral. Also, the Witten index Δ_W takes the value -1 and the supersymmetry is not spontaneously broken. The other model has two individual localized modes which localized at different sides of the interval. It should be stressed that the two localized modes are not massless but have a tiny mass. Therefore, the lowest KK modes are not 4D chiral. It is easy to verify that the Witten index Δ_W is equal to 0 and there are no massless KK modes. Hence, the supersymmetry is spontaneously broken.

We emphasize that the $\mathcal{N} = 2$ SQM holds in any models even the $\mathcal{N}$ -extended SQM is broken. It is natural because the $\mathcal{N} = 2$ SQM provides the 4D Dirac mass term. Moreover, the existence of such the $\mathcal{N} = 2$ SQM is equivalent to the existence of the higher dimensional theory. Namely, if the nontrivial mode functions (or the higher dimensional theory) exist, the $\mathcal{N} = 2$ SQM holds.

These are only toy models but indicate the possibility to solve the three problems of the standard model, the generation problem, the fermion mass hierarchy problem, and the chiral asymmetry problem. There are multiple 4D chiral localized massless KK modes in the 4D spectrum. As we showed, the hidden symmetry such as the $\mathcal{N} = 2$ SQM and the $\mathcal{N}$ -extended SQM control the 4D spectrum. The deeper understanding of such the hidden symmetry will help the classification of the higher dimensional theory.

We have not discussed the geometric phase in this chapter even treating the explicit models because there is a technical difficulty to solve the equation of the mode functions with the boundary condition which has nontrivial parameter space. However, such the restricted huge parameter space is one of the important features of the extra-dimensional theory with boundary.

¹⁰ These are might not independent. However, we do not prove it.

Chapter 6

Summary

The standard model is one of the most successful theories which describe the nature. It is no exaggeration to say that the physics below the TeV scale with low density is completely understood. On the other hand, there are still reminded mysteries; the generation problem, the fermion mass hierarchy problem, and the chiral asymmetry problem.

A higher dimensional theory is a candidate which solves the problems simultaneously. The Kaluza–Klein (KK) decomposition is the method of inducing the 4D theory from a higher dimensional theory. The standard model's matter sector appears as the low energy effective fields of the higher dimensional Dirac fields after the KK decomposition, e.g., lowest KK modes.

We were interested in the fact that degeneracy exists between 4D fields with the same 4D mass in KK decomposition of high dimensional Dirac field. In general, the degeneracy corresponds to the symmetry, and the symmetry controls the spectrum and the solutions of the theory. Therefore, we expect that the hidden symmetry corresponding to the degeneracy will control the 4D spectrum and the lowest modes which provide the SM matter sector.

In this thesis, we revealed the hidden symmetry in the KK decomposition of the higher dimensional free Dirac field consists of two classes of the quantum mechanical supersymmetry. These supersymmetries control the 4D spectrum and the mode functions of the higher dimensional theory.

The first one is the higher dimensional realization of the $\mathcal{N} = 2$ quantum mechanical supersymmetry. The $\mathcal{N} = 2$ quantum mechanical supersymmetry is the crucial symmetry for the KK decomposition of the free higher dimensional Dirac field because the $\mathcal{N} = 2$ supersymmetry ensures the existence of the chiral partners of the mode functions and generates it. We showed the existence of such the $\mathcal{N} = 2$ quantum mechanics supersymmetry by only requiring the variable separation of the high dimensional Dirac field and the mass eigenstate of the 4D field ([Chapter 2](#) and [Chapter 3](#)). Therefore, the $\mathcal{N} = 2$ quantum mechanical supersymmetry will exist for various generalizations of the higher dimensional Dirac theory.

The second one is the $\mathcal{N}$ -extended quantum mechanical supersymmetry which is a generalizations of the $\mathcal{N} = 2$ supersymmetry. Due to the high degrees of freedom of the higher dimensional Dirac field from the viewpoint of the 4D theory, there are, typically, the additional isospectral pairs in the 4D spectrum. The $\mathcal{N}$ -extended quantum mechanical

supersymmetry generates the additional isospectral pairs in the 4D spectrum. In contrast to the $\mathcal{N} = 2$ supersymmetry, the $\mathcal{N}$ -extended supersymmetry is highly sensitive to the boundary condition. We showed various models which have nontrivial supersymmetry in [Chapter 5](#). In [Chapter 4](#), we discuss the boundary condition consistent with the $\mathcal{N} = 2$ quantum mechanical supersymmetry for preparation of [Chapter 5](#).

We emphasize that there exist(s) (multiple) localized massless (or tiny mass) KK modes in the models with boundary ([Chapter 5](#)). The models which we introduced are only toy models. However, the models indicate the possibility to solve the three problems of the standard model, the generation problem, the fermion mass hierarchy problem, and the chiral asymmetry problem, because of appearing multiple 4D chiral localized massless KK modes.

In our model, the $\mathcal{N}$ -extended supersymmetry partially controls the 4D spectrum and the mode functions. There are still additional isospectral pairs which are not described by the hidden symmetry. It implies that there may exist another hidden symmetry. If another hidden symmetry is found, the hidden symmetry will expand by gathering with the $\mathcal{N}$ -extended supersymmetry and another symmetry.

We cannot completely classify the boundary condition. On the other hand, the parameter space of the boundary condition which provides the nontrivial mode functions is phenomenologically important. The boundary condition will control the profiles of the mode functions also. Such the parameter space should be completely classified with the explicit form of the mode functions to construct realistic models and quantitatively analyze the 4D spectrum.

We should bring in the scalar and the (non-Abelian) gauge fields to construct a realistic model. If the internal space has the boundary, we should also classify the boundary condition of the scalar and the gauge fields. It is known that the general boundary condition of the scalar field at each point on the boundary is the Robin boundary condition [36]. However, the general boundary condition of the (non-Abelian) gauge fields at each point on the boundary is not disclosed. It is also very important to classify the boundary condition of the (non-Abelian) gauge fields for constructing realistic models.

The parameter space of the boundary condition is defined at each point of the boundary. To be precise, the boundary condition is a map U from the boundary $\partial\Omega$ to the BC_y , where the unitary matrix U characterizes the projection operator P .¹ If we can introduce natural dynamics into the “field” U defined on the boundary (e.g. gauging the “field” U , a path integral of U , etc.), it may be possible to dynamically determine the boundary condition.

One of the important generalization of our result is the internal space depended “mass” $M(y)$. In general, the vacuum expectation value (VEV) of the scalar field can have the internal space dependence. Such the VEV can be regarded as the internal space depending “mass” $M(y)$. In the left-handed and right-handed massless KK modes’ equation, the “mass” $M(y)$ will have opposite signs. Therefore, this generalization may provide the 4D chiral massless KK modes and may solve the mystery of the standard model.

We had revealed the hidden supersymmetries by discussion independent of model’ details. Therefore, it is easy to generalize the results to the theory with the curved internal space. It will be expected that the supercharges are still the combination of the two Dirac

¹ This is the bundle structure itself. Namely, a set $BC = \bigcup_{y \in \partial\Omega} BC_y$ will be a “boundary condition bundle”.

operators of the curved internal space.² And, the Witten index Δ_W characterizes the 4D chirality of the massless KK modes with the solution of the Dirac equation on the internal space. It is known that the Witten index of the differential operator relates to the topological quantities [31–33, 43]. It implies that we may be able to completely classify the extra-dimensional theory based on topological quantities. The properties of the Dirac operator which defined on the (pseudo-)Riemannian with no boundary is well studied, e.g. spectrum, index, response of the Weyl transformation, etc. [43, 44]. It is very interesting to generalized it on orbifold and the space with boundary.

² ? Because the KK decomposition should work well even the internal space is curved.

Appendix A

Higher dimensional Dirac action

In this chapter, we note the supplement topics of the higher dimensional Dirac action.

A.1 Two component spinor

In this section, we introduce two component spinor $\bar{\phi}(x)$ and $\chi(x)$. These are spinors of the irreducible representation of the 4D Lorentz group $SO(1, 3)$.

A.1.1 Pauli matrices

The Pauli matrices σ^k ($k = 1, 2, 3$) are given by

$$\sigma^1 = \begin{pmatrix} & 1 \\ 1 & \end{pmatrix}, \quad \sigma^2 = \begin{pmatrix} & -i \\ i & \end{pmatrix}, \quad \sigma^3 = \begin{pmatrix} 1 & \\ & -1 \end{pmatrix}, \quad (\text{A.1})$$

which have the properties,

$$[\sigma^i, \sigma^j] = 2i \sum_{k=1}^3 \varepsilon^{ijk} \sigma^k, \quad \{\sigma^i, \sigma^j\} = 2\delta^{ij} \mathbf{1}_2, \quad (\text{A.2})$$

for $i, j = 1, 2, 3$. The Levi–Civita tensor ε^{ijk} ($i, j, k = 1, 2, 3$) is normalized as $\varepsilon^{123} = 1$ and δ^{ij} is the Kronecker delta.

The Pauli matrices σ^k ($k = 1, 2, 3$) satisfy the anti-commutation relation in (A.2). Therefore, one can construct the representation space of the Pauli matrices by the creation/annihilation operator $\sigma^\pm$ given by,

$$\sigma^\pm = \frac{1}{\sqrt{2}}(\sigma^1 \pm i\sigma^2), \quad (\text{A.3})$$

which has the properties,

$$\{\sigma^\pm, \sigma^\pm\} = 0, \quad \{\sigma^+, \sigma^-\} = 2, \quad [\sigma^3, \sigma^\pm] = \pm 2\sigma^\pm. \quad (\text{A.4})$$

Here, we note the construction of the representation space of the Pauli matrices. Given a vacuum $|0\rangle$ which satisfy $\sigma^- |0\rangle = 0$. Due to the properties (A.4), there exists only two states $|0\rangle$ and $|1\rangle = \sigma^+ |0\rangle$. If we take the states as,

$$|0\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \quad |1\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad (\text{A.5})$$

the representation of the Pauli matrices in (A.1) is reproduced up to phase factor.¹

A.1.2 Two component spinor

We used the two components spinor fields $\phi(x)$ and $\bar{\chi}(x)$ without any notion [24, 28]. We introduce them and discuss of the property here. For convenient, we use the anti-symmetrize operator,

$$A^{[\mu} B^{\nu]} := \frac{1}{2}(A^\mu B^\nu - A^\nu B^\mu). \quad (\text{A.6})$$

The two component spinor fields $\bar{\phi}(x)$ and $\chi(x)$ are the spinor fields of the irreducible representation of the 4D Lorentz group $SO(1, 3)$ respectively. The Lorentz group acts on the spinor fields as,

$$\bar{\phi}(x) \rightarrow \bar{\phi}'(x') = \exp[i\omega_{\mu\nu}\bar{\sigma}^{\mu\nu}]\bar{\phi}(x), \quad (\text{A.7})$$

$$\chi(x) \rightarrow \chi'(x') = \exp[i\omega_{\mu\nu}\sigma^{\mu\nu}]\chi(x), \quad (\text{A.8})$$

where the generators $\bar{\sigma}^{\mu\nu}$ and $\sigma^{\mu\nu}$ ($\mu, \nu = 0, 1, 2, 3$) of the 4D Lorentz group are given by

$$\bar{\sigma}^{\mu\nu} = \frac{i}{2}\bar{\sigma}^{[\mu}\bar{\sigma}^{\nu]}, \quad \sigma^{\mu\nu} = \frac{i}{2}\sigma^{[\mu}\sigma^{\nu]}, \quad (\text{A.9})$$

with $\sigma^\mu = (1, \sigma^k)$ and $\bar{\sigma}^\mu = (1, -\sigma^k)$ ($k = 1, 2, 3$).

The convention in this thesis follows [24, 28],

$$\phi\bar{\sigma}^\mu\bar{\phi} = \sum_{\dot{\alpha}, \alpha}^{1,2} \phi^\alpha (\bar{\sigma}^\mu)_{\alpha\dot{\alpha}} \bar{\phi}^{\dot{\alpha}}, \quad \bar{\chi}\sigma^\mu\chi = \sum_{\dot{\alpha}, \alpha}^{1,2} \bar{\chi}_{\dot{\alpha}} (\sigma^\mu)^{\dot{\alpha}\alpha} \chi_\alpha, \quad (\text{A.10})$$

$$\phi\chi = \sum_{\alpha}^{1,2} \phi^\alpha \chi_\alpha, \quad \bar{\phi}\bar{\chi} = \sum_{\dot{\alpha}}^{1,2} \bar{\phi}_{\dot{\alpha}} \bar{\chi}^{\dot{\alpha}}. \quad (\text{A.11})$$

Raising and lowering indexes is done by the Levi-Civita tensor $\varepsilon^{\alpha\beta}$ and $\varepsilon_{\dot{\alpha}\dot{\beta}}$ ($\alpha, \beta, \dot{\alpha}, \dot{\beta} = 1, 2$) normalized as $\varepsilon^{12} = 1$ as,

$$\phi^\alpha = \varepsilon^{\alpha\beta} \phi_\beta, \quad \bar{\chi}_{\dot{\alpha}} = \varepsilon_{\dot{\alpha}\dot{\beta}} \bar{\chi}^{\dot{\beta}}. \quad (\text{A.12})$$

¹ Given finite vector spaces $\mathcal{V}$ and $\mathcal{W}$ and the linear map $A : \mathcal{V} \rightarrow \mathcal{W}$. The component $(A)_{ij}$ of A is given by ${}_{\mathcal{W}}\langle i|A|j\rangle_{\mathcal{V}}$ where $|j\rangle_{\mathcal{V}}$ ($|i\rangle_{\mathcal{W}}$) are the normalized basis of $\mathcal{V}$ ($\mathcal{W}$).

We note properties of the product of the two component spinor fields,

$$\phi\chi = \chi\phi, \quad (\phi\chi)^\dagger = \bar{\phi}\bar{\chi}, \quad \phi\sigma^\mu\bar{\chi} = -\bar{\chi}\bar{\sigma}^\mu\phi, \quad (\phi\sigma^\mu\bar{\chi})^\dagger = \chi\sigma^\mu\bar{\phi}, \quad (\text{A.13})$$

and $\psi\psi \neq 0$ even $\psi^\alpha\psi^\alpha = 0$.

We should note that one can define the left-handed spinor $\bar{\phi}$ as a complex conjugate of the right-handed spinor ϕ ,

$$\bar{\phi}^{\dot{\alpha}} = (\phi^*)_{\dot{\beta}}\epsilon^{\dot{\beta}\dot{\alpha}}. \quad (\text{A.14})$$

or, frankly, $\bar{\phi} = -i\sigma^2\phi^*$. This is the property of the complex spinor representation of the 4D Lorentz group $SO(1, 3)$.

At the end of this section, we mention about the 4D Dirac spinor in the 4D Weyl representation. The 4D Dirac spinor ψ is the reducible spinor of the 4D Lorentz group $SO(1, 3)$ which is usually given as,

$$\psi = \begin{pmatrix} \bar{\phi} \\ \chi \end{pmatrix} = \bar{\phi}^{\dot{\alpha}} \oplus \chi_\alpha. \quad (\text{A.15})$$

A.2 Higher dimensional gamma matrices

A.2.1 The higher dimensional gauge field

It is a significant property that there are multiple 4D fields in the higher dimensional Dirac field. It is provided from the algebraic property. In this section, we discuss the degree of freedom of the higher dimensional gauge (vector) field from the viewpoint of the 4D theory for comparison of the Dirac field.

Given a higher dimensional $U(1)$ gauge (vector) field $A_N(x, y)$ ($N = \mu, y$) which belongs to a fundamental representation of the higher dimensional Lorentz group $SO(1, 3 + d)$. The higher dimensional gauge field $A_N(x)$ ($N = \mu, y$) is transformed into its linear combination by the action of the higher dimensional Lorentz group

$$A_N(x, y) \rightarrow A'_N(x, y) = \Lambda_N^{N'} A_{N'}(x, y), \quad (\text{A.16})$$

where $\Lambda_N^{N'}$ is the element of the fundamental representation of the higher dimensional Lorentz group $SO(1, 3 + d)$. The 4D Lorentz subgroup $SO(1, 3)$ of the higher dimensional Lorentz group $SO(1, 3 + d)$ transforms the 4D components $A_\mu(x, y)$ but has trivial action on the extra-dimensional component(s) $A_y(x, y)$,

$$A_\mu(x, y) \rightarrow A'_\mu(x, y) = \Lambda_\mu^{\mu'} A_{\mu'}(x, y), \quad (\text{A.17})$$

$$A_y(x, y) \rightarrow A'_y(x, y) = A_y(x, y), \quad (\text{A.18})$$

where $\Lambda_\mu^{\mu'}$ is the element of the 4D Lorentz group subgroup $SO(1, 3)$. It implies that the higher dimensional component(s) $A_y(x, y)$ ($y = y_1, \dots, y_d$) seems to be the “scalar” field from the view point of the 4D theory.

The higher dimensional gauge (vector) field also has the higher degrees of freedom more than 4D one from the view point of the 4D theory. However, the increased degrees of freedom is not devoted to the 4D vector field sector, i.e., there is an only single 4D gauge field.

We discuss again in different perspectives, the representation of the higher dimensional Lorentz group. A fundamental representation of the higher dimensional Lorentz group $SO(1, 3 + d)$ is a set of real $(4 + d)$ -by- $(4 + d)$ matrices satisfying

$$(R^\top)^N_{N'} \eta^{N'M'} R_{M'}^M = \eta^{NM}, \quad R^\top = R^{-1}, \quad \det R = 1. \quad (\text{A.19})$$

It is easy to verify that any element of the $SO(1, 3) \times SO(d)$ subgroup in this representation is block diagonalized. Denoting Λ by the element of the higher dimensional Lorentz group $SO(1, 3 + d)$ and $\Lambda_{(1,3)}$ and $\Lambda_{(d)}$ by the element of the fundamental representation of the 4D Lorentz group $SO(1, 3)$ and the $SO(d)$ group respectively, one can decompose Λ as,

$$\Lambda = \begin{pmatrix} \Lambda^{(1,3)} & \\ & \Lambda^{(d-4)} \end{pmatrix}. \quad (\text{A.20})$$

That is, the subgroup is a direct sum of a fundamental representation of the 4D Lorentz group $SO(1, 3)$ and a fundamental representation of the $SO(d)$,

$$SO(1, 3) \times SO(d) = SO(1, 3) \oplus SO(d), \quad (\text{A.21})$$

in this representation. This implies that, in contrast to the Dirac field, any element of the generators of the 4D Lorentz subgroup $SO(1, 3)$ is given by the direct sum of $\Lambda^{(1,3)}$ and the identity matrix,

$$\Lambda^{(1,3)} \oplus \mathbf{1}_{d-4}. \quad (\text{A.22})$$

Of course, the generators of the subgroup $SO(1, 3) \times SO(d)$ are the direct sums of the generators of the 4D Lorentz group $SO(1, 3)$ and the $SO(d)$.

In contrast to the Dirac field, it is clear that there is only one 4D gauge field in the higher dimensional gauge field. In the gauge (vector) field, the increased degrees of freedom is devoted to provide the extra “scalar” fields $A_y(x, y)$ but not degeneracy of the “4D gauge field” $A_\mu(x, y)$ ($\mu = 0, 1, 2, 3$).

We note that the increased degree of freedom provides the degeneracy on the 4D field in the higher dimensional Dirac field.

A.2.2 Formal construction of the gamma matrices

In this section, we prove that there exists the spinor representation of the higher dimensional Lorentz group $SO(1, 3 + d)$ for any $1 \leq d$. It is easy that the quadratics of the gamma matrices generates $SO(1, 3 + d)$ group if the gamma matrices exist. Therefore, we will show the existence of the gamma matrices in any dimension [16, 25–27].

We will show a construction of the $d + 1$ dimensional gamma matrices Γ^N ($N = 0, 1, \dots, d$) from the d dimensional gamma matrices γ^μ ($\mu = 0, 1, \dots, d - 1$) satisfying,

$$\{\gamma^\mu, \gamma^\nu\} = -2\eta^{\mu\nu} \mathbf{1}, \quad \gamma^0 \gamma^\mu \gamma^0 = (\gamma^\mu)^\dagger, \quad (\text{A.23})$$

for $\mu, \nu = 0, 1, \dots, d-1$, where $\eta^{\mu\nu}$ is the d dimensional metric given by,

$$\eta^{\mu\nu} = \eta_{\mu\nu} = \text{diag}(-1, +1, \dots, +1). \quad (\text{A.24})$$

Even to Odd Given the d (even) dimensional gamma matrices γ^μ ($\mu = 0, 1, \dots, d-1$) satisfying the Clifford algebra and Hermiticity (A.23), one can define the chiral matrix γ by,

$$\gamma = i^{d/2-2} \gamma^0 \gamma^1 \dots \gamma^{d-1}, \quad (\text{A.25})$$

which satisfy,

$$\gamma^\dagger = \gamma, \quad \gamma^2 = \mathbf{1}, \quad \{\gamma, \gamma^\mu\} = 0. \quad (\text{A.26})$$

Therefore, one can define the $d+1$ (odd) dimensional gamma matrices Γ^N ($\mu = 0, 1, \dots, d$) as,

$$\Gamma^\mu = \gamma^\mu, \quad \Gamma^d = i\gamma. \quad (\text{A.27})$$

for $\mu = 0, 1, \dots, d-1$. It is obvious that the $d+1$ dimensional gamma matrices Γ^N ($N = 0, 1, \dots, d$) satisfy the Clifford algebra and Hermiticity (A.23) with the $d+1$ dimensional metric,

$$\eta^{NM} = \eta_{NM} = \text{diag}(-1, +1, \dots, +1), \quad (\text{A.28})$$

where $N, M = 0, 1, \dots, d$.

Odd to Even Given the d (odd) dimensional gamma matrices γ^μ ($\mu = 0, 1, \dots, d-1$) satisfying the Clifford algebra and Hermiticity (A.23), one can define the $d+1$ (even) dimensional gamma matrices Γ^N ($\mu = 0, 1, \dots, d$) by the tensor product of the Pauli matrices and the gamma matrices as,

$$\Gamma^0 = \sigma^1 \otimes \mathbf{1}, \quad \Gamma^1 = i\sigma^2 \otimes \gamma^0, \quad \Gamma^{k+1} = \sigma^2 \otimes \gamma^k, \quad (\text{A.29})$$

for $k = 1, \dots, d-1$. It is obvious that the $d+1$ dimensional gamma matrices Γ^N ($N = 0, 1, \dots, d$) satisfy the Clifford algebra and Hermiticity (A.23) with the $d+1$ dimensional metric,

$$\eta^{NM} = \eta_{NM} = \text{diag}(-1, +1, \dots, +1), \quad (\text{A.30})$$

where $N, M = 0, 1, \dots, d-1, d$. Especially, this representation of the $d+1$ (even) dimensional gamma matrices Γ^N is chiral (or $d+1$ dimensional Weyl representation). Define the $d+1$ dimensional chiral matrix Γ in the same manner as (A.25), the chiral matrix Γ will be block diagonalized as,

$$\Gamma = i^{(d+1)/2-2} \Gamma^0 \Gamma^1 \dots \Gamma^{d-1} \Gamma^d = -\sigma^3 \otimes \mathbf{1}. \quad (\text{A.31})$$

In the four-dimensions, the gamma matrices exist. Thus, one can obtain any dimensional gamma matrices by using this construction recursively.

A.2.3 Explicit construction of the gamma matrix

In the previous section, we show the existence of the spinor representation by proofing the existence of the gamma matrices for any dimension. However, it is not clear how to construct and label the spinor (representation space). In this section, we introduce the explicit representation of the gamma matrix which is easy to handle the spinor (the representation space). For simplicity, we introduce the explicit representation for the internal gamma matrices γ^{y_k} ($k = 1, \dots, d$) defined in (2.41).

One easily finds that the Pauli matrices satisfy the three-dimensional Clifford algebra and wish to extend them into the d dimensional one. In fact, we can give the gamma matrices as tensor products of the Pauli matrices in a simple form. For even dimension, the internal gamma matrices γ^{y_k} ($k = 1, \dots, d$) are given as,

$$\begin{aligned}\gamma^{y_1} &= i\mathbf{1}_2 \otimes \mathbf{1}_2 \otimes \cdots \otimes \mathbf{1}_2 \otimes \mathbf{1}_2 \otimes \sigma^1, \\ \gamma^{y_2} &= i\mathbf{1}_2 \otimes \mathbf{1}_2 \otimes \cdots \otimes \mathbf{1}_2 \otimes \mathbf{1}_2 \otimes \sigma^2, \\ \gamma^{y_3} &= i\mathbf{1}_2 \otimes \mathbf{1}_2 \otimes \cdots \otimes \mathbf{1}_2 \otimes \sigma^1 \otimes \sigma^3, \\ \gamma^{y_4} &= i\mathbf{1}_2 \otimes \mathbf{1}_2 \otimes \cdots \otimes \mathbf{1}_2 \otimes \sigma^2 \otimes \sigma^3, \\ \gamma^{y_5} &= i\mathbf{1}_2 \otimes \mathbf{1}_2 \otimes \cdots \otimes \sigma^1 \otimes \sigma^3 \otimes \sigma^3, \\ \gamma^{y_6} &= i\mathbf{1}_2 \otimes \mathbf{1}_2 \otimes \cdots \otimes \sigma^2 \otimes \sigma^3 \otimes \sigma^3, \\ &\vdots \\ \gamma^{y_{d-1}} &= i\sigma^1 \otimes \sigma^3 \otimes \cdots \otimes \sigma^3 \otimes \sigma^3 \otimes \sigma^3, \\ \gamma^{y_d} &= i\sigma^2 \otimes \sigma^3 \otimes \cdots \otimes \sigma^3 \otimes \sigma^3 \otimes \sigma^3.\end{aligned}$$

For odd dimension, the internal space gamma matrices γ^y ($y = y_1, \dots, y_d$) are given as,

$$\begin{aligned}\gamma^{y_1} &= i\mathbf{1}_2 \otimes \mathbf{1}_2 \otimes \cdots \otimes \mathbf{1}_2 \otimes \mathbf{1}_2 \otimes \sigma^1, \\ \gamma^{y_2} &= i\mathbf{1}_2 \otimes \mathbf{1}_2 \otimes \cdots \otimes \mathbf{1}_2 \otimes \mathbf{1}_2 \otimes \sigma^2, \\ \gamma^{y_3} &= i\mathbf{1}_2 \otimes \mathbf{1}_2 \otimes \cdots \otimes \mathbf{1}_2 \otimes \sigma^1 \otimes \sigma^3, \\ \gamma^{y_4} &= i\mathbf{1}_2 \otimes \mathbf{1}_2 \otimes \cdots \otimes \mathbf{1}_2 \otimes \sigma^2 \otimes \sigma^3, \\ \gamma^{y_5} &= i\mathbf{1}_2 \otimes \mathbf{1}_2 \otimes \cdots \otimes \sigma^1 \otimes \sigma^3 \otimes \sigma^3, \\ \gamma^{y_6} &= i\mathbf{1}_2 \otimes \mathbf{1}_2 \otimes \cdots \otimes \sigma^2 \otimes \sigma^3 \otimes \sigma^3, \\ &\vdots \\ \gamma^{y_{d-2}} &= i\sigma^1 \otimes \sigma^3 \otimes \cdots \otimes \sigma^3 \otimes \sigma^3 \otimes \sigma^3, \\ \gamma^{y_{d-1}} &= i\sigma^2 \otimes \sigma^3 \otimes \cdots \otimes \sigma^3 \otimes \sigma^3 \otimes \sigma^3, \\ \gamma^{y_d} &= i\sigma^3 \otimes \sigma^3 \otimes \cdots \otimes \sigma^3 \otimes \sigma^3 \otimes \sigma^3.\end{aligned}$$

The size of the gamma matrices is $2^{\lfloor d/2 \rfloor}$ because these are $\lfloor d/2 \rfloor$ times the tensor products of 2-by-2 matrices.

These representations respect following construction of the spinor (the representation space). The space of the spinor (the representation space of the spinor representation of

$SO(1, 3 + d)$ is generated by the creation/annihilation operators $\gamma_{(m)}^\pm$ and is labelled by the “fermion number” operators $\gamma_{(m)}$ which are given by,

$$\gamma_{(m)}^\pm = -\frac{i}{\sqrt{2}}(\gamma^{y_{2m-1}} \pm i\gamma^{y_{2m}}), \quad \gamma_{(m)} = i\gamma^{y_{2m-1}}\gamma^{y_{2m}}, \quad (\text{A.34})$$

for $m = 1, \dots, d$ ($m = 1, \dots, d-1$) in even (odd) dimensions.² The operators satisfy the “fermionic” creation/annihilation algebra,

$$\{\gamma_{(n)}^\pm, \gamma_{(m)}^\pm\} = 0, \quad \{\gamma_{(n)}^+, \gamma_{(m)}^-\} = \delta_{nm}, \quad [\gamma_{(m)}, \gamma_{(n)}^\pm] = \pm\gamma_{(n)}^\pm\delta_{nm}, \quad (\text{A.35})$$

for all m, n . The creation (annihilation) operator $\gamma_{(m)}^+$ ($\gamma_{(m)}^-$) increases (decreases) the eigenvalues of the “fermion number” operators $\gamma_{(m)}$ by one. And, these are nilpotent. Hence, the space of the spinor (the representation space) has the dimensions 2^m . The finite series $\{ |s_1, \dots, s_m\rangle \}_{m=1}^d$ (or $\{ |s_1, \dots, s_m\rangle \}_{m=1}^{d-1}$) where $s_m = 0, 1$ completes the space of the spinor. The state $|0, \dots, 0\rangle$ is a vacuum which vanishes by multiplying for all annihilation operators $\gamma_{(m)}^-$,

$$\gamma_{(m)}^- |0, \dots, 0\rangle = 0, \quad (\text{A.36})$$

and the others are the “exited states” which given by multiplying the creation operators $\gamma_{(m)}^+$,

$$|0, \dots, 0, \underset{m\text{-th}}{1}, 0, \dots, 0\rangle = \gamma_{(m)}^+ |0, \dots, 0\rangle, \quad (\text{A.37})$$

$$|0, \dots, 0, \underset{m\text{-th}}{1}, 0, \dots, 0, \underset{n\text{-th}}{1}, 0, \dots, 0\rangle = \gamma_{(n)}^+ \gamma_{(m)}^+ |0, \dots, 0\rangle, \quad (\text{A.38})$$

$\vdots$

for all $m < n$. In above representation, the m -th creation/annihilation operators $\gamma_{(m)}^\pm$ have the creation/annihilation operators $\sigma^\pm$ at the m -th component and the “fermion number” operator $\gamma_{(m)}$ has σ^3 at the m -th component,

$$\gamma_{(m)}^\pm = \mathbf{1}_2 \otimes \mathbf{1}_2 \otimes \dots \otimes \mathbf{1}_2 \otimes \overset{m\text{-th}}{\sigma^\pm} \otimes \sigma^3 \otimes \dots \otimes \sigma^3 \otimes \sigma^3, \quad (\text{A.39})$$

$$\gamma_{(m)} = \mathbf{1}_2 \otimes \mathbf{1}_2 \otimes \dots \otimes \mathbf{1}_2 \otimes \overset{m\text{-th}}{\sigma^3} \otimes \sigma^3 \otimes \dots \otimes \sigma^3 \otimes \sigma^3. \quad (\text{A.40})$$

Therefore, in this representation, a vacuum $|0, \dots, 0\rangle$ is given as,

$$|0, \dots, 0\rangle = |0\rangle \otimes \dots \otimes |0\rangle, \quad |0\rangle = (0, 1)^\top, \quad (\text{A.41})$$

and the “exited states” are given by replacing some $|0\rangle$ to $|1\rangle = (1, 0)^\top$. We figure the diagram of generating the space of spinor in [Figure A.1](#).

² One should take care of the factors of the operators for convenient.

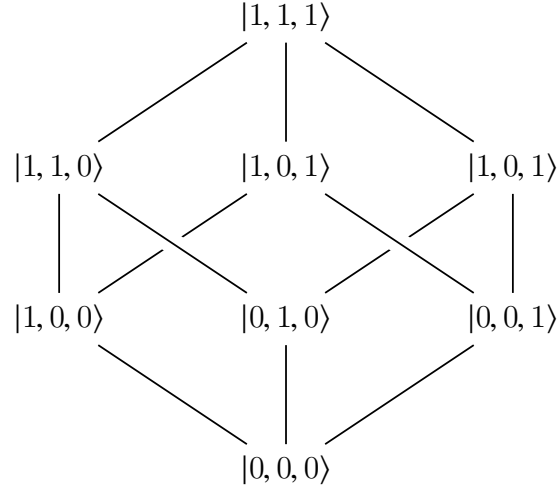


Figure A.1: The figure of the spinor space generated by the generators $\gamma_{(m)}^\pm$.

A.2.4 List of the explicit representations of the gamma matrices

We listed the explicit representations of the gamma matrices up to ten-dimensions. Here, we list the higher dimensional gamma matrices explicitly. The five- or higher dimensional gamma matrices are based on the 4D Weyl representation for simplicity. In addition, we note the charge conjugation $C_\pm$ which generates transpose of the gamma matrices,

$$C_\pm \Gamma^N C_\pm^{-1} = \pm (\Gamma^N)^\top, \quad (\text{A.42})$$

if it exists.³

2D Gamma matrices

$$\begin{aligned} \Gamma^0 &= \sigma^1, & \Gamma &= \sigma^3, \\ \Gamma^1 &= -i\sigma^2, & C_+ &= \sigma^1, \\ & & C_- &= i\sigma^2. \end{aligned}$$

3D Gamma matrices

$$\begin{aligned} \Gamma^0 &= \sigma^1, & \Gamma &= \text{non.} \\ \Gamma^1 &= -i\sigma^2, & C_+ &= \text{non.} \\ \Gamma^3 &= -i\sigma^3, & C_- &= i\sigma^2. \end{aligned}$$

³ Even if the charge congregations exist, the Majorana spinor does not always exist. See [16, 25, 27].

4D Gamma matrices: Weyl representation

$$\gamma^0 = \begin{pmatrix} & \sigma^\mu \\ \bar{\sigma}^\mu & \end{pmatrix} \quad \Gamma = \begin{pmatrix} -\mathbf{1}_2 & \\ & \mathbf{1}_2 \end{pmatrix}$$

$$C_+ = i\gamma^2\gamma^0\gamma^5$$

$$C_- = i\gamma^2\gamma^0$$

4D Gamma matrices: Majorana representation

$$\gamma^0 = \sigma^1 \otimes \sigma^2, \quad \Gamma = \sigma^3 \otimes \sigma^2$$

$$\gamma^1 = i\mathbf{1}_2 \otimes \sigma^3, \quad C_+ = i\gamma^0\gamma^5,$$

$$\gamma^2 = -i\sigma^2 \otimes \sigma^2, \quad C_- = i\gamma^0,$$

$$\gamma^3 = -i\mathbf{1}_2 \otimes \sigma^1,$$

5D Gamma matrices

$$\Gamma^\mu = \gamma^\mu, \quad \Gamma_{4D} = \gamma^5,$$

$$\Gamma^{y_1} = i\gamma^5, \quad \Gamma_{\text{in}} = \text{non.}$$

$$C_+ = C_+^{4d}$$

$$C_- = \text{non.}$$

6D Gamma matrices

$$\Gamma^\mu = \mathbf{1}_2 \otimes \gamma^\mu \quad \Gamma_{4D} = \mathbf{1}_2 \otimes \gamma^5$$

$$\Gamma^{y_1} = i\sigma^1 \otimes \gamma^5 \quad \Gamma_{\text{in}} = \sigma^3 \otimes \gamma^5$$

$$\Gamma^{y_2} = i\sigma^2 \otimes \gamma^5 \quad C_+ = \sigma^1 \otimes C_+^{4d}$$

$$C_- = \sigma^2 \otimes iC_-^{4d}$$

7D Gamma matrices

$$\Gamma^\mu = \mathbf{1}_2 \otimes \gamma^\mu \quad \Gamma_{4D} = \mathbf{1}_2 \otimes \gamma^5$$

$$\Gamma^{y_1} = i\sigma^1 \otimes \gamma^5 \quad \Gamma_{\text{in}} = \text{non.}$$

$$\Gamma^{y_2} = i\sigma^2 \otimes \gamma^5 \quad C_+ = \text{non.}$$

$$\Gamma^{y_3} = i\sigma^2 \otimes \gamma^5 \quad C_- = \sigma^2 \otimes iC_-^{4d}$$

8D Gamma matrices

$$\Gamma^\mu = \mathbf{1}_2 \otimes \mathbf{1}_2 \otimes \gamma^\mu \quad \Gamma_{4D} = \mathbf{1}_2 \otimes \mathbf{1}_2 \otimes \gamma^5$$

$$\Gamma^{y_1} = i\mathbf{1}_2 \otimes \sigma^1 \otimes \gamma^5 \quad \Gamma_{\text{in}} = \sigma^3 \otimes \sigma^3 \otimes \gamma^5$$

$$\Gamma^{y_2} = i\mathbf{1}_2 \otimes \sigma^2 \otimes \gamma^5 \quad C_+ = \sigma^2 \otimes \sigma^1 \otimes iC_+^{4d}$$

$$\Gamma^{y_3} = i\sigma^1 \otimes \sigma^3 \otimes \gamma^5 \quad C_- = \sigma^1 \otimes \sigma^2 \otimes iC_-^{4d}$$

$$\Gamma^{y_4} = i\sigma^2 \otimes \sigma^3 \otimes \gamma^5$$

9D Gamma matirices

$$\Gamma^\mu = \mathbf{1}_2 \otimes \mathbf{1}_2 \otimes \gamma^\mu$$

$$\Gamma^{y_1} = i\mathbf{1}_2 \otimes \sigma^1 \otimes \gamma^5$$

$$\Gamma^{y_2} = i\mathbf{1}_2 \otimes \sigma^2 \otimes \gamma^5$$

$$\Gamma^{y_3} = i\sigma^1 \otimes \sigma^3 \otimes \gamma^5$$

$$\Gamma^{y_4} = i\sigma^2 \otimes \sigma^3 \otimes \gamma^5$$

$$\Gamma^{y_5} = i\sigma^3 \otimes \sigma^3 \otimes \gamma^5$$

$$\Gamma_{4D} = \mathbf{1}_2 \otimes \mathbf{1}_2 \otimes \gamma^5$$

$$\Gamma_{\text{in}} = \text{non.}$$

$$C_+ = \sigma^2 \otimes \sigma^1 \otimes iC_+^{4d}$$

$$C_- = \text{non.}$$

10D Gamma matirices

$$\Gamma^\mu = \mathbf{1}_2 \otimes \mathbf{1}_2 \otimes \mathbf{1}_2 \otimes \gamma^\mu,$$

$$\Gamma^{y_1} = i\mathbf{1}_2 \otimes \mathbf{1}_2 \otimes \sigma^1 \otimes \gamma^5,$$

$$\Gamma^{y_2} = i\mathbf{1}_2 \otimes \mathbf{1}_2 \otimes \sigma^2 \otimes \gamma^5,$$

$$\Gamma^{y_3} = i\mathbf{1}_2 \otimes \sigma^1 \otimes \sigma^3 \otimes \gamma^5,$$

$$\Gamma^{y_4} = i\mathbf{1}_2 \otimes \sigma^2 \otimes \sigma^3 \otimes \gamma^5,$$

$$\Gamma^{y_5} = i\sigma^1 \otimes \sigma^3 \otimes \sigma^3 \otimes \gamma^5,$$

$$\Gamma^{y_6} = i\sigma^2 \otimes \sigma^3 \otimes \sigma^3 \otimes \gamma^5,$$

$$\Gamma_{4D} = \mathbf{1}_2 \otimes \mathbf{1}_2 \otimes \mathbf{1}_2 \otimes \gamma^5,$$

$$\Gamma_{\text{in}} = \sigma^3 \otimes \sigma^3 \otimes \sigma^3 \otimes \gamma^5,$$

$$C_+ = \sigma^1 \otimes \sigma^2 \otimes \sigma^1 \otimes iC_+^{4d},$$

$$C_- = \sigma^2 \otimes \sigma^1 \otimes \sigma^2 \otimes C_-^{4d},$$

Appendix B

Supersymmetric quantum mechanics

In this chapter, we note the supplement topics of the supersymmetric quantum mechanics.

B.1 $\mathcal{N} = 2$ SQM supercharges

We listed the representations of the higher dimensional gamma matrices in [Section A.2](#). Just to be sure, in this section, we note the explicit form of the supercharges of the higher dimensional realization of the $\mathcal{N} = 2$ SQM ([3.20a](#)) up to ten-dimensions explicitly.

The supercharges are constructed by the operators A and $A^\dagger$,

$$A = i\gamma^y \partial_y - M, \quad A^\dagger = -i\gamma^y \partial_y - M. \quad (\text{B.1})$$

Therefore, we list the operator A and $A^\dagger$ in this section.¹

We use the complex partial differential operators ∂_m and $\bar{\partial}_m$,

$$\partial_m = \partial_{y_{2m+1}} - i\partial_{y_{2m}}, \quad \bar{\partial}_m = \partial_{y_{2m+1}} + i\partial_{y_{2m}}, \quad (\text{B.2})$$

for $m = 1, \dots, d$ ($m = 1, \dots, d-1$) in even (odd) dimensions, which have the properties,

$$\partial_m^\dagger = -\bar{\partial}_m, \quad \bar{\partial}_m^\dagger = -\partial_m, \quad (\text{B.3})$$

The operators respect the complex coordinates $z_m = y_{2m+1} + iy_{2m}$ and $\bar{z}_m = y_{2m+1} - iy_{2m}$ as

$$\partial_m = \partial_{z_m}, \quad \bar{\partial}_m = \partial_{\bar{z}_m}. \quad (\text{B.4})$$

5D Gamma matrices In the five-dimensions, the internal gamma matrices are trivial. That is, the higher dimensional gamma matrices have the same size as the 4D one. As we mentioned in [Section 2.4](#), the operators A and $A^\dagger$ are given as,

$$A = +\partial_y + M, \quad A^\dagger = -\partial_y + M. \quad (\text{B.5})$$

¹ As we mentioned in [Section 3.2](#), the operators A and $A^\dagger$ are the Dirac operator (with mass M) on the internal space itself. So we just list the Dirac operator of mass M .

6D Gamma matrices As we mentioned in [Chapter 5](#), the internal gamma matrices are the Pauli matrices in the six-dimensions. Thus, the operators A and $A^\dagger$ are given as,

$$A = - \begin{pmatrix} M & \partial_1 \\ \bar{\partial}_1 & M \end{pmatrix}, \quad A^\dagger = - \begin{pmatrix} M & -\partial_1 \\ -\bar{\partial}_1 & M \end{pmatrix}. \quad (\text{B.6})$$

7D Gamma matrices The y_3 direction cannot be complex coordinates. Therefore, the bare differential operator ∂_{y_3} appears in the operator A and $A^\dagger$,

$$A = - \begin{pmatrix} \partial_{y_3} + M & \partial_1 \\ \bar{\partial}_1 & -\partial_{y_3} + M \end{pmatrix}, \quad A^\dagger = - \begin{pmatrix} -\partial_{y_3} + M & -\partial_1 \\ -\bar{\partial}_1 & \partial_{y_3} + M \end{pmatrix}. \quad (\text{B.7})$$

8D Gamma matrices We note only the operator A represented both tensor product and matrix form,

$$-A = \begin{pmatrix} & \partial_2 \\ \bar{\partial}_2 & \end{pmatrix} \otimes \sigma^3 + \mathbf{1}_2 \otimes \begin{pmatrix} & \partial_1 \\ \bar{\partial}_1 & \end{pmatrix} + M \quad (\text{B.8})$$

$$= \begin{pmatrix} M & \partial_1 & \partial_2 & \\ \bar{\partial}_1 & M & & -\partial_2 \\ \bar{\partial}_2 & & M & \partial_1 \\ & -\bar{\partial}_2 & \bar{\partial}_1 & M \end{pmatrix}. \quad (\text{B.9})$$

9D Gamma matrices We note only the operator A with both tensor product and matrix form,

$$-A = \begin{pmatrix} & \partial_2 \\ \bar{\partial}_2 & \end{pmatrix} \otimes \sigma^3 + \mathbf{1}_2 \otimes \begin{pmatrix} & \partial_1 \\ \bar{\partial}_1 & \end{pmatrix} + \sigma_3 \otimes \sigma_3 \partial_{y_9} + M \quad (\text{B.10})$$

$$= \begin{pmatrix} \partial_{y_9} + M & \partial_1 & \partial_2 & & \\ \bar{\partial}_1 & -\partial_{y_9} + M & & -\partial_2 & \\ \bar{\partial}_2 & & -\partial_{y_9} + M & \partial_1 & \\ & -\bar{\partial}_2 & \bar{\partial}_1 & \partial_{y_9} + M & \end{pmatrix}. \quad (\text{B.11})$$

10D Gamma matrices The operators A and $A^\dagger$ is the 8-by-8 matrices. We also note only the operator A with both tensor product and matrix form,

$$-A = \begin{pmatrix} & \partial_3 \\ \bar{\partial}_3 & \end{pmatrix} \otimes \sigma^3 \otimes \sigma^3 + \mathbf{1}_2 \otimes \begin{pmatrix} & \partial_2 \\ \bar{\partial}_2 & \end{pmatrix} \otimes \sigma^3 \\ + \mathbf{1}_2 \otimes \mathbf{1}_2 \otimes \begin{pmatrix} & \partial_1 \\ \bar{\partial}_1 & \end{pmatrix} + M \quad (\text{B.12})$$

$$= \begin{pmatrix} M & \partial_1 & \partial_2 & \partial_3 & & & & \\ \bar{\partial}_1 & M & & -\partial_2 & -\partial_3 & & & \\ \bar{\partial}_2 & & M & \partial_1 & & -\partial_3 & & \\ & -\bar{\partial}_2 & \bar{\partial}_1 & M & & & \partial_3 & \\ \bar{\partial}_3 & & & & M & \partial_1 & \partial_2 & \\ & -\bar{\partial}_3 & & & \bar{\partial}_1 & M & & -\partial_2 \\ & & -\bar{\partial}_3 & & \bar{\partial}_2 & M & \partial_1 & \\ & & & \bar{\partial}_3 & -\bar{\partial}_2 & \bar{\partial}_1 & M & \end{pmatrix}. \quad (\text{B.13})$$

Appendix C

Boundary condition

We note the supplement of discussion of the boundary condition.

C.1 Derivation of the surface integral term

In [Chapter 4](#), we introduce the non-local boundary condition (4.1) without any notion. In this chapter, we derive the non-local boundary condition, i.e. the surface integral, in various method. It should be note that it is easy to derive the local boundary condition at each point on boundary form the non-local boundary condition.

In this section, for saving spaces, we use shorthanded notations as,

$$\int_{M^4} dx = \int_{M_4} d^4x, \quad \int_{\Omega} dy = \int_{\Omega} dy^d, \quad \oint_{\partial\Omega} dy = \oint_{\partial\Omega} d^{d-1}y. \quad (\text{C.1})$$

Before the discussion, we introduce the Stokes' theorem (also known as the Gauss' law, the Green' theorem and the divergence theorem).

Theorem (The Stokes' theorem). *Given a smooth vector $\omega^y(y)$ defined on a d dimensional orientated manifold Ω with a boundary $\partial\Omega$. The integration of the divergence of the vector $\partial_y \omega^y(y)$ over the whole manifold Ω is equal to the integration of $n_y \omega^y(y)$ over the surface $\partial\Omega$,*

$$\int_{\Omega} dy \partial_y \omega^y(y) = \oint_{\partial\Omega} dy n_y \omega^y(y),$$

where n_y is a normal vector to the surface $\partial\Omega$.

This section organized as follow; in [Section C.1.1](#), we derive the non-local boundary condition by only assuming the variation principle. We show the Hermiticity of the action provide the non-local boundary condition in [Section C.1.2](#), and the conservation law (the continuity equation) can also provide it in [Section C.1.3](#).

C.1.1 The variation principle

The variation principle, $\delta S = 0$, provides the equation of motion and the surface term in general. Thus, one can derive the non-local boundary condition through the variation principle.

We take a variation δS of the higher dimensional Dirac field $\Psi(x, y)$ as,

$$\delta S = S[\Psi + \tilde{\Psi}, \bar{\Psi} + \tilde{\bar{\Psi}}] - S[\Psi, \bar{\Psi}], \quad (\text{C.2})$$

where $\tilde{\Psi}(x, y)$ is any variation function. We assume that the higher dimensional Dirac field satisfies some boundary condition and we take the variation function from the same function space. In other words, we assume that the variation function $\tilde{\Psi}(x, y)$ satisfies the same boundary condition as the higher dimensional Dirac field $\Psi(x, y)$.

The variation principle provides the equation of motion of the higher dimensional Dirac field $\Psi(x, y)$,

$$(i\Gamma^\mu \partial_\mu + i\Gamma^y \partial_y - M)\Psi(x, y) = 0, \quad (\text{C.3})$$

and the surface term,

$$\int_{M^4} dx \int_{\Omega} dy i \partial_y (\bar{\Psi}(x, y) \Gamma^y \tilde{\Psi}(x, y)) = 0. \quad (\text{C.4})$$

By the Stokes' theorem, the surface term is equivalent to a surface integral of the higher dimensional Dirac field $\Psi(x, y)$ and the variation function $\tilde{\Psi}(x, y)$,

$$\oint_{\partial\Omega} dy i n_y \bar{\Psi}(x, y) \Gamma^y \tilde{\Psi}(x, y) = 0. \quad (\text{C.5})$$

We note that The 4D Minkowski space M^4 has no boundary. Thus, there is no integral over the boundary of the 4D Minkowski space M^4 .¹

If the surface integral of the higher dimensional Dirac field vanishes,

$$\oint_{\partial\Omega} dy i n_y \bar{\Psi}(x, y) \Gamma^y \Psi(x, y) = 0, \quad (\text{C.6})$$

the surface integral with the variation function (C.7) vanishes because of the assumption that the higher dimensional Dirac field $\Psi(x, y)$ and the variation function $\tilde{\Psi}(x, y)$ satisfy the same boundary condition.

Hence, it is enough to discuss this surface integral (C.7). With the representation of the higher dimensional gamma matrices which respects the $SO(1, 3) \times SO(d)$ subgroup,² one finds that the surface integral has only off-diagonal components,

$$\oint_{\Omega} dy \begin{pmatrix} \Psi_L(x, y) \\ \Psi_R(x, y) \end{pmatrix}^\dagger \begin{pmatrix} i n_y (\mathbf{1}_2 \otimes \gamma^y) \\ -i n_y (\mathbf{1}_2 \otimes \gamma^y) \end{pmatrix} \begin{pmatrix} \Psi_L(x, y) \\ \Psi_R(x, y) \end{pmatrix} = 0, \quad (\text{C.7})$$

where the higher dimensional field $\Psi_L(x, y)$ and $\Psi_R(x, y)$ are defined as,

$$\Psi(x, y) = \mathbf{e}_L \otimes \Psi_L(x, y) + \mathbf{e}_R \otimes \Psi_R(x, y). \quad (\text{C.8})$$

Therefore, we have shown that the variation principle provides the non-local boundary condition. It is not difficult to show the local boundary condition given in [Chapter 4](#).

¹ Strictly speaking, this is an technical assumption because the 4D Minkowski space is not compact.

² The representation is given in (2.22).

C.1.2 The Hermiticity of the action

One can also derive the non-local boundary condition from the Hermiticity of the action, $S^\dagger = S$.

If the action is Hermitian, $S^\dagger = S$, without any surface term, the following surface term should vanish,

$$\oint_{\partial\Omega} dy n_y \bar{\Psi}(x, y) i\Gamma^y \Psi(x, y) = 0. \quad (\text{C.9})$$

Therefore, the Hermiticity of the action provides the non-local boundary condition.

C.1.3 The conservation law (the continuity equation) of the $U(1)$ current

Also, the surface integral can be derived from the conservation law (the continuity equation) of the $U(1)$ current,

$$\partial_\mu J^\mu(x, y) + \partial_y J^y(x, y) = 0, \quad (\text{C.10})$$

for all point of the space-time $M^4 \times \Omega$ (including the boundary). The higher dimensional $U(1)$ current, $J^\mu(x, y)$ and $J^y(x, y)$, of the higher dimensional Dirac field $\Psi(x, y)$ is defined as,

$$J^\mu(x, y) = \bar{\Psi}(x, y) \Gamma^\mu \Psi(x, y), \quad J^y(x, y) = \bar{\Psi}(x, y) \Gamma^y \Psi(x, y). \quad (\text{C.11})$$

The conservation law (the continuity equation) holds over the whole space-time. Therefore, an integration of the equation also vanishes,

$$\int_{M^4} dx \int_{\Omega} dy^{d-4} \partial_N J^N = 0. \quad (\text{C.12})$$

By the Stokes' theorem, one finds the surface integral term,

$$\int_{M^4} dx \int_{\Omega} dy \partial_N J^N = \oint_{\partial\Omega} dy i n_y \bar{\Psi}(x, y) \Gamma^y \Psi(x, y) = 0. \quad (\text{C.13})$$

Therefore, the conservation law (the continuity equation) of the $U(1)$ current provides the non-local boundary condition.

C.1.4 Summary: Derivation of the surface integral term

In this section, we showed multiple ways deriving the non-local boundary condition, i.e. the surface integral term. The last two ways are very easy to understand because of its clear logic. However, it is difficult to apply to the real-valued field. If the field values real (for example, the real scalar field and the Majorana spinor field), the surface term will vanishes trivially. On the other hand, the variation principle is highly applicable to various models. However, the surface term (C.6) of the real valued field trivially vanishes. Therefore, we should solve the equation (C.5) directly.

Appendix D

Models

In this chapter, we note the supplement discussion of the models.

D.1 One-dimensional interval internal space

In this model, any massive mode functions are given as a trigonometric functions driven from the sine function. We note the explicit form in this section [18].

In the five-dimensional models with an interval internal space, any massive mode function can be given as a sine function. Thus, we prepare a normalized sine function $h(y)$ on an interval $[0, L]$,

$$h(y) = \sqrt{\frac{4p}{2p\pi L - \sin(2p\pi L)}} \sin(py) . \quad (\text{D.1})$$

The function $h(y)$ is normalized in L^2 norm,

$$\int_0^L dy h(y)^2 = 1 . \quad (\text{D.2})$$

The boundary conditions of each side restrict the value of the momentum p (see [Section 5.4](#)).

In the five-dimensional models, the operators A and $A^\dagger$ are just the first-degree deferential operators $+\partial_y + M$ and $-\partial_y + M$ respectively (see [Section 2.4](#)). One can obtain the chiral partners by operating A and $A^\dagger$ onto the function $h(y)$,

$$\begin{aligned} & \frac{1}{\sqrt{M^2 + p^2}} (\pm \partial_y + M) h(y) \\ &= \frac{1}{\sqrt{M^2 + p^2}} \sqrt{\frac{4p}{2p\pi L - \sin(2p\pi L)}} (\pm \cos(py) + M \sin(py)) . \end{aligned} \quad (\text{D.3})$$

The mass M shifts the origin (or the phase) of the mode functions. If $M = 0$, the chiral partners of the mode functions $h(y)$ will be the cosine function.

We plot the some mode functions appeared in [Section 5.4](#) in [Figure D.1](#) and [Figure D.2](#).

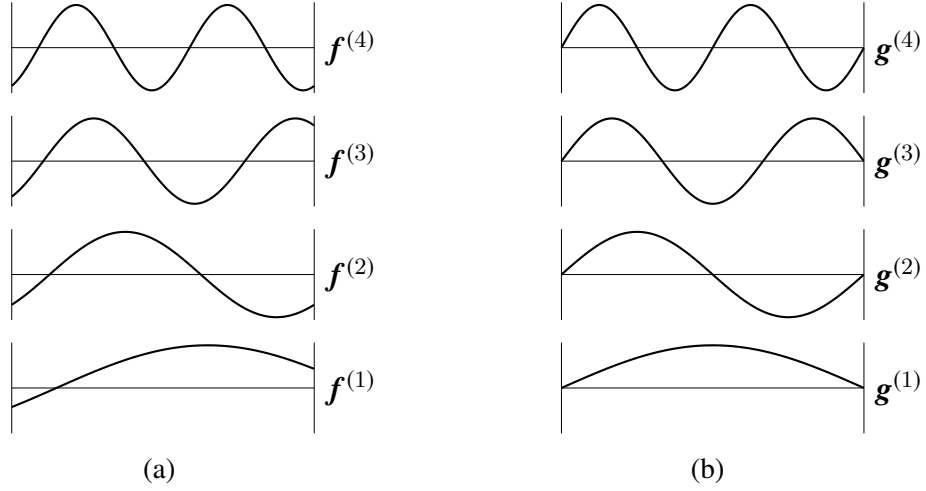


Figure D.1: The plot of the mode functions with RH and RH boundary condition. The right-handed mode functions are the sine functions itself (figure (a)). The left-handed mode functions are the shifted cosine functions (figure (b)).

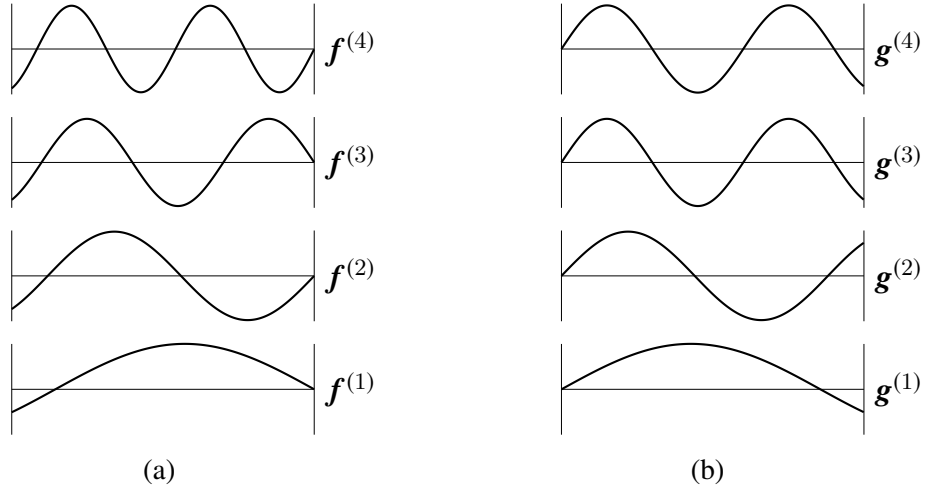


Figure D.2: The plot of the mode functions with RH and LH boundary condition. The left-handed and right-handed mode functions are the shifted trigonometric functions (figure (a) and figure (b) respectively). The shift happens by the mass M dependence of the momentum p though the boundary condition.

Appendix E

Mathematical Supplement

In this chapter, we note the mathematical supplement.

E.1 Solution of the screened Poisson equation

In general, the massless mode function is a solution of the d ($1 < d$) dimensional screened Poisson equation [34],

$$(-\partial_y^2 + M^2)h(y) = 0. \quad (\text{E.1})$$

In this section, we note the solution in the hyperspherical coordinate [34, 45–50].

The screened Poisson operator $-\partial_y^2 + M^2$ is the linear combination of the Laplace operator $-\partial_y^2$ and the mass term M^2 . By taking the hyperspherical coordinates, one can separate the variables into the radial part and the angular part,

$$-\partial_y^2 + M^2 = -\sum_{k=1}^d \frac{\partial^2}{\partial y_k^2} = -\frac{1}{|y|^{d-1}} \frac{\partial}{\partial |y|} |y|^{d-1} \frac{\partial}{\partial |y|} - \frac{1}{|y|^2} \Delta_{S^{d-1}} + M^2, \quad (\text{E.2})$$

where $-\Delta_{S^{d-1}}$ is a Laplace operator on a $(d-1)$ -sphere which is recursively defined as,

$$\Delta_{S^{d-1}} = \frac{1}{\sin^{2-d} \theta_{d-1}} \frac{\partial}{\partial \theta_{d-1}} \sin^{d-2} \theta_{d-1} \frac{\partial}{\partial \theta_{d-1}} + \frac{1}{\sin^2 \theta_{d-1}} \Delta_{S^{d-2}}. \quad (\text{E.3})$$

The symbol θ_k ($k = 1, \dots, d-1$) are the coordinates of the $(d-1)$ -sphere.

The Laplace operator on a $(d-1)$ -sphere $-\Delta_{S^{d-1}}$ has positive-definite eigenvalues $\ell(\ell + d - 2)$ where $\ell = 1, 2, \dots$. By denoting the eigenfunctions as $Y_m^\ell(\vec{\theta})$, the eigenvalue equation is given as,

$$-\Delta_{S^{d-1}} Y_m^\ell(\vec{\theta}) = \ell(\ell + d - 2) Y_m^\ell(\vec{\theta}), \quad (\text{E.4})$$

where $Y_m^\ell(\vec{\theta})$ are the hyperspherical harmonics functions defined by the Legendre functions

P_m^ℓ as,

$$Y_m^\ell(\vec{\theta}) = \frac{1}{\sqrt{2\pi}} e^{im_1\theta_1} {}_{d-1}\bar{P}_\ell^{m_{d-2}}(\theta_{d-1}) \prod_{j=2}^{d-2} {}_j\bar{P}_{m_j}^{m_{j-1}}(\theta_j), \quad (\text{E.5})$$

$${}_j\bar{P}_L^\ell(\theta) = \sqrt{\frac{2L+j-1}{2} \frac{(L+\ell+j-2)!}{(L-\ell)!}} \sin^{\frac{2-j}{2}}(\theta) P_{L+\frac{j-2}{2}}^{-\left(\ell+\frac{j-2}{2}\right)}(\cos(\theta)). \quad (\text{E.6})$$

The “magnetic quantum numbers” m_k ($k = 1, \dots, d-2$) value integer and are bounded by the “azimuthal quantum number” ℓ as,

$$|m_1| \leq m_2 \leq \dots \leq m_{d-2} \leq \ell. \quad (\text{E.7})$$

Hence, the variables of the solution of the screened Poisson equation will be separated as,

$$h(y) = |y|^{2-d/2} R(|y|) Y_m^\ell(\vec{\theta}), \quad (\text{E.8})$$

where $R(Mx)$ is a radial part which is the solution of the modified Bessel equation with $n = (d + 2\ell - 2)/2$,

$$\left[(Mx)^2 \frac{\partial^2}{\partial (Mx)^2} + (Mx) \frac{\partial}{\partial (Mx)} - \left((Mx)^2 + \frac{(d + 2\ell - 2)^2}{2^2} \right) \right] R(Mx) = 0, \quad (\text{E.9})$$

where

$$\frac{(d + 2\ell - 2)^2}{2^2} = \frac{(d - 4)d}{4} + \ell(\ell + d - 2) + 1. \quad (\text{E.10})$$

There are two individual solutions $K_n(x)$ and $I_n(x)$ of the modified Bessel equation called modified Bessel functions. We note the asymptotic behavior of them,

$$\lim_{x \rightarrow +0} I_n(x) = 0, \quad \lim_{x \rightarrow \infty} I_n(x) = \infty, \quad (\text{E.11})$$

$$\lim_{x \rightarrow +0} K_n(x) = \infty, \quad \lim_{x \rightarrow \infty} K_n(x) = 0, \quad (\text{E.12})$$

for all $0 < n$ except $I_0(x)$,

$$\lim_{x \rightarrow +0} I_0(x) = 1. \quad (\text{E.13})$$

E.2 Direct sum and tensor product

We define direct sum and tensor product belfry [51].

We denote $\text{Mat}(n, m, ; \mathbf{K})$ by the set of n -by- m matrix on the field $\mathbf{K}$. Given a vector space $\mathcal{V}$ and its subspace D , the set $\{ \sum_{i=1}^k a_i u_i \mid k \in \mathbf{N}, u_i \in D, a_i \in \mathbf{K}, i = 1, \dots, k \}$ is called a subspace generated by D .

E.2.1 Cartesian product, Direct sum and Direct product

Frankly speaking, the direct sum is a way to enlarge the vector space by *adding* two vector spaces. The dimensions of the new vector space is equal to the sum of the dimensions of original vector spaces.

On the vector space, we can identify the direct product to the direct sum. Therefore, we only introduce Cartesian product and direct sum independently of basis.

Definition (Cartesian product/direct product). Given sets A and B , the Cartesian product $A \times B$ is the set of all ordered pairs (a, b) where $a \in A$ and $b \in B$,

$$A \times B := \{ (a, b) \mid a \in A, b \in B \}.$$

Definition (direct sum). Given $\mathcal{V}$ and $\mathcal{W}$ are vector spaces over the field $\mathbf{K}$, the Cartesian product $\mathcal{V} \times \mathcal{W}$ can be given the structure of a vector space over $\mathbf{K}$ by defining the operations component-wise,

$$\begin{aligned} (v_1, w_1) + (v_2, w_2) &= (v_1 + v_2, w_1 + w_2), \\ \alpha(v, w) &= (\alpha v, \alpha w), \end{aligned}$$

where $v_1, v_2 \in \mathcal{V}$, $w_1, w_2 \in \mathcal{W}$ and $\alpha \in \mathbf{K}$. The resulting vector space is denoted by $\mathcal{V} \oplus \mathcal{W}$. The component is also denoted as $v \oplus w$.

Corollary. Given finite dimensional vector spaces $\mathcal{V}$ and $\mathcal{W}$, $\dim \mathcal{V} \oplus \mathcal{W} = \dim \mathcal{V} + \dim \mathcal{W}$.

Example ($\mathbf{R}^n$). The vector space $\mathbf{R}^2$ is defined by $\mathbf{R} \oplus \mathbf{R}$. Repeatedly using it, the vector space $\mathbf{R}^n$ for $1 < n$ is defined.

Example ($A \oplus B$). Given matrices $A \in \text{Mat}(n_A, m_A; \mathbf{K})$ and $B \in \text{Mat}(n_B, m_B; \mathbf{K})$, the direct sum $A \oplus B$ can be identified with $\text{diag}(A, B) \in \text{Mat}(n_A + n_B, m_A + m_B; \mathbf{K})$.

E.2.2 Tensor product

Frankly speaking, tensor product gives a new vector space whose dimensions are the *product* of dimensions of the original spaces. We note that the essence of tensor product is the bilinear function.

First, we introduce the definition of the tensor product depending on the basis.

Definition (tensor product). Given bases $\{e_i^{\mathcal{V}}\}$ ($i = 1, \dots, \dim \mathcal{V}$) and $\{e_j^{\mathcal{W}}\}$ ($j = 1, \dots, \dim \mathcal{W}$) for vector spaces $\mathcal{V}$ and $\mathcal{W}$ over the field $\mathbf{K}$ respectively, the tensor product $\mathcal{V} \otimes \mathcal{W}$ is the vector space over $\mathbf{K}$ generated by the Cartesian product $\{e_i^{\mathcal{V}}\} \times \{e_j^{\mathcal{W}}\}$. The pair is denoted by $e_i^{\mathcal{V}} \otimes e_j^{\mathcal{W}} = (e_i^{\mathcal{V}}, e_j^{\mathcal{W}})$ and the vector space is also denoted by $\mathcal{V} \otimes \mathcal{W}$.

We also denote the tensor product of $v = \sum_{i=1}^{\dim \mathcal{V}} v_i e_i^{\mathcal{V}} \in \mathcal{V}$ and $w = \sum_{j=1}^{\dim \mathcal{W}} w_j e_j^{\mathcal{W}} \in \mathcal{W}$ as

$$v \otimes w = \sum v_i w_j e_i^{\mathcal{V}} \otimes e_j^{\mathcal{W}}.$$

By the definition, it is clear that the tensor product is a bilinear function,

$$\begin{aligned}(v_1 + v_2) \otimes w &= (v_1 + v_2) \otimes w, \\ v \otimes (w_1 + w_2) &= v \otimes w_1 + v \otimes w_2, \\ \alpha(v_1 \otimes w_2) &= (\alpha v_1) \otimes w_2 = v_1 \otimes (\alpha w_2),\end{aligned}$$

where $v_1, v_2 \in \mathcal{V}$, $w_1, w_2 \in \mathcal{W}$ and $\alpha \in \mathbf{K}$. We note that the tensor product is not non-commutative in general, $v \otimes w \neq w \otimes v$.

We introduce the basis independent definition of tensor product.

Definition (dual vector space). Given vector space $\mathcal{V}$ over a field $\mathbf{K}$, the dual vector space $\mathcal{V}^*$ is defined as the set of all linear functionals $\phi : \mathcal{V} \rightarrow \mathbf{K}$ satisfying,

$$\begin{aligned}(\phi + \psi)(v) &= \phi(v) + \psi(v), \\ (\alpha\phi)(v) &= \alpha(\phi(v)),\end{aligned}$$

for all $v \in \mathcal{V}$ and $\alpha \in \mathbf{K}$.

Definition (tensor product). A tensor product $\mathcal{V}$ and $\mathcal{W}$ over a field $\mathbf{K}$ is the vector space $\mathcal{V} \otimes \mathcal{W}$ generated by $v \otimes w$ mapping $v \otimes w : \mathcal{V}^* \times \mathcal{W}^* \rightarrow \mathbf{K}$ by,

$$(v \otimes w)(\phi, \psi) = \phi(v)\psi(w).$$

Corollary. Given finite dimensional vector spaces $\mathcal{V}$ and $\mathcal{W}$, that is, $\dim \mathcal{V} \otimes \mathcal{W} = \dim \mathcal{V} \dim \mathcal{W}$.

Example ($\mathbf{R}^n \otimes \mathbf{R}^m$). There is an isomorphism between $\mathbf{R}^n \otimes \mathbf{R}^m$ and $\text{Mat}(n, m; \mathbf{R})$, that is, the tensor product $\mathbf{R}^n \otimes \mathbf{R}^m$ can be identified with $\text{Mat}(n, m; \mathbf{R})$.

Example ($A \otimes B$). Given matrices $A \in \text{Mat}(n_A, m_A; \mathbf{K})$ and $B \in \text{Mat}(n_B, m_B; \mathbf{K})$, tensor product $A \otimes B$ can be identified with matrix belonging to $\text{Mat}(n_A n_B, m_A m_B; \mathbf{K})$ by

$$A \otimes B = \begin{pmatrix} a_{11}B & a_{12}B & \cdots \\ a_{21}B & a_{22}B & \\ \vdots & & \ddots \end{pmatrix} = \begin{pmatrix} a_{11}b_{11} & a_{11}b_{12} & \cdots \\ a_{11}b_{21} & a_{11}b_{22} & \\ \vdots & & \ddots \end{pmatrix}.$$

We use that rule of the example as nesting rule, in this thesis.

Bibliography

- [1] G. Aad *et al.*, [Physics Letters B **716**, 1 \(2012\)](#), [arXiv:1207.7214](#) .
- [2] S. Chatrchyan *et al.*, [Physics Letters B **716**, 30 \(2012\)](#), [arXiv:1207.7235](#) .
- [3] C. Patrignani *et al.*, [Chinese Physics C **40**, 100001 \(2016\)](#).
- [4] C. S. Wu, E. Ambler, R. W. Hayward, D. D. Hoppes, and R. P. Hudson, [Physical Review **105**, 1413 \(1957\)](#).
- [5] N. Cabibbo, [Physical Review Letters **10**, 531 \(1963\)](#).
- [6] M. Kobayashi and T. Maskawa, [Progress of Theoretical Physics **49**, 652 \(1973\)](#).
- [7] J.-M. Frère, M. Libanov, and S. Troitsky, [Physics Letters B **512**, 169 \(2001\)](#).
- [8] M. Libanov and S. Troitsky, [Nuclear Physics B **599**, 319 \(2001\)](#), [arXiv:0011095 \[hep-ph\]](#) .
- [9] Z. Horvath, L. Palla, E. Cremmer, and J. Scherk, [Nuclear Physics B **127**, 57 \(1977\)](#).
- [10] S. Randjbar-Daemi, A. Salam, and J. Strathdee, [Nuclear Physics B **214**, 491 \(1983\)](#).
- [11] D. Cremades, L. Ibanez, and F. Marchesano, [Journal of High Energy Physics **2004**, 079 \(2004\)](#), [arXiv:0404229 \[hep-th\]](#) .
- [12] L. Randall and R. Sundrum, [Physical Review Letters **83**, 9 \(1999\)](#), [arXiv:9905221 \[hep-ph\]](#) .
- [13] Y. Kawamura, [Progress of Theoretical Physics **105**, 999 \(2001\)](#), [arXiv:0012125 \[hep-ph\]](#) .
- [14] Y. Kawamura, [Progress of Theoretical Physics **103**, 613 \(2000\)](#), [arXiv:9902423v2 \[hep-ph\]](#) .
- [15] J. Polchinski, *String Theory: Volume 1, An Introduction to the Bosonic String* (Cambridge University Press, 1998) p. 424.
- [16] J. Polchinski, *String Theory: Volume 2, Superstring Theory and Beyond* (Cambridge University Press, 2005) p. 531.

- [17] C. S. Lim, T. Nagasawa, M. Sakamoto, and H. Sonoda, *Physical Review D* **72**, 064006 (2005), [arXiv:0502022 \[hep-th\]](#) .
- [18] Y. Fujimoto, K. Hasegawa, T. Nagasawa, K. Nishiwaki, M. Sakamoto, and K. Tatsumi, *Journal of High Energy Physics* **2016**, 178 (2016), [arXiv:1601.05265](#) .
- [19] Y. Fujimoto, K. Hasegawa, K. Nishiwaki, M. Sakamoto, and K. Tatsumi, *Progress of Theoretical and Experimental Physics* **2017** (2017), 10.1093/ptep/ptx092, [arXiv:1609.04565](#) .
- [20] Y. Fujimoto, K. Hasegawa, K. Nishiwaki, M. Sakamoto, and K. Tatsumi, *Nuclear Physics B* **922**, 186 (2017), [arXiv:1609.01413](#) .
- [21] E. Witten, *Nuclear Physics B* **188**, 513 (1981).
- [22] M. E. Peskin and D. V. Schroeder, *An Introduction to Quantum Field Theory*, first edit ed. (Westview Press, 1995) p. 864.
- [23] M. Srednicki, *Quantum Field Theory*, Vol. 130 (Cambridge University Press, Cambridge, 2007) pp. 83–94.
- [24] M. D. Schwartz, *Quantum field theory and the Standard Model* (Cambridge University Press, 2013) p. 863.
- [25] B. de Wit and J. Smith, *Field Theory in Particle Physics* (North-Holland Personal Library, 1986) p. 490.
- [26] A. Van Proeyen, [arXiv:9910030 \[hep-th\]](#) .
- [27] J.-H. Park, in *Modave Summer School in Mathematical Physics, Belgium* (2005) p. 22.
- [28] J. Wess and J. Bagger, *Supersymmetry and Supergravity*, 2nd ed. (Princeton University Press, 1992) p. 272.
- [29] M. de Crombrugghe and V. Rittenberg, *Annals of Physics* **151**, 99 (1983).
- [30] F. Cooper, A. Khare, and U. Sukhatme, *Physics Reports* **251**, 267 (1995), [arXiv:9405029 \[hep-th\]](#) .
- [31] E. Witten, *Journal of Differential Geometry* **17**, 661 (1982).
- [32] L. Alvarez-Gaumé, *Communications in Mathematical Physics* **90**, 161 (1983).
- [33] L. Alvarez-Gaumé, *Journal of Physics A: Mathematical and General* **16**, 4177 (1983).
- [34] P. K. Kythe, *Green's Functions and Linear Differential Equations: Theory, Applications, and Computation* (Chapman and Hall/CRC, 2011) p. 382.
- [35] Y. Fujimoto, K. Hasegawa, K. Nishiwaki, M. Sakamoto, I. Ueba, and K. Tatsumi, in preparation .

- [36] K. Gustafson, in *Domain Decomposition Methods 10*, Contemporary Mathematics, Vol. 218, edited by J. Mandel, C. Farhat, and X.-C. Cai (American Mathematical Society, Providence, Rhode Island, 1998) pp. 432–437.
- [37] C. Gordon, D. Webb, and S. Wolpert, *Inventiones Mathematicae* **110**, 1 (1992).
- [38] C. Chevalley, *Theory of Lie Groups* (Princeton University Press, 1946) p. 232.
- [39] J. Adams, *Lectures on Lie groups* (University of Chicago Press, 1969) p. 192.
- [40] S. Helgason, *Differential Geometry, Lie Groups, and Symmetric Spaces*, Graduate Studies in Mathematics, Vol. 34 (American Mathematical Society, Providence, Rhode Island, 2001) pp. xiv+486.
- [41] F. Wilczek and A. Shapere, *Geometric Phases in Physics*, Advanced Series in Mathematical Physics, Vol. 5 (WORLD SCIENTIFIC, 1989) p. 528.
- [42] T. Cheon, T. Fülöp, and I. Tsutsui, *Annals of Physics* **294**, 1 (2001), [arXiv:0008123 \[quant-ph\]](https://arxiv.org/abs/0008123) .
- [43] T. Friedrich, *Dirac Operators in Riemannian Geometry*, Graduate Studies in Mathematics, Vol. 25 (American Mathematical Society, Providence, Rhode Island, 2000) p. 195.
- [44] J. Jost, *Riemannian Geometry and Geometric Analysis*, Universitext (Springer-Verlag, Berlin/Heidelberg, 2005) p. 566.
- [45] H. Cohen, *Number Theory Volume II: Analytic and Modern Tools*, Graduate Texts in Mathematics, Vol. 240 (Springer-Verlag New York, New York, NY, 2007) p. 596.
- [46] G. Teschl, *Ordinary differential equations and dynamical systems*, Vol. 140 (American Mathematical Society, 2012) p. 356.
- [47] D. Zwillinger, *Table of Integrals, Series, and Products*, 8th ed., edited by D. Zwillinger (Academic Press, 2014) p. 1184.
- [48] F. W. J. Olver, D. W. Lozier, R. F. Boisvert, and C. W. Clark, *NIST Handbook of mathematical functions* (Cambridge University Press, 2010) p. 968.
- [49] F. W. J. Olver, D. W. Lozier, R. F. Boisvert, and C. W. Clark, “NIST Digital Library of Mathematical Functions,” <http://dlmf.nist.gov/>.
- [50] J. S. Avery, *Journal of Computational and Applied Mathematics* **233**, 1366 (2010).
- [51] N. Bourbaki, *Algebra*.