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博　士　論　文

Essays on Unconventional Monetary Policy in Japan

(日本の非伝統的金融政策に関する研究)

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Rui WANG
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Abstract

Japan economy has experienced the low interest rate environment since 1999. As the authority of monetary policy in Japan, Bank of Japan (BoJ) has implemented several unconventional monetary policy programs to stimulate the real economy since 1999 when the zero lower bound (ZLB) of policy rate became binding and the policy rate lost its function as a policy instrument. Compared with the central banks in other developed economies such as Fed Reserve Board (FRB) or Bank of England (BoE) that started unconventional monetary policy due to the Great Recession from 2007 to 2013 incurred by the 2007-2008 global financial crisis, Bank of Japan has more than 15 years' experience on the implementation of unconventional monetary policy. As the global economy is still in slow recovery and the low interest rate environment still prevails in major developed economies, in the foreseeable future for some time, the unconventional monetary policy will be the main policy regime of the central banks, also be the major concern of macroeconomics, it is necessary to give some temporary summary of the unconventional monetary policy conducted by Bank of Japan. In this doctoral thesis, unconventional monetary policy in Japan is my main research topic and this topic will be discussed from both empirical and theoretical views. This research will provide some tentative conclusions for the questions such as how we evaluate the policy stance of Bank of Japan, or what the macroeconomic performance would have be if the unconventional monetary policy hadn't been implemented. The main methodology of the research is to estimate the shadow rate from a shadow rate term structure model, and then use the shadow rate as a proxy variable of unconventional monetary policy in some non-structural or structural econometric procedures, including Vector Autoregression (VAR) or Dynamic Stochastic General Equilibrium (DSGE) models, to check the policy effect empirically. Also, we model the main transmission mechanism of unconventional monetary policy, portfolio rebalance mechanism, in a theoretical-consistent way to provide some theory-based insights.

This research will not be the end of the exploration of the unconventional monetary policy, but a comprehensive summary as a reference for the future study. The brief summary of each chapter is given as follows.

Chapter 1: This chapter is the introduction of this doctoral thesis. Firstly, I give a brief introduction of unconventional monetary policy including the

theoretical background and empirical evidence. Our understanding of the unconventional monetary policy is still very limited, but there exists some theoretical consensus about why and how liquidity trap happens, what central banks can do to stimulate the economy facing the zero lower bound constraint. The theoretical background of unconventional monetary policy is explained in a unified fashion, a standard small-scale New Keynesian DSGE model with some numerical examples. Then we do some literature review about the empirical evidence about the unconventional monetary policy. While we introduce the theatrical and empirical issues about the unconventional monetary policy, we also review the history of the policy programs of Bank of Japan, so we can understand what Bank of Japan has done since 1999 in a theoretical and empirical context.

Chapter 2: In this chapter, we use a shadow rate term structure model and yield curve data of Japan government bond to estimate the shadow short interest rate, or more simply, shadow rate. Shadow rate is same as the short-term policy rate in the normal non-ZLB environment but can take negative value in the ZLB environment. The estimated shadow rate is explained in detail by an event study fashion in the time line context of the policy programs conducted by Bank of Japan since 1999. By check the policy decisions officially declared by the Bank of Japan, we can find the shadow rate is a good approximation to describe the policy stance of monetary authority. The main conclusion is that by monitoring the estimated shadow rate, we can obtain a consistent and comparable insight of the different policy programs. This chapter provides the foundation of the whole research that the shadow rate is a reasonable and consistent measure of monetary policy in both non-ZLB and ZLB environment and we can use the shadow rate for the analysis of unconventional monetary policy.

Chapter 3: We use the estimated shadow rate in Chapter 2 to conduct some econometric exercises. We run two kinds of time series econometric procedures, Time Varying Parameter-Stochastic Volatility Vector Autoregression (TVP-SV VAR) model and standard New Keynesian DSGE model to check the empirical relationship of shadow rate and macroeconomic variables. The empirical results show that the shadow rate can be used in both structural model and non-structural model. The estimation results of structural parameters and impulse response are also very robust for the monetary policy analysis.

Chapter 4: Following the analysis in Chapter 3, we build a medium-scale DSGE model and estimate it with the shadow rate for Japan economy. For a long time, how to deal with the ZLB in the estimation of DSGE models has been a major difficulty in macroeconomic research. Standard procedures for DSGE modeling such as local linear approximation and Kalman filter can't handle the nonlinearity incurred by the ZLB and the nonlinear methods such as global approximation and particle filter are highly complicated and computation-demanding. In this chapter, we show that with the shadow rate, we can still get reasonable estimation results and policy implications implied by the model dynamics. The merit that we use a DSGE model is that we can conduct a counterfactual simulation to find what the macroeconomic performance would have be if the unconventional monetary policy hadn't been implemented. The simulation shows that without the unconventional monetary policy since 1999, Japan economy would have had bad performance compared with its actual realization.

Chapter 5: This chapter is relatively independent from other chapters. In other chapters, we use the shadow rate as a comprehensive measure of the unconventional monetary policy without considering the specific transmission mechanism. In this chapter, we map the portfolio rebalance mechanism of quantitative easing (QE) in a medium-scale DSGE model. The model is calibrated to fit the balance sheet of Bank of Japan and the structure of Japan government bond. Through some policy experiments, we find the different policy effects caused by different scenarios of quantitative easing. These policy experiments provide some insights about the future policy direction of quantitative easing.

Chapter 6: The review of the whole research, about what we have known through this doctoral thesis and what we should do in the next step, will be summarized in Chapter 6. The main originality in this doctor thesis is to use the shadow rate as a clue to review the full history of the unconventional monetary policy programs conducted by Bank of Japan, which makes the whole research consistent and tractable.

Appendix: Model derivation, mathematical proof, some figures and tables and other related technical details are provided in Appendix for reference. All results in this doctoral thesis can be replicated. Data and program code are available upon request.

Chapter 1

Introduction of Unconventional Monetary Policy: Theoretical Background and Empirical Evidence

1.1 Introduction of Unconventional Monetary Policy

In December 2008, the FRB cut policy rate target range to 0-0.25%, and this range remained there until December 2015. Between this period, the FRB employed balance sheet policy to simulate economic activity. This policy is referred to as Large-Scale Asset Purchases (LASP) programs, or more simply as Quantitative Easing (QE). During the same period, monetary authorities in other major developed countries also adopted the similar actions to conduct the monetary policy, facing the lower bound of nominal short-term policy rate. Monetary policy that is implemented by a monetary authority in low interest rate environment is generally called as unconventional monetary policy. In contrast, we commonly call the monetary policy which is conducted in the way of adjustment of short-term policy rate by the open market operation as the conventional monetary policy. In such sense, monetary policy conducted by adjusting the balance sheet of central bank, is unconventional. Among all central banks of developed countries, as the first central bank that begun to implement unconventional monetary policy, since 1999, Bank of Japan (BoJ) has more than 15 years' experience of the practice of unconventional monetary policy. Table 2.3 summarizes the major regimes of monetary policy adopted by BoJ since 1999. In this doctoral thesis, the main research objective is the unconventional monetary policy conducted by BoJ from 1999/Feb/2 to 2016/Jan/29. Although we have already known that the zero is not the actual lower bound of nominal interest rate given the fact that several central banks have already allowed negative interest rates, it is better to refer the minimum possible level of the nominal interest rate as the effective lower bound (ELB) than the more common ZLB, but in this doctoral thesis, the negative interest rate policy and the yield curve control policy are out of our research scope. We still use the ZLB as the only terminology of the lower bound on nominal interest rate through whole doctoral thesis consistently.

Note that the adjustment of central bank's balance sheet is not the only way to implement unconventional monetary policy. Claudio and Anna (2016) provides a good summary of unconventional monetary policy from both academic view and practical view. Table 1.1 gives a brief summary of all options of unconventional monetary policy and the central banks' actions. In this doctoral, we mainly focus on the balance sheet policy conducted by BoJ. The data used in all empirical works in this doctoral thesis are before the start of the negative interest rate policy of BoJ. Let us emphasize once more in advance that the negative interest rate policy and yield curve control policy are out of our research scope.

Unconventional Monetary Policy			Central banks			
			FRB	BoE	ECB	BoJ
Balance sheet policy	Credit policy		yes	yes	yes	yes
	Quasi-debt management policy		yes	yes	yes	yes
	Bank reserve policy		no	no	no	yes
Forward guidance on interest rates	Calender-based	Qualitative	yes	no	no	no
		Quantitative	yes	no	yes	no
	State-contingent	Qualitative	yes	yes	no	yes
		Quantitative	no	yes	no	yes
Negative interest rates			no	no	yes	yes
Yield curve control			no	no	no	yes

Table 1.1: Policy Options of Unconventional Monetary Policy

The central bank in ZLB environment can't lower the policy rate, but can make announcements about the forward guidance of future path of the policy rate, expanding its balance sheet, and change its balance sheet's portfolio, it may be difficult to develop a summary measure of monetary policy. Our approach in this doctoral thesis follows Wu and Xia (2016) and Krippner (2015) that use the estimated shadow rate as a comprehensive and consistent measure of monetary policy stance. When the actual short-term rate is positive, the estimated shadow rate corresponds to the actual short-term rate. When the actual short-term rate is fixed at its lower bound, given the condition that unconventional monetary policy can be effective at reducing long-term interest rates, even though the actual policy rate has not changed, the shadow rate will be below the policy rate and its level can proxy for the impact of the unconventional monetary policy. Than we can use the estimated shadow rate in econometric procedures to evaluate the effects of unconventional monetary policy. This is the basic logic of this doctoral thesis.

1.2 Theoretical Background of Unconventional Monetary Policy

We give some explanations about the theoretical background of unconventional monetary policy in this section.

1.2.1 Balance Sheet Policy

We survey two recently developed DSGE models that are designed to assess the impact of balance sheet policy. In Harrison (2011)'s model, a portfolio rebalance mechanism is used to capture the effect of QE. It implies that even the policy rate is constrained by the ZLB, the monetary authority has some ability to reduce the premium by purchasing the long-term government bonds. In the Chapter 5 of this doctoral thesis, we also use a similar calibrated DSGE model with portfolio rebalance mechanism to quantify the different scenarios of QE policy conducted by BoJ. The simulation shows that a long-lasting QE policy has stronger initial impact on real variables and the decreasing of long-term interest rate is also long-lasting. More details are given in Chapter 5. Since 1999, even the name of unconventional monetary policy conducted by BoJ has changed many times, "Quantitative Easing", "Comprehensive Monetary Easing", "Quantitative and Qualitative Easing", but the nature of these programs are almost same, purchase of government bond and enlargement of the balance sheet.

The Gertler and Karadi (2011) model is a model of unconventional monetary policy with capital and financial intermediation. The unconventional monetary policy in this model is interpreted as expanding central bank credit intermediation to offset a disruption of private financial intermediation, in another word, so-called "credit easing" , "lending faculty" or "capital injection". Within their framework, the central bank is less efficient than private intermediaries at making loans but it has the advantage of being able to elastically obtain funds by issuing risk-less government debt. Unlike private financial intermediaries, the central bank is not balance sheet constrained. During a crisis, the balance sheet constraints on private intermediaries tighten, raising the net benefits from central bank intermediation. These benefits may be substantial even if the ZLB constraint on the nominal interest rate is not binding. But in the case that the ZLB constraint is binding, these net benefits may be significantly enhanced. The reason that why we don't adopt the framework of Gertler and Karadi (2011) is that the case for BoJ is little different from the FRB or BoE. Financial intermediaries in Japan don't suffer much from the influence of financial crisis, the reason that why BoJ implements unconventional monetary policy is to stimulate the real economy and push-up the low price level that lasts for many years. But for FRB and BoE, the situation is different.

1.2.2 Forward Guidance Policy

As the standard methodology of macroeconomics, New Keynesian Dynamic Stochastic General Equilibrium (NK-DSGE[1]) model provides some theoretical insights about how to conduct monetary policy in the ZLB environment. We present a numerical example here to demonstrate the "optimal discretion versus optima commitment" in the situation that the nominal policy rate is constrained by the ZLB.

$$x_t = \mathbb{E}_t x_{t+1} - \frac{1}{\sigma}\left(i_t - \mathbb{E}_t \pi_{t+1} - r_t^N\right)$$
$$\pi_t = \beta \mathbb{E}_t \pi_{t+1} + \kappa x_t$$
$$i_t \geq 0$$

x_t represents the output gap, which is defined as the difference between actual output level with price rigidity and natural output level under flexible price. π_t is inflation rate whose dynamics follow New Keynesian Phillips Curve (NKPC) derived under the price rigidity mechanism of Calvo (1983). i_t is nominal interest rate, which is generally considered as a controllable policy instrument of central bank. But under the ZLB environment, the central bank can't lower nominal interest rate i_t further, which can be considered as a constraint condition $i_t \geq 0$. r_t^N is natural rate which is decided by exogenous technology progress. $i = r^N = \beta^{-1} - 1$ holds at the steady state with zero inflation and closed output gap. At the initial period, the economy is at its steady state with $x_t = \pi_t = 0$. Let us consider a situation that economic recession caused by the unexpected slowdown of exogenous technology progress. Such unexpected slowdown of exogenous technology progress leads to the decreasing of r_t^N to a negative value $-\varepsilon$. r_t^N remains at that negative value from period 1 to period t_T. From period $t_T + 1$, r_t^N takes again its steady state value. Under the perfect foresight equilibrium, what should the central bank do to conduct is monetary policy in response to such shock given the non-negativity constraint on i_t?

Given $i_t \geq 0$, $i_t = \mathbb{E}_t \pi_{t+1} + r_t^N - \sigma\left(x_t - \mathbb{E}_t x_{t+1}\right) \geq 0$. The non-negativity constraint on i_t can be written as $x_t \leq \frac{1}{\sigma}\left(\mathbb{E}_t \pi_{t+1} + r_t^N + \sigma \mathbb{E}_t x_{t+1}\right)$. Given $\sum_{t=0}^{\infty} \beta^t \mathbb{E}_t\left(\pi_t^2 + \lambda x_t^2\right)$ as the objective loss function of central bank[2], solving this problem under optimal discretionary policy and optimal commitment policy leads to the equilibrium paths of model variables showed in following figures[3].

[1]In Appendix 1, we derive a standard NK-DSGE model. Here we use i_t to represent the nominal interest rate in replace of r_t^B used in Appendix 1.

[2]For the micro-foundation of central bank's quadratic loss function, please refer to Rotemberg and Woodford (1999), Woodford (2003), Galí (2015), Miao (2014) and Walsh (2017).

[3]For the algorithm of solving rational expectation model under perfect foresight, please refer to Chapter 5 of Galí (2015). The numerical example demonstrated here also uses the same method and calibration of Galí (2015).

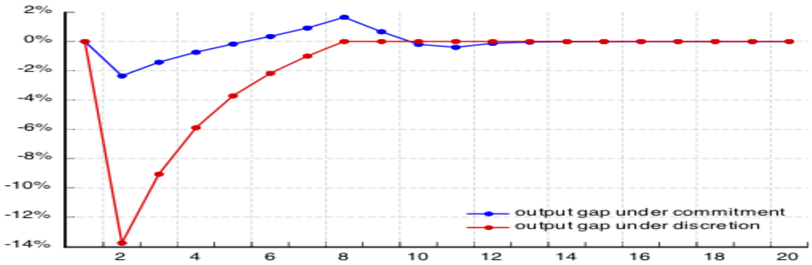


Figure 1.1: Equilibrium Path of Inflation Rate

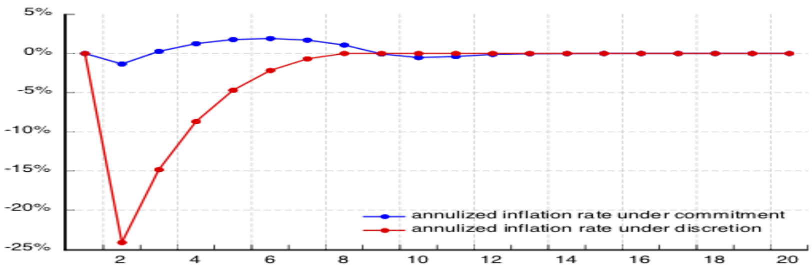


Figure 1.2: Equilibrium Path of Output Gap

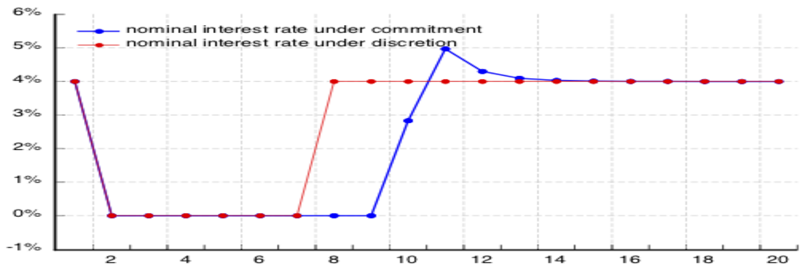


Figure 1.3: Equilibrium Path of Nominal Interest Rate

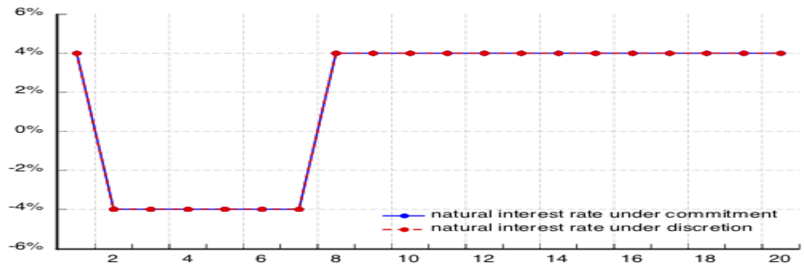


Figure 1.4: Equilibrium Path of Natural Interest Rate

From these figures, we can find the equilibrium paths of inflation, output gap, nominal interest rate given the same path of natural interest rate. These figures show that the optimal commitment policy has better performance than the optimal discretion policy. An unexpected drop in natural interest rate leads to the decreasing of nominal interest rate to its lower bound 0%. Commitment on the future path of nominal interest rate, a credible promise made by the central bank to keep nominal interest rate low for an extended long period even beyond the time when the adverse demand shock[4] is gone and inflation and output gap would start recovering, reduces the initial impact of such adverse shock, which means the less loss of welfare. This finding provides the theoretical background of the so-called "forward guidance" adopted by the FRB, ECB, BoE, BoJ and other central banks during the Great Recession from 2007 to 2013 when policy rates were constrained by the lower bound. There are many times that BoJ made a statement that says "keeping interest rate low for an extended period even though the economy starts recovery". This is called as "Jikanjiku" policy in Japanese. The economic foundation can be found in the previous numerical example.

1.3 Empirical Evidence of Unconventional Monetary Policy

Most of empirical works focus on the effects of QE policy on the bond yields and asset prices in financial market and on the real variables of macroeconomy. The effects on the financial market are investigated by the non-structural econometric procedures. Most of these works confirm the effect of QE that QE can lower the term spread of long-term bond yield and risk premium of risk asset, also push up the stock price and exchange rate (depreciation). Claudio and Anna (2016) summaries the related empirical

[4] The decreasing of natural interest rate caused by the slowdown of exogenous technology process can be considered as an adverse demand shock here, because r_t^N appears in New Keynesian IS equation.

results. About the effects of QE policy on the macroeconomy, several authors have applied DSGE models to simulate the effects of QE policy. Chen, Curdía and Ferrero (2012) simulated the effects of a QE program in an estimated DSGE model with segmented financial markets and a transaction cost that limits arbitrage. The mechanism in their model is similar to the model of Chapter 5 that the arbitrage can't hold when the management cost of portfolio exists. Del Negro et al. (2016) developed a DSGE model to assess the FRB's policy. They found that the liquidity injection provided by the FRB during the global financial crisis did help avert another Great Depression. Baumeister and Benati (2012) estimated a time-varying VAR model with stochastic volatility to evaluate the impact of a decreasing in the long-term interest rate on the inflation and output. Wu and Xia (2016) estimated the shadow rate and used the estimated shadow rate in a FAVAR model with many real variables. They found that the impact of the estimated shadow rate on macroeconomic variables was similar to the estimated impact of the funds rate target in the prior zero interest rate period. Ugai (2007) provides a comprehensive survey about the empirical works of 2001-2006 QE policy of BoJ. Most of these empirical works use VAR models as the methodology. Given different specifications of VAR models, although there exists differences about the effects of QE in the scale, the effects of QE have been confirmed.

1.4 Summary

Since the financial crisis, there have been many literatures that have studied unconventional monetary policy from either empirical perspective or theoretical perspective. We can't survey all of these studies in this introductory chapter. These exists a consensus that the unconventional monetary policy does have effect on real economy, but the exact channels through which the unconventional monetary policy affects the activity of macroeconomy are not very clear.

Balance sheet policy is more convenient to be specified in a structural model, but the forward guidance is very difficult to model in an economic fashion because it affects the public's belief and expectation about the future path of monetary policy. But how the expectation affects the current economic decisions is hard to describe in a structural economic model. Even the central bank can announce the future path of monetary policy, there exists uncertainty. So in this doctoral thesis, we use the estimated shadow rate as a measure of monetary policy without distinguishing the specified mechanism of all these unconventional monetary policies.

Chapter 2

Estimating the Monetary Policy Measures of Japan in Shadow/ZLB Term Structure Model

2.1 Introduction

Before the 2007-2008 global financial crisis, nominal short interest rates generally evolved above the zero lower bound (ZLB) and the central banks in major developed countries generally chose short interest rate, which is also known as policy rate such as Federal Funds Rate (FFR) in US, as the direct operating target of the monetary policy. But after the global financial crisis, central banks in major countries, expect China, have to face the untraditional challenges from the ZLB constraint. The conventional monetary policy, for example, the most typical one is the adjustment of short policy rate, is not practical and operative anymore in the ZLB environment and the unconventional monetary policies, such as large scale of asset purchases of government bonds and forward guidance on public expectation formation, have been adopted to provide further stimulus to economy, stopping recession and stabilizing financial system.

In the non-ZLB environment assumed by the orthodox macroeconomic theory, short interest rate is a useful measure for monitoring the stance of monetary policy[1]. The practice of monetary policy in the central banks of major developed countries also have proved this point. But facing the ZLB constraint, short interest rate evolves near the neighborhood of ZLB and it can't provide useful information about the stance of monetary policy.

Gaussian Affine Term Structure Model (GATSM) which is a widely used non-structural econometric methodology to fit the term structure of interest rates can provide good description of the dynamics of the yield curve in many macro-financial applications, but when short interest rates are near the ZLB, the performance of GATSM is deficient and unable to provide satisfied fitness of data, just like DSGE modeling, another standard methodology in macroeconomics. Also, in the ZLB environment, the short interest

[1]This is true for the most of central banks in developed countries that choose the short interest rate as the operating target of monetary policy.

rates are ”sticky” which means that they tend to keep approximately static around the ZLB with lower volatilities for an extended long period of time. This phenomenon can be visually confirmed in Figure 2.1 and Figure 2.2 in Section 2.3.1, which show that the short interest rates in Japan have experienced a long period of the ZLB since 1999. Ordinary GATSMs are unable to accommodate this kind of stickiness. In the ordinary GATSM framework, interest rates can evolve below the ZLB to take negative values implicitly, but it is inconsistent with the actual observed data and has less economic implications for practical purpose.

In the ZLB environment, the economic agents generally choose to hold currency and receive zero return actually rather than accept a negative interest rate. This provides the intuition of how to solve the issue of negative interest rates in GATSM framework. Shadow/ZLB-GATSM proposed by Krippner (2012) use a GATSM to represent the shadow term structure which is the term structure that would exist if the currencies were not available. If short interest rates are not restricted by the ZLB and evolve above the ZLB, shadow/ZLB-GATSM provides same results as general GATSMs do. But shadow/ZLB-GATSM can provide an mathematical-consistent adjustment to ensure that interest rates are constrained by the ZLB explicitly. According to Krippner (2015c), this adjustment is effectively the optionality form the availability of physical currency as an alternative to the negative interest rates below the ZLB[2]. Another important reason is that the shadow/ZLB-GATSM can provide three useful indicators, Shadow Short Interest Rate (SSR), Expected Time to Zero (ETZ) and Effective Monetary Stimulus (EMS) which can show the stance of monetary policy in the ZLB environment. These indicators can be used as the measures of monetary policy instead of short policy rate because the short policy rate has already lost its effectiveness in the ZLB environment.

Given this background, we estimated a shadow/ZLB-GATSM for Japan and used the estimation results to evaluate the stance of monetary policy in Japan. The remaining of this chapter is organized as follows. We review the general GATSM and introduce its extension in the ZLB environment in Section 2.2.1. Then we specify a two-factor shadow/ZLB GATSM in Section 2.2.2. In Section 2.2.3, we talk about the data and estimation and then give the estimation results. In Section 2.4, we use the main results, three measures of monetary policy to evaluate the stance of monetary policy in Japan. We also check the relationships between these measures and other macroeconomic variables. The Section 2.5 concludes the chapter. Some technical details such as mathematical derivations will be provided in Appendix for reference.

[2]For detailed discussions, please refer to Krippner (2015c), p44.

2.2 GATSM and its Extension in the ZLB Environment

In this section, we review the derivation of GATSM and the related calculations in a general continuous time specification[3]. Then we turn to illustrate its extension in the ZLB environment, the shadow/ZLB-GATSM framework which can represent the shadow term structure in the ZLB environment.

2.2.1 Generic Specification of GATSM

The short interest rate r_t at time t can be represented as a linear function of the vector of state variables x_t,

$$r_t = a_0 + b_0^\top x_t$$

where r_t is the shortest maturity[4] interest rate of the yield curve as a scalar and a_0 is a constant scalar. x_t is a $N \times 1$ vector containing N state variables $[x_{1,t}, x_{2,t}, \cdots, x_{N,t}]^\top$. b_0 is a $N \times 1$ vector containing the weight[5] for each state variable[6]. Under the objective $\mathbb{P}$ measure[7], x_t follows a correlated vector Ornstein-Uhlenbeck process[8]:

$$dx_t = -\kappa^{\mathbb{P}} \left(x_t - \theta^{\mathbb{P}} \right) dt + \sigma dW_t^{\mathbb{P}}$$

where $\theta^{\mathbb{P}}$ is a $N \times 1$ constant vector which represents the mean level of x_t in long-run. $\kappa^{\mathbb{P}}$ is a $N \times N$ constant parameter matrix that controls the deterministic mean reversion of x_t to its long-run mean level $\theta^{\mathbb{P}}$. σ is a $N \times N$ constant variance-covariance matrix of innovations to x_t. $dW_t^{\mathbb{P}}$ is a $N \times 1$ vector containing independent Wiener process $[dW_{1,t}, \cdots, dW_{N,t}]^\top$ with each component $dW_{n,t}^{\mathbb{P}} \sim \mathcal{N}(0,1)\sqrt{dt}$ where $\mathcal{N}(0,1)$ is the standard normal distribution. This stochastic process can be solved by using method of variation of constants[9]. Define a function $f(x_t, t) = e^{\kappa^{\mathbb{P}} t} x_t$ and differentiate two sides with respect to t, then apply Ito's lemma.

$$\frac{df(x_t,t)}{dt} = \kappa^{\mathbb{P}} e^{\kappa^{\mathbb{P}} t} x_t dt + e^{\kappa t} dx_t = \kappa^{\mathbb{P}} e^{\kappa^{\mathbb{P}} t} \theta^{\mathbb{P}} dt + e^{\kappa^{\mathbb{P}} t} \sigma dW_t^{\mathbb{P}}$$

[3]Continuous time specification can give a closed-form solution with higher traceability.

[4]Generally, we use 3-months bond interest rate and it is essentially equal to the short policy interest rate.

[5]In GATSM, each state variable can be recognized as a factor which can capture the dynamics of short interest rate. Each element in b_0 is also known as a factor loading.

[6]We use superscript $^\top$ to represent the transposition of a matrix or vector.

[7]The parameters and variables with $^{\mathbb{P}}$ represent the objective $\mathbb{P}$ measure.

[8]Ornstein-Uhlenbeck process can be considered as a first-order autoregressive stochastic process in continuous time.

[9]To simplify the notation, we illustrate the solution by assuming x_t has one factor.

Integrating two sides of this equation[10] from t to $t+\tau$ leads to

$$x_{t+\tau} = \theta^{\mathbb{P}} + e^{-\kappa^{\mathbb{P}}\tau}(x_t - \theta^{\mathbb{P}}) + \int_t^{t+\tau} e^{-\kappa^{\mathbb{P}}(\tau-u)} \sigma dW_u^{\mathbb{P}}$$

where we use u as a dummy variable to evaluate the integral over time[11]. Under the risk-adjusted $\mathbb{Q}$ measure, the market prices of risk Π_t can be represented as a linear function of x_t as

$$\Pi_t = \sigma^{-1}\left[\gamma + \Gamma x_t\right]$$

where γ is the constant component of the market prices of risks and Γ determines the variations in market prices of risks with respect to the state variables x_t. Since the bonds or securities in financial markets are priced under the risk-adjusted $\mathbb{Q}$ measure, the process of state variables x_t must be adjusted to represent the observed term structure of interest rates by modification of parameters[12] as $\kappa^{\mathbb{Q}} = \kappa^{\mathbb{P}} + \Gamma$, $\theta^{\mathbb{Q}} = \left(\kappa^{\mathbb{Q}}\right)^{-1}\left(\kappa^{\mathbb{P}}\theta^{\mathbb{P}} - \gamma\right)$ and $dW_t^{\mathbb{Q}} = dW_t^{\mathbb{P}} + \Pi_t dt$. Under risk-adjusted $\mathbb{Q}$ measure, x_t still evolves as a correlated vector Ornstein-Uhlenbeck process[13].

$$dx_t = \kappa^{\mathbb{Q}}\left(\theta^{\mathbb{Q}} - x_t\right)dt + \sigma dW_t^{\mathbb{Q}}$$

The dynamics of x_t under the risk-adjusted $\mathbb{Q}$ measure can be solved as

$$x_{t+\tau} = \theta^{\mathbb{Q}} + e^{-\kappa^{\mathbb{Q}}\tau}\left(x_t - \theta^{\mathbb{Q}}\right) + \int_t^{t+\tau} e^{-\kappa^{\mathbb{Q}}(\tau-u)} \sigma dW_u^{\mathbb{Q}}$$

which is analogous to the counterpart under the objective $\mathbb{P}$ measure. Eigen-decompose $N \times N$ parameter matrix $\kappa^{\mathbb{Q}}$ as

$$\kappa^{\mathbb{P}} = V\kappa_D V^{-1}$$

---

[10] To avoid the confusion of notation, here we use dummy variable u to evaluate the integral over time from t to $t+\tau$.

$$\int_t^{t+\tau} \kappa^{\mathbb{P}} x_u e^{\kappa^{\mathbb{P}}u} du + \int_t^{t+\tau} e^{\kappa^{\mathbb{P}}u} dx_u = \int_t^{t+\tau} e^{\kappa^{\mathbb{P}}u}\kappa^{\mathbb{P}}\theta^{\mathbb{P}} du + \int_t^{t+\tau} \sigma e^{\kappa^{\mathbb{P}}u} dW_u^{\mathbb{P}}$$

$$x_u e^{\kappa^{\mathbb{P}}u}\Big|_{u=t}^{u=t+\tau} - \int_t^{t+\tau} e^{\kappa^{\mathbb{P}}u} dx_u + \int_t^{t+\tau} e^{\kappa^{\mathbb{P}}u} dx_u = \theta^{\mathbb{P}}\left(e^{\kappa^{\mathbb{P}}(t+\tau)} - e^{\kappa^{\mathbb{P}}t}\right) + \int_t^{t+\tau} \sigma e^{\kappa^{\mathbb{P}}u} dW_u^{\mathbb{P}}$$

$$x_{t+\tau}e^{\kappa^{\mathbb{P}}(t+\tau)} - x_t e^{\kappa^{\mathbb{P}}t} = \theta e^{\kappa^{\mathbb{P}}(t+\tau)} - \theta e^{\kappa^{\mathbb{P}}t} + \int_t^{t+\tau} \sigma e^{\kappa^{\mathbb{P}}u} dW_u^{\mathbb{P}}$$

[11] If we need to calculate the double integral over time, we use u and s to evaluate them.

[12] To avoid confusion, note that Π_t and γ are both $N \times 1$ vector and Γ is $N \times N$ matrix. Here we consider the general specification of GATSM with N state variables in vector x_t, so the dimension of variables and parameters are all N-dimensions.

[13] The parameters and variables with $^{\mathbb{Q}}$ mean the modification under risk-adjusted $\mathbb{Q}$ measure.

where V contains the eigenvectors of $\kappa^{\mathbb{P}}$ in columns and κ_D is diagonal matrix containing eigenvalues $[\kappa_1, \kappa_2, \cdots, \kappa_N]$ in its diagonal. The matrix exponential $e^{-\kappa^{\mathbb{P}}\tau}$ can be represented as follows[14].

$$e^{-\kappa^{\mathbb{P}}\tau} = e^{-V\kappa_D V^{-1}\tau} = Ve^{-\kappa_D\tau}V^{-1} = V\mathbf{diag}[e^{-\kappa_1\tau}, \cdots, e^{-\kappa_N\tau}]V^{-1}$$

Under objective $\mathbb{P}$ measure, the expectation and variance of state variables $x_{t+\tau}$ at time t are given as $\mathbb{E}_t^{\mathbb{P}}(x_{t+\tau}|x_t)$ and $\mathbb{VAR}_t^{\mathbb{P}}(x_{t+\tau}|x_t)$,

$$\mathbb{E}_t^{\mathbb{P}}(x_{t+\tau}|x_t) = \theta^{\mathbb{P}} + e^{-\kappa^{\mathbb{P}}\tau}\left(x_t - \theta^{\mathbb{P}}\right)$$

$$\mathbb{VAR}_t^{\mathbb{P}}(x_{t+\tau}|x_t) = \int_0^\tau e^{-\kappa^{\mathbb{P}}u}\sigma\sigma^\top e^{-\left(\kappa^{\mathbb{P}}\right)^\top u}du = V\Theta(\tau)V^\top$$

where the element at row $i = 1, 2, \cdots, N$ and column $j = 1, 2, \cdots, N$ is given as follows[15],

$$[\Theta(\tau)]_{ij} = \frac{\Sigma_{ij}}{\kappa_i + \kappa_j}\left[1 - e^{-(\kappa_i+\kappa_j)\tau}\right]$$

where $\Sigma = V^{-1}\sigma\sigma^\top\left(V^{-1}\right)^\top$ is a $N \times N$ matrix. Under risk-adjusted $\mathbb{Q}$ measure, x_t has a similar formulation as its counterpart under objective $\mathbb{P}$

[14] The exponential of $N \times N$ matrix can be calculated analogously as a scalar. For any scalar x, e^x can be expanded as a Taylor polynomial.

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots = \sum_{i=0}^{\infty}\frac{x^i}{i!}$$

For $N \times N$ matrix $\kappa^{\mathbb{P}}$, $e^{\kappa^{\mathbb{P}}}$ can be calculated as follows.

$$e^{\kappa^{\mathbb{P}}} = \sum_{i=0}^{\infty}\frac{\kappa^i}{i!} = I + \kappa + \frac{\kappa^2}{2!} + \frac{\kappa^3}{3!} + \cdots$$

$$e^{-\kappa^{\mathbb{P}}\tau} = \sum_{i=0}^{\infty}\frac{\left[-V\kappa_D V^{-1}\tau\right]^i}{i!} = \sum_{i=0}^{\infty}\frac{V\left[-\kappa_D\tau\right]^i V^{-1}}{i!} = V\left[\sum_{i=0}^{\infty}\frac{(-\kappa_D\tau)^i}{i!}\right]V^{-1} = Ve^{-\kappa_D\tau}V^{-1}$$

$$e^{-\kappa_D\tau} = e^{-\tau\mathbf{diag}[\kappa_1,\cdots,\kappa_N]} = \mathbf{diag}\left[e^{-\kappa_1\tau}, \cdots, e^{-\kappa_N\tau}\right]$$

$$e^{-\kappa^{\mathbb{P}}\tau} = V\mathbf{diag}\left[e^{-\kappa_1\tau}, \cdots, e^{-\kappa_N\tau}\right]V^{-1}$$

[15] Conditional variance $\mathbb{VAR}_t^{\mathbb{P}}(x_{t+\tau}|x_t)$ can be calculated as follows.

$$e^{-\kappa^{\mathbb{P}}\tau}\sigma\sigma^\top e^{-\left(\kappa^{\mathbb{P}}\right)^\top\tau} = V\mathbf{diag}[e^{-\kappa_1\tau}, \cdots, e^{-\kappa_N\tau}]V^{-1}\sigma\sigma^\top\left(V^{-1}\right)^\top\mathbf{diag}[e^{-\kappa_1\tau}, \cdots, e^{-\kappa_N\tau}]V^\top$$

Define $V^{-1}\sigma\sigma^\top\left(V^{-1}\right)^\top$ as Σ, which is a $N \times N$ matrix with each element Σ_{ij}. Perform integral for each element of $e^{-\kappa^{\mathbb{P}}u}\sigma\sigma^\top e^{-\left(\kappa^{\mathbb{P}}\right)^\top u}$.

$$\int_0^\tau \Sigma_{ij}e^{-(\kappa_i+\kappa_j)u}du = \frac{\Sigma_{ij}}{\kappa_i + \kappa_j}\left[1 - e^{-(\kappa_i+\kappa_j)\tau}\right]$$

measure,

$$x_{t+\tau} = \theta^{\mathbb{Q}} + e^{-\kappa^{\mathbb{Q}}\tau}\left(x_t - \theta^{\mathbb{Q}}\right) + \int_t^{t+\tau} e^{-\kappa^{\mathbb{Q}}(\tau-u)}\sigma dW_u^{\mathbb{Q}}$$

so do its expectation and variance[16] conditional on state variables x_t.

$$\mathbb{E}_t^{\mathbb{Q}}(x_{t+\tau}|x_t) = \theta^{\mathbb{Q}} + e^{-\kappa^{\mathbb{Q}}\tau}\left(x_t - \theta^{\mathbb{Q}}\right)$$

$$\mathbb{VAR}_t^{\mathbb{Q}}(x_{t+\tau}|x_t) = \int_0^{\tau} e^{-\kappa^{\mathbb{Q}}\tau}\sigma\sigma^{\top}e^{-\left(\kappa^{\mathbb{Q}}\right)^{\top}\tau}du$$

Substituting $\mathbb{E}_t^{\mathbb{Q}}(x_{t+\tau}|x_t)$ into $r_t = a_0 + b_0^{\top}x_t$ evaluated at time point $t+\tau$ leads to the following expression for the expected short interest rate $\mathbb{E}_t^{\mathbb{Q}}(r_{t+\tau}|x_t)$.

$$\mathbb{E}_t^{\mathbb{Q}}(r_{t+\tau}|x_t) = a_0 + b_0^{\top}\mathbb{E}_t^{\mathbb{Q}}(x_{t+\tau}|x_t) = a_0 + b_0^{\top}\mathbb{E}_t^{\mathbb{Q}}\left[\theta^{\mathbb{Q}} + e^{-\kappa^{\mathbb{Q}}\tau}\left(x_t - \theta^{\mathbb{Q}}\right)\right]$$

The conditional variance $\mathbb{VAR}_t^{\mathbb{Q}}(r_{t+\tau}|x_t)$ under risk-adjusted $\mathbb{Q}$ measure, here we use notation ω_τ^2 (given other parameters, the conditional variance is a function of maturity τ) also has the analogous form as its counterpart under objective $\mathbb{P}$ measure.

$$\omega_\tau^2 = \mathbb{VAR}_t^{\mathbb{Q}}(r_{t+\tau}|x_t) = b_0^{\top}\mathbb{VAR}_t^{\mathbb{Q}}(x_{t+\tau}|x_t)b_0 = \int_0^{\tau} b_0^{\top}e^{-\kappa^{\mathbb{Q}}u}\sigma\sigma^{\top}e^{-\left(\kappa^{\mathbb{Q}}\right)^{\top}u}b_0du$$

To calculate the forward interest rate $f_{t,\tau}$, we have to know the volatility effect V_τ which captures the influence from the volatility in the short interest rate on the expected returns. Due to Jensen's inequality, the expected compounded return from investing in a volatile short interest rate over time t to $t+\tau$ is less than the compounded return from investing in the expected short interest rate over same period. The volatility effect V_τ can be calculated by the double integral as follows.

$$V_\tau = \int_0^{\tau} b_0^{\top}e^{-\kappa^{\mathbb{Q}}(\tau-s)}\sigma\left[\sigma^{\top}\int_s^{\tau} e^{-\left(\kappa^{\mathbb{Q}}\right)^{\top}(u-s)}b_0du\right]ds$$

Given all above results, forward interest rate $f_{t,\tau}$, the expected path of the short interest rate under risk-adjusted $\mathbb{Q}$ measure is given by

$$f_{t,\tau} = \mathbb{E}_t^{\mathbb{Q}}(r_{t+\tau}|x_t) - V_\tau$$

and the GATSM interest rate $R_{t,\tau}$ can be obtained using the standard term structure relationship in continuous time.

$$R_{t,\tau} = \frac{1}{\tau}\int_0^{\tau} f_{t,u}du = \frac{1}{\tau}\int_0^{\tau}\mathbb{E}_t^{\mathbb{Q}}(r_{t+\tau}|x_t)du - \frac{1}{\tau}\int_0^{\tau}V_\tau du = a_\tau + b_\tau^{\top}x_t$$

Finally, the bond pricing with maturity τ in GATSM takes an exponential affine form.

$$P_{t,\tau} = e^{-\tau R_{t,\tau}} = e^{-\tau a_\tau - \tau b_\tau^{\top}x_t}$$

[16]Here we use $\mathbb{E}_t^{\mathbb{Q}}$ and $\mathbb{VAR}_t^{\mathbb{Q}}$ to represent the conditional expectation and conditional variance under the risk-adjusted $\mathbb{Q}$ measure.

2.2.2 Shadow/ZLB-GATSM

We extend general GATSM by a intuitive modification to adapt GATSM to the ZLB environment[17]. The extension of general GATSM in the ZLB environment is firstly proposed by Krippner (2012), which is known as shadow/ZLB-GATSM framework. In this framework, imposing ZLB restriction can be represented by a max operator.

$$\underline{\mathrm{r}}_t = \max\{0, r_t\} = r_t + \max\{-r_t, 0\}$$

Here $\underline{\mathrm{r}}_t$ means the actual short interest rate and the r_t is the shadow short interest rate. In general non-ZLB environment, $r_t \geq 0$, the economic agent invests at the instantaneous interest rate r_t and $\underline{\mathrm{r}}_t = r_t$. But in the ZLB environment, $r_t < 0$, the economic agent will choose to hold physical currency and obtain zero return actually with $\underline{\mathrm{r}}_t = 0$. The short interest rate under forward $t+\tau$ risk-adjusted $\mathbb{Q}$ measure follows the normal distribution

$$r_{t+\tau}|x_t \sim \mathcal{N}\left(f_{t,\tau}, \omega_\tau^2\right)$$

where $f_{t,\tau}$ and ω_τ^2 have been derived in previous section. Under forward $t+\tau$ risk-adjusted $\mathbb{Q}$ measure, the mean of this distribution $f_{t,\tau}$ satisfies

$$f_{t,\tau} = \mathbb{E}_{t+\tau}^{\mathbb{Q}}\left(r_{t+\tau}|x_t\right)$$

where $\mathbb{E}_{t+\tau}^{\mathbb{Q}}$ represents the expectation under forward $t+\tau$ risk-adjusted $\mathbb{Q}$ measure. The conditional variance of $r_{t+\tau}|x_t$ is time-invariant.

$$\mathbb{VAR}_{t+\tau}^{\mathbb{Q}}(r_{t+\tau}|x_t) = \mathbb{VAR}_t^{\mathbb{Q}}(r_{t+\tau}|x_t) = \omega_\tau^2$$

Given the distribution of $r_{t+\tau}|x_t$, its probability density function $\mathbb{PDF}(r_{t+\tau}|x_t)$ is given as follows.

$$\mathbb{PDF}(r_{t+\tau}|x_t) = \frac{1}{\omega_\tau\sqrt{2\pi}}\exp\left[-\frac{1}{2}\left(\frac{r_{t+\tau} - f_{t,\tau}}{\omega_\tau}\right)^2\right]$$

In practice, we allow a non-zero lower bound r_L to represent the ZLB which may be a very small number, approximately equal to zero but not zero actually.

$$\underline{\mathrm{r}}_{t+\tau} = \max\{r_L, r_{t+\tau}\} = r_{t+\tau} + \max\{r_L - r_{t+\tau}, 0\}$$

The forward interest rate can be modified as

$$\underline{\mathrm{f}}_{t,\tau} = \mathbb{E}_{t+\tau}^{\mathbb{Q}}\left(\underline{\mathrm{r}}_{t+\tau}|x_t\right) = \mathbb{E}_{t+\tau}^{\mathbb{Q}}\left(r_{t+\tau}|x_t\right) + \mathbb{E}_{t+\tau}^{\mathbb{Q}}\left(\max\{r_L - r_{t+\tau}, 0\}|x_t\right) = f_{t,\tau} + z_{t,\tau}$$

and if we set $r_L = 0$,

$$\underline{\mathrm{f}}_{t,\tau} = \mathbb{E}_{t+\tau}^{\mathbb{Q}}\left(\underline{\mathrm{r}}_{t+\tau}|x_t\right) = \mathbb{E}_{t+\tau}^{\mathbb{Q}}\left(r_{t+\tau}|x_t\right) + \mathbb{E}_{t+\tau}^{\mathbb{Q}}\left(\max\{-r_{t+\tau}, 0\}|x_t\right) = f_{t,\tau} + z_{t,\tau}$$

[17] In this section, the notation with underbar _ means the restriction of the ZLB.

where $\mathbb{E}^{\mathbb{Q}}_{t+\tau}\left(r_{t+\tau}|x_t\right) = f_{t,\tau}$ has been obtained in previous analysis. We now evaluate another part[18] $z_{t,\tau}$ in forward interest rate $\underline{\mathrm{f}}_{t,\tau}$.

$$
\begin{aligned}
z_{t,\tau} = \mathbb{E}^{\mathbb{Q}}_{t+\tau}\left(\max\{r_L - r_{t+\tau}, 0\}|x_t\right) &= \int_{-\infty}^{\infty} \max\{r_L - r_{t+\tau}, 0\} \cdot \mathbb{PDF}(r_{t+\tau}) dr_{t+\tau} \\
&= \int_{-\infty}^{r_L} (r_L - r_{t+\tau}) \cdot \mathbb{PDF}(r_{t+\tau}) dr_{t+\tau} + \int_{r_L}^{\infty} 0 \cdot \mathbb{PDF}\left(r_{t+\tau}\right) dr_{t+\tau} \\
&= \int_{-\infty}^{r_L} (r_L - r_{t+\tau}) \frac{1}{\omega_\tau \sqrt{2\pi}} \exp\left[-\frac{1}{2}\left(\frac{r_{t+\tau} - f_{t,\tau}}{\omega_\tau}\right)^2\right] dr_{t+\tau} \\
&= (r_L - f_{t,\tau}) \cdot \left[1 - \Phi\left(\frac{f_{t,\tau} - r_L}{\omega_\tau}\right)\right] + \omega_\tau \frac{1}{\sqrt{2\pi}} \exp\left[-\frac{1}{2}\left(\frac{f_{t,\tau} - r_L}{\omega_\tau}\right)^2\right]
\end{aligned}
$$

$\Phi\left(\frac{f_{t,\tau} - r_L}{\omega_\tau}\right)$ is the cumulative density function of standard normal distribution. Substituting $z_{t,\tau}$ into $\underline{\mathrm{f}}_{t,\tau} = f_{t,\tau} + z_{t,\tau}$ leads to the expression of $\underline{\mathrm{f}}_{t,\tau}$.

$$
\underline{\mathrm{f}}_{t,\tau} = f_{t,\tau} + z_{t,\tau} = r_L + (f_{t,\tau} - r_L)\,\Phi\left(\frac{f_{t,\tau} - r_L}{\omega_\tau}\right) + \omega_\tau \frac{1}{\sqrt{2\pi}} \exp\left[-\frac{1}{2}\left(\frac{f_{t,\tau} - r_L}{\omega_\tau}\right)^2\right]
$$

Using the results from Priebsch (2013)[19], the expected value of $\underline{\mathrm{r}}_{t+\tau} = \max\{r_L, r_{t+\tau}\}$ can be calculated as follows, which is identical to previous result.

$$
\begin{aligned}
\underline{\mathrm{f}}_{t,\tau} = \mathbb{E}^{\mathbb{Q}}_{t+\tau}\left(\underline{\mathrm{r}}_{t+\tau}|x_t\right) = &\int_{-\infty}^{r_L} \frac{r_L}{\omega_\tau\sqrt{2\pi}} \exp\left[-\frac{1}{2}\left(\frac{r_{t+\tau} - f_{t,\tau}}{\omega_\tau}\right)^2\right] dr_{t+\tau} \\
&+ \int_{r_L}^{\infty} \frac{r_{t+\tau}}{\omega_\tau\sqrt{2\pi}} \exp\left[-\frac{1}{2}\left(\frac{r_{t+\tau} - f_{t,\tau}}{\omega_\tau}\right)^2\right] dr_{t+\tau}
\end{aligned}
$$

Given all results derived above, the interest rate $\underline{\mathrm{R}}_{t,\tau}$ in the shadow/ZLB-GATSM framework still has the analogous expression of the counterpart in general GATSM framework[20].

$$
\underline{\mathrm{R}}_{t,\tau} = \frac{1}{\tau}\int_0^{\tau} \underline{\mathrm{f}}_{t,u} du
$$

2.2.3 A Two-Factor Shadow/ZLB-GATSM

In practice, there are many specifications of GATSM which all have analogous structure[21]. Arbitrage-free Nelson and Siegel (1987) Model (ANSM)

[18] See Appendix for the derivation.

[19] For the mathematical background of this calculation, please refer to Priebsch (2013, Appendix A.2) for how to handle a max operator in the calculation of expectation under censored normal distribution.

[20] This expression does not have analytic solution in closed-form and need to be evaluated numerically.

[21] See Filipović (2009, Chapter 5) for introduction of this class of models.

is one of the widely used GATSMs. Here we derive a two-factor GATSM which has ANSM specification and then extend it in the ZLB environment in the way introduced in previous section. Note that although we can use three or more factors to track the dynamics of yield curve, two-factor model can produce the level and slope components of term structure which can explain 99.9% variation in the yield curve data and provide a realistic representation of the yield curve in many applications with the most parsimonious parameters and variables in all GATSMs which have more than one state variable. Another reason that we choose the two-factor specification is that according to Kim and Singleton (2012), if short interest rate is near the ZLB, the information for estimation GATSM with three state variables (level, slope and bow) is not enough, it is better to use two-factor model for short interest rates in the ZLB environment. We specify the function forms, parameters and and give the results of related calculations in this section and use Japan government bond yield curve data to estimate it in next section.

According to Singleton (2009), the number of parameters in a general GATSM with two factors is 19, but the maximum number of parameters that can be uniquely identified with econometric estimation is 12. To ensure the identification of parameters in estimation, a two-factor model have some parameters to be estimated and other parameters to be calibrated.

A two-factor shadow/ZLB-GATSM has tow factors (state variables), level component L_t and slope component S_t with the same weight $b_0 = \begin{bmatrix} 1 & 1 \end{bmatrix}^\top$ and a calibrated constant $a_0 = 0$.

$$r_t = b_0^\top x_t = \begin{bmatrix} 1 & 1 \end{bmatrix} \begin{bmatrix} L_t \\ S_t \end{bmatrix} = L_t + S_t$$

$\kappa^{\mathbb{Q}}$ is a diagonal matrix $\begin{bmatrix} 0 & 0 \\ 0 & \varphi \end{bmatrix}$, and its exponential $e^{-\kappa^{\mathbb{Q}}\tau}$ is given as

$$e^{-\kappa^{\mathbb{Q}}\tau} = \exp\left(-\begin{bmatrix} 0 & 0 \\ 0 & \varphi \end{bmatrix}\tau\right) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \exp\left(-\begin{bmatrix} 0 & 0 \\ 0 & \varphi \end{bmatrix}\tau\right) \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & e^{-\varphi\tau} \end{bmatrix}$$

which can be calculated by the methods described in Section 2.2.1. Given $\kappa = \begin{bmatrix} \kappa_{11} & \kappa_{12} \\ \kappa_{21} & \kappa_{22} \end{bmatrix}$, Γ can be identified from relationships of parameters between $\mathbb{Q}$ and $\mathbb{P}$ measures, which has been introduced in Section 2.2.1. Recall that state variables $x_t = \begin{bmatrix} L_t & S_t \end{bmatrix}^\top$ follow a continuous-time first-order vector-autoregression process

$$x_{t+\tau} = \theta^{\mathbb{Q}} + e^{-\kappa^{\mathbb{Q}}\tau}\left(x_t - \theta^{\mathbb{Q}}\right) + \int_t^{t+\tau} e^{-\kappa^{\mathbb{Q}}(\tau-u)}\sigma dW_u$$

where standard error of innovation

$$\sigma = \begin{bmatrix} \sigma_1 & 0 \\ \rho_{12}\sigma_2 & \sigma_2\sqrt{1-\rho_{12}^2} \end{bmatrix}$$

is a 2×2 matrix. We can rewrite this equation in a small time interval Δt

$$\begin{bmatrix} L_{t+\Delta t} \\ S_{t+\Delta t} \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & e^{-\varphi\tau} \end{bmatrix} \begin{bmatrix} L_t \\ S_t \end{bmatrix} + \begin{bmatrix} \varepsilon_{L,t+\Delta t} \\ \varepsilon_{S,t+\Delta t} \end{bmatrix} = \begin{bmatrix} L_t \\ e^{-\varphi\tau} S_t \end{bmatrix} + \begin{bmatrix} \varepsilon_{L,t+\Delta t} \\ \varepsilon_{S,t+\Delta t} \end{bmatrix}$$

where $\theta^{\mathbb{Q}}$ and θ are given[22] as $\theta^{\mathbb{Q}} = \begin{bmatrix} 0 & 0 \end{bmatrix}^\top$ and $\theta = \begin{bmatrix} \theta_1 & \theta_2 \end{bmatrix}^\top$. From this result, we can note that L_t follows a random-walk process and S_t evolves as a mean-reverting process. The remaining job is just to calculate[23] the expected path of the short interest rate (mean and variance), volatility effect and forward interest rate in the fashion introduced in Section 2.2.1 and extend the results in the ZLB environment in the way introduced in Section 2.2.2.

- **Expectation of short interest rate**:

$$\mathbb{E}_t^{\mathbb{Q}}\left(r_{t+\tau}|x_t\right) = L_t + e^{-\varphi\tau} S_t$$

- **Variance of short interest rate**:

$$\omega_\tau^2 = \sigma_1^2 \tau + \sigma_2^2 \frac{1 - e^{-2\varphi\tau}}{2\varphi} + 2\rho_{12}\sigma_1\sigma_2 \frac{1 - e^{-\varphi\tau}}{\varphi}$$

- **Volatility effect**:

$$V_t = \frac{\sigma_1^2 \tau^2}{2} + \frac{\rho_{12}\sigma_1\sigma_2\tau}{\varphi}\left(1 - e^{-\varphi\tau}\right) + \frac{\sigma^2}{2\varphi^2}\left(1 - 2e^{-\varphi\tau} + e^{-2\varphi\tau}\right)$$

- **Forward interest rate**:

$$f_{t,\tau} = L_t + e^{-\varphi\tau} S_t - \left[\frac{\sigma_1^2 \tau^2}{2} + \frac{\rho_{12}\sigma_1\sigma_2\tau}{\varphi}\left(1 - e^{-\varphi\tau}\right) + \frac{\sigma^2}{2\varphi^2}\left(1 - 2e^{-\varphi\tau} + e^{-2\varphi\tau}\right)\right]$$

- **Interest rate with maturity τ**[24]:

$$R_{t,\tau} = a_\tau + L_t + \frac{1}{\varphi\tau}\left(1 - e^{-\varphi\tau}\right) S_t$$

[22] Other parameters and variables such as γ and Π_t can also be identified by the relationships between $\mathbb{Q}$ and $\mathbb{P}$ measures introduced in Section 2.2.1.

[23] The details of calculation are given in Appendix for reference.

[24] See Appendix for deviations.

$$a_\tau = -\frac{\sigma_1^2\tau^2}{6} - \frac{\sigma_2^2}{2\varphi^2}\left[1 - \frac{1}{2\varphi\tau}e^{-2\varphi\tau} + \frac{2}{\varphi\tau}e^{-\varphi\tau} - \frac{3}{2\varphi\tau}\right] - \frac{\rho_{12}\sigma_1\sigma_2}{\varphi^2}\left[-\frac{1 - e^{-\varphi\tau}}{\varphi\tau} + \frac{\varphi\tau}{2} + e^{-\varphi\tau}\right]$$

$$b_\tau = \begin{bmatrix} 1 \\ \frac{1}{\varphi\tau}\left(1 - e^{-\varphi\tau}\right) \end{bmatrix}$$

- **Shadow short interest rate**:

$$\underline{\mathrm{r}}_t = \max\{r_L, L_t + S_t\}$$

- **Forward interest rate in the ZLB environment**:

$$\underline{\mathrm{f}}_{t,\tau} = r_L + (f_{t,\tau} - r_L)\,\Phi\left(\frac{f_{t,\tau} - r_L}{\omega_\tau}\right) + \omega_\tau \frac{1}{\sqrt{2\pi}} \exp\left[-\frac{1}{2}\left(\frac{f_{t,\tau} - r_L}{\omega_\tau}\right)^2\right]$$

- **Interest rate with maturity τ in the ZLB environment**:

$$\underline{\mathrm{R}}_{t,\tau} = \frac{1}{\tau}\int_0^\tau \underline{\mathrm{f}}_{t,u} du$$

2.3 Data and Estimation

In this section, we use the model derived in previous section and Japan's yield curve data to estimate this model. The results from estimation are parameters, estimated values of two factors and fitted values of short interest rate. Three monetary policy measures can be directly calculate from the these results, which will be introduced in next section.

2.3.1 Data

The yield curve data set of Japan is Japan government bond yield data from 1992/07/10 (Friday) to 2016/11/24 (Thursday), daily frequency of 5-business days week with 6360 observations, obtained from Bloomberg database. To reduce the burden of computation, we chose the end of month data as one observation for one month, then we get 293 observations from 1992M7 to 2016M11, monthly frequency. The maturity of yield curve is from 3-months to 30-years and 3-months interest rate is adapted to be the short interest rate r_t. Interest rates with other maturities are equivalent to be $R_{t,\tau}$ in term structure models. The summary statistics of all 12 series are given in Table 2.1.

Maturity τ	3M[25]	6M	1Y	2Y	3Y	4Y
Mean	0.46	0.47	0.51	0.63	0.77	0.94
Med	0.12	0.13	0.15	0.25	0.40	0.57
Max	4.29	4.17	4.16	4.24	4.41	4.95
Min	-0.42	-0.43	-0.37	-0.36	-0.36	-0.36
Std. Dev.	0.85	0.83	0.85	0.90	0.97	1.06
Obs.	6360	6360	6360	6360	6360	6360
Maturity τ	5Y	7Y	10Y	15Y	20Y	30Y
Mean	1.10	1.40	1.81	2.13	2.53	2.70
Med	0.73	1.03	1.49	1.83	2.20	2.48
Max	5.25	5.71	5.64	6.16	6.44	6.24
Min	-0.37	-0.39	-0.28	-0.13	0.03	0.05
Std. Dev.	1.13	1.22	1.26	1.29	1.26	1.16
Obs.	6360	6360	6360	6360	6360	6360

Table 2.1: Summary Statistics of Japan Yield Curve Data

We can find that with the increasing of maturity, the standard deviation also increases, which means the increasing of uncertainty of long maturity. Note that the minimum values of yield curve data from 3-months to 15-years are all negative. The time series plot of yield curve data is given in Figure 2.1a and Figure 2.1b.

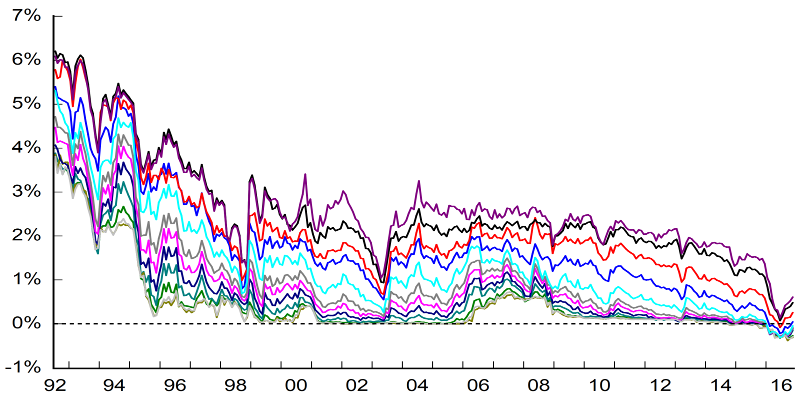


Figure 2.1a: Japan government Bond Yield Curve Data

From the 3D view of yield curve data, we can confirm that the yield curves of all maturities have shifted down since 1999.

[25]M=Month and Y=Year.

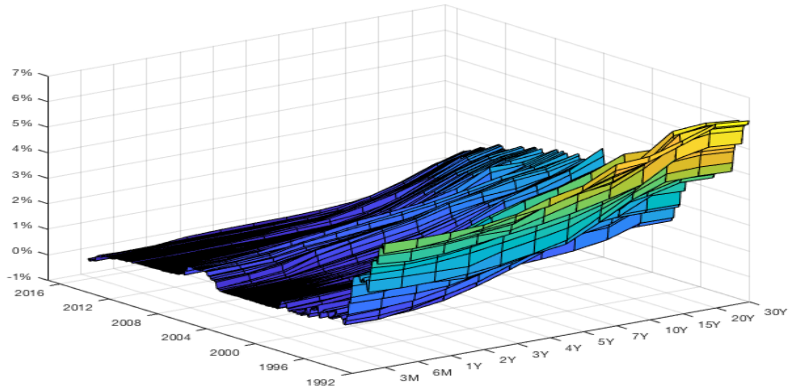


Figure 2.1b: Japan government Bond Yield Curve Data (3D vision)

Figure 2.2 shows the plot of 3-months bond interest rate and uncollateralized overnight call rate, which is generally recognized as the official short policy interest rate of Bank of Japan. We can find that after 1999, these two kinds of typical short interest rates have been evolving near the ZLB. Although during the ZLB period, Bank of Japan has adopted several unconventional monetary policy programs and during the same period, the macroeconomy in Japan also has experienced dramatic fluctuations, we can't get further information from these kinds of short interest rate if we still consider them as the measure of monetary policy. This is why we need shadow/ZLB-GATSM to model the short interest rate in the ZLB environment and estimate the corresponding shadow short interest rate from shadow/ZLB-GATSM.

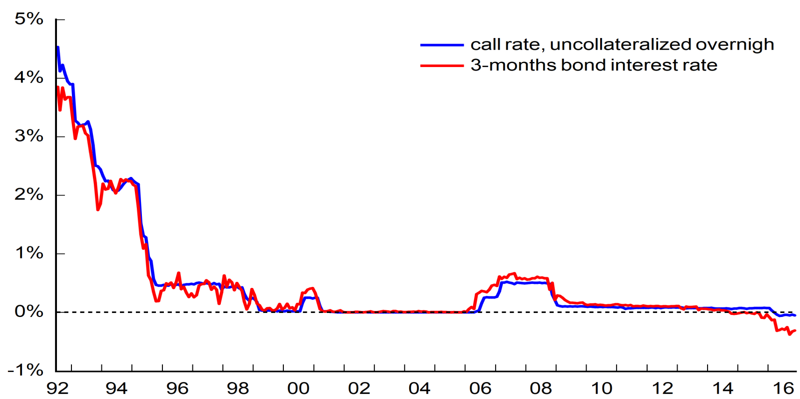


Figure 2.2: Short Interest Rates in Japan

Note that money statistics and long-maturity interest rates can be potentially used to quantify the stance of monetary policy in the ZLB environment. But there are some fatal defects of them. For money statistics such as money supply or money growth, history has told us that the relationships between money statistics and macroeconomic variables haven't been stable and reliable since 1980s, especially in developed economies. For long-maturity interest rates such as 10-years interest rate of government bond, even long-maturity interest rates are not constrained by the ZLB and there exists research that supports the view that the long-maturity interest rates can response to monetary policy events, the fluctuations in long-maturity interest rates can be the results of other reasons such as neutral interest rates, inflation expectations and so on. In another word, long-maturity interest rates are noisy measures of stance of monetary policy. Also, the amount of large-scale asset purchase can be used as the measure of stimulative degree of monetary easing, but from the results of this chapter, the first round of monetary easing from 2001Q1 to 2006Q1, the following round from 2010Q3 to 2013Q1 and the latest QQE from 2013Q4, the estimated stimulative degree of monetary policy doesn't have much difference even the scale of asset purchase is quite different during different policy programs.

2.3.2 Estimation Methodology

GATSM is generally estimated in state-space form with Kalman filter and maximum likelihood method. Compared to general GATSM used in the non-ZLB environment, shadow/ZLB-GATSM has nonlinear functions of state variables x_t via the CDF or PDF of normal distribution. Iterated Extended Kalman Filter (IEKF) can handle the nonlinearity. For technical details of IEKF, please refer to Krippner (2015, pp.117-126) which provides the instruction of IEKF in the estimation of shadow/ZLB term structure model. Wu and Xia (2016), Baure and Rudebusch (2016) also used the same algorithm.

A two-factor model has 11 parameters to estimate. Denote the set of all estimated parameters as $\Omega = \{r_L, \varphi, \theta_1, \theta_2, \kappa_{11}, \kappa_{12}, \kappa_{21}, \kappa_{22}, \sigma_1, \sigma_2, \rho_{12}\}$. Here we consider the lower bound of nominal interest rate r_L as a parameter to be estimated instead of setting $r_L = 0$ directly. The estimation can be conducted by using the Matlab code[26] provided by Krippner (2015). The state equation in state-space under objective $\mathbb{P}$ measure is given in discrete

[26]The Matlab code for estimation is available from http://www.rbnz.govt.nz/research-and-publications/research-programme/additional-research/measures-of-the-stance-of-united-states-monetary-policy/matlab-code-for-krippner-2015-shadow-zlb-term-structure-model.

time where Δt is time span between two observations.

$$x_t = \theta^{\mathbb{P}} + e^{-\kappa^{\mathbb{P}}\Delta t}\left(x_{t-1} - \theta^{\mathbb{P}}\right) + \varepsilon_t$$

Variance of innovation ε_t is calculated as follows.

$$\mathbb{VAR}^{\mathbb{P}}[\varepsilon_t] = \int_0^{\Delta t} e^{-\kappa^{\mathbb{P}} u}\sigma\sigma^{\top} e^{-\left(\kappa^{\mathbb{P}}\right)^{\top} u} du$$

Note that $\underline{\mathrm{R}}_{t,\tau} = \frac{1}{\tau}\int_0^{\tau} \underline{\mathrm{f}}_{t,u} du$ is a function of state variables x_t, maturity τ and other parameters, so measurement equation can be represented as

$$\underline{\mathrm{R}}_{t,\tau} = \underline{\mathrm{R}}(x_t, \tau, \Omega) + \eta_t$$

where η_t is a 12×1 vector of measurement errors[27]. The variance of measurement errors which is a 12×12 diagonal matrix $\mathbb{VAR}(\eta_t) = \mathbf{diag}[\sigma^2_{\eta,\tau}]$ are assumed to have homoscedasticity $\sigma^2_{\eta,\tau}$ and to be independent to each other. ε_t in state equation and η_t in measurement equation are also assumed to be uncorrelated to each other.

2.3.3 Estimation Results

We conducted three groups of estimation using the daily frequency data, monthly frequency data and weekly frequency data. The monthly data and weekly data were extracted from the original daily frequency data by choosing end-monthly (293 observations) and end-weekly (1273 observations) observations of yield curve. To avoid the negative values in dataset, which are not compatible with mathematical specification of the model, we excluded the data from 2016/02/01 to 2016/11/24 for the period of QQE with a negative interest rate[28]. Monthly estimates are used to match other data which is only available in monthly frequency such as Consumer Price Index (CPI) or uncollateralized overnight call rate.

[27] Because we have 12 series yield curve data with different maturities, we have to specify 12 measurement equations.

[28] The yield curve data from 3-months maturity to 10-years maturity begun to take negative values since the start or QQE with a negative interest rate policy, which made data inapplicable to the computation of model estimation.

	Daily		Weely		Monthly	
Parameters	Estimates	Std.Error	Estimates	Std.Error	Estimates	Std.Error
r_L	0.000706	0.000011	0.000730	0.000025	0.000648	0.000064
φ	0.143037	0.000768	0.138777	0.001671	0.129480	0.003335
κ_{11}	0.078780	0.002108	0.109261	0.002665	0.061399	0.001233
κ_{12}	-0.173436	0.012115	0.001283	0.000024	0.010122	0.001473
κ_{21}	0.035443	0.001775	0.059224	0.001202	0.040958	0.008651
κ_{22}	0.000358	0.000011	0.001065	0.000018	0.007225	0.000809
θ_1	-0.003166	0.000167	0.039420	0.002259	0.074089	0.014680
θ_2	-0.049843	0.003068	-0.649511	0.012159	-0.355374	0.006008
σ_1	0.012395	0.000103	0.011765	0.000222	0.011904	0.000438
σ_2	0.017150	0.000201	0.014753	0.000432	0.013264	0.000702
ρ_{12}	-0.954813	0.001711	-0.925731	0.005326	-0.891964	0.013401

Table 2.2: Estimated Parameters of shadow/ZLB-GATSM

From Table 2.2, we can find that r_L is a very small number, approximately equal to zero. Other parameters from three groups of estimation all have similar values. Estimated two state variables, shadow short interest rate are plotted in Figure 2.3. Note that each shaded area near level, slope and shadow short interest rate are calculated from the point estimates plus or minus their 1.96 unit estimated standard error. We plot short interest rates in the same figure for ease of comparison. During the non-ZLB period, the shadow interest rate has almost same path as actual short interest rates does during non-ZLB period. But during the ZLB period, short interest rates remain static near the ZLB and the shadow short interest rate still evolves to negative values. Given these estimated series, we put them in a time line of monetary policy events and then evaluate the stance of monetary policy in Japan.

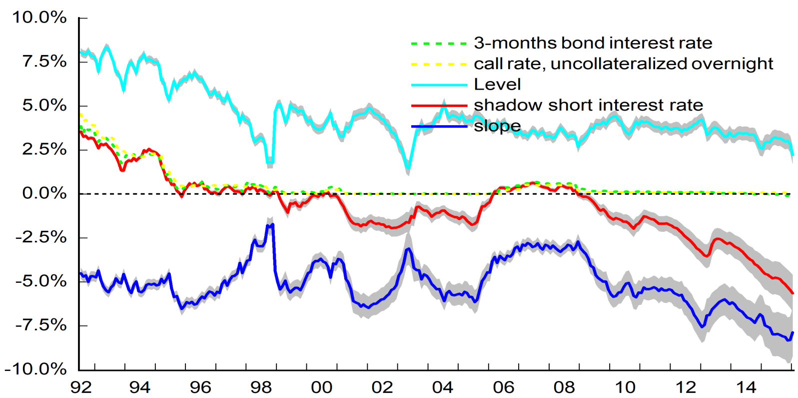


Figure 2.3: Estimated Factors and SSR

2.4 Quantitative Measures of Monetary Policy in Japan

Three quantitative measures can be calculated from the estimation results and then can be used as quantitative indicators of the stance of monetary policy. We explain these indicators one by one in Section 2.4 in a context of monetary policy events.

2.4.1 Shadow Short Interest Rate

The concept of shadow short interest rate (SSR) was firstly proposed by Black (1995). If the ZLB wouldn't exist and nominal interest rate could decrease to negative value freely, the economic agents would hold physical currency rather than invest in government bonds. The value of this call option that the economic agents could choose to hold currency in hand with zero return plus the SSR is equal to zero theoretically. In another word, the call option of holding currency would have positive value when the economic agents expect the deep decrease of interest rates in future.

As we have already mentioned previously, the SSR is the shortest maturity interest rate from the estimated shadow yield curve, which can take negative values in the ZLB environment. We can confirm from Figure 2.4 that the SSR is approximately equal to the 3-months bond interest rate and policy rate during the non-ZLB period. Note that the estimated results from this paper are similar to Ichiue and Ueno (2013) or Imakubo and Nakajima (2015).

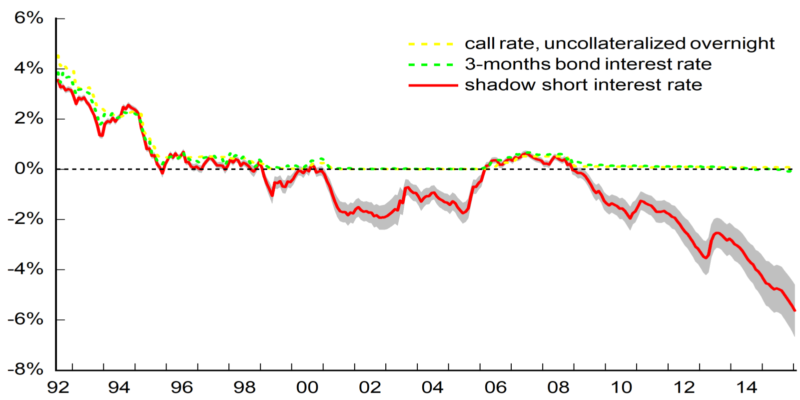


Figure 2.4: Estimated SSR

Figure 2.5 shows the major monetary policy regimes officially announced by BoJ since 1999 where each colored area indicates the period of each policy regime. We can at least confirm the negative SSR levels are correlated with the evolution of monetary policy events. Although the SSR shows different response to each policy event, the main trend of the SSR is the decrease with the evolution of unconventional monetary policy from Zero Interest Rate Policy to QQE in Japan.

The Policy Evolution of BoJ since 1999	
1999-Feb-2 to 2000-Aug-11	Zero Interest Rate Policy (pink shaded area in Figure 2.5)
2001-Mar-19 to 2006-Mar-9	Quantitative Easing (yellow shaded area in Figure 2.5)
2010-Oct-5 to 2013-Mar-20	Comprehensive Monetary Easing (green shaded area in Figure 2.5)
2013-Apr-4 to 2016-Jan-29	Price Stability Target of 2% and Quantitative and Qualitative Monetary Easing (blue shaded area in Figure 2.5)
2016-Feb-16 to 2016-Sep-19	Price Stability Target of 2% and Quantitative and Qualitative Monetary Easing with a Negative Interest Rate
2016-Sep-21 to Now	Price Stability Target of 2% and Quantitative and Qualitative Monetary Easing with Yield Curve Control

Table 2.3: Monetary Policy Regimes of BoJ

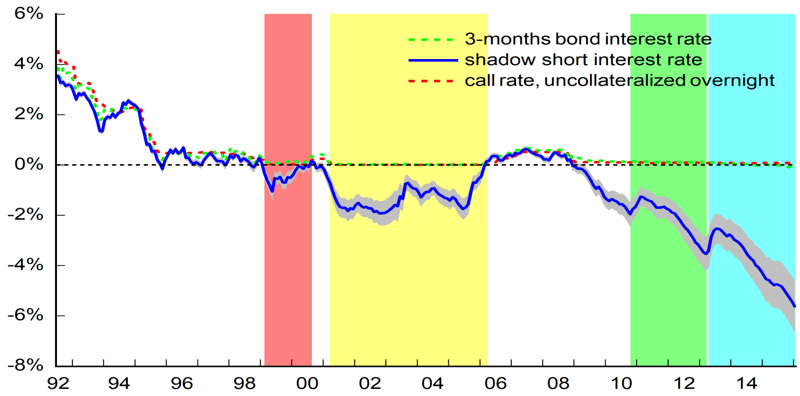


Figure 2.5: SSR in Different Monetary Policy Regimes

Note that it is better to consider the SSR as an ordinal measure of monetary policy. The SSR can be used to track the unconventional monetary policy events in a consistent way. Generally, more lower values are consistent with more monetary stimulus, and vice versa. But in real world, the economic agents can't transact with negative nominal interest rates, the change of the SSR doesn't mean that it can have real effect to economy in the same way as the change of policy rate in the conventional non-ZLB environment.

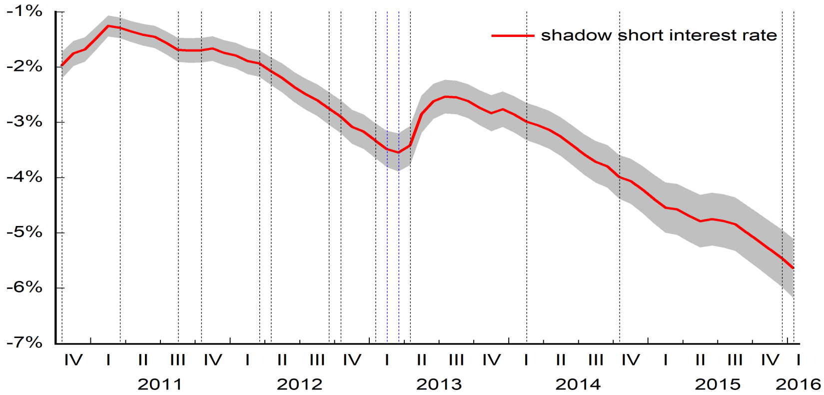


Figure 2.6: Monetary Policy Events from 2010M10 to 2016M1

Figure 2.6 is a close-up of Figure 2.5, zooming up from 2010M10 to 2016M1, the start of Comprehensive Monetary Easing to 2016M1, the end of first phase of Quantitative and Qualitative Monetary Easing. The vertical dashed lines plotted in Figure 2.6 are the indicators for the major policy decisions made by policy board of BoJ. For most of this period, the policy decisions are the "*maintain the status quo*" or "*additional monetary easing*", and the shadow short interest rate shows response to the policy decisions, decreasing with more negative values. We located two policy decisions which are "*send-off of additional monetary easing*" by two blue dashed lines at 2013/2/14 and 2013/3/7. The shadow short interest rate increased after these two decisions in a consistent way of short interest rate in general environment. But during the same period, call rate and 3-months interest rate all didn't have much response to the policy changes and we can't get any information from these general monetary policy indicators. This is why we should use the shadow short interest rate as a proxy of policy rate in the ZLB environment.

Then we check the empirical relations between the shadow short interest rate and the balance sheet of BoJ.

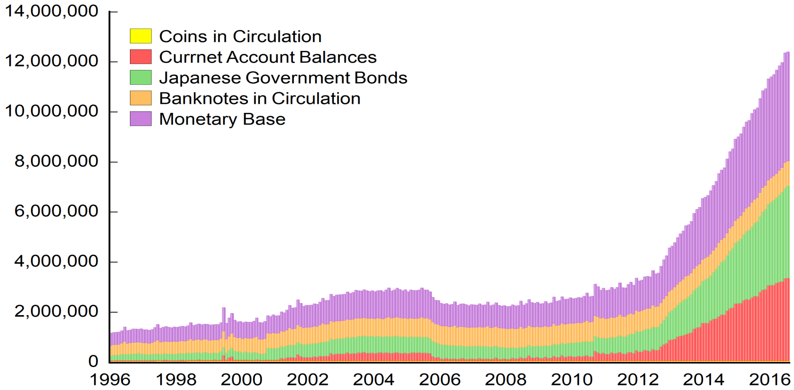


Figure 2.7: Balance Sheet of BoJ

Bank of Japan is the first central bank which introduced unconventional monetary policy among major advanced economies. The essence of the policy programs conducted by BoJ is the large scale purchase of Japan government bond. From the Figure 2.7, we can find since 2010M10, the balance sheet of BoJ has increased aggressively. What is the relation between the SSR and the size of central bank's balance sheet? The Figure 2.8 shows the time series plot of SSR, minus log of bond holdings and minus log of monetary base[29].

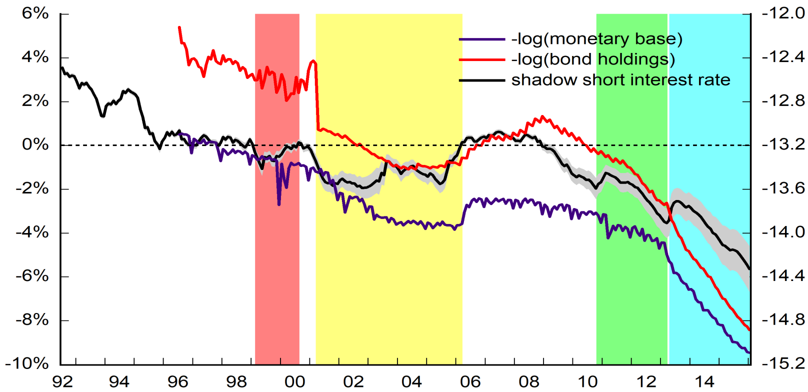


Figure 2.8: SSR and Balance Sheet of BoJ

[29] The pink shaded area in Figure 2.8 represents the Zero Interest Rate Policy from 1999M3 to 2000M8. The yellow shaded area shows the first round of QE from 2001M3 to 2006M3. The green shaded area represents the comprehensive monetary easing from 2010M10 to 2013M3. The blue shaded area represents the QQE from 2013M4.

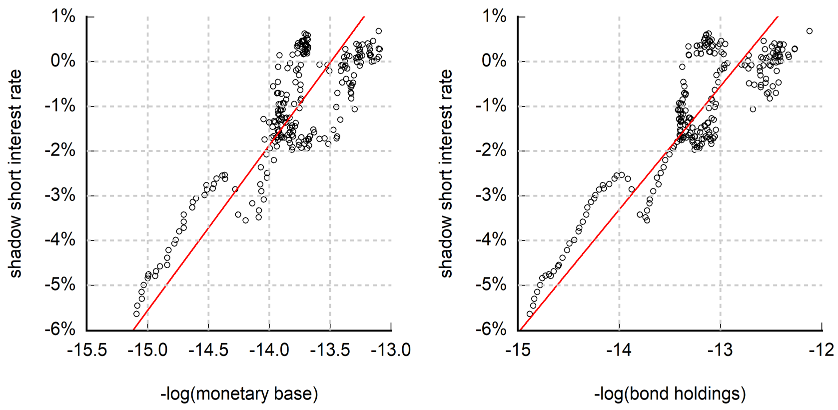


Figure 2.9: SSR and Balance Sheet of BoJ (scatter plot)

The correlation between SSR and bond holdings (in log term) is -0.87 and the correlation between SSR and monetary base (in log term) is -0.86. The X-Y scatter plots in Figure 2.9 also show obvious high correlation of the SSR and the variables of balance sheet. Note that the SSR can't be controlled directly by the central bank in the ZLB environment[30]. What the central bank can manipulate is its balance sheet. The SSR just summarizes the stance of monetary policy. According to the empirical relation between the SSR and the balance sheet, we can map the manipulation of the balance sheet into the change of the SSR.

2.4.2 Expected Time to Zero

Figure 2.10 shows the another monetary policy measure in the ZLB environment, Expected Time to Zero (ETZ). According to Krippner (2015), ETZ can provide an implied market-based expectation of when the actual short interest rate is expected to lift off from zero and return to its normal value in the non-ZLB environment.

Note that from Section 2.2.2, the expected path of shadow short interest rate is

$$\mathbb{E}_t^{\mathbb{Q}}(r_{t+\tau_0}) = L_t + e^{-\varphi\tau_0} S_t = 0$$

where the τ_0 is $\text{ETZ}_t = \tau_0$ and can be calculated from the estimated values of state variables L_t, S_t and parameter φ by solving the equation as $\tau_0 = -\frac{1}{\varphi}\ln\left(-\frac{L_t}{S_t}\right)$.

[30]In the non-ZLB environment, the SSR takes positive value which is same to the general short policy rate. The short policy rate can be controlled by the central bank through open market operations.

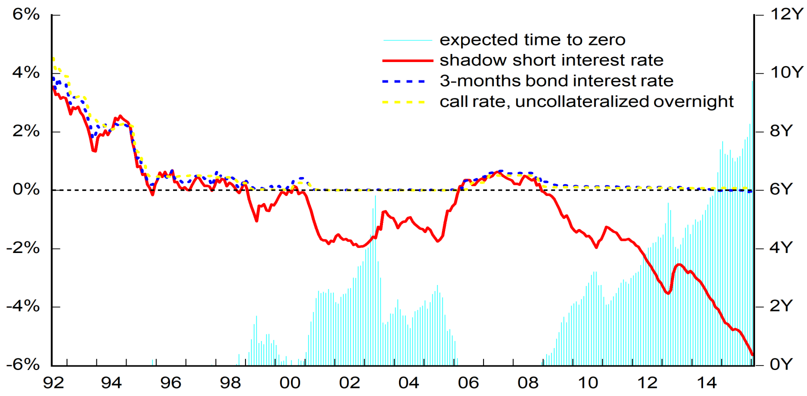


Figure 2.10a: ETZ from 1992M7 to 2016M1

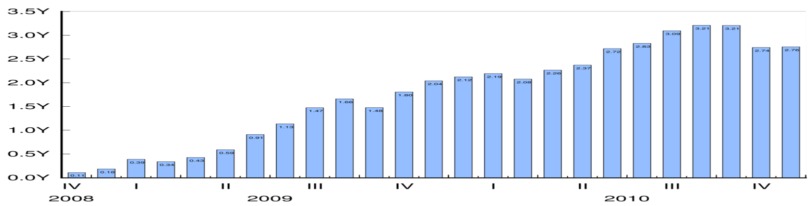


Figure 2.10b: ETZ from 2008M12 to 2010M12

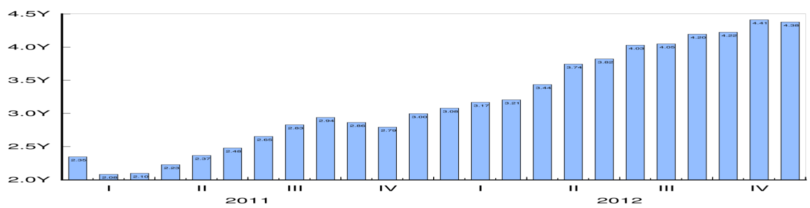


Figure 2.10c: ETZ from 2011M1 to 2012M12

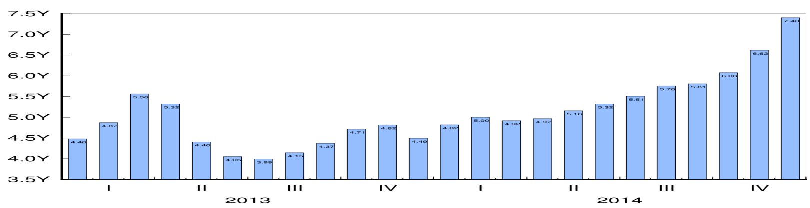


Figure 2.10d: ETZ from 20013M1 to 2014M12

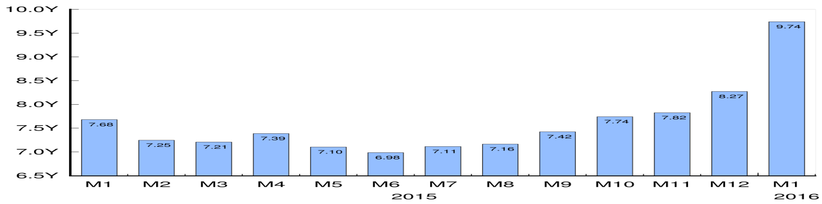


Figure 2.10e: ETZ from 2015M1 to 2016M1

Figure 2.10a provides the ETZ (right axis) in Japan. During the Zero Interest Rate Policy period , Qualitative Easing period and Comprehensive Monetary Easing period, the ETZ shows consistent path with the evolution of monetary policy, especially the start and end of each policy. Since the QQE is still ongoing, the ETZ is a good measure providing the information about how long economic agents are likely to face the ZLB in the future and when and how the central bank should consider the exit strategy or forward guidance. For example, at the end point of the sample, 2016M1, the ETZ is almost 10 years. Adding this expected horizon of the ZLB environment provides a range from 2016M1 to 2026M1. This range may be the reference for BoJ's forward guidance about the time point of the end of the ZLB environment. Though BoJ doesn't officially announce the time point of the end of the ZLB environment, the ETZ can still provide the market-implied expectation about how long the ZLB will continue. Since there doesn't exist market survey about this expectation, the ETZ is the only available measure if we want to know it. From Figure 2.10, we can imply that the ZLB environment in Japan will last for quite long time, especially from 2013M4, the start of first phase of QQE. There exists an obvious trend that the ETZ will increase with the deepening of QQE.

2.4.3 Effective Monetary Stimulus

Before we introduce the concept of Effective Monetary Stimulus (EMS), let we firstly consider the neutral interest rate of economy. The neutral interest rate is the short interest rate when the economy will achieve its long-run equilibrium level, the balance of investment and saving. If policy rate is equal to the neutral interest rate, the monetary policy is neutral to economy, neither easy and stimulative, nor tight and suppressive. In the non-ZLB environment, if the policy rate is below the neutral rate with an expectation that the policy rate will finally revert to the neutral rate level as time evolves, the EMS can be calculated as the area between the expected path of the short interest rate and the neutral interest rate. In the ZLB environment, the explicit stance of monetary policy is a zero interest rate policy but with the expectation that the short interest rate will rise to normal positive level at some horizon in the future and finally return to its neutral level. The

EMS has consistent interpretation in both non-ZLB and ZLB environment.

Now we explain the EMS. We have already showed that the short interest rate $r_t = L_t + S_t$ has its expected path $\mathbb{E}_t^{\mathbb{Q}}(r_{t+\tau}|x_t) = L_t + e^{-\varphi\tau}S_t$ in Section 2.2.3. For the long-horizon $\tau \to \infty$, the expectation of the shadow short interest rate is

$$\lim_{\tau\to\infty} \mathbb{E}_t^{\mathbb{Q}}(r_{t+\tau}|x_t) = L_t$$

and the expected path of short interest rate relative to its long-run expected value is

$$\lim_{\tau\to\infty} \mathbb{E}_t^{\mathbb{Q}}(r_{t+\tau}|x_t) - \max\{0, \mathbb{E}_t^{\mathbb{Q}}(r_{t+\tau}|x_t)\} = \begin{cases} L_t, \text{if } \mathbb{E}_t^{\mathbb{Q}}(r_{t+\tau}|x_t) < 0 \\ -e^{-\varphi\tau}S_t, \text{if } \mathbb{E}_t^{\mathbb{Q}}(r_{t+\tau}|x_t) \geq 0 \end{cases}$$

where L_t is the estimate of the neutral interest rate at time t. The stance of monetary policy can be identified from this policy-neutral interest rate gap. The EMS is the area that can be calculated from the integral of this gap with respect to τ with a range $(0, \infty)$.

$$EMS_t = \int_0^\infty \left(L_t - \max\{0, \mathbb{E}_t^{\mathbb{Q}}(r_{t+\tau}|x_t)\}\right) d\tau$$

If $\mathbb{E}_t^{\mathbb{Q}}(r_{t+\tau}|x_t) \geq 0$ at all horizons $\tau \in (0, \infty)$, the EMS is

$$EMS_t = \int_0^\infty \left(-e^{-\varphi\tau}S_t\right) d\tau = -\frac{S_t}{\varphi}$$

and if $\mathbb{E}_t^{\mathbb{Q}}(r_{t+\tau}|x_t) < 0$, for $\mathbb{E}_t^{\mathbb{Q}}(r_{t+\tau}|x_t)$ is a monotonic function of horizon τ, if $\mathbb{E}_t^{\mathbb{Q}}(r_{t+\tau}|x_t)$ has an intersection with zero at horizon τ_0, it has only one intersection and $\mathbb{E}_t^{\mathbb{Q}}(r_{t+\tau}|x_t) < 0$ holds for values of $\tau < \tau_0$ when $r_t = S_t + L_t < 0$. τ_0 can be solved as follows by setting $\mathbb{E}_t^{\mathbb{Q}}(r_{t+\tau}|x_t) = 0$.

$$L_t = -S_t e^{-\varphi\tau_0} \Rightarrow \tau_0 = -\frac{1}{\varphi}\log\left(-\frac{L_t}{S_t}\right)$$

The EMS in this case is given by following integral.

$$EMS_t = \int_0^{\tau_0} L_t d\tau + \int_{\tau_0}^\infty \left(-S_t e^{-\varphi\tau}\right) d\tau = \tau_0 L_t - \frac{S_t}{\varphi}e^{-\varphi\tau_0}$$

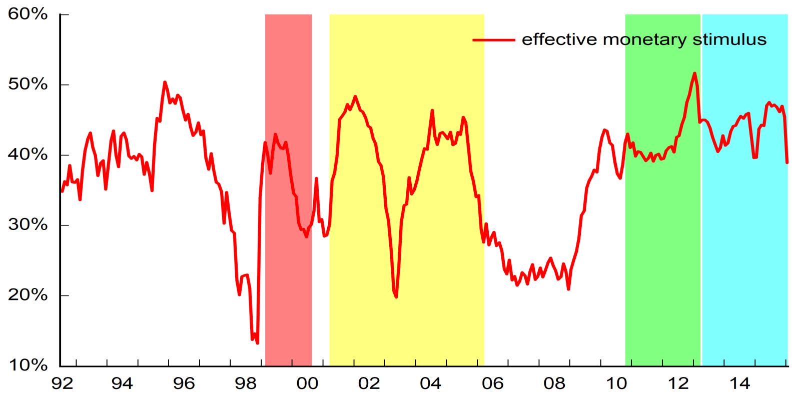


Figure 2.11: Effective Monetary Stimulus

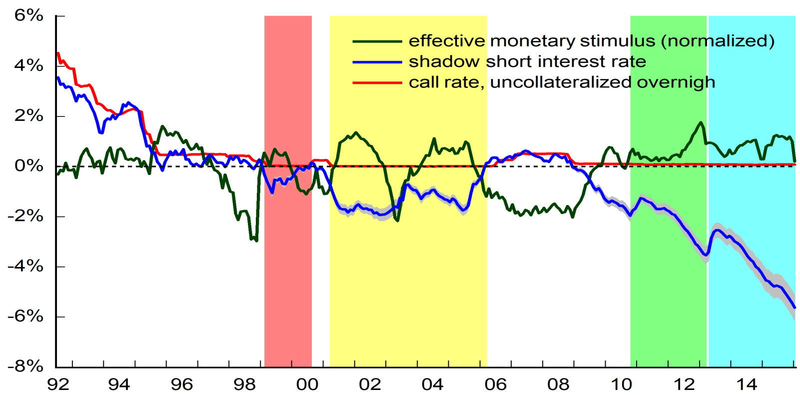


Figure 2.12: Effective Monetary Stimulus (normalized)

Figure 2.11 shows the monthly estimate of EMS. To get a intuitive view of the EMS, we normalized the monthly estimate of EMS by $\frac{EMS_t - mean(EMS_t)}{std.er(EMS_t)}$ and plot the SSR and call rate together in Figure 2.12, we can find that the SSR and EMS have a consistent relationship in the ZLB environment, lower SSR coincides with higher EMS, just like that the policy rate and EMS have in the non-ZLB environment. When SSR rises, the EMS decreases and vice versa. The estimated EMS shows very good traceability of monetary policy in Japan. We also calculated the mean of EMS in different schemes of monetary policy. The mean of EMS in QE (and comprehensive monetary easing) from 2001M3 to 2013M3 is 36% and the mean of EMS in QQE from 2013M4 to 2016M1 is 44%, which shows the QQE is more aggressive than the previous QE, but only 8% of increasing. We can conclude that both EMS and ETZ show very accurate and consistent traceability of monetary policy.

$x_t = \beta_0 + \beta_1 x_{t-1} + \beta_2 EMS_{t-3} + \varepsilon_t$				
Coefficient	Point Estimate	Std.Error	t-Statistic	P-value
β_0	-0.741^{***}	0.180	-4.116	0.0005
β_1	0.574^{***}	0.055	10.525	0.0000
β_2	0.686^{***}	0.189	3.638	0.0015
$R^2 = 0.86$		Sample: 2010Q1 2016Q1		
x_t = Output Gap		EMS_t = normalized EMS		

Table 2.4: Regression of Output Gap on EMS

$x_t = \beta_0 + \beta_1 x_{t-1} + \beta_2 EMS_{t-3} + \varepsilon_t$				
Coefficient	Point Estimate	Std.Error	t-Statistic	P-value
β_0	-0.164	0.115	-1.422	0.1689
β_1	0.210^{*}	0.111	1.896	0.0711
β_2	0.297^{**}	0.139	2.130	0.0446
$R^2 = 0.16$		Sample: 2010Q1 2016Q1		
x_t = Inflation		EMS_t = normalized EMS		

Table 2.5: Regression of Inflation on EMS

If the stimulative degree of monetary policy can be represented by EMS, what is the empirical relationship between EMS and macroeconomic variables[31]? To figure out this question, we run simple regressions of output gap and inflation on normalized EMS. As the results of estimation showed in Table 2.4, from 2010Q1 to 2016Q1, the stimulative degree of monetary policy represented by normalized EMS has a positive effect on output gap. 1% increasing of the EMS leads to 0.686% increasing of output gap. We also tried the regression of normalized EMS and GDP deflator-based inflation. From Table 2.5, we can find that 1% increasing of the EMS leads to 0.297% increasing of inflation, which is less than the effect of EMS on output gap. This may imply that the unconventional monetary policy of BoJ is more sensitive to output, but less sensitive to inflation.

[31]The data of output gap and inflation is same as that used in the estimation of NK-DSGE and TVP-SV VAR model in Chapter 3. Please refer to Figure 3.3.

2.5 Concluding Remarks

In this chapter, we estimated a two-factor shadow/ZLB-GATSM that can be used to model short interest rate in the ZLB environment. We can derive three useful monetary policy measures from this model which can provide the consistent view of the stance of monetary policy in Japan.

The SSR acts as a proxy of general short interest rate in the ZLB environment. We can find that the SSR has already decreased to -6%, which means that the value of holding physical currency is +6%. The ETZ shows that the expected horizon of the ZLB in Japan will last for at least 8 years at the end of 2015. As far as we are concerned, this research firstly provided the answer about how long the ZLB will last in Japan because there doesn't exist any research or survey about this expectation of the market. The EMS provides a consistent way to track the stance of monetary policy in Japan. Generally, lower SSR means further stimulus of monetary policy, but from the Figure 2.11 and Figure 2.12, we can find that the peak stimulus of QQE is not stronger than the first time of monetary easing conducted by BoJ from 2001 to 2006, but the mean of EMS in QQE is still 8% larger than the mean of EMS in previous scheme of policy. This may imply that the monetary easing policy, large-scale purchasing of the government bond and increasing money supply, is more aggressive than before, but approaching its limit, no matter how many government bond have been purchased, the whole volume of government bond is finite and it is not far from the limit of QQE. We also found that the stimulative degree of monetary policy represented by EMS has a positive effect (+0.686%) on output gap and a relatively small positive effect (+0.294%) on inflation averagely from 2010Q1 to 2016Q1. Note that we have already confirmed that the SSR and EMS can be used as measures of monetary policy, these measures can be directly used in other econometric procedures to evaluate the effects of monetary policy.

We also expanded the estimation period from 2016/1/29, the start of QQE with a negative interest rate policy to latest available data, but the mathematical specification of the model in this chapter has some defects when dealing with negative values of interest rate data. The modification is still ongoing. This chapter provides purely empirical results from a two-factor shadow/ZLB-GATSM and we will use the results from this chapter for further research.

Chapter 3

Empirical Investigation of Shadow Rate

3.1 Introduction

When the short nominal interest rate is at or near zero, central banks have to face the problems incurred by the ZLB because the ZLB invalidates the implementation of conventional monetary policy, the adjustment of short policy rate. Facing the constraint of ZLB, central banks conduct unconventional monetary policy to stabilize and stimulate the economy. This is what Japan economy has experienced and Bank of Japan has done since 1999 when the call rate decreased to a very low level near zero. The Great Recession incurred by the 2007-2008 global financial crisis brought the same problem to US, UK and Euro area. Besides the practical policy issues faced by the monetary authorities in advanced economies, the ZLB and the related unconventional monetary policy also pose academic issues and new challenges for macroeconomic research.

New Keynesian Dynamic Stochastic General Equilibrium (NK-DSGE) model is one of the main workhorses in modern monetary macroeconomics. But when the ZLB constraint on the short nominal interest rate binds, unfortunately, NK-DSGE models have deficient performance and undesired economic implications, leading to implausible and weird policy paradoxes.

As a standard methodology for monetary policy analysis, NK-DSGE model in the ZLB environment predicts that positive temporary supply shocks have contractionary effects and vice versa, negative supply shocks have expansionary effects. Also, fiscal and forward guidance multipliers can be implausibly larger than one. All these conclusions from the standard NK-DSGE models are inconsistent with economic intuition and empirical facts[1]. Besides the misleading policy implications, the ZLB also brings many technical problems in DSGE methodology. The explicit introduction of the ZLB constraint into DSGE models accompanies with structural break or nonlinear kink. Such kind of nonlinearity invalidates the linear

[1] Wieland (2015) and Garín et al. (2016) showed similar impulse responses of output to a supply shock in both ZLB and non-ZLB environment.

approximation and the Kalman filter. Some researchers use global projection method and the particle filter to deal with the nonlinearity in solution and estimation of DSGE models, but these methods are technically difficult and demand for numerous computation.

Wu and Zhang (2016) established the equivalence between shadow rate and unconventional monetary policy in a standard NK-DSGE model. The equivalence between shadow rate and unconventional monetary policy is established on the empirical findings that have shown the highly correlation between the quantity of government bond purchase and the estimated shadow rate. For the case of Bank of Japan, these empirical findings can be confirmed in Figure 2.8 and Figure 2.9. The shadow rate can take both positive and negative values and show consistent response to monetary policy events in both non-ZLB and ZLB environment. Introducing the shadow rate into a DSGE model can provide more insights for the propagation and amplification mechanism of unconventional monetary policy without introducing the complications incurred by the ZLB constraint.

We use the shadow rate estimated from a shadow/ZLB-GATSM in Chapter 2 as the data for the estimation of a NK-DSGE model where the general policy rate is replaced by the shadow rate during ZLB period. Then we use the NK-DSGE model with shadow rate to do some monetary policy analysis for Japan economy. This may be a circuitous route, but the logic is valid and consistent from the beginning to the end. Also, this approach salvages the DSGE models from the nonlinearity incurred by the ZLB. Standard procedures such as the linear approximation and the Kalman filter can be used instead of complicated nonlinear solution and estimation techniques.

To check the applicability of shadow rate in the non-structural econometric model, we also estimate a Time Varying Parameter-Stochastic Volatility Vector Autoregression (TVP-SV VAR) model with the same data used in previous DSGE analysis. By using TVP-SV VAR model, we can plot the impulse response function for each time point to check the empirical dynamic relationship of macroeconomic variable and monetary policy.

The remaining of this paper is organized as follows. Section 3.2 shows the equivalence between shadow rate and unconventional monetary policy. In Section 3.2.1, we estimate a NK-DSGE model with shadow rate. Section 3.2.2 checks the empirical results such as historical decomposition and impulse response from the estimated NK-DSGE model. Section 3.3 shows another similar analysis by TVP-SV VAR model. This exercise also proves that even though the shadow rate during ZLB period is negative, it is still robust to use the shadow rate in the econometric procedures of monetary policy. Section 3.4 concludes this chapter and gives the prospect for further research.

3.2 Shadow Rate in NK-DSGE Model

According to the empirical evidence of shadow rate presented in Section 2.4.1, we introduce shadow rate into the standard NK-DSGE model. For general NK-DSGE model, the economic agents face risk-free short rate r_t and hold risk-free bond. r_t is generally recognized as the short policy rate which can be controlled by the central bank. In actual, the relevant interest rates affecting economic agents' decisions are private interest rates r_t^B, through which both conventional and unconventional monetary policies transmit into the economy.

Generally, the private interest rates r_t^B can be represented as the sum of risk-free short rate r_t plus a time-varying risk premium r_t^P

$$r_t^B = r_t + r_t^P$$

where r_t is assumed that can be adjusted by the conventional monetary policy of the central bank. Empirical works such as Gagnon et al. (2011), Krishnamurthy and Vissing-Jorgensen (2012) and Hamilton and Wu (2012) advocate that the large-scale asset purchase by the central banks can reduce the risk premium which means

$$\frac{\partial r_t^P}{\partial b_t^G} < 0$$

where b_t^G is the log of bond holdings of the central bank. This is known as the risk premium channel of QE.

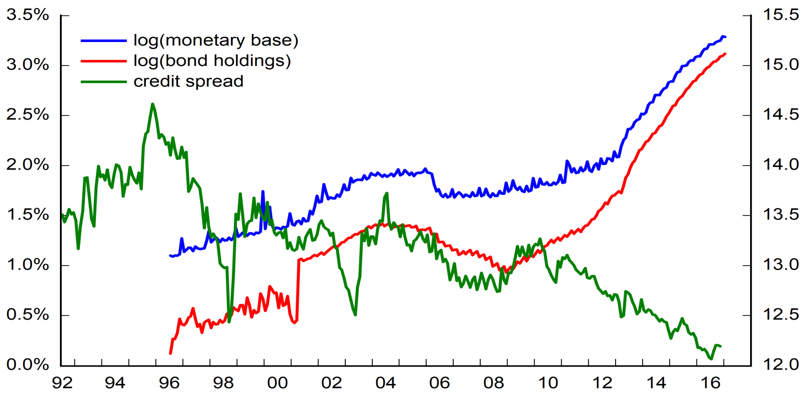


Figure 3.1: Credit Spread and Balance Sheet of BoJ

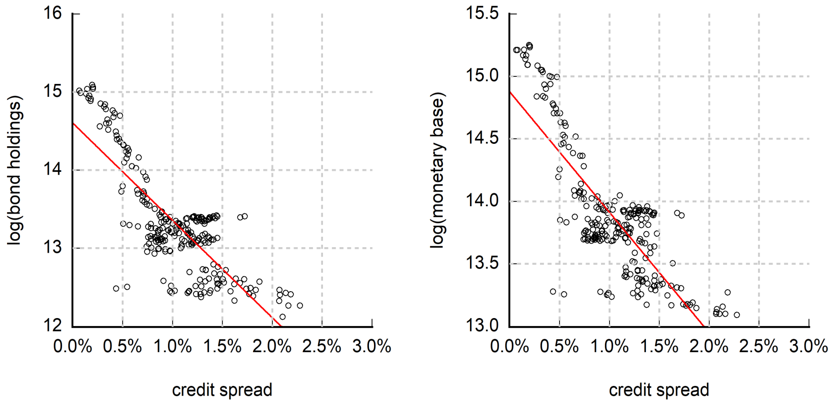


Figure 3.2: Credit Spread and Balance Sheet of BoJ (scatter plot)

The Figure 3.1 and Figure 3.2 show the relation between credit spread and the balance sheet of BoJ. The credit spread used here is defined as the difference between the S&P Japan Corporate Bond Index and S&P Japan Government Bond Index. The correlation of credit spread and bond holdings (in log term) is -0.80 and the correlation of credit spread and monetary base (in log term) is -0.81.

According to the regression lines in Figure 3.2, we assume that the response of risk premium r_t^P to bond holdings b_t^G follows a simple linear form

$$r_t^P = r^P - \gamma \left(b_t^G - b^G \right) + \varepsilon_t^P$$

where $-\gamma = \frac{\partial r_t^P}{\partial b_t^G} < 0$, r^P is the constant component of risk premium and ε_t^P is the exogenous time-varying component of risk premium which is interpreted as the liquidity preference shock in Campbell et al. (2017). In the non-ZLB environment, $b_t^G = b^G$, $r_t^P = r^P + \varepsilon_t^P$ such that

$$r_t^B = r_t + r_t^P = r_t + r^P + \varepsilon_t^P$$

which means that the private interest rate is the short rate controlled by the central bank plus risk premium. When r_t is restricted by the ZLB, approximately $r_t = 0$ and

$$r_t^B = r^P - \gamma \left(b_t^G - b^G \right) + \varepsilon_t^P$$

through which the unconventional monetary policy affects risk premium to reduce private interest rate and stimulate the economy. According to the empirical evidence of the shadow rate showed by Figure 2.8 and Figure 2.9, we also assume that the shadow rate has a same response to the log of bond holdings in a linear form like

$$s_t = -\gamma \left(b_t^G - b^G \right)$$

then

$$r_t^B = s_t + r^P + \varepsilon_t^P$$

can capture the both conventional and unconventional monetary policies.

In the non-ZLB environment, $s_t = r_t > 0$, $b_t^G = b^G$ and $r_t^B = r_t + r^P + \varepsilon_t^P$, the New Keynesian IS curve[2] is

$$\begin{aligned}
x_t &= \mathbb{E}_t x_{t+1} - \frac{1}{\sigma}\left(r_t^B - \mathbb{E}_t \pi_{t+1} - r_t^N\right) \\
&= \mathbb{E}_t x_{t+1} - \frac{1}{\sigma}\left(r_t + r^P + \varepsilon_t^P - \mathbb{E}_t \pi_{t+1} - r_t^N\right) \\
&= \mathbb{E}_t x_{t+1} - \frac{1}{\sigma}\left(r_t - \mathbb{E}_t \pi_{t+1}\right) + \varepsilon_t^x \\
&= \mathbb{E}_t x_{t+1} - \frac{1}{\sigma}\left(s_t - \mathbb{E}_t \pi_{t+1}\right) + \varepsilon_t^x
\end{aligned}$$

where $\varepsilon_t^x = -\frac{1}{\sigma}\left(r^P + \varepsilon_t^P - r_t^N\right)$ is a compound of exogenous shocks. The risk premium shock ε_t^P and $r_t^N = -\ln\beta + \frac{\sigma(1+\eta)}{\sigma(1-\alpha)+\alpha+\eta}\left(\mathbb{E}_t \hat{A}_{t+1} - \hat{A}_t\right)$ can't be identified separately, so we denote the compound of these exogenous shocks as a demand shock ε_t^x. In the ZLB environment, the New Keynesian IS curve is

$$\begin{aligned}
x_t &= \mathbb{E}_t x_{t+1} - \frac{1}{\sigma}\left(r_t^B - \mathbb{E}_t \pi_{t+1} - r_t^N\right) \\
&= \mathbb{E}_t x_{t+1} - \frac{1}{\sigma}\left(s_t + r^P + \varepsilon_t^P - \mathbb{E}_t \pi_{t+1} - r_t^N\right) \\
&= \mathbb{E}_t x_{t+1} - \frac{1}{\sigma}\left(s_t - \mathbb{E}_t \pi_{t+1}\right) + \varepsilon_t^x
\end{aligned}$$

which is same as its counterpart in the non-ZLB environment. Finally, we define a *Taylor rule* of shadow rate

$$s_t = \varphi_s s_{t-1} + (1-\varphi_s)\left(\varphi_x x_t + \varphi_\pi \pi_t\right) + \varepsilon_t^s$$

where ε_t^s is the monetary policy shock and φ_s is a smoothing parameter of interest rate. $\varphi_\pi > 1$ guarantees the existence a unique, non-explosive equilibrium[3].

3.2.1 Estimation of NK-DSGE Model with Shadow Rate

From the analysis in Section 3.2, we can find the NK-DSGE model with shadow rate has the same formulation in both ZLB and non-ZLB environment. Because we have three observable variables, output gap, inflation

[2] For the derivation of NK-DSGE model, please refer to Appendix for Chapter 1.
[3] See Bullard and Mitra (2002) for a proof.

rate and shadow rate, we add a shock term to the New Keynesian Phillips Curve to avoid the stochastic singularity in estimation[4].

$$
\begin{aligned}
x_t &= \mathbb{E}_t x_{t+1} - \frac{1}{\sigma}\left(s_t - \mathbb{E}_t \pi_{t+1}\right) + \varepsilon_t^x \\
\pi_t &= \beta \mathbb{E}_t \pi_{t+1} + \kappa x_t + \varepsilon_t^\pi \\
s_t &= \varphi_s s_{t-1} + (1-\varphi_s)\left(\varphi_x x_t + \varphi_\pi \pi_t\right) + \varepsilon_t^s
\end{aligned}
$$

All shocks follow autoregressive processes

$$
\varepsilon_t^{\text{shock}} = \rho_{\text{shock}} \varepsilon_{t-1}^{\text{shock}} + \mu_t^{\text{shock}}, \ \text{shock} \in (x, \pi, s)
$$

where $\mu_t^{\text{shock}} \sim \mathcal{N}\left(0, \sigma^2_{\text{shock}}\right)$ is normal-distributed exogenous innovation term.

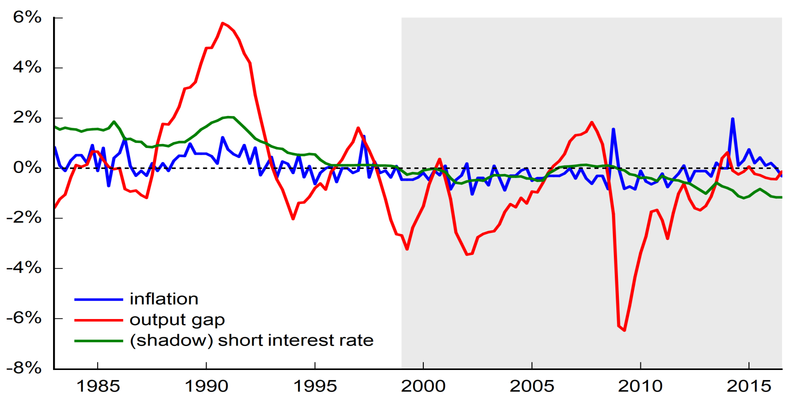


Figure 3.3: Data for the Estimation of NK-DSGE Model

The data used for output gap x_t which is official estimate[5] obtained from BoJ is from 1983Q1 to 2016Q3. The data series of inflation π_t is GDP deflator-based inflation rate from 1980Q3 to 2016Q3. The data series of shadow rate is from 1999Q1 to 2016Q3. For non-ZLB period from 1983Q1 to 1998Q4, the shadow rate is replaced by the non-ZLB constrained call rate to complete the full data series of interest rate.

Some structural parameters are calibrated[6] as $\alpha = 0$, $\beta = 0.9975$, $\varepsilon = 6$

[4]According to Galí (2014, chapter 5), the shock term ε_t^π can be explained as a cost-push shock which may come from the exogenous variations in desired price markups or exogenous variations in wage markups.

[5]This series can be downloaded from https://www.boj.or.jp/research/research_data/gap/index.htm/.

[6]The elasticity of substitution ε is calibrated to be 6 which means an average 20% markup charged by intermediate good firms at steady state. σ and η are calibrated at 1 because these parameters can't be identified in the estimation. $\alpha = 0$ means the model economy has the constant scale to return.

and $\sigma = \eta = 1$. κ is a composite of other structural parameters and we specify its prior distribution as non-informative uniform prior distribution $U(0,1)$. The prior and posterior distributions of parameters and standard deviations are given in Table 3.1. We estimate the NK-DSGE in a DSGE-VAR[7] style to compare the theoretical impulse response from DSGE model and corresponding empirical impulse response from Bayesian VAR model. The basic idea of the DSGE-VAR(λ) is to use the implied moments of a DSGE model as the prior distribution for a Bayesian VAR model. When choosing the prior distribution of a Bayesian VAR model, λ is the weight of this constraint of moments implied by the DSGE model. Following Adjemian et al. (2008), we treat λ as a parameter which can be jointly estimated from the Bayesian estimation of other structural parameters and specify the prior distribution of λ as non-informative uniform prior distribution $U(0,2)$. Also, we compare the impulse response and calculate the historical decomposition to check the contribution of the shadow rate monetary policy shock to output gap and inflation.

	Prior Distribution			Posterior Distribution		
Parameters	Mean	St.Dev.	Prior	Mean	St.Dev.	90% HPD Interval
φ_π	1.5	0.1	G	1.4895	0.1264	[1.2786, 1.6923]
φ_x	0.375	0.1	G	0.5896	0.1994	[0.2472, 0.9084]
φ_s	0.8	0.1	B	0.8088	0.0341	[0.7557, 0.8669]
ρ_π	0.8	0.1	B	0.4835	0.1393	[0.2694, 0.7190]
ρ_x	0.8	0.1	B	0.7713	0.0800	[0.6460, 0.9060]
ρ_s	0.8	0.1	B	0.4868	0.0989	[0.3250, 0.6457]
μ^π	0.5	0.5	IG	0.2011	0.0353	[0.1475, 0.2613]
μ^x	0.5	0.5	IG	0.2057	0.0318	[0.1491, 0.2517]
μ^s	0.5	0.5	IG	0.1347	0.0160	[0.1096, 0.1611]
Parameters	Prior Distribution			Mean	St.Dev.	90% HPD Interval
κ	U(0, 1)			0.1920	0.1741	[-0.0039, 0.4726]
λ	U(0, 2)			0.5174	0.1001	[0.3518, 0.6619]

Table 3.1: Prior and Posterior Distribution of Structural Parameters

The posterior distribution of structural parameters are consistent with most of related literature on the NK-DSGE estimation with non-ZLB constrained policy rate. For example, given posterior mean of κ and other calibrated parameters, we can calculate θ as $\theta = \frac{1}{2}\left(\Theta - \sqrt{\Theta^2 - 4}\right)$ where $\Theta = 1 + \frac{1}{\beta} + \frac{\kappa}{\beta(\eta+\sigma)}$. θ is the probability that an intermediate good firm can't optimize its price in the Calvo (1983) framework. We find that $\theta = 0.7313$

[7] Please refer to Del Negro and Shorfheide (2004), Adjemian et al. (2008), Del negro et al. (2007) for the technical details of DSGE-VAR model.

means the duration of price optimization is almost 11 months (3.7 quarters), which is quite reasonable. The multivariate convergence diagnostic also shows good convergence for all parameters. In Appendix for Chapter 3, as a check of robustness, we also give the estimation results of a 1999Q1-2016Q3 sub-sample. The structural parameters still have reasonable values and impulse response of monetary policy also shows consistent dynamics of variables. But if we use the ZLB-constrained call rate for subsample estimation in the same period, we can't get any reasonable results because it is impossible for a linear NK-DSGE model, without using nonlinear techniques, to capture the information of monetary policy from the ZLB-constrained call rate with less dynamics.

3.2.2 Application of NK-DSGE Model with Shadow Rate

Figure 3.4 gives the impulse response of monetary policy shock.

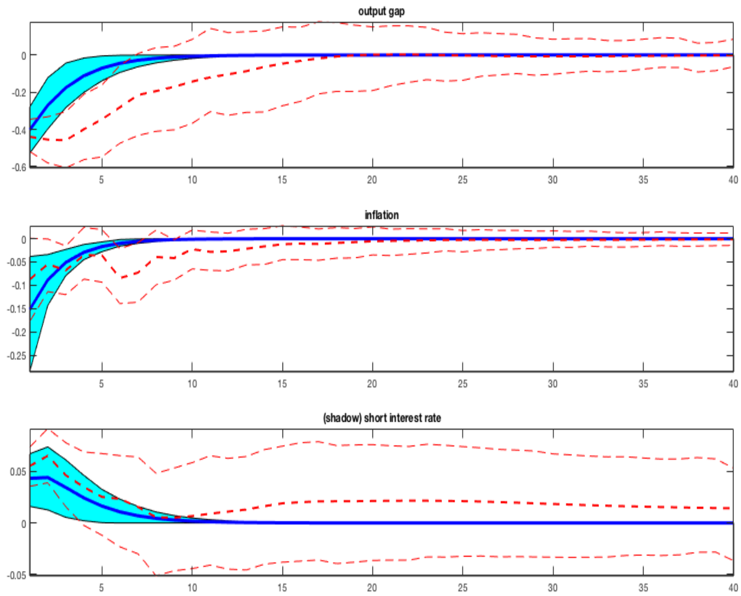


Figure 3.4: DSGE-VAR Impulse Response of Monetary Policy Shock

In Figure 3.4, the blue shaded area represents the 90% Highest Posterior Density Interval (HPDI) of posterior impulse response calculated from DSGE model. The thick blue line is the medium of posterior impulse response of DSGE model. The corresponding impulse response calculated from Bayesian VAR is also plotted in red dashed lines with 90% HPDI and the medium of impulse response inside the two red dashed lines. We can find the posterior impulse response from DSGE model and VAR model has the almost same dynamics, similar range and direction, especially for inflation and interest rate. For impulse response of output gap to monetary policy shock, both models show the output gap decreases due to the contractionary monetary policy shock.

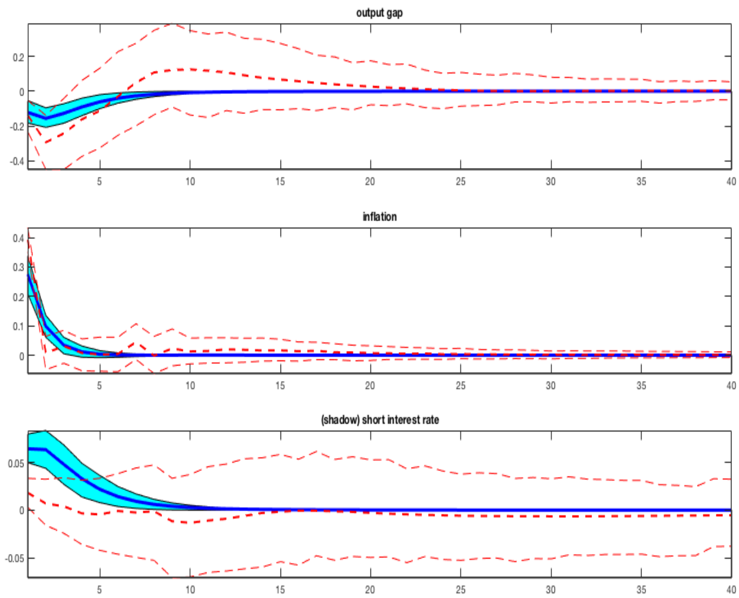


Figure 3.5: DSGE-VAR Impulse Response of Supply Shock

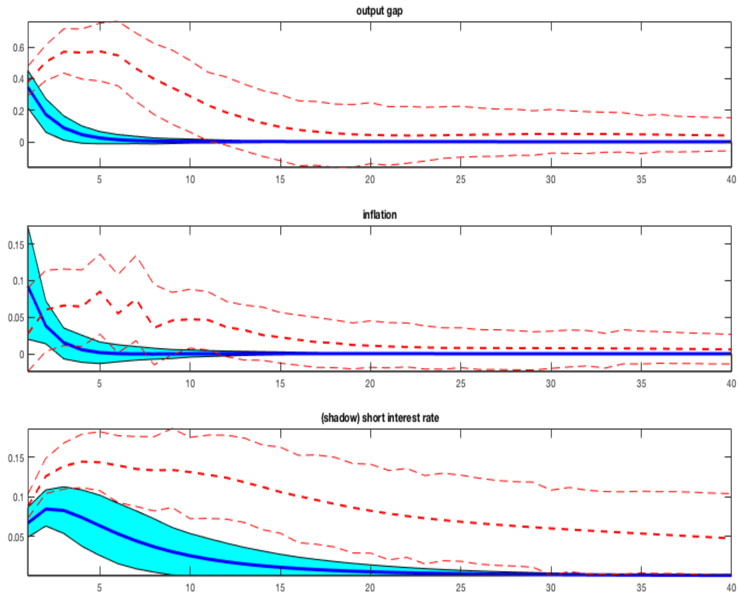


Figure 3.6: DSGE-VAR Impulse Response of Demand Shock

For supply shock, Figure 3.5 shows the consistent impulse response from both DSGE and VAR models. Figure 3.6 shows the impulse response triggered by demand shock. Though two kinds of impulse response are not completely consistent in the range, but the instantaneous response of each variable to demand shock still has the same direction. The impulse response analysis from DSGE-VAR model may imply that even if we introduce the shadow rate into estimate a structural DSGE model and estimate it, the impulse response still can be trusted in the sense that the DSGE-implied impulse response is similar to the non-structural VAR model which is estimated with the same data.

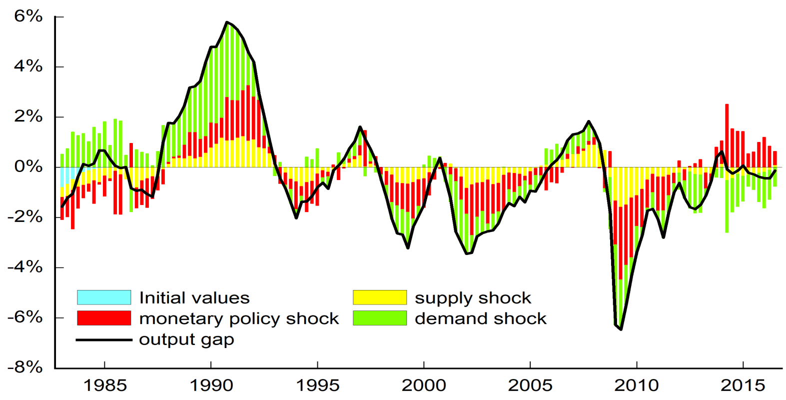


Figure 3.7: Historical Decomposition of Output Gap

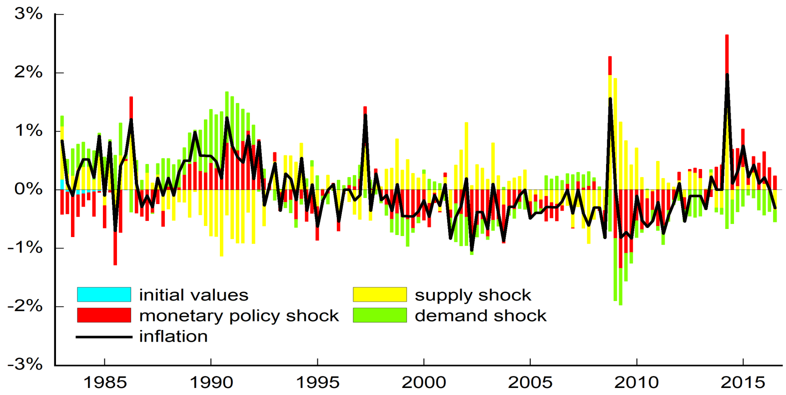


Figure 3.8: Historical Decomposition of Inflation

Figure 3.7 shows the historical decomposition of output gap where the red area shows the contribution of monetary policy shock. The contribution of monetary policy shock to the positive improvement of output gap since 2013Q3 is very obvious. Also, for the bubble economy of Japan from 1985-1990, the monetary policy had contribution of the positive output gap. The similar pattern can be confirmed in the Figure 3.8 where the monetary policy shock begun to have positive contribution to inflation since 2013Q3, but less significant than the contribution to the output gap. Recall the regression results showed by Table 2.4 and Table 2.5 in Section 2.4.3, the monetary policy stance represented by the normalized EMS is more sensitive to the output gap than to the inflation. This result is same in the historical decomposition of DSGE analysis.

3.3 Shadow Rate in TVP-SV VAR Model

Following Nakajima (2011) and Primiceri (2005), Del Nergo and Primiceri (2015)[8], we estimate a TVP-SV VAR model with 3 lags to check the impulse response of monetary policy shock at different time points. The dataset used here has 3 variables, output gap, inflation and short interest rate that are same as the data used in the estimation of DSGE-VAR model in Section 3.2.1. From Figure 3.9 and Figure 3.10, 1% increase in short interest rate leads to the decreasing of output gap and inflation. Time-varying impulse response shows that the decreasing of output gap in the period of 2007-2009, the global financial crisis, is more serious than other periods. The time-varying nature of the parameters capture the structure of economy very accurately.

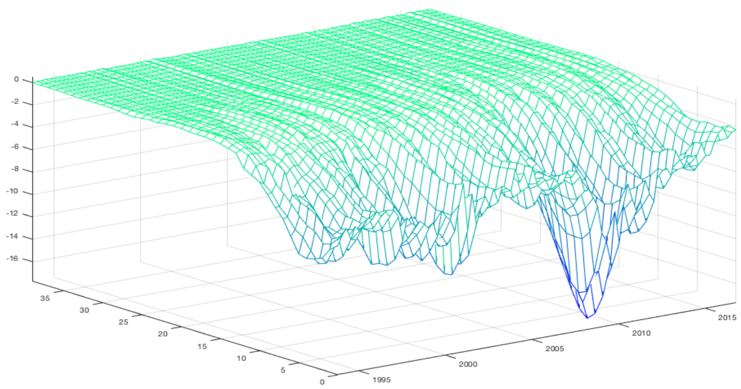


Figure 3.9: Impulse Response of Output Gap to Monetary Policy Shock

[8]Del Nergo and Primiceri (2015) corrects the misspecification of estimation algorithm in Primiceri (2005).

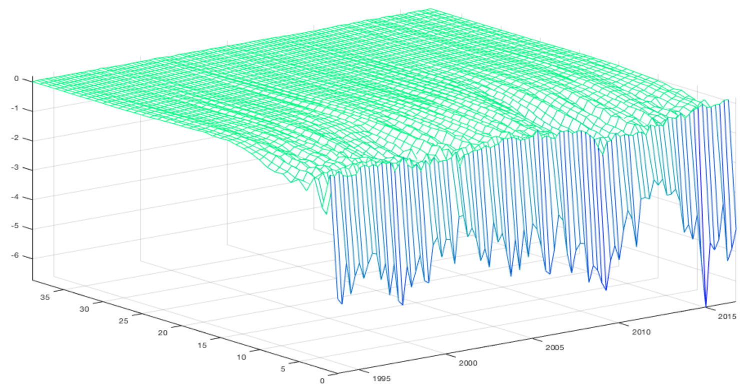


Figure 3.10: Impulse Response of Inflation to Monetary Policy Shock

Figure 3.10 shows the impulse response of inflation to monetary policy shock. Increasing of short interest rate leads to the decreasing of inflation. This is the standard result in such literature of monetary policy analysis with VAR models.

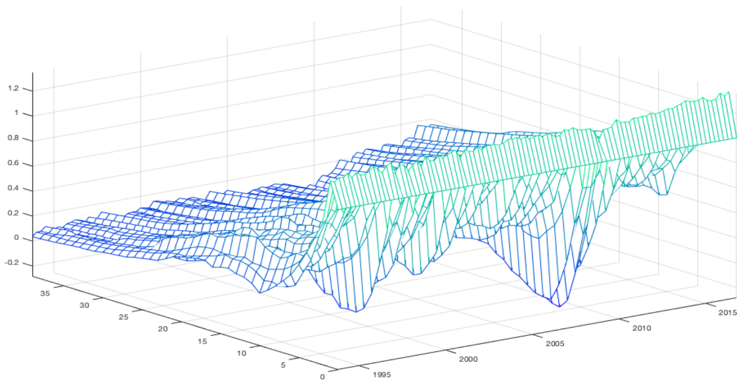


Figure 3.11: Impulse Response of Interest Rate to Monetary Policy Shock

Figure 3.12 shows the stochastic volatility of each structural shock. Note that the second peak in the structural shock of inflation comes from the increasing of consumption tax.

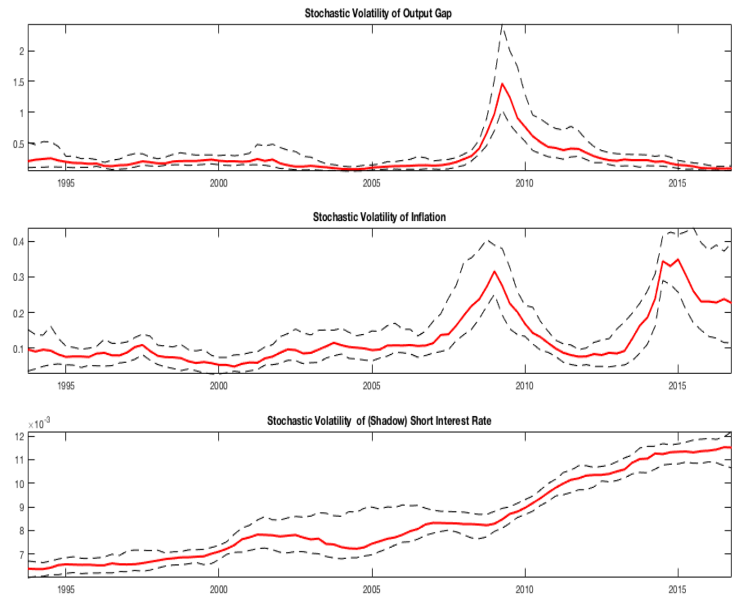


Figure 3.12: Time-Varying Stochastic Volatility of Structural Shock

From Section 3.3, we can find that the shadow rate can be used in non-structural VAR models. But we can't use the ZLB-constrained short policy rate in the similar exercise because data of ZLB-constrained is near zero without dynamics from where we can get the information of monetary policy stance. This is biggest advantage of using shadow rate as a measure of monetary policy in the econometric analysis.

3.4 Concluding Remarks

In this chapter, we adopt the shadow rate as a measure of monetary policy of BoJ because since 1999Q1 when the general policy rate of BoJ, call rate, has kept been near zero and already lost its function as a operating target for the conduct of monetary policy. The concept of shadow rate is not new, but for a long time, it is not widely used in macroeconomics. Since the ZLB has become a common issue for monetary authorities in the advanced economies, using the shadow rate to observe and analyze the monetary policy has been applied in many empirical works. We also confirmed that the shadow rate does have credible traceability of the BoJ's policy in the ZLB environment, providing us a new perspective to check the unconventional monetary policy of BoJ. Most of existing literatures use the shadow rate in reduced-form time series econometric models such as FAVAR model or TVP-VAR model to find the empirical evidences of unconventional monetary policy, but these models are not structural. The introduction of the shadow rate into the DSGE framework is a new attempt. Using the shadow rate does relieve us from the technical difficulties incurred by the ZLB, but whether this method is robust or not is still unclear. But at least, the structural parameters from the full-sample estimation have reasonable values, which is also true for the sub-sample estimation. Impulse response of monetary policy shock also shows consistent dynamics of macroeconomic variables. As far as we know, there doesn't exist other similar works that use the shadow rate in the estimation of DSGE model. However, the historical decomposition of output gap and inflation advocates the effectiveness of the monetary easing conducted by BoJ. The equivalence between shadow rate and the monetary easing policy which are assumed in this chapter are based on the empirical findings and these empirical evidences have been mapped into the DSGE model. Than we find that the impulse response functions of NK-DSGE model are similar to VAR model as its empirical counterpart in a DSGE-VAR framework. TVP-SV VAR model in Section 3.3 also proves the application of shadow rate in such kind of non-structural model is appropriate and quite robust.

For further research, we want to introduce the shadow rate to a Smets and Wouters (2003, 2007) type medium-scale DSGE model with more dynamics than the simple NK-DSGE model used in this chapter. Smets and Wouters (2003, 2007) type medium-scale DSGE model is the prototype of the DSGE models used in major central banks. If we can show that the shadow rate can be used in such medium-scale DSGE model, the shadow rate will be more valuable because how to deal with the ZLB-constrained short policy rate is the most difficult part in the DSGE estimation. If we can replace the ZLB-constrained short policy rate by the negative shadow rate and still get reasonable estimation of structural parameters and empirical results, the ZLB constraint will no longer be a problem. Also we shouldn't forget that the shadow rate only exists as a economic concept and the central bank can't control the shadow rate as a operating target, but we can use the information from it as a guidance for monetary policy operation.

Chapter 4

Estimation of Medium-Scale DSGE Model with Shadow Rate

4.1 Introduction

In recent years, since the nominal policy rates in major economies have been constraint by the ZLB and the policy regime has been changed from conventional monetary policy to unconventional monetary policy, how to evaluate the effects of unconventional monetary policy in the ZLB environment has been very important for macroeconomic research. Due to the difficulty and unreliability incurred by the nonlinearity of ZLB in linear solution and estimation techniques of DSGE models, researchers have made various types of attempts. One approach is to take the ZLB constraint explicitly by nonlinear techniques, but dealing with the ZLB by nonlinear techniques is much more demanding than widely-used linear techniques (Fernández-Villaverde (2015), Del Negro et al. (2015), Lindé et al (2016), Aruoba et al. (2017), Gust et al. (2017)), especially for medium-scale models. Another approach is to make some kind of compromise to allow a shortcut by abstracting from the ZLB or to assume that there doesn't exist significant structural break before and after the ZLB becomes binding (Benati (2008), Ireland (2011), Chen et al. (2012)).

Even in the real world, the short nominal interest rates are constrained by the ZLB and not allowed to be negative, many recent empirical works (Bullard (2012), Kim and Singleton (2012), Bauer and Rudebusch (2016), Christensen and Rudebusch (2014, 2016), Lombardi and Zhu (2014), Krippner (2015), Wu and Xia (2016)) use the shadow rate as a consistent and compatible measure of monetary policy stance to quantify the effects of unconventional monetary policy. The concept of shadow rate as a tractable method to account for the ZLB is proposed by Black (1995). The shadow rate estimated from shadow rate term structure model can take positive values that are same as the short nominal interest rates in non-ZLB period, but negative values in the ZLB environment when the short nominal interest rates are static and near zero.

Compared with the many empirical literatures, because the shadow rate is estimated from the reduced-form factor models that use two or three factors to fit the yield curve, it is difficult to incorporate the shadow rate into structural models in a structurally consistent fashion, there are few applications of the shadow rate in structural macroeconomic models. Wu and Zhang (2016) documented the strong empirical relationship between the shadow rate and unconventional monetary policy (quantitative easing and lending facilities) in the case of the United States. They also provide a theoretical foundation to introduce the shadow rate into the structural macroeconomic models. The counterintuitive puzzles such as large government spending multiplier and stimulative effect of negative supply shock in DSGE models with the presence of the ZLB also disappear in their framework. In Chapter 3 we confirmed similar empirical relationship in the case of Japan and also confirmed positive effects of unconventional monetary policy conducted by BoJ in a small-scale NK-DSGE model estimated by using Japan's output gap, inflation and shadow rate. Considering that the small-scale NK-DSGE model is highly stylized and has less dynamics than widely-used medium-scale DSGE model, we estimate a medium-scale NK-DSGE model to check the reliability of shadow rate's application in the estimation of DSGE models by comparing the structural parameters and model dynamics implied by estimated models with a pre-ZLB sub-sample (1980Q1-1998Q4) and a full-sample (1980Q1-2016Q3). We also conduct counterfactual simulation exercises to see the macroeconomic effects of unconventional monetary policy.

The remaining of this chapter is organized as follows. In Section 4.2, we specify a medium-scale NK-DSGE model. In Section 4.3, we compare the results in pre-ZLB sub-sample estimation and full-sample estimation. For the full-sample estimation, the data for nominal interest rate from 1999Q1 to 2016Q3 is replaced by the shadow rate that is estimated in Chapter 2. In Section 4.4, we follow the simulation methodology designed by Sarah and Jean-Guillaume (2016) to generate the path of macroeconomic variables in the absence of unconventional monetary policy. Finally, Section 4.5 concludes this chapter and gives the prospect for further research.

4.2 A Medium-Scale DSGE Model

According to the equivalence between the shadow rate and quantitative easing (QE) showed by Wu and Zhang (2016), there are two key points that we need to confirm if we want to introduce the shadow rate into DSGE models through the New Keynesian IS curve based on the theoretical foundation proposed by Wu and Zhang (2016).

1. The shadow rate has high correlation with QE variables such as the balance sheet of central bank.

2. The QE variables have high correlation with credit spread between risky assets such as corporate bond and safe assets such as government bond.

These empirical evidences are already confirmed by Figure 2.8, Figure 2.9 in Section 2.4.1 and Figure 3.1, Figure 3.2 in Section 3.2. Based on these empirical foundation, we proceed to build the medium-scale DSGE model.

The seminal medium-scale NK-DSGE models constructed by Christiano et al. (2005) and Smets and Wouters (2003, 2007) are the benchmark models used by the central banks for policy analysis and macroeconomic forecast. In this section, we build a medium-scale NK-DSGE model with features such as monopolistic competition, Calvo (1983) type price and wage rigidity, variable capital utilization, consumption habit formation. Additionally, stochastic balanced growth trend

$$\log Z_t = \log z + \log Z_{t-1} + \varepsilon_t^z$$

is introduced into the model to improve the data fit of long run economic growth and short run business cycles simultaneously. We define the gross growth rate of technology level as

$$\log \frac{Z_t}{Z_{t-1}} = \log z_t = \log z + \varepsilon_t^z$$

where ε_t^z is an exogenous technology shock process. We can rewrite this as $z_t = ze^{\varepsilon_t^z}$. From this result, we can find that at the steady state, the balanced growth rate in model economy is z. Under this specification, we don't need to isolate the trend and cycles of macroeconomic data, avoiding information loss due to the filtering of raw data.

4.2.1 Household

The representative household exists continuously in interval $h \in [0, 1]$ where h is an index. The utility function is additively separable. The representative household derives utility from consumption $C_{t,h} - \theta C_{t-1,h}$ and disutility from labor supply $L_{t,h}$, where θ is the degree of habit formation. Structural parameters β, σ and χ represent the discount factor, risk aversion and the inverse of labor supply elasticity respectively. ε_t^b and ε_t^l are a preference shock and a labor supply shock respectively, which affects the household's consumption demand and labor supply exogenously. $Z_t^{1-\sigma}$ multiplied on labor disutility $e^{\varepsilon_t^l} \frac{L_t^{1+\chi}}{1+\chi}$ is to satisfy the balanced growth path of model economy[1] .

$$E_t \sum_{t=0}^{\infty} \beta^t e^{\varepsilon_t^b} \left[\frac{(C_{t.h} - \theta C_{t-1,h})^{1-\sigma}}{1-\sigma} - e^{\varepsilon_t^l} \frac{Z_t^{1-\sigma} L_{t,h}^{1+\chi}}{1+\chi} \right]$$

[1]Please refer to Erceg et al. (2006) for the specification of utility function in DSGE models with balanced growth path. For the conditions of balanced growth path in macroeconomic models, please refer to King et al. (1988).

The budget constraint of the representative household is given as

$$C_{t,h} + I_{t,h} + \frac{B_{t,h}}{P_t} = \frac{R_{t-1}^N B_{t-1,h}}{P_t} + R_t^K U_{t,h} K_{t-1,h} + W_t L_{t,h} + \frac{T_{t,h}}{P_t}$$

where P_t is general price level in the model economy. R_t^K and W_t are competitive factor prices, capital rental and wage respectively. The representative household purchases one-period maturity government bond $B_{t,h}$ which has a nominal gross return R_t^N and receives a transfer $T_{t,h}$. The capital accumulation law is

$$K_{t,h} = [1 - \delta (U_{t,h})] K_{t-1,h} + \left[1 - \Gamma\left(\frac{I_{t,h}}{I_{t-1,h}} \frac{e^{\varepsilon_t^i}}{z}\right)\right] I_{t,h} \tag{4.1}$$

where $\Gamma\left(\frac{I_{t,h}}{I_{t-1,h}} \frac{e^{\varepsilon_t^i}}{z}\right) = \frac{1}{2\zeta}\left(\frac{I_{t,h}}{I_{t-1,h}} \frac{e^{\varepsilon_t^i}}{z} - 1\right)^2$ is a convex function of investment adjustment cost. Capital utilization $U_{t,h}$ is assumed to be variable which satisfies $\delta^{'}(\cdot) > 0$, $\delta^{''}(\cdot) > 0$. At the steady state, $\delta(U = 1) = \delta \in (0, 1)$ and we define $\mu = \delta^{'}(U = 1)/\delta^{''}(U = 1)$ as a steady state parameter. ε_t^i is an investment adjustment cost shock that affects the investment installation cost and then investment decision. The representative household optimizes life utility subject to budget constraint and capital accumulation law by deciding $C_{t-1,h}$, $C_{t,h}$, $B_{t,h}$, $U_{t,h}$, $K_{t,h}$ and $I_{t,h}$. Since all households are identical and make same optimal decisions, we can drop index h to aggregate the variables due to the symmetric equilibrium. The equilibrium conditions are given as follows.

$$\Lambda_t = e^{\varepsilon_t^b} (C_t - \theta C_{t-1})^{-\sigma} - \beta\theta E_t e^{\varepsilon_{t+1}^b} (C_{t+1} - C_t)^{-\sigma} \tag{4.2}$$

$$\Lambda_t = \beta E_t \Lambda_{t+1} \frac{R_t^N}{\Pi_{t+1}} \tag{4.3}$$

$$R_t^K = \frac{\Lambda_t^K}{\Lambda_t} \delta^{'}(U_t) = Q_t \delta^{'}(U_t) \tag{4.4}$$

$$Q_t = \beta E_t \frac{\Lambda_{t+1}}{\Lambda_t}\left[R_t^K U_{t+1} + Q_{t+1}(1 - \delta(U_{t+1}))\right] \tag{4.5}$$

$$\begin{aligned} 1 = Q_t &\left[1 - \Gamma\left(\frac{I_t}{I_{t-1}} \frac{e^{\varepsilon_t^i}}{z}\right) - \Gamma^{'}\left(\frac{I_t}{I_{t-1}} \frac{e^{\varepsilon_t^i}}{z}\right) \frac{I_t}{I_{t-1}} \frac{e^{\varepsilon_t^i}}{z}\right] \\ &+ \beta E_t \frac{\Lambda_{t+1}}{\Lambda_t} Q_{t+1} \Gamma^{'}\left(\frac{I_{t+1}}{I_t} \frac{e^{\varepsilon_{t+1}^i}}{z}\right)\left(\frac{I_{t+1}}{I_t}\right)^2 \frac{e^{\varepsilon_{t+1}^i}}{z} \end{aligned} \tag{4.6}$$

Λ_t and Λ_t^K are the Lagrange multipliers associated with budget constraint and capital accumulation law respectively. The Tobin's Q can be defined as $Q_t = \Lambda_t^K/\Lambda_t$ which represents real capital price measured by household's consumption marginal utility. $\Pi_{t+1} = P_{t+1}/P_t$ is the gross inflation rate from period t to $t + 1$. Households provide heterogenous kinds of labor to intermediate good firms. The wage rigidity is introduced according to Erceg et al. (2000). An intermediate good firm f aggregate the heterogenous kinds of labor $L_{t,(f,h)}$ to homogenous labor $L_{t,f}$ by a Dixit-Sriglitz production technology

$$L_{t,f} = \left(\int_0^1 L_{t,(f,h)}^{\frac{1}{1+\lambda_t^w}} dh \right)^{1+\lambda_t^w}$$

where λ_t^w is the wage markup of heterogenous labors, which has a relationship $\lambda_t^w = \frac{1}{\theta_t^w - 1}$ with elasticity of substitute of heterogenous labors θ_t^w. Intermediate good firm minimizes the cost of labor inputs $\int_0^1 L_{t,(f,h)} W_{t,(f,h)} dh$ subject to its labor aggregate technology. Since all intermediate good firms are assumed to make same optimal decision, so the index of intermediate good firm f can be omitted here. The first order conditions are

$$W_t = \left(\int_0^1 W_{t,h}^{-\frac{1}{\lambda_t^w}} dh \right)^{-\lambda_t^w}$$

$$L_{t,h} = \left(\frac{W_{t,h}}{W_t} \right)^{-\frac{1+\lambda_t^w}{\lambda_t^w}} L_t$$

where W_t is the aggregate level wage. For each period, $1 - \xi_w$ fraction of all households can optimize their wage $W_{t,h}$. ξ_w fraction of all households can't optimize the wage and just index the wage according to

$$P_{t+j} W_{t+j,h} = z \Pi_{t+j-1}^{\gamma_w} \Pi^{1-\gamma_w} P_{t+j-1} W_{t+j-1,h} \Rightarrow W_{t+j,h} = z^j W_{t,h} \left[\prod_{k=1}^{j} \left(\frac{\Pi_{t+k-1}}{\Pi} \right)^{\gamma_w} \frac{\Pi}{\Pi_{t+k}} \right]$$

where Π is the gross inflation rate at steady state and γ_w is the wage indexation weight on the gross inflation rate of previous period. Note that z^j means that the wage will increase at each period with a growth rate z from period t to period $t+j$. The representative household has a probability $1 - \xi_w$ to set optimal wage to maximize the lifetime utility at each period. So the optimization problem is

$$\max E_t \sum_{j=0}^{\infty} (\beta \xi_w)^j \left[\Lambda_{t+j} L_{t+j,h} z^j W_{t,h} \left[\prod_{k=1}^{j} \left(\frac{\Pi_{t+k-1}}{\Pi} \right)^{\gamma_w} \frac{\Pi}{\Pi_{t+k}} \right] - U_{L,t+j} \right]$$

$$\text{s.t. } L_{t+j,h} = \left[\frac{1}{W_{t+j}} z^j W_{t,h} \left[\prod_{k=1}^{j} \left(\frac{\Pi_{t+k-1}}{\Pi} \right)^{\gamma_w} \frac{\Pi}{\Pi_{t+k}} \right] \right]^{-\frac{1+\lambda_{t+j}^w}{\lambda_{t+j}^w}} L_{t+j}$$

where $U_{L,t+j} = e^{\varepsilon_{t+j}^l + \varepsilon_{t+j}^b} \frac{Z_{t+j}^{1-\sigma}}{1+\chi} L_{t+j,h}^{1+\chi}$ represents the disutility from labor supply. The first order condition is

$$E_t \sum_{j=0}^{\infty} \left\{ \begin{array}{l} (\beta \xi_w)^j \dfrac{\Lambda_{t+j} L_{t+j}}{\lambda_{t+j}^w} \left[\dfrac{z^j W_t^*}{W_{t+j}} \left[\prod_{k=1}^{j} \left(\dfrac{\Pi_{t+k-1}}{\Pi} \right)^{\gamma_w} \dfrac{\Pi}{\Pi_{t+k}} \right] \right]^{-\frac{1}{\lambda_{t+j}^w} - 1} \\ \times \left[\begin{array}{l} z^j W_t^* \left[\prod_{k=1}^{j} \left(\dfrac{\Pi_{t+k-1}}{\Pi} \right)^{\gamma_w} \dfrac{\Pi}{\Pi_{t+k}} \right] - \left(1 + \lambda_{t+j}^w \right) \dfrac{e^{\varepsilon_{t+j}^b + \varepsilon_{t+j}^l} Z_{t+j}^{1-\sigma}}{\Lambda_{t+j}} \\ \times \left[L_{t+j} \left[\dfrac{z^j W_t^*}{W_{t+j}} \left[\prod_{k=1}^{j} \left(\dfrac{\Pi_{t+k-1}}{\Pi} \right)^{\gamma_w} \dfrac{\Pi}{\Pi_{t+k}} \right] \right]^{-\frac{1}{\lambda_{t+j}^w} - 1} \right]^{\chi} \end{array} \right] \end{array} \right\} = 0 \quad (4.7)$$

where W_t^* is the optimal wage set by the household. The aggregate level of wage W_t is the result of all households' optimization.

$$W_t = \left[(1-\xi_w)(W_t^*)^{-\frac{1}{\lambda_t^w}} + \xi_w (1-\xi_w) \left[W_{t-1}^* z \left(\frac{\Pi_{t-1}}{\Pi} \right)^{\gamma_w} \frac{\Pi}{\Pi_t} \right]^{-\frac{1}{\lambda_t^w}} + \dots \right]^{-\lambda_t^w} \Rightarrow$$

$$W_t^{-\frac{1}{\lambda_t^w}} = (1-\xi_w) \left[(W_t^*)^{-\frac{1}{\lambda_t^w}} + \sum_{j=1}^{\infty} \xi_w^j \left[z^j W_{t-1}^* \left[\prod_{k=1}^{j} \left(\frac{\Pi_{t-k}}{\Pi} \right)^{\gamma_w} \frac{\Pi}{\Pi_{t-k+1}} \right] \right]^{-\frac{1}{\lambda_t^w}} \right] \tag{4.8}$$

Log linearization of (4.7) and (4.8) leads to a wage version of Hybrid New Keynesian Phillips Curve (NKPC) which describes the dynamics of inflation and wage. The time-variant wage markup λ_t^w and labor supply shock ε_t^l will become unidentified after log-linearization and be redefined as a wage shock ε_t^w.

4.2.2 Intermediate Good Firms and Final Good Firms

Final good firm has a Dixit-Stiglitz production technology

$$Y_t = \left(\int_0^1 Y_{t,f}^{\frac{1}{1+\lambda_t^p}} df \right)^{1+\lambda_t^p}$$

where λ_t^p is the price markup of intermediate good, which has a relationship $\lambda_t^p = \frac{1}{\theta_t^p - 1}$ with elasticity of substitute of intermediate good θ_t^p. Final good firm minimizes the cost of intermediate good inputs $\int_0^1 Y_{t,f} P_{t,f} df$ subject to its production technology. The cost minimization leads to the general price level P_t and the demand curve of intermediate good $Y_{t,f}$.

$$P_t = \left(P_{t,f}^{-\frac{1}{\lambda_t^p}} df \right)^{-\lambda_t^p}$$

$$Y_{t,f} = \left(\frac{P_{t,f}}{P_t} \right)^{-\frac{1+\lambda_t^p}{\lambda_t^p}} Y_t$$

The price setting of intermediate good firm is according to Calvo (1983) mechanism. For each period, $1-\xi_p$ fraction of all intermediate good firms can optimize their price $P_{t,f}$. ξ_p fraction of all intermediate good firms can't optimize the price and just index the price according to

$$P_{t+j,f} = P_{t+j-1,f} \Pi_{t+j-1}^{\gamma_p} \Pi^{1-\gamma_p} = P_{t,f} \left[\prod_{k=1}^{j} \left(\frac{\Pi_{t+k-1}}{\Pi} \right)^{\gamma_p} \Pi \right]$$

where γ_p is the price indexation weight on the gross inflation rate of previous period. The intermediate good firm has a Cobb-Douglas production function

$$Y_{t,f} = \left(Z_t L_{t,f} \right)^{1-\alpha} \left(U_{t,f} K_{t-1,f} \right)^{\alpha} - \varphi Z_t$$

where φ is fixed cost in intermediate good production. Cost minimization of intermediate good firm $\min W_t L_{t,f} + R_t^K U_{t,f} K_{t-1,f}$ leads to the marginal cost and the optimal factor input ratio. Since the optimal decision is identical among all intermediate good firms, we can drop the index of intermediate good firm f.

$$MC_t = \left[\frac{W_t}{(1-\alpha)Z_t}\right]^{1-\alpha} \left(\frac{R_t^K}{\alpha}\right)^{\alpha} \tag{4.9}$$

$$\frac{U_t K_{t-1}}{L_t} = \frac{\alpha W_t}{(1-\alpha)R_t^K} \tag{4.10}$$

Using the demand curve of intermediate good $Y_{t,f}$ derived previously to aggregating the production function of all intermediate good firms leads to aggregate production function

$$Y_t \Theta_t = (Z_t L_t)^{1-\alpha} (U_t K_{t-1})^{\alpha} - \varphi Z_t \tag{4.11}$$

where $\Theta_t = \int_0^1 \left(\frac{P_{t,f}}{P_t}\right)^{-\frac{1+\lambda_t^p}{\lambda_t^p}} df$ is the price dispersion. Since all intermediate good firms choose same optimal price, Θ_t is approximately equal to unit at steady state. Intermediate good firm optimizes $P_{t,f}$ to maximizes discounted present value of profit.

$$\max E_t \sum_{j=0}^{\infty} (\beta \xi_p)^j \left(\frac{\Lambda_{t+j}}{\Lambda_t}\right) \left[\frac{P_{t,f}}{P_{t+j}} \left[\prod_{k=1}^{j} \left(\frac{\Pi_{t+k-1}}{\Pi}\right)^{\gamma_p} \Pi\right] - MC_{t+j}\right] Y_{t+j,f}$$

$$\text{s.t. } Y_{t+j,f} = \left[\frac{P_{t,f}}{P_{t+j}} \left[\prod_{k=1}^{j} \left(\frac{\Pi_{t+k-1}}{\Pi}\right)^{\gamma_p} \Pi\right]\right]^{-\frac{1+\lambda_{t+j}^p}{\lambda_{t+j}^p}} Y_{t+j}$$

The FOC of this optimization is given as

$$E_t \sum_{j=0}^{\infty} \left\{ \begin{array}{c} (\beta\xi_p)^j \dfrac{\Lambda_{t+1}}{\Lambda_t} \dfrac{1}{\lambda_{t+j}^p} \left[\dfrac{P_t^*}{P_t} \left(\prod_{k=1}^{j} \left(\dfrac{\Pi_{t+k-1}}{\Pi}\right)^{\gamma_p} \dfrac{\Pi}{\Pi_{t+k}}\right)\right]^{-\frac{1+\lambda_{t+j}^p}{\lambda_{t+j}^p}} Y_{t+j} \\ \times \left[\dfrac{P_t^*}{P_t} \left(\prod_{k=1}^{j} \left(\dfrac{\Pi_{t+k-1}}{\Pi}\right)^{\gamma_p} \dfrac{\Pi}{\Pi_{t+k}}\right) - (1+\lambda_{t+j}^p) MC_{t+j}\right] \end{array} \right\} = 0 \tag{4.12}$$

where P_t^* is the optimal price decided by the intermediate good firm. Aggregating the price of all intermediate goods leads to the law of motion of general price level.

$$\begin{aligned} 1 &= \left[\int_0^1 \left(\frac{P_{f,t}}{P_t}\right)^{-\frac{1}{\lambda_t^p}} df\right]^{-\lambda_t^p} \\ &= \left[(1-\xi_p)\left(\frac{P_t^*}{P_t}\right)^{-\frac{1}{\lambda_t^p}} + \xi_p(1-\xi_p)\left(\frac{P_{t-1}^* \Pi_{t-1}^{\gamma_p} \Pi^{1-\gamma_p}}{P_t}\right)^{-\frac{1}{\lambda_t^p}} + \dots\right]^{-\lambda_t^p} \Rightarrow \end{aligned}$$

$$1 = (1-\xi_p)\left\{\left(\frac{P_t^*}{P_t}\right)^{-\frac{1}{\lambda_t^p}} + \sum_{j=1}^{\infty} (\xi_p)^j \left[\frac{P_{t-j}^*}{P_{t-j}} \left(\prod_{k=1}^{j} \left(\frac{\Pi_{t-k}}{\Pi}\right)^{\gamma_p} \frac{\Pi}{\Pi_{t-k+1}}\right)\right]^{-\frac{1}{\lambda_t^p}}\right\} \tag{4.13}$$

Log linearization of (4.12) and (4.13) leads to a Hybrid NKPC which describes the dynamics of inflation.

4.2.3 Monetary Authority

We assume that the monetary authority follows a Taylor Rule by gradually adjusting the nominal interest rate in response to inflation, and output gap.

$$\log\frac{R_t^N}{R^N} = \varphi_r \log\frac{R_{t-1}^N}{R^N} + (1-\varphi_r)\left[\varphi_\pi\left(\frac{1}{4}\sum_{j=0}^{3}\log\frac{\Pi_{t-j}}{\Pi}\right) + \varphi_y \log\frac{Y_t}{Y_t^*}\right] + \varepsilon_t^r \quad (4.14)$$

φ_r is interest rate smoothing parameter of Taylor Rule. φ_π and φ_y are the parameters which represent the response degree of shadow rate to inflation and output gap respectively. ε_t^r is a monetary policy shock. Y_t^* is the potential output when price dispersion Θ_t is unit and capital utilization U_t is 100% at steady state.

$$Y_t^* = (Z_t L_t)^{1-\alpha}(K_{t-1})^\alpha - \varphi Z_t \quad (4.15)$$

For the steady state, de-trending and log-linearization of model, please refer Appendix. In the estimation of model, we replace the data of nominal interest rate with the shadow rate data after the ZLB becomes binding. Note that the shadow rate is not the operation target of the central bank, but it is a good summary statistic of monetary policy in both ZLB and non-ZLB environment. The usefulness of the shadow rate has been proved in many empirical applications. The theoretical foundation established by Wu and Zhang (2016) shows that the impact of unconventional policy (quantitative easing and lending facilities) is identical to that of a negative shadow rate that enters directly into the New Keynesian IS curve. This validates our approach to introduce the shadow rate into a medium-scale model.

4.2.4 Market Equilibrium and Exogenous Shocks

Aggregate resource constraint is given as

$$Y_t = C_t + I_t + gZ_t e^{\varepsilon_t^g} \quad (4.16)$$

where $gZ_t e^{\varepsilon_t^g}$ represents the government expenditure. ε_t^g is the government expenditure shock. g is a scale parameter which can be calibrated from the data. In this model, we have 7 exogenous shocks which all follow first-order autoregression processes.

$$\varepsilon_t^x = \rho_x \varepsilon_{t-1}^x + \mu_t^x,\ x \in (b, i, g, p, w, r, z) \quad (4.17)$$

ε_t^p is a cost-push shock process which comes from the time-variant price markup of intermediate good λ_t^p. This shock appears in NKPC. Regarding the dynamic properties of the technology shock ε_t^z and monetary policy shock ε_t^r, these two shocks are assumed to be I.I.D processes which means $\rho_r = \rho_z = 0$. μ_t^x is an I.I.D innovation term which follows standard nominal distribution $\mathcal{N}(0, \sigma_x^2)$. The log-linearized version of model is given in Appendix.

4.3 Estimation and Model Properties

The variables defined in model and the actual observed data can be linked through the observation equations. Data used for estimation is same as Ueda and Sudo

(2008) and Hirose and Kurozumi (2012) in the estimation of medium-scale DSGE of Japan economy. The data of nominal interest rate from 1999Q1 to 2016Q3 is replaced by the shadow rate estimated previously in full-sample estimation.

$$\begin{bmatrix} 100\Delta\log GDP_t \\ 100\Delta\log C_t \\ 100\Delta\log I_t \\ 100\Delta\log W_t \\ L_t \\ 100\Delta\log DEF_t \\ R_t^N \end{bmatrix} = \begin{bmatrix} z^* \\ z^* \\ z^* \\ z^* \\ L^* \\ \pi^* \\ r^* + \pi^* \end{bmatrix} + \begin{bmatrix} \widehat{y}_t - \widehat{y}_{t-1} + \varepsilon_t^z \\ \widehat{c}_t - \widehat{c}_{t-1} + \varepsilon_t^z \\ \widehat{i}_t - \widehat{i}_{t-1} + \varepsilon_t^z \\ \widehat{w}_t - \widehat{w}_{t-1} + \varepsilon_t^z \\ \widehat{L}_t \\ \pi_t \\ \widehat{R}_t^N \end{bmatrix}$$

$z^* = 100\log z$, $r^* = 100\log R$, $\pi^* = 100\log\Pi$ are the state steady of the labor, the net balance growth rate, net real interest rate and net inflation rate respectively. For the choice of prior distribution, we follow the similar research that focuses on the Bayesian estimation of a medium-scale DSGE model for Japan economy, such as Ueda and Sudo (2008) and Hirose and Kurozumi (2012). Price markup λ_p and wage markup λ_w are calibrated to be 0.2 and 0.15 respectively. Capital depreciation rate δ is calibrated to be 0.15. Capital share α and government expenditure ratio g are calibrated to be 0.37 and 0.31.

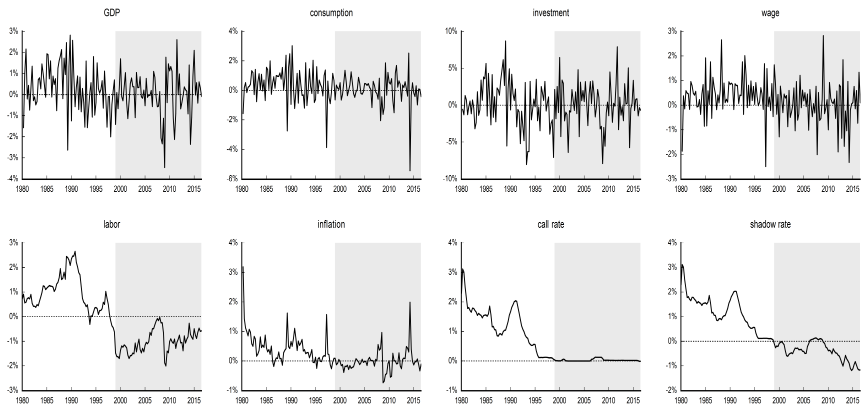


Figure 4.1: Data for Estimation of Medium-Scale DSGE Model

Figure 4.1 shows the data used for estimation. The shaded area in Figure 4 indicates 1999Q1 when the ZLB became binding. The call rate is near zero and static since 1999Q1, from which we can't get too much information about the stance of monetary policy. But the shadow rate from 1999Q1 still keeps decreasing to show the simulative stance of monetary policy conducted by BoJ. We firstly ran the estimation with a sub-sample 1980Q1-1998Q4 because the call rate (policy rate of BoJ) decreased to 0.0375% in 1999Q1 and the effect of ZLB on policy rate emerged since then. Then we ran the estimation of full-sample 1980Q1-2016Q3 with shadow rate from 1999Q1 to 2016Q3. Prior distribution and results of posterior distribution are given in Table 4.1.

Prior Distribution				Posterior Distribution estimation with call rate 1980Q1-1998Q4				Posterior Distribution estimation with shadow rate 1980Q1-2016Q3			
Parameter	Distribution	Mean	Std	Mean	5%	95%	Std	Mean	5%	95%	Std
σ	G	1.000	0.375	2.091	1.539	2.611	0.329	2.065	1.724	2.413	0.212
θ	B	0.700	0.150	0.341	0.231	0.453	0.067	0.333	0.253	0.413	0.049
χ	G	2.000	0.750	3.062	1.639	4.362	0.862	4.423	2.893	5.956	0.945
$1/\zeta$	G	4.000	1.500	6.081	2.959	8.913	1.851	10.215	6.422	13.851	2.284
μ	G	1.000	1.000	0.755	0.051	1.432	0.513	0.551	0.153	0.931	0.249
φ/y	B	0.075	0.013	0.067	0.048	0.086	0.012	0.066	0.047	0.085	0.012
γ_p	B	0.500	0.250	0.300	0.006	0.574	0.192	0.191	0.006	0.361	0.124
ξ_p	B	0.375	0.100	0.646	0.573	0.717	0.044	0.725	0.673	0.775	0.031
γ_w	B	0.500	0.250	0.706	0.465	0.981	0.168	0.674	0.441	0.940	0.154
ξ_w	B	0.375	0.100	0.497	0.377	0.617	0.074	0.513	0.421	0.610	0.057
φ_r	B	0.800	0.100	0.776	0.724	0.829	0.032	0.857	0.831	0.883	0.016
φ_π	G	1.700	0.100	1.705	1.560	1.846	0.088	1.812	1.664	1.963	0.091
φ_y	G	0.125	0.050	0.034	0.012	0.055	0.014	0.037	0.018	0.055	0.012
π^*	G	0.400	0.050	0.398	0.319	0.477	0.048	0.320	0.234	0.406	0.053
z^*	G	0.200	0.050	0.161	0.098	0.223	0.038	0.117	0.072	0.162	0.028
r^*	G	0.500	0.050	0.600	0.517	0.682	0.050	0.383	0.276	0.483	0.063
L^*	N	0.000	0.050	-0.002	-0.085	0.081	0.050	0.000	-0.082	0.085	0.050
ρ_b	B	0.500	0.200	0.899	0.833	0.970	0.044	0.942	0.900	0.987	0.029
ρ_i	B	0.500	0.200	0.568	0.376	0.748	0.115	0.342	0.165	0.513	0.106
ρ_w	B	0.500	0.200	0.116	0.014	0.216	0.070	0.053	0.007	0.100	0.032
ρ_g	B	0.500	0.200	0.968	0.946	0.990	0.015	0.976	0.965	0.988	0.007
ρ_p	B	0.500	0.200	0.951	0.913	0.989	0.026	0.935	0.902	0.971	0.023
μ^b	IG	0.500	inf	3.532	2.424	4.643	0.694	3.726	2.648	4.777	0.703
μ^i	IG	0.500	inf	4.930	3.547	6.333	0.993	3.808	3.208	4.383	0.393
μ^w	IG	0.500	inf	0.798	0.662	0.934	0.084	0.863	0.760	0.959	0.061
μ^g	IG	0.500	inf	2.261	1.901	2.592	0.212	2.581	2.283	2.861	0.178
μ^p	IG	0.500	inf	0.354	0.201	0.516	0.105	0.241	0.165	0.315	0.051
μ^r	IG	0.500	inf	0.188	0.159	0.215	0.017	0.176	0.156	0.195	0.012
μ^z	IG	0.500	inf	1.967	1.698	2.238	0.167	1.898	1.704	2.092	0.118

Table 4.1: Estimation Results of Structural Parameters

For most of the parameters, the posterior distributions of the full-sample estimation with the shadow rate data are still very reasonable and similar to the corresponding posterior distribution obtained from the estimation with pre-ZLB sub-sample. This may suggest that a DSGE model can be still applied in the ZLB environment by estimating the model with shadow rate. The posterior distribution of some structural parameters, for instance, the probability that an intermediate good firm is not allowed to optimize its price is $\xi_p \approx 0.725$, which implies average duration of price contracts of about 11 months. For the probability of Calvo (1983) type wage contract, $\xi_w \approx 0.497$ in pre-ZLB estimation and $\xi_w \approx 0.513$ in full-sample estimation with shadow rate means the rigidity of nominal wage has increased since ZLB period, but still in a reasonable range. Monetary policy parameters $(\varphi_r, \varphi_\pi, \varphi_y)$ don't have significant changes in both groups of estimation. Other parameters are consistent with the most of medium-scale DSGE literature. We than compare the dynamic properties implied by both estimated models. Table 4.2 gives the variance decomposition of 7 observable variables calculated with the posterior distributions obtained from two groups of estimation.

Variance Decomposition at $T = 8$	μ_t^b		μ_t^i		μ_t^g		μ_t^w		μ_t^p		μ_t^z		μ_t^r	
GDP	3.68%	3.89%	23.15%	25.67%	9.87%	13.85%	0.56%	0.33%	7.39%	7.39%	54.93%	49.49%	0.42%	0.80%
consumption	34.64%	36.54%	2.89%	1.64%	3.69%	7.70%	0.73%	0.52%	5.65%	5.65%	51.19%	45.27%	1.22%	2.71%
investment	3.35%	3.44%	77.21%	81.48%	0.80%	1.10%	0.54%	0.29%	10.16%	10.16%	7.77%	5.61%	0.17%	0.28%
wage	1.05%	0.94%	0.75%	0.84%	0.04%	0.05%	24.18%	32.01%	44.69%	44.69%	29.25%	21.13%	0.05%	0.11%
labor	3.16%	2.86%	28.68%	27.09%	7.60%	11.98%	4.78%	4.17%	36.74%	36.74%	18.15%	13.28%	0.89%	2.33%
inflation	7.54%	7.76%	16.50%	10.08%	0.81%	1.33%	6.53%	5.06%	50.59%	50.59%	15.42%	9.94%	2.59%	5.73%
interest rate	9.34%	9.03%	28.19%	15.72%	1.52%	2.36%	4.26%	3.29%	34.85%	34.85%	8.49%	5.13%	13.35%	23.80%
Variance Decomposition at $T = 32$	μ_t^b		μ_t^i		μ_t^g		μ_t^w		μ_t^p		μ_t^z		μ_t^r	
GDP	3.70%	3.93%	23.44%	25.63%	9.78%	13.74%	0.61%	0.37%	7.82%	6.60%	54.24%	48.93%	0.41%	0.80%
consumption	34.91%	36.82%	3.41%	1.96%	3.65%	7.60%	0.73%	0.54%	5.71%	5.92%	50.38%	44.50%	1.20%	2.66%
investment	3.92%	3.84%	76.24%	80.11%	0.82%	1.12%	0.61%	0.35%	10.81%	8.84%	7.43%	5.44%	0.16%	0.29%
wage	1.27%	1.21%	0.78%	0.88%	0.06%	0.09%	23.54%	30.67%	45.68%	46.55%	28.62%	20.47%	0.05%	0.12%
labor	4.99%	7.18%	21.41%	19.84%	9.30%	13.68%	3.60%	3.26%	34.60%	34.99%	25.44%	19.34%	0.65%	1.70%
inflation	7.44%	7.80%	15.87%	9.67%	0.87%	1.37%	6.26%	4.78%	48.47%	56.60%	18.63%	14.46%	2.47%	5.32%
interest rate	10.14%	13.01%	30.25%	16.79%	1.89%	3.40%	3.47%	2.59%	28.70%	32.74%	14.81%	13.88%	10.74%	17.59%
Variance Decomposition at $T = \infty$	μ_t^b		μ_t^i		μ_t^g		μ_t^w		μ_t^p		μ_t^z		μ_t^r	
GDP	3.70%	3.94%	23.43%	25.62%	9.78%	13.74%	0.61%	0.37%	7.85%	6.61%	54.22%	48.92%	0.41%	0.80%
consumption	34.88%	36.82%	3.41%	1.95%	3.66%	7.61%	0.73%	0.54%	5.71%	5.91%	50.40%	44.50%	1.19%	2.66%
investment	3.93%	3.93%	76.16%	79.97%	0.85%	1.17%	0.61%	0.35%	10.86%	8.86%	7.43%	5.43%	0.16%	0.29%
wage	1.27%	1.22%	0.78%	0.88%	0.07%	0.10%	23.44%	30.56%	45.83%	46.65%	28.56%	20.47%	0.04%	0.12%
labor	5.20%	7.26%	18.23%	16.50%	11.78%	18.35%	3.00%	2.69%	29.42%	29.08%	31.82%	24.72%	0.54%	1.40%
inflation	7.45%	8.53%	15.13%	8.97%	1.58%	3.02%	5.90%	4.38%	46.31%	52.14%	21.31%	18.09%	2.32%	4.87%
interest rate	9.77%	13.46%	26.58%	13.25%	3.54%	7.68%	3.02%	2.01%	25.98%	25.89%	21.84%	24.19%	9.27%	13.52%

Table 4.2: Variance Decomposition of 7 Observable Variables

The first column of each shock's variance decomposition is from the pre-ZLB sub-sample estimation. The second column of each shock's variance decomposition is from the full-sample estimation with shadow rate.

For the variance decomposition of GDP, except monetary policy shock, other structural shocks have similar explanatory power. The full-sample estimation with the shadow rate shows 2 times stronger explanatory power than the pre-ZLB estimation. For most periods after 1999Q1, call rate is static and near zero, but the impact of monetary policy on macroeconomic variables can be still confirmed by using the shadow rate.

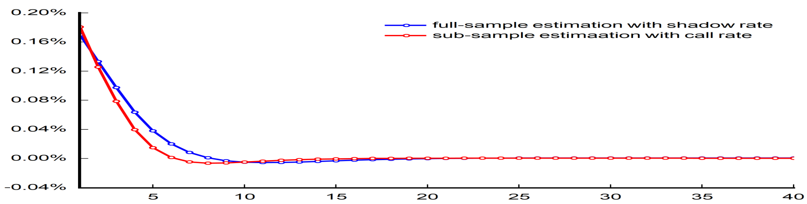


Figure 4.2: Impulse Response of Interest Rate on Monetary Policy Shock

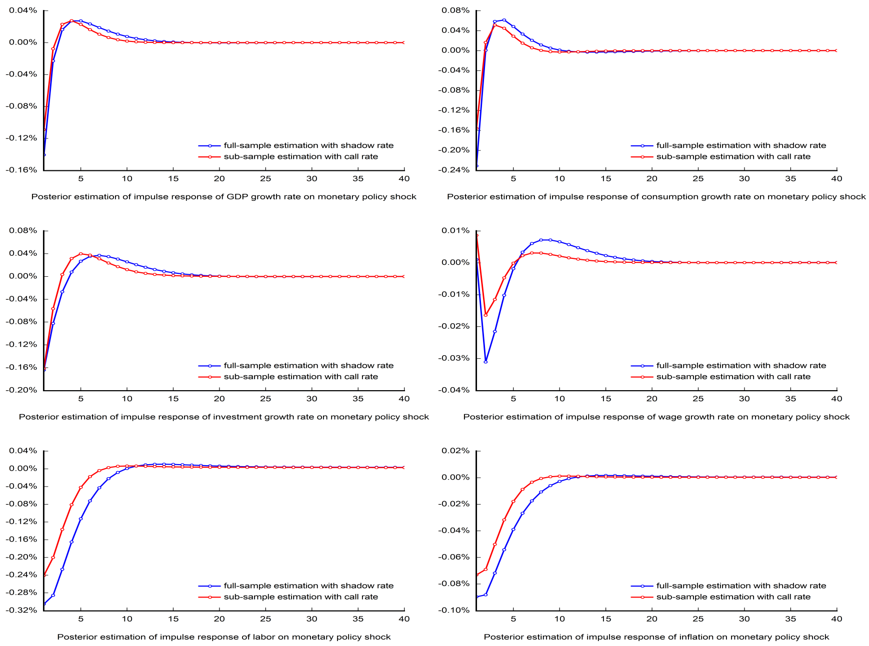


Figure 4.3: Impulse Response of Observable Variables on Monetary Policy Shock

Figure 4.2 gives the impulse response of interest rate on monetary policy shock calculated from posterior mean of three groups of estimation. We omit the credit interval because the 95% band is very thin. Two lines are very close and we can find that the dynamics of impulse response implied by both groups of estimation don't have too much difference, quantitatively and qualitatively. Figure 4.3 shows that a positive monetary policy shock has similar mechanism even we use the shadow rate to estimate the model. But the impulse response calculated from the posterior mean of estimation with shadow rate is quantitatively more significant than another groups of estimation. This may imply the monetary policy stance of unconventional policies is more aggressive.

4.4 Counterfactual Simulation

This section presents the quantitative evaluation of unconventional monetary policy of BoJ since 1999Q1 from the view of counterfactual simulation. We follow the simulation methodology designed by Sarah and Jean-Guillaume (2016).

1. **Identification of actual structural shocks with unconventional monetary policy implied by shadow rate**: We firstly take the posterior mean estimates of structural parameters obtained from full-sample estimation with shadow rate and compute the structural shocks $\{\mu_t^b, \mu_t^z, \mu_t^i, \mu_t^g, \mu_t^w, \mu_t^p, \mu_t^r\}$ by Kalman filter. These shocks are the realizations from all monetary policy decisions ("observed").

2. **Identification of counterfactual structural shocks with conventional monetary policy implied by ZLB-constrained call rate**: We then re-estimate the standard deviation of monetary policy shock μ_t^r by replacing the shadow rate by the usual call rate, all other structural parameters are calibrated to be their values obtained in the previous step 1. These shocks are those realizations that come from the conventional part of monetary policy ("counterfactual").

3. **Simulation**: We then compute the simulated time series of variables from the estimated model using the first and second set of monetary policy shocks, given other structural shocks obtained in step 1.

We explain the procedure of counterfactual simulation in more details.

1. Estimation with shadow rate full-sample dataset.
   - smoothed structural shocks $\{\mu_t^b, \mu_t^i, \mu_t^g, \mu_t^w, \mu_t^p, \mu_t^z, \mu_t^{r,\text{shadow rate}}\}_{t=1}^T$
   - posterior distribution of parameters

2. Estimation of standard error of monetary policy shock with call rate full-sample dataset given other parameters fixed at the posterior mean obtained in step 1.
   - smoothed structural shocks $\{\mu_t^b, \mu_t^i, \mu_t^g, \mu_t^w, \mu_t^p, \mu_t^z, \mu_t^{r,\text{call rate}}\}_{t=1}^T$

3. Setup of two groups of structural shocks.
   - "observed" structural shocks $\{\mu_t^b, \mu_t^i, \mu_t^g, \mu_t^w, \mu_t^p, \mu_t^z, \mu_t^{r,\text{shadow rate}}\}_{t=1}^T$
   - "counterfactual" structural shocks $\{\mu_t^b, \mu_t^i, \mu_t^g, \mu_t^w, \mu_t^p, \mu_t^z, \mu_t^{r,\text{call rate}}\}_{t=1}^T$

4. Calibration of model given the structural parameters in step 1.

5. Simulation of calibrated model in step 4 given same initial condition and two groups of shocks as exogenous driven force.
   - "observed" endogenous variables
   - "counterfactual" endogenous variables

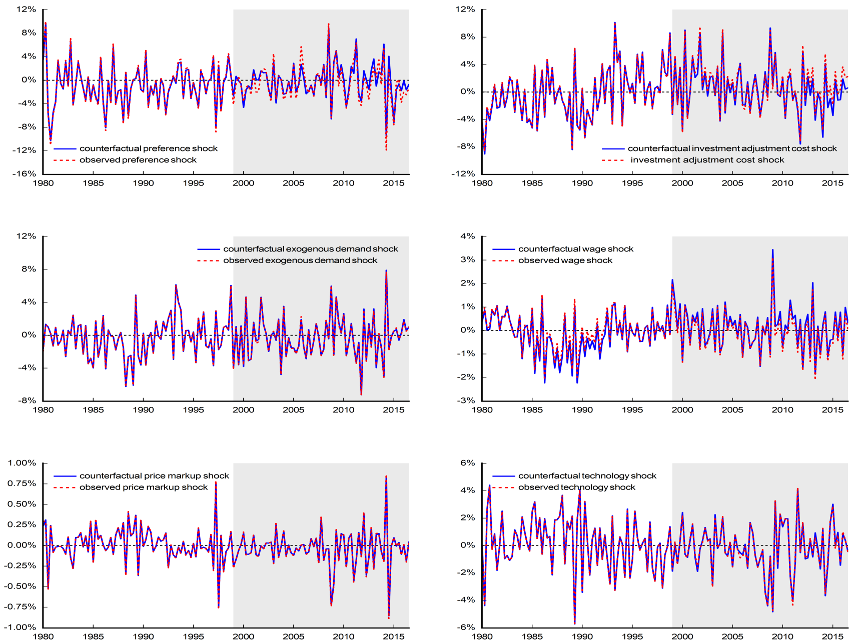


Figure 4.4: Smoothed Structural Shocks

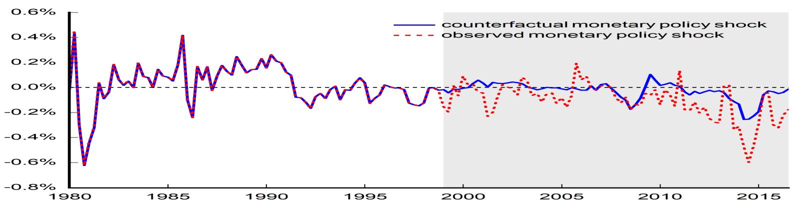


Figure 4.5: Identified Monetary Policy Shock

As shown in Figure 4.4, except the monetary policy shock, other 6 structural shocks identified in the step 1 and step 2 have almost same paths. Figure 4.5 shows the monetary policy shocks identified from the shadow rate and call rate. We use "observed" to mean the monetary policy shock identified from the estimation using the shadow rate because we consider the shadow rate can accurately represent the stance of monetary policy, but call rate constrained by the ZLB can't given the information about the monetary policy stance. Since 1999Q1, the estimated monetary policy shocks are negative in most periods. Even though these shocks can't change the observed call rate but it can represent a commitment to expansionary monetary policy stance. The unconventional monetary policy shock can affect the future expectation of interest rate and then the whole yield curve, then affect the real economy. The monetary policy shocks identified here are very similar to those of Aoki and Ueno (2012) with another approach.

Figure 4.6 shows the simulated paths of selected model variables. The blue line in each subplot means the simulated path of counterfactual monetary policy shock, given other structural shocks fixed. Counterfactual simulated paths for $\widehat{y}_t$, $\widehat{c}_t$ and $\widehat{i}_t$ are all below their actual "observed" realizations. Without the implementation of unconventional monetary policy from 1999Q1, Japan economy would have experienced poorer performance. The policy impact on labor $\widehat{L}_t$, investment $\widehat{i}_t$ and capital price $\widehat{Q}_t$ is especially obvious. The policy impact on inflation is also significant, which implies that unconventional monetary policy seems to have some effect on the resolution of deflation.

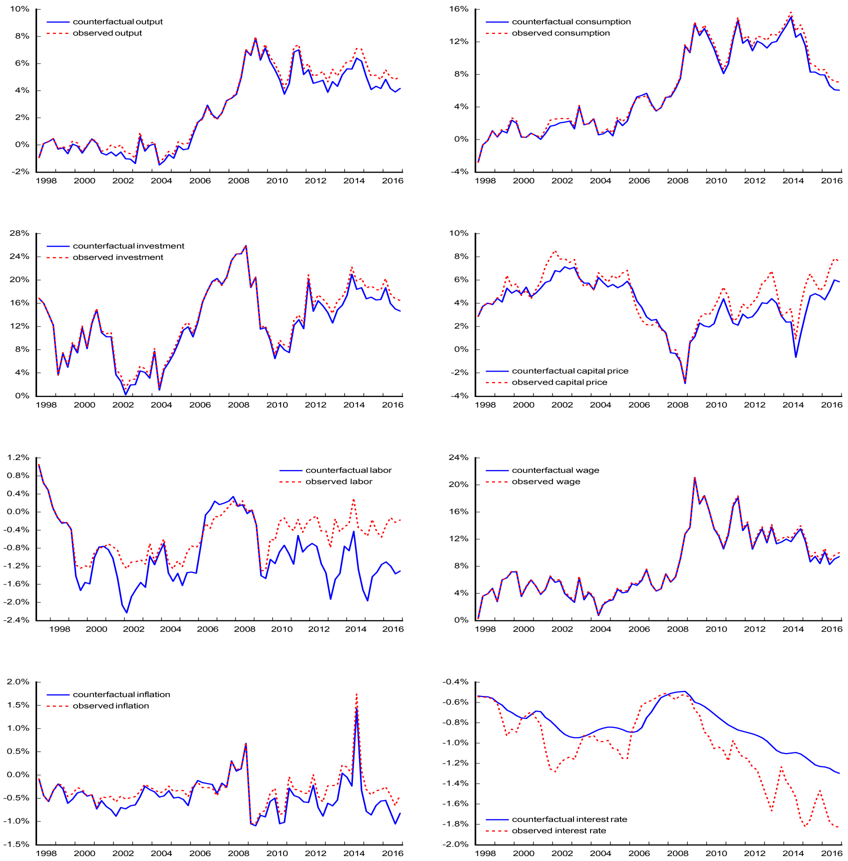


Figure 4.6: Simulated Percentage Deviation Path of Macroeconomic Variables

Note that the simulated paths given in the Figure 4.6 are the paths of model variables, which means that these variables are the percentage deviation from their steady states. To get a better visualization of counterfactual simulation, we normalize the values of main macroeconomic real variables at 1998Q1 as 100 and use the observation equation to reconstruct the level paths of GDP, consumption, investment, wage.

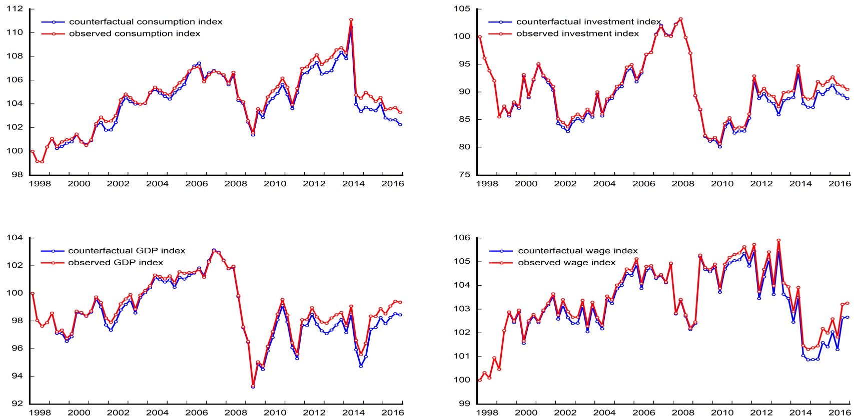


Figure 4.7: Simulated Level Path of Macroeconomic Variables

Macroeconomic variable	GDP	consumption	investment	wage
Relative cumulative loss	25.80	27.67	53.17	15.79

Table 4.3: Relative Cumulative Loss from 1999Q1 to 2016Q3

The cumulative loss associated with the variable x_t is $\sum\left(\frac{x_t^o}{x_t^c}-1\right)\times 100$, where x_t^o is the observed level and x_t^c is the counterfactual level.

Based on the level paths of real macroeconomic variables, we can calculate relative cumulative loss taken back by the implementation of the unconventional monetary policy since 1999Q1. Without the implementation of the unconventional monetary policy since 1999Q1, macroeconomic variables would suffered lost relatively compared to their actual realizations, especially for investment. Figure 4.8, 4.9, 4.10, 4.11, 4.12, 4.13 give the historical decomposition of observable GDP growth rate, consumption growth rate and investment growth rate from 1999Q1 to 2016Q3. For most of periods, the monetary policy shock has made positive contribution to these variables. Figure 14 gives the historical decomposition of capital price $\widehat{Q}_t$, the monetary policy shock has very obvious contribution on the capital price. Figure 4.12 shows the historical decomposition of labor from which we can find the obviously positive contribution of monetary policy shock. The positive contribution of monetary policy shock on inflation can also be confirmed in Figure 4.13. These two figures are consistent with the counterfactual simulation. The impact of unconventional monetary policy on labor and inflation became larger since 2013Q2 when the Quantitative Qualitative Easing of BoJ begun.

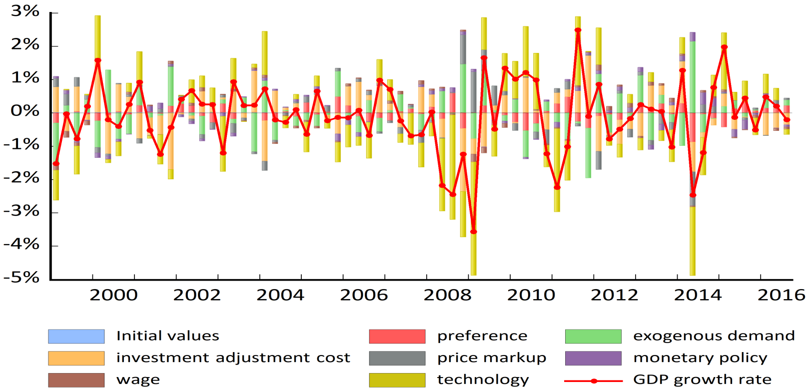


Figure 4.8: Historical Decomposition of GDP Growth Rate

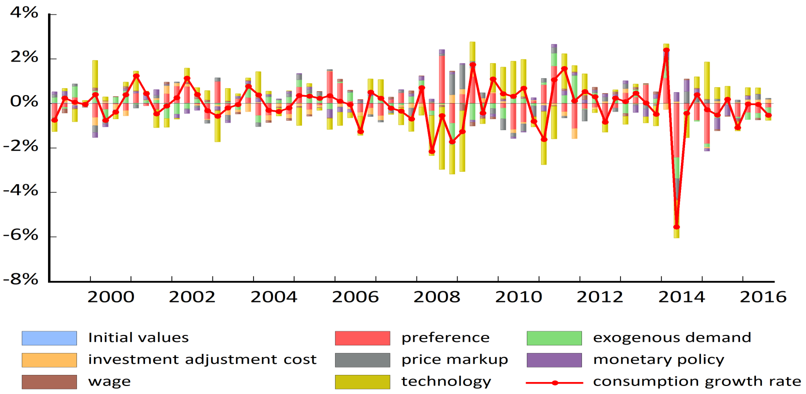


Figure 4.9: Historical Decomposition of Consumption Growth Rate

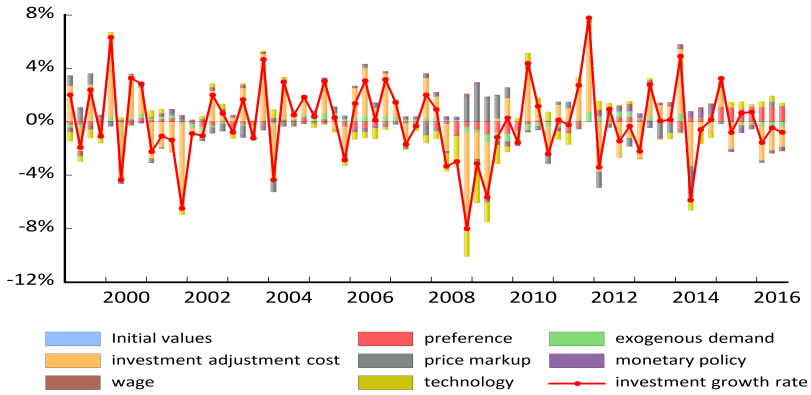


Figure 4.10: Historical Decomposition of Investment Growth Rate

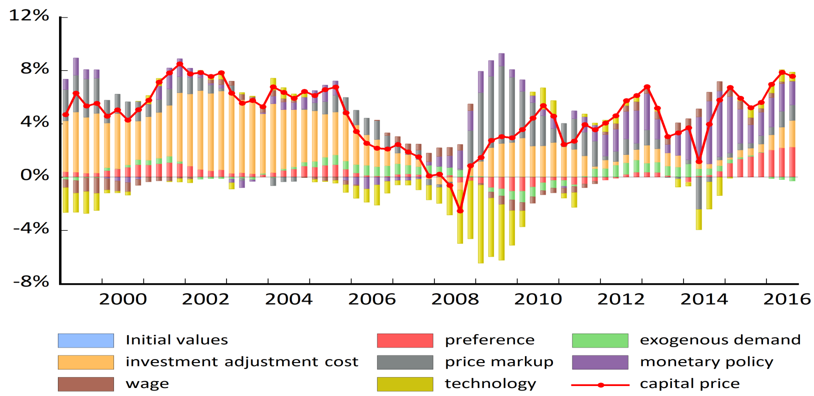


Figure 4.11: Historical Decomposition of Capital Price

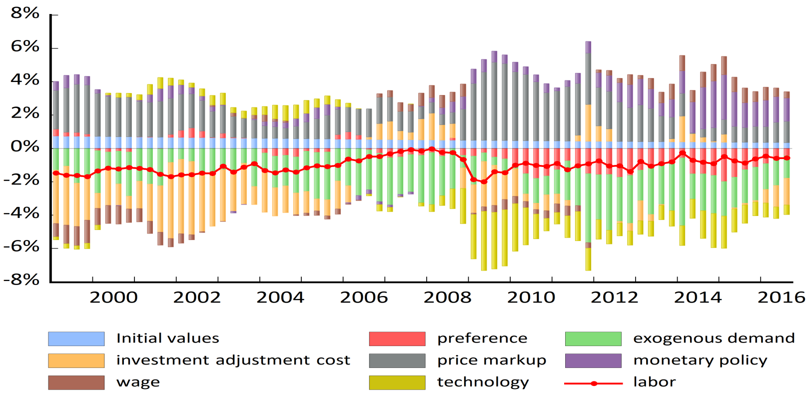


Figure 4.12: Historical Decomposition of Labor

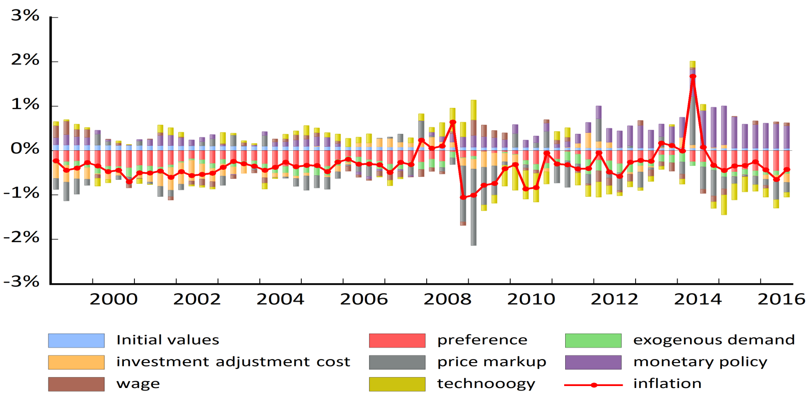


Figure 4.13: Historical Decomposition of Inflation

4.5 Concluding Remarks

We give some concluding remarks for this chapter. In this chapter, we confirmed the availability of shadow rate in the estimation of DSGE models for Japan economy. Firstly, following the theoretical foundation proposed by Wu and Zhang (2016), we estimated a medium-scale DSGE model with the shadow rate. By comparing the results from pre-ZLB sub-sample estimation and full-sample estimation, we find that the structural parameters estimated from full-sample with shadow rate are still reasonable and consistent compared with the pre-ZLB sample estimation. The model dynamics implied by these estimated results are also similar and very close. This result is consistent with Sarah and Jean-Guillaume (2016) which estimated a similar DSGE model with the shadow rate for Euro area. The advantage of using the shadow rate in the estimation of DSGE models is that we can still use all techniques for linear rational expectation models, such as Kamlan filter and general MCMC algorithm of Bayesian estimation, without explicitly dealing with the ZLB by difficult nonlinear techniques. Secondly, following the simulation methodology designed by Sarah and Jean-Guillaume (2016), we identified the observed monetary policy shocks under the implementation of unconventional monetary policy and the counterfactual monetary policy shocks without the implementation of unconventional monetary policy. Figure 4.5 shows shat in the conventional regime of monetary policy before 1999Q1, identified shocks are almost same, because the shadow rate is same as the call rate when the ZLB is not binding. But after 1999Q1 when the ZLB became binding, these shocks are quite different. We also identified other structural shocks through the "observed" estimation and "counterfactual" estimation, these shocks are almost same under different regimes because they are exogenous and not affected by different policy regimes. Given the only difference in monetary policy shock and other structural shocks and parameters fixed, we simulated two set of paths for model variables. The simulation results show that, without the implementation of unconventional monetary policy, Japan would have experienced worse economic performance since 1999Q1. We confirmed the policy effects on most macroeconomic variables such as GDP, investment and capital price.

Note that there may exist one defect in the estimation of DSGE models with shadow rate. It is that the shadow rate is not endogenously derived from a structural model, but exogenously estimated by a statistical model. Short rate term structure model is a factor model and the factors are used to fit and describe the yield curve in a ZLB environment. But the factors have less economic interpretation. Krippner (2015) gives a structural interpretation of these factors in a linear economy framework, but this framework is highly stylized and has less dynamics than DSGE models. The estimation of the shadow rate by shadow rate term structure model needs some nonlinear numerical calculations, this makes the endogenous determination of the shadow rate in a structural DSGE framework very difficult.

Chapter 5

Portfolio Rebalancing Mechanism of QE in DSGE Model

5.1 Introduction

From February 1999, when the Bank of Japan announced the commitment to the zero interest rate policy, to April 2013, when it started Quantitative Qualitative Easing (QQE), BoJ has implemented unconventional monetary policy for almost 15 years. Especially after the global financial crisis, many advanced economies had to depart from conventional ways of conducting monetary policy as they faced the Zero Lower Bound and systemic risk. The importance of unconventional monetary policy has been realized by macroeconomists and central banks both theoretically and practically. Krugman (1998), Svensson (2003) and Bernanke and Reinhart (2004) are the early contributors in this area. Unconventional monetary policy can take many forms besides those that are generally publicly recognized. For example, during the global financial crisis, the Danish National Bank permitted the use of negative interest rates. In general, as one main option of unconventional monetary policy, Quantitative Easing (QE) can be defined as the change in the composition and size of the central bank's balance sheet. The change can be the result of the large asset purchases of private assets or government debt, and it can also occur through direct lending or capital injection from the central bank to the private sector or the financial system.

Joyce et al. (2012) comprehensively introduced the QE conducted by the Federal Reserve (Fed), Bank of England (BoE) and European Central Bank (ECB) with a theoretical background of unconventional monetary policy. In the US, from December 2008 to the end of 2009, the Fed conducted the first phase of QE (QE1, or officially Large-Scale Asset Purchases or LSAP) by expanding its portfolio assets to provide liquidity to the financial system and reduce the risk premium. Following QE1, QE2 lasted from October 2010 to June 2011 and was conducted by Feb through the large purchase of US treasury securities. Bernanke also announced the purchase of mortgage-backed securities (MBS) in September 2012, which is known as QE3, with the objective of pushing down the long-term yield curve to support financial system reconstruction and stimulate aggregate demand. During the same period, in the UK, BoE also started a QE program by establishing the Asset Purchase Facility (APF), the operations of which are conducted by purchasing medium- and long-term UK government bonds.

In Japan, the first phase of QE started 15 years ago, beginning in March 2001 and until March 2006. After a paused in operations from April 2006 to September 2010, the QE program, known as the Comprehensive Easing Policy, was restarted from October 2010 and lasted until March 2013. Its purpose was to stimulate the real economy and protect the financial system from the global financial crisis by purchasing a variety of assets, including commercial papers (CP), Exchange Traded Funds (ETF) and Japan Real Estate Investment Trusts (J-REITs). With the advent of BoJ's new president Haruhiko Kuroda, the new stage of QE known as Quantitative Qualitative Easing (QQE) has started, with a more aggressive scale of balance sheet expansion and with more varieties of asset purchases than in the past. QQE is positioned as one arrow of Abenomics' three arrows. At the same time, BoJ clearly declared a 2% inflation target to shape the formation of expectations. Since the start of QQE from April 2013, two and a half years have pasted. It is still ongoing, so a comprehensive evaluation and final conclusion about QQE may be inappropriate at this time. However, we still recognize the significance of a temporary evaluation of QQE. Most existing research about BoJ's QE take a nonstructural approach, including VAR analysis or event study to obtain empirical evidence about the effectiveness of QE. Especially in VAR analysis, as surveyed by Ugai (2007), different choices of variables and specifications of models lead to different results. In contrast to the nonstructural econometric approach, the DSGE framework has inherent advantages for policy evaluation. The transmission mechanism of monetary policy can be identified with clear explanation based on economic theory. In addition, to the best of my knowledge, no trials have been done in this area. For these reasons, we conduct an empirical project to evaluate the QQE of BoJ in this study.

The remainder of the study is organized as follows. Section 5.2 describes the derivation of the model. Section 5.3 is the calibration of the model's structural parameters and the steady state. Section 5.4 presents the results of simulation with sensitivity analysis. Section 5.5 concludes this chapter.

5.2 DSGE with Portfolio Rebalancing Mechanism

As noted by Meier (2009), there are different approaches to unconventional monetary policy, which can be motivated by alternative views of the transmission channels and their effect on the economy. The model developed here has the standard structure and specification of the New Keynesian DSGE model, but the bond trading market proposed by Ljungqvist and Sargent (2012, Chapter 13, Section 8) is incorporated to isolate the *portfolio rebalancing mechanism* of large asset purchases by the central bank. Tobin (1969) initially described this mechanism, whereby variation in relative supplies of financial assets with different maturities and liquidities triggered by large asset purchases of the central bank can have a real effect on the yield curve due to imperfect asset substitutability. Tobin and Brainard (1963) define the imperfect substitution assumption as follows:

> *Assets are assumed to be imperfect substitutes for each other in wealth-owners' portfolios. That is, an increase in the rate of return on any one asset will lead to an increase in the fraction of wealth held in that asset, and to a decrease or at most no change in the fraction held in every other asset.*

Relating this assumption to unconventional monetary policy, the basic idea is that the central bank's purchase of assets held by the private sector increases the prices of these assets. As asset prices increase, yields fall, stimulating aggregate demand. Even when the short-term nominal interest rate faces ZLB, asset purchases can be a practical policy instrument for the central bank. As described later, large-scale purchases of government bonds by BoJ can be evaluated using this approach in a dynamic stochastic general equilibrium framework.

5.2.1 Household

There is a continuum of representative households, existing continuously in $i \in (0,1)$, where i is indexation[1]. The representative household derives utility from consumption C_t and real money balance $\frac{M_t}{P_t}$ and disutility from labor supply L_t. The utility function is additively separable,

$$U_t = \frac{(C_t - \theta C_{t-1})^{1-\sigma}}{1-\sigma} + \frac{1}{1-\xi}\left(\frac{M_t}{P_t}\right)^{1-\xi} - \frac{\eta_L}{1+\chi}L_t^{1+\chi}$$

where σ is inverse of elasticity of inter-temporal substitution, θ is degree of habit formation, ξ is the interest rate semielasticity of money demand and χ is the inverse of the Frisch elasticity of labor supply. η_L is a preference parameter which measures the relative weight of disutility from labor supply. The household maximizes the discounted infinite stream of utility subject to the inter-temporal budget constraint and the standard law of motion of capital accumulation. ε_t^u is a preference shock process following $\varepsilon_t^u = \rho_u \varepsilon_{t-1}^u + \mu_t^u$ and $\mu_t^u \sim \mathcal{N}(0, \sigma_u^2)$ is an i.i.d exogenous shock.

$$\mathbb{E}_t \sum_{t=s}^{\infty} \beta^t e^{\varepsilon_t^u} U_t \left(C_t, \frac{M_t}{P_t}, L_t\right)$$

$$\begin{aligned}
&\frac{B_{S,t}}{P_t R_{S,t}} + \frac{B_{L,t}^H(1+AC_{B,t})}{P_t R_{L,t}} + \frac{M_t}{P_t} + I_t(1+AC_{K,t}) \\
&= \frac{B_{S,t-1}}{P_t} + \frac{B_{L,t-1}^H}{P_t R_{S,t}} + \frac{M_{t-1}}{P_t} + w_t L_t + q_t K_t - C_t - T_t \\
&K_t = I_t + (1-\delta)K_{t-1}
\end{aligned}$$

The household allocates wealth among real money holdings $\frac{M_t}{P_t}$, capital K_t with rental rate q_t and two types of government bonds[2], short-term bonds $B_{S,t}$, whose maturities are equal to or shorter than 1 year with yield $R_{S,t}$, and long-term bonds $B_{L,t}^H$[3], whose maturities are equal to or longer than 10 years with yield $R_{L,t}$. The household supplies labor L_t and receives real wages w_t and pays a real lump-sum tax T_t at the general aggregate price level P_t. Investment I_t and capital accumulation processes occur with adjustment cost

$$AC_{K,t} = \frac{\varphi_K}{2}\left(\frac{I_t}{K_t}\right)^2$$

[1] Indexation of each household is omitted because they are homogenous and identical.

[2] This kind of classification in also used in the calibration of model, steady-state ratio of two kinds of bonds with different maturities relative to the total amount of government bonds.

[3] $B_{L,t}^H$ means the long-term bonds held by households.

and the portfolio adjustment between two kinds of bonds also accompanies with cost,

$$AC_{B,t} = \frac{\varphi_B}{2}\left(\kappa_B \frac{B_{S,t}}{B_{L,t}^H} - 1\right)^2 Y_t$$

where κ_B is the steady-state ratio of long-term bond holdings of the household to short-term bond holdings $\frac{B_L^H}{B_S}$, so at the steady state, the portfolio is adjusted to its optimal allocation and adjustment cost, which is paid in terms of the household's income of zero. The first order conditions of the household's maximization with respect to consumption, labor supply, real money, short-term bond, long-term bond, capital and investment are given by Equations (5.1) to (5.7).

$$e^{\varepsilon_t^u}(C_t - \theta C_{t-1})^{-\sigma} - \beta\theta \mathbb{E}_t e^{\varepsilon_{t+1}^u}(C_{t+1} - \theta C_t)^{-\sigma} = \lambda_t \tag{5.1}$$

$$e^{\varepsilon_t^u}\eta_L L_t^{\chi} = \lambda_t w_t \tag{5.2}$$

$$e^{\varepsilon_t^u}(m_t)^{-\xi} + \beta\mathbb{E}_t \frac{\lambda_{t+1}}{\Pi_{t+1}} = \lambda_t \tag{5.3}$$

$$\beta\mathbb{E}_t \frac{\lambda_{t+1}}{\Pi_{t+1}} = \frac{\lambda_t}{R_{S,t}} + \frac{\kappa_B \varphi_B \lambda_t Y_t}{R_{L,t}}\left(\kappa_B \frac{b_{S,t}}{b_{L,t}^H} - 1\right) \tag{5.4}$$

$$\begin{aligned}\mathbb{E}_t \frac{\beta\lambda_{t+1}}{\Pi_{t+1}R_{S,t+1}} - \frac{\lambda_t}{R_{L,t}} &= \frac{\varphi_B \lambda_t Y_t}{2R_{L,t}}\left(\kappa_B \frac{b_{S,t}}{b_{L,t}^H} - 1\right)^2 \\ &- \frac{\kappa_B \varphi_B \lambda_t Y_t b_{S,t}}{R_{L,t} b_{L,t}^H}\left(\kappa_B \frac{b_{S,t}}{b_{L,t}^H} - 1\right)\end{aligned} \tag{5.5}$$

$$\beta(1-\delta)\mathbb{E}_t \mu_{t+1} = \mu_t - \lambda_t \left[q_t + \varphi_K \left(\frac{I_t}{K_t}\right)^3\right] \tag{5.6}$$

$$\beta\mathbb{E}_t \mu_{t+1} = \lambda_t \left[1 + \frac{3\varphi_K}{2}\left(\frac{I_t}{K_t}\right)^2\right] \tag{5.7}$$

λ_t and μ_t are two Lagrange multipliers corresponding with budget constraint and law of motion of capital accumulation respectively. $\Pi_{t+1} = \frac{P_{t+1}}{P_t}$ is the gross inflation rate at $t+1$ period. Bonds and money are rewritten in real terms $b_{L,t}^H = \frac{B_{L,t}^H}{P_t}$, $b_{S,t} = \frac{B_{S,t}}{P_t}$ and $m_t = \frac{M_t}{P_t}$ by lower case letters for convenience.

Before proceeding, it is worth discussing the adjustment cost of the portfolio introduced above. There are necessary conditions under which the purchase of private sector assets or government securities by the central bank can be effective. As discussed by Eggertsson and Woodford (2004), if representative agents who have rational expectations with an infinite time horizon and face no credit frictions or restrictions consider the assets held by the government and by the central bank as indistinguishable from assets held by themselves, then asset purchases by the central bank change nothing. This proposition is similar and analogous to the Ricardian Equivalence in fiscal theory. But if credit or financial frictions and borrowing constraints do exist, then this proposition no longer holds. In Cúrdia and Woodford (2011), one unconventional monetary policy, direct facility lending from the central bank to the private sector (credit easing), does affect the aggregate economy. Kiyotaki and Moore (2012) described a monetary economy with heterogeneous liquidity of financial assets. In their model, when entrepreneurs want to undertake new investment projects, they can only finance a limited proportion by issuing new equities. Therefore, purchases of such less-liquid equities by the central bank can change their prices, leading to real effects on investment decisions. This is known as the credit channel of QE. Gertler and Kiyotaki (2010) and Gertler and Karadi (2011) also made contributions in this area. The framework in the research mentioned above is highly complicated as it includes the full sketch of financial intermediaries or the banking sector. In this study, we focus only on the portfolio rebalancing channel of QE. This approach is more appropriate for the QE implemented by BoJ[4]. Falagiarda and Marzo (2012), Falagiarda (2014) and Chen et al. (2012) take the same approach to evaluating the QE of the Fed and BoE. Rationale for including portfolio adjustment frictions is intuitional. As mentioned by Falagiarda (2014), long-term bond holdings have less liquidity. Households realize this risk and hold short-term bonds as precautionary liquidity holdings relative to their longer-term investments. Another justification for this adjustment cost comes from the theory of preferred habit. Vayanos and Vila (2009) emphasised that agents have preferences for different bond maturities, and any deviation from the preferred portfolio allocation is costly. More simply, management of the portfolio itself is costly.

5.2.2 Intermediate Good Firm and Final Good Firm

In the same way as the standard New Keynesian DSGE models, final good firms produce homogenous final goods by bundling differentiated intermediate goods with CES technology, so the intermediate goods market is monopolistic. We use Calvo (1983) type of staggered price setting to replicate rigidity of price. As pointed to by Wordfood (2003), the output of all intermediate good firms is equal to the output of all final good firms, and the aggregate production function holds at the steady state when the dispersion of price is unity.

$$Y_t = \left(\int_0^1 Y_{f,t}^{\frac{1}{1+\epsilon_t}} df \right)^{1+\epsilon_t}$$

$f \in (0,1)$ is the indexation of each intermediate good firm and ϵ_t is time-varying price mark-up that has a relationship $\epsilon_t = \frac{1}{\sigma_t - 1} > 0$ with the elasticity of substitution $\sigma_t > 1$ between different intermediate goods. After log-linearizing the model, time varying price mark-up ϵ_t can be represented as a cost-push mark-up shock process that follows $\varepsilon_t^p = \rho_p \varepsilon_{t-1}^p + \mu_t^p$ where $\mu_t^p \sim N(0, \sigma_p^2)$ is an i.i.d shock. Cost minimization of final good firms leads to the intermediate good demand function

[4]BoJ also purchases risky assets such as ETFs and J-REITs from the private sector, but the quantity of these purchases is very less than the quantity purchased of Japanese government bonds.

and aggregate price index.

$$P_t = \left(\int_0^1 P_{f,t}^{-\frac{1}{\epsilon_t}} df \right)^{-\epsilon_t}$$

$$Y_{f,t} = \left(\frac{P_{f,t}}{P_t} \right)^{-\frac{1+\epsilon_t}{\epsilon_t}} Y_t$$

Under Calve (1983) type price setting, each period, $1-\eta$ fraction of all intermediate good firms have the chance to adjust price to their optimal level and the others just index their prices to a weighted average of inflation of last period and steady state with the weight $1-\gamma$ and γ respectively.

$$P_{f,t+j} = P_{f,t+j-1} \Pi_{t+j-1}^{\gamma} \Pi^{1-\gamma} = P_{f,t} \left[\prod_{k=1}^{j} \left(\frac{\Pi_{t+k-1}}{\Pi} \right)^{\gamma} \Pi \right]$$

Intermediate good firms first minimize the cost of production $w_t L_{f,t} + q_t K_{f,t}$ subject to its production technology,

$$Y_{f,t} = e^{\varepsilon_t^a} L_{f,t}^{1-\alpha} K_{f,t}^{\alpha} - \phi$$

where ϕ is fixed cost keeping the all intermediate good firms' profits zero at steady state. ε_t^a represents the TFP that follows $\varepsilon_t^a = \rho_a \varepsilon_{t-1}^a + \mu_t^a$. $\mu_t^a \sim \mathcal{N}(0.\sigma_a^2)$ is an I.I.D shock driving TFP precess.

$$\frac{K_{f,t}}{L_{f,t}} = \frac{\alpha w_t}{(1-\alpha) q_t}$$

Aggregating the first-order condition of cost minimization over each intermediate good firm by $\int_0^1 K_{f,t} df = K_t$ and $\int_0^1 L_{f,t} df = L_t$ leads to the relationship aggregate capital stock and labor supply.

$$\frac{K_t}{L_t} = \frac{\alpha w_t}{(1-\alpha) q_t} \tag{5.8}$$

Marginal cost is identical among all intermediate good firms.

$$MC_t = \left(\frac{w_t}{1-\alpha} \right)^{1-\alpha} \left(\frac{q_t}{\alpha} \right)^{\alpha} \tag{5.9}$$

Then intermediate good firms set the optimal price to maximize the discounted profits.

$$\max \mathbb{E}_t \sum_{j=0}^{\infty} \eta^j \left(\beta^j \frac{\lambda_{t+j}}{\lambda_t} \right) \left[\frac{P_{f,t}}{P_{t+j}} \left(\prod_{k=1}^{j} \left(\frac{\Pi_{t+k-1}}{\Pi} \right)^{\gamma} \Pi \right) - MC_{t+j} \right] Y_{f,t+j}$$

$$s.t. Y_{f,t+j} = \left[\frac{P_{f,t}}{P_{t+j}} \left(\prod_{k=1}^{j} \left(\frac{\Pi_{t+k-1}}{\Pi} \right)^{\gamma} \Pi \right) \right]^{-\frac{1+\epsilon_t}{\epsilon_t}} Y_{t+j}$$

First order condition is given by (5.10).

$$\mathbb{E}_t \sum_{j=0}^{\infty} \left\{ \begin{array}{c} (\beta\eta)^j \dfrac{\lambda_{t+1}}{\lambda_t} \dfrac{1}{\epsilon_{t+j}} \left[\dfrac{P_t^*}{P_t} \left(\prod_{k=1}^{j} \left(\dfrac{\Pi_{t+k-1}}{\Pi} \right)^{\gamma} \dfrac{\Pi}{\Pi_{t+k}} \right) \right]^{-\frac{1+\epsilon_{t+j}}{\epsilon_{t+j}}} Y_{t+j} \\ \times \left[\dfrac{P_t^*}{P_t} \left(\prod_{k=1}^{j} \left(\dfrac{\Pi_{t+k-1}}{\Pi} \right)^{\gamma} \dfrac{\Pi}{\Pi_{t+k}} \right) - (1+\epsilon_{t+j}) MC_{t+j} \right] \end{array} \right\} = 0 \quad (5.10)$$

Here P_t^* represents the optimal price set at period t. The law of motion of general price level is given by aggregating the optimal prices set by all intermediate good firms in each period.

$$\begin{aligned} 1 &= \left[\int_0^1 \left(\frac{P_{f,t}}{P_t} \right)^{-\frac{1}{\epsilon_t}} df \right]^{-\epsilon_t} \\ &= \left[(1-\eta) \left(\frac{P_t^*}{P_t} \right)^{-\frac{1}{\epsilon_t}} + \eta(1-\eta) \left(\frac{P_{t-1}^* \Pi_{t-1}^{\gamma} \Pi^{1-\gamma}}{P_t} \right)^{-\frac{1}{\epsilon_t}} + \dots \right]^{-\epsilon_t} \qquad (5.11) \\ &= (1-\eta) \left\{ \left(\frac{P_t^*}{P_t} \right)^{-\frac{1}{\epsilon_t}} + \sum_{j=1}^{\infty} \eta^j \left[\frac{P_{t-j}^*}{P_{t-j}} \left(\prod_{k=1}^{j} \left(\frac{\Pi_{t-k}}{\Pi} \right)^{\gamma} \frac{\Pi}{\Pi_{t-k+1}} \right) \right]^{-\frac{1}{\epsilon_t}} \right\} \end{aligned}$$

Log-linearization of (5.10) and (5.11) leads to the hybrid NKPC. Final aggregate output with price dispersion[5] $\Theta_t = \int_0^1 \left(\frac{P_{f,t}}{P_t} \right)^{-\frac{1+\epsilon_t}{\epsilon_t}} df$ equal to the aggregate of all intermediate output.

$$\int_0^1 Y_{f,t} df = \int_0^1 \left(\frac{P_{f,t}}{P_t} \right)^{-\frac{1+\epsilon_t}{\epsilon_t}} Y_t df = \int_0^1 \left(e^{\varepsilon_t^a} L_{f,t}^{1-\alpha} K_{f,t}^{\alpha} - \phi \right) df = e^{\varepsilon_t^a} L_t^{1-\alpha} K_t^{\alpha} - \phi$$

$$Y_t \Theta_t = e^{\varepsilon_t^a} L_t^{1-\alpha} K_t^{\alpha} - \phi \qquad (5.12)$$

5.2.3 Fiscal and Monetary Authorities

The government-central bank budget constraint is given by:

$$\frac{B_{S,t}}{P_t R_{S,t}} + \frac{B_{L,t}}{P_t R_{L,t}} + \frac{\Delta S_t}{P_t} = \frac{B_{S,t-1}}{P_t} + \frac{B_{L,t-1}}{P_t R_{S,t}} + G_t - T_t \qquad (5.13)$$

where $B_{L,t}$ and $B_{S,t}$ are the total amount of long-term government bond and short-term government bond respectively. The central bank holds long-term government bonds $B_{L,t}^{CB}$ as an asset and supplies money as a liability, so its balance sheet

[5] As proved in Galí (2015, Chapter 3), at steady state, price dispersion Θ_t is approximate to unity at first-order and zero at second-order, which means that all intermediate good firms choose the same price, and price dispersion disappears at steady state.

variation ΔS_t can be represented as the change of these two parts.

$$\frac{\Delta S_t}{P_t} = \frac{M_t - M_{t-1}}{P_t} - \left(\frac{B^{CB}_{L,t}}{P_t R_{L,t}} - \frac{B^{CB}_{L,t-1}}{P_t R_{S,t}} \right) \tag{5.14}$$

Central bank holdings of long-term governments bonds are a fraction x_t of the total amount of long-term bonds. All households hold the remaining long-term bonds[6]. The asset purchase by the central bank can be described by the variation of this fraction variable x_t that we assume it as an AR (1) process.

$$B^{CB}_{L,t} = x_t B_{L,t} \tag{5.15}$$

Combing (5.13), (5.14) and (5.15) by cancelling $\frac{\Delta S_t}{P_t}$ and $B^{CB}_{L,t}$ and rewriting nominal terms in real terms leads to the government-central bank budget constraint represented by (5.16).

$$\frac{b_{S,t}}{R_{S,t}} + \frac{b_{L,t}}{R_{L,t}} + m_t - \frac{m_{t-1}}{\Pi_t} - \left(x_t \frac{b_{L,t}}{R_{L,t}} - x_{t-1} \frac{b_{L,t-1}}{\Pi_t R_{S,t}} \right) = \frac{b_{S,t-1}}{\Pi_t} + \frac{b_{L,t-1}}{\Pi_t R_{S,t}} + G_t - T_t \tag{5.16}$$

$$B^{H}_{L,t} = (1 - x_t) B_{L,t} \tag{5.17}$$

$$\log\left(\frac{x_t}{x}\right) = \rho_x \log\left(\frac{x_{t-1}}{x}\right) + \mu^x_t \tag{5.18}$$

x is the fraction of the central bank's long-term bond holdings $\frac{B^{CB}_L}{B_L}$ at steady state. $\mu^x_t \sim \mathcal{N}(0, \sigma^2_x)$ is an i.i.d shock to drive the asset purchase process. By calibrating the size of μ^x_t and ρ_x, we can simulate the effect of asset purchase by the central bank on aggregate economic activity. ρ_x needs to be calibrated carefully because it represents the exit strategy of the central bank about when the central bank stops the QE and returns to the normal amount of government debt holdings.

Government spending is assumed to follow an AR (1) process with shock term $\mu^g_t \sim \mathcal{N}(0, \sigma^2_g)$. Long-term bonds supplied by government is assumed to be AR (1) process, as in Zagaglia (2013), where $\mu^{b_L}_t \sim \mathcal{N}(0, \sigma^2_{b_L})$.

$$\log\left(\frac{G_t}{G}\right) = \rho_g \log\left(\frac{G_{t-1}}{G}\right) + \mu^g_t \tag{5.19}$$

$$\log\left(\frac{b_{L,t}}{b_L}\right) = \rho_{b_L} \log\left(\frac{b_{L,t-1}}{b_L}\right) + \mu^{b_L}_t \tag{5.20}$$

As proposed by Leeper (1991), to prevent the inflation triggered by the fiscal expansion, a passive fiscal policy rule is introduced by Falagiarda (2014) to characterize the tax collection as a function of total government's debt,

$$T_t = \tau + \tau_S \left(\frac{b_{S,t-1}}{\Pi_t} - \frac{b_S}{\Pi} \right) + \tau_L \left(\frac{b_{L,t-1}}{R_{S,t}\Pi_t} - \frac{b_L}{R_S \Pi} \right) \tag{5.21}$$

where τ_S and τ_L are parameters that represent the reaction to the bonds' deviation from the steady-state value. Lump-sum tax T_t at steady state is τ. Because T_t is

[6] This is not true for real economy because other financial institutions can hold government debt. In this model, financial intermediaries are neglected and we can explain that all private sector households hold the remaining long-term bonds.

real tax income of government, the bonds are also represented in real terms $b_{L,t}$ and $b_{S,t}$. From this specification we can know that deviation of government debt from long-run steady state can be offset or compensated by the lump-sum tax collection from households.

The central bank is assumed to follow a Taylor (1993) rule.

$$\log\left(\frac{R_{S,t}}{R_S}\right) = \rho_R \log\left(\frac{R_{S,t-1}}{R_S}\right) + (1-\rho_R)\left\{\begin{array}{l}\log\left(\frac{\Pi_t^*}{\Pi}\right) + \varphi_Y \log\left(\frac{Y_t}{Y}\right) + \\ \varphi_\pi\left[\log\left(\frac{\Pi_t}{\Pi}\right) - \log\left(\frac{\Pi_t^*}{\Pi}\right)\right]\end{array}\right\} + \varepsilon_t^r \tag{5.22}$$

Inflation target Π_t^* is assumed to be AR (1) process where $\mu_t^{\pi^*} \sim \mathcal{N}(0, \sigma_{\pi^*}^2)$ is an i.i.d shock.

$$\log\left(\frac{\Pi_t^*}{\Pi}\right) = \rho_{\pi^*} \log\left(\frac{\Pi_{t-1}^*}{\Pi}\right) + \mu_t^{\pi^*} \tag{5.23}$$

Monetary policy shock is also assumed to be AR (1) process with disturbance term $\mu_t^r \sim \mathcal{N}(0, \sigma_r^2)$.

$$\varepsilon_t^r = \rho_r \varepsilon_{t-1}^r + \mu_t^r \tag{5.24}$$

5.2.4 Equilibrium and Asset Market

Finally, we close the model by imposing aggregate resource constraint.

$$Y_t = C_t + G_t + I_t(1 + AC_{K,t}) + \frac{b_{L,t}^H}{R_{L,t}} AC_{B,t} \tag{5.25}$$

The total output is allocated to consumption, investment government expenditure and two types of adjustment cost. This completes the description of the model. Steady state and log-linearization are given in Appendix.

Before proceeding to to numerical simulation, we do some analytical investigation about asset market to check the transmission mechanism of QE. Log-linearizing first order condition[7] (5.4) and (5.5) and combing them by cancelling $\tilde{\lambda}_t$, $\tilde{\lambda}_{t+1}$ and π_{t+1} leads to (5.26):

$$\tilde{R}_{L,t} = \tilde{R}_{S,t} + \mathbb{E}_t \tilde{R}_{S,t+1} - \left(\frac{\kappa_b \varphi_B Y}{R_S} + \varphi_B Y\right) \tilde{b}_{S,t} + \left(\frac{\kappa_b \varphi_B Y}{R_S} + \varphi_B Y\right) \tilde{b}_{L,t}^H \tag{5.26}$$

where parameters in (5.4) and (5.5) can be cancelled using the steady-state values[8]. From (5.26), we can find that long-term interest rate is positively related to the short-term interest rate, the expectation of short-term interest rate and long-term bonds held by private sector, but negatively related to short-term bonds because of the imperfect substitution of two kinds of bond assets. When central bank purchases long-term bond from private sector, long-term interest rate can be reduced to stimulate the economy. Conversely, when the central bank reduces long-term

[7] See Appendix for log-linearization of the model.

[8] See Appendix for steady state of the model.

bond holdings, less liquid asset (long-term bond) holdings of private sector increases, leading to increasing of the interest rate spread. This mechanism, *portfolio rebalancing channel* of QE can be summarized as below.

$$b^{CB}_{L,t} \uparrow \Rightarrow b^{H}_{L,t} \downarrow, b_{S,t} \uparrow \Rightarrow \tilde{b}^{H}_{L,t} < 0, \tilde{b}_{S,t} > 0 \Rightarrow \tilde{R}_{L,t} < 0 \Rightarrow R_{L,t} \downarrow \Rightarrow R_{L,t} - R_{S,t} \downarrow$$

$$b^{CB}_{L,t} \downarrow \Rightarrow b^{H}_{L,t} \uparrow, b_{S,t} \downarrow \Rightarrow \tilde{b}^{H}_{L,t} > 0, \tilde{b}_{S,t} < 0 \Rightarrow \tilde{R}_{L,t} > 0 \Rightarrow R_{L,t} \uparrow \Rightarrow R_{L,t} - R_{S,t} \uparrow$$

Note that the parameter φ_B represents the degree of adjustment cost in portfolio management. The existence of the adjustment cost makes the standard arbitrage condition invalid. When this friction disappears, $\varphi_B = 0$, the first order condition (5.4) and (5.5)'s log-linearization simplifies to the standard Euler equation, arbitrage equation and the term structure between long-term interest rate and short-term interest rate which are familiar in the standard DSGE models without adjustment cost of assets with different maturities.

$$\tilde{\lambda}_t = \tilde{R}_{S,t} + \mathbb{E}_t(\tilde{\lambda}_{t+1} - \pi_{t+1})$$

$$\tilde{\lambda}_t = \tilde{R}_{L,t} + \mathbb{E}_t(\tilde{\lambda}_{t+1} - \pi_{t+1} - \tilde{R}_{S,t+1})$$

$$\tilde{R}_{L,t} = \tilde{R}_{S,t} + \mathbb{E}_t\tilde{R}_{S,t+1}$$

To check the QE's transmission mechanism from asset market to real economy, combing the log-linearization of (5.4) and (5.5) by cancelling bond variables $\tilde{b}_{S,t} - \tilde{b}^{H}_{L,t}$ yields the Euler equation of consumption.

$$\tilde{\lambda}_t = \mathbb{E}_t(\tilde{\lambda}_{t+1} + \pi_{t+1}) + \frac{R_S}{\kappa_B + R_S}\tilde{R}_{S,t} + \frac{\kappa_B}{\kappa_B + R_S}(\tilde{R}_{L,t} - \mathbb{E}_t\tilde{R}_{S,t+1}) \tag{5.27}$$

Following the analysis of transmission mechanism inside asset market, the transmission mechanism from asset market to real economy can be summarized as below.

$$R_{L,t} \downarrow \Rightarrow \tilde{R}_{L,t} < 0 \Rightarrow \tilde{\lambda}_t < 0 \Rightarrow \lambda_t \downarrow \Rightarrow C_t \uparrow \Rightarrow Y_t \uparrow$$

Summarizing the whole analysis given previously, the story of QE in this model can be described as follows:

> *Long-term bond purchases by the central bank leads to the change of assets with different maturities, so does the assets returns (from (5.26)). Consequently, the real economy is stimulated through the general equilibrium (from (5.27).*

Analytical investigation given above describes the whole scenario. To check accurate dynamics triggered by asset purchase by the central bank, we conduct a calibration exercise.

5.3 Calibration

This model is developed to simulate the effects of QQE conducted by BoJ from April, 2013. The benchmark calibration of the steady state is to adjust to match the quarterly data over the most recent periods prior to the April, 2013. Steady-state values can be calculated from the data. GDP at steady state is normalized to unit.

Total government debt $b_S + b_L$, short-term debt[9] b_S and long-term debt[10] b_L, long-term debt held by private sector b_L^H and the central bank b_L^{CB}, are obtained from OECD Statistical Database, Ministry of Finance Japan[11] and BoJ, and calculated as the relative ratio to output.

Notation	Description	Steady-State Value[12]
Y	Output	1 (normalization)
C	Consumption	0.6114
I	Investment	0.2173
L	Labor Supply[13]	0.2308
G	Government Expenditure	0.119
T	Lump-sum Tax	0.1196
R_S	Gross Short-Term Interest Rate	1.01
R_L	Gross Long-Term Interest Rate	1.0201
Π	Gross Inflation Rate	1.0039
$b_S + b_L$	Total Debt	1.5493
b_S	Total Short-Term Debt	0.0869
b_L	Total Long-Term Debt	1.4624
b_L^{CB}	Long-Term Debt held by Central Bank	0.2296
b_L^H	Long-Term Debt held by Private Sector	1.2328
κ_B	Steady-State Ratio of $\frac{b_L^H}{b_S}$	14.1864
x	Steady-State Ratio of $\frac{b_L^{CB}}{b_L}$	0.1570

Table 5.1: Calibration for Steady State

Structural parameters and policy parameters are directly obtained from other DSGE literature. Parameters like discount factor β, capital share α and depreciation rate δ are set to their general values. Average mark-up rate in economy is set to 0.2. Calvo (1983) type price rigidity set equal to 0.75 implies an average price duration of 4 quarters, a value consistent with much of the empirical evidence. Parameters in monetary policy rule equation take the standard values in a way consistent with Taylor's original rule. To reflect a situation similar to ZLB, ρ_R is set at a highly persistent value 0.995 to prevent the short-term interest rate from responding to inflation and output change, also as proposed by Falagiarda (2014), to avoid indeterminacy of model's solution.

[9]Short-term debt b_S includes bond held by the central bank as the operation instrument in interbank market plus bonds with maturity less than or equal to one year.

[10]Long-term debt b_L is calculated by subtracting its amount from total debt.

[11]http://www.mof.go.jp/jgbs/reference/appendix/index.htm

[12]For other steady-state values, see Appendix.

[13]The steady state of labor supply is calculated by assuming that the share of representative household's time endowment spent on labor supply $\frac{L}{1-L}$ is equal to 0.3.

Notation	Description	Value
α	Capital Share	0.36
δ	Depreciation Rate	0.025
β	Discount Factor	0.994
θ	Habit Formation	0.7
ϕ	Fixed Cost in Production	0.2
χ	Inverse of Frisch Elasticity of Labor Supply	5
σ	Inverse of Inter-temporal Substitution (Risk Aversion)	2
ξ	Interest Rate Semielasticity of Money Demand	4
η	Calvo (1983) type Price Rigidity	0.75
γ	Price Indexation	0.5
ϵ	Steady-State Mark-up Rate	0.2
φ_B	Portfolio Adjustment Friction[14]	0.01
φ_K	Investment Adjustment Friction[15]	770.6056
τ	Steady-State Lump-sum Tax	0.1196
τ_S	Response to Short-Term Debt Deviation	0.3
τ_L	Response to Long-Term Debt Deviation	0.3
φ_Y	Response to Output	0.25
φ_π	Response to Inflation	1.5
ρ_R	Monetary Policy Smoothing	0.995

Table 5.2: Calibration for Structural and Policy Parameters

Two key parameters ρ_x and σ_x^2 are calibrated to replicate QE's persistence and scale[16]. Recall that BoJ has announced at April 4, 2013 that the long-term bond held by BoJ will be increased from 89 trillion yen to 190 trillion yen from the end of 2012 to end of 2014, which means 113.48% increasing of long-term bond holdings. Considering the inaccuracy of calibration, the σ_x is set to be 1 to simulate the effect of long-term bond purchase by BoJ. σ_{π^*} is set to 0.02 which means a 2% inflation target is introduced when this shock happens. Other exogenous shock parameters are set to usually used values.

[14]In other similar research, this parameter is set to different values such as Chen et al. (2012) (0.015), Andrés et al. (2004) (0.045), Harrison (2011, 2012) (0.1, 0.09). Following Falagiarda (2014), φ_B is set to 0.01 which means that 1% of household's income is paid for the portfolio adjustment cost. Sensitivity analysis given in next section checks the role of this parameter in *portfolio rebalancing channel* of QE.

[15]Note that φ_K is derived from the steady state of first order conditions (5.6) and (5.7). See Appendix.

[16]This calibration is conducted by checking the impulse response of $\tilde{x}_t = \rho_x \tilde{x}_{t-1} + \mu_t^x$ through trial and error. Just like parameter φ_B, these two parameters ρ_x and σ_x are also assumed to be important in the *portfolio rebalancing channel* of QE. Sensitivity analysis will be given in next section.

Description	AR (1)	Value	S.D.	Value
Technology	ρ_a	0.95	σ_a	0.01
Asset Purchase	ρ_x	0.83	σ_x	1
Long-Term Bond Supply	ρ_{b_L}	0.9	σ_{b_L}	0.01
Government Expenditure	ρ_g	0.9	σ_g	0.01
Inflation Target	ρ_{π^*}	0.9	σ_{π^*}	0.02
Preference	ρ_u	0.9	σ_u	0.01
Price Mark-up	ρ_p	0.9	σ_p	0.01
Monetary Policy	ρ_r	0.9	σ_r	0.01

Table 5.3: Calibration for Shock Process Parameters

5.4 Simulation of QE Policy

Using the benchmark calibration, now we report the simulation results of long-term bond purchase by BoJ. We consider a scenario that the central bank increases its long-term bond holdings 100% and takes 6 years to gradually return to its normal level.

5.4.1 Benchmark Simulation of QE Policy

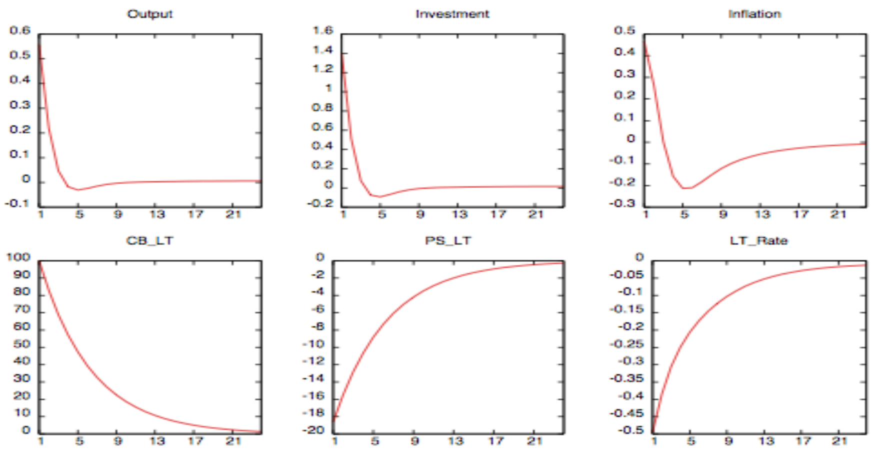


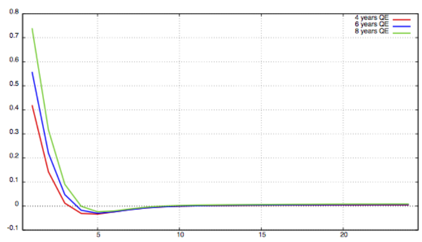
Figure 5.1: Impulse Response of QE Policy Shock

From Figure 5.1, QE has a strong effect on output and investment. Peak impact on output and investment is almost 0.56% and 1.41%. The effect stimulated by QE lasts for almost 5 quarters. As set up in the scenario, the central bank increases its long-term bond holdings (In Figure 5.1, panel `CB_LT`) on its balance sheet by 100% and will return to a normal level 6 years later. During the same period, long-term bonds held by the private sector (In Figure 5.1, panel `PS_LT`) decrease 18.62% and will return to a normal level 6 years later. The inflation rate increases

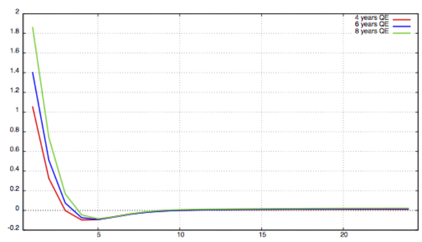
0.47% from the stimulation from QE. The long-term interest rate, which is critical to the investment, decreases 0.5%. Considering the low interest rate environment existing in Japan economy, 0.5% decreasing of yield curve is not a small number. As long as the QE has its effect, the long-term interest rate is suppressed to a low level. From the simulation results, we can conclude that the mechanism analyzed in Section 5.2.4 is appropriate. In addition, the effect stimulated by asset purchase is limited because it merely lasts for just more than one year. In this study, we do not explicitly introduce the balance sheet of the central bank, and the operations by BoJ are more complicated than what we simulated, but the positive effect of QE on the real economy can be identified with rigorous structural explanation.

5.4.2 Sensitivity Simulation of QE Policy

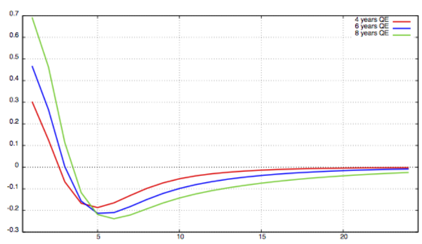
We consider different exit strategies of the central bank's QE and its effects. The benchmark simulation is set to be a 6-years QE policy for $\rho_x = 0.83$ (blue line in Figure 5.2). As the sensitivity analysis done in the Falagiarda (2014), we run two more simulations for a long-lasting QE policy (8-years and $\rho_x = 0.88$, green line in Figure 5.2) and short-lasting QE policy (4-years and $\rho_x = 0.76$, red line in Figure 5.2). Figure 5.2 shows that the longer the duration of QE, the stronger its effect. Especially for long-term interest rate, we can conclude that the push-down effect of QE to long-term interest rate lasts longer when the QE policy has a high persistence.



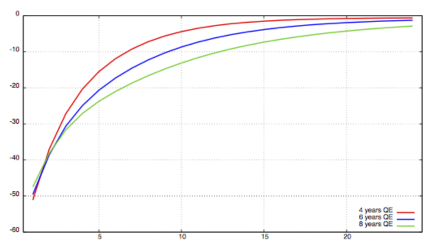
(a) IRF of Output to QE



(b) IRF of Investment to QE



(c) IRF of Inflation to QE



(d) IRF of Long-Term Interest Rate to QE

Figure 5.2: Sensitivity of QE Policy Shock

QE Persistence	Output	Investment	Inflation	Long-Term Interest Rate
$\rho_x = 0.76$ (4 years)	0.42%	1.06%	0.30%	-51bp
$\rho_x = 0.83$ (6 years)	0.56%	1.41%	0.47%	-50bp
$\rho_x = 0.88$ (8 years)	0.74%	1.87%	0.70%	-47bp

Table 5.4: Simulated Peak Impact of QE

As mentioned above in Section 5.3, φ_B is also considered to have a critical role in the effect of QE. Similar sensitivity analysis has been done for other two cases, higher portfolio adjustment cost ($\varphi_B = 0.02$) and lower portfolio adjustment cost ($\varphi_B = 0.005$), and compared with the benchmark case ($\varphi_B = 0.01$). The results is similar to Figure 5.2 so we don't report the IRF again here. Sensitivity analysis shows that with higher portfolio adjustment cost, short-term bond and long-term bond become less substitutable. The asset purchase conducted by the central bank consequently has macroeconomic effects. The effects are also amplified as φ_B increases. Also, when $\varphi_B = 0$, two kinds of bond are perfectly substitutable and no effects can be generated by QE.

5.5 Concluding Remarks

In this chapter, a DSGE model has been developed to capture the portfolio rebalancing channel of QE, and the model is calibrated to match the Japanese economy and BoJ's policy. There are two main conclusions from the simulation. The first is that the QE policy conducted by BoJ does have an effect on the real economy, pushing up output and inflation and pushing down long-term interest rates to stimulate investment. The peak impact on output is moderate for the benchmark case 0.42%, and the pushing-up effect lasts for merely 5 quarters, but the pushing-down effect on long-term rates is persistent, lasting for the whole period when the policy is effective. As QE's period becomes longer, the effect becomes larger. Under the same level of asset purchases, the central bank should announce a long-lasting time frame for QE policy. The second conclusion is that the key assumption in this study, imperfect substitution of different assets and the corresponding cost of portfolio adjustment cost, is critical to the effectiveness of QE. The key parameter φ_B is not a policy-controlled variable. If the central bank wishes to increase the effectiveness of QE, a larger scale and longer period are two options.

Another contribution is that the model developed here can be extended to more rigorous specifications of economic agents, such as the balance sheet of the central bank and the introduction of different assets. Recall that another important channel of QE, the *credit channel*, can be verified with the incorporation of financial intermediaries and other financial frictions.

Chapter 6

Summary and Outlook

Let we get a brief summary of this doctoral thesis and then talk about the outlook for future research. The basic idea for Chapter 2, Chapter 3 and Chapter 4 is that we consider the shadow rate, which has been estimated from a shadow/ZLB-GATSM in Chapter 2, as a consistent quantitative measure of monetary policy stance in ZLB environment, then we use this measure to evaluate the effect of unconventional monetary policy conducted by Bank of Japan.

In Chapter 2, we firstly introduce the basic specification of term structure model, GATSM, and then following the method proposed by Krippner (2012), we extend this basic framework to adapt the ZLB environment, explicitly allowing the zero lower bound for short interest rate. We put the estimated shadow rate and other two related quantitative measures in a context of monetary policy regime of BoJ along the time line. The estimated shadow rate shows very good traceability of monetary policy stance. There exists one problem that since the shadow/ZLB-GATSM is not compatible with negative interest rate, so we can't get the accurate estimates of shadow rate since 2016M2. We also firstly derive the information about how long the ZLB environment will last from the estimated Expected Time to Zero (ETZ). Though this measure is less important than the shadow rate, but it provides the expectation of market which may be a valuable reference for the policy decisions of central bank. In Chapter 2, we also calculated Effective Monetary Stimulus (EMS), the area of gap between observed yield curve and neutral interest rate. The stimulative degree of monetary policy among different policy regimes can be compared through the lens of the EMS.

In Chapter 3, we check the capability of shadow rate by using two typical econometric methods in empirical macroeconomics, a NK-DSGE model and a TVP-SV VAR model. The integration of shadow rate and NK-DSGE model proposed by Wu and Zhang (2016) is based on the empirical relationship between shadow rate and the balance sheet of central bank. Strictly speaking, this is not a rigorous micro-foundation, but it is convenient to use the shadow rate in DSGE model without considering the technical difficulties of ZLB. The empirical results, estimated structural parameters, impulse response and historical decomposition, are also very robust even we use the estimated shadow rate in the framework of NK-DSGE. Historical decomposition shows the positive contribution of monetary policy to the improvement of both output gap and inflation since 2013Q3. TVP-SV VAR model is non-structural, only providing statistical results about the time-varying dynamics when we use the shadow rate in ZLB period. We find that even the increasing of negative shadow rate can still trigger the decreasing of output gap and inflation. This finding convinces us to use the shadow rate to analyze the monetary policy in ZLB environment. Following Chapter 3, we estimate a medium-scale DSGE model in Chapter 4 and conduct counterfactual simulation. The structural parameters estimated with the shadow rate are quite reasonable. Counterfactual simulation shows that without the implementation of unconventional monetary policy, Japan economy would have worse performance that its actual realization.

Chapter 5 uses a calibrated medium-scale DSGE model with endogenous term structure to simulate the effect the QE policy on the holding structure of government bond. Sensitive simulation shows that higher persistence of QE policy can lead to longer-lasting decreasing of long-term rate, along with higher instantaneous impact on the real economy. We don't estimate the model in Chapter 5 because it is difficult to describe the path of bond purchase that is consistent with the actual policy actions of BoJ. But we can still get some policy implications from the experimental simulation, announcement of a long-lasting QE policy can trigger more effective macroeconomic effect.

We must admit that there still exists many flaws in this research. For example, the time span is limited to the end of 2016M1. After the start of QQE with negative interest rate, we can't get accurate estimate of shadow rate and we have to give up the further analysis beyond 2016M1. Also, declaring again, the empirical relationship which we rely on to incorporate the shadow rate into structural models is not rigorously theoretical. We have to admit that this approach has some arbitrariness, but on the other hand, this approach is very convenient and simple.

There exists some perspectives for future research. Firstly, given the realization of negative interest rate, we have to figure out a way to deal with negative interest rate in term structure model, in a reasonable, mathematical-consistent fashion. Secondly, we should establish a theoretical-consistent estimation of shadow rate, not from a factor model with less economic interpretation, but something like a joint estimation of structural macroeconomic model and shadow rate with rigorous theoretical foundation, which can explain the existence of negative interest rate reasonably with rigorous theoretical foundation. Thirdly, this research takes a totally macroeconomic vision with less attention on the effects of unconventional monetary policy effects from a microeconomic vision. Is it easy for firms and households to borrow money from banks due to the implementation of unconventional monetary policy? Dose the behavior of microeconomic agents change due to the different monetary policy regimes? Such questions can be clarified only in micro-econometric procedures.

Finally, we have to say that what we know about the unconventional monetary policy is still limited, not only for macroeconomics from an academic view, but also for central banks from a practical view, so it is worth to keep tracking this topic in the future.

Appendix

Appendix for Chapter 1

In Appendix for Chapter 1, we derive a standard New Keynesian Dynamic Stochastic General Equilibrium (NK-DSGE) model which is used through this doctoral thesis. This model is also used in Chapter 3.

A representative infinitely-living household maximizes lifetime utility

$$\mathbb{E}_t \sum_{t=0}^{\infty} \beta^t \left[\frac{C_t^{1-\sigma}}{1-\sigma} - \frac{\chi L_t^{1+\eta}}{1+\eta} \right]$$

subject to the budget constraint

$$C_t + \frac{B_t}{P_t} \leq \frac{R_{t-1}^B B_{t-1}}{P_t} + W_t L_t + T_t$$

where C_t and L_t denote household's consumption and labor supply. P_t is the price level. The nominal gross bond return paid for bonds B_t is R_{t-1}^B. W_t is real wage and T_t is the real transfer. Two first-order conditions decide the optimal decision of consumption and labor supply

$$C_t^{-\sigma} = \beta \mathbb{E}_t R_t^B \left(\frac{C_{t+1}^{-\sigma}}{\Pi_{t+1}} \right)$$

$$W_t = \frac{\chi L_t^{\eta}}{C_t^{-\sigma}}$$

where $\Pi_{t+1} = \frac{P_{t+1}}{P_t}$ is the gross inflation from t period to $t+1$ period. The specification of intermediate good firms and final good firms is same as the standard NK-DSGE model. A continuum of intermediate good firms exist, producing heterogenous intermediate goods and selling them into final good firms. Let Y_t be the output of the final good which is produced using inputs of the intermediate goods according to a bundle production function

$$Y_t = \left[\int_0^1 Y_{j,t}^{\frac{\varepsilon-1}{\varepsilon}} dj \right]^{\frac{\varepsilon}{\varepsilon-1}}$$

where $\varepsilon > 1$ is the elasticity of substitution among differentiated intermediate goods and $Y_{j,t}$ is the input of intermediate good $j \in [0,1]$. Final good firms in completely competitive market maximize profits

$$\max_{Y_{j,t}} P_t Y_t - \int_0^1 P_{j,t} Y_{j,t} dj$$

subject to the bundle production function. The optimal input for intermediate good $Y_{j,t}$ is

$$Y_{j,t} = \left(\frac{P_{j,t}}{P_t}\right)^{-\varepsilon} Y_t$$

and the zero profit condition of final good market leads to the general price level index P_t.

$$P_t = \left(\int_0^1 P_{j,t}^{1-\varepsilon} dj\right)^{\frac{1}{1-\varepsilon}}$$

Intermediate good firms produce and sell differentiated products to final good firms in monopolistically competitive markets. A intermediate good firm j minimizes the total cost $W_t L_{j,t}$ subject to a Cobb-Douglas production function given by

$$Y_{j,t} = A_t L_{j,t}^{1-\alpha}$$

where A_t is aggregate productivity shock and $L_{j,t}$ is labor input. Cost minimization leads to the real marginal cost.

$$MC_{j,t} = \frac{W_t}{A_t(1-\alpha)(Y_{j,t}/A_t)^{-\alpha/(1-\alpha)}}$$

Define the economy-wide average real marginal cost as

$$MC_t = \frac{W_t}{A_t(1-\alpha)(Y_t/A_t)^{-\alpha/(1-\alpha)}}$$

and the relation between intermediate good firm j's marginal cost and the economy-wide average real marginal cost is

$$MC_{j,t} = MC_t \left(\frac{Y_t}{Y_{j,t}}\right)^{\frac{-\alpha}{1-\alpha}} = MC_t \left(\frac{P_{j,t}}{P_t}\right)^{\frac{\varepsilon\alpha}{\alpha-1}}$$

which can be derived by the demand curve of intermediate good j. Intermediate good firm's objective is to maximize the discounted present value of real profits according to Calvo (1983) price setting mechanism

$$\max_{P_t^*} \mathbb{E}_t \sum_{t=0}^{\infty} (\beta\theta)^k \frac{\Lambda_{t+k}}{\Lambda_t} \left(\frac{P_t^* Y_{j,t+k|t}}{P_{t+k}} - MC_{j,t+k} Y_{j,t+k|t}\right)$$

subject to demand curve of $Y_{j,t+k|t}$,

$$Y_{j,t+k|t} = \left(\frac{P_t^*}{P_{t+k}}\right)^{-\varepsilon} Y_{t+k}$$

where θ is the probability that the intermediate good firm can't adjust its price and Λ_t is the Lagrange multiplier in the optimization of household which represents the marginal utility of consumption. The first-order condition leads to the optimal price setting for intermediate firm

$$P_t^* = \frac{\varepsilon}{\varepsilon-1} \frac{\mathbb{E}_t \sum_{k=0}^{\infty} (\beta\theta)^k \Lambda_{t+k} P_{t+k}^{\varepsilon} Y_{t+k} MC_{j,t+k|t}}{\mathbb{E}_t \sum_{k=0}^{\infty} (\beta\theta)^k \Lambda_{t+k} P_{t+k}^{\varepsilon-1} Y_{t+k}}$$

where $MC_{j,t+k|t} = MC_{t+k}\left(\frac{P_t^*}{P_{t+k}}\right)^{\frac{\varepsilon\alpha}{\alpha-1}}$. By a law of large number, a fraction θ of intermediate good firms can't adjust prices and keep prices at the previous period price level P_{t-1} and the remaining fraction $1-\theta$ of intermediate good firms adjust to the new level P_t^*, so the price level $P_t = \left(\int_0^1 P_{j,t}^{1-\varepsilon} dj\right)^{\frac{1}{1-\varepsilon}}$ can be rewritten as a weighted sum of all intermediate good firms' prices.

$$P_t^{1-\varepsilon} = \theta P_{t-1}^{1-\varepsilon} + \theta\left(P_t^*\right)^{1-\varepsilon}$$

Under flexible price equilibrium, $\theta = 0$ and price rigidity disappears. The optimal price setting of intermediate good firm is the standard result in microeconomics

$$P_t^* = P_t \frac{\varepsilon}{\varepsilon - 1} MC_{j,t}$$

where $\frac{\varepsilon}{\varepsilon-1}$ can be explained as a markup charged by the intermediate good firm. When prices are flexible, all intermediate good firms are symmetric and charge the same price such that $P_t^* = P_t$, $MC_{j,t} = \frac{\varepsilon-1}{\varepsilon}$, $Y_{j,t} = Y_t$ and $L_{j,t} = L_t$ for all j. Real wage is equal to marginal productivity of labor.

$$W_t = \frac{\varepsilon - 1}{\varepsilon}(1-\alpha)A_t L_t^{-\alpha}$$

Combining this equation to the first-order condition of household's labor supply leads to

$$\frac{\chi L_t^{\eta}}{C_t^{-\sigma}} = \frac{\varepsilon - 1}{\varepsilon}(1-\alpha)A_t L_t^{-\alpha}$$

from which we can solve the output Y_t^f with the resource constraint $Y_t = C_t = A_t L_t^{1-\alpha}$ under flexible price equilibrium.

$$Y_t^f = \left[\frac{(\varepsilon-1)(1-\alpha)}{\varepsilon\chi}\right]^{\frac{1-\alpha}{\sigma(1-\alpha)+\eta+\alpha}} A_t^{\frac{1+\eta}{\sigma(1-\alpha)+\alpha+\eta}}$$

Log-linearization of all equilibrium conditions of final good and intermediate good firms leads to the standard New Keynesian Phillips Curve

$$\pi_t = \beta\mathbb{E}_t\pi_{t+1} + \frac{(1-\theta)(1-\beta\theta)(1-\alpha)}{\theta(1-\alpha+\varepsilon\alpha)}\widehat{MC}_t$$

where

$$\widehat{MC}_t = \frac{\eta + \alpha + \sigma(1-\alpha)}{1-\alpha}\hat{Y}_t - \frac{1+\eta}{1-\alpha}\hat{A}_t$$

is the percentage deviation of economy-wide average real marginal cost. Using the output under flexible price equilibrium, this can be written as

$$\widehat{MC}_t = \frac{\eta + \alpha + \sigma(1-\alpha)}{1-\alpha}\left(\hat{Y}_t - \hat{Y}_t^f\right) = \frac{\eta + \alpha + \sigma(1-\alpha)}{1-\alpha}x_t$$

where $x_t = \hat{Y}_t - \hat{Y}_t^f$ can be explained as the output gap. So relation between the inflation and the output gap can be rewritten as

$$\pi_t = \beta\mathbb{E}_t\pi_{t+1} + \kappa x_t$$

where $\kappa = \frac{(1-\theta)(1-\beta\theta)[\eta+\alpha+\sigma(1-\alpha)]}{\theta(1-\alpha+\varepsilon\alpha)}$. Similarly, log-linearization of the household's Euler equation leads to

$$\hat{Y}_t = \mathbb{E}_t \hat{Y}_{t+1} - \frac{1}{\sigma}\left(r_t^B + \ln\beta - \mathbb{E}_t \pi_{t+1}\right)$$

which also holds at the flexible price equilibrium as

$$\hat{Y}_t^f = \mathbb{E}_t \hat{Y}_{t+1}^f - \frac{1}{\sigma}\left(r_t^N + \ln\beta\right)$$

where $r_t^N = -\ln\beta + \sigma\left(\mathbb{E}_t \hat{Y}_{t+1}^f - \hat{Y}_t^f\right)$ is the natural interest rate. Using the definition of output gap, the New Keynesian IS curve is

$$x_t = \mathbb{E}_t x_{t+1} - \frac{1}{\sigma}\left(r_t^B - \mathbb{E}_t \pi_{t+1} - r_t^N\right)$$

where $r_t^N = -\ln\beta + \frac{\sigma(1+\eta)}{\sigma(1-\alpha)+\alpha+\eta}\left(\mathbb{E}_t \hat{A}_{t+1} - \hat{A}_t\right)$ only depends on exogenous productivity shock.

Appendix for Chapter 2

In Appendix for Chapter 2, we give the derivation of a two-factor shadow/ZLB-GATSM. Expectation of short interest rate:

$$\mathbb{E}_t^{\mathbb{Q}}\left(r_{t+\tau}|x_t\right) = a_0 + b_0^\top\left[\hat{\theta} + e^{-\hat{\kappa}\tau}\left(x_t - \hat{\theta}\right)\right] = \begin{bmatrix}1 & 1\end{bmatrix}\begin{bmatrix}1 & 0\\ 0 & e^{-\varphi\tau}\end{bmatrix}\begin{bmatrix}L_t\\ S_t\end{bmatrix} = L_t + e^{-\varphi\tau}S_t$$

Variance of short interest rate:

$$\begin{aligned}
\omega_\tau^2 &= \int_0^\tau b_0^\top e^{-\hat{\kappa}u}\sigma\sigma^\top e^{-\hat{\kappa}^\top u}b_0 du = \int_0^\tau \begin{bmatrix}1 & 1\end{bmatrix}\begin{bmatrix}1 & 0\\ 0 & e^{-\varphi u}\end{bmatrix}\begin{bmatrix}\sigma_1^2 & \rho_{12}\sigma_1\sigma_2\\ \rho_{12}\sigma_1\sigma_2 & \sigma_2^2\end{bmatrix}\begin{bmatrix}1 & 0\\ 0 & e^{-\varphi u}\end{bmatrix}\begin{bmatrix}1\\ 1\end{bmatrix}du\\
&= \int_0^\tau\left(\sigma_1^2 + \sigma_2^2 e^{-2\varphi u} + 2\rho_{12}\sigma_1\sigma_2 e^{-\varphi u}\right)du = \sigma_1^2\tau + \sigma_2^2\frac{1-e^{-2\varphi\tau}}{2\varphi} + 2\rho_{12}\sigma_1\sigma_2\frac{1-e^{-\varphi\tau}}{\varphi}
\end{aligned}$$

Volatility effect:

$$\begin{aligned}
V_\tau &= \int_0^\tau b_0^\top e^{-\hat{\kappa}(\tau-s)}\sigma\left[\sigma^\top\int_s^\tau e^{-\hat{\kappa}^\top(u-s)}b_0 du\right]ds\\
&= \int_0^\tau\left\{\begin{bmatrix}1 & 1\end{bmatrix}\begin{bmatrix}1 & 0\\ 0 & e^{-\varphi(\tau-s)}\end{bmatrix}\begin{bmatrix}\sigma_1^2 & \rho_{12}\sigma_1\sigma_2\\ \rho_{12}\sigma_1\sigma_2 & \sigma_2^2\end{bmatrix}\left[\int_s^\tau\begin{bmatrix}1 & 0\\ 0 & e^{-\varphi(u-s)}\end{bmatrix}\begin{bmatrix}1\\ 1\end{bmatrix}du\right]ds\right\}\\
&= \int_0^\tau\left\{\begin{bmatrix}\sigma_1^2 + \rho_{12}\sigma_1\sigma_2 e^{-\varphi(\tau-s)} & \rho_{12}\sigma_1\sigma_2 + e^{-\varphi(\tau-s)}\sigma_2^2\end{bmatrix}\left[\int_s^\tau\begin{bmatrix}1\\ e^{-\varphi(u-s)}\end{bmatrix}du\right]ds\right\}\\
&= \int_0^\tau\left\{\begin{bmatrix}\sigma_1^2 + \rho_{12}\sigma_1\sigma_2 e^{-\varphi(\tau-s)} & \rho_{12}\sigma_1\sigma_2 + e^{-\varphi(\tau-s)}\sigma_2^2\end{bmatrix}\begin{bmatrix}\tau - s\\ \frac{1}{\varphi}\left(1-e^{\varphi(s-\tau)}\right)\end{bmatrix}ds\right\}\\
&= \int_0^\tau\left[\left(\sigma_1^2 + \rho_{12}\sigma_1\sigma_2 e^{-\varphi(\tau-s)}\right)(\tau-s) + \left(\rho_{12}\sigma_1\sigma_2 + e^{-\varphi(\tau-s)}\sigma_2^2\right)\frac{1}{\varphi}\left(1-e^{\varphi(s-\tau)}\right)\right]ds\\
&= \frac{\sigma_1^2\tau^2}{2} + \frac{\rho_{12}\sigma_1\sigma_2\tau}{\varphi}\left(1-e^{-\varphi\tau}\right) + \frac{\sigma^2}{2\varphi^2}\left(1-2e^{-\varphi\tau}+e^{-2\varphi\tau}\right)
\end{aligned}$$

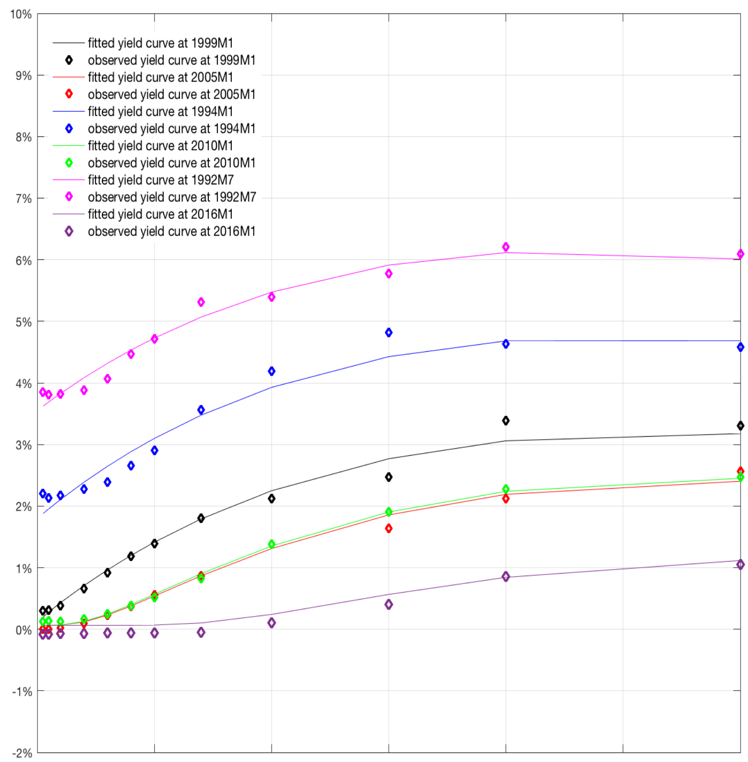
Forward interest rate:

$$f_{t,\tau} = \mathbb{E}_t^{\mathbb{Q}}\left(r_{t+\tau}|x_t\right) - V_\tau = L_t + e^{-\varphi\tau}S_t - V_\tau$$

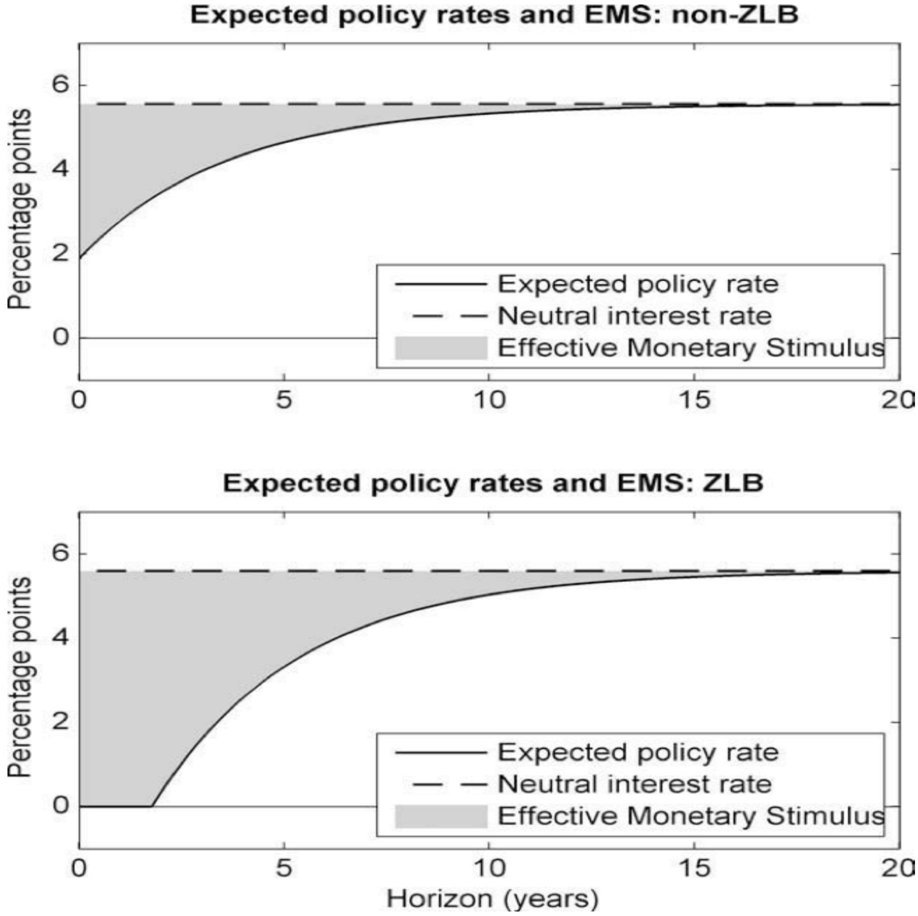
Interest rate:

$$\begin{aligned}
R_{t,\tau} &= \frac{1}{\tau}\int_0^\tau \mathbb{E}_t^{\mathbb{Q}}(r_{t+\tau}|x_t)du - \frac{1}{\tau}\int_0^\tau V_\tau d\tau = \frac{1}{\tau}\int_0^\tau\begin{bmatrix}1 & e^{-\varphi u}\end{bmatrix}\begin{bmatrix}L_t\\ S_t\end{bmatrix}du\\
&\quad - \frac{1}{\tau}\int_0^\tau\left[\frac{\sigma_1^2\tau^2}{2} + \frac{\rho_{12}\sigma_1\sigma_2\tau}{\varphi}\left(1-e^{-\varphi\tau}\right) + \frac{\sigma_2^2}{2\varphi^2}\left(1-2e^{-\varphi\tau}+e^{-2\varphi\tau}\right)\right]d\tau\\
&\quad = a_\tau + \begin{bmatrix}1 & \frac{1}{\tau\varphi}\left(1-e^{-\varphi\tau}\right)\end{bmatrix}\begin{bmatrix}L_t\\ S_t\end{bmatrix} = a_\tau + L_t + \frac{1}{\varphi\tau}\left(1-e^{-\varphi\tau}\right)S_t
\end{aligned}$$

$$a_\tau = -\frac{\sigma_1^2\tau^2}{6} - \frac{\sigma_2^2}{2\varphi^2}\left[1 - \frac{1}{2\varphi\tau}e^{-2\varphi\tau} + \frac{2}{\varphi\tau}e^{-\varphi\tau} - \frac{3}{2\varphi\tau}\right] - \frac{\rho_{12}\sigma_1\sigma_2}{\varphi^2}\left[-\frac{1-e^{-\varphi\tau}}{\varphi\tau} + \frac{\varphi\tau}{2} + e^{-\varphi\tau}\right]$$



Selected Fitted and Observed Yield Curves



Conceptual Image of Effective Monetary Stimulus from Krippner (2015, P256)

Appendix for Chapter 3

TVP-SV VAR Model

In Appendix for Chapter 3, we give some technical details of TVP-SV VAR model. We consider the VAR model with following specification,

$$Y_t = c_t + B_{1,t}Y_{t-1} + \cdots B_{s,t}Y_{t-s} + u_t$$

where $u_t \sim \mathcal{N}(0, \Omega_t)$ is the disturbance term. $Y_t = (Y_{1,t}, Y_{2,t}, \cdots Y_{k,t})'$ is a vector with k variables. Parameters $c_t, B_{1,t}, \cdots B_{s,t}, \Omega_t$ are all time-varying. The variance-covariance matrix of the disturbance term Ω_t can be represented in following structure by Cholesky decomposition.

$$\Omega_t = A_t^{-1}\Sigma_t\Sigma_t' A_t^{-1'}$$

A_t is a $k \times k$ lower triangular matrix with 1 on all diagonal elements. Σ_t is a $k \times k$ diagonal matrix.

$$A_t = \begin{bmatrix} 1 & 0 & \cdots & 0 \\ a_{21,t} & 1 & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ a_{k1,t} & \cdots & a_{k,k-1,t} & 1 \end{bmatrix}, \Sigma_t = \begin{bmatrix} \sigma_{1,t} & 0 & \cdots & 0 \\ 0 & \sigma_{2,t} & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & \sigma_{k,t} \end{bmatrix}$$

$\sigma_{i,t}$ is the time-varying variance of i-th variable's structural shock and $a_{ij,t}$ is a time-varying coefficient which represents the instantaneous effect from structural shock of j-th variable to i-th variable. Define

$$X_t = I_k \otimes \left(1, y_{t-1}', y_{t-2}', \cdots, y_{t-s}'\right)$$

$$\beta_t = [c_t, B_{1,t}, B_{2,t}, \cdots, B_{s,t}]'$$

and rewrite the model in stacked form,

$$Y_t = X_t\beta_t + A_t^{-1}\Sigma_t e_t$$

where $e_t = (e_{1,t}, e_{2,t}, \cdots e_{k,t})'$ is the vector of normalized structural shocks. Given $e_t \sim \mathcal{N}(0, I_k)$, we have $A_t^{-1}\Sigma_t e_t \sim \mathcal{N}\left(0, A_t^{-1}\Sigma_t I_k \Sigma_t' A_t^{-1'}\right)$, so u_t can be written as $u_t = A_t^{-1}\Sigma_t e_t$. Define the lower triangular elements in A_t as a vector

$$a_t = (a_{21,t}, a_{31,t}, a_{32,t}, \cdots a_{k,k-1,t})'$$

and the diagonal elements in Σ_t as a vector

$$h_t = (h_{1,t}, h_{2,t} \cdots h_{k,t})'$$

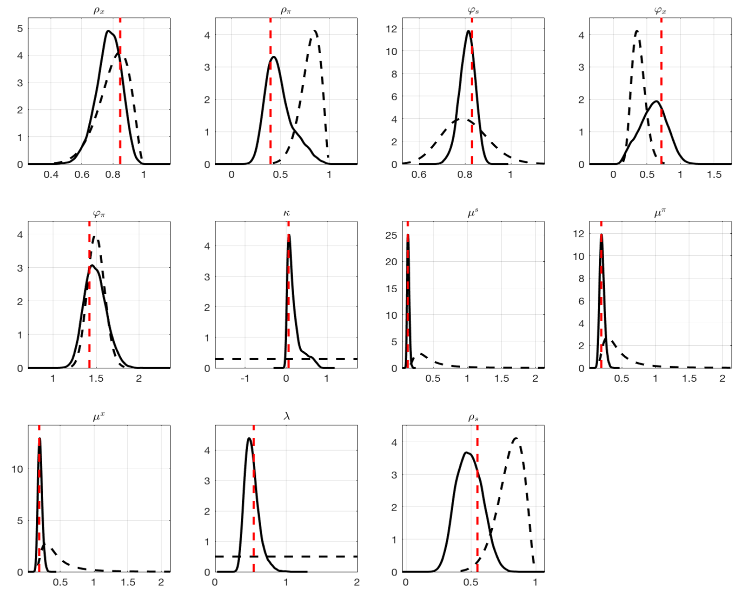
where $h_{i,t} = \log \sigma_{i,t}^2$. The parameters in TVP-SV VAR model (β_t, a_t, h_t) are assumed to follow random walk process,

$$\beta_{t+1} = \beta_t + u_{\beta,t}$$
$$a_{t+1} = a_t + u_{a,t}$$
$$h_{t+1} = h_t + u_{h,t}$$

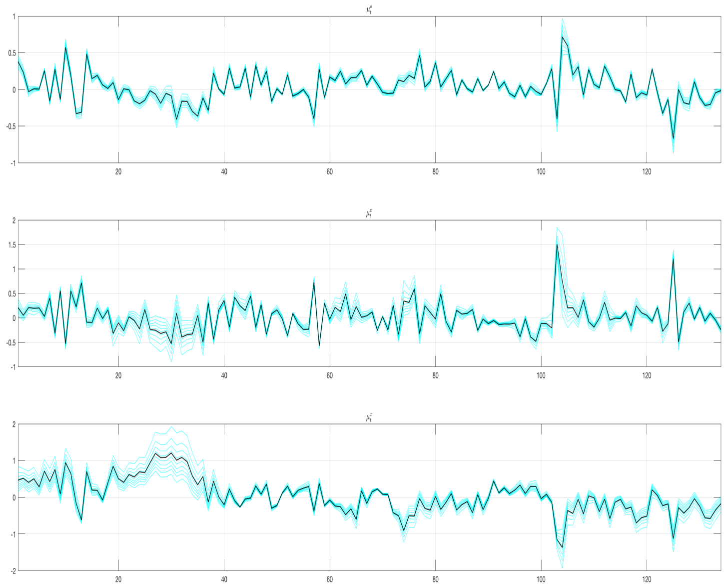
where $u_{\beta,t} \sim N(0, \Sigma_\beta), u_{a,t} \sim N(0, \Sigma_a), u_{h,t} \sim N(0, \Sigma_h)$ are also parameters for estimation.

The estimation of TVP-SV VAR model is based on Bayesian Theorem. The joint posterior distribution's kernel is given by $P(\beta_t, a_t, h_t, \Sigma_\beta, \Sigma_a, \Sigma_h \mid X_t, Y_t) \propto P(Y_t \mid X_t, \beta_t, a_t, h_t, \Sigma_\beta, \Sigma_a, \Sigma_h) \times P(\beta_t, a_t, h_t, \Sigma_\beta, \Sigma_a, \Sigma_h)$ where $P(Y_t \mid X_t, \beta_t, a_t, h_t, \Sigma_\beta, \Sigma_a, \Sigma_h)$ is likelihood function and $P(\beta_t, a_t, h_t, \Sigma_\beta, \Sigma_a, \Sigma_h)$ is prior distribution. For the selection of prior distribution and the algorithm of MCMC sampling, please refer to Nakajima (2011).

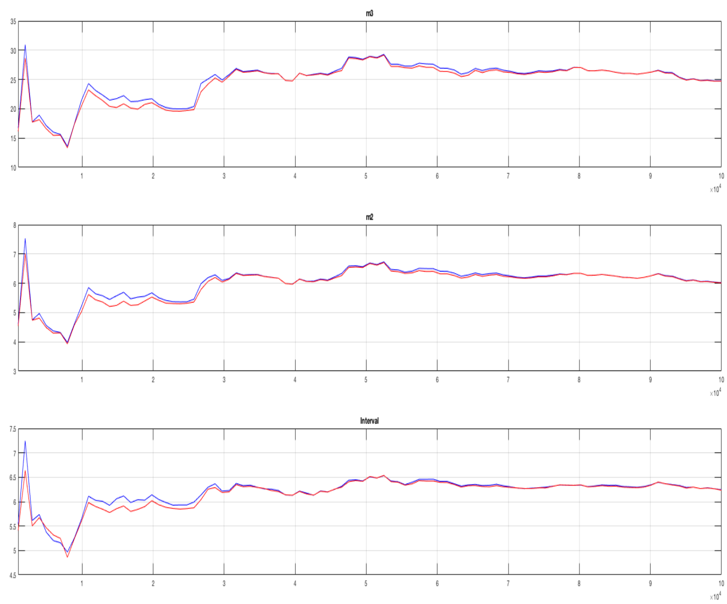
Figures from NK-DSGE Estimation with Shadow Rate



Prior and Posterior Distribution of Structural Parameters



Smoothed Structural Shocks

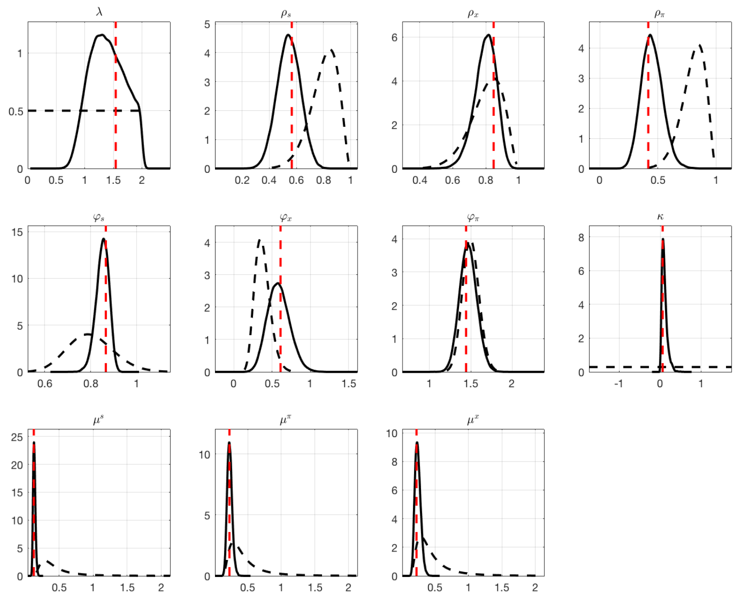


Multivariate Convergence Diagnostic

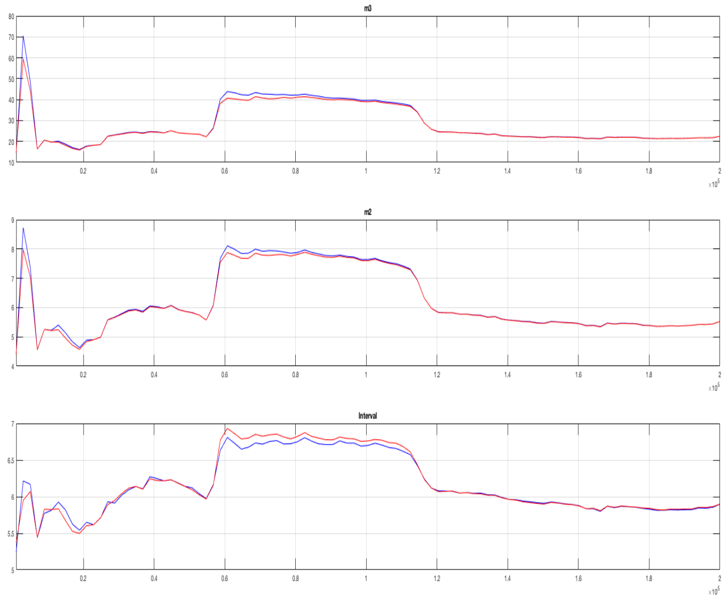
Estimation Results of 1999Q1-2016Q3 Sample

	Prior Distribution			Posterior Distribution		
Parameters	Mean	St.Dev.	Prior	Mean	St.Dev.	90% HPD Interval
φ_π	1.5	0.1	G	1.4679	0.1035	[1.2925, 1.6328]
φ_x	0.375	0.1	G	0.5839	0.1434	[0.3520, 0.8209]
φ_s	0.8	0.1	B	0.8506	0.0281	[0.8050, 0.8959]
ρ_π	0.8	0.1	B	0.4475	0.0906	[0.2967, 0.5921]
ρ_x	0.8	0.1	B	0.7949	0.0658	[0.6906, 0.9022]
ρ_s	0.8	0.1	B	0.5421	0.0843	[0.4001, 0.6784]
μ^π	0.5	0.5	IG	0.2424	0.0367	[0.1730, 0.2912]
μ^x	0.5	0.5	IG	0.2325	0.0432	[0.1749, 0.3130]
μ^s	0.5	0.5	IG	0.1364	0.0168	[0.1088, 0.1629]
Parameters	Prior Distribution			Mean	St.Dev.	90% HPD Interval
κ	U(0, 1)			0.1058	0.0679	[0.0102, 0.2029]
λ	U(0, 2)			1.3958	0.3007	[0.9527, 1.9328]

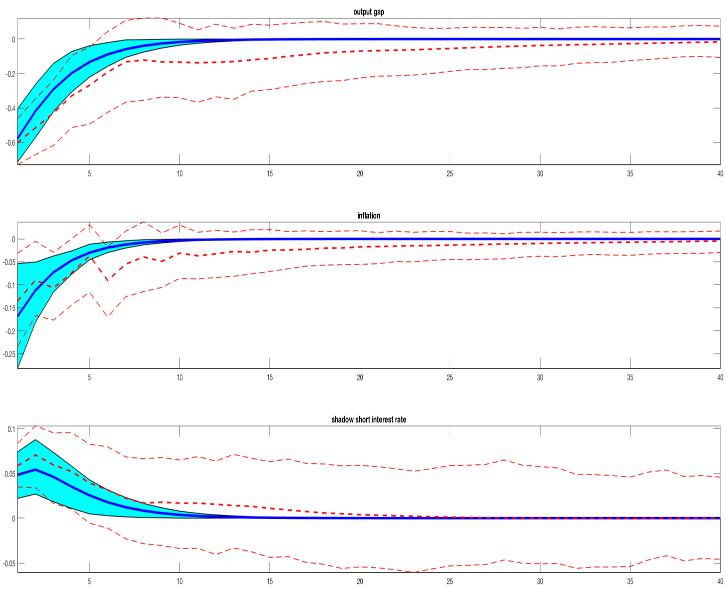
Prior and Posterior Distribution of Structural Parameters



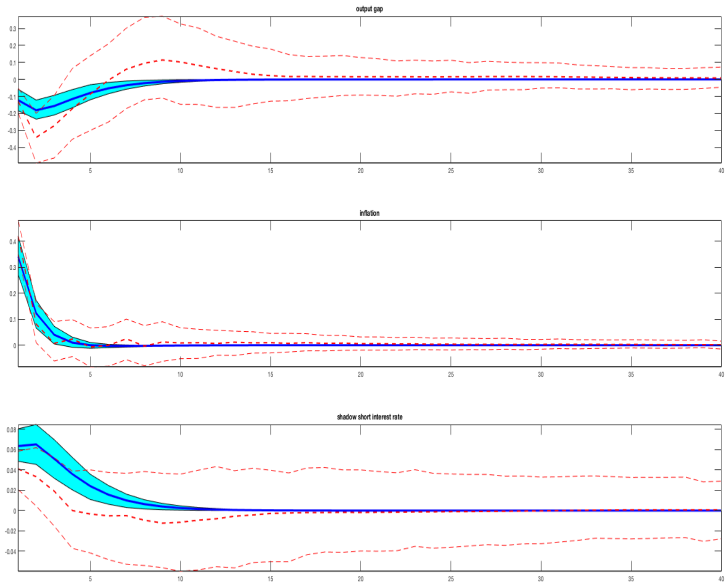
Prior and Posterior Distribution of Structural Parameters



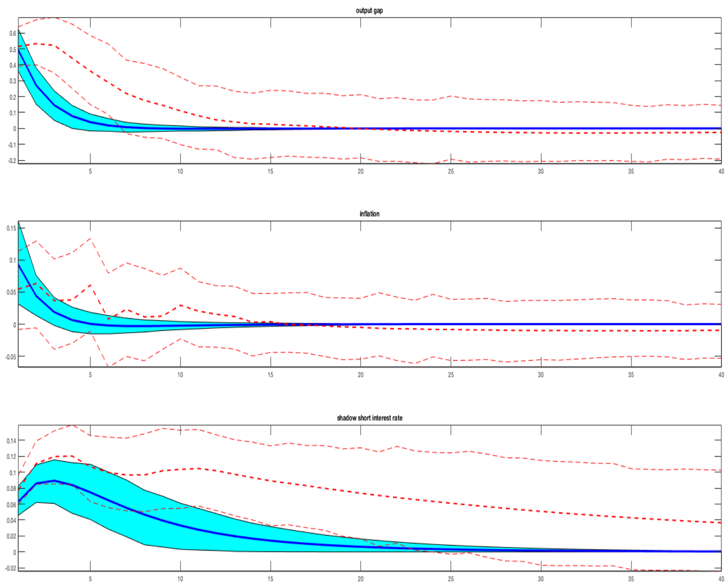
Multivariate Convergence Diagnostic



DSGE-VAR Impulse Response of Monetary Policy Shock



DSGE-VAR Impulse Response of Supply Shock



DSGE-VAR Impulse Response of Demand Shock

Appendix for Chapter 4

We show the de-trend, steady state and log-linearization of the medium-scale DSGE model used in Chapter 4. Since the model has a long run growth trend, it is necessary to de-trend model to make it stationary. De-trending $\{Y_t, Y_t^*, C_t, W_t, K_t, I_t\}$ by dividing Z_t leads to stationary variables $\{y_t, y_t^*, c_t, w_t, k_t, i_t\}$. Consumption marginal utility Λ_t can be de-trend as $\lambda_t = \frac{\Lambda_t}{Z_t^{-\sigma}}$. De-trend equilibrium conditions (4.1)-(4.16) and log-linearizing around steady state leads to the stationary model. $\widehat{X}_t \approx \frac{X_t - X}{X}$ means the deviation from its steady state. $\pi_t = \frac{\Pi_t - \Pi}{\Pi}$ is the net inflation rate.

We have 14 endogenous variables $\{\widehat{\lambda}_t, \widehat{c}_t, \widehat{i}_t, \widehat{y}_t, \widehat{y}_t^*, \widehat{k}_t, \widehat{w}_t, \widehat{L}_t, \pi_t, \widehat{U}_t, \widehat{Q}_t, \widehat{R}_t^K, \widehat{R}_t^N, \widehat{MC}_t\}$, 7 shock processes $\{\varepsilon_t^b, \varepsilon_t^z, \varepsilon_t^i, \varepsilon_t^g, \varepsilon_t^w, \varepsilon_t^p, \varepsilon_t^r\}$ with 7 innovation terms $\{\mu_t^b, \mu_t^z, \mu_t^i, \mu_t^g, \mu_t^w, \mu_t^p, \mu_t^r\}$.

The Log-linearized Model

$$\widehat{k}_t = \frac{1-\delta}{z}\left(\widehat{k}_{t-1} - \varepsilon_t^z\right) + \frac{R^K}{z}\widehat{U}_t + \left(1 - \frac{1-\delta}{z}\right)\widehat{i}_t$$

$$\left(1-\frac{\theta}{z}\right)\left(1-\frac{\beta\theta}{z^\sigma}\right)\widehat{\lambda}_t = -\sigma\left(\widehat{c}_t - \frac{\theta}{z}\widehat{c}_{t-1} + \frac{\theta}{z}\varepsilon_t^z\right) + \left(1-\frac{\theta}{z}\right)\varepsilon_t^b$$
$$+ \frac{\beta\theta}{z^\sigma}\left[\sigma\left(E_t\widehat{c}_{t+1} - \frac{\theta}{z}\widehat{c}_t + E_t\varepsilon_{t+1}^z\right) - \left(1-\frac{\theta}{z}\right)E_t\varepsilon_{t+1}^b\right]$$

$$\widehat{\lambda}_t = E_t\widehat{\lambda}_{t+1} + \widehat{R}_t^N - E_t\pi_{t+1} - \sigma E_t\varepsilon_{t+1}^z$$

$$\widehat{U}_t = \frac{\delta'(U=1)}{\delta''(U=1)}\left(\widehat{R}_t^K - \widehat{Q}_t\right) = \mu\left(\widehat{R}_t^K - \widehat{Q}_t\right)$$

$$\widehat{Q}_t = E_t\widehat{\lambda}_{t+1} - \widehat{\lambda}_t - \sigma E_t\varepsilon_{t+1}^z + \frac{\beta}{z^\sigma}\left[R^K E_t\widehat{R}_{t+1}^K + (1-\delta)E_t\widehat{Q}_{t+1}\right]$$

$$\widehat{Q}_t + \frac{\beta z^{1-\sigma}}{\zeta}\left(E_t\widehat{i}_{t+1} - \widehat{i}_t + E_t\varepsilon_{t+1}^z + E_t\varepsilon_{t+1}^i\right) = \frac{1}{\zeta}\left(\widehat{i}_t - \widehat{i}_{t-1} + \varepsilon_t^z + \varepsilon_t^i\right)$$

$$\widehat{MC}_t = (1-\alpha)\widehat{w}_t + \alpha\widehat{R}_t^K$$

$$\widehat{w}_t + \widehat{L}_t = \widehat{R}_t^K + \widehat{U}_t + \widehat{k}_{t-1} - \varepsilon_t^z$$

$$\widehat{y}_t = \left(1+\frac{\varphi}{y}\right)\left[(1-\alpha)\widehat{L}_t + \alpha\left(\widehat{U}_t + \widehat{k}_{t-1} - \varepsilon_t^z\right)\right]$$

$$\pi_t - \gamma_p\pi_{t-1} = \beta z^{1-\sigma}\left(E_t\pi_{t+1} - \gamma_p\pi_t\right) + \frac{(1-\xi_p)\left(1-\beta\xi_p z^{1-\sigma}\right)}{\xi_p}\widehat{MC}_t + \varepsilon_t^p$$

$$\widehat{w}_t - \widehat{w}_{t-1} + \pi_t - \gamma_w\pi_{t-1} + \varepsilon_t^z = \beta z^{1-\sigma}\left(E_t\widehat{w}_{t+1} - \widehat{w}_t + E_t\pi_{t+1} - \gamma_w\pi_t + E_t\varepsilon_{t+1}^z\right)$$
$$+ \frac{1-\xi_w}{\xi_w}\frac{\left(1-\beta\xi_w z^{1-\sigma}\right)\lambda^w}{\lambda^w + \chi(1+\lambda^w)}\left(\chi\widehat{L}_t - \widehat{\lambda}_t - \widehat{w}_t + \varepsilon_t^b\right) + \varepsilon_t^w$$

$$\widehat{y}_t = \frac{c}{y}\widehat{c}_t + \frac{i}{y}\widehat{i}_t + \frac{g}{y}\varepsilon_t^g$$

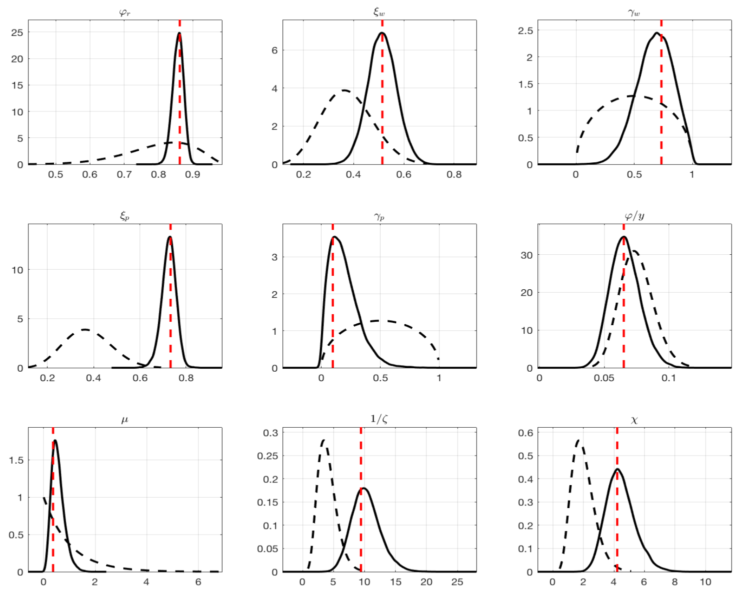
$$\widehat{y}_t^* = -\alpha\left(1+\frac{\varphi}{y}\right)\varepsilon_t^z$$

$$\widehat{R}_t^N = \varphi_r \widehat{R}_{t-1}^N + (1-\varphi_r)\left[\varphi_\pi \left(\frac{1}{4}\sum_{j=0}^{3} \pi_{t-j}\right) + \varphi_y \left(\widehat{y} - \widehat{y}_t^*\right)\right] + \varepsilon_t^r$$

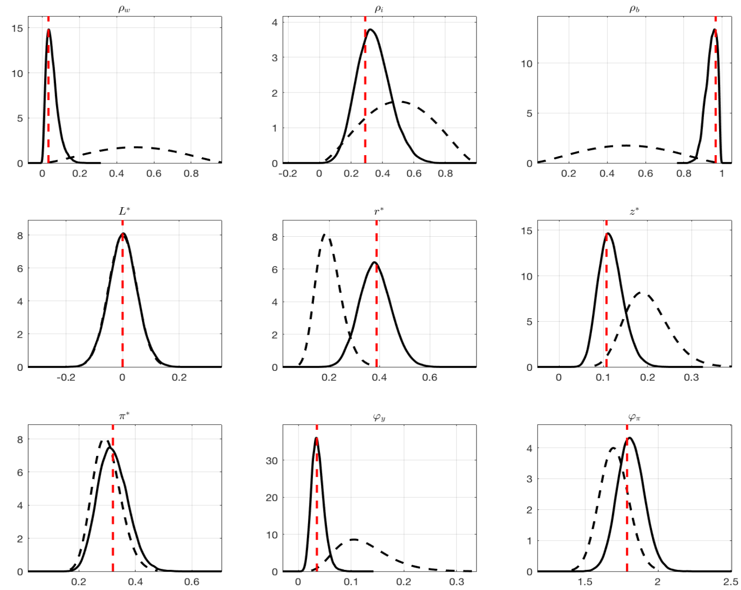
Steady State

$R = \frac{R^N}{\Pi}$, $1 = \frac{\beta R^N}{z^\sigma \Pi} = \frac{\beta R}{z^\sigma}$, $Q = 1$, $R^K = \frac{z^\sigma}{\beta} - 1 + \delta$, $\frac{P^*}{P} = 1$, $MC = \frac{1}{1+\lambda^p}$, $w = (1-\alpha)\left(\frac{1}{1+\lambda^p}\right)^{\frac{1}{1-\alpha}}\left(\frac{z^\sigma - \beta + \beta\delta}{\alpha\beta}\right)^{-\frac{\alpha}{1-\alpha}}$, $\frac{k}{l} = \frac{\alpha}{1-\alpha}\frac{zw}{R^K}$, $\frac{k}{y} = \left(1 + \frac{\varphi}{y}\right) z^\alpha \left(\frac{k}{l}\right)^{1-\alpha}$, $\frac{i}{y} = \left(1 - \frac{1-\delta}{z}\right)\frac{k}{y}$, $\frac{c}{y} = 1 - \frac{i}{y} - \frac{g}{y}$

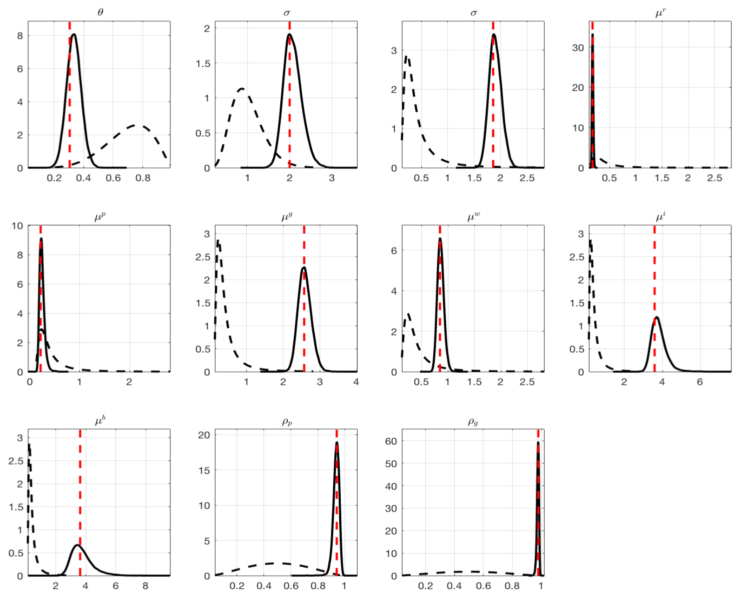
Figures from Medium-Scale DSGE Estimation with Shadow Rate



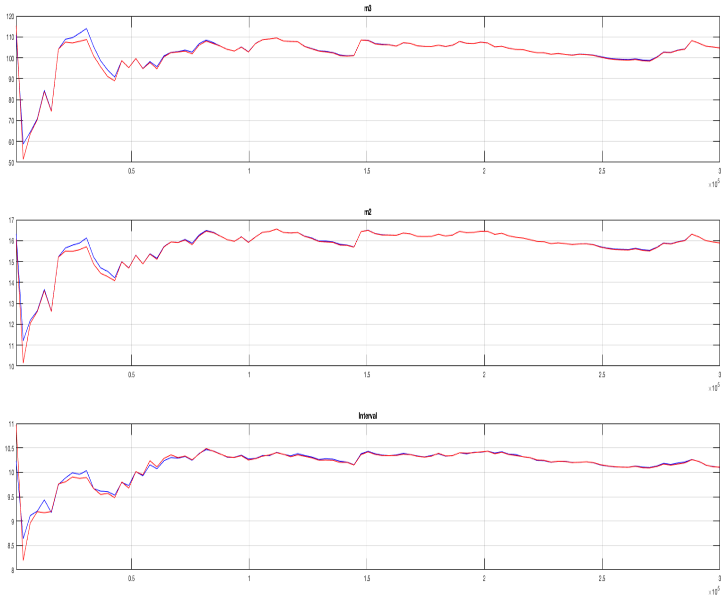
Prior and Posterior Distribution of Structural Parameters



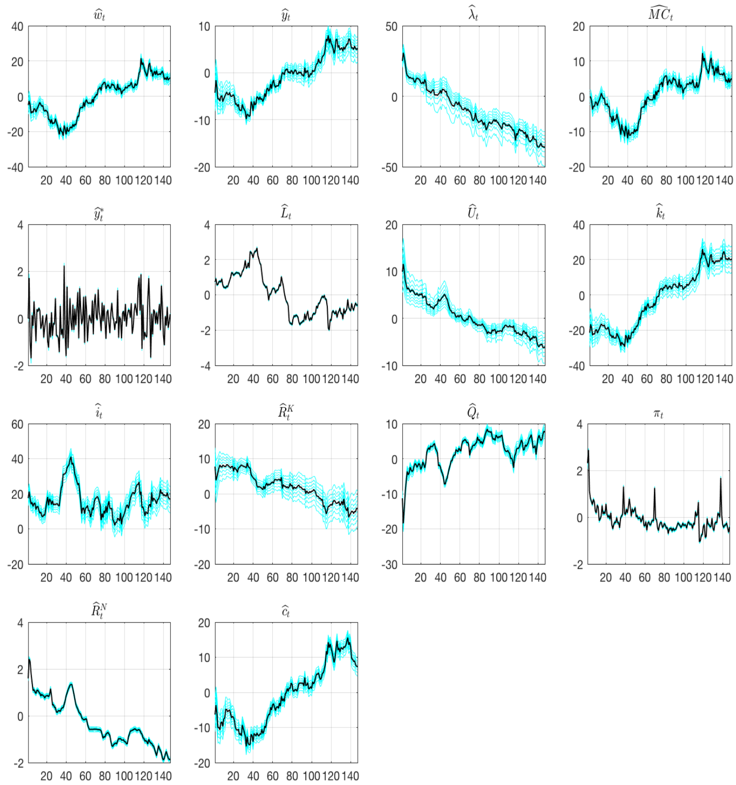
Prior and Posterior Distribution of Structural Parameters



Prior and Posterior Distribution of Structural Parameters



Multivariate Convergence Diagnostic



Smoothed Endogenous Variables

Appendix for Chapter 5

Steady State

$I = \delta K$, $(1-\beta\theta)(C-\theta C)^{-\sigma} = \lambda$, $R_L = R_S^2$, $\beta R_S = \Pi$, $m^{-\xi} = \lambda\left(1-\frac{\beta}{\Pi}\right)$, $\lambda(q+\varphi_K\delta^3) = \mu[1-\beta(1-\delta)]$, $2\beta\mu = \lambda(2+3\delta^2\varphi_K)$, $\varphi_K = \frac{2(1-\beta+\beta\delta-q\beta)}{\delta^2(3\beta-3-\beta\delta)}$, $x = \frac{b_L^{CB}}{b_L} = \frac{b_L-b_L^H}{b_L}$, $Y = C + I\left(1+\frac{\varphi_K}{2}\delta^2\right) + G$, $MC = \frac{1}{1+\epsilon}$, $w = (1-\alpha)MC\frac{Y+\phi}{L}$, $q = \alpha MC\frac{Y+\phi}{K}$, $\frac{Y}{Y+\phi} = \frac{1}{1+\epsilon}$, $Y = qK + wL$, $\frac{b_S}{R_S} + \frac{b_L^H}{R_L} + m + I(1+\frac{\varphi_K}{2}\delta^2) = \frac{b_S}{\Pi} + \frac{b_L^H}{\Pi R_S} + \frac{m}{\Pi} + wL + qK - C - T$

The Log-linearized Model

$$\tilde{K}_t = \delta\tilde{I}_t + (1-\delta)\tilde{K}_{t-1}$$

$$\tilde{\lambda}_t = \frac{1}{(1-\beta\theta)(1-\theta)}\left[\beta\theta\sigma\mathbb{E}_t\tilde{C}_{t+1} - \sigma(\beta\theta^2+1)\tilde{C}_t + \sigma\theta\tilde{C}_{t-1}\right] + \frac{1}{1-\beta\theta}(\varepsilon_t^u - \beta\theta\mathbb{E}_t\varepsilon_{t+1}^u)$$

$$\varepsilon_t^u + \chi\tilde{L}_t = \tilde{w}_t + \tilde{\lambda}_t$$

$$\xi\tilde{m}_t = \frac{\beta}{\Pi-\beta}\mathbb{E}_t(\tilde{\lambda}_{t+1} - \pi_{t+1}) - \frac{\Pi}{\Pi-\beta}\tilde{\lambda}_t + \varepsilon_t^u$$

$$\frac{\beta}{\Pi}\mathbb{E}_t(\tilde{\lambda}_{t+1} - \pi_{t+1}) = \frac{1}{R_S}(\tilde{\lambda}_t - \tilde{R}_{S,t}) + \frac{\kappa_B\varphi_B Y}{R_L}(\tilde{b}_{S,t} - \tilde{b}_{L,t}^H)$$

$$\frac{\beta}{R_S\Pi}\mathbb{E}_t(\tilde{\lambda}_{t+1} - \pi_{t+1} - \tilde{R}_{S,t+1}) = \frac{1}{R_L}(\tilde{\lambda}_t - \tilde{R}_{L,t}) - \frac{\varphi_B Y}{R_L}(\tilde{b}_{S,t} - \tilde{b}_{L,t}^H)$$

$$\beta(1-\delta)\mathbb{E}_t\tilde{\mu}_{t+1} = \tilde{\mu}_t - [1-\beta(1-\delta)]\tilde{\lambda}_t - \frac{2\beta q}{2+3\delta^2\varphi_K}\tilde{q}_t + \frac{6\varphi_K\delta^3}{2+3\varphi_K\delta^2}(\tilde{K}_t - \tilde{I}_t)$$

$$\mathbb{E}_t\tilde{\mu}_{t+1} = \tilde{\lambda}_t + \frac{6\varphi_K\delta^2}{2+3\varphi_K\delta^2}(\tilde{I}_t - \tilde{K}_t)$$

$$\tilde{K}_t - \tilde{w}_t = \tilde{L}_t - \tilde{q}_t$$

$$\widetilde{MC}_t = (1-\alpha)\tilde{w}_t + \alpha\tilde{q}_t$$

$$\pi_t = \frac{\beta}{1+\beta\gamma}\mathbb{E}_t\pi_{t+1} + \frac{\gamma}{1+\beta\gamma}\pi_{t-1} + \frac{(1-\eta)(1-\beta\eta)}{\eta(1+\beta\gamma)}(\widetilde{MC}_t + \varepsilon_t^p)$$

$$\tilde{Y}_t = \left(1+\frac{\phi}{Y}\right)[\varepsilon_t^a + \alpha\tilde{K}_t + (1-\alpha)\tilde{L}_t]$$

$$\frac{b_S}{R_S}(\tilde{b}_{S,t} - \tilde{R}_{S,t}) + \frac{b_L}{R_L}(\tilde{b}_{L,t} - \tilde{R}_{L,t}) + m\tilde{m}_t - \frac{m}{\Pi}(\tilde{m}_{t-1} - \pi_t) - \frac{xb_L}{R_L}(\tilde{b}_{L,t} - \tilde{R}_{L,t}) +$$
$$\frac{xb_L}{\Pi R_L}(\tilde{b}_{L,t} + \tilde{x}_{t-1} - \pi_t - \tilde{R}_{S,t}) = \frac{b_S}{\Pi}(\tilde{b}_{S,t-1} - \pi_t) + \frac{b_L}{R_S\Pi}(\tilde{b}_{L,t-1} - \pi_t - \tilde{R}_{S,t}) + G\tilde{G}_t - T\tilde{T}_t$$

$$\tilde{b}_{L,t}^H = \tilde{b}_{L,t} + \frac{x}{x-1}\tilde{x}_t$$

$$\tilde{b}_t^{CB} = \tilde{x}_t + \tilde{b}_t^L$$

$$
\begin{aligned}
T\tilde{T}_t &= \tau_S \frac{b_S}{\Pi}(\tilde{b}_{S,t-1} - \pi_t) + \tau_L \frac{b_L}{R_S \Pi}(\tilde{b}_{L.t-1} - \tilde{R}_{S,t} - \pi_t) \\
\tilde{R}_{S,t} &= \rho_R \tilde{R}_{S,t-1} + (1-\rho_R)\left[\pi_t^* + \varphi_\pi(\pi_t - \pi_t^*) + \varphi_Y \tilde{Y}_t\right] + \varepsilon_t^r
\end{aligned}
$$

$$
\frac{b_S}{R_S}(\tilde{b}_{S,t} - \tilde{R}_{S,t}) + \frac{b_L^H}{R_L}(\tilde{b}_{L,t}^H - \tilde{R}_{L,t}) + m\tilde{m}_t + I\tilde{I}_t + \varphi_K I \delta^2 \left(\frac{3}{2}\tilde{I}_t - \tilde{K}_t\right) =
$$

$$
\frac{b_S}{\Pi}(\tilde{b}_{S,t-1} - \pi_t) + \frac{b_L^H}{R_S \Pi}(\tilde{b}_{L,t-1}^H - \pi_t - \tilde{R}_{S,t}) + \frac{m}{\Pi}(\tilde{m}_t - \pi_t) + Y\tilde{Y}_t - C\tilde{C}_t - T\tilde{T}_t
$$

$$
\begin{aligned}
\pi_t^* &= \rho_{\pi^*}\pi_{t-1}^* + \mu_t^{\pi^*} \\
\tilde{x}_t &= \rho_x \tilde{x}_{t-1} + \mu_t^x \\
\tilde{b}_{L,t} &= \rho_{b_L}\tilde{b}_{L,t-1} + \mu_t^{b_L} \\
\tilde{G}_t &= \rho_g \tilde{G}_{t-1} + \mu_t^g \\
\varepsilon_t^u &= \rho_u \varepsilon_{t-1}^u + \mu_t^u \\
\varepsilon_t^a &= \rho_a \varepsilon_{t-1}^a + \mu_t^a \\
\varepsilon_t^p &= \rho_u \varepsilon_{t-1}^p + \mu_t^p \\
\varepsilon_t^r &= \rho_r \varepsilon_{t-1}^r + \mu_t^r
\end{aligned}
$$

Appendix for Data

Data	Source
Yield Curve of Japanese Government Bond	Bloomberg
Call Rate	Bank of Japan, Time-Series Data Search
Balance Sheet Variables	Bank of Japan, Time-Series Data Search
S&P Japan Corporate Bond Index	S&P
S&P Japan Government Bond Index	S&P
Output Gap	Bank of Japan, Research Data
Core-CPI and Core-CPI Inflation	Ministry of Internal Affairs and Communications, Statistics Bureau
GDP Growth Rate	Cabinet Office, System of National Accounts
Consumption Growth Rate	Cabinet Office, System of National Accounts
Investment Growth Rate	Cabinet Office, System of National Accounts
Wage Growth Rate	Ministry of Health, Labour and Welfare, Labour Statistics
Bond Holding Structure for Calibration	Ministry of Finance

Appendix for Program Code

Program code is available upon request.

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