Integrable structures of the Killing equation

Miyamoto, Kentaro

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Integrable structures of the Killing equation

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Graduate School of Science, Kobe University

Kentaro Miyamoto

宮本　健太郎

Abstract

The problem of determining whether a Hamiltonian system is completely integrable has been long discussed since the early days of development of celestial mechanics. A Hamiltonian system with N degrees of freedom is said to be completely integrable in the Liouville sense if there exist N smooth first integrals in involution which are functionally independent. In spite of the fact that we have a well-defined notion of complete integrability, the above problem still remains an open question.

Our purpose in this thesis is to construct a systematic method to enumerate first integrals of a given Hamiltonian system. The main idea is that many Hamiltonian systems can be captured through geodesic problems in curved space(-time)s: Euler's equations for a rigid body emerge from geodesic flow on the special orthogonal group with to a left-invariant metric; In general relativity, the motion of a free particle in a gravitational field can be formulated as geodesic flow on a curved spacetime; In general setting, the trajectories of a *natural Hamiltonian systems*, that are given as the sum of a curved kinetic and a potential energy with the kinetic term being quadratic in momenta, can always be described as geodesics in enlarged spaces, i.e. interactions are *geometrised* by introducing one or more extra dimensions.

In this thesis, we assume that first integrals of a geodesic flow are polynomial in momenta. In this setting, the Hamiltonian function is constructed out of the metric on a (pseudo-)Riemannian manifold, and thus the first integrals must be associated with *Killing tensor fields* obeying the *Killing equation*. It is a simple fact, usually known as Noether's first theorem, that if there is a first integral linear in momenta, then the metric admits a one-parameter group of isometries generated by a Killing vector field. In an analogous way, polynomial first integrals lead to Killing tensor fields whose orders are equal to the degree of the polynomials. A Killing tensor field generates a canonical transformation which maps the original Hamiltonian system into itself. In general relativity, Carter's constant in the Kerr black hole spacetime directly stems from a second order Killing tensor field.

Our assumptions reduce the integrability problem in Hamiltonian systems to the problem how to solve the Killing equation. Then it is natural to ask the following questions:

- Are there any solutions of the Killing equation for given metrics?
- If the answer is yes, then how many solutions are there?
- What quantities are sufficient to determine the number of solutions?

In this thesis, we study the above issues and give partial answers. In particular, we introduce a systematic method to analyse the Killing equation and to study its properties. A key ingredient here is projection operators called *Young symmetrisers*. Main results are as follows:

(i) We construct an effective way to analyse the Killing equation and to study its properties based on Young symmetrisers. We particularly establish a prolongation procedure which

transforms the Killing equation of a specified order into a closed system dubbed the *prolonged system* by introducing new variables. Then the explicit form of the prolonged system was written out up to the third order.

(ii) We give a formula for the integrability conditions of the prolonged system that put tough restrictions on the Riemann curvature tensor and its derivatives. We also derive the concrete form of the integrability conditions up to the third order. Moreover, we make a conjecture on the Young symmetries of the integrability conditions of a general order. Furthermore, we provide a method for computing the dimension of the solution space of the Killing equation with a specific example.

(iii) We characterise metrics which admit Killing vector fields by local curvature obstructions. The obstructions have been obtained by analysing the integrability condition and the original Killing equation. In particular, we provide the algorithm that tells us exactly how many Killing vector fields exist for a given metric.

(iv) Killing tensor fields arise out of an assumption that first integrals of a geodesic flow are polynomial in momenta. We relax this assumption and conceive of first integrals that are meromorphic in momenta. We then define gauged Killing tensor fields in order to describe first integrals that are meromorphic in momenta. We also study their properties in detail and construct several metrics admitting a nontrivial rational first integral.

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It is believed that in 1675 Isaac Newton once said [1],

> "If I have seen further, it is by standing on the shoulders of *Giants*."

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Chapter 1

Introduction

The problem of determining whether a Hamiltonian system is completely integrable has its roots in the classical literature. In the late 18th century, d'Alambert, Euler, Lagrange, Clairaut had already discussed exact integration of Hamilton's equations. Afterwards, Hamilton and Jacobi had developed an elegant method for solving Hamilton's equations. Against such an interesting background, we shall begin with a historical perspective of the fundamental problem identified by Henri Poincaré to grasp main problem in this thesis. After some twists and turns, we will arrive at the Killing equation that is a main subject in this thesis.

Poincaré's fundamental problem

At the end of the 19th century, Poincaré considered a system given by the Hamiltonian with N degrees of freedom

$$H(I,\theta) \;=\; H_0(I)+\varepsilon H_1(I,\theta)+\varepsilon^2 H_2(I,\theta)+\cdots\,, \qquad \varepsilon \;\ll\; 1\,, \tag{1.1}$$

and promoted the study of this Hamiltonian as the *fundamental problem* of Hamiltonian dynamics. Here H_0 is an *integrable* Hamiltonian depending only on the action variables $I=(I_1,\dots,I_N)$ and εH_1 is a perturbation term periodic in the angle variables $\theta=(\theta_1,\dots,\theta_N)$. A classic example of eq. (1.1) is the planetary motion around the sun. The integrable Hamiltonian represents a sum of two-body Keplerian Hamiltonians (the sun and each planet) and the perturbation results from the mutual interactions between the planets. The small parameter is typically estimated by the ratio of the mass of the heaviest planet (that is, Jupiter) to that of the sun as $\varepsilon \sim 0.5\times 10^{-3}$.

Here and in what follows, a Hamiltonian system with N degrees of freedom is said to be completely integrable in the Liouville sense if there exists N smooth first integrals $(Q^{(1)},\dots,Q^{(N)})$ such that

$$\{Q^{(a)},Q^{(b)}\} \;=\; 0\,, \qquad \text{for any } a,b \;=\; 1,\cdots,N\,. \tag{1.2}$$

We have denoted the Poisson bracket by $\{,\}$. A function Q is said to be a first integral of motion if $\{H,Q\}=0$ holds true. Any autonomous Hamiltonian system has at least one first integral as the Hamiltonian itself. The important feature of completely integrable Hamiltonian systems is that exact solutions of Hamilton's equations can be obtained by quadratures. There are physically significant examples such as the Kepler problem, the Kowalevskaya top, the Toda lattice and geodesic flows in the Kerr black hole spacetime.

Poincaré himself contributed to the problem in his celebrated theorem [2] on the non-existence of first integrals that are analytic with respect to the small parameter. To be more precise, Poincaré's theorem states that there is in general no first integral which can be expressed in the form

$$\Phi(I,\theta) \;=\; \Phi_0(I) + \varepsilon\Phi_1(I,\theta) + \varepsilon^2\Phi_2(I,\theta) + \cdots . \tag{1.3}$$

This result had been applied at once to the restricted three-body problem. Afterwards his theorem is widely quoted as saying: Henri Poincaré proved that the three-body problem cannot be solved analytically; or an integrable Hamiltonian system inevitably fails to be completely integrable if the perturbation exists. However, it should be noted that his theorem is less strong than we imagine. This is because

- Poincaré assumed in his proof that the perturbation term can be expanded in the *infinite* Fourier series, that is

$$H_1(I,\theta) \;=\; \sum_k^{\infty} \hat{H}_1(I)\, e^{ik\theta}\,, \tag{1.4}$$

- We certainly expect that any first integral is analytic with respect to I and θ. However, it does not necessarily need to be analytic with respect to ε.

So if the perturbation term can be written by the finite Fourier series, Poincaré's theorem does not work. More importantly, nowdays we already know completely integrable Hamiltonian systems are very rare. Therefore, nobody is desirous of getting first integrals that are analytic with respect to ε. The important thing will be the discussion for a given value of $\varepsilon = \varepsilon_0$. It should be also stressed that Poincaré did not weed out the possible existence of locally valid first integrals for a certain value of ε.

On the other hand, Poincaré's fundamental problem is intrinsically based on known integrable Hamiltonian systems. In the 19th century, as there were a few known integrable Hamiltonian, this does not matter. Over the years since the 19th century, despite the fact that integrable Hamiltonian systems are very rare, many new and important integrable Hamiltonian systems have been discovered. Apart from Poincaré's fundamental problem, single out integrable Hamiltonian systems remains an open question. In fact, this question had been posed by astronomers in the later of the 20th century, which are known as the problem of the *third integral of motion*.

The third integral of motion

Let us consider the motion of the Milky Way. Our observations allow us to assume that the gravitational potential of our galaxy is time-independent and has an axial symmetry. We are interested in the motion of a star in such a potential. According to S. Chandrasekar [3], the mean collision time can be conservatively estimated as 10^{14}yr whilst the estimated age of the Milky Way is about 10^{10}yr. Hence, the one-body distribution function obeys the collisionless Boltzmann equation

$$\frac{\partial f}{\partial t} + \{f, H\} \;=\; 0\,. \tag{1.5}$$

Here the Hamiltonian with three degrees of freedom is taken to be

$$H \;=\; \frac{1}{2}\left(p_r^2 + \frac{p_\theta^2}{r^2} + p_z^2\right) + V(r,z)\,, \tag{1.6}$$

where (r, θ, z) is cylindrical coordinates. For simplicity we further assume it is stationary and consequently $\partial f/\partial t = 0$. In this case, the one-body distribution function must be a first integral of the Hamiltonian system (1.6).

Clearly, the Hamiltonian $Q^{(1)} = H$ and the angular momentum $Q^{(2)} = p_\theta$ are first integrals. It seems that other first integrals do not exist for a general potential $V(r,z)$. Therefore, we deduce that $f = f(H, p_\theta)$ which depends on p_r and p_z via the p_r and p_z dependences of H, thereby allowing us to conclude that $\langle p_r^2 \rangle = \langle p_z^2 \rangle$. However, this conclusion is in conflict with the observed distribution of stellar velocities near the sun. Particularly, the observed dispersions of velocities have approximately a $\langle p_r^2 \rangle : \langle p_z^2 \rangle = 2 : 1$ ratio.

Over the years, it had been believed that on the Hamiltonian (1.6) no third integral of motion exists (see e.g. Refs [4, 5, 6]). However, quite unexpectedly, numerical results for a number of galactic orbits implied the existence of the third integral [8, 7]. As a result, many efforts were made to prove analytically the existence of the last integral, $Q^{(3)}$ [7, 9]

In order to sketch the nature of the third integral, we follow Henon–Heiles's simplification. By expanding the Hamiltonian near a circular orbit $r = r_0, z = 0$ and p_θ =const., we obtain a 2-dimensional nonlinear oscillator

$$H = \frac{1}{2}(p_r^2 + p_z^2) + \frac{1}{2}\left[\omega_r^2 (r - r_0)^2 + \omega_z^2 z^2\right] + O(r - r_0, z)^3 , \tag{1.7}$$

where ω_r and ω_z are the constants determined by the derivatives of $V(r,z)$. G. Contopoulos showed the perturbative expression for the third integral of the Hamiltonian (1.7). However, the third integral can be understood in a much simpler way: We take the Hamiltonian as a toy model of eq. (1.7)

$$\bar{H} = \frac{1}{2}(p_1^2 + p_2^2) + \frac{1}{2}(q_1^2 + q_2^2) + V(q_1, q_2) , \tag{1.8}$$

where $V(q_1, q_2)$ is a homogeneous potential of degree three

$$V(q_1, q_2) = \mu q_1^3 + \nu q_1^2 q_2 + \rho q_2^3 , \tag{1.9}$$

where μ, ν, ρ are constant parameters. For the Hamiltonian system (1.8), it is not widely known that the following theorem holds true [10, 11, 12, 51].

Theorem 1 (Hietarinta)**.** *The Hamiltonian system* (1.8) *is completely integrable if and only if the potential V belongs to the following list:*

$$V(q_1, q_2) = q_1^3 , \qquad V(q_1, q_2) = \frac{1}{3} q_1^3 + \frac{\mu}{3} q_2^3 , \qquad V(q_1, q_2) = \frac{2\mu}{3} q_1^3 + q_1^2 q_2 + \frac{1}{3} q_2^3 ,$$
$$V(q_1, q_2) = q_1^2 q_2 + 2 q_2^3 , \quad V(q_1, q_2) = q_1^2 q_2 + \frac{16}{3} q_2^3 .$$

For instance, in the case $V = q_1^2 q_2 + (16/3) q_2^3$, the third integral can be written explicitly as

$$Q^{(3)} = 9(p_2^2 + q_2^2)^2 + 12 p_2 q_2^2 (3 q_1 p_2 - q_2 p_1) - 2 q_2^4 (6 q_1^2 + q_2^2) - 12 q_1 q_2^4 . \tag{1.10}$$

The above discussion indicates that the behaviour of the Hamiltonian system (1.6) is very complicated, and thus there is no simple answer to the question of the existence of the third integral of motion. However, the final answer must be based not on a numerical analysis but on a mathematically rigorous proof. Consequently, the problem of the third integral of motion has not been completely resolved yet, and is still being discussed.

Painlevé analysis

Integrable Hamiltonian systems often exist discretely among a family of Hamiltonian systems, against our expectations. A natural question arises out of the above discussion is how to recognise if a given Hamiltonian has a first integral. We will mention a method called the the *Painlevé analysis*, or alternatively *singularity analysis* [14, 15, 16].

An ordinary differential equation is said to have the Painlevé property if all its solutions have no movable singular points other than poles. For Hamiltonian systems, a rigorous relation between the Liouville integrability and the Painlevé property was established by an elegant and powerful way [17]. A number of integrable Hamiltonian systems has been detected by imposing the Painlevé property on their solutions (the Painlevé analysis). However, it should be stated that for the application of the Painlevé analysis or its extension called the Morales-Ramis theory [18], we definitely need a particular solution of the Hamiltonian system under consideration. As there is no direct method for finding particular solutions to Hamilton's equations, this requirement limits the range of applicability of the Painlevé analysis. Nevertheless, historically, the Painlevé analysis has helped lead us to new integrable Hamiltonian systems. The first example was made by S. Kowalevskaya.

In 1889, S. Kowalevskaya studied the motion of a rotating rigid body under a constant gravitational force. More precisely, She explored the possible connection between the Liouville integrability and the presence of movable poles in the solutions to the Euler–Poisson equations

$$A\frac{d\omega_1}{dt} = (B-C)\omega_2\omega_3 + z_0\gamma_2 - y_0\gamma_3\,, \qquad \frac{d\gamma_1}{dt} = \omega_3\gamma_2 - \omega_2\gamma_3\,, \tag{1.11}$$

$$B\frac{d\omega_2}{dt} = (C-A)\omega_3\omega_1 + x_0\gamma_3 - z_0\gamma_1\,, \qquad \frac{d\gamma_2}{dt} = \omega_1\gamma_3 - \omega_3\gamma_1\,, \tag{1.12}$$

$$C\frac{d\omega_3}{dt} = (A-B)\omega_1\omega_2 + y_0\gamma_1 - x_0\gamma_2\,, \qquad \frac{d\gamma_3}{dt} = \omega_2\gamma_1 - \omega_1\gamma_2\,, \tag{1.13}$$

where (A,B,C) denote the principal momenta of inertia and (x_0,y_0,z_0) is the center of mass. The angular velocity vector and direction cosines are denoted by $(\omega_1,\omega_2,\omega_3)$ and $(\gamma_1,\gamma_2,\gamma_3)$, respectively. Generally, the Euler–Poisson equations have the two first integrals

$$\frac{1}{2}\left(A\omega_1^2 + B\omega_2^2 + C\omega_3^2\right) + x_0\gamma_1 + y_0\gamma_1 + z_0\gamma_3 = \text{const.}\,, \tag{1.14}$$

$$A\omega_1\gamma_1 + B\omega_2\gamma_2 + C\omega_3\gamma_3 = \text{const.}\,. \tag{1.15}$$

Therefore, the existence of an additional integral guarantees that the Euler–Poisson equations is completely integrable. Kowalevskaya examined a particular solution to the Euler–Poisson equations and found that there are only 3 cases

$$\begin{aligned}
&x_0 = y_0 = z_0 = 0\,, && && \text{(Euler)}\\
&x_0 = y_0 = 0\,, && A = B\,, && \text{(Lagrange)}\\
&z_0 = 0\,, && A = B = 2C\,, && \text{(Kowalevskaya)}
\end{aligned}$$

which are free from the movable singular points. It was already known from the work of Euler and Lagrange (see [19] for review) that the first two cases are completely integrable with the additional integral

$$\begin{aligned}
A^2\omega_1^2 + B^2\omega_2^2 + C^2\omega_3^2 &= \text{const.}\,, && \text{(Euler)}\\
C\omega_3 &= \text{const.}\,. && \text{(Lagrange)}
\end{aligned}$$

Fortunately, Kowalevskaya also found the additional integral

$$\left(\omega_1^2-\omega_2^2-x_0\gamma_1\right)^2+\left(2\omega_1\omega_2-x_0\gamma_2\right)^2 \;=\; \text{const.}\,,$$

and then it became known as Kowalevskaya's top.

After her success, some integrable Hamiltonian systems were discovered by the Painlevé analysis. On the other hand, a mathematical rigorous relation between the integrability and the analytic properties of the solutions had been extensively studied. For Hamiltonian systems, S. L. Ziglin established a well-organised description of the Painlevé analysis by using the monodromy group [17]. We avoid in-depth discussion of it.

Main problem and strategy

In spite of the fact that we have a well-defined notion of complete integrability in the Liouville sense, deciding whether a given Hamiltonian system is completely integrable is still an open question. Looking at only Poincaré's fundamental problem, a few mathematical statements were made. This has motivated many authors to develop various methods to investigate the Liouville integrability of Hamiltonian systems. One candidate is the Painlevé analysis or its extension. However, it requires particular solutions to Hamilton's equations. Finding a particular solution is equivalent to solving Hamilton's equations itself, and thus the Painlevé analysis is sometimes not constructive. As a result, the range of its applicability is highly limited.

Our purpose in this thesis is to construct a systematic method to enumerate first integrals of a given Hamiltonian system. The main idea is laid out in the following.

Throughout this thesis, we only deal with systems described by a *natural Hamiltonian*

$$H \;=\; \frac{1}{2}\sum_{a,b=1}^{N} g^{ab}(q)p_a p_b+V(q)\,, \tag{1.16}$$

where (q,p) are canonical variables. The first and second terms respectively denote a curved kinetic and a potential energy. A symmetric matrix $g^{ab}=g^{ba}$ amounts to a Riemannian metric on configuration space. The equations of motion take the form

$$\frac{dq^a}{dt} \;=\; \frac{\partial H}{\partial p_a} \;=\; \sum_{b=1}^{N} g^{ab}p_b\,, \qquad \frac{dp_a}{dt} \;=\; -\frac{\partial H}{\partial q^a} \;=\; -\frac{1}{2}\sum_{b,c=1}^{N}\partial_a g^{bc}p_b p_c-\partial_a V\,. \tag{1.17}$$

The main idea is to use the so-called *Eisenhart lift* to a natural Hamiltonian (1.16) (for recent review [20]): Introducing a new momentum p_{N+1}, we make a natural Hamiltonian H a homogeneous quadratic in momenta

$$\tilde{H} \;=\; \frac{1}{2}\sum_{a,b=1}^{N} g^{ab}(q)p_a p_b+V(q)p_{N+1}^2 \;=\; \frac{1}{2}\sum_{A,B=1}^{N+1}\tilde{g}^{AB}(q)p_A p_B\,, \tag{1.18}$$

where $p_A=(p_a,p_{N+1})$, $\tilde{g}^{ab}=g^{ab}$, $\tilde{g}^{N+1\,a}=0$ and $\tilde{g}^{N+1\,N+1}=2V$. This Hamiltonian can reduce to its original counterpart (1.16) when p_{N+1} is set unity. We think of $\tilde{H}$ as new Hamiltonian with additional canonical variables p_{N+1} and q^{N+1}. The new equations of motion are written out as

$$\frac{dq^a}{dt} = \sum_{b=1}^{N} g^{ab} p_b \,, \qquad \frac{dp_a}{dt} = -\frac{1}{2}\sum_{b,c=1}^{N} \partial_a g^{bc} p_b p_c - p_{N+1}^2 \partial_a V \,, \tag{1.19}$$

$$\frac{dq^{N+1}}{dt} = 2p_{N+1} V \,, \qquad \frac{dp_{N+1}}{dt} = 0 \,, \tag{1.20}$$

confirming that we can set $p_{N+1} = 1$. The important point is that new Hamiltonian (1.18) describes a *geodesic flow*. This implies that it is enough to consider a geodesic Hamiltonian as long as we are concerned only with a natural Hamiltonian.

Moreover, we restrict our attention to first integrals that are polynomial in momenta. For geodesic Hamiltonian systems, such integrals can be written as

$$Q(q,p) = K^{a_1 \cdots a_p}(q) p_{a_1} \cdots p_{a_p} \,, \tag{1.21}$$

where $K^{a_1\cdots a_p} = K^{(a_1\cdots a_p)}$ is a $(p,0)$-type symmetric tensor field. The round brackets $(\cdots)$ denote symmetrisation over the enclosed indices. Square brackets over indices $[\cdots]$ will be used for antisymmetrisation. Here and in what follows, we will use Einstein's summation convention which means any repeated Latin index is to be summed from 0 to N. Requiring Q to be a first integral, $\{H,Q\} = 0$, we are led to the *Killing equation*

$$\nabla_{(b} K_{a_1 \cdots a_p)} = 0 \,, \tag{1.22}$$

where ∇ is the Levi–Civita connection. The symmetric tensor filed $K^{a_1\cdots a_p}$ is referred to as *Killing tensor fields* (KTs). A Riemannian metric is a trivial KT and is always a solution of the Killing equation (1.22). This corresponds to the fact that the Hamiltonian $(1/2)g^{ab}p_a p_b$ is surely a first integral of the geodesic flow. The first order KTs are known as Killing vector fields (KVs) that have been actively studied as the isometry group. The second order KTs have also been considerably studied in connection with separation of variables in Hamilton–Jacobi equations [21, 22, 23]. In general relativity, a nontrivial KT of second-order was discovered in the Kerr spacetime [24, 25] which describes an isolated stationary rotating black hole in a vacuum. In the Kerr spacetime, the geodesic equations can be solved by separation of variables due to the presence of a KT.

It should be noted that: The presence of first integrals represents an *intrinsic character* of Hamiltonian systems. On the one hand, the presence of first integrals that are a certain degree in momenta is an *extrinsic character* since it depends on a certain choice of canonical variables. Therefore, we in principle need to find out whether there is any KT of a general order. In this thesis, we will focus on KTs up to the third order; After solving the Killing equation (1.22), Poisson commutativity (1.2) must be examined separately. In terms of KTs, Poisson commutativity can be written as

$$[K,\bar{K}]_{\mathrm{SN}}^{a_1\cdots a_{p+q-1}} = 0 \,, \tag{1.23}$$

where $K^{a_1\cdots a_p}, \bar{K}^{a_1\cdots a_q}$ are KTs and $[,]_{\mathrm{SN}}$ is Schouten-Nijenhuis bracket defined by

$$[K,\bar{K}]_{\mathrm{SN}}^{a_1\cdots a_{p+q-1}} \equiv p\, K^{b(a_1\cdots a_{p-1}} \nabla_b \bar{K}^{a_p\cdots a_{p+q-1})} - q\, \bar{K}^{b(a_1\cdots a_{q-1})} \nabla_b K^{a_q\cdots a_{p+q-1})} \,. \tag{1.24}$$

It can be confirmed that if the Schouten-Nijenhuis bracket $[K,\bar{K}]_{\mathrm{SN}}^{a_1\cdots a_{p+q-1}}$ vanishes, first integrals $K^{a_1\cdots a_p} p_{a_1}\cdots p_{a_p}$ and $\bar{K}^{a_1\cdots a_q} p_{a_1}\cdots p_{a_q}$ are Poisson commutating.

As described above, the problem of determining whether a Hamiltonian system is completely integrable is reduced to the problem how to solve the Killing equation. Then it is natural to ask the following questions:

- Are there any solutions of the Killing equation (1.22) for given metrics?
- If the answer is yes, then how many solutions are there?
- What quantities are sufficient to determine the number of solutions?

In this thesis, we study the above issues and give partial answers. In particular, we introduce a systematic method to analyse the Killing equation and to study its properties. A key ingredient here is projection operators called *Young symmetrisers*.

Organisation of the thesis

The remainder of this thesis is organised as follows.

Chapter 2 In order to introduce Young symmetrisers which give a diagrammatic method to decompose the irreducible repserentations of the general linear group, we begin with some basic concepts in representation theory. After then, we introduce Young symmetrisers and study their properties to keep this thesis readable independently of any reference. Young symmetrisers are the basic building block of the covariant tensor calculus in the subsequent chapters.

Chapter 3 We give a procedure which transforms the Killing equation into a closed system called the prolonged system by introducing new variables. Young symmetriser plays essential roles in the procedure. It will be a first step towards better understanding of the integrability of the Killing equation. In particular, the closed system serves to put a maximum upper bound on the number of linearly independent solutions to the Killing equation.

Chapter 4 We formulate the integrability conditions of of the prolonged system. It provides a concrete way to enumerate the number of the solutions to the Killing equation. Our analysis here is also based on Young symmetrisers. We also demonstrate a method for computing the dimension of the space of KTs with a specific example.

Chapter 5 We characterise metrics which admit Killing vector fields by local curvature obstructions. The obstructions will be obtained by analysing the integrability condition and the original Killing equation. As a consequence, the algorithm that tells us exactly how many Killing vector fields exist for 3-dimensional Riemannian metrics will be formulated.

Chapter 6 Killing tensor fields arise out of an assumption that first integrals of a geodesic flow are polynomial in momenta. It is then natural to relax this assumption and conceive of first integrals that are meromorphic in momenta. We call them the rational first integrals. As a consequence, we are naturally led to introduce gauged Killing tensor fields.

Chapter 7 We summarise this thesis and conclude our study with a summary and outlook.

Chapter 2

Young symmetriser

The aim of this chapter is to introduce projection operators called *Young symmetrisers*. Young symmetrisers are the basic building block of the covariant tensor calculus in the subsequent chapters since they make it a snap to take multi-term symmetries (such as the first Bianchi identity $R_{[abc]}{}^{d} = 0$) into account. In our notation the Latin letters $(a, b, c, \ldots)$ are identified as a naturally ordered set $(1, 2, 3, \ldots)$. Therefore, for instance, the standard Young tableau $Y_{\begin{array}{|c|c|}\hline 1&2\\\hline 3&4\\\hline\end{array}}$ is equated with $Y_{\begin{array}{|c|c|}\hline a&b\\\hline c&d\\\hline\end{array}}$ which is more suitable for the calculus. We also order the subscripted Latin letters $(a_1, a_2, \ldots, b_1, b_2, \ldots)$ as $a_1 < a_2 < \cdots < b_1 < b_2 < \cdots$.

This chapter consists of four sections: In Section 2.1 we commence by some basic concepts in representation theory. In Section 2.2 we then introduce the definition and properties of Young symmetrisers. Section 2.3 is devoted to resolving a technical issue of Young symmetriser. In Section 2.4 we collect some useful theorems without any proof. See Ref. [26] for more on Young tableaux and the representation theory of symmetric groups.

2.1 Preliminaries

In order to introduce Young symmetrisers which give a diagrammatic method to decompose the irreducible repserentations of the general linear group, we begin with some basic concepts in representation theory.

Definition 2 (partition)**.** *A partition of a positive integer k is a set of integers* $(\lambda_1, \lambda_2, \ldots \lambda_k)$ *such that*

$$\lambda_1 \, \geq \, \lambda_2 \, \geq \, \cdots \, \geq \, \lambda_k \, \geq \, 0 \,, \qquad\qquad \lambda_1 + \lambda_2 + \cdots + \lambda_k \, = \, k \,. \tag{2.1}$$

For instance, the natural number 3 can be partitioned in three distinct ways

$$3 \, = \, 3 \, = \, 2+1 \, = \, 1+1+1 \,.$$

Any partition can be graphically represented by using Young diagrams.

Definition 3 (Young diagram)**.** *A Young diagram is a finite collection of boxes arranged in k rows corresponding to a partition of k.*

For instance, the following three diagrams corresponding to partitions of the natural number 3.

$$\underbrace{\begin{array}{|c|c|c|}\hline \;&\;&\;\\\hline\end{array}}_{3}\,, \qquad\qquad \underbrace{\begin{array}{|c|c|}\hline \;&\;\\\hline \;\\\cline{1-1}\end{array}}_{2+1}\,, \qquad\qquad \underbrace{\begin{array}{|c|}\hline \;\\\hline \;\\\hline \;\\\hline\end{array}}_{1+1+1}\,.$$

Subsequently we introduce a standard Young tableau which denotes the irreducible representation of the general linear groups.

Definition 4 (standard Young tableau)**.** *A standard Young tableau is a filling of a Young diagram with the natural numbers* $1, 2, \ldots, k$ *so that entries are increasing along rows and columns.*

For instance, the possible standard Young tableaux with three boxes are written as follows.

$$\begin{array}{|c|c|c|}\hline 1&2&3\\\hline\end{array}\,, \qquad \begin{array}{|c|c}\hline 1&\multicolumn{1}{c|}{2}\\\hline \multicolumn{1}{|c|}{3}\\\cline{1-1}\end{array}\,, \qquad \begin{array}{|c|c}\hline 1&\multicolumn{1}{c|}{3}\\\hline \multicolumn{1}{|c|}{2}\\\cline{1-1}\end{array}\,, \qquad \begin{array}{|c|}\hline 1\\\hline 2\\\hline 3\\\hline\end{array}\,.$$

At the end, we define the hook length of a box in a Young diagram. It will be used to determine the normalisation of Young symmetrisers.

Definition 5 (hook length)**.** *The hook length* $h(i, j)$ *of a box* (i, j) *in a Young diagram is the number of boxes that are in the same row* i *to the right of it, plus the number of boxes in the same column* j *below it, plus one.*

$$\begin{array}{|c|c|c|}\hline 3&2&1\\\hline\end{array}\,, \qquad \begin{array}{|c|c}\hline 3&\multicolumn{1}{c|}{1}\\\hline \multicolumn{1}{|c|}{1}\\\cline{1-1}\end{array}\,, \qquad \begin{array}{|c|}\hline 3\\\hline 2\\\hline 1\\\hline\end{array}\,.$$

2.2 Definition and properties

A Young symmetriser Y_Θ is defined as the projection operator corresponding to a standard Young tableau Θ which makes sequential row-by-row symmetrisation and column-by-column antisymmetrisation. To be precise, for a Young diagram θ (this is a partition of the integer k) and one of its standard Young tableaux Θ, the Young symmetriser reads

$$Y_\Theta \;\equiv\; \alpha_\theta \prod_{C_i \in \mathrm{col}(\Theta)} \hat{A}_{C_i} \prod_{R_i \in \mathrm{row}(\Theta)} \hat{S}_{R_i}\,, \tag{2.2}$$

where $\hat{S}_{R_i}$ ($\hat{A}_{C_i}$) denotes the (anti-)symmetrisation of the slots corresponding to the entries in the i^{th} row (column) of the tableau Θ; α_θ is the normalisation factor determined so as to satisfy $Y_\Theta^2 = Y_\Theta$ and depends only on the shape of the Young diagram not the particular tableau.

For instance, let $\theta = \begin{array}{|c|c|}\hline \ & \ \\\hline \ & \ \\\hline\end{array}$ and $\Theta = \begin{array}{|c|c|}\hline a&b\\\hline c&d\\\hline\end{array}$, then the corresponding Young symmetriser reads

$$Y_{\begin{array}{|c|c|}\hline a&b\\\hline c&d\\\hline\end{array}} \;=\; \alpha_{\begin{array}{|c|c|}\hline \ & \ \\\hline \ & \ \\\hline\end{array}} \hat{A}_{ac}\hat{A}_{bd}\hat{S}_{cd}\hat{S}_{ab}\,, \tag{2.3}$$

with

$$\alpha_{\begin{array}{|c|c|}\hline \ & \ \\\hline \ & \ \\\hline\end{array}} \;=\; \frac{(2!)^4}{|\theta|} \;=\; \frac{4}{3}\,, \tag{2.4}$$

where $|\theta|$ the product of hook lengths of the boxes of θ, i.e.

$$|\theta| \;=\; 3\cdot 2\cdot 2\cdot 1 \;=\; 12\,, \tag{2.5}$$

since the tableau listing the hook length of each box in θ is given by $\begin{array}{|c|c|}\hline 3&2\\\hline 2&1\\\hline\end{array}$. The numerator of eq. (2.4) comes from the normalisation of $\hat{A}_{ac}\hat{A}_{bd}\hat{S}_{cd}\hat{S}_{ab}$. Notice that both $\hat{S}_{R_i}$ and $\hat{A}_{C_i}$ are also idempotent, e.g. $\hat{S}_{ab}$ if defined by $1/(2!)(\mathrm{id}_2 + (a,b))$ and not $(\mathrm{id}_2 + (a,b))$ where id_2 is the identity operator and (a,b) denotes the permutation that swaps indices a and b.

Young symmetrisers are endowed with the following three properties.

Idempotence

$$Y_\Theta^2 \;=\; Y_\Theta\,, \qquad\qquad {}^\forall\Theta \in \mathscr{Y}_k\,, \tag{2.6}$$

Orthogonality

$$Y_\Theta\, Y_\Phi \;=\; \delta_{\Theta\Phi}\, Y_\Phi\,, \qquad {}^\forall\Theta\,,\Phi \in \mathscr{Y}_k\,, \qquad (k=1,2,3,4) \tag{2.7}$$

Completeness

$$\sum_{\Theta\in\mathscr{Y}_k} Y_\Theta \;=\; \mathrm{id}_k\,, \qquad\qquad (k=1,2,3,4) \tag{2.8}$$

where $\mathscr{Y}_k$ denotes the set of all standard Young tableaux with k boxes, e.g.

$$\mathscr{Y}_2 \;=\; \left\{\ytableaushort{ab}\,,\ytableaushort{a,b}\right\}, \qquad\qquad \mathscr{Y}_3 \;=\; \left\{\ytableaushort{abc}\,,\ytableaushort{ab,c}\,,\ytableaushort{ac,b}\,,\ytableaushort{a,b,c}\right\}.$$

It is worth mentioning that the orthogonality (2.7) holds for general k if the shapes of the tableaux Θ and Φ are different.

Throughout the remaining part of this thesis, it is stipulated that the Young symmetriser with k boxes acts only on covariant not contravariant indices of a type (n,m) tensor field T for $m \geq k$, e.g. $Y_{\ytableaushort{ab}}\,T_{ab}$ is well-defined and yields $1/2(T_{ab}+T_{ba})$ but $Y_{\ytableaushort{ab}}\,T^{ab}$ is an ill-defined operation. Note that any Young symmetriser does not commute with the trace operation, e.g. $g^{ab}Y_{\ytableaushort{ab}}\,T_{ab}$ is well-defined but $Y_{\ytableaushort{ab}}\,g^{ab}T_{ab}$ is an ill-defined.

In the stipulation, Young symmetrisers provide a decomposition of the space of all tensor fields into its irreducible representations for the action of the general linear group. For instance, the decomposition of a type $(0,2)$ tensor field T into the irreducible representations can be done as

$$T_{ab} \;=\; \mathrm{id}_2\, T_{ab} \;=\; \left(Y_{\ytableaushort{ab}} + Y_{\ytableaushort{a,b}}\right)T_{ab} \;=\; T_{(ab)} + T_{[ab]}\,,$$

where in the second equality we have used the compleness relation (2.8). For a type $(0,3)$ tensor field T, a similar calculation reads

$$\begin{aligned} T_{abc} \;&=\; \left(Y_{\ytableaushort{abc}} + Y_{\ytableaushort{ab,c}} + Y_{\ytableaushort{ac,b}} + Y_{\ytableaushort{a,b,c}}\right)T_{abc} \\ &=\; T_{(abc)} + \tfrac{4}{3}\,\hat{A}_{ac}T_{(ab)c} + \tfrac{4}{3}\,\hat{A}_{ab}T_{(a|b|c)} + T_{[abc]}\,. \end{aligned}$$

But at $k=5$ the subsequent calculation reaches a deadlock since the completeness relation (2.8) no longer holds for $k \geq 5$. The standard example of this is

$$Y_{\ytableaushort{ab,cd,e}} \;=\; 0 \qquad\qquad \text{but} \qquad\qquad Y_{\ytableaushort{ad,be,c}} \;\neq\; 0\,,$$

implying that the orthogonality (2.7) is broken down.

2.3 Littlewood's correction

The failure of the orthogonality and completeness of Young symmetrisers with $k \geq 5$ boxes is fatally shot in practical use. Fortunately, it is known that this can be complemented by *Littlewood's correction* [27]. We shall present it here. For other prescriptions, see Refs. [28, 29].

Before going into the details, we introduce the following two definitions:

Definition 6 (row-word of a Young tableau). *Let $\Theta \in \mathscr{Y}_k$ be a Young tableau. The row-word of Θ, say $\mathrm{row}(\Theta)$, is defined as the row vector whose entries are those of Θ read row-wise from top to bottom.*

For instance, suppose $\Theta = \begin{array}{|c|c|}\hline a & b \\ \hline c & d \\ \hline e \\ \cline{1-1}\end{array}$. Then the row-word of Θ reads $\mathrm{row}(\Theta) = (a,b,c,d,e)$.

Definition 7 (row-order relation). *Let Θ and Φ be two Young tableaux of the same shape. Denoting $\mathrm{row}(\Theta)_i$ be the i^{th} component of $\mathrm{row}(\Theta)$, it is said that Θ precedes Φ and write $\Theta \prec \Phi$ if $\mathrm{row}(\Theta)_i < \mathrm{row}(\Phi)_i$ for the leftmost i where $\mathrm{row}(\Theta)_i$ and $\mathrm{row}(\Phi)_i$ differ.*

Using the row-order relation, we can order the Young tableaux of the same shape, e.g.

$$\begin{array}{|c|c|}\hline a & b \\ \hline c & d \\ \hline e \\ \cline{1-1}\end{array} \quad \prec \quad \begin{array}{|c|c|}\hline a & b \\ \hline c & e \\ \hline d \\ \cline{1-1}\end{array} \quad \prec \quad \begin{array}{|c|c|}\hline a & c \\ \hline b & d \\ \hline e \\ \cline{1-1}\end{array} \quad \prec \quad \begin{array}{|c|c|}\hline a & c \\ \hline b & e \\ \hline d \\ \cline{1-1}\end{array} \quad \prec \quad \begin{array}{|c|c|}\hline a & d \\ \hline b & e \\ \hline c \\ \cline{1-1}\end{array}\,.$$

The following result is an easy consequence of the row-order relation. Let $\{\Theta_1,\ \Theta_2, \Theta_3, \ldots\}$ be the set of all Young tableaux in $\mathscr{Y}_k$ with a particular shape. Suppose this set be ordered as $\Theta_i \prec \Theta_j$ whenever $i < j$, one can see by inspection that the one-sided orthogonality

$$Y_{\Theta_i} Y_{\Theta_j} = 0\,, \tag{2.9}$$

holds true.

We are now able to state Littlewood's correction. The Young symmetriser with Littlewood's correction, say L_{Θ_i}, corresponding the tableau $\Theta_i \in \{\Theta_1, \Theta_2, \Theta_3, \ldots\}$ is iteratively defined by

$$L_{\Theta_i} \equiv Y_{\Theta_i} \left(1 - \sum_{j=1}^{i-1} L_{\Theta_j}\right), \tag{2.10}$$

or the factorised form

$$L_{\Theta_i} = Y_{\Theta_i} \prod_{j=1}^{i-1} \left(1 - Y_{\Theta_{i-j}}\right). \tag{2.11}$$

It is possible to prove [27] that the Young symmetrisers with correction (2.10) recover the orthogonality,

$$L_\Theta\, L_\Phi = \delta_{\Theta\Phi}\, L_\Phi\,, \qquad {}^\forall\Theta\,, \Phi \in \mathscr{Y}_k\,, \tag{2.12}$$

and the completeness,

$$\sum_{\Theta \in \mathscr{Y}_k} L_\Theta = \mathrm{id}_k\,, \tag{2.13}$$

for general k. Here, we only prove the orthogonality (2.12). If L_{Θ_i} and L_{Θ_j} correspond to tableaux of the same shape with $\Theta_i \prec \Theta_j$ whenever $i < j$, then

$$L_{\Theta_i}\, L_{\Theta_j} = 0\,,$$

holds by the one-sided orthogonality (2.9). Moreover, we have

$$\begin{aligned} L_{\Theta_j}\, L_{\Theta_i} &= Y_{\Theta_j}\,(1-Y_{\Theta_{j-1}})\,\cdots\,(1-Y_{\Theta_i})\,\cdots\,(1-Y_{\Theta_1})Y_{\Theta_i}\,(1-Y_{\Theta_{i-1}})\,\cdots\,(1-Y_{\Theta_1}) \\ &= Y_{\Theta_j}\,(1-Y_{\Theta_{j-1}})\,\cdots\,(Y_{\Theta_i}-Y_{\Theta_i}^2)\,(1-Y_{\Theta_{i-1}})\,\cdots\,(1-Y_{\Theta_1}) \;=\; 0\,, \end{aligned}$$

confirming eq. (2.12).

It is advisable to note that all the corrections in (2.10) vanish for the tableaux with $k \leq 4$ boxes, then L_{Θ_i} reduces to Y_{Θ_i}. Even for $k \geq 5$, many corrections would vanish, e.g. At $k = 5$ the only two symmetrisers

$$L_{\begin{array}{l}\scriptstyle a\,c\,e\\ \scriptstyle b\,d\end{array}} \;=\; Y_{\begin{array}{l}\scriptstyle a\,c\,e\\ \scriptstyle b\,d\end{array}}\left(1-Y_{\begin{array}{l}\scriptstyle a\,b\,c\\ \scriptstyle d\,e\end{array}}\right)\,, \qquad\qquad L_{\begin{array}{l}\scriptstyle a\,d\\ \scriptstyle b\,e\\ \scriptstyle c\end{array}} \;=\; Y_{\begin{array}{l}\scriptstyle a\,d\\ \scriptstyle b\,e\\ \scriptstyle c\end{array}}\left(1-Y_{\begin{array}{l}\scriptstyle a\,b\\ \scriptstyle c\,d\\ \scriptstyle e\end{array}}\right)\,,$$

differ from their original counterparts. Thus it is useful for practical use to make it clear what kinds of the Young symmetrisers with Littlewood's correction are equivalent to the original counterparts. Since the tableau $\begin{array}{l}\scriptstyle a_1\cdots\cdots a_p\\ \scriptstyle b_1\cdots b_q\\ \scriptstyle c\end{array}$ is row-ordered, it follows from the definition

$$L_{\begin{array}{l}\scriptstyle a_1\cdots\cdots a_p\\ \scriptstyle b_1\cdots b_q\\ \scriptstyle c\end{array}} \;=\; Y_{\begin{array}{l}\scriptstyle a_1\cdots\cdots a_p\\ \scriptstyle b_1\cdots b_q\\ \scriptstyle c\end{array}}\,. \tag{2.14}$$

It is also shown that

$$L_{\begin{array}{l}\scriptstyle a_1\cdots\cdots\cdots a_p\,b_2\\ \scriptstyle b_1\,b_3\cdots b_q\,c\end{array}} \;=\; Y_{\begin{array}{l}\scriptstyle a_1\cdots\cdots\cdots a_p\,b_2\\ \scriptstyle b_1\,b_3\cdots b_q\,c\end{array}}\left(1-Y_{\begin{array}{l}\scriptstyle a_1\cdots\cdots\cdots a_p\,b_1\\ \scriptstyle b_2\,b_3\cdots b_q\,c\end{array}}\right) \;=\; Y_{\begin{array}{l}\scriptstyle a_1\cdots\cdots\cdots a_p\,b_2\\ \scriptstyle b_1\,b_3\cdots b_q\,c\end{array}}\,, \tag{2.15}$$

$$L_{\begin{array}{l}\scriptstyle a_1\cdots\cdots\cdots a_p\,b_3\\ \scriptstyle b_1\,b_2\,b_4\cdots b_q\,c\end{array}} \;=\; Y_{\begin{array}{l}\scriptstyle a_1\cdots\cdots\cdots a_p\,b_3\\ \scriptstyle b_1\,b_2\,b_4\cdots b_q\,c\end{array}}\left(1-Y_{\begin{array}{l}\scriptstyle a_1\cdots\cdots\cdots a_p\,b_1\\ \scriptstyle b_2\,b_3\cdots b_q\,c\end{array}}\right)\left(1-Y_{\begin{array}{l}\scriptstyle a_1\cdots\cdots\cdots a_p\,b_2\\ \scriptstyle b_1\,b_3\cdots b_q\,c\end{array}}\right) \;=\; Y_{\begin{array}{l}\scriptstyle a_1\cdots\cdots\cdots a_p\,b_3\\ \scriptstyle b_1\,b_2\,b_4\cdots b_q\,c\end{array}}\,, \tag{2.16}$$

and so on, where we have only used the relations $\hat{S}_{a_1b_2}\,\hat{A}_{a_1b_2} = 0$ and $\hat{S}_{a_2b_3}\,\hat{A}_{a_2b_3} = 0$. In general, the corrected symmetriser

$$L_{\begin{array}{l}\scriptstyle a_1\cdots\cdots\cdots a_p\,b_i\\ \scriptstyle b_1\cdots\cancel{b_i}\cdots b_q\,c\end{array}}\,, \qquad \text{with} \qquad p \geq q \geq 1\,, \qquad q \geq i \geq 2\,, \tag{2.17}$$

is coincident with its original counterpart by a trivial relation $\hat{S}_{a_{i-1}b_i}\,\hat{A}_{a_{i-1}b_i} = 0$.

2.4 Theorems

In this section, we collect some theorems that are of fundamental importance in the next chapter. We also carry out sample calculations which will be helpful to readers to acquire a better understanding of the theorems.

First we show the result referred to as Schur's lemma in the context of the representation theory of symmetric groups.

Theorem 8 (Schur)**.** *Let Θ and Φ be Young tableaux with k boxes. If Y_Θ and Y_Φ are orthogonal, that is $Y_\Theta\, Y_\Phi = 0$, then*

$$Y_\Theta\;\sigma\;Y_\Phi = 0\,, \tag{2.18}$$

holds true for an arbitrary permutation σ.

Take as an example the product $Y_{\young(ab,cd)}\, Y_{\young(ab,c)}$ and expand it to

$$Y_{\young(ab,cd)}\, Y_{\young(ab,c)}\, \mathrm{id}_4 \;=\; Y_{\young(ab,cd)}\, Y_{\young(ab,c)} \sum_{\Theta\in\mathscr{Y}_4} Y_\Theta \;=\; Y_{\young(ab,cd)}\, Y_{\young(ab,c)}\, Y_{\young(ab,cd)}\,,$$

where in the last equality we have used Schur's lemma.

Second, we state Raicu's theorem.

Theorem 9 (Raicu)**.** *Let $\Theta \in \mathscr{Y}_k$ and $\Phi \in \mathscr{Y}_{k+1}$. Suppose that the unique entry in Φ outside Θ is located in the right edge of the tableau Φ, then*

$$Y_\Theta\, Y_\Phi \;=\; Y_\Phi\,, \tag{2.19}$$

holds true.

Using Raicu's theorem, the product $Y_{\young(ab,cd)}\, Y_{\young(ab,c)}\, Y_{\young(ab,cd)}$ can be simplified to

$$Y_{\young(ab,cd)}\, Y_{\young(ab,c)}\, Y_{\young(ab,cd)} \;=\; Y_{\young(ab,cd)}\, Y_{\young(ab,cd)} \;=\; Y_{\young(ab,cd)}\,.$$

To be precise, the above theorem is merely an example of Raicu's theorem. A complete wording of Raicu's theorem can be found in Ref. [30].

At last, we state an important result, called Pieri's formula, from the representation theory of symmetric groups.

Theorem 10 (Pieri)**.** *Let θ and ϕ be two Young diagrams with k and $k+\ell\,(\ell\geq 1)$ boxes respectively. It is said that ϕ includes θ if θ is a subdiagram of ϕ. Let Θ and Φ be Young tableaux of shapes θ and ϕ respectively, then*

$$Y_\Theta\, Y_\Phi \;=\; Y_\Phi\, Y_\Theta \;=\; 0\,, \tag{2.20}$$

holds if ϕ does not include θ.

It should be noted that the first Bianchi identity, $R_{[abc]}{}^{d} = 0$, can be recaptured by Pieri's formula. We know that R_{abcd} belongs to $\young(ac,bd)$, and hence the first Bianchi identity can be written in terms of Young symmetrisers as

$$Y_{\young(a,b,c)}\, Y_{\young(ac,bd)} \;=\; 0\,,$$

which is clearly a type of Pieri's formula. Therefore we can say that Pieri's formula is a generalisation of the Bianchi identity.

Before closing this chapter, we look at an example of the actual application of Pieri's formula. Take the product $Y_{\young(ab,c)} \sum_{\Theta\in\mathscr{Y}_4} Y_\Theta$ and expand it to

$$Y_{\young(ab,c)} \sum_{\Theta\in\mathscr{Y}_4} Y_\Theta \;=\; Y_{\young(ab,c)} \sum \Big(Y_{\yng(2,2)} + Y_{\yng(2,1,1)} + Y_{\yng(3,1)} \Big) \;=\; Y_{\young(ab,c)} \Big(Y_{\young(ab,cd)} + Y_{\young(ab,c,d)} + Y_{\young(abc,d)} + Y_{\young(abd,c)} \Big)\,,$$

where in the first equality we have used Pieri's formula (2.20).

Chapter 3

Prolongation of the Killing equation

In this chapter, we give a procedure which transforms the Killing equation into a closed system by introducing new variables. Young symmetriser plays essential roles in the procedure. It will be a first step towards better understanding of the integrability of the Killing equation. In particular, the closed system serves to put a maximum upper bound on the number of linearly independent solutions to the Killing equation.

This chapter consists of four sections: In Section 3.1 we take a brief look at a procedure of prolongation of the Killing equation in classical literature. In Section 3.2 we improve the procedure of prolongation of the Killing equation by using Young symmetrisers introduced in Chapter 2. In Section 3.3 we give the explicit forms of the prolonged system for the Killing equation up to the third order. In Section 3.4 we comment on a geometric interpretation of the prolonged system. We also show the Barbance–Delong–Takeuchi–Thompson formula which gives the upper bound on the number of linearly independent solutions to the Killing equation of the p^{th} order.

3.1 Preliminaries

Before proceeding, it is instructive to take the simplest case in order to grasp the intuitive significance of the Killing equation. In particular, we aim to illustrate a procedure of prolongation in classical literature. Thus we here consider a Killing vector field satisfying the equation

$$\nabla_{(a}K_{b)} \;=\; 0\,. \tag{3.1}$$

One immediate observation is that eq. (3.1) is an overdetermined linear system since there are $N(N+1)/2$ equations for N variables, and consequently might not have any solutions. We then follow the well-known procedure of *prolongation* to find a maximum upper bound on the dimension of the solution space.

Prolongation of an overdetermined system of partial differential equations proceeds by introducing new dependent variables for unknown higher derivatives to establish a first order *closed system*, in which all the first derivatives of all the dependent variables are completely expressed in terms of the variables themselves.

For the Killing equation of the first order, the prolongation procedure had been established in classical literature (e.g. Ref. [31]). In fact we can derive a closed system by the following way: Let K_a be a Killing (co-)vector field and consider definition of the Riemann curvature tensor

$$\nabla_a \nabla_b K_c - \nabla_b \nabla_a K_c \;=\; R_{abc}{}^d K_d \,. \tag{3.2}$$

By using the Killing equation (3.1), we can rewrite eq. (3.2) as

$$\nabla_a \nabla_b K_c + \nabla_b \nabla_c K_a \;=\; R_{abc}{}^d K_d \,, \tag{3.3}$$

and call the same equations with cyclic permutations of the indices *abc* as

$$\nabla_b \nabla_c K_a + \nabla_c \nabla_a K_b \;=\; R_{bca}{}^d K_d \,, \tag{3.4}$$

$$\nabla_c \nabla_a K_b + \nabla_a \nabla_b K_c \;=\; R_{cab}{}^d K_d \,. \tag{3.5}$$

Adding eq. (3.3) to eq. (3.4) and then subtracting eq. (3.5), we obtain

$$2\nabla_b \nabla_c K_a \;=\; (R_{abc}{}^d + R_{bca}{}^d - R_{cab}{}^d) K_d \;=\; 2R_{acb}{}^d K_d \,. \tag{3.6}$$

By combining the above equation (3.6) and the Killing equation (3.1), we are led to a closed system of the form

$$\nabla_a K_b \;=\; K^{(1)}_{ab} \,, \tag{3.7a}$$

$$\nabla_a K^{(1)}_{bc} \;=\; R_{cba}{}^d K_d \,, \tag{3.7b}$$

where $K^{(1)}_{ab} \equiv K^{(1)}_{[ab]}$. An overdetermined system that can be transformed into a closed system is called of *finite type*. Hereafter, the closed system obtained by prolongation is referred to as the *prolonged system*.

We remark that the Killing equation (3.1) is first order PDE but the prolonged system (3.7) necessarily involves second derivatives of a Killing vector field K^a. As we will see in the next section, the Killing equation of the p^{th}-order is also an overdetermined system of finite type and needs $(p+1)^{\text{th}}$-order derivatives of a Killing tensor field $K^{a_1 \cdots a_p}$ to complete the prolongation procedure.

We also remark that if we have the values of $(K_a, K^{(1)}_{ab})$ at any point, then $(K_a, K^{(1)}_{ab})$ at any other point is in principle determined by integration of the prolonged system (3.7). Consequently, the upper bound on the dimension of the solution space on a N-dimensional space M is given by

$$\dim K^1(M) \;\leq\; N + \frac{N(N-1)}{2} \;=\; \frac{N(N+1)}{2} \,, \tag{3.8}$$

where $K^1(M)$ denotes the solution space of the Killing equation of the first order. This bound is saturated if and only if M is of constant curvature.

3.2 Prolongation procedure

As indicated in the preceding section, even in the first order case, the procedure of prolongation of the Killing equation is rather complicated. Such a procedure is quickly becoming more and more complicated as the order of the Killing tensor field grows. We therefore refine the prolongation procedure of the Killing equation. A key ingredient here is Young symmetrisers introduced in Chapter 2.

Our procedure of prolongation is based on a decomposition of the space of all tensor fields into its irreducible representations for the action of the general linear group: Let K_a be a Killing (co-)vector field and consider its derivatives. Since $\nabla_b K_a$ is a type $(0,2)$ tensor field, its irreducible representations reads

$$\nabla_b K_a \;=\; \mathrm{id}_2\, \nabla_b K_a \;=\; \big(Y_{\ytableaushort{ab}} + Y_{\ytableaushort{a,b}}\big)\nabla_b K_a \;=\; Y_{\ytableaushort{a,b}}\, \nabla_b K_a \;\equiv\; K^{(1)}_{ba}\,, \tag{3.9a}$$

where we have inserted the completeness relation (2.8) in the second equality and have used the Killing equation (3.1) in the third equality. We next consider $\nabla_c K^{(1)}_{ba}$ as the above equation (3.9a) is not yet closed. The irreducible representations reads

$$\begin{aligned}
\nabla_c K^{(1)}_{ba} \;&=\; Y_{\ytableaushort{a,b}}\, \nabla_{cb} K_a \;=\; Y_{\ytableaushort{a,b}}\big(Y_{\ytableaushort{abc}} + Y_{\ytableaushort{ab,c}} + Y_{\ytableaushort{ac,b}} + Y_{\ytableaushort{a,b,c}}\big)\nabla_{cb} K_a \;=\; Y_{\ytableaushort{a,b}}\, Y_{\ytableaushort{ac,b}}\, \nabla_{cb} K_a \\
&=\; Y_{\ytableaushort{a,b}}\, Y_{\ytableaushort{ac,b}}\big(2\nabla_{[cb]} K_a + \nabla_{bc} K_a\big) \;=\; Y_{\ytableaushort{a,b}}\, Y_{\ytableaushort{ac,b}}\, R_{cba}{}^d K_d\,,
\end{aligned} \tag{3.9b}$$

where $\nabla_{ab\cdots c} \equiv \nabla_a \nabla_b \cdots \nabla_c$. In the third equality, all the Young symmetrisers except for the third one vanish because of Pieri's formula (2.20), the Killing equation (3.1) and the first Bianchi identity $R_{[abc]}{}^d = 0$. A system of linear differential equations (3.9) is now closed. This implies that we are at the completion of the procedure of prolongation.

Our prolongation procedure can be extended to the higher-order cases. We take as another example Killing tensor fields of the second order obeying

$$\nabla_{(a} K_{bc)} \;=\; 0\,, \qquad\qquad \text{with} \qquad\qquad K_{ab} \;=\; K_{(ab)}\,, \tag{3.10}$$

and skip more higher-order cases due to space considerations. Let us consider $\nabla_c K_{ba}$. Its irreducible representations reads

$$\begin{aligned}
\nabla_c K_{ba} \;&=\; Y_{\ytableaushort{ab}}\, \nabla_c K_{ba} \;=\; Y_{\ytableaushort{ab}}\big(Y_{\ytableaushort{abc}} + Y_{\ytableaushort{ab,c}} + Y_{\ytableaushort{ac,b}} + Y_{\ytableaushort{a,b,c}}\big)\nabla_c K_{ba} \;=\; Y_{\ytableaushort{ab}}\, Y_{\ytableaushort{ab,c}}\, \nabla_c K_{ba} \\
&\equiv\; Y_{\ytableaushort{ab}}\, K^{(1)}_{cba}\,,
\end{aligned} \tag{3.11a}$$

where the third equality follows from the Killing equation (3.10) and a trivial relation $Y_{\ytableaushort{ab}}\, Y_{\ytableaushort{a{},b,{}}} = 0$. We subsequently consider $\nabla_d K^{(1)}_{cba}$ because eq. (3.11a) is not closed. The irreducible representations of $\nabla_d K^{(1)}_{cba}$ reads

$$\begin{aligned}
\nabla_d K^{(1)}_{cba} \;&=\; Y_{\ytableaushort{ab,c}}\, \nabla_{dc} K_{ba} \;=\; Y_{\ytableaushort{ab,c}}\big(Y_{\ytableaushort{ab,c,d}} + Y_{\ytableaushort{abd,c}} + Y_{\ytableaushort{ab,cd}}\big)\nabla_{dc} K_{ba} \\
&=\; Y_{\ytableaushort{ab,c}}\Big(\big[Y_{\ytableaushort{ab,c,d}} + 2Y_{\ytableaushort{abd,c}}\big]\nabla_{[dc]} K_{ba} + K^{(2)}_{dcba}\Big) \\
&=\; Y_{\ytableaushort{ab,c}}\Big(\big[Y_{\ytableaushort{ab,c,d}} + 2Y_{\ytableaushort{abd,c}}\big] R_{dcb}{}^m K_{ma} + K^{(2)}_{dcba}\Big)\,,
\end{aligned} \tag{3.11b}$$

where $K^{(2)}_{dcba} \equiv Y_{\ytableaushort{ab,cd}}\, \nabla_{dc} K_{ba}$. We have used Pieri's formula (2.20) and the Killing equation (3.10) in the second equality. We further consider $\nabla_e K^{(2)}_{dcba}$ as eq. (3.11b) is not yet closed.

Notice in advance that the number of boxes of Young tableaux exceeds 4, so we need to take the Littlewood's correction (2.10) into account.

$$\begin{aligned}
\nabla_e K^{(2)}_{dcba} &= Y_{\ytableaushort{ab,cd}}\,\nabla_{edc}K_{ba} = Y_{\ytableaushort{ab,cd}}\left(L_{\ytableaushort{ab,cd,e}} + L_{\ytableaushort{abd,ce}} + L_{\ytableaushort{abe,cd}}\right)\nabla_{edc}K_{ba} \\
&= Y_{\ytableaushort{ab,cd}}\left(Y_{\ytableaushort{ab,cd,e}} + Y_{\ytableaushort{abd,ce}}\left[1 - Y_{\ytableaushort{abc,de}}\right] + Y_{\ytableaushort{abe,cd}}\left[1 - Y_{\ytableaushort{abd,ce}}\right]\left[1 - Y_{\ytableaushort{abc,de}}\right]\right)\nabla_{edc}K_{ba} \\
&= Y_{\ytableaushort{ab,cd}}\left(Y_{\ytableaushort{ab,cd,e}} + Y_{\ytableaushort{abd,ce}} + Y_{\ytableaushort{abe,cd}}\right)\nabla_{edc}K_{ba}\,,
\end{aligned} \tag{3.11c}$$

where all the Littlewood's corrections did not affect the result (3.11c) because of trivial relations $Y_{\ytableaushort{a{}d,{}{}}}\,Y_{\ytableaushort{a{}{},d{}}} = 0$ and $Y_{\ytableaushort{{}be,{}{}}}\,Y_{\ytableaushort{{}b{},{}e}} = 0$. Prolongation is still pursued by

$$\begin{aligned}
\nabla_e K^{(2)}_{dcba} &= Y_{\ytableaushort{ab,cd}}\left(Y_{\ytableaushort{ab,cd,e}}\,\nabla_{edc}K_{ba} + 2Y_{\ytableaushort{abd,ce}}\,\nabla_{e[dc]}K_{ba} + Y_{\ytableaushort{abe,cd}}\left[2\nabla_{[ed]c}K_{ba} + 2\nabla_{d[ec]}K_{ba}\right]\right) \\
= {}& Y_{\ytableaushort{ab,cd}}\Bigg(Y_{\ytableaushort{ab,cd,e}}\,\nabla_{edc}K_{ba} + Y_{\ytableaushort{abd,ce}}\left[2(\nabla_e R_{dcb}{}^m)K_{ma} + R_{dcb}{}^m K^{(1)}_{mea} - 2R_{dcb}{}^m K^{(1)}_{mae}\right] \\
&+ Y_{\ytableaushort{abe,cd}}\left[R_{edc}{}^m K^{(1)}_{mba} + 2R_{edb}{}^m K^{(1)}_{mca} - 4R_{edb}{}^m K^{(1)}_{mac} + 2(\nabla_d R_{ecb}{}^m)K_{ma}\right]\Bigg)\,.
\end{aligned} \tag{3.11d}$$

A straightforward calculation makes the first term explicit as

$$\begin{aligned}
Y_{\ytableaushort{ab,cd,e}}\,\nabla_{edc}K_{ba} = \tfrac{1}{6}\Big(&4R_{ecd}{}^m K^{(1)}_{mab} - 9R_{acd}{}^m K^{(1)}_{meb} - 9R_{eac}{}^m K^{(1)}_{mdb} + 5R_{acd}{}^m K^{(1)}_{mbe} \\
&+ 5R_{eca}{}^m K^{(1)}_{mdb} + 2(\nabla_c R_{eda}{}^m)K_{mb} + 2(\nabla_c R_{dab}{}^m)K_{me} - 2(\nabla_c R_{eab}{}^m)K_{md}\Big)\,.
\end{aligned} \tag{3.11e}$$

The results (3.11a)–(3.11e) imply that we are at the completion of the procedure of prolongation.

3.3 Prolonged systems

In this section, we first provide the prolonged system for the Killing equation of the p^{th} order

$$\nabla_{(a}K_{b_1\cdots b_p)} = 0\,, \tag{3.12}$$

without any proof. A sketch of proof is shown later in this section. To provide the prolonged system, we introduce the prolongation variables,

$$K^{(q)}_{b_q\cdots b_1 a_p\cdots a_1} \equiv Y_{\ytableaushort{{a_1}{\cdots}{\cdots}{a_p},{b_1}{\cdots}{b_q}}}\,\nabla_{b_q\cdots b_1}K_{a_p\cdots a_1}\,, \qquad (1 \le q \le p) \tag{3.13}$$

where $\nabla_{ab\cdots c} = \nabla_a\nabla_b\cdots\nabla_c$, $K_{a_p\cdots a_1}$ is a KT of the p^{th} order. We remark that one needs $(p+1)$ prolongation variables to carry out the prolongation for KTs, while that for Killing-Yano tensor fields involves only two prolongation variables for any order (see [32, 33] or Section 4.4). This fact complicates the prolongation for the Killing equation (3.12).

We are now ready to provide the prolonged system. The prolongation for the Killing equation of the p^{th}order can be achieved as follows:

$$
\begin{aligned}
\nabla_c K_{a_p\cdots a_1} &= Y_{\begin{smallmatrix} a_1 & \cdots & a_p \end{smallmatrix}}\, K^{(1)}_{ca_p\cdots a_1}\,, && (3.14)\\
\nabla_c K^{(q)}_{b_q\cdots b_1 a_p\cdots a_1} &= Y_{\begin{smallmatrix} a_1 & \cdots & \cdots & a_p \\ b_1 & \cdots & b_q \end{smallmatrix}} \Bigg(\Bigg[Y_{\begin{smallmatrix} a_1 & \cdots & \cdots & a_p \\ b_1 & \cdots & b_q \\ c \end{smallmatrix}} + Y_{\begin{smallmatrix} a_1 & \cdots & \cdots & a_p & c \\ b_1 & \cdots & b_q \end{smallmatrix}} + \sum_{i=2}^{q} Y_{\begin{smallmatrix} a_1 & \cdots & \cdots & \cdots & a_p & b_i \\ b_1 & \cdots & \not{b}_i & \cdots & b_q & c \end{smallmatrix}} \Bigg] \nabla_{cb_q\cdots b_1} K_{a_p\cdots a_1} \\
&\quad + K^{(q+1)}_{cb_q\cdots b_1 a_p\cdots a_1} \Bigg)\,, \qquad (1 \le q \le p-1) && (3.15)\\
\nabla_c K^{(p)}_{b_p\cdots b_1 a_p\cdots a_1} &= Y_{\begin{smallmatrix} a_1 & \cdots & a_p \\ b_1 & \cdots & b_p \end{smallmatrix}} \Bigg[Y_{\begin{smallmatrix} a_1 & \cdots & a_p \\ b_1 & \cdots & b_p \\ c \end{smallmatrix}} + Y_{\begin{smallmatrix} a_1 & \cdots & a_p & c \\ b_1 & \cdots & b_p \end{smallmatrix}} + \sum_{i=2}^{p} Y_{\begin{smallmatrix} a_1 & \cdots & \cdots & a_p & b_i \\ b_1 & \cdots & \not{b}_i & \cdots & b_p & c \end{smallmatrix}} \Bigg] \nabla_{cb_p\cdots b_1} K_{a_p\cdots a_1}\,, && (3.16)
\end{aligned}
$$

where the slashed index $\not{b}_i$ is deleted from the Young tableau.

It is noteworthy to comment that the derivative terms look like being left on the right-hand side. However, by virtue of the properties of Young symmetrisers, those terms can be replaced with non-derivative terms whose coefficients consist of the Riemann curvature tensor and its derivatives. The proof is given by induction with respect to q as follows: For a fixed q ($1 \le q \le p$), the first term in the parenthesis of eqs. (3.15) and (3.16) reads

$$
Y_{\begin{smallmatrix} a_1 & \cdots & \cdots & a_p \\ b_1 & \cdots & b_q \\ c \end{smallmatrix}} \nabla_{cb_q\cdots b_1} K_{a_p\cdots a_1} \propto \hat{A}_{a_q b_q} \cdots \hat{A}_{a_2 b_2} \left(\hat{A}_{a_1 b_1 c} \nabla_{c(b_q\cdots b_1)} K_{a_p\cdots a_1} \right). \tag{3.17}
$$

Performing the symmetrisation over the indices $b_1, \cdots, b_q$ in the above expression, we obtain the $q!$ terms. For each term, we then exchange b_1 with the index immediately to the left repeatedly as

$$
\nabla_{cb_q\cdots b_2 b_1} = \nabla_{cb_q\cdots b_1 b_2} + \nabla_{cb_q\cdots [b_2 b_1]} = \nabla_{cb_q\cdots b_1 b_3 b_2} + \nabla_{cb_q\cdots [b_3 b_1] b_2} + \nabla_{cb_q\cdots [b_2 b_1]} = \cdots\,,
$$

until b_1 comes next to c. After that, we act $\hat{A}_{a_1 b_1 c}$ on the resulting terms so as to replace the outer two derivatives ∇_{cb_1} with the Riemann curvature tensor, confirming that eq. (3.17) can be cast in the prolongation variables of lower orders than q with the coefficients of the Riemann curvature tensor and its derivatives. Similarly, the summands in eqs. (3.15) and (3.16) can read

$$
\begin{aligned}
Y_{\begin{smallmatrix} a_1 & \cdots & \cdots & \cdots & a_p & b_2 \\ b_1 & \cdots & b_q & c \end{smallmatrix}} \nabla_{cb_q\cdots b_1} K_{a_p\cdots a_1} &= Y_{\begin{smallmatrix} a_1 & \cdots & \cdots & \cdots & a_p & b_2 \\ b_1 & \cdots & b_q & c \end{smallmatrix}}\, 2\nabla_{cb_q\cdots [b_2 b_1]} K_{a_p\cdots a_1}\,,\\
Y_{\begin{smallmatrix} a_1 & \cdots & \cdots & \cdots & a_p & b_3 \\ b_1 & \cdots & b_q & c \end{smallmatrix}} \nabla_{cb_q\cdots b_1} K_{a_p\cdots a_1} &= Y_{\begin{smallmatrix} a_1 & \cdots & \cdots & \cdots & a_p & b_3 \\ b_1 & \cdots & b_q & c \end{smallmatrix}} \left(2\nabla_{cb_q\cdots [b_3 b_2] b_1} K_{a_p\cdots a_1} + 2\nabla_{cb_q\cdots b_2 [b_3 b_1]} K_{a_p\cdots a_1} \right),
\end{aligned}
$$

and so on. We deduce that all the summands can also be cast in the prolongation variables of lower orders than q with the coefficients of the Riemann curvature tensor and its derivatives. We therefore conclude that eqs. (3.14)–(3.16) are sufficient to state that the prolongation has been completed.

We show the explicit forms of the prolonged system for $p = 1$, 2 and 3. The prolonged system for KVs is given by

$$
\begin{aligned}
\nabla_b K_a &= K^{(1)}_{ba}\,, && (3.18)\\
\nabla_c K^{(1)}_{ba} &= Y_{\begin{smallmatrix} a \\ b \end{smallmatrix}}\, Y_{\begin{smallmatrix} a & c \\ b \end{smallmatrix}}\, R_{cba}{}^{d} K_d\,, && (3.19)
\end{aligned}
$$

which completely agree with eqs. (3.7); for the second-order, the prolonged system is given by

$$
\begin{aligned}
\nabla_c K_{ba} &= Y_{\ytableaushort{ab}}\, K^{(1)}_{cba}\,, && (3.20)\\
\nabla_d K^{(1)}_{cba} &= Y_{\ytableaushort{ab,c}} \left[K^{(2)}_{dcba} - \tfrac{5}{2} R_{dac}{}^{m} K_{mb} - 2R_{dab}{}^{m} K_{mc} + \tfrac{1}{2} R_{acb}{}^{m} K_{md}\right], && (3.21)\\
\nabla_e K^{(2)}_{dcba} &= Y_{\ytableaushort{ab,cd}} \Big[-\tfrac{4}{3}(\nabla_a R_{bcd}{}^{m})K_{me} - \tfrac{2}{3}(\nabla_e R_{cab}{}^{m})K_{md} - \tfrac{8}{3}(\nabla_a R_{bde}{}^{m})K_{mc} - 12R_{eac}{}^{m} K^{(1)}_{mdb} \\
&\qquad - 4R_{eab}{}^{m} K^{(1)}_{mcd} - \tfrac{2}{3} R_{cab}{}^{m} K^{(1)}_{mde} + \tfrac{7}{3} R_{cab}{}^{m} K^{(1)}_{med}\Big]. && (3.22)
\end{aligned}
$$

Compared with the results of [34, 35, 36, 37], our results have simpler forms; taking one more step, we can write out the prolonged system for the third-order explicitly.

$$
\begin{aligned}
\nabla_d K_{cba} &= Y_{\ytableaushort{abc}}\, K^{(1)}_{dcba}\,, && (3.23)\\
\nabla_e K^{(1)}_{dcba} &= Y_{\ytableaushort{abc,d}} \left[K^{(2)}_{edcba} - 3R_{ead}{}^{m} K_{mbc} - 5R_{eab}{}^{m} K_{mdc} - R_{dab}{}^{m} K_{mce}\right], && (3.24)\\
\nabla_f K^{(2)}_{edcba} &= Y_{\ytableaushort{abc,de}} \Big[K^{(3)}_{fedcba} + 2(\nabla_b R_{ead}{}^{m})K_{mcf} + 2(\nabla_b R_{eac}{}^{m})K_{mdf} + 2(\nabla_b R_{fac}{}^{m})K_{mde} \\
&\qquad + 6(\nabla_b R_{fad}{}^{m})K_{mce} - 2(\nabla_b R_{fda}{}^{m})K_{mce} - \tfrac{32}{3} R_{fbc}{}^{m} K^{(1)}_{meda} - \tfrac{16}{3} R_{fba}{}^{m} K^{(1)}_{mced} \\
&\qquad - 20R_{fbe}{}^{m} K^{(1)}_{mcda} + \tfrac{2}{3} R_{cea}{}^{m} K^{(1)}_{mbdf} - \tfrac{10}{3} R_{bea}{}^{m} K^{(1)}_{mfcd} + \tfrac{2}{3} R_{bea}{}^{m} K^{(1)}_{mfdc}\Big], && (3.25)\\
\nabla_g K^{(3)}_{fedcba} &= Y_{\ytableaushort{abc,def}} \Big[-24R_{gfc}{}^{m} K^{(2)}_{mdbea} - 6R_{gfd}{}^{m} K^{(2)}_{mecba} + 4R_{fcd}{}^{m} K^{(2)}_{mgbae} - 12R_{fcd}{}^{m} K^{(2)}_{mgeba} \\
&\qquad - 20(\nabla_d R_{aeg}{}^{m})K^{(1)}_{mbcf} + 12(\nabla_d R_{age}{}^{m})K^{(1)}_{mbcf} - 2(\nabla_f R_{gde}{}^{m})K^{(1)}_{mcba} \\
&\qquad - \tfrac{3}{2}(\nabla_d R_{aef}{}^{m})K^{(1)}_{mbcg} - 16(\nabla_d R_{aeb}{}^{m})K^{(1)}_{mgcf} - 3(\nabla_d R_{aef}{}^{m})K^{(1)}_{mgbc} \\
&\qquad + \tfrac{9}{2}(\nabla_{fe} R_{bda}{}^{m})K_{mgc} - \tfrac{9}{2}(\nabla_{ed} R_{fcg}{}^{m})K_{mba} + 3(\nabla_{ge} R_{afc}{}^{m})K_{mdb} \\
&\qquad + 6R_{gfc}{}^{m} R_{ebd}{}^{n} K_{mna} + 5R_{gfe}{}^{m} R_{dac}{}^{n} K_{mnb} + 6R_{gfe}{}^{m} R_{mac}{}^{n} K_{ndb} \\
&\qquad - 4R_{fbe}{}^{m} R_{mgd}{}^{n} K_{nac} + \tfrac{1}{2} R_{fbe}{}^{m} R_{mdg}{}^{n} K_{nac} - 9R_{dbf}{}^{m} R_{mcg}{}^{n} K_{nea} \\
&\qquad - 2R_{fce}{}^{m} R_{mbd}{}^{n} K_{nag} + \tfrac{11}{2} R_{fce}{}^{m} R_{mdb}{}^{n} K_{nag}\Big]. && (3.26)
\end{aligned}
$$

Let us provide a sketch of the proof for the results (3.14)–(3.16). Since $K_{a_p \cdots a_1}$ is totally symmetric, we have

$$
\nabla_c K_{a_p \cdots a_1} = Y_{\ytableaushort{{a_1}{\cdots}{a_p}}}\, \mathrm{id}_{p+1}\, \nabla_c K_{a_p \cdots a_1}\,,
$$

where id_{p+1} is the identity operator. Using the completeness of the Young symmetrisers with Littlewood's correction (2.13) yields

$$
Y_{\ytableaushort{{a_1}{\cdots}{a_p}}}\, \mathrm{id}_{p+1}\, \nabla_c K_{a_p \cdots a_1} = Y_{\ytableaushort{{a_1}{\cdots}{a_p}}} \left(L_{\ytableaushort{{a_1}{\cdots}{a_p},c}} + \cdots\right) \nabla_c K_{a_p \cdots a_1}\,.
$$

The round brackets contain a lot of the Young symmetrisers. However, most of these symmetrisers vanish due to Pieri's formula (2.20) and the Killing equation (3.12), leaving only $L_{\ytableaushort{{a_1}{\cdots}{a_p},c}}$. Thus we obtain

$$
\nabla_c K_{a_p \cdots a_1} = Y_{\ytableaushort{{a_1}{\cdots}{a_p}}}\, L_{\ytableaushort{{a_1}{\cdots}{a_p},c}}\, \nabla_c K_{a_p \cdots a_1}\,. \qquad (3.27)
$$

The tableau $\begin{smallmatrix} a_1 & \cdots & a_p \\ c \end{smallmatrix}$ is row-ordered and then $L_{\begin{smallmatrix} a_1 & \cdots & a_p \\ c \end{smallmatrix}}$ is equal to $Y_{\begin{smallmatrix} a_1 & \cdots & a_p \\ c \end{smallmatrix}}$, confirming eq. (3.14). Similarly, differentiating the q^{th} prolongation variable (3.13) for $1 \leq q \leq p$ gives

$$\begin{aligned} \nabla_c K^{(q)}_{b_q \cdots b_1 a_p \cdots a_1} &= Y_{\begin{smallmatrix} a_1 & \cdots & \cdots & \cdots & a_p \\ b_1 & \cdots & b_q \end{smallmatrix}} \nabla_{c b_q \cdots b_1} K_{a_p \cdots a_1} \\ = Y_{\begin{smallmatrix} a_1 & \cdots & \cdots & \cdots & a_p \\ b_1 & \cdots & b_q \end{smallmatrix}} \Big[L_{\begin{smallmatrix} a_1 & \cdots & \cdots & \cdots & \cdots & a_p \\ b_1 & \cdots & b_q & c \end{smallmatrix}} &+ L_{\begin{smallmatrix} a_1 & \cdots & \cdots & \cdots & a_p \\ b_1 & \cdots & b_q \\ c \end{smallmatrix}} + L_{\begin{smallmatrix} a_1 & \cdots & \cdots & \cdots & a_p & c \\ b_1 & \cdots & b_q \end{smallmatrix}} + \sum_{i=2}^{q} L_{\begin{smallmatrix} a_1 & \cdots & \cdots & \cdots & \cdots & \cdots & a_p & b_i \\ b_1 & \cdots & \cancel{b_i} & \cdots & b_q & c \end{smallmatrix}} \Big] \nabla_{c b_q \cdots b_1} K_{a_p \cdots a_1} \,. \end{aligned} \tag{3.28}$$

As we see from eq. (2.15)–(2.16), all Littlewood's corrections in the above expression vanish and thus L_Θ equals Y_Θ. We therefore obtain eq. (3.15), concluding the proof. Note that the expression (3.28) is also valid for $q = p$ if $L_{\begin{smallmatrix} a_1 & \cdots & \cdots & \cdots & a_p \\ b_1 & \cdots & b_q & c \end{smallmatrix}}$ is omitted.

3.4 Geometric interpretation

Once the prolonged system (3.14)–(3.16) has been formulated, one may forget the definitions of the prolongation variables (3.13) because one can reconstruct eqs. (3.12) and (3.13) from the prolonged system (3.14)–(3.16) under the assumption

$$K^{(q)}_{b_q \cdots b_1 a_p \cdots a_1} = Y_{\begin{smallmatrix} a_1 & \cdots & \cdots & \cdots & a_p \\ b_1 & \cdots & b_q \end{smallmatrix}} K^{(q)}_{b_q \cdots b_1 a_p \cdots a_1} \,, \tag{3.29}$$

which means that the prolonged system (3.14)–(3.16) with the assumption (3.29) are equivalent to the Killing equation (3.12). A proof of this assertion is given as follows: Suppose the prolonged system (3.14)–(3.16) with the assumption (3.29) hold. First, multiplying both sides of eq. (3.14) by $Y_{\begin{smallmatrix} a_1 & \cdots & a_p & c \end{smallmatrix}}$ from the left gives

$$\nabla_{(c} K_{a_p \cdots a_1)} = Y_{\begin{smallmatrix} a_1 & \cdots & a_p & c \end{smallmatrix}} Y_{\begin{smallmatrix} a_1 & \cdots & a_p \\ c \end{smallmatrix}} K^{(1)}_{c a_p \cdots a_1} = 0 \,, \tag{3.30}$$

confirming the Killing equation (3.12). We have used the orthogonality of Young symmetrisers (2.7) here. Next, multiplying both sides of eq. (3.15) by $Y_{\begin{smallmatrix} a_1 & \cdots & \cdots & \cdots & a_p \\ b_1 & \cdots & b_q & c \end{smallmatrix}}$ from the left yields

$$Y_{\begin{smallmatrix} a_1 & \cdots & \cdots & \cdots & a_p \\ b_1 & \cdots & b_q & c \end{smallmatrix}} \nabla_c K^{(q)}_{b_q \cdots b_1 a_p \cdots a_1} = Y_{\begin{smallmatrix} a_1 & \cdots & \cdots & \cdots & a_p \\ b_1 & \cdots & b_q & c \end{smallmatrix}} Y_{\begin{smallmatrix} a_1 & \cdots & \cdots & \cdots & a_p \\ b_1 & \cdots & b_q \end{smallmatrix}} K^{(q+1)}_{c b_q \cdots b_1 a_p \cdots a_1} = K^{(q+1)}_{c b_q \cdots b_1 a_p \cdots a_1} \,, \tag{3.31}$$

which leads to

$$K^{(q+1)}_{c b_q \cdots b_1 a_p \cdots a_1} = Y_{\begin{smallmatrix} a_1 & \cdots & \cdots & \cdots & a_p \\ b_1 & \cdots & b_q & c \end{smallmatrix}} \cdots Y_{\begin{smallmatrix} a_1 & \cdots & a_p \\ b_1 \end{smallmatrix}} \nabla_{c b_q \cdots b_1} K_{a_p \cdots a_1} = Y_{\begin{smallmatrix} a_1 & \cdots & \cdots & \cdots & a_p \\ b_1 & \cdots & b_q & c \end{smallmatrix}} \nabla_{c b_q \cdots b_1} K_{a_p \cdots a_1} \,, \tag{3.32}$$

where we have used the identity

$$Y_{\begin{smallmatrix} a_1 & \cdots & \cdots & \cdots & a_p \\ b_1 & \cdots & b_i & c \end{smallmatrix}} Y_{\begin{smallmatrix} a_1 & \cdots & \cdots & \cdots & a_p \\ b_1 & \cdots & b_i \end{smallmatrix}} = Y_{\begin{smallmatrix} a_1 & \cdots & \cdots & \cdots & a_p \\ b_1 & \cdots & b_i & c \end{smallmatrix}} \,, \tag{3.33}$$

which follows from Schur's lemma (2.18) and Raicu's formula (2.19).

Geometrically, the set of the variables (3.29) can be viewed as a section of the vector bundle $E^{(p)}$ over M

$$E^{(p)} = \underbrace{\begin{smallmatrix} \square & \square & \square & \cdots & \square \end{smallmatrix}}_{p\text{ boxes}} \oplus \underbrace{\begin{smallmatrix} \square & \square & \square & \cdots & \square \\ \square \end{smallmatrix}}_{(p+1)\text{ boxes}} \oplus \cdots \oplus \underbrace{\begin{smallmatrix} \square & \square & \square & \cdots & \square \\ \square & \square & \square & \cdots & \square \end{smallmatrix}}_{2p\text{ boxes}} \,, \tag{3.34}$$

where the fibers are irreducible representations of $\mathrm{GL}(N)$ corresponding to the Young diagrams. Moreover, the prolonged system (3.14)–(3.16) can be viewed as the parallel equation for a section of $E^{(p)}$,

$$D_a \boldsymbol{K} \;=\; 0\,, \tag{3.35}$$

where $D_a \equiv \nabla_a - \Omega_a$ is the connection on $E^{(p)}$ and $\boldsymbol{K}$ is a section of $E^{(p)}$. $\Omega_a \in \mathrm{End}(E^{(p)})$ depends on the Riemann curvature tensor and its derivatives up to $(p-1)$th order which can be read off from the right-hand side of the prolonged system (3.14)–(3.16). Hence it turns out that there is a one-to-one correspondence between KTs of the p^{th} order and the parallel sections. This leads to the Barbance-Delong-Takeuchi-Thompson (BDTT) formula [38, 39, 40, 41]

$$\dim K^p(M) \;\leq\; \frac{1}{n}\binom{N+p}{p+1}\binom{N+p-1}{p} \;=\; \mathrm{rank}\, E^{(p)}\,, \tag{3.36}$$

where $K^p(M)$ denotes the space of KTs of the p^{th} order in an N-dimensional space(-time) M. The equality is attained if and only if M is of constant curvature.

Chapter 4

Integrability conditions of the Killing equation

In the previous chapter, we have formulated the prolonged system of the Killing equation (3.12) in a manner that uses Young symmetrisers. We have also seen the BDDT formula (3.36) that gives a maximum upper bound on the number of linearly independent solutions to the Killing equation. In this chapter, we formulate the integrability conditions of the prolonged system. It provides a concrete way to enumerate the number of the solutions to the Killing equation. Our analysis here is also based on Young symmetrisers introduced in Chapter 2.

This chapter consists of four sections: In Section 4.1 we provide the explicit forms of the integrability condition up to the third order. We also make a conjecture on the integrability condition for a general order. In Section 4.2 we demonstrate a method for computing the dimension of the space of KTs with a specific example. A derivation of the formula for the integrability condition (4.8) have been posted in Section 4.3. In Section 4.3, we make a slight digression to discuss the Killing-Yano equation by using our analysis.

4.1 Main results

This section is devoted to investigating the integrability conditions of the prolonged system for KTs of the p^{th} order, which arises as a consistency condition:

$$0 \;=\; 2\,I^{(p,q)}_{a_1\cdots a_p b_1\cdots b_q cd} \;\equiv\; \nabla_{dc}K^{(q)}_{b_q\cdots b_1 a_p\cdots a_1} - \nabla_{cd}K^{(q)}_{b_q\cdots b_1 a_p\cdots a_1} - 2\nabla_{[dc]}K^{(q)}_{b_q\cdots b_1 a_p\cdots a_1}\,, \tag{4.1}$$

where it is understood that; the first and second terms in eq. (4.1) are evaluated by the equation for the q^{th} prolonged variable (3.15); on the one hand, the last term in eq. (4.1) is described by the defining equation of the Riemann curvature tensor.

Calculating the integrability condition (4.1) up to $p = 3$, we obtain the following results:

$\underline{p=1}$

$$I^{(1,0)}_{abc} = 0\,, \tag{4.2}$$

$$I^{(1,1)}_{abcd} = Y_{\tiny\begin{array}{|c|c|}\hline a&c\\\hline b&d\\\hline\end{array}}\left[(\nabla_d R_{cba}{}^{m})K_m - 2R_{cba}{}^{m}K^{(1)}_{md}\right], \tag{4.3}$$

$\underline{p=2}$

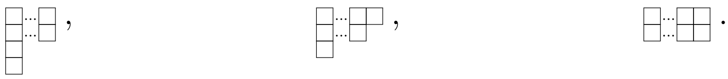
$$
\begin{aligned}
I^{(2,0)}_{abcd} &= I^{(2,1)}_{abcde} = 0\,, &(4.4)\\
I^{(2,2)}_{abcdef} &= Y_{\begin{array}{|c|c|c|}\hline a&b&e\\\hline c&d&f\\\hline\end{array}} \Big[3(\nabla_{fd}R_{ecb}{}^{m})K_{ma} + 2R_{edc}{}^{m}R_{mbf}{}^{n}K_{na} - 5R_{edc}{}^{m}R_{mfb}{}^{n}K_{na}\\
&\quad + 3(\nabla_{f}R_{eda}{}^{m})K^{(1)}_{mbc} - 9(\nabla_{f}R_{eda}{}^{m})K^{(1)}_{mcb} - 8R_{fed}{}^{m}K^{(2)}_{mcba}\Big]\,, &(4.5)
\end{aligned}
$$

$\underline{p=3}$

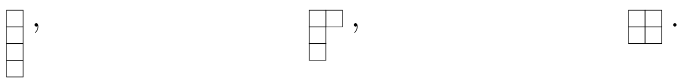
$$
\begin{aligned}
I^{(3,0)}_{abcde} &= I^{(3,1)}_{abcdef} = I^{(3,2)}_{abcdefg} = 0\,, &(4.6)\\
I^{(3,3)}_{abcdefgh} &= Y_{\begin{array}{|c|c|c|c|}\hline a&b&c&g\\\hline d&e&f&h\\\hline\end{array}} \Big[6(\nabla_{hfe}R_{gdc}{}^{m})K_{mba} - 27(\nabla_{h}R_{gfe}{}^{m})R_{mdc}{}^{n}K_{nba} - 34R_{ghf}{}^{m}(\nabla_{e}R_{mdc}{}^{n})K_{nba}\\
&\quad - 15(\nabla_{h}R_{gfe}{}^{m})R_{mcb}{}^{n}K_{nda} + 15(\nabla_{h}R_{gfe}{}^{m})R_{dcb}{}^{n}K_{mna} - 20R_{ghf}{}^{m}(\nabla_{e}R_{mcb}{}^{n})K_{nda}\\
&\quad + 20R_{ghf}{}^{m}(\nabla_{e}R_{dcb}{}^{n})K_{mna} - 24(\nabla_{hf}R_{ceg}{}^{m})K^{(1)}_{mbad} + 12(\nabla_{hf}R_{ceg}{}^{m})K^{(1)}_{mdba}\\
&\quad + 50R_{ghf}{}^{m}R_{mec}{}^{n}K^{(1)}_{nbad} + 40R_{ghf}{}^{m}R_{mce}{}^{n}K^{(1)}_{ndba} - \tfrac{74}{3}R_{ghf}{}^{m}R_{med}{}^{n}K^{(1)}_{ncba}\\
&\quad - \tfrac{40}{3}R_{ghf}{}^{m}R_{ced}{}^{n}K^{(1)}_{mnba} - 35(\nabla_{h}R_{fgc}{}^{m})K^{(2)}_{mbaed} - 5(\nabla_{h}R_{fgc}{}^{m})K^{(2)}_{medba}\\
&\quad - 20R_{hgf}{}^{m}K^{(3)}_{medcba}\Big]\,. &(4.7)
\end{aligned}
$$

From eqs. (4.2)–(4.7), it is observed that in the cases $q<p$ the integrability condition of q^{th} prolongation variable is automatically satisfied. More precisely, we can confirm that all of these conditions vanish identically, up to the first and second Bianchi identities, $R_{[abc]}{}^{d} = \nabla_{[a}R_{bc]de} = 0$. In contrast, the integrability condition at $q=p$ provides nontrivial relations among all the prolongation variables. This is consistent with the result in [42].

It is intriguing to note that the integrability condition of the p^{th} prolongation variable belongs to the Young diagram of shape $(p+1,p+1)$. If we act the curvature operator on the p^{th} prolongation variable, we obtain a $(2p+2)^{\text{th}}$ order tensor belonging to the representation $(1,1)\otimes(p,p)$. It can be decomposed into the irreducible representations $(p,p,1,1)$, $(p+1,p,1)$ and $(p+1,p+1)$ that are respectively described by the Young diagrams

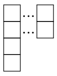
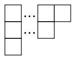


However, we observe from eqs. (4.3), (4.5) and (4.7) that the integrability condition of the p^{th} prolongation variable makes non-trivial contribution only for the representation $(p+1,p+1)$. For instance the integrability condition $I^{(1,1)}_{abcd}$ could have the representations

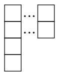
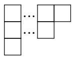


However, the result (4.3) claims that the first two representations do not appear for some reason. Thus we are led to make the following conjecture:

Conjecture 1. *The integrability condition of the p^{th} prolongation variable belongs to the representation described by the rectanguler Young diagram $(p+1,p+1)$.*

In general, it is not easy to write out the integrability condition of the p^{th} prolongation variable. This difficulty becomes more prominent as the order of KTs increases. However, if this conjecture holds true for $p \geq 4$, then there is no need to calculate the terms that belong to the representations  and , thereby allowing us to obtain the formula

$$
\begin{aligned}
I^{(p,p)}_{a_1\cdots a_p b_1\cdots b_p cd} \;=\; & Y_{\begin{smallmatrix} a_1\cdots a_p & c \\ b_1\cdots b_p & d\end{smallmatrix}} \Big[2\nabla_{d[cb_p]\cdots b_1} K_{a_p\cdots a_1} + 3\nabla_{db_p[cb_{p-1}]\cdots b_1} K_{a_p\cdots a_1} \\
& + 4\nabla_{db_p b_{p-1}[cb_{p-2}]\cdots b_1} K_{a_p\cdots a_1} \cdots + (p+1)\nabla_{db_p b_{p-1} b_{p-2}\cdots [cb_1]} K_{a_p\cdots a_1} - \nabla_{[dc]} K^{(p)}{}_{b_p\cdots b_1 a_p\cdots a_1} \Big] \\
& + (\text{the terms that belong to the representations } \square \text{ and } \square)\,. \qquad (4.8)
\end{aligned}
$$

We have confirmed that for the cases $p \leq 3$, the last term exactly vanish up to the first and second Bianchi identities, $R_{[abc]}{}^d = \nabla_{[a} R_{bc]de} = 0$. The proof of the formula (4.8) is given by 4.3.

As is the case with the prolonged system (3.14)–(3.16), there are still a lot of derivative terms left in the right-hand side of eq. (4.8). Then again, we can rewrite all these terms in eq. (4.8) to non-derivative terms by using the prolonged system. For $p \geq 4$, this is a challenging and daunting task which is beyond our scope here and will be considered in the future.

4.2 Application

As an application of the integrability conditions, we show a method for computing the number of linearly independent solutions to the Killing equation.

Let us recall the parallel equation (3.35). We introduce the curvature of the connection D_a as $R^D_{ab}\boldsymbol{K} \equiv [D_a, D_b]\boldsymbol{K}$. We call this the *Killing curvature*. All the integrability conditions of a KT of order p can be collectively expressed as

$$
R^D_{ab}\boldsymbol{K} \;=\; 0\,. \qquad (4.9)
$$

By repeatedly differentiating the condition (4.9), we obtain the set of linear algebraic equations

$$
R^D_{ab}\boldsymbol{K} \;=\; 0\,, \qquad (D_a R^D_{bc})\boldsymbol{K} \;=\; 0\,, \qquad (D_a D_b R^D_{cd})\boldsymbol{K} \;=\; 0\,, \qquad \cdots \qquad (4.10)
$$

After working out r differentiations, we are led to the system

$$
\boldsymbol{R}^D_r\boldsymbol{K} \;=\; 0\,, \qquad (4.11)
$$

where the coefficient matrix $\boldsymbol{R}^D_r$ depends on the Killing curvature and its derivatives. For example,

$$
\boldsymbol{R}^D_0 = \left(R^D_{ab}\right)\,, \qquad \boldsymbol{R}^D_1 = \begin{pmatrix} R^D_{ab} \\ D_a R^D_{bc} \end{pmatrix}\,, \qquad \boldsymbol{R}^D_2 = \begin{pmatrix} R^D_{ab} \\ D_a R^D_{bc} \\ D_a D_b R^D_{cd} \end{pmatrix}\,. \qquad (4.12)
$$

It is known that by applying the Frobenius theorem to the condition (4.11), the following theorem holds true (see, e.g. Ref. [43]).

Theorem 11 (Bryant–Dunajski–Eastwood)**.** *If we find the smallest natural number r_0 such that*

$$\mathrm{rank}\,\boldsymbol{R}^D_{r_0} \;=\; \mathrm{rank}\,\boldsymbol{R}^D_{r_0+1}\,, \tag{4.13}$$

then it follows that $\mathrm{rank}\,\boldsymbol{R}^D_{r_0} = \mathrm{rank}\,\boldsymbol{R}^D_{r_0+r}$ *for any natural number r and consequently the dimension of the space of the KT reads*

$$\dim K^p \;=\; \mathrm{rank}\,E^{(p)} - \mathrm{rank}\,\boldsymbol{R}^D_{r_0}\,, \tag{4.14}$$

where $\mathrm{rank}\,E^{(p)}$ *is given by the BDTT formula* (3.36).

The condition (4.13) means that the components of the $(r_0+1)^{\text{th}}$ order derivatives of the Killing curvature, $D_{a_1}\cdots D_{a_{r_0+1}}R^D_{bc}$, can be expressed as the linear combinations of the components of the lower order derivatives than r_0+1. Hence, the components of the one-higher order derivatives, $D_{a_1}\cdots D_{a_{r_0+2}}R^D_{bc}$, can also be expressed as the linear combinations of the lower order derivatives than r_0+1. By induction, we can conclude that the theorem holds true.

It should be remarked that computing the rank of the matrix $\boldsymbol{R}^D_r$ boils down to solve a system of the linear algebraic equations

$$I^{(p,p)}_{a_1\cdots a_p b_1\cdots b_p cd} \;=\; 0\,, \qquad \cdots\,, \qquad \nabla_{e_1\cdots e_r} I^{(p,p)}_{a_1\cdots a_p b_1\cdots b_p cd}\Big|_{D\boldsymbol{K}=0} \;=\; 0\,, \tag{4.15}$$

where $I^{(p,p)}_{a_1\cdots a_p b_1\cdots b_p cd}$ is the integrability condition of the p^{th} prolongation variable of a KT of p^{th}-order defined by eq. (4.1). If r_0 exists, eq. (4.14) allows us to have the value of $\dim K^p$. Otherwise differentiating the integrability condition (4.9) reveals a large number of additional conditions. We can stop the differentiation and conclude that no KT of p^{th}-order exists if $\mathrm{rank}\,\boldsymbol{R}^D_{r_0}$ is equal to $\mathrm{rank}\,E^{(p)}$. Based on this fact, we can determine the dimension of the space of KTs.

To demonstrate the efficacy of our method, let us take the Kerr metric in Boyer-Lindquist coordinates:

$$\begin{aligned} ds^2 \;=\; & -\left(1-\frac{2Mr}{\Sigma}\right)dt^2 - \frac{4aMr\sin^2\theta}{\Sigma}dt\,d\phi + \frac{\Sigma}{\Delta}dr^2 + \Sigma\,d\theta^2 \\ & + \left(r^2+a^2+\frac{2a^2Mr\sin^2\theta}{\Sigma}\right)\sin^2\theta\,d\phi^2\,, \end{aligned} \tag{4.16}$$

with

$$\Sigma \;=\; r^2 + a^2\cos^2\theta\,, \qquad \Delta \;=\; r^2 - 2Mr + a^2\,, \tag{4.17}$$

and determine the number of the solutions to the Killing equation up to $p=2$. As a higher order KT includes reducible ones, e.g. $\xi_{(a}\zeta_{b)}$ is a trivial KT if ξ^a and ζ^a are KVs, at first we must solve the integrability condition for $p=1$.

For $p=1$ case, a section of the bundle $E^{(1)}$ can be written as

$$\boldsymbol{K} \;=\; \begin{pmatrix} K_a \\ K^{(1)}_{ba} \end{pmatrix}\,, \qquad \text{with} \qquad K^{(1)}_{ba} \in \begin{array}{|c|}\hline a \\ \hline b \\ \hline \end{array}\,. \tag{4.18}$$

By solving the linear systems $\boldsymbol{R}^D_1\boldsymbol{K}=0$ and $\boldsymbol{R}^D_2\boldsymbol{K}=0$, that is

$$I^{(1,1)}_{abcd} \;=\; 0\,, \qquad \nabla_e I^{(1,1)}_{abcd}\Big|_{D\boldsymbol{K}=0} \;=\; 0\,, \tag{4.19}$$

4-dimensional metrics / order of KTs	1	2
Maximally symmetric	10	50
Schwarzschild	4	11
Kerr	2	5
Reissner-Nordstrom	4	11

Table 4.1: The number of the first and second order KTs in several regular black hole metrics.

and

$$I^{(1,1)}_{abcd} = 0\,, \qquad \nabla_e I^{(1,1)}_{abcd}\Big|_{D\boldsymbol{K}=0} = 0\,, \qquad \nabla_{ef} I^{(1,1)}_{abcd}\Big|_{D\boldsymbol{K}=0} = 0\,, \tag{4.20}$$

we find that $\operatorname{rank}\boldsymbol{R}^D_1 = \operatorname{rank}\boldsymbol{R}^D_2 = 8$. In other words, the adjoined equation in eq. (4.20), $\nabla_{ef} I^{(1,1)}_{abcd} = 0$, does not change the rank. Since the maximal number of the KVs is $\operatorname{rank}E^{(1)} = 10$, we can conclude that $\dim K^1 = 2$. This is consistent with our knowledge: the two vector fields $\xi^a = (\partial_t)^a$ and $\zeta^a = (\partial_\phi)^a$ are the only KVs in the Kerr metric (4.16). Similarly, a section of the bundle $E^{(2)}$ is given by

$$\boldsymbol{K} = \begin{pmatrix} K_{ba} \\ K^{(1)}_{cba} \\ K^{(2)}_{dcba} \end{pmatrix}, \qquad \text{with} \qquad K^{(1)}_{cba} \in \begin{array}{|c|c|}\hline a & b \\ \hline c \\ \cline{1-1}\end{array}\,, \qquad K^{(2)}_{dcba} \in \begin{array}{|c|c|}\hline a & b \\ \hline c & d \\ \hline\end{array}\,. \tag{4.21}$$

After solving the linear systems $\boldsymbol{R}^D_1 \boldsymbol{K} = 0$ and $\boldsymbol{R}^D_2 \boldsymbol{K} = 0$, we find that $\operatorname{rank}\boldsymbol{R}^D_1 = \operatorname{rank}\boldsymbol{R}^D_2 = 45$. It also follows from eq. (3.36) that $\operatorname{rank}E^{(2)} = 50$. This amounts to $\dim K^2 = 5$. We know that four of them

$$g_{ab}\,, \qquad \xi_{(a}\xi_{b)}\,, \qquad \xi_{(a}\zeta_{b)}\,, \qquad \zeta_{(a}\zeta_{b)}\,, \tag{4.22}$$

are reducible KTs while the only one

$$\begin{aligned} K_{ab} = {} & \frac{a^2}{\Sigma}\Big[\Delta\cos^2\theta + r^2\sin^2\theta\Big](dt)^2_{ab} - \frac{a^2\,\Sigma\cos^2\theta}{\Delta}(dr)^2_{ab} + r^2\,\Sigma\,(d\theta)^2_{ab} \\ & + \frac{\sin^2\theta}{\Sigma}\Big[r^2(a^2+r^2)^2 + a^4\,\Delta\cos^2\theta\sin^2\theta\Big]\,(d\phi)^2_{ab} \\ & - \frac{2a\sin^2\theta}{\Sigma}\Big[r^2(a^2+r^2) + a^2\Delta\cos^2\theta\Big]\,(dt)_{(a}(d\phi)_{b)}\,. \end{aligned} \tag{4.23}$$

Using the same method, we investigate the first and second order KTs in several regular black hole metrics, as shown in Table 4.1. It would be of great interest to make a systematic investigation of higher-order KTs in various spacetimes. We leave it as a future work.

4.3 Supplement: derivation of the integrability condition

In this section we shall verify the integrability conditions (4.8). We begin with the condition of the p^{th} prolonged variable. Evaluating the expression (4.1) at $q = p$, one finds that

$$\begin{aligned} I^{(p,p)}_{a_1\cdots a_p b_1\cdots b_p cd} = {} & Y_{\substack{c\\d}}\, Y_{\substack{a_1\cdots a_p\\ b_1\cdots b_p}} \Bigg[Y_{\substack{a_1\cdots a_p\\ b_1\cdots b_p\\ c}} \nabla_{dcb_p\cdots b_1} K_{a_p\cdots a_1} + Y_{\substack{a_1\cdots a_p\, c\\ b_1\cdots b_p}} \nabla_{dcb_p\cdots b_1} K_{a_p\cdots a_1} \\ & + \sum_{i=2}^{p} Y_{\substack{a_1\cdots\cdots\cdots a_p\, b_i\\ b_1\cdots \cancel{b_i}\cdots b_p\, c}} \nabla_{dcb_p\cdots b_1} K_{a_p\cdots a_1} - \nabla_{[dc]} K^{(p)}_{b_p\cdots b_1 a_p\cdots a_1} \Bigg]. \end{aligned} \tag{4.24}$$

The last term in eq. (4.24) can be treated as

$$\begin{aligned} Y_{\substack{c\\d}}\, Y_{\substack{a_1\cdots a_p\\ b_1\cdots b_p}} \nabla_{[dc]} K^{(p)}_{b_p\cdots b_1 a_p\cdots a_1} &= Y_{\substack{c\\d}}\, Y_{\substack{a_1\cdots a_p\\ b_1\cdots b_p}} \sum_{\Theta\in\mathscr{Y}_{2p+1}} L_\Theta\, \nabla_{[dc]} K^{(p)}_{b_p\cdots b_1 a_p\cdots a_1} \\ &= Y_{\substack{c\\d}}\, Y_{\substack{a_1\cdots a_p\\ b_1\cdots b_p}} \Bigg(Y_{\substack{a_1\cdots a_p\\ b_1\cdots b_p\\ c}} + Y_{\substack{a_1\cdots a_p\, c\\ b_1\cdots b_p}} \Bigg) \nabla_{[dc]} K^{(p)}_{b_p\cdots b_1 a_p\cdots a_1}, \end{aligned}$$

where Pieri's formula (2.20) is used and Littlewood's correction (2.10) is dropped by a relation $\hat{S}_{a_p c}\,\hat{A}_{a_p c} = 0$. Hence, eq. (4.24) can be rewritten as

$$\begin{aligned} I^{(p,p)}_{a_1\cdots a_p b_1\cdots b_p cd} = {} & Y_{\substack{c\\d}}\, Y_{\substack{a_1\cdots a_p\\ b_1\cdots b_p}} \Bigg[\Bigg(Y_{\substack{a_1\cdots a_p\\ b_1\cdots b_p\\ c}} + Y_{\substack{a_1\cdots a_p\, c\\ b_1\cdots b_p}} \Bigg) \Big(\nabla_{dcb_p\cdots b_1} K_{a_p\cdots a_1} - \nabla_{[dc]} K^{(p)}_{b_p\cdots b_1 a_p\cdots a_1} \Big) \\ & + \sum_{i=2}^{p} Y_{\substack{a_1\cdots\cdots\cdots a_p\, b_i\\ b_1\cdots \cancel{b_i}\cdots b_p\, c}} \nabla_{dcb_p\cdots b_1} K_{a_p\cdots a_1} \Bigg]. \end{aligned} \tag{4.25}$$

Suppose now that the conjecture in Section 4 holds true. We then ignore the first symmetriser $Y_{\substack{a_1\cdots a_p\\ b_1\cdots b_p\\ c}}$ since it does not induce the representations belonging to $(p+1, p+1)$. The products of Young symmetrisers in eq. (4.25) can be simplified to

$$Y_{\substack{a_1\cdots a_p\\ b_1\cdots b_p}}\, Y_{\substack{a_1\cdots a_p\, c\\ b_1\cdots b_p}} = Y_{\substack{a_1\cdots a_p\, c\\ b_1\cdots b_p}}\,, \tag{4.26}$$

$$Y_{\substack{a_1\cdots a_p\\ b_1\cdots b_p}}\, Y_{\substack{a_1\cdots\cdots\cdots a_p\, b_i\\ b_1\cdots \cancel{b_i}\cdots b_p\, c}} = \tfrac{1}{2} Y_{\substack{a_1\cdots a_p\, c\\ b_1\cdots b_p}} (c, b_p) \prod_{j=1}^{p-i} (b_{p+1-j}, b_{p-j})\,. \tag{4.27}$$

The first result (4.26) follows immediately from Raicu's formula (2.19). The second result (4.27) can be confirmed by a direct calculation. By using the relations (4.26) and (4.27), the equation (4.25) can be rewritten as

$$\begin{aligned} I^{(p,p)}_{a_1\cdots a_p b_1\cdots b_p cd} = {} & Y_{\substack{c\\d}}\, Y_{\substack{a_1\cdots a_p\, c\\ b_1\cdots b_p}} \mathrm{id}_{2p+2} \Bigg[\nabla_{dcb_p\cdots b_1} K_{a_p\cdots a_1} + \frac{1}{2} \nabla_{db_p c b_{p-1}\cdots b_1} K_{a_p\cdots a_1} \\ & + \frac{1}{2} \nabla_{db_p b_{p-1} c b_{p-2}\cdots b_1} K_{a_p\cdots a_1} + \cdots + \frac{1}{2} \nabla_{db_p b_{p-1} b_{p-2}\cdots c b_1} K_{a_p\cdots a_1} - \nabla_{[dc]} K^{(p)}_{b_p\cdots b_1 a_p\cdots a_1} \Bigg]. \end{aligned} \tag{4.28}$$

Expanding id_{2p+2} and the antisymmetrisations of the operands yields the result (4.8).

4.4 Supplement: the Killing-Yano equation

Our analysis based on Young symmetrisers has effective applications to other types of overdetermined PDE systems. In this section we make a slight digression to discuss the Killing-Yano equation

$$\nabla_{(b}F_{a_1)\cdots a_p} \;=\; 0\,, \tag{4.29}$$

where $F_{a_1\cdots a_p} = F_{[a_1\cdots a_p]}$ is a Killing-Yano tensor field (KY). If we have a KY, then we can obtain a KT of order 2 as

$$K_{ab} \;\equiv\; F_{ac_1\cdots c_{p-1}}\, F_b{}^{c_1\cdots c_{p-1}}\,, \tag{4.30}$$

but the converse is not generally true. While the prolonged system of the Killing-Yano equation and their integrability conditions have been known in [33, 44, 45], we revisit the results by using Young symmetrisers.

Let F_{ab} be a KY and consider its derivatives. Since $\nabla_c F_{ba}$ is a type $(0,3)$ tensor field, its decomposition to the irreducible representations reads

$$\nabla_c F_{ba} \;=\; Y_{\ytableaushort{a,b}}\,\mathrm{id}_3\,\nabla_c F_{ba} \;=\; Y_{\ytableaushort{a,b}}\left(Y_{\ytableaushort{a,b,c}} + Y_{\ytableaushort{ab,c}} + Y_{\ytableaushort{ac,b}}\right)\nabla_c F_{ba} \;=\; Y_{\ytableaushort{a,b,c}}\,\nabla_c F_{ba} \;\equiv\; F^{(1)}_{cba}\,, \tag{4.31}$$

where we have used Pieri's formula (2.20) and the Killing-Yano equation (4.29). We next consider $\nabla_d F^{(1)}_{cba}$ as the above result is not yet closed. Its decomposition to the irreducible representations reads

$$\begin{aligned}
\nabla_d F^{(1)}_{cba} \;&=\; Y_{\ytableaushort{a,b,c}}\,\mathrm{id}_4\,\nabla_{dc}F_{ba} \;=\; Y_{\ytableaushort{a,b,c}}\left(Y_{\ytableaushort{a,b,c,d}} + Y_{\ytableaushort{ab,c,d}} + Y_{\ytableaushort{ac,b,d}} + Y_{\ytableaushort{ad,b,c}}\right)\nabla_{dc}F_{ba} \;=\; Y_{\ytableaushort{ad,b,c}}\,\nabla_{dc}F_{ba} \\
&=\; Y_{\ytableaushort{ad,b,c}}\left(2\nabla_{[dc]}F_{ba} + \nabla_{cd}F_{ba}\right) \;=\; 2Y_{\ytableaushort{ad,b,c}}\,Y_{\ytableaushort{a,b}}\,R_{dcb}{}^m F_{ma}\,,
\end{aligned} \tag{4.32}$$

which is now closed. This implies that we are at the completion of the procedure of prolongation. A similar calculation, taking into account Littlewood's corrections, yields the conclusion that the Killing-Yano equation (4.29) is equivalent to the prolonged system

$$\nabla_b F_{a_p\cdots a_1} \;=\; F^{(1)}_{ba_p\cdots a_1}\,, \tag{4.33}$$

$$\nabla_c F^{(1)}_{ba_p\cdots a_1} \;=\; p\,Y_{\ytableaushort{{a_1}c,{\vdots},{a_p},b}}\,Y_{\ytableaushort{{a_1},{\vdots},{a_p}}}\,R_{cba_p}{}^m F_{ma_{p-1}\cdots a_1}\,, \tag{4.34}$$

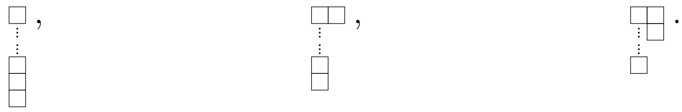
where

$$F^{(1)}_{ba_p\cdots a_1} \;\equiv\; Y_{\ytableaushort{{a_1},{\vdots},{a_p},b}}\,\nabla_b F_{a_p\cdots a_1}\,. \tag{4.35}$$

After a calculation analogous to that in Section 4.3, we obtain the integrability condition for eq. (4.33)

$$Y_{\ytableaushort{{a_1}b,{\vdots}c,{a_p}}}\left[R^m{}_{ca_1\ldots}F_{\ldots a_p bm}\right] \;=\; 0\,, \qquad \text{for} \qquad p > 1\,. \tag{4.36}$$

It can be confirmed that the integrability condition for eq. (4.34) is involved in the derivative of eq. (4.36). Therefore, eq. (4.36) and its derivatives are enough to discuss the integrability condition of the Killing–Yano equation. Once again, we face a situation similar to the one just discussed in Section 4.1. Namely, there is only a representation in eq. (4.36), even though the possible representations of eq. (4.36) are three



Chapter 5

Cartan's test for the Killing equation

So far we have discussed the integrability conditions that are necessary, but not sufficient to ensure the existence of Killing tensor fields. In this chapter we will restrict our attention to Killing vector fields, and derive necessary and sufficient conditions for admitting a solution to the Killing equation of the first order. In particular, we characterise metrics which admit Killing vector fields by local curvature obstructions. The obstructions will be obtained by analysing the integrability condition and the original Killing equation. As a consequence, the algorithm that tells us how many Killing vector fields exist for 3-dimensional Riemannian metrics will be formulated.

This chapter consists of three sections: In Section 5.1 we begin with a brief history to understand our place in the literature. In Section 5.2 we show our result for 3-dimensional Riemannian metrics. In Section 5.6 we demonstrate this result for a Hamiltonian system.

5.1 Some history

As we have seen in Chapter 4, the existence of first integrals of a geodesic flow puts tough restrictions on the Riemann curvature tensor. It is then natural to ask what is a major obstruction for their existence. A more restricted question is whether, conversely, the existence of the first integrals can be guaranteed by only several components of the Riemann curvature tensor and its derivatives. This classical problem reaches back at least to a partial answer provided by J. G. Darboux [46]. As mentioned above, we only deal with Killing vector fields (KVs) obeying the Killing equation

$$\nabla_{(a}K_{b)} = 0. \tag{5.1}$$

In 1887, J. G. Darboux had implicitly solved our problem in two dimensions. He characterised 2-dimensional metrics by *curvature invariants* that are a set of scalars constructed out of the Riemann curvature tensor R_{abcd} and possibly operations on it. His result is shown in Figure 5.1 and interpreted as follows:

(i) Noticing that in two dimensions $R_{abcd} = (1/2)Rg_{a[c}g_{d]b}$, we firstly evaluate the 1-form dR. If dR vanishes, we can conclude that 3 KVs exist and the space(-time) is maximally symmetric; Otherwise we go to the next step.

(ii) If the 2-form $dR \wedge d[(\nabla_a R)(\nabla^a R)]$ vanishes, we can go to the next step; Otherwise no KV exists.

(iii) If the 2-form $dR \wedge d\Box R$ vanishes, there are only one KV; Otherwise no KV exists.

$$
\begin{tikzcd}
\blacktriangleright\ dR = 0 \arrow[r, "\text{yes}"] \arrow[d, "\text{no}"] & \boxed{\text{3 KVs}} \\
dR \wedge d[(\nabla_a R)(\nabla^a R)] = 0 \arrow[r, "\text{no}"] \arrow[d, "\text{yes}"] & \boxed{\text{no KV}} \\
dR \wedge d\Box R = 0 \arrow[r, "\text{yes}"] \arrow[ur, "\text{no}"] & \boxed{\text{1 KV}}
\end{tikzcd}
$$

Figure 5.1: The algorithm for a 2-dimensinal space(-time). A triangle symbol ▶ stands for a root of this algorithm. A box denotes d'Alambertian, $\Box = \nabla_a \nabla^a$.

For 4-dimensional Einstein space(-time)s, R. Kerr also shown that there exists an algorithm which tells us that how many KVs exist. However, a specific form of the algorithm is still unknown. Meanwhile, an analogous algorithm for three or more higher-dimensional metrics remains largely unexplored. The reason for this may lie in the fact that in three or more dimensions, curvature invariants have a poor predictive for the existence of KVs. Namely, for *vanishing scalar invariant spacetimes* in which all curvature invariants vanish identically, they have no role in understanding of the existence of KVs.

In this Chapter, we do not stick to curvature invariants and characterise metrics admitting KVs by curvature and *curvilinear* invariants. As we will see, the curvilinear invariants result from an analysis of the integrability condition of the Killing equation. Although we will only consider 3-dimensional Riemannian metrics, our result would be extended to more higher-dimensional Riemannian or Lorentzian metrics. Concrete steps are based on the Cartan's prolongation: Solving the integrability condition, we obtain an ansatz for the solutions to the Killing equation. Using this, we subsequently rewrite the prolonged system and write out the integrability condition for it. If this is trivially satisfied, we can stop the procedure. Otherwise we once again solve the integrability condition and then repeat the above procedure.

5.2 Result for 3-dimensional Riemannian cases

We consider 3-dimensional Riemannian metrics. Our algorithm is outlined in Figure 5.2. In the following, we give the proof for this algorithm.

In 3-dimensional spaces, the integrability condition for KVs (4.3) can be written by

$$I_{ab} \;=\; \tfrac{1}{2}(\nabla_m R_{ab})K^m + R_{(a}{}^m K^{(1)}_{b)m}\,, \tag{5.2}$$

where $I_{ab} = g^{cd} I^{(1,1)}_{acbd}$. Multiplying I_{ab} by g^{ab}, R^{ab} and $R^{ac}R^b{}_c$, we obtain the necessary conditions

$$M^{(b)}{}_a K^a \;=\; \begin{pmatrix} \nabla_a I^{(1)} \\ \nabla_a I^{(2)} \\ \nabla_a I^{(3)} \end{pmatrix} K^a \;=\; 0\,, \tag{5.3}$$

where $I^{(1)} = R$, $I^{(2)} = R_{ab}R^{ab}$ and $I^{(3)} = R_{ac}R_b{}^c R^{ab}$. Note that the determinant of M

$$\det M \;=\; dI^{(1)} \wedge dI^{(2)} \wedge dI^{(3)}\,, \tag{5.4}$$

must be zero. Otherwise the kernel of M will only consist of $\{0\}$ and consequently there is not a KV. In what follows, $\mathrm{rank}M = 0,1,2$ cases are respectively called case $0,1,2$ and in turn considered separately.

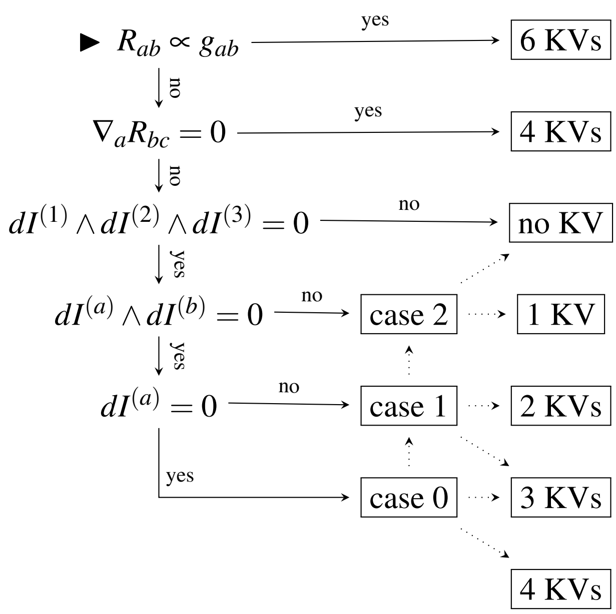


Figure 5.2: The algorithm for a 3-dimensinal space. A triangle symbol ▶ stands for a root of the algorithm. $I^{(a)}$ $(a = 1,2,3)$ denotes $I^{(1)} = R$, $I^{(2)} = R_{ab}R^{ab}$ and $I^{(3)} = R_{ab}R^{bc}R^a{}_c$, respectively. Several dotted lines involve some procedures shown in Figures 5.3–5.5.

5.3 Details of case 2

In this case, it follows from the rank-nullity theorem that $\dim\ker M = 1$. So if we have an annihilator of M, KVs can be written by

$$K^a = \omega\, U^a\,, \tag{5.5}$$

where ω and U^a are respectively an unknown function and the annihilator. We here take the annihilator as

$$U^a \;\equiv\; U\,\varepsilon^{abc}(\nabla_b I^{(1)})(\nabla_c I^{(2)})\,, \tag{5.6}$$

where the normalisation factor U is determined by

$$U^{-2} \;=\; 2(\nabla_{[a} I^{(1)})(\nabla_{b]} I^{(2)})(\nabla^{[a} I^{(1)})(\nabla^{b]} I^{(2)})\,, \tag{5.7}$$

so as to satisfy $U_a U^a = 1$. If U^a vanishes identically, two scalars (I_1, I_2) in the definition (5.6) must be replaced by (I_1, I_3) or (I_2, I_3). The condition $\mathrm{rank}M = 2$ guaratees that at least one of the 2-forms constructed from (dI_1, dI_2, dI_3) is non-zero.

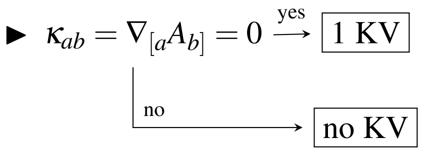


Figure 5.3: The sub algorithm for case 2. The curvature obstructions κ_{ab} and A_a are defined in eq. (5.13).

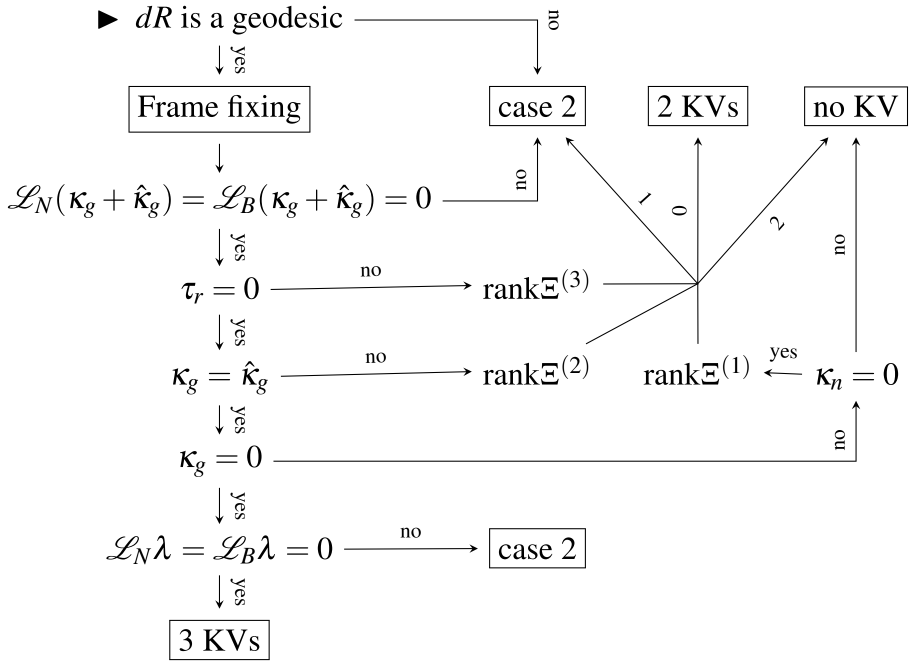


Figure 5.4: The sub algorithm for case 1. Several undefined quantities are defined in Subsection 5.4.

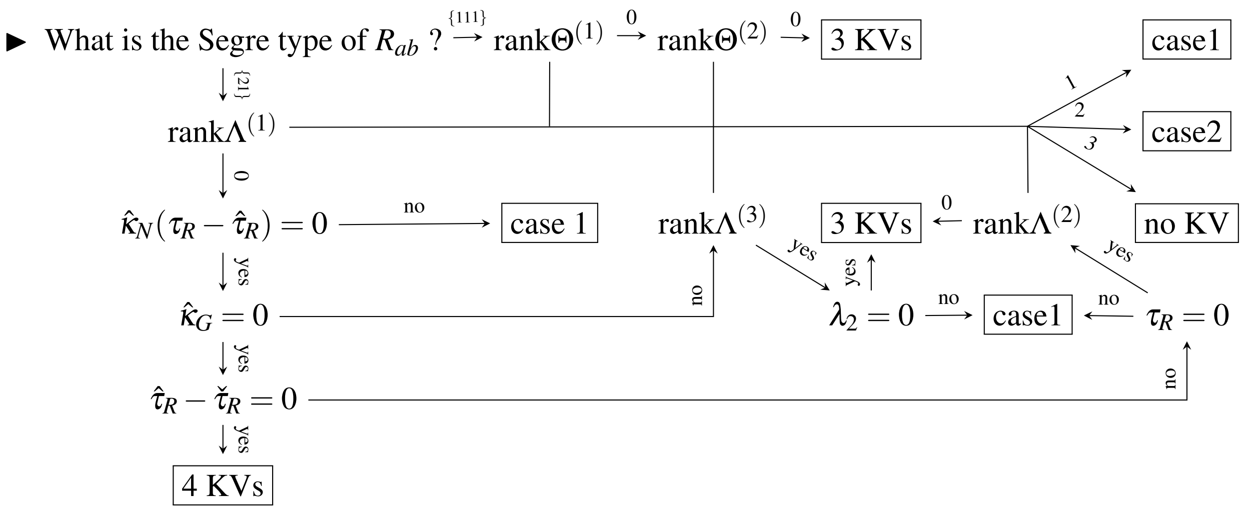


Figure 5.5: The sub algorithm for case 0. Several undefined quantities are defined in Subsection 5.5.

Using the concrete form (5.5), we perform prolongation of the Killing equation (5.1). To write out the components of the Killing equation, we introduce the projection tensor onto the hyperplanes orthogonal to U^a as

$$q_{ab}(U) \equiv g_{ab} - U_a U_b \,, \tag{5.8}$$

that is endowed with a metric property and an orthogonality

$$q_{ac}q^c{}_b = q_{ab} \,, \qquad q_{ab}\,U^b = 0 \,. \tag{5.9}$$

The (UU), (Uq) and (qq)-parts of the Killing equation have respectively 1, 2 and 3 components as follows.

$$2U^aU^b\nabla_{(a}K_{b)} = 2\mathscr{L}_U\,\omega = 0 \,, \tag{5.10}$$

$$2U^aq^b{}_c\nabla_{(a}K_{b)} = \omega\,(\nabla_c \ln\omega + A_c) \,, \tag{5.11}$$

$$2q^a{}_cq^b{}_d\nabla_{(a}K_{b)} = 2\omega\kappa_{cd} \,, \tag{5.12}$$

where we have defined the acceleration vector A^a and the extrinsic curvature κ_{ab} as

$$A^a(U) \equiv U^b\nabla_bU^a \,, \qquad \kappa_{ab}(U) \equiv \tfrac{1}{2}\mathscr{L}_U q_{ab} = q^c{}_aq^d{}_b\nabla_{(c}U_{d)} \,. \tag{5.13}$$

It can be concluded that the Killing equation (5.1) can be rewritten as

$$\kappa_{ab} = 0 \,, \qquad \nabla_a \ln 1/\omega = A_a \,, \tag{5.14}$$

with its compatibility condition

$$\nabla_{[a}A_{b]} = 0 \,. \tag{5.15}$$

If the annihilator U^a passes the first-order and second-order tests,

$$\kappa_{ab} = 0 \,, \qquad \nabla_{[a}A_{b]} = 0 \,, \tag{5.16}$$

then there are no extra conditions that must be satisfied by any solution to the Killing equation, thereby allowing us to confirm that one KV exists.

5.4 Details of case 1

Again by the rank-nullity theorem, $\dim\ker M = 2$. Then KVs take the form

$$K^a = \omega_1\,N^a + \omega_2\,B^a \,, \tag{5.17}$$

where ω_1 and ω_2 are two unknown functions. N^a and B^a must be two annihilators of M, $N^a\nabla_aR = B^a\nabla_aR = 0$. For now $\{N^a, B^a\}$ remain undetermined and will be fixed in the branches we will see below. We assume, however, that $\{N^a, B^a\}$ are unit vector fields satisfying an orthogonality $N^aB_a = 0$. We further introduce a unit vector field T^a as

$$T^a \equiv \frac{\nabla^aR}{\sqrt{(\nabla_mR)(\nabla^mR)}} \,, \tag{5.18}$$

so that a triad $\{T^a, N^a, B^a\}$ forms an orthogonal frame, i.e.

$$\delta^a{}_b \;=\; T^a T_b + N^a N_b + B^a B_b \,. \tag{5.19}$$

By using eqs. (5.17) and (5.19), the (TT)-part of the Killing equation can formally be written by

$$0 \;=\; (\omega_1 N^a + \omega_2 B^a) T^b \nabla_b T_a \,, \tag{5.20}$$

which gives a first-order test for the Ricci scalar. If the two functions ω_1 and ω_2 are independent, then $T^b \nabla_b T_a = 0$. Depending on whether the gradient of the Ricci scalar satisfies the geodesic equation

$$(\nabla^b R) \nabla_b \nabla_a R \;\propto\; \nabla_a R \,, \tag{5.21}$$

our algorithm branches off.

Branch where $\nabla_a R$ is not a geodesic

In this branch T^a and its acceleration $T^b \nabla_b T^a$ are linearly independent. It is therefore possible to define the Frenet–Serret frame as

$$T^a \;\equiv\; \frac{\nabla^a R}{\sqrt{(\nabla_m R)(\nabla^m R)}} \,, \qquad N^a \;\equiv\; \frac{T^b \nabla_b T^a}{\sqrt{(T^m \nabla_m T^k)(T^n \nabla_n T_k)}} \,, \qquad B^a \;\equiv\; \varepsilon^{abc} T_b N_c \,. \tag{5.22}$$

This triad obeys the Frenet–Serret formulas

$$T^b \nabla_b \begin{pmatrix} T^a \\ N^a \\ B^a \end{pmatrix} \;=\; \begin{pmatrix} 0 & \kappa & 0 \\ -\kappa & 0 & \tau \\ 0 & -\tau & 0 \end{pmatrix} \begin{pmatrix} T^a \\ N^a \\ B^a \end{pmatrix} , \tag{5.23}$$

where

$$\kappa \;\equiv\; N^a T^b \nabla_b T_a \,, \qquad \tau \;\equiv\; B^a T^b \nabla_b N_a \,, \tag{5.24}$$

are respectively the geodesic curvature and torsion of an integral curve of T^a.

Now, the (TT)-part of the Killing equation reads

$$\kappa\, \omega_1 \;=\; 0 \,. \tag{5.25}$$

Since $\kappa = 0$ contradicts $T^b \nabla_b T^a \neq 0$, ω_1 must be zero. As KVs take the form $K^a = \omega_2 B^a$, our algorithm reduces to the case 2 with the identification of $U^a = B^a$. Thus the acceleration $A^a(B)$ and the extrinsic curvature $\kappa_{ab}(B)$ defined in eq. (5.13) give the first-order and second-order tests. In this branch, there is at most one KV.

Branch where $\nabla_a R$ is a geodesic

We firstly have to fix the orthogonal frame $\{T^a, N^a, B^a\}$. There are two natural frame depending on the property of T^a: If T^a is an eigenvector of the Ricci tensor, we take the orthogonal frame as the eigensystem of the Ricci tensor. Otherwise N^a and B^a are taken to be

$$N^a \;\equiv\; N\, \varepsilon^{abc} T_b (R_{cd} T^d) \,, \qquad B^a \;\equiv\; \varepsilon^{abc} T_b N_c \,, \tag{5.26}$$

where N is the normalisation factor. Our following analysis does not depend on the choice of the frame explicitly.

In this branch, the (TT)-part of the Killing equation is identically satisfied. To write out the remaining parts, we introduce the *curvilinear invariants* as

$$T^b\nabla_b\begin{pmatrix}T^a\\N^a\\B^a\end{pmatrix} = \begin{pmatrix}0&0&0\\0&0&\tau\\0&-\tau&0\end{pmatrix}\begin{pmatrix}T^a\\N^a\\B^a\end{pmatrix}, \tag{5.27}$$

$$N^b\nabla_b\begin{pmatrix}T^a\\N^a\\B^a\end{pmatrix} = \begin{pmatrix}0&-\kappa_g&\tau_r\\\kappa_g&0&\kappa_n\\-\tau_r&-\kappa_n&0\end{pmatrix}\begin{pmatrix}T^a\\N^a\\B^a\end{pmatrix}, \tag{5.28}$$

$$B^b\nabla_b\begin{pmatrix}T^a\\N^a\\B^a\end{pmatrix} = \begin{pmatrix}0&\tau_r&-\hat\kappa_g\\-\tau_r&0&-\hat\kappa_n\\\hat\kappa_g&\hat\kappa_n&0\end{pmatrix}\begin{pmatrix}T^a\\N^a\\B^a\end{pmatrix}, \tag{5.29}$$

where

$$\begin{aligned}\kappa_g &\equiv T^aN^b\nabla_bN_a\,, & \kappa_n &\equiv B^aN^b\nabla_bN_a\,, & \tau_r &\equiv B^aN^b\nabla_bT_a\,,\\ \hat\kappa_g &\equiv T^aB^b\nabla_bB_a\,, & \hat\kappa_n &\equiv N^aB^b\nabla_bB_a\,.\end{aligned} \tag{5.30}$$

$\kappa_g(\hat\kappa_g)$, $\kappa_n(\hat\kappa_n)$ and τ_r are respectively the geodesic curvature, normal curvature and relative torsion of an integral curve of $N^a(B^a)$. Note that since $T^b\nabla_bT^a=0$, the derivative of T^a is symmetric; the derivatives of the curvilinear invariants are not independent due to the analyticity. These relations are listed in Appendix 5.7.

Using the curvilinear invariants, the remaining parts of the Killing equation read

$$\mathscr{L}_T\,\omega_1 = -\kappa_g\omega_1+(\tau+\tau_r)\omega_2\,, \tag{5.31a}$$
$$\mathscr{L}_N\,\omega_1 = \kappa_n\omega_2\,, \tag{5.31b}$$
$$\mathscr{L}_B\,\omega_1 = -\kappa_n\omega_1-\hat\kappa_n\omega_2-\hat\omega\,, \tag{5.31c}$$
$$\mathscr{L}_T\,\omega_2 = -(\tau-\tau_r)\omega_1-\hat\kappa_g\omega_2\,, \tag{5.31d}$$
$$\mathscr{L}_N\,\omega_2 = \hat\omega\,, \tag{5.31e}$$
$$\mathscr{L}_B\,\omega_2 = \hat\kappa_g\omega_1\,, \tag{5.31f}$$

where eq. (5.31e) defines new variable $\hat\omega$. Clearly, the above equations are not closed. We thus need the derivatives of the Killing equation. It can be seen several parts of the equation $\nabla_{[a}\nabla_{b]}\omega_1=\nabla_{[a}\nabla_{b]}\omega_2=0$ tell us that

$$0 = -2\tau_r\hat\omega+(\mathscr{L}_N\kappa_g)\omega_1+(\mathscr{L}_B\kappa_g-2\tau_r\hat\kappa_n)\omega_2\,, \tag{5.32}$$
$$0 = 2\tau_r\hat\omega+(\mathscr{L}_N\hat\kappa_g)\omega_1+(\mathscr{L}_B\hat\kappa_g+2\tau_r\hat\kappa_n)\omega_2\,, \tag{5.33}$$
$$0 = (\kappa_g-\hat\kappa_g)\hat\omega+(\mathscr{L}_N\tau_r)\omega_1+(\mathscr{L}_B\tau_r+\hat\kappa_n(\kappa_g-\hat\kappa_g))\omega_2\,, \tag{5.34}$$

Adding eq. (5.32) and (5.33) gives

$$0 = [\mathscr{L}_N(\kappa_g+\hat\kappa_g)]\omega_1+[\mathscr{L}_B(\kappa_g+\hat\kappa_g)]\omega_2\,. \tag{5.35}$$

So $\mathscr{L}_N(\kappa_g+\hat\kappa_g)$ and $\mathscr{L}_B(\kappa_g+\hat\kappa_g)$ must be zero. Otherwise we can write KVs as $K^a=\omega_1\,(N^a-\mu B^a)$ or $K^a=\omega_2\,(B^a-\mu^{-1}N^a)$ with $\mu=\mathscr{L}_N(\kappa_g+\hat\kappa_g)/\mathscr{L}_B(\kappa_g+\hat\kappa_g)$. Thus, our algorithm reduces to the case 2 with the identification of

$$U^a \propto (N^a-\mu B^a)\,, \qquad\text{or}\qquad U^a \propto (B^a-\mu^{-1}N^a)\,, \tag{5.36}$$

where the proportionality factor is determined by the normalisation.

In the sub-branch where

$$\mathscr{L}_N(\kappa_g+\hat{\kappa}_g) \;=\; \mathscr{L}_B(\kappa_g+\hat{\kappa}_g) \;=\; \tau_r \;=\; \kappa_g-\hat{\kappa}_g \;=\; 0\,, \tag{5.37}$$

the set of functions $\{\omega_1,\omega_2,\hat{\omega}\}$ can possibly be independent. It also follows from eqs. (5.37) and (5.106) that $\mathscr{L}_N\kappa_g=\mathscr{L}_N\hat{\kappa}_g=\mathscr{L}_B\kappa_g=\mathscr{L}_B\hat{\kappa}_g=0$ and

$$0 \;=\; R_{ab}T^aN^b \;=\; R_{ab}T^aB^b\,, \qquad\qquad \lambda_N \;=\; \lambda_B \;\equiv\; \lambda\,. \tag{5.38}$$

The above equations implies that T^a has to be an eigenvector of $R^a{}_b$ whose Segre type is $\{21\}$. Now we look at the Killing equation

$$\nabla_a\boldsymbol{\omega} \;=\; \boldsymbol{\Omega}_a\boldsymbol{\omega}\,, \qquad\qquad \boldsymbol{\omega} \;\equiv\; \begin{pmatrix}\omega_1\\ \omega_2\\ \hat{\omega}\end{pmatrix}\,, \tag{5.39}$$

where

$$\begin{aligned}\boldsymbol{\Omega}_a \;\equiv\; & T_a\begin{pmatrix}-\kappa_g & \tau & 0\\ -\tau & -\kappa_g & 0\\ \tau\hat{\kappa}_n-\mathscr{L}_N\tau & -\tau\kappa_n & 0\end{pmatrix}+N_a\begin{pmatrix}0 & \kappa_n & 0\\ 0 & 0 & 1\\ -\mathscr{L}_N\kappa_n & \hat{\kappa}_n^2-\mathscr{L}_N\hat{\kappa}_n-\mathscr{L}_B\kappa_n & 0\end{pmatrix}\\ & +B_a\begin{pmatrix}-\kappa_n & -\hat{\kappa}_n & -1\\ \hat{\kappa}_n & 0 & 0\\ \mathscr{L}_N\hat{\kappa}_n-\hat{\kappa}_n^2 & \hat{\kappa}_n\kappa_n & \kappa_n\end{pmatrix}\,.\end{aligned} \tag{5.40}$$

Notice that the equation for $\hat{\omega}$ arises from the some parts of $\nabla_{[a}\nabla_{b]}\omega_1=\nabla_{[a}\nabla_{b]}\omega_2=0$. The integrability condition for eq. (5.39) reads

$$\big(\nabla_{[a}\boldsymbol{\Omega}_{b]}-\boldsymbol{\Omega}_{[a}\boldsymbol{\Omega}_{b]}\big)\boldsymbol{\omega} \;=\; 0\,. \tag{5.41}$$

or equivalently,

$$\kappa_g(\mathscr{L}_N\kappa_n)\omega_1 \;=\; 0\,, \qquad \kappa_g\hat{\kappa}_n\omega_1 \;=\; 0\,, \qquad \kappa_g\kappa_n\omega_2 \;=\; 0\,, \tag{5.42}$$

$$\kappa_g\hat{\kappa}_n\omega_2+\kappa_g\hat{\omega} \;=\; 0\,, \qquad (\mathscr{L}_N\lambda)\omega_1+(\mathscr{L}_B\lambda)\omega_2 \;=\; 0\,. \tag{5.43}$$

So if $\kappa_g=\mathscr{L}_N\lambda=\mathscr{L}_B\lambda=0$, then there are no extra conditions, thereby allowing us to conclude that three KVs exist. It should be noted that the third KV can be obtained by the Lie bracket of the two KVs,

$$\mathscr{L}_{\omega_1 N}(\omega_2B^a) \;=\; (\hat{\omega}+\hat{\kappa}_n\omega_2)(\omega_1\,B^a+\omega_2\,N^a)\,, \tag{5.44}$$

which satisfies the Killing equation (5.1); if $\kappa_g=0$ but $\mathscr{L}_N\lambda\neq 0$ or $\mathscr{L}_B\lambda\neq 0$, our algorithm again reduces to the case 2 with the identification of

$$U^a \;\propto\; (N^a-\nu B^a)\,, \qquad\qquad \text{or} \qquad\qquad U^a \;\propto\; (B^a-\nu^{-1}N^a)\,, \tag{5.45}$$

where $\nu=(\mathscr{L}_N\lambda)/(\mathscr{L}_B\lambda)$ and the proportionality factor takes care of the normalisation of U^a; if $\kappa_g\neq 0$ but $\kappa_n=0$, we can write $\hat{\omega}=-\hat{\kappa}_n\omega_2$ and the rank of the 1st obstruction matrix

$$\Xi^{(1)} \;\equiv\; \begin{pmatrix}\hat{\kappa}_n & 0\\ \mathscr{L}_N\lambda & \mathscr{L}_B\lambda\end{pmatrix}\,, \tag{5.46}$$

controls the number of KVs. If $\Xi^{(1)}$ is identically zero, two KVs exist. If $\det\Xi^{(1)} \neq 0$, no KV exists. If $\det\Xi^{(1)} = 0$ but $\Xi^{(1)} \neq 0$, then, once again, our algorithm reduces to the case 2 with an appropriate identification of U^a; we can see that, after re-prolongation of the Killing equation, the conditions $\kappa_g \neq 0$ and $\kappa_n \neq 0$ are in conflict with the integrability condition and thus there is no KV. We are at the end of this sub-branch.

In the sub-branch where

$$\mathscr{L}_N(\kappa_g + \hat{\kappa}_g) \;=\; \mathscr{L}_B(\kappa_g + \hat{\kappa}_g) \;=\; \tau_r \;=\; 0\,, \qquad \text{but} \qquad \kappa_g - \hat{\kappa}_g \neq 0\,, \tag{5.47}$$

the function $\hat{\omega}$ takes the form

$$\hat{\omega} = -\hat{\kappa}_n \omega_2\,. \tag{5.48}$$

Substituting this form into eqs. (5.31), we obtain

$$\nabla_a \hat{\boldsymbol{\omega}} \;=\; \hat{\boldsymbol{\Omega}}_a \hat{\boldsymbol{\omega}}\,, \qquad \hat{\boldsymbol{\omega}} \;\equiv\; \begin{pmatrix} \omega_1 \\ \omega_2 \end{pmatrix}\,, \tag{5.49}$$

where

$$\hat{\boldsymbol{\Omega}}_a \;\equiv\; T_a \begin{pmatrix} -\kappa_g & \tau \\ -\tau & -\hat{\kappa}_g \end{pmatrix} + N_a \begin{pmatrix} 0 & \kappa_n \\ 0 & -\hat{\kappa}_n \end{pmatrix} + B_a \begin{pmatrix} -\kappa_n & 0 \\ \hat{\kappa}_n & 0 \end{pmatrix}\,. \tag{5.50}$$

Its integrability condition

$$\left(\nabla_{[a}\hat{\boldsymbol{\Omega}}_{b]} - \hat{\boldsymbol{\Omega}}_{[a}\hat{\boldsymbol{\Omega}}_{b]}\right)\hat{\boldsymbol{\omega}} \;=\; 0\,. \tag{5.51}$$

leads to

$$0 \;=\; (\mathscr{L}_N\tau)\omega_1 + (\mathscr{L}_B\tau)\omega_2\,, \qquad 0 \;=\; (\mathscr{L}_N\kappa_g)\omega_1 + (\mathscr{L}_B\kappa_g)\omega_2\,, \tag{5.52a}$$

$$0 \;=\; (\mathscr{L}_N\kappa_n)\omega_1 + (\mathscr{L}_B\kappa_n)\omega_2\,, \qquad 0 \;=\; (\mathscr{L}_N\hat{\kappa}_n)\omega_1 + (\mathscr{L}_B\hat{\kappa})\omega_2\,. \tag{5.52b}$$

Therefore, the rank of the 2nd obstruction matrix

$$\Xi^{(2)} \;\equiv\; \begin{pmatrix} \mathscr{L}_N\tau & \mathscr{L}_B\tau \\ \mathscr{L}_N\kappa_g & \mathscr{L}_B\kappa_g \\ \mathscr{L}_N\kappa_n & \mathscr{L}_B\kappa_n \\ \mathscr{L}_N\hat{\kappa}_n & \mathscr{L}_B\hat{\kappa}_n \end{pmatrix}\,, \tag{5.53}$$

reveals the number of KVs. If $\mathrm{rank}\Xi^{(2)} = 0$, two KVs exist; If $\mathrm{rank}\Xi^{(2)} = 1$, our algorithm reduces to the case 2 with an appropriate identification of U^a; If $\mathrm{rank}\Xi^{(2)} = 2$, there is no KV.

Similarly, in the sub-branch where

$$\mathscr{L}_N(\kappa_g + \hat{\kappa}_g) \;=\; \mathscr{L}_B(\kappa_g + \hat{\kappa}_g) \;=\; 0\,, \qquad \text{but} \qquad \tau_r \neq 0\,, \tag{5.54}$$

eq. (5.32) tells us that

$$\hat{\omega} \;=\; \tfrac{1}{2\tau_r}(\mathscr{L}_N\kappa_g)\omega_1 + \left(\tfrac{1}{2\tau_r}\mathscr{L}_B\kappa_g - \hat{\kappa}_n\right)\omega_2\,. \tag{5.55}$$

Thus the Killing equation (5.31) can be rewritten by

$$\nabla_a \hat{\boldsymbol{\omega}} \;=\; \check{\boldsymbol{\Omega}}_a \hat{\boldsymbol{\omega}}\,, \tag{5.56}$$

where

$$\check{\mathbf{\Omega}}_a \equiv T_a \begin{pmatrix} -\kappa_g & \tau+\tau_r \\ -\tau+\tau_r & -\hat{\kappa}_g \end{pmatrix} + N_a \begin{pmatrix} 0 & \kappa_n \\ \frac{1}{2\tau_r}\mathscr{L}_N\kappa_g & \frac{1}{2\tau_r}\mathscr{L}_B\kappa_g - \hat{\kappa}_n \end{pmatrix}$$
$$+ B_a \begin{pmatrix} -\kappa_n - \frac{1}{2\tau_r}\mathscr{L}_N\kappa_g & -\frac{1}{2\tau_r}\mathscr{L}_B\kappa_g \\ \hat{\kappa}_n & 0 \end{pmatrix}. \tag{5.57}$$

Its integrability condition gives the relations between ω_1 and ω_2

$$0 = [\mathscr{L}_T\mathscr{L}_N\kappa_g + 2\tau_r\mathscr{L}_N(\tau-\tau_r) - (2\kappa_g + \mathscr{L}_T\ln\tau_r)\mathscr{L}_N\kappa_g + \hat{\kappa}_g\mathscr{L}_N\kappa_g - (\tau-\tau_r)\mathscr{L}_B\kappa_g]\,\omega_1$$
$$+ [\mathscr{L}_T\mathscr{L}_B\kappa_g + 2\tau_r\mathscr{L}_B(\tau-\tau_r) - (2\kappa_g + \mathscr{L}_T\ln\tau_r)\mathscr{L}_B\kappa_g - \kappa_g\mathscr{L}_B\hat{\kappa}_g + (\tau+\tau_r)\mathscr{L}_B\kappa_g]\,\omega_2\,, \tag{5.58}$$

$$0 = \left[\mathscr{L}_N\mathscr{L}_N\kappa_g + 2\tau_r\mathscr{L}_N\kappa_n - \left(\hat{\kappa}_n - \frac{\mathscr{L}_B\kappa_g}{2\tau_r}\right)\mathscr{L}_N\kappa_g - (\mathscr{L}_N\ln\tau_r)(\mathscr{L}_N\kappa_g)\right]\omega_1$$
$$+ \left[\mathscr{L}_N\mathscr{L}_B\kappa_g + 2\tau_r\mathscr{L}_B\kappa_n - \left(2\hat{\kappa}_n - \frac{\mathscr{L}_B\kappa_g}{2\tau_r}\right)\mathscr{L}_B\kappa_g - (\mathscr{L}_N\ln\tau_r)(\mathscr{L}_B\kappa_g) + \kappa_n\mathscr{L}_N\kappa_g\right]\omega_2\,, \tag{5.59}$$

$$0 = \left[-\mathscr{L}_B\mathscr{L}_N\kappa_g + 2\tau_r\mathscr{L}_N\hat{\kappa}_n + \left(2\kappa_n + \frac{\mathscr{L}_N\kappa_g}{2\tau_r}\right)\mathscr{L}_N\kappa_g + (\mathscr{L}_B\ln\tau_r)(\mathscr{L}_N\kappa_g) - \hat{\kappa}_n\mathscr{L}_B\kappa_g\right]\omega_1$$
$$+ \left[-\mathscr{L}_B\mathscr{L}_B\kappa_g + 2\tau_r\mathscr{L}_B\hat{\kappa}_n + \left(\kappa_n + \frac{\mathscr{L}_N\kappa_g}{2\tau_r}\right)\mathscr{L}_B\kappa_g + (\mathscr{L}_B\ln\tau_r)(\mathscr{L}_B\kappa_g)\right]\omega_2\,, \tag{5.60}$$

Writing eqs. (5.58)–(5.60) as

$$\Xi^{(3)}\hat{\boldsymbol{\omega}} = 0\,, \tag{5.61}$$

the rank of the 3rd obstruction matrix $\Xi^{(3)}$ governs the number of KVs in a way analogous to that of $\Xi^{(1)}$ and $\Xi^{(2)}$. It is remarkable that if $\kappa_g - \hat{\kappa}_g = 0$, then we obtain the relation

$$\mathscr{L}_N\kappa_g = \mathscr{L}_B\kappa_g = 0\,, \tag{5.62}$$

from eq. (5.54), which makes the 3rd obstruction matrix much simpler form

$$\Xi^{(3)} = 2\tau_r \begin{pmatrix} \mathscr{L}_N(\tau-\tau_r) & \mathscr{L}_B(\tau-\tau_r) \\ \mathscr{L}_N\kappa_n & \mathscr{L}_B\kappa_n \\ \mathscr{L}_N\hat{\kappa}_n & \mathscr{L}_B\hat{\kappa}_n \end{pmatrix}. \tag{5.63}$$

5.5 Details of case 0

In this case, any eigenvalue of the Ricci tensor is a constant. Therefore, our algorithm may depend on Segre types of the Ricci tensor. As the Segre type $\{3\}$ implies a manifold is maximally symmetric, there are possibly two Segre types $\{21\}$ and $\{111\}$. In both cases, any KV can be written as

$$K^a = \gamma_1 V_1^a + \gamma_2 V_2^a + \gamma_3 V_3^a\,, \tag{5.64}$$

where γ_1, γ_2 and γ_3 are unknown functions. Here the orthogonal frame $\{V_1^a, V_2^a, V_3^a\}$ is taken as the eigensystem of the Ricci tensor.

Introducing the curvilinear invariants

$$V_1^b\nabla_b\begin{pmatrix}V_1^a\\V_2^a\\V_3^a\end{pmatrix} = \begin{pmatrix}0 & \kappa_G & \kappa_N\\ -\kappa_G & 0 & \tau_R\\ -\kappa_N & -\tau_R & 0\end{pmatrix}\begin{pmatrix}V_1^a\\V_2^a\\V_3^a\end{pmatrix}, \tag{5.65}$$

$$V_2^b\nabla_b\begin{pmatrix}V_1^a\\V_2^a\\V_3^a\end{pmatrix} = \begin{pmatrix}0 & -\hat\kappa_G & \hat\tau_R\\ \hat\kappa_G & 0 & \hat\kappa_N\\ -\hat\tau_R & -\hat\kappa_N & 0\end{pmatrix}\begin{pmatrix}V_1^a\\V_2^a\\V_3^a\end{pmatrix}, \tag{5.66}$$

$$V_3^b\nabla_b\begin{pmatrix}V_1^a\\V_2^a\\V_3^a\end{pmatrix} = \begin{pmatrix}0 & -\check\tau_R & -\check\kappa_G\\ \check\tau_R & 0 & -\check\kappa_N\\ \check\kappa_G & \check\kappa_N & 0\end{pmatrix}\begin{pmatrix}V_1^a\\V_2^a\\V_3^a\end{pmatrix}, \tag{5.67}$$

we can write out the Killing equation as

$$\mathscr{L}_{V_1}\gamma_1 = \kappa_G\gamma_2+\kappa_N\gamma_3\,, \tag{5.68a}$$
$$\mathscr{L}_{V_2}\gamma_1 = \hat\gamma_1\,, \tag{5.68b}$$
$$\mathscr{L}_{V_3}\gamma_1 = -\kappa_N\gamma_1-(\tau_R+\check\tau_R)\gamma_2-\check\kappa_G\gamma_3-\hat\gamma_3\,, \tag{5.68c}$$
$$\mathscr{L}_{V_1}\gamma_2 = -\kappa_G\gamma_1-\hat\kappa_G\gamma_2+(\tau_R+\hat\tau_R)\gamma_3-\hat\gamma_1\,, \tag{5.68d}$$
$$\mathscr{L}_{V_2}\gamma_2 = \hat\kappa_G\gamma_1+\hat\kappa_N\gamma_3\,, \tag{5.68e}$$
$$\mathscr{L}_{V_3}\gamma_2 = \hat\gamma_2\,, \tag{5.68f}$$
$$\mathscr{L}_{V_1}\gamma_3 = \hat\gamma_3\,, \tag{5.68g}$$
$$\mathscr{L}_{V_2}\gamma_3 = -(\hat\tau_R-\check\tau_R)\gamma_1-\hat\kappa_N\gamma_2-\check\kappa_N\gamma_3-\hat\gamma_2\,, \tag{5.68h}$$
$$\mathscr{L}_{V_3}\gamma_3 = \check\kappa_G\gamma_1+\check\kappa_N\gamma_2\,, \tag{5.68i}$$

where eqs. (5.68b), (5.68f) and (5.68g) define new variables $\{\hat\gamma_1,\hat\gamma_2,\hat\gamma_3\}$. Moreover, the integrability conditions of eqs. (5.68) reveal that

$$0 = [(\lambda_1-\lambda_2)\hat\kappa_G+(\lambda_1-\lambda_3)\check\kappa_G]\,\gamma_1\,, \tag{5.69a}$$
$$0 = [(\lambda_1-\lambda_2)\kappa_G+(\lambda_3-\lambda_2)\check\kappa_N]\,\gamma_2\,, \tag{5.69b}$$
$$0 = [(\lambda_3-\lambda_2)\hat\kappa_N+(\lambda_3-\lambda_1)\kappa_N]\,\gamma_3\,, \tag{5.69c}$$
$$0 = (\lambda_1-\lambda_2)\,[\kappa_G\gamma_1-(\hat\tau_R+\check\tau_R)\gamma_3+\hat\gamma_1]\,, \tag{5.69d}$$
$$0 = (\lambda_1-\lambda_3)\,[(\tau_R-\hat\tau_R)\gamma_2+\check\kappa_G\gamma_3+\hat\gamma_3]\,, \tag{5.69e}$$
$$0 = (\lambda_3-\lambda_2)\,[(\tau_R-\check\tau_R)\gamma_1+\hat\gamma_2]+(\lambda_1-\lambda_3)\kappa_N\gamma_2\,, \tag{5.69f}$$
$$\begin{aligned}0 &= (\lambda_3-\lambda_2)\,[(\tau_R-\check\tau_R)\gamma_1+\hat\kappa_N\gamma_2+\check\kappa_N\gamma_3+\hat\gamma_2]\\ &\quad+(\lambda_1-\lambda_2)\kappa_G\gamma_3\,,\end{aligned} \tag{5.69g}$$
$$0 = [(\lambda_3-\lambda_2)\hat\kappa_N+(\lambda_3-\lambda_1)\kappa_N]\,\gamma_1\,, \tag{5.69h}$$

where λ_1,λ_2 and λ_3 are the eigenvalues of the Ricci tensor defined by

$$R^a{}_bV_1^b = \lambda_1V_1^a\,,\qquad R^a{}_bV_2^b = \lambda_1V_2^a\,,\qquad R^a{}_bV_3^b = \lambda_1V_3^a\,. \tag{5.70}$$

Notice that the above conditions are evidently satisfied if the Segre type is $\{3\}$, $\lambda_1=\lambda_2=\lambda_3$. In the remaining parts of this subsection, we discuss the Segre types $\{21\}$ and $\{111\}$ separately.

Branch where the Segre type is $\{21\}$

In this branch, we can assume $\lambda_2 = \lambda_3$ without loss of generality. The integrability conditions eqs. (5.69a)–(5.69c) are simplified to

$$0 = (\hat{\kappa}_G + \check{\kappa}_G)\gamma_1\,, \qquad 0 = \kappa_G\gamma_2\,, \qquad 0 = \kappa_N\gamma_3\,. \tag{5.71}$$

So if the rank of the 1st obstruction matrix

$$\Lambda^{(1)} \equiv \begin{pmatrix} \hat{\kappa}_G + \check{\kappa}_G & 0 & 0 \\ 0 & \kappa_G & 0 \\ 0 & 0 & \kappa_N \end{pmatrix} \tag{5.72}$$

is 1 or more, our algorithm reduces to the case 1 or 2 with the appropriate identifications: For instance, if $\hat{\kappa}_G + \check{\kappa}_G \neq 0$ then γ_1 must be zero and consequently any KV can be written as

$$K^a = \gamma_2 V_2^a + \gamma_3 V_3^a\,, \tag{5.73}$$

thereby allowing us to go back to the case 1.

In the following, we asssume rank$\Lambda^{(1)} = 0$. Under this assumption, the remaining parts of the integrabiliy conditions (5.69) lead to

$$\hat{\gamma}_1 = (\hat{\tau}_R + \check{\tau}_R)\gamma_3\,, \qquad \hat{\gamma}_3 = -(\tau_R - \hat{\tau}_R)\gamma_2 - \check{\kappa}_G\gamma_3\,. \tag{5.74}$$

Using this, the Killing equation (5.68) can be rewritten as

$$\nabla_a \boldsymbol{\gamma} = \boldsymbol{\Gamma}_a \boldsymbol{\gamma}, \qquad \boldsymbol{\gamma} \equiv \begin{pmatrix} \gamma_1 \\ \gamma_2 \\ \gamma_3 \\ \hat{\gamma}_2 \end{pmatrix}, \tag{5.75}$$

where

$$\begin{aligned}\boldsymbol{\Gamma}^a \equiv\; & V_1^a \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & -\hat{\kappa}_G & \tau_R - \check{\tau}_R & 0 \\ 0 & -(\tau_R - \hat{\tau}_R) & -\check{\kappa}_G & 0 \\ \mathscr{L}_{V_1}\check{\tau}_R & \check{\kappa}_N(\tau_R - \hat{\tau}_R) & \mathscr{L}_{V_3}\tau_R - \hat{\kappa}_N(\tau_R - \check{\tau}_R) & 0 \end{pmatrix} \\ & + V_2^a \begin{pmatrix} 0 & 0 & \hat{\tau}_R + \check{\tau}_R & 0 \\ \hat{\kappa}_G & 0 & \hat{\kappa}_N & 0 \\ -(\hat{\tau}_R - \check{\tau}_R) & -\hat{\kappa}_N & -\check{\kappa}_N & -1 \\ \mathscr{L}_{V_2}\check{\tau}_R + \check{\kappa}_N(\hat{\tau}_R - \check{\tau}_R) & \hat{\kappa}_N\check{\kappa}_N & \mathscr{L}_{V_3}\hat{\kappa}_N - \hat{\kappa}_N^2 - (\tau_R - \check{\tau}_R)(\hat{\tau}_R + \check{\tau}_R) & \check{\kappa}_N \end{pmatrix} \\ & + V_3^a \begin{pmatrix} 0 & -(\hat{\tau}_R + \check{\tau}_R) & 0 & 0 \\ 0 & 0 & 0 & 1 \\ \check{\kappa}_G & \check{\kappa}_N & 0 & 0 \\ \mathscr{L}_{V_3}\check{\tau}_R - \check{\kappa}_G\check{\kappa}_N & \hat{\kappa}_N^2 - (\mathscr{L}_{V_3}\hat{\kappa}_N + \mathscr{L}_{V_2}\check{\kappa}_N) + (\tau_R - \check{\tau}_R)(\hat{\tau}_R + \check{\tau}_R) & -\mathscr{L}_{V_3}\check{\kappa}_N & 0 \end{pmatrix}.\end{aligned} \tag{5.76}$$

After some gymnastics, we are led to the integrability conditions of (5.75)

$$0 = \hat{\kappa}_G\check{\tau}_R\gamma_1 + \tfrac{1}{2}[\check{\kappa}_N(\hat{\tau}_R - \check{\tau}_R) - \mathscr{L}_{V_3}\hat{\kappa}_G]\gamma_2 - \tfrac{1}{2}[\hat{\kappa}_N(\tau_R - \hat{\tau}_R) + \mathscr{L}_{V_3}\check{\tau}_R]\gamma_3 - \hat{\kappa}_G\hat{\gamma}_2\,, \tag{5.77a}$$

$$0 = -\check{\tau}_R(\hat{\tau}_R - \check{\tau}_R)\gamma_1 + (2\hat{\kappa}_G\check{\kappa}_N + \mathscr{L}_{V_3}\hat{\tau}_R)\gamma_2 + (\mathscr{L}_{V_3}\hat{\kappa}_G)\gamma_3 + (\hat{\tau}_R - \check{\tau}_R)\hat{\gamma}_2\,, \tag{5.77b}$$

$$0 = \hat{\kappa}_N(\tau_R - \hat{\tau}_R)\gamma_3\,. \tag{5.77c}$$

Thus if $\hat{\kappa}_N(\tau_R - \hat{\tau}_R) \neq 0$ then $\gamma_3 = 0$ and we go back to the case 1 with the identification

$$\omega_1 = \gamma_1, \qquad N^a = V_1^a, \qquad \omega_2 = \gamma_2, \qquad B^a = V_2^a. \tag{5.78}$$

In the sub-branch where

$$\hat{\kappa}_N(\tau_R - \hat{\tau}_R) = \hat{\kappa}_G = \hat{\tau}_R - \check{\tau}_R = 0, \tag{5.79}$$

the integrability conditions (5.77) are identically satisfied. It follows from eqs. (5.108) that $\mathscr{L}_{V_3}\hat{\tau}_R = 0$. As there are no extra conditions, four KVs exist. Notice that the fourth KV can be obtained by the Lie bracket of the two KVs,

$$\mathscr{L}_{\gamma_3 V_3}(\gamma_2 V_2^a) = 2\gamma_2\gamma_3\hat{\tau}_R V_1^a + (\hat{\gamma}_2 + \gamma_2\hat{\kappa}_N)(\gamma_3 V_2^a + \gamma_2 V_3^a). \tag{5.80}$$

In the sub-branch where

$$\hat{\kappa}_N(\tau_R - \hat{\tau}_R) = \hat{\kappa}_G = 0, \qquad \text{but} \qquad \hat{\tau}_R - \check{\tau}_R \neq 0, \tag{5.81}$$

the integrability condition (5.77a) gives us that

$$\hat{\gamma}_2 = \check{\tau}_R\gamma_1 - \hat{\kappa}_N\gamma_2. \tag{5.82}$$

So we can rewrite the Killing equation (5.75) as

$$\nabla_a\hat{\boldsymbol{\gamma}} = \hat{\boldsymbol{\Gamma}}_a\hat{\boldsymbol{\gamma}}, \qquad \hat{\boldsymbol{\gamma}} \equiv \begin{pmatrix}\gamma_1\\ \gamma_2\\ \gamma_3\end{pmatrix}, \tag{5.83}$$

where

$$\hat{\boldsymbol{\Gamma}}^a \equiv V_1^a\begin{pmatrix}0 & 0 & 0\\ 0 & 0 & \tau_R - \check{\tau}_R\\ 0 & -(\tau_R - \hat{\tau}_R) & 0\end{pmatrix} + V_2^a\begin{pmatrix}0 & 0 & \hat{\tau}_R + \check{\tau}_R\\ 0 & 0 & \hat{\kappa}_N\\ -\hat{\tau}_R & 0 & -\check{\kappa}_N\end{pmatrix} + V_3^a\begin{pmatrix}0 & -(\hat{\tau}_R + \check{\tau}_R) & 0\\ \check{\tau}_R & -\hat{\kappa}_N & 0\\ 0 & \check{\kappa}_N & 0\end{pmatrix}. \tag{5.84}$$

The integrability conditions of eq. (5.83) reads

$$0 = \tau_R\gamma_1, \tag{5.85a}$$
$$0 = (\mathscr{L}_{V_2}\hat{\tau}_R)\gamma_2 + (\mathscr{L}_{V3}\hat{\tau}_R)\gamma_3, \tag{5.85b}$$
$$0 = (\mathscr{L}_{V_2}\check{\tau}_R)\gamma_2 + (\mathscr{L}_{V3}\check{\tau}_R)\gamma_3, \tag{5.85c}$$
$$0 = (\mathscr{L}_{V_1}\hat{\kappa}_N)\gamma_1 + (\mathscr{L}_{V_2}\hat{\kappa}_N)\gamma_2 + (\mathscr{L}_{V3}\hat{\kappa}_N)\gamma_3, \tag{5.85d}$$
$$0 = (\mathscr{L}_{V_1}\check{\kappa}_N)\gamma_1 + (\mathscr{L}_{V_2}\check{\kappa}_N)\gamma_2 + (\mathscr{L}_{V3}\check{\kappa}_N)\gamma_3, \tag{5.85e}$$

If $\tau_R \neq 0$, then γ_1 must be zero and thus our algorithm goes back to the case 1. Otherwise $\tau_R = 0$ and the rank of the 2nd obstruction matrix

$$\Lambda^{(2)} \equiv \begin{pmatrix}0 & \mathscr{L}_{V_2}\hat{\tau}_R & \mathscr{L}_{V3}\hat{\tau}_R\\ 0 & \mathscr{L}_{V_2}\check{\tau}_R & \mathscr{L}_{V3}\check{\tau}_R\\ \mathscr{L}_{V_1}\hat{\kappa}_N & \mathscr{L}_{V_2}\hat{\kappa}_N & \mathscr{L}_{V3}\hat{\kappa}_N\\ \mathscr{L}_{V_1}\check{\kappa}_N & \mathscr{L}_{V_2}\check{\kappa}_N & \mathscr{L}_{V3}\check{\kappa}_N\end{pmatrix}, \tag{5.86}$$

controls the number of KVs. If $\Lambda^{(2)}$ is a zero matrix, three KVs exist. If $\mathrm{rank}\Lambda^{(2)} = 3$, no KV exists. If $\mathrm{rank}\Lambda^{(2)} = 1$ or 2, our algorithm reduces to the case 1 or case 2 with appropriate identifications.

In the sub-branch where

$$\hat{\kappa}_N(\tau_R - \hat{\tau}_R) \;=\; 0\,, \qquad\qquad \text{but} \qquad\qquad \hat{\kappa}_G \;\neq\; 0\,, \tag{5.87}$$

the integrability condition (5.77a) reads

$$\hat{\gamma}_2 \;=\; \check{\tau}_R\gamma_1 + \tfrac{1}{2\hat{\kappa}_G}\left[\check{\kappa}_N(\hat{\tau}_R - \check{\tau}_R) - \mathscr{L}_{V_3}\hat{\kappa}_G\right]\gamma_2 - \tfrac{1}{2\hat{\kappa}_G}(\mathscr{L}_{V_3}\check{\tau}_R)\gamma_3\,. \tag{5.88}$$

Using this, we can rewrite the Killing equation as

$$\nabla_a\hat{\boldsymbol{\gamma}} \;=\; \check{\boldsymbol{\Gamma}}_a\hat{\boldsymbol{\gamma}}\,, \tag{5.89}$$

where

$$\begin{aligned}
\check{\boldsymbol{\Gamma}}^a \;\equiv\; & V_1^a\begin{pmatrix} 0 & 0 & 0 \\ 0 & -\hat{\kappa}_G & \tau_R - \check{\tau}_R \\ 0 & -(\tau_R - \hat{\tau}_R) & \hat{\kappa}_G \end{pmatrix} \\
& + V_2^a\begin{pmatrix} 0 & 0 & \hat{\tau}_R + \check{\tau}_R \\ \hat{\kappa}_G & 0 & \hat{\kappa}_N \\ -\hat{\tau}_R & -\hat{\kappa}_N - \tfrac{1}{2\hat{\kappa}_G}\left[\check{\kappa}_N(\hat{\tau}_R - \check{\tau}_R) - \mathscr{L}_{V_3}\hat{\kappa}_G\right] & -\check{\kappa}_N + \tfrac{1}{2\hat{\kappa}_G}\mathscr{L}_{V_3}\check{\tau}_R \end{pmatrix} \\
& + V_3^a\begin{pmatrix} 0 & -(\hat{\tau}_R + \check{\tau}_R) & 0 \\ \check{\tau}_R & \tfrac{1}{2\hat{\kappa}_G}\left[\check{\kappa}_N(\hat{\tau}_R - \check{\tau}_R) - \mathscr{L}_{V_3}\hat{\kappa}_G\right] & -\tfrac{1}{2\hat{\kappa}_G}\mathscr{L}_{V_3}\check{\tau}_R \\ \check{\kappa}_G & \check{\kappa}_N & 0 \end{pmatrix}\,.
\end{aligned} \tag{5.90}$$

Its integrability condition leads to

$$0 \;=\; (\mathscr{L}_{V_2}\hat{\tau}_R + \mathscr{L}_{V_2}\check{\tau}_R)\gamma_2 + (\mathscr{L}_{V_3}\hat{\tau}_R + \mathscr{L}_{V_3}\check{\tau}_R)\gamma_3\,, \tag{5.91}$$

$$\begin{aligned}
0 \;=\; & \left[\mathscr{L}_{V_2}\hat{\kappa}_G^2 + (\hat{\tau}_R - \check{\tau}_R)\mathscr{L}_{V_2}\hat{\tau}_R\right]\gamma_2 \\
& + \left[(\hat{\tau}_R - \check{\tau}_R)\mathscr{L}_{V_2}\hat{\kappa}_G - 2\hat{\kappa}_G\mathscr{L}_{V_2}\hat{\tau}_R + \hat{\kappa}_N\left(4\hat{\kappa}_G^2 - (\hat{\tau}_R - \check{\tau}_R)(\tau_R - 2\hat{\tau}_R + \check{\tau}_R)\right)\right]\gamma_3\,.
\end{aligned} \tag{5.92}$$

Rewriting eqs. (5.91) and (5.92) as

$$\Lambda^{(3)}\hat{\boldsymbol{\gamma}} \;=\; 0\,, \tag{5.93}$$

the rank of the 3rd obstruction matrix $\Lambda^{(3)}$ controls the number of KVs in a way analogous to that of $\Lambda^{(1)}$ and $\Lambda^{(2)}$. If $\Lambda^{(3)}$ is a zero matrix, we come to grips with

$$\lambda_2\gamma_2 \;=\; 0\,. \tag{5.94}$$

Therefore, in this sub-branch there are 3 KVs if $\mathrm{rank}\Lambda^{(3)} = \lambda_2 = 0$.

Branch where the Segre type is $\{111\}$

In this branch, the eigenvalues of the Ricci tensor differ from each other. The integrability conditions eqs. (5.69a)–(5.69c) read

$$0 \;=\; \left[\alpha\hat{\kappa}_G + \check{\kappa}_G\right]\gamma_1\,, \qquad 0 \;=\; \left[\beta\kappa_G + \check{\kappa}_N\right]\gamma_2\,, \qquad 0 \;=\; \left[\alpha\beta^{-1}\hat{\kappa}_N - \kappa_N\right]\gamma_3\,, \tag{5.95}$$

where $\alpha = (\lambda_1 - \lambda_2)/(\lambda_1 - \lambda_3)$ and $\beta = (\lambda_1 - \lambda_2)/(\lambda_3 - \lambda_2)$. So if the rank of the 1st obstruction matrix

$$\Theta^{(1)} \equiv \begin{pmatrix} \alpha\hat{\kappa}_G + \check{\kappa}_G & 0 & 0 \\ 0 & \beta\kappa_G + \check{\kappa}_N & 0 \\ 0 & 0 & \alpha\beta^{-1}\hat{\kappa}_N + \kappa_N \end{pmatrix} \tag{5.96}$$

is 1 or more, we can go back to the case 1 or 2 with the appropriate identifications: For instance, if the conditions $\alpha\hat{\kappa}_G + \check{\kappa}_G \neq 0$ and $\beta\kappa_G + \check{\kappa}_N \neq 0$ lead us to the conclusion that $\gamma_1 = \gamma_2 = 0$ and thus any KV can be written as

$$K^a = \gamma_3 V_3^a . \tag{5.97}$$

We thus go back to the case 2 with the identification $U^a = V_3^a$.

In the following, we asssume rank$\Theta^{(1)} = 0$. Under this assumption, the remaining parts of the integrabiliy conditions (5.69) read

$$\begin{aligned} \hat{\gamma}_1 &= -\kappa_G\gamma_1 + (\hat{\tau}_R + \check{\tau}_R)\gamma_3 , \qquad & \hat{\gamma}_2 &= -(\tau_R - \check{\tau}_R)\gamma_1 - \hat{\kappa}_N\gamma_2 , \\ \hat{\gamma}_3 &= -(\tau_R - \hat{\tau}_R)\gamma_2 - \check{\kappa}_G\gamma_3 . \end{aligned} \tag{5.98}$$

Thus, in this branch, there are at most 3 KVs. Using eqs. (5.98), the Killing equation (5.68) can be rewritten by

$$\nabla_a \hat{\boldsymbol{\gamma}} = \bar{\boldsymbol{\Gamma}}_a \hat{\boldsymbol{\gamma}} , \tag{5.99}$$

where

$$\begin{aligned} \bar{\boldsymbol{\Gamma}}^a \equiv{}& V_1^a \begin{pmatrix} 0 & \kappa_G & \kappa_N \\ 0 & -\hat{\kappa}_G & (\tau_R - \check{\tau}_R) \\ 0 & -(\tau_R - \hat{\tau}_R) & \alpha\hat{\kappa}_G \end{pmatrix} + V_2^a \begin{pmatrix} -\kappa_G & 0 & \hat{\tau}_R + \check{\tau}_R \\ \hat{\kappa}_G & 0 & \alpha^{-1}\beta\kappa_N \\ \tau_R - \hat{\tau}_R & 0 & \beta\kappa_G \end{pmatrix} \\ &+ V_3^a \begin{pmatrix} -\kappa_N & -(\hat{\tau}_R + \check{\tau}_R) & 0 \\ -(\tau_R - \check{\tau}_R) & -\alpha^{-1}\beta\kappa_N & 0 \\ -\alpha\hat{\kappa}_G & -\beta\kappa_G & 0 \end{pmatrix} . \end{aligned} \tag{5.100}$$

After some algebra, we can see that the integrability conditions of eqs. (5.99) take the form

$$\Theta^{(2)}\hat{\boldsymbol{\gamma}} = 0 , \tag{5.101}$$

where

$$\Theta^{(2)} = \begin{pmatrix} \mathscr{L}_{V_1}\kappa_G & \mathscr{L}_{V_2}\kappa_G & \mathscr{L}_{V_3}\kappa_G \\ \mathscr{L}_{V_1}\hat{\kappa}_G & \mathscr{L}_{V_2}\hat{\kappa}_G & \mathscr{L}_{V_3}\hat{\kappa}_G \\ \mathscr{L}_{V_1}\kappa_N & \mathscr{L}_{V_2}\kappa_N & \mathscr{L}_{V_3}\kappa_N \\ \mathscr{L}_{V_1}(\tau_R - \hat{\tau}_R) & \mathscr{L}_{V_2}(\tau_R - \hat{\tau}_R) & \mathscr{L}_{V_3}(\tau_R - \hat{\tau}_R) \\ \mathscr{L}_{V_1}(\check{\tau}_R - \tau_R) & \mathscr{L}_{V_2}(\check{\tau}_R - \tau_R) & \mathscr{L}_{V_3}(\check{\tau}_R - \tau_R) \\ \mathscr{L}_{V_1}(\hat{\tau}_R + \check{\tau}_R) & \mathscr{L}_{V_2}(\hat{\tau}_R + \check{\tau}_R) & \mathscr{L}_{V_3}(\hat{\tau}_R + \check{\tau}_R) \end{pmatrix} . \tag{5.102}$$

We therefore are at the conclusion that the rank of the 2nd obstruction matrix $\Theta^{(2)}$ controls the number of KVs.

5.6 Example

This section is devoted to the application of our algorithm. As we have demonstrated in Chapter 1, a natural Hamiltonian with two degrees of freedom

$$\bar{H}(p,x) \;=\; \tfrac{1}{2}(p_1^2+p_2^2)+V(x_1,x_2)\,, \tag{5.103}$$

can be lifted to the geodesic Hamiltonian with three degrees of freedom

$$H(p,x) \;=\; \tfrac{1}{2}g^{ab}p_a p_b\,, \qquad \text{where} \qquad g^{ab} \;=\; \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2V \end{pmatrix}. \tag{5.104}$$

The metric g_{ab} is the Eisenhart metric. Using this metric, we can calculate the Ricci tensor and its derivatives. We examine the Eisenhart metric with

$$V(x_1,x_2) \;=\; \tfrac{1}{4}(x_1^4+x_2^4)+\tfrac{\varepsilon}{2}x_1^2x_2^2\,, \tag{5.105}$$

by our algorithm.

After simple algebra, we can see that the first two tests $R_{ab} \propto g_{ab}$ and $\nabla_a R_{bc}=0$ do not hold for any ε. However, the metric have no trouble passing the third test $dI^{(1)}\wedge dI^{(2)}\wedge dI^{(3)}=0$. Interestingly, the result of the fourth test $dI^{(a)}\wedge dI^{(b)}=0$ depends on the value of ε: If $\varepsilon=1$, the metric arrive at the case 1. Otherwise the metric drifts to the case 2 and then passes the last two test $\kappa_{ab}=\nabla_{[a}A_{b]}=0$. So there is only one KV, $(\partial_3)^a$.

In the case $\varepsilon=1$, $\nabla_a R$ is to be a geodesic as well as an eigenvector of $R^a{}_b$ with the eigenvalue $-6/(x_1^2+x_2^2)$. The metric passes the tests $\mathscr{L}_N(\kappa_g+\hat{\kappa}_g)=\mathscr{L}_B(\kappa_g+\hat{\kappa}_g)=0$ and $\tau_r=0$. However, it fails to be $\kappa_g=\hat{\kappa}_g$. As its 2nd obstruction matrix $\Xi^{(2)}$ has rank 2, we can conclude that 2 KVs exist.

5.7 Supplement: Relations between the curvilinear invariants

Due to their analyticity, the derivatives of the curvilinear invariants are not independent. In this section, we record the relations between the curvilinear invariants.

For case 1

When $\nabla_a R$ is a geodesic, the following relations hold true:

$$R_{ab}T^aT^b \;=\; -(\kappa_g^2+\hat{\kappa}_g^2+2\tau_r^2)+\mathscr{L}_T(\kappa_g+\hat{\kappa}_g) \;\equiv\; \lambda_T\,, \tag{5.106a}$$
$$R_{ab}N^aN^b \;=\; 2\tau\tau_r-\kappa_g^2-\kappa_n^2-\hat{\kappa}_n^2-\kappa_g\hat{\kappa}_g+\mathscr{L}_T\kappa_g+\mathscr{L}_B\kappa_n+\mathscr{L}_N\hat{\kappa}_n \;\equiv\; \lambda_N\,, \tag{5.106b}$$
$$R_{ab}B^aB^b \;=\; -2\tau\tau_r-\hat{\kappa}_g^2-\kappa_n^2-\hat{\kappa}_n^2-\kappa_g\hat{\kappa}_g+\mathscr{L}_T\hat{\kappa}_g+\mathscr{L}_B\kappa_n+\mathscr{L}_N\hat{\kappa}_n \;\equiv\; \lambda_B\,, \tag{5.106c}$$
$$R_{ab}T^aN^b \;=\; \hat{\kappa}_n(\kappa_g-\hat{\kappa}_g)-2\kappa_n\tau_r+\mathscr{L}_B\tau_r+\mathscr{L}_N\hat{\kappa}_g\,, \tag{5.106d}$$
$$\;=\; \mathscr{L}_B\tau-\kappa_n(\tau+\tau_r)-\hat{\kappa}_n\hat{\kappa}_g+\mathscr{L}_T\hat{\kappa}_n\,, \tag{5.106e}$$
$$R_{ab}T^aB^b \;=\; -2\tau_r\hat{\kappa}_n-\kappa_n(\kappa_g-\hat{\kappa}_g)+\mathscr{L}_N\tau_r+\mathscr{L}_B\kappa_g\,, \tag{5.106f}$$
$$\;=\; \hat{\kappa}_n(\tau-\tau_r)-\mathscr{L}_N\tau-\kappa_g\kappa_n+\mathscr{L}_T\kappa_n\,, \tag{5.106g}$$
$$R_{ab}N^aB^b \;=\; \tau(\kappa_g-\hat{\kappa}_g)+\tau_r(\kappa_g+\hat{\kappa}_g)-\mathscr{L}_T\tau_r\,. \tag{5.106h}$$

The commutators of the Lie derivatives can be written as follows.

$$
\begin{aligned}
\mathscr{L}_T\mathscr{L}_N f-\mathscr{L}_N\mathscr{L}_T f &= \tau\mathscr{L}_B f+\kappa_g\mathscr{L}_N f-\tau_r\mathscr{L}_B f\,, &\text{(5.107a)}\\
\mathscr{L}_B\mathscr{L}_T f-\mathscr{L}_T\mathscr{L}_B f &= \tau_r\mathscr{L}_N f-\hat{\kappa}_g\mathscr{L}_B f+\tau\mathscr{L}_N f\,, &\text{(5.107b)}\\
\mathscr{L}_N\mathscr{L}_B f-\mathscr{L}_B\mathscr{L}_N f &= \tau_r\mathscr{L}_T f+\hat{\kappa}_n\mathscr{L}_B f-\tau_r\mathscr{L}_T f-\kappa_n\mathscr{L}_N f\,, &\text{(5.107c)}
\end{aligned}
$$

where f is an arbitrary scalar function.

For case 0

In the case 0, the following relations hold true

$$
\begin{aligned}
R_{ab}V_1^aV_1^a &= -\kappa_G^2-\hat{\kappa}_G^2-\check{\kappa}_G^2-\kappa_G\check{\kappa}_N-\kappa_N^2-\hat{\kappa}_N\kappa_N &\text{(5.108a)}\\
&\quad +2\hat{\tau}_R\check{\tau}_R+\mathscr{L}_{V_1}\hat{\kappa}_G+\mathscr{L}_{V_1}\check{\kappa}_G+\mathscr{L}_{V_2}\kappa_G+\mathscr{L}_{V_3}\kappa_N\,,\\
R_{ab}V_2^aV_2^a &= -\kappa_G^2-\hat{\kappa}_G^2-\hat{\kappa}_G\check{\kappa}_G-\hat{\kappa}_N^2-\check{\kappa}_N^2-\hat{\kappa}_N\kappa_N\\
&\quad -2\tau_R\check{\tau}_R+\mathscr{L}_{V_1}\hat{\kappa}_G+\mathscr{L}_{V_2}\kappa_G+\mathscr{L}_{V_2}\check{\kappa}_N+\mathscr{L}_{V_3}\hat{\kappa}_N\,, &\text{(5.108b)}\\
R_{ab}V_3^aV_3^a &= -\check{\kappa}_G^2-\check{\kappa}_G\hat{\kappa}_G-\hat{\kappa}_N^2-\kappa_G\check{\kappa}_N-\check{\kappa}_N^2-\kappa_N^2 &\text{(5.108c)}\\
&\quad -2\tau_R\hat{\tau}_R+\mathscr{L}_{V_1}\check{\kappa}_G+\mathscr{L}_{V_2}\check{\kappa}_N+\mathscr{L}_{V_3}\kappa_N+\mathscr{L}_{V_3}\hat{\kappa}_N\,,\\
R_{ab}V_1^aV_2^a &= \check{\kappa}_N(\hat{\kappa}_G-\check{\kappa}_G)-\hat{\kappa}_N(\hat{\tau}_R-\check{\tau}_R)-\kappa_N(\hat{\tau}_R+\check{\tau}_R)+\mathscr{L}_{V_2}\check{\kappa}_G+\mathscr{L}_{V_3}\hat{\tau}_R\,, &\text{(5.108d)}\\
&= \check{\kappa}_G(\kappa_G-\check{\kappa}_N)-\hat{\kappa}_N(\tau_R-\check{\tau}_R)-\kappa_N(\tau_R+\check{\tau}_R)+\mathscr{L}_{V_1}\check{\kappa}_N+\mathscr{L}_{V_3}\tau_R\,, &\text{(5.108e)}\\
R_{ab}V_1^aV_3^a &= -\hat{\kappa}_N(\hat{\kappa}_G-\check{\kappa}_G)+\hat{\tau}_R(\kappa_G-\check{\kappa}_N)+\check{\tau}_R(\kappa_G+\check{\kappa}_N)-\mathscr{L}_{V_2}\check{\tau}_R+\mathscr{L}_{V_3}\hat{\kappa}_G\,, &\text{(5.108f)}\\
&= -\hat{\kappa}_G\hat{\kappa}_N+\tau_R(\kappa_G+\check{\kappa}_N)+\hat{\tau}_R(\kappa_G-\check{\kappa}_N)+\kappa_N\hat{\kappa}_G+\mathscr{L}_{V_1}\hat{\kappa}_N-\mathscr{L}_{V_2}\tau_R\,, &\text{(5.108g)}\\
R_{ab}V_2^aV_3^a &= \tau_R(\hat{\kappa}_G-\check{\kappa}_G)-\check{\tau}_R(\hat{\kappa}_G+\check{\kappa}_G)-\kappa_N(\kappa_G-\check{\kappa}_N)+\mathscr{L}_{V_1}\check{\tau}_R+\mathscr{L}_{V_3}\kappa_G\,, &\text{(5.108h)}\\
&= \kappa_G\hat{\kappa}_N-\kappa_G\kappa_N+\tau_R(\hat{\kappa}_G-\check{\kappa}_G)+\hat{\tau}_R(\hat{\kappa}_G+\check{\kappa}_G)+\mathscr{L}_{V_2}\kappa_N-\mathscr{L}_{V_1}\hat{\tau}_R\,. &\text{(5.108i)}
\end{aligned}
$$

The commutators of the Lie derivatives can be written as follows.

$$
\begin{aligned}
\mathscr{L}_{V_1}\mathscr{L}_{V_2}f-\mathscr{L}_{V_2}\mathscr{L}_{V_1}f &= -\kappa_G\mathscr{L}_{V_1}f+\hat{\kappa}_G\mathscr{L}_{V_2}f+(\tau_R-\hat{\tau}_R)\mathscr{L}_{V_3}f\,, &\text{(5.109a)}\\
\mathscr{L}_{V_1}\mathscr{L}_{V_3}f-\mathscr{L}_{V_3}\mathscr{L}_{V_1}f &= -\kappa_N\mathscr{L}_{V_1}f-(\tau_R-\check{\tau}_R)\mathscr{L}_{V_2}f+\check{\kappa}_G\mathscr{L}_{V_3}f\,, &\text{(5.109b)}\\
\mathscr{L}_{V_2}\mathscr{L}_{V_3}f-\mathscr{L}_{V_3}\mathscr{L}_{V_2}f &= -(\hat{\tau}_R+\check{\tau}_R)\mathscr{L}_{V_1}f-\hat{\kappa}_N\mathscr{L}_{V_2}f+\check{\kappa}_N\mathscr{L}_{V_3}f\,, &\text{(5.109c)}
\end{aligned}
$$

where f is an arbitrary scalar function.

Chapter 6

Rational first integrals and gauged Killing tensor fields

As we have seen in Chapter 1, Killing tensor fields (KTs) arise out of an assumption that first integrals of a geodesic flow are polynomial in momenta. It is then natural to relax this assumption and conceive of first integrals that are *meromorphic* in momenta. We call them the *rational first integrals*. As a consequence, we are naturally led to introduce *gauged Killing tensor fields* (GKTs).

The study of rational first integrals was already initiated by Darboux [47]. After then, several works have been concerned with concrete examples of models admitting rational first integrals of the geodesic equations. An example is the Collinson–O'Donnell solution [48] which is a solution to the vacuum Einstein equations. However, as we will see below that the Collinson-O'Donnell solution is a *trivial example* in the sense that a rational first integral results from a pair of polynomial first integrals. Therefore, another aim in this chapter is to obtain a nontrivial model admitting rational first integrals.

This chapter consists of two sections: In Section 6.1 we formulate rational first integrals. Consequently, GKTs are naturally introduced. After introducing the notion of *pure* GKTs, we provide a method for checking whether a GKT is pure. We also show that the defining equation of GKTs can be written in the same form as the ordinary Killing equation with replacing the Levi–Civita connection by a certain connection. Moreover, we provide the integrability condition for GKVs. In Section 6.2, we show the rational first integral of the Collinson-O'Donnell solution is trivial. We then construct several metrics admitting a nontrivial rational first integral in two and four dimensions.

6.1 Formulation

As it is for Chapter 1, we only deal with a geodesic Hamiltonian

$$H(q,p) = \frac{1}{2}g^{ab}(q)p_a p_b\,, \tag{6.1}$$

where (q,p) are canonical variables. A first integral F that is meromorphic in canonical momenta p_a takes the form

$$F(q,p) = \frac{U(q,p)}{V(q,p)}\,, \tag{6.2}$$

where U and V are nonzero polynomials of degree u and v in p_a. Without loss of generality, we can assume $u \geq v$ since if F is a first integral, the same is true of F^{-1}. We also assume that U and V have no common root. A rational first integral F is said to be *irreducible* if the degrees of U and V cannot be reduced by using other first integrals. For instance if p_y/p_x is a first integral, the linear combinations such as $(p_y + Hp_x)/p_x$ and $(p_x^2 + p_y^2)/p_x p_y$ are also first integrals. However, they are not irreducible.

It is obvious that if U and V are first integrals, then F is also a first integral. This suggests that a rational first integral is not so meaningful if it can be constructed out of a pair of polynomial first integrals. We must distinguish it from the others. Therefore, we introduce the notion of *inconstructible* rational first integrals: A rational first integral is said to be inconstructible if it does not result from a pair of polynomial first integrals. Otherwise it is *constructible*.

Requiring F to be a first integral, $\{H, Q\} = 0$, we obtain

$$\{U,H\}V - \{V,H\}U \;=\; 0\,, \tag{6.3}$$

where $\{\ ,\ \}$ denotes the Poisson bracket. Introducing an auxiliary function W, the above condition is equivalent to

$$\{U,H\} \;=\; WU\,, \qquad\qquad \{V,H\} \;=\; WV\,. \tag{6.4}$$

When U and V are homogeneous polynomials in momenta, we can write

$$U \;=\; K^{a_1\cdots a_u} p_{a_1}\cdots p_{a_u}\,, \qquad\qquad V \;=\; \bar{K}^{a_1\cdots a_v} p_{a_1}\cdots p_{a_v}\,, \tag{6.5}$$

where $K^{a_1\cdots a_u}$ and $\bar{K}^{a_1\cdots a_v}$ are totally symmetric tensor fields. Substituting eqs. (6.5) into eqs. (6.4) with the Hamiltonian (6.1), we find that W must be a linear function of p_a, $W = A^a p_a$. We are then able to rewrite eqs. (6.5) as

$$\nabla_{(a}K_{b_1\cdots b_u)} \;=\; A_{(a}K_{b_1\cdots b_u)}\,, \qquad\qquad \nabla_{(a}\bar{K}_{b_1\cdots b_v)} \;=\; A_{(a}\bar{K}_{b_1\cdots b_v)}\,, \tag{6.6}$$

where ∇ denotes the Levi–Civita connection and the round brackets $(\cdots)$ denote symmetrisation over the enclosed indices. As the Hamiltonian (6.1) is homogeneous with respect to the momenta. The above results are valid order by order in momenta, even supporsing that U and V are inhomogeneous polynomials. The equations (6.6) motivate us to introduce the following definition for *gauged Killing tensor fields*.

Definition 12. *A symmetric tensor $K_{a_1\cdots a_p}$ is called a gauged Killing tensor field (GKT) if there exists a 1-form A_a satisfying the differential equation*

$$\nabla_{(a}K_{b_1\cdots b_p)} \;=\; A_{(a}K_{b_1\cdots b_p)}\,, \tag{6.7}$$

where A_a is called the associated 1-form of $K_{a_1\cdots a_p}$. A GKT is equivalent to a KT if $A_a = 0$.

It should be noted that C. D. Collinson had already introduced gauged Killing vector fields [50]. To obtain a rational first integral of the geodesic equations, we need to find a pair of GKTs with a common associated 1-form A_a. This pair is referred to as a Killing pair in Refs [49, 48].

It is worth commenting here that the associated 1-form A_a can be determined uniquely: If there exist two different associated 1-forms $A_a^{(1)}$ and $A_a^{(2)}$, we deduce that

$$\delta A_{(a}K_{b_1\cdots b_p)} \;=\; 0\,, \tag{6.8}$$

where $\delta A_a \equiv A_a^{(1)} - A_a^{(2)}$. However, it is possible to show $\delta A_a = 0$ in the following way. Let δA_1 be a nonzero component of δA_a. Then it follows from eq. (6.8) that $\delta A_1 K_{11\cdots 1} = 0$, then $K_{11\cdots 1} = 0$. It also follows from eq. (6.8) that $p\,\delta A_1 K_{1\cdots 12} + \delta A_2 K_{1\cdots 1} = 0$, then $K_{1\cdots 12} = 0$. The same is true for $K_{1\cdots 13}, K_{1\cdots 14}, \cdots K_{1\cdots 1N}$. Moreover, $(p-1)\delta A_1 K_{1\cdots 123} + \delta A_3 K_{1\cdots 12} + \delta A_2 K_{1\cdots 13} = 0$, then $K_{1\cdots 123} = 0$. The same is true for $K_{1\cdots 124}, K_{1\cdots 125}, \cdots K_{1\cdots 12N}$. After a lot of repetition, it is concluded that all the components of $K_{a_1\cdots a_p}$ vanish. This completes the proof by contradiction.

We can confirm that the following properties hold true:

- Given two GKTs $K_{a_1\cdots a_p}, C_{a_1\cdots a_q}$, a symmetric tensor product $K_{(a_1\cdots a_p} C_{a_1\cdots a_q)}$ is also a GKT.
- Given two GKTs $K_{a_1\cdots a_p}, \bar{K}_{a_1\cdots a_q}$ which have a common associated 1-form, their linear combinations such as $K_{a_1\cdots a_p} - \bar{K}_{a_1\cdots a_q}$ and $K_{a_1\cdots a_p} + \bar{K}_{a_1\cdots a_q}$ are also GKTs.

We particularly find the following property.

Proposition 13. *Suppose $K_{a_1\cdots a_p}$ is a GKT. Then, $\bar{K}_{a_1\cdots a_p} \equiv \omega K_{a_1\cdots a_p}$ is also a GKT for an arbitrary function ω.*

Proof. Since $K_{a_1\cdots a_p}$ satisfies eq. (6.7), we have

$$\nabla_{(a}\bar{K}_{b_1\cdots b_p)} = K_{(b_1\cdots b_p}\nabla_{a)}\omega + \omega A_{(a}K_{b_1\cdots b_p)} = \bar{A}_{(a}\bar{K}_{b_1\cdots b_p)}\,.$$

where $\bar{A}_a = A_a + \nabla_a \ln\omega$. □

It follows from this proposition that $\bar{K}_{a_1\cdots a_p} = \omega K_{a_1\cdots a_p}$ is a GKT if $K_{a_1\cdots a_p}$ a KT. Obviously, not all GKTs take this form. In what follows, a GKT $K_{a_1\cdots a_p}$ is said to be *pure* if it there exists a function ω such that $\bar{K}_{a_1\cdots a_p} \equiv \omega K_{a_1\cdots a_p}$ is a KT. Otherwise, it is said to be *impure*. If we construct a rational first integral from two pure GKTs with a common associated 1-form, the resulting first integral becomes constructible.

We offer a criterion to decide whether a GKT is pure.

Proposition 14. *A GKT is pure if and only if the associated 1-form is closed.*

Proof. Let $K_{a_1\cdots a_p}$ be a GKT. ($\Leftarrow$) If the associated 1-form A_a is closed, $\nabla_{[a}A_{b]} = 0$, there exists a function ψ such that $A_a = \nabla_a \ln\psi$. Using this, we define $\bar{K}_{a_1\cdots a_p} \equiv \psi^{-1}K_{a_1\cdots a_p}$ and consequently find that $\bar{K}_{a_1\cdots a_p}$ is a KT. ($\Rightarrow$) As $K_{a_1\cdots a_p}$ is pure, there exists a function ψ such that $\bar{K}_{a_1\cdots a_p} \equiv \psi^{-1}K_{a_1\cdots a_p}$ is a KT. Using this, we obtain

$$\nabla_{(a}K_{b_1\cdots b_p)} = \nabla_{(a}\Big(\psi\bar{K}_{b_1\cdots b_p)}\Big) = \Big(\nabla_{(a}\ln\psi\Big)K_{b_1\cdots b_p)}\,, \tag{6.9}$$

so it follows from the uniqueness of the associated 1-form that $A_a = \nabla_a \ln\psi$. Thus if $K_{a_1\cdots a_p}$ is pure, then A_a is closed. □

Proposition 14 states that by investigating whether the associated 1-form is closed, we can check whether a rational frist integral is inconstructible. Using this fact, we investigate several concrete examples of rational first integrals in the next section.

Geometric interpretation

Let us introduce the connection $\hat{\nabla}_a$ on $\bigotimes^p T^*M$ which acts on a tensor $T_{a_1 \ldots a_p}$ as

$$\hat{\nabla}_a T_{b_1 \cdots b_p} \;=\; \nabla_a T_{b_1 \cdots b_p} - \sum_{i=1}^{n} B_{(a} T_{|b_1 \cdots b_{i-1}|b_i) b_{i+1} \cdots b_p} \,, \tag{6.10}$$

where B_a is a 1-form. This connection is torsion-free but not metric-compatible, $\hat{\nabla}_a g_{bc} \neq 0$. The curvature tensor of $\hat{\nabla}_a$ defined by $\hat{R}_{abc}{}^d V_d \equiv (\hat{\nabla}_a \hat{\nabla}_b - \hat{\nabla}_b \hat{\nabla}_a) V_c$ has antisymmetry with respect to the initial two indices, $\hat{R}_{abc}{}^d = -\hat{R}_{bac}{}^d$, and the Bianchi identities $\hat{R}_{[abc]}{}^d = 0$. Whilst antisymmetry of the latter two indices does not hold. Now the gauged Killing equation (6.7) is written as

$$\hat{\nabla}_{(a} K_{b_1 \cdots b_p)} \;=\; 0 \,, \tag{6.11}$$

where $B_a = pA_a$. This equation takes the same form as the ordinary Killing equation with replacing ∇ by $\hat{\nabla}$. It turns out by using the torsion-free connection (6.10) and Proposition 14 that a GKT is pure if the *gauge potential* B_a is locally pure gauge.

Integrability condition

By using the torsion-free connection (6.10), the integrability conditions for GKTs can be written in a simple form. For instance, let us consider GKVs obeying

$$\hat{\nabla}_{(a} K_{b)} \;=\; 0 \,. \tag{6.12}$$

The integrability condition for GKVs were already provided by C. D. Collinson [50]. However, the expression provided by him is rather complicated as he used the Riemann curvature tensor and the associated 1-form. On the other hand, it follows from eq. (6.12) that

$$\hat{\nabla}_b K_a = K^{(1)}_{ba} \,, \tag{6.13}$$

$$\hat{\nabla}_c K^{(1)}_{ba} = Y_{\substack{a\\b}} \, Y_{\substack{a\,c\\b}} \, \hat{R}_{cba}{}^d K_d \,, \tag{6.14}$$

where $K^{(1)}_{ba} \equiv Y_{\substack{a\\b}} \, \hat{\nabla}_b K_a$. Its integrability condition reads

$$0 \;=\; Y_{\substack{a\,c\\b\,d}} \left[(\hat{\nabla}_d \hat{R}_{cba}{}^m) K_m - 2 \hat{R}_{cba}{}^m K^{(1)}_{md} \right] . \tag{6.15}$$

Once again, we confirmed that this takes the same form as that of the ordinary Killing equation with replacing the Levi–Civita connection ∇ and the Riemann curvature tensor R_{abcd} by $\hat{\nabla}$ and $\hat{R}_{abcd}$, respectively. It should be noted that the procedure of prolongation developed in Chapter 3 can be applied for GKTs of a general order; however, the explicit forms of the prolonged system (3.20)–(3.26) and its integrability conditions for the second and third order (4.4)–(4.7) cannot available for the GKTs because we had implicitly used the Bianchi identity $R_{a[bcd]} = 0$ in the derivation

6.2 Examples

6.2.1 Collinson-O'Donnell solution

Vaz and Collinson [49] have found the canonical form of the 4-dimensional metrics admitting a pair of GKVs. They assumed that one of the GKVs is hypersurface orthogonal. Following this result, Collinson and O'Donnell [48] have obtained the solutions of the vacuum Einstein equations and have classified them into two cases. The solution of Case 2 was given in the form

$$ds^2 = -\frac{y}{x}dtdx + \frac{yt}{x^2}dx^2 + \frac{\alpha^2}{2\sqrt{y}}(dy^2+dz^2) - \frac{2\sqrt{y}\alpha^2}{x}dx\left(\frac{f}{\sqrt{y}}dy - g\sqrt{y}dz\right), \qquad (6.16)$$

where α is a constant, f and g are arbitrary functions of y and z obeying the Einstein equations

$$\partial_y f - y\partial_z g = -\frac{C^2}{y^2\sqrt{y}}, \qquad \partial_z f + y\partial_y g = \frac{C}{y\sqrt{y}}, \qquad (6.17)$$

where C is a constant. The geodesic equations in the metric admit an irreducible rational first integral

$$F = \frac{p_x}{p_t}. \qquad (6.18)$$

and a pair of GKVs $(\partial_t)^a$ and $(\partial_x)^a$ with the common associated 1-form $(2/x)dx_a$. We note that F is a rational first integral even if f and g do not satisfy eqs. (6.17). Since the associated 1-from is closed, we find from Proposition 14 that the rational first integral is constructible. Indeed, we find that $x(\partial_t)^a, x(\partial_x)^a$ are independent KVs, and the rational first integral is given by $F = Q_2/Q_1$ with two independent polynomial first integrals $Q_1 = xp_t$ and $Q_2 = xp_x$.

The solution of Case 1 is obtained as the limiting case: If we take $y \to 1+\varepsilon y$, $z \to \varepsilon z$ with $f \to \varepsilon f$, $g \to \varepsilon g$, $\alpha^2 \to \alpha^2/\varepsilon^2$. Subsequently taking $\varepsilon \to 0$ gives the metric

$$ds^2 = -\frac{1}{x}dtdx + \frac{t}{x^2}dx^2 + \frac{\alpha^2}{2}(dy^2+dz^2) - \frac{2\alpha^2}{x}dx(fdy - gdz), \qquad (6.19)$$

where f and g are functions of y and z obeying the Einstein equations

$$\partial_y f - \partial_z g = -C^2, \qquad \partial_z f + \partial_y g = C. \qquad (6.20)$$

As this metric still has two independent KVs $x(\partial_x)^a$, $x(\partial_t)^a$, two GKVs $(\partial_t)^a$, $(\partial_x)^a$ are constructible. Thus it follows that the rational first integral is constructible.

6.2.2 Metrics admitting an inconstructible rational first integral

Two dimensions

To construct metrics admitting an inconstructible rational first integral, we consider the Maciejewski-Przybylska system [51]. The Hamiltonian is given by

$$H = \frac{1}{2}(p_x^2+p_y^2) + f(p_x,p_y)(xp_x - \alpha yp_y), \qquad (6.21)$$

where α is a constant and $f(p_x, p_y)$ is an arbitrary function. This Hamiltonian admits a first integral of the form

$$F = p_x^\alpha p_y \,, \tag{6.22}$$

for arbitrary α and f. To obtain a rational first integral, we assume that α is a negative rational number. Then by setting $\alpha = -s/r$ with positive integers r and s, we have a rational first integral $F^r = p_y^r/p_x^s$. Moreover, we take $f = p_x + p_y$ to make the Hamiltonian quadratic. As a result, the Hamiltonian describes geodesic flows on a 2-dimensional surface with the metric

$$ds^2 = \frac{(1-2\alpha y)dx^2 - 2(x-\alpha y)dxdy + (1+2x)dy^2}{Q(x,y)} \,, \tag{6.23}$$

where α is a constant and

$$Q(x,y) = 1 + 2x - 2\alpha y - (x+\alpha y)^2 \,. \tag{6.24}$$

We firstly focus on the case $\alpha = -1$. In this case, the metric (6.23) is flat. As the first integral is given by $F = p_y/p_x$, $(\partial_x)^a$ and $(\partial_y)^a$ are GKVs. The common associated 1-form is $-g_{ab}((\partial_x)^b + (\partial_y)^b)$. We can confirm that the the associated 1-form is closed. Thus it turns out from Proposition 14 that $(\partial_x)^a$ and $(\partial_y)^a$ are constructible. More explicitly, we perform the coordinate transformation

$$x = u + v + \frac{1}{2}(u^2+v^2) \,, \qquad y = u - v + \frac{1}{2}(u^2+v^2) \,. \tag{6.25}$$

In the (u,v) coordinates, the metric is given by $ds^2 = du^2 + dv^2$ and the GKVs are given by

$$(\partial_x)^a = \frac{1-v}{1+u}(\partial_u)^a + (\partial_v)^a = \frac{1}{1+u}\Big[(\partial_u)^a + (\partial_v)^a - v(\partial_u)^a + u(\partial_v)^a\Big] \,, \tag{6.26}$$

$$(\partial_y)^a = \frac{1+v}{1+u}(\partial_u)^a - (\partial_v)^a = \frac{1}{1+u}\Big[(\partial_u)^a - (\partial_v)^a + v(\partial_u)^a - u(\partial_v)^a\Big] \,, \tag{6.27}$$

confirming they are constructible.

For general $\alpha = -s/r$, since $F^r = p_y^r/p_x^s$ is a rational first integral, $(\partial_x)^{a_1}\cdots(\partial_x)^{a_s}$ and $(\partial_y)^{a_1}\cdots(\partial_y)^{a_r}$ are respectively GKTs with the common associated 1-form $-sg_{ab}((\partial_x)^b + (\partial_y)^b)$. After some algebra, it follows that the associated 1-form is not closed except for $\alpha = -1$. Thus the rational first integral is inconstructible for $\alpha \neq -1$. This implies that we have constructed the metric (6.23) admitting an inconstructible rational first integral of the geodesic equations in two dimensions.

Four dimensions

We are able to generalise the Maciejewski-Przybylska system (6.21) to the N-dimensional system. The Hamiltonian is given by

$$H = \frac{1}{2}\sum_{i=1}^N p_i^2 + f(p_1, \ldots, p_N)\sum_{i=1}^N \alpha_i x^i p_i \,, \tag{6.28}$$

where $\alpha_1, \ldots, \alpha_N$ are constants and $f(p_1, \ldots, p_N)$ is a function. This Hamiltonian admits a first integral

$$F = p_1^{\beta_1} p_2^{\beta_2} \cdots p_N^{\beta_N} \,, \tag{6.29}$$

where $\beta_1,\ldots,\beta_N$ are constants satisfying the condition

$$\alpha_1\beta_1+\alpha_2\beta_2+\cdots+\alpha_N\beta_N \;=\; 0\,. \tag{6.30}$$

It should be noted that this system is completely integrable because first integral (6.29) indicates there are $(N-1)$ constants of motion.

Taking f as $f=a_1p_1+a_2p_2+\cdots+a_Np_N$ with constants $(a_1,\ldots,a_N)$, the Hamiltonian (6.28) describes geodesic flows on the N-dimensional curved space with the inverse metric

$$g^{ii} \;=\; 1+2a_i\alpha_i x^i\,, \qquad\qquad g^{ij} \;=\; a_j\alpha_i x^i+a_i\alpha_j x^j\,. \tag{6.31}$$

In what follows, we consider the Maciejewski-Przybylska system in four dimensions. For brevity, we adopt the following setup: $a_1=a_2=a_3=1$, $a_4=-\sqrt{-1}$, $\alpha_1=1$, $\alpha_2=-\alpha$ and $\alpha_3=\alpha_4=0$. Under this setup, the Hamiltonian is independent of the coordinates x^3 and x^4, so that p_3 and p_4 are first integrals. As another first integral is given by $F=p_1^{\beta_1}p_2^{\beta_2}$ with $\beta_1-\beta_2\alpha=0$, we normalise β_2 as $\beta_2=1$ and then obtain $\beta_1=\alpha$. Moreover, identifying the coordinates x^1,x^2,x^3 as x,y,z and x^4 as $\sqrt{-1}\,w$, we obtain the Hamiltonian

$$H \;=\; \frac{1}{2}(p_x^2+p_y^2+p_z^2-p_w^2)+(p_x+p_y+p_z+p_w)(xp_x-\alpha yp_y)\,, \tag{6.32}$$

with the first integrals p_z, p_w and $F=p_x/p_y^{\alpha}$. As a result, we obtain one-parameter family of 4-dimensional metrics admitting integrable geodesic flows. In particular, when we take $\alpha=-1$, the metric becomes scalar-flat, i.e. $R=g^{ab}R_{ab}=0$ whilst $R_{ab}\neq 0$. The components of the scalar-flat metric are given by

$$\begin{aligned}
g_{xx} &= \frac{1+2y}{K(x,y)}\,, & g_{yy} &= \frac{1+2x}{K(x,y)}\,, & g_{xy} &= \frac{-x-y}{K(x,y)}\,,\\
g_{zz} &= \frac{1+2x+2y+2xy}{K(x,y)}\,, & g_{zw} &= \frac{-x^2-y^2}{K(x,y)}\,, & g_{xz} &= \frac{y^2-xy-x}{K(x,y)}\,,\\
g_{yz} &= \frac{x^2-xy-y}{K(x,y)}\,, & g_{ww} &= \frac{2(x-y)^2-1-2x-2y+2xy}{K(x,y)}\,, & g_{xw} &= \frac{-y^2+xy+x}{K(x,y)}\,,\\
g_{yw} &= \frac{-x^2+xy+y}{K(x,y)}\,,
\end{aligned} \tag{6.33}$$

where $K(x,y)=1+2x+2y+2xy-x^2-y^2$. This metric admits a rational first integral $F=p_y/p_x$ which is inconstructible. Thus, we have constructed the scalar-flat metric (6.33) admitting an inconstructible rational first integral of the geodesic equations in four dimensions.

Chapter 7

Summary and outlook

In this thesis, we have explored the classical problem of determining whether a Hamiltonian system is completely integrable. More precisely, as is explained in Chapter 1, we assumed that: The Hamiltonian under consideration is given by a natural Hamiltonian

$$\bar{H} \;=\; \frac{1}{2}\bar{g}^{AB}(q)p_A p_B + V(q)\,, \qquad\qquad (A,B \;=\; 1,\ldots,N-1)$$

so as to reduce a geodesic Hamiltonian

$$H \;=\; \frac{1}{2}g^{ab}(q)p_a p_b\,; \qquad\qquad (a,b \;=\; 1,\ldots,N)$$

First integrals are polynomial in momenta. For a geodesic Hamiltonian, such integrals can be written as

$$Q(q,p) \;=\; K^{a_1\cdots a_p}(q)p_{a_1}\cdots p_{a_p}\,,$$

where $K^{a_1\cdots a_p} = K^{(a_1\cdots a_p)}$ is a Killing tensor field obeying the Killing equation

$$\nabla_{(b}K_{a_1\cdots a_p)} \;=\; 0\,. \tag{7.1}$$

Under the assumptions, we studied the Killing equation (7.1) and undertook the following questions:

- Are there any solutions of the Killing equation (7.1) for given metrics?
- If the answer is yes, then how many solutions are there?
- What quantities are sufficient to determine the number of solutions?

In this thesis, we study the above issues and give partial answers. In particular, we introduce a systematic method to analyse the Killing equation and to study its properties. A key ingredient here is projection operators called Young symmetrisers. Main results are as follows:

(i) We have constructed an effective way to analyse the Killing equation and to study its properties based on Young symmetrisers Whilst several studies [52] had focused on the role of Young symmetry in the tensor calculus, Young symmetrisers have not generally been recognised so far. We have established a prolongation procedure which transforms the Killing equation of a specified order into a closed system called the prolonged system by introducing new variables. Then the explicit form of the prolonged system was written out up to the third order.

(ii) We gave a formula for the integrability conditions of the prolonged system that put tough restrictions on the Riemann curvature tensor and its derivatives. We also derived the concrete form of the integrability conditions up to the third order. Moreover, we made a conjecture on the Young symmetries of the integrability conditions of a general order and provided a method for computing the dimension of the solution space of the Killing equation with a specific example.

(iii) We characterised metrics which admit Killing vector fields by local curvature obstructions. The obstructions had been obtained by analysing the integrability condition and the original Killing equation. In particular, we provided the algorithm that tells us exactly how many Killing vector fields exist for a given metric. This results would be extended to 4 or more higher-dimensional metrics.

(iv) Killing tensor fields arise out of an assumption that first integrals of a geodesic flow are polynomial in momenta. We relaxed this assumption and conceive of first integrals that are meromorphic in momenta. We then defined gauged Killing tensor fields in order to describe rational first integrals. We also studied their properties in detail and constructed several metrics admitting a nontrivial rational first integral.

We close our study with some questions and outlook.

For the integration of the geodesic equations with a constraint $H = 0$, it is sufficient to find a first integral in the zero energy level set. Such integrals F must satisfy the condition $\{H, F\} = LH$ for a certain function L. If F are polynomial in momenta, this condition leads to the conformal Killing equation

$$\nabla_{(a}C_{b_1\cdots b_p)} \;=\; g_{(ab_1}\phi_{b_2\cdots b_p)}\,, \tag{7.2}$$

where $\phi_{a_1\cdots a_{p-1}}$ is a symmetric tensor. As shown in Section 4.4, our analysis based on Young symmetrisers has effective applications to other types of overdetermined PDE systems. So a natural question to ask is whether our analysis is valid for the conformal Killing equation. The immediate answer to this question is *No* since in order to analyse the conformal Killing equation, we need not irreducible representations of $\mathrm{GL}(N)$ but those of $\mathrm{SO}(N)$. So it will be necessary to incorporate the trace operation into our analysis. Such modifications have not been pursued in this thesis but will be considered in the future.

Despite the fact that integrability conditions are simply a consequence of the requirement that mixed partial derivatives must commute, the explicit forms of them have brought us essential insights into physics and mathematics. Classic examples are the Gauss-Codazzi equations in the Hamiltonian formulation of general relativity and the Raychaudhuri equations in the derivation of the singularity theorems. Similarly, the integrability conditions of the Killing equation for $p = 1$ lead to an immediate corollary (see, e.g. [53]): After some algebra, we can show that

$$\nabla_{a_1\cdots a_r} I^{(1,1)}_{bcde}\Big|_{D\boldsymbol{K}=0} \;=\; 0 \qquad\Leftrightarrow\qquad \mathscr{L}_K \nabla_{a_1\cdots a_r} R_{bcde}\big|_{D\boldsymbol{K}=0} \;=\; 0\,. \tag{7.3}$$

where $\mathscr{L}_K$ is the Lie derivative along a KV K^a. This implies that if Q is the scalar constructed out of the Riemann curvature tensor and its derivatives, then $\mathscr{L}_K Q$ must be zero. So if the set of the 1-form $\{dQ^{(1)}, \ldots, dQ^{(n)}\}$ are linearly independent, the N-form

$$dQ^{(1)} \wedge \cdots \wedge dQ^{(N)}\,, \tag{7.4}$$

must also be zero for N-dimensional metrics. If any of the possible obstructions is not vanishing, such a metric admits no KV. We explicitly used this fact in Chapter 5. We therefore expect that further analysis of the explicit forms of the integrability condition for $p > 1$ will lead to similar corollaries for KTs.

If we establish analogous algorithms for KVs in 4 or more higher dimensions, it is not an overstatement that we do not need to solve the Killing equation of the first order anymore. However, it would need an intricate and elaborate calculation. On the other hand, it is not clear how the algorithm extends to the second or more higher order KTs. Perhaps this results from not understanding of the integrability condition completely. We expect that the conjecture we made in Chapter 4 provides a clue to establish the algorithms for KTs.

Another question to ask is whether we can formulate the existence condition or hopefully the value of r_0 in eq. (4.13). Answering this question may be linked to the conjecture we made in Chapter 4. In fact, if the conjecture holds true, we obtain a criteria

$$C = \frac{\operatorname{rank} I^{(p,p)}}{\operatorname{rank} E^{(p)}} = \frac{N(N-1)}{(p+2)(p+1)}, \tag{7.5}$$

where N and p are the dimension of space(-time) M and the order of KTs, respectively. Here, rank $E^{(p)}$ agrees with the upper limit of the BDTT formula (3.36); rank $I^{(p,p)}$ denotes the number of linearly independent components of the integrability condition (4.8). If $C \leq 1$, we definitely need the derivatives of $I^{(p,p)}$ to determine the dimension of the space of KTs and thus $r_0 > 0$. The equality is attained when $N = p+2$. Namely, our conjecture serves to formulate the lower bound on the value of r_0.

It would be worthwhile to comment the significance of our results in application to computer program. In recent years various softwares implemented with a computer algebraic system, such as *Mathematica* and *Maple*, have been developed. Each software prepares many packages available for solving individual problems in mathematics and physics, and we then find packages for solving the Killing equation for Killing vector fields as well as Killing and Killing–Yano tensor fields. However most of these packages do not solve the Killing equations efficiently, as they merely use a built-in PDE solver without the integrability conditions. For fairness it should be mentioned that we found one package (e.g. *KillingVectors* in *Maple*) which does use the integrability conditions, albeit only for Killing vector fields and not for Killing and Killing-Yano tensor fields. Hence, in order to make such packages more efficient especially for Killing tensor fields, our results in Chapter 4 are significant to provide the formulas of the integrability conditions to be implemented.

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