Some quantum scattering problems in external electromagnetic fields

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Doctoral Dissertation

Some quantum scattering problems in external electromagnetic fields
（外電磁場内での量子散乱問題について）

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Contents

Chapter 1

Introduction

Chapter 2 is based on the accepted paper [5]. For $N \geq 2$, we give a natural derivation of the Avron-Herbst type formula for the time evolution generated by an N-body Hamiltonian with constant electric and magnetic fields. By virtue of the formula, some scattering problems can be reduced to those in the case where the constant electric and magnetic fields are parallel to each other. As an application of the formula, we give the result of the asymptotic completeness for the systems which have the only charged particle and some neutral ones in crossed constant electric and magnetic fields. In §2.1 and 2.2, we first prove an Avron-Herbst type formula (Theorem 2.1.1) for an N-body system in a constant electric field $\boldsymbol{E} = (E_1, E_2, E_3) \in \boldsymbol{R}^3 \setminus \{0\}$ and a constant magnetic field $\boldsymbol{B} = (0, 0, B) \in \boldsymbol{R}^3 \setminus \{0\}$, $B > 0$, which says that the time evolution generated by the total Hamiltonian $\tilde{H}(\boldsymbol{E})$ is transformed, by a family of unitary operators $\tilde{\mathscr{T}}(t)$, into that of the Hamiltonian $\tilde{H}(\boldsymbol{E}_\parallel)$ only with the magnetic field $\boldsymbol{B}$ and the electric field parallel to the magnetic field $\boldsymbol{E}_\parallel = (0, 0, E_3) \parallel \boldsymbol{B}$:

$$
\begin{aligned}
e^{-it\tilde{H}(\boldsymbol{E})} &= \tilde{\mathscr{T}}(t) e^{-it\tilde{H}(\boldsymbol{E}_\parallel)} \tilde{\mathscr{T}}(0)^*, \\
\tilde{\mathscr{T}}(t) &= e^{-itM\alpha^2/2} e^{iM\alpha\cdot x_{\mathrm{cm}}} e^{-it\alpha\cdot k_{\mathrm{total}}}.
\end{aligned}
$$

In this case, Skibsted [33] has already obtained such a formula, but his formula is written under the assumption that all the particles are charged, and our formula is represented so naturally that the system may contain some neutral particles, which is a good feature of our formula. As an application of Theorem 2.1.1, in §2.3, we consider the problem of the asymptotic completeness for a N-body system consisting of $N - 1$ neutral particles and only one charged particle under the assumption that the component of the electric field parallel to the magnetic field is zero; in this case, the wave operators are unitarily equivalent to that of the system with the magnetic field alone, for which Adachi [1],[2] has already obtained the asymptotic completeness, so the asymptotic completeness for our wave operators follows immediately (Theorem 2.3.1). Also in the case where the space dimension is not three but two, that is, if the electric field $E = (E_1, E_2) \in \boldsymbol{R}^2 \setminus \{0\}$ lies in the plane $\boldsymbol{R}^2$ parpendicular to the constant magnetic field $\boldsymbol{B}$ and the N-body system is restricted to this plane, we can prove the corresponding Avron-Herbst type formula (Theorem 2.1.2), which says that the time evolution generated by the total Hamiltonian $\tilde{H}_\perp(E)$ is transformed, by a family of unitary operators $\tilde{\mathscr{T}}_\perp(t)$, into that of the Hamiltonian $\tilde{H}_\perp(0)$ only with the magnetic field $\boldsymbol{B}$:

$$e^{-it\tilde{H}_\perp(E)} = \tilde{\mathscr{T}}_\perp(t)e^{-it\tilde{H}_\perp(0)}\tilde{\mathscr{T}}_\perp(0)^*,$$
$$\tilde{\mathscr{T}}_\perp(t) = e^{-itM\alpha_\perp^2/2}e^{iM\alpha_\perp\cdot x_{\mathrm{cm},\perp}}e^{-it\alpha_\perp\cdot k_{\mathrm{total},\perp}}.$$

Hence, in the same way as the case where the space dimension is three, if the number of charged particle is one, the asymptotic completeness (Theorem 2.3.4) follows from the result of Adachi. Here we remark that the result of Adachi has a strong connection with the fact that charged particles are bounded in the direction parpendicular to the magnetic field but the neutral particles are not so, and the presence of the neutral particles are crucial, especially in this case where the system is restricted to the plane. Lastly, in §2.4, we make some remarks on the extention to the case where the electric field is time-dependent.

Chapter 3 is based on the submitted paper [6]. In the spectral and scattering theory for a Schrödinger operator with a time-periodic potential $H(t) = p^2/2 + V(t,x)$, the Floquet Hamiltonian $K = -i\partial_t + H(t)$ associated with $H(t)$ plays an important role frequently, by virtue of the Howland-Yajima method. In this chapter, we introduce a new conjugate operator for K in the standard Mourre theory, that is different from the one due to Yokoyama, in order to relax a certain smoothness condition on V. As a conjugate operator for K, Yokoyama [39] introduced

$$\tilde{A}_1 = \frac{1}{2}\{x\cdot p(1+p^2)^{-1} + (1+p^2)^{-1}p\cdot x\}.$$

Roughly speaking, the usual conjugate operator

$$\hat{A}_0 = \frac{1}{2}(x\cdot p + p\cdot x)$$

makes $i[K, \hat{A}_0] = p^2$ and fails to let this commutator be K-bounded. To avoid this problem, he multiplied $\hat{A}_0$ by the resolvent of p^2 and made $i[K, \hat{A}_1] = p^2(p^2/2+1)^{-1}$ bounded. However, in this chapter, we multiply $\hat{A}_0$ by the resolvent of $D_t = -i\partial_t$ and introduce a new conjugate operator

$$A_{\lambda_0,\delta} = (\lambda_0 - \delta - D_t)^{-1} \otimes \hat{A}_0.$$

Unlike the conjugate operator due to Yokoyama, in this case, the potential $V(t,x)$ needs some assumption on the derivatives with respect to the time variable t (see

Condition (V)), but referring to the assumptions of Yajima [38] which guarantees the existence and uniqueness of the unitary propagator, this condition (V) is found to be more natural than that of Yokoyama. Moreover, while Yokoyama imposed an infinite differentiability on the regular part of the potential for the sake of a pseudo-differential calculus, our condition (V) may relax this to the extent that $V^{\mathrm{reg}}(t,x) \in C^2(\boldsymbol{R} \times \boldsymbol{R}^d)$. The main result of this chapter is the Mourre estimate for K with $A_{\lambda_0,\delta}$ (Theorem 3.1.1). We give that proof in §3.2. In §3.3, as an application, we consider the one-body system in the time-periodic electric field $E(t) \in \boldsymbol{R}^d$. By an Avron-Herbst type formula, the time evolution generated by the Hamiltonian of this system

$$\hat{H}(t) = \hat{H}_0(t) + V(x), \quad \hat{H}_0(t) = \frac{1}{2}p^2 - E(t) \cdot x$$

is transformed, by a family of unitary operators, into that of the Hamiltonian with the potential made time-periodic by a time-periodic function $c(t)$ and the free Hamiltonian made time-independent:

$$H(t) = H_0 + V(x + c(t)), \quad H_0 = \frac{1}{2}p^2;$$

if V is short-range, $V(x + c(t))$ satisfies our condition (V). Then, for the Floquet Hamiltonians associated with H_0 and $H(t)$, the result of the asymptotic completeness can be derived from the Mourre theory in §3.1. By virtue of the Howland-Yajima method, the asymptotic completeness of the wave operators for H_0 and $H(t)$ also follows, and consequently that of the original wave operators for $\hat{H}_0(t)$ and $\hat{H}(t)$ follows. Lastly, in §3.4, we make some remarks on the many body case. Unlike the congugate operator due to Yokoyama, $A_{\lambda_0,\delta}$ is expected to have an extension to the many body systems, but this is an issue in the future.

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Chapter 2

Remarks on the Avron-Herbst type formula for N-body quantum systems in constant electric and magnetic fields

2.1 Introduction

In this chapter, we study the scattering theory for N-body quantum systems in constant electric and magnetic fields.

Let $N \geq 2$. Consider the system of N particles moving in the Euclidean space $\boldsymbol{R}^3$ on which the constant electric field $\boldsymbol{E} = (E_1, E_2, E_3) \in \boldsymbol{R}^3 \setminus \{0\}$ and the constant magnetic field $\boldsymbol{B} = (0, 0, B) \in \boldsymbol{R}^3 \setminus \{0\}$ with $B > 0$ are impressed. Denote by $m_j > 0$, $q_j \in \boldsymbol{R}$ and $x_j = (x_{j,1}, x_{j,2}, x_{j,3}) \in \boldsymbol{R}^3$ $(j = 1, \ldots, N)$ the mass, the charge, and the position of the j-th particle, respectively. We assume that for some $N_{\mathrm{c}} \in \boldsymbol{N}$ such that $N_{\mathrm{c}} \leq N$, the last N_{c} particles are charged and the rest are neutral. In other words, we suppose

$$q_j \neq 0 \quad (\text{if } j \geq N_{\mathrm{n}} + 1), \qquad q_j = 0 \quad (\text{otherwise}), \tag{2.1.1}$$

where $N_{\mathrm{n}} := N - N_{\mathrm{c}} \geq 0$. $N = N_{\mathrm{n}} + N_{\mathrm{c}}$. Then the total Hamiltonian $\tilde{H}(\boldsymbol{E})$ of the system is defined by

$$\begin{aligned} \tilde{H}(\boldsymbol{E}) &= \tilde{H}_0(\boldsymbol{E}) + V, \\ \tilde{H}_0(\boldsymbol{E}) &= \sum_{j=1}^{N} \left(\frac{1}{2m_j}(p_j - q_j \boldsymbol{A}(x_j))^2 - q_j \boldsymbol{E} \cdot x_j \right), \\ V &= \sum_{1 \leq j < k \leq N} V_{jk}(x_j - x_k), \end{aligned} \tag{2.1.2}$$

on $L^2(\boldsymbol{R}^{3\times N})$, where $p_j = -i\nabla_{x_j} = (p_{j,1}, p_{j,2}, p_{j,3})$ is the canonical momentum of the j-th particle, $V_{jk}(x_j - x_k)$'s are pair potentials, and $\boldsymbol{A}(r)$ is the vector potential associated with the magnetic field $\boldsymbol{B}$. In the symmetric gauge, $\boldsymbol{A}(r)$ is written as

$$\boldsymbol{A}(r) = \frac{1}{2}\boldsymbol{B} \times r = \frac{B}{2}(-r_2, r_1, 0), \quad r = (r_1, r_2, r_3) \in \boldsymbol{R}^3.$$

We will use the symmetric gauge in this paper. Put

$$D_j := p_j - q_j \boldsymbol{A}(x_j)$$

for the sake of brevity. D_j is called the kinetic momentum of the j-th particle. Here we note that if $j \leq N_{\mathrm{n}}$, then $p_j = D_j$ because of $q_j = 0$. For the sake of simplicity, we impose the following condition $(V0)_d$ with $d = 3$ on V at first:

$(V0)_d$ For $1 \leq j < k \leq N$, V_{jk} belongs to $C(\boldsymbol{R}^d; \boldsymbol{R})$, and satisfies the decaying condition

$$|V_{jk}(r)| \leq C\langle r\rangle^{-\rho}$$

with some $\rho > 0$.

Here $\langle r\rangle = \sqrt{1 + r^2}$. Under the condition $(V0)_3$, $\tilde{H}(\boldsymbol{E})$ is self-adjoint.

Put $\boldsymbol{E}_\perp := (E_1, E_2, 0)$ and $\boldsymbol{E}_\parallel := (0, 0, E_3)$. Then $\boldsymbol{E}_\perp \perp \boldsymbol{B}$, $\boldsymbol{E}_\parallel \parallel \boldsymbol{B}$, and $\boldsymbol{E}$ can be decomposed into the direct sum $\boldsymbol{E}_\perp \oplus \boldsymbol{E}_\parallel$. Now we would like to give the relation between $e^{-it\tilde{H}(\boldsymbol{E})}$ and $e^{-it\tilde{H}(\boldsymbol{E}_\parallel)}$ in terms of the Avron-Herbst type formula. Let us introduce the total mass M, the total charge Q, the position of the center of mass x_{cm}, the total pseudomomentum k_{total} of the system, and the $\boldsymbol{E} \times \boldsymbol{B}$ drift velocity α by

$$M = \sum_{j=1}^N m_j, \quad Q = \sum_{j=1}^N q_j, \quad x_{\mathrm{cm}} = \frac{1}{M}\sum_{j=1}^N m_j x_j,$$
$$k_{\mathrm{total}} = \sum_{j=1}^N (p_j + q_j\boldsymbol{A}(x_j)), \quad \alpha = \frac{\boldsymbol{E} \times \boldsymbol{B}}{\boldsymbol{B}^2} = \left(\frac{E_2}{B}, -\frac{E_1}{B}, 0\right).$$

Put

$$k_j := p_j + q_j \boldsymbol{A}(x_j)$$

for the sake of brevity. k_j is called the pseudomomentum of the j-th particle. Here we note that if $j \leq N_{\mathrm{n}}$, then $p_j = k_j$ because of $q_j = 0$, and that

$$k_{\mathrm{total}} = \sum_{j=1}^N k_j$$

holds. Then we obtain the following Avron-Herbst type formula for $e^{-it\tilde{H}(\boldsymbol{E})}$:

Theorem 2.1.1. *Assume V satisfies $(V0)_3$. Then the Avron-Herbst type formula for $e^{-it\tilde{H}(\boldsymbol{E})}$*

$$e^{-it\tilde{H}(\boldsymbol{E})} = \tilde{\mathscr{T}}(t)e^{-it\tilde{H}(\boldsymbol{E}_\parallel)}\tilde{\mathscr{T}}(0)^*, \\ \tilde{\mathscr{T}}(t) = e^{-itM\alpha^2/2}e^{iM\alpha\cdot x_{\mathrm{cm}}}e^{-it\alpha\cdot k_{\mathrm{total}}} \tag{2.1.3}$$

holds.

We note $\tilde{\mathscr{T}}(0) = e^{iM\alpha\cdot x_{\mathrm{cm}}}$. $e^{-it\alpha\cdot k_{\mathrm{total}}}$ in the definition of $\tilde{\mathscr{T}}(t)$ is called a magnetic translation generated by k_{total}. It is well-known that $e^{-it\alpha\cdot k_{\mathrm{total}}}$ can be written as

$$e^{-it\alpha\cdot k_{\mathrm{total}}} = e^{-it\alpha\cdot \boldsymbol{A}(\tilde{x}_{\mathrm{cc}})}e^{-it\alpha\cdot p_{\mathrm{total}}} \tag{2.1.4}$$

(see e.g. [17]), where $\tilde{x}_{\mathrm{cc}}$ and the total canonical momentum p_{total} are given by

$$\tilde{x}_{\mathrm{cc}} = \sum_{j=1}^N q_j x_j, \quad p_{\mathrm{total}} = \sum_{j=1}^N p_j.$$

If $Q \neq 0$, then the position of the center of charge x_{cc} can be given by

$$x_{\mathrm{cc}} = \frac{1}{Q}\tilde{x}_{\mathrm{cc}},$$

and (2.1.4) can be written as

$$e^{-it\alpha\cdot k_{\mathrm{total}}} = e^{-it\alpha\cdot Q\boldsymbol{A}(x_{\mathrm{cc}})}e^{-it\alpha\cdot p_{\mathrm{total}}}. \tag{2.1.5}$$

Hence $e^{-it\alpha\cdot k_{\mathrm{total}}}$ should be called a magnetic translation of the center of charge.

The Avron-Herbst type formula for $e^{-it\tilde{H}(\boldsymbol{E})}$ like (2.1.3) was already obtained by Skibsted [33]. In fact, he introduced

$$U_1(t) = \prod_{j=1}^N G_j(t), \quad G_j(t) = e^{itm_j\alpha^2/2}e^{-it\alpha\cdot p_j}e^{i(tq_j\boldsymbol{A}(\alpha)+m_j\alpha)\cdot x_j}, \tag{2.1.6}$$

where $G_j(t)$ is the Galilei transform associated with the j-th particle which reflects the effect of the constant magnetic field $\boldsymbol{B}$. One of the basic properties of $G_j(t)$ is that

$$G_j(t)^* x_j G_j(t) = x_j + t\alpha, \quad G_j(t)^* D_j G_j(t) = D_j + m_j\alpha$$

hold. Thus $G_j(t)$ transforms the expectation of the position of the j-th particle by $t\alpha$, and that of the kinetic momentum by $m_j\alpha$ respectively. Then he claimed that the Avron-Herbst type formula

$$e^{-it\tilde{H}(\boldsymbol{E})} = U_1(t)e^{-it\tilde{H}(\boldsymbol{E}_\parallel)}U_1(0)^* \tag{2.1.7}$$

holds. Since charged particles drift with the $\boldsymbol{E}\times\boldsymbol{B}$ drift velocity α, it is natural to consider the Galilei transform $G_j(t)$ for each charged particle. However, it is not certain whether the Galilei transforms $G_j(t)$'s must be introduced also for neutral particles. Notice that neutral particles can move freely independent of the drift velocity α. One of the purposes of this paper is to give a natural definition of an equivalent of $U_1(t)$ even if the system under consideration has some neutral particles, that is, $N_{\mathrm{n}} \geq 1$.

Also in the case where the space dimension d is not three but two, the Avron-Herbst type formula can be obtained quite similarly: We suppose that the constant magnetic field $\boldsymbol{B}$ is perpendicular to the plane $\boldsymbol{R}^2$, and that the constant electric field $E = (E_1, E_2) \in \boldsymbol{R}^2 \setminus \{0\}$ lies in the plane. We use the notation

$$
\begin{aligned}
&x_{j,\perp} = (x_{j,1}, x_{j,2}), \quad p_{j,\perp} = (p_{j,1}, p_{j,2}), \quad A(x_{j,\perp}) = \frac{B}{2}(-x_{j,2}, x_{j,1}),\\
&D_{j,\perp} = p_{j,\perp} - q_j A(x_{j,\perp}), \quad k_{j,\perp} = p_{j,\perp} + q_j A(x_{j,\perp}), \quad \alpha_\perp = \left(\frac{E_2}{B}, -\frac{E_1}{B}\right),\\
&x_{\mathrm{cm},\perp} = \frac{1}{M}\sum_{j=1}^N m_j x_{j,\perp}, \quad \tilde{x}_{\mathrm{cc},\perp} = \sum_{j=1}^N q_j x_{j,\perp}, \quad p_{\mathrm{total},\perp} = \sum_{j=1}^N p_{j,\perp},\\
&D_{\mathrm{total},\perp} = \sum_{j=1}^N D_{j,\perp} = p_{\mathrm{total},\perp} - A(\tilde{x}_{\mathrm{cc},\perp}),\\
&k_{\mathrm{total},\perp} = \sum_{j=1}^N k_{j,\perp} = p_{\mathrm{total},\perp} + A(\tilde{x}_{\mathrm{cc},\perp}).
\end{aligned}
$$

Then the total Hamiltonian $\tilde{H}_\perp(E)$ of the system is defined by

$$
\begin{aligned}
&\tilde{H}_\perp(E) = \tilde{H}_{0,\perp}(E) + V,\\
&\tilde{H}_{0,\perp}(E) = \sum_{j=1}^N \left(\frac{1}{2m_j} D_{j,\perp}^2 - q_j E\cdot x_{j,\perp}\right),\\
&V = \sum_{1\leq j<k\leq N} V_{jk}(x_{j,\perp} - x_{k,\perp}),
\end{aligned}
\tag{2.1.8}
$$

on $L^2(\boldsymbol{R}^{2\times N})$. Under the condition $(V0)_2$, $\tilde{H}_\perp(E)$ is self-adjoint. Then we obtain the following Avron-Herbst type formula for $e^{-it\tilde{H}_\perp(E)}$:

Theorem 2.1.2. *Assume V satisfies $(V0)_2$. Then the Avron-Herbst type formula for $e^{-it\tilde{H}_\perp(E)}$*

$$
\begin{aligned}
&e^{-it\tilde{H}_\perp(E)} = \tilde{\mathscr{T}}_\perp(t) e^{-it\tilde{H}_\perp(0)} \tilde{\mathscr{T}}_\perp(0)^*,\\
&\tilde{\mathscr{T}}_\perp(t) = e^{-itM\alpha_\perp^2/2} e^{iM\alpha_\perp\cdot x_{\mathrm{cm},\perp}} e^{-it\alpha_\perp\cdot k_{\mathrm{total},\perp}}
\end{aligned}
\tag{2.1.9}
$$

holds.

Since $\alpha = (\alpha_\perp, 0)$, $\tilde{\mathscr{T}}(t)$ in Theorem 2.1.1 can be represented as

$$\tilde{\mathscr{T}}(t) = \tilde{\mathscr{T}}_\perp(t) \otimes \mathrm{Id} \tag{2.1.10}$$

on $L^2(\boldsymbol{R}^{3\times N}) \cong L^2(\boldsymbol{R}^{2\times N}) \otimes L^2(\boldsymbol{R}^N)$. Hence we have only to show Theorem 2.1.2 essentially. We will give the proof in §2.2.

When $N = 1$, the Avron-Herbst type formula for the free propagator was already obtained by Adachi-Kawamoto [4], even if the homogeneous electric field is strictly time-dependent. Here we note that before the work [4], a different but meaningful factorization of the free propagator was given by Chee [8]. In the case where the homogeneous electric field is constant, as for some spectral problems for perturbed Hamiltonians, see Wang [36], Dimassi-Petkov [11], [12], [13], Ferrari-Kovařík [14], [15], and Kawamoto [19]; while in the homogeneous electric field is time-dependent, Lawson and Avossevou [25] have recently studied a certain spectral problem for the free Hamiltonian with time-dependent mass (see also the references therein).

On the other hand, when $N \geq 2$, in general, it seems hard to obtain a certain effective Avron-Herbst type formula if the homogeneous electric field is time-dependent, except in the case where all the specific charges of particles are the same; that is, $x_{\mathrm{cm}} = x_{\mathrm{cc}}$: if $q_j/m_j = c$, $(j = 1, \ldots, N)$, then

$$Q = \sum_{j=1}^N q_j = \sum_{j=1}^N m_j c = cM$$

holds; this gives

$$x_{\mathrm{cc}} = \frac{1}{Q}\sum_{j=1}^N q_j x_j = \frac{1}{Q}\sum_{j=1}^N c m_j x_j = \frac{1}{M}\sum_{j=1}^N m_j x_j = x_{\mathrm{cm}}.$$

We will mention it in §2.4.

The plan of this chapter is as follows: In §2.2, we will give the proof of Theorem 2.1.2. In §2.3, as an application of our results, we will deal with the problem of the asymptotic completeness for the systems which have the only charged particle and some neutral ones in crossed constant electric and magnetic fields, mainly in the short-range case. In §2.4, we will make some remarks on the extension to the case where the homogeneous electric field is strictly time-dependent.

2.2 Proof of Theorem 2.1.2

In this section, we will show Theorems 2.1.1 and 2.1.2. As mentioned in §2.1, we have only to give the proof of Theorem 2.1.2.

First of all, we note that

$$k_{\mathrm{total},\perp} - D_{\mathrm{total},\perp} = 2\sum_{j=1}^{N} q_j A(x_{j,\perp}) = 2A(\tilde{x}_{\mathrm{cc},\perp}),$$

$$A(A(r_\perp)) = -\left(\frac{B}{2}\right)^2 r_\perp, \quad \hat{r}_\perp \cdot A(r_\perp) = -A(\hat{r}_\perp)\cdot r_\perp$$

hold for $r_\perp$, $\hat{r}_\perp \in \boldsymbol{R}^2$. Then $\tilde{H}_{0,\perp}(E)$ can be represented as

$$\begin{aligned}
\tilde{H}_{0,\perp}(E) &= \sum_{j=1}^{N} \frac{1}{2m_j} D_{j,\perp}^2 - E\cdot \tilde{x}_{\mathrm{cc},\perp} \\
&= \sum_{j=1}^{N} \frac{1}{2m_j} D_{j,\perp}^2 + \frac{2}{B^2} E\cdot A(k_{\mathrm{total},\perp} - D_{\mathrm{total},\perp}) \\
&= \sum_{j=1}^{N} \frac{1}{2m_j} D_{j,\perp}^2 - \frac{2}{B} A\left(\frac{E}{B}\right)\cdot (k_{\mathrm{total},\perp} - D_{\mathrm{total},\perp}).
\end{aligned}$$

Noticing

$$\alpha_\perp = -\frac{2}{B} A\left(\frac{E}{B}\right),$$

we have

$$\begin{aligned}
\tilde{H}_{0,\perp}(E) &= \sum_{j=1}^{N} \frac{1}{2m_j} D_{j,\perp}^2 + \alpha_\perp \cdot (k_{\mathrm{total},\perp} - D_{\mathrm{total},\perp}) \\
&= \sum_{j=1}^{N} \frac{1}{2m_j} (D_{j,\perp} - m_j\alpha_\perp)^2 + \alpha_\perp \cdot k_{\mathrm{total},\perp} - \frac{M}{2}\alpha_\perp^2.
\end{aligned}$$

Putting

$$\tilde{T}_\perp := \sum_{j=1}^{N} \frac{1}{2m_j} (D_{j,\perp} - m_j\alpha_\perp)^2 + V,$$

we see that $\tilde{T}_\perp$ does commute with $\alpha_\perp \cdot k_{\mathrm{total},\perp}$, and that

$$\tilde{T}_\perp = e^{iM\alpha_\perp\cdot x_{\mathrm{cm},\perp}} \tilde{H}_\perp(0) e^{-iM\alpha_\perp\cdot x_{\mathrm{cm},\perp}}$$

holds.

$$e^{-iM\alpha_\perp\cdot x_{\mathrm{cm},\perp}} e^{-it\alpha_\perp\cdot k_{\mathrm{total},\perp}} e^{iM\alpha_\perp\cdot x_{\mathrm{cm},\perp}} = e^{-itM\alpha_\perp^2} e^{-it\alpha_\perp\cdot k_{\mathrm{total},\perp}}$$

can be also verified. Hence we have

$$
\begin{aligned}
e^{-it\tilde{H}_\perp(E)} &= e^{iMt\alpha_\perp^2/2} e^{-it\alpha_\perp\cdot k_{\mathrm{total},\perp}} e^{-it\tilde{T}_\perp} \\
&= e^{iMt\alpha_\perp^2/2} e^{-it\alpha_\perp\cdot k_{\mathrm{total},\perp}} e^{iM\alpha_\perp\cdot x_{\mathrm{cm},\perp}} e^{-it\tilde{H}_\perp(0)} e^{-iM\alpha_\perp\cdot x_{\mathrm{cm},\perp}} \\
&= e^{-itM\alpha_\perp^2/2} e^{iM\alpha_\perp\cdot x_{\mathrm{cm},\perp}} e^{-it\alpha_\perp\cdot k_{\mathrm{total},\perp}} e^{-it\tilde{H}_\perp(0)} e^{-iM\alpha_\perp\cdot x_{\mathrm{cm},\perp}},
\end{aligned}
$$

which yields (2.1.9). Thus the proof is completed.

2.3 Application

In this section, we will apply the Avron-Herbst type formula to some scattering problems for N-body quantum systems in constant electric and magnetic fields, which have neutral particles. Here we suppose that $\boldsymbol{E}_\parallel = 0$. The case where $\boldsymbol{E}_\parallel \neq 0$ can be treated by the results due to Skibsted [33]. We impose the following condition $(V1)_{d,\mathrm{SR}}$ on V, which is stronger than $(V0)_d$, with $d = 3$:

$(V1)_{d,\mathrm{SR}}$ For $1 \leq j < k \leq N$, V_{jk} belongs to $C^2(\boldsymbol{R}^d;\boldsymbol{R})$, and satisfies the decaying condition

$$
|\partial_r^\beta V_{jk}(r)| \leq C_\beta \langle r\rangle^{-\rho-|\beta|}, \quad |\beta| \leq 2
$$

with some $\rho > 1$.

We consider the problem of the asymptotic completeness for the N-body quantum system consisting of $N-1$ neutral particles and one charged particle in the constant electric field $\boldsymbol{E}_\perp = (E, 0) = (E_1, E_2, 0) \in \boldsymbol{R}^3 \setminus \{0\}$ and the constant magnetic field $\boldsymbol{B} = (0, 0, B) \in \boldsymbol{R}^3 \setminus \{0\}$ with $B > 0$; suppose $N_{\mathrm{c}} = 1$ and $N_{\mathrm{n}} = N-1 \geq 1$. Then the total Hamiltonian $\tilde{H}(\boldsymbol{E}_\perp)$ on $L^2(\boldsymbol{R}^{3\times N})$ is represented as

$$
\begin{aligned}
&\tilde{H}(\boldsymbol{E}_\perp) = \tilde{H}_0(\boldsymbol{E}_\perp) + V, \\
&\tilde{H}_0(\boldsymbol{E}_\perp) = \sum_{j=1}^{N-1} \frac{1}{2m_j} p_j^2 + \left(\frac{1}{2m_N} D_N^2 - q_N \boldsymbol{E}_\perp \cdot x_N \right), \\
&V = \sum_{1\leq j<k\leq N} V_{jk}(x_j - x_k)
\end{aligned}
$$

with $q_N \neq 0$. Since there is no external force in the direction parallel to the magnetic field, the motion of the center of mass in that direction is uniform linear motion and can be reduced first; more precisely, both of the electromagnetic potentials $\boldsymbol{A}(x_N)$ and $\boldsymbol{E}_\perp \cdot x_N$ are independent of $x_{N,3}$, and the scalar potential V is invariant under the translation in that direction:

$$
\boldsymbol{A}(x_N) = \frac{B}{2}(-x_{N,2}, x_{N,1}, 0) = (A(x_{N,\perp}), 0),
$$

$$
\begin{aligned}
D_N &= (p_{N,\perp} - q_N A(x_{N,\perp}), p_{N,3}) = (D_{N,\perp}, p_{N,3}), \quad \boldsymbol{E}_\perp \cdot x_N = E \cdot x_{N,\perp}, \\
V(x) &= \sum_{1 \le j < k \le N} V_{jk}((x_{j,\perp}, x_{j,3} - x_{\mathrm{cm},3}) - (x_{k,\perp}, x_{k,3} - x_{\mathrm{cm},3})) \\
&= \sum_{1 \le j < k \le N} V_{jk}((x_{j,\perp}, (\pi_{X_\parallel^{\mathrm{cm}}} x_\parallel)_j) - (x_{k,\perp}, (\pi_{X_\parallel^{\mathrm{cm}}} x_\parallel)_k)) \\
&= \sum_{1 \le j < k \le N} V_{jk}((x_\perp, \pi_{X_\parallel^{\mathrm{cm}}} x_\parallel)_j - (x_\perp, \pi_{X_\parallel^{\mathrm{cm}}} x_\parallel)_k).
\end{aligned}
$$

Here $x_\perp = (x_{1,\perp}, \ldots, x_{N,\perp})$ and $x_\parallel = (x_{1,3}, \ldots, x_{N,3})$ are the first and second components and the third components of $x \in \boldsymbol{R}^{3\times N}$, respectively, $\boldsymbol{R}^{3\times N}$ is identified with $\boldsymbol{R}^{2\times N} \times \boldsymbol{R}^N$, that is,

$$
(x_\perp, x_\parallel)_j \equiv (x_{j,\perp}, x_{j,3}) = x_j, \quad (j = 1, \ldots, N),
$$

for $x \in \boldsymbol{R}^{3\times N}$, $\boldsymbol{R}^N$ is equipped with the metric

$$
\langle \zeta, \tilde{\zeta} \rangle = \sum_{j=1}^N m_j x_{j,\parallel} \tilde{x}_{j,\parallel}
$$

for $\zeta = (x_{1,\parallel}, \ldots, x_{N,\parallel})$, $\tilde{\zeta} = (\tilde{x}_{1,\parallel}, \ldots, \tilde{x}_{N,\parallel}) \in \boldsymbol{R}^N$, $X_\parallel^{\mathrm{cm}}$ and $X_{\mathrm{cm},\parallel}$ are its two subspaces

$$
\begin{aligned}
X_\parallel^{\mathrm{cm}} &= \left\{ (x_{1,\parallel}, \ldots, x_{N,\parallel}) \in \boldsymbol{R}^N \,\middle|\, \sum_{j=1}^N m_j x_{j,\parallel} = 0 \right\}, \\
X_{\mathrm{cm},\parallel} &= \boldsymbol{R}^N \ominus X_\parallel^{\mathrm{cm}},
\end{aligned}
$$

which denotes the configuration space of the relative positions with respect to the center of mass in the direction parallel to the magnetic field and that of the center of mass in that direction, respectively, the orthogonal projections onto these subspaces are given by

$$
\pi_{X_{\mathrm{cm},\parallel}} \zeta = \left(\frac{1}{M} \sum_{k=1}^N m_k x_{k,\parallel} \right)_{j=1}^N, \quad \pi_{X_\parallel^{\mathrm{cm}}} \zeta = \zeta - \pi_{X_{\mathrm{cm},\parallel}} \zeta
$$

for $\zeta = (x_{1,\parallel}, \ldots, x_{N,\parallel}) \in \boldsymbol{R}^N$. Hence V operates only on $L^2(\boldsymbol{R}^{2\times N} \times X_\parallel^{\mathrm{cm}})$, that is, for any $\varphi_1 \in \mathscr{S}(\boldsymbol{R}^{2\times N})$, $\varphi_2 \in \mathscr{S}(X_\parallel^{\mathrm{cm}})$, $\varphi_3 \in \mathscr{S}(X_{\mathrm{cm},\parallel})$,

$$
V(x)(\varphi_1 \otimes \varphi_2 \otimes \varphi_3)(x) = \sum_{1 \le j < k \le N} V_{jk}((x_\perp, \pi_{X_\parallel^{\mathrm{cm}}} x_\parallel)_j - (x_\perp, \pi_{X_\parallel^{\mathrm{cm}}} x_\parallel)_k)
$$

$$\times\, \varphi_1(x_\perp)\varphi_2(\pi_{X_\parallel^{\mathrm{cm}}} x_\parallel)\varphi_3(\pi_{X_{\mathrm{cm},\parallel}} x_\parallel)$$
$$= \left(V \,\big|_{\boldsymbol{R}^{2\times N}\times X_\parallel^{\mathrm{cm}}} \varphi_1 \otimes \varphi_2 \right)(x_\perp, \pi_{X_\parallel^{\mathrm{cm}}} x_\parallel)\varphi_3(\pi_{X_{\mathrm{cm},\parallel}} x_\parallel),$$

so

$$V = V \,\big|_{\boldsymbol{R}^{2\times N}\times X_\parallel^{\mathrm{cm}}} \otimes \mathrm{Id}$$

on $L^2(\boldsymbol{R}^{3\times N}) \cong L^2(\boldsymbol{R}^{2\times N} \times X_\parallel^{\mathrm{cm}}) \otimes L^2(X_{\mathrm{cm},\parallel})$. The third component of $\tilde{H}(\boldsymbol{E}_\perp)$ can be decomposed into the relative motion with respect to the center of mass in the direction parallel to the magnetic field and the motion of the center of mass in that direction:

$$\begin{aligned}
\sum_{j=1}^{N-1} \frac{1}{2m_j} p_{j,3}^2 + \frac{1}{2m_N} D_{N,3}^2 &= \sum_{j=1}^{N} \frac{1}{2m_j} p_{j,3}^2 = \mathrm{Id} \otimes \left(-\frac{1}{2}\Delta_{\boldsymbol{R}^N} \right) \\
&= \mathrm{Id} \otimes \left(-\frac{1}{2}\Delta_{X_\parallel^{\mathrm{cm}}} - \frac{1}{2}\Delta_{X_{\mathrm{cm},\parallel}} \right) \\
&= \left(\mathrm{Id} \otimes \left(-\frac{1}{2}\Delta_{X_\parallel^{\mathrm{cm}}} \right) \right) \otimes \mathrm{Id} + (\mathrm{Id} \otimes \mathrm{Id}) \otimes \left(-\frac{1}{2}\Delta_{X_{\mathrm{cm},\parallel}} \right),
\end{aligned}$$

on $L^2(\boldsymbol{R}^{3\times N}) \cong L^2(\boldsymbol{R}^{2\times N}) \otimes L^2(\boldsymbol{R}^N) \cong L^2(\boldsymbol{R}^{2\times N}) \otimes L^2(X_\parallel^{\mathrm{cm}}) \otimes L^2(X_{\mathrm{cm},\parallel}) \cong L^2(\boldsymbol{R}^{2\times N} \times X_\parallel^{\mathrm{cm}}) \otimes L^2(X_{\mathrm{cm},\parallel})$, where for any finite-dimensional real inner product space W, Δ_W is the Laplace-Beltrami operator on W. Therefore the total Hamiltonian $\tilde{H}(\boldsymbol{E}_\perp)$ can be decomposed as

$$\begin{aligned}
\tilde{H}(\boldsymbol{E}_\perp) = &\sum_{j=1}^{N-1} \frac{1}{2m_j} p_{j,\perp}^2 + \left(\frac{1}{2m_N} D_{N,\perp}^2 - q_N E \cdot x_{N,\perp} \right) + V \,\big|_{\boldsymbol{R}^{2\times N}\times X_\parallel^{\mathrm{cm}}} \otimes \mathrm{Id} \\
&+ \left(\mathrm{Id} \otimes \left(-\frac{1}{2}\Delta_{X_\parallel^{\mathrm{cm}}} \right) \right) \otimes \mathrm{Id} + (\mathrm{Id} \otimes \mathrm{Id}) \otimes \left(-\frac{1}{2}\Delta_{X_{\mathrm{cm},\parallel}} \right) \\
= &\,\tilde{H}(E) \otimes \mathrm{Id} + (\mathrm{Id} \otimes \mathrm{Id}) \otimes \left(-\frac{1}{2}\Delta_{X_{\mathrm{cm},\parallel}} \right),
\end{aligned}$$

on $L^2(\boldsymbol{R}^{3\times N}) \cong L^2(\boldsymbol{R}^{2\times N} \times X_\parallel^{\mathrm{cm}}) \otimes L^2(X_{\mathrm{cm},\parallel})$, where the reduced Hamiltonian $\tilde{H}(E)$ on $L^2(\boldsymbol{R}^{2\times N} \times X_\parallel^{\mathrm{cm}}) \cong L^2(\boldsymbol{R}^{2\times N}) \otimes L^2(X_\parallel^{\mathrm{cm}})$ is represented as

$$\begin{aligned}
\tilde{H}(E) &= \tilde{H}_0(E) + V \,\big|_{\boldsymbol{R}^{2\times N}\times X_\parallel^{\mathrm{cm}}}, \\
\tilde{H}_0(E) &= \left(\sum_{j=1}^{N-1} \frac{1}{2m_j} p_{j,\perp}^2 + \left(\frac{1}{2m_N} D_{N,\perp}^2 - q_N E \cdot x_{N,\perp} \right) \right) \otimes \mathrm{Id} \\
&\qquad + \mathrm{Id} \otimes \left(-\frac{1}{2}\Delta_{X_\parallel^{\mathrm{cm}}} \right);
\end{aligned}$$

see e.g. [3] and [17] for details. Hereafter we simply write as

$$\begin{aligned} \tilde{H}(E) &= \tilde{H}_0(E) + V, \\ \tilde{H}_0(E) &= \sum_{j=1}^{N-1} \frac{1}{2m_j} p_{j,\perp}^2 + \left(\frac{1}{2m_N} D_{N,\perp}^2 - q_N E \cdot x_{N,\perp} \right) - \frac{1}{2} \Delta_{X_\parallel^{\mathrm{cm}}}. \end{aligned} \tag{2.3.1}$$

$\tilde{H}(E)$ has a pure absolutely continuous spectrum, that is,

$$L^2_{\mathrm{ac}}(\tilde{H}(E)) = L^2(\boldsymbol{R}^{2\times N} \times X_\parallel^{\mathrm{cm}}) \tag{2.3.2}$$

where $L^2_{\mathrm{ac}}(\tilde{H}(E))$ is the absolutely continuous spectral subspace associated with $\tilde{H}(E)$. In fact, putting $\tilde{A} = q_N E \cdot k_{\mathrm{total},\perp}$ as in Adachi-Kawamoto [4],

$$i[\tilde{H}(E), \tilde{A}] = q_N^2 E^2 > 0 \tag{2.3.3}$$

holds even if $V \neq 0$, which implies the above property.

When $E = 0$, the results of the asymptotic completeness for

$$\tilde{H}(0) = \sum_{j=1}^{N-1} \frac{1}{2m_j} p_{j,\perp}^2 + \frac{1}{2m_N} D_{N,\perp}^2 - \frac{1}{2} \Delta_{X_\parallel^{\mathrm{cm}}} + V \tag{2.3.4}$$

were already obtained by Adachi [1] and [2]. We will show the asymptotic completeness for $\tilde{H}_\perp(E)$ by using those. For the sake of explanation, we introduce some notation in the many body scattering theory: A non-empty subset of the set $\{1, \ldots, N\}$ is called a cluster. Let C_j, $1 \leq j \leq m$, be clusters. If $\cup_{1\leq j\leq m} C_j = \{1, \ldots, N\}$ and $C_j \cap C_k = \emptyset$ for $1 \leq j < k \leq m$, then $a = \{C_1, \ldots, C_m\}$ is called a cluster decomposition. $\#(a)$ denotes the number of clusters in a. Let $\mathscr{A}$ be the set of all cluster decompositions. Suppose a, $b \in \mathscr{A}$. If b is obtained as a refinement of a, that is, if each cluster in b is a subset of a cluster in a, we say $b \subset a$, and its negation is denoted by $b \not\subset a$. Any a is regarded as a refinement of itself. The one- and N-cluster decompositions are denoted by $a_{\max}$ and $a_{\min}$, respectively. The pair (j, k) is identified with the $(N-1)$-cluster decomposition $\{(j, k), (1), \ldots, \widehat{(j)}, \ldots, \widehat{(k)}, \ldots, (N)\}$. For $a \in \mathscr{A}$, the cluster Hamiltonian $\tilde{H}_a(E)$ and the intercluster potential I_a are defined by

$$\begin{aligned} \tilde{H}_a(E) &= \tilde{H}_0(E) + V^a, \quad V^a = \sum_{(j,k)\subset a} V_{jk}(x_j - x_k), \\ I_a &= V - V^a = \sum_{(j,k)\not\subset a} V_{jk}(x_j - x_k). \end{aligned} \tag{2.3.5}$$

Here we note

$$\tilde{H}_{a_{\max}}(E) = \tilde{H}(E), \quad \tilde{H}_{a_{\min}}(E) = \tilde{H}_0(E).$$

Of course, $\tilde{H}_a(0)$ can be defined similarly. Let $a = \{C_1, \dots, C_{\#(a)}\} \in \mathscr{A}$. For the sake of simplicity, we suppose $N \in C_{\#(a)}$. For each cluster C_l in a, the innercluster Hamiltonian $\tilde{H}^{C_l}(0)$ is defined by

$$\begin{aligned} \tilde{H}^{C_l}(0) &= \sum_{j \in C_l} \frac{1}{2m_j} D_{j,\perp}^2 - \frac{1}{2}\Delta_{X_\parallel^{C_l}} + V^{C_l}, \\ V^{C_l} &= \sum_{\{j,k\} \subset C_l} V_{jk}(x_j - x_k), \end{aligned} \tag{2.3.6}$$

on $L^2(\boldsymbol{R}^{2\times\#(C_l)} \times X_\parallel^{C_l})$, where $\#(C_l)$ denotes the number of elements in C_l and the configuration space $X_\parallel^{C_l}$ is defined by

$$X_\parallel^{C_l} = \left\{ (x_{c_l(1),\parallel}, \dots, x_{c_l(\#(C_l)),\parallel}) \in \boldsymbol{R}^{\#(C_l)} \;\middle|\; \sum_{k=1}^{\#(C_l)} m_{c_l(k)} x_{c_l(k),\parallel} = 0 \right\},$$

which is equipped with the metric defined by

$$\langle \zeta, \tilde{\zeta} \rangle = \sum_{k=1}^{\#(C_l)} m_{c_l(k)} x_{c_l(k),\parallel} \tilde{x}_{c_l(k),\parallel}$$

for $\zeta = (x_{c_l(1),\parallel}, \dots, x_{c_l(\#(C_l)),\parallel})$, $\tilde{\zeta} = (\tilde{x}_{c_l(1),\parallel}, \dots, \tilde{x}_{c_l(\#(C_l)),\parallel}) \in \boldsymbol{R}^{\#(C_l)}$. In particular, when $l = \#(a)$, $\tilde{H}^{C_{\#(a)}}(0)$ is represented as

$$\tilde{H}^{C_{\#(a)}}(0) = \sum_{\substack{j \in C_{\#(a)} \\ j < N}} \frac{1}{2m_j} p_{j,\perp}^2 + \frac{1}{2m_N} D_{N,\perp}^2 - \frac{1}{2}\Delta_{X_\parallel^{C_{\#(a)}}} + V^{C_{\#(a)}}.$$

If $N = \#(C_{\#(a)})$, that is, $a = a_{\max}$, then $X_\parallel^{C_{\#(a)}} = X_\parallel^{\mathrm{cm}}$, so $\tilde{H}^{C_{\#(a)}}(0)$ is just equal to $\tilde{H}(0)$. On the other hand, when $l < \#(a)$,

$$\tilde{H}^{C_l}(0) = \sum_{j \in C_l} \frac{1}{2m_j} p_{j,\perp}^2 - \frac{1}{2}\Delta_{X_\parallel^{C_l}} + V^{C_l}$$

is just a $\#(C_l)$-body Schrödinger operator without external electromagnetic fields. We also define two subspaces $X_\parallel^a$ and $X_{a,\parallel}$ of $X_\parallel^{\mathrm{cm}}$ by

$$\begin{aligned} X_\parallel^a &= \left\{ (x_{1,\parallel}, \dots, x_{N,\parallel}) \in X_\parallel^{\mathrm{cm}} \;\middle|\; \sum_{k=1}^{\#(C_l)} m_{c_l(k)} x_{c_l(k),\parallel} = 0,\ (l = 1, \dots, \#(a)) \right\}, \\ X_{a,\parallel} &= X_\parallel^{\mathrm{cm}} \ominus X_\parallel^a. \end{aligned}$$

As is well-known, one can identify $X_{\parallel}^{a}$ with $X_{\parallel}^{C_1} \oplus \cdots \oplus X_{\parallel}^{C_{\#(a)}}$. Since $V^a = \sum_{l=1}^{\#(a)} V^{C_l}$ and $X_{\parallel}^{\mathrm{cm}} = (X_{\parallel}^{C_1} \oplus \cdots \oplus X_{\parallel}^{C_{\#(a)}}) \otimes X_{a,\parallel}$, the cluster Hamiltonian $\tilde{H}_a(0)$ can be decomposed into the sum of all the inner cluster Hamiltonian $\tilde{H}^{C_l}(0)$ and the free motion along $X_{a,\parallel}$:

$$
\begin{aligned}
\tilde{H}_a(0) &= \sum_{j=1}^{N} \frac{1}{2m_j} D_{j,\perp}^2 + \sum_{l=1}^{\#(a)} \left(-\frac{1}{2} \Delta_{X_{\parallel}^{C_l}} \right) - \frac{1}{2} \Delta_{X_{a,\parallel}} + \sum_{l=1}^{\#(a)} V^{C_l} \\
&= \sum_{l=1}^{\#(a)} \left(\sum_{j \in C_l} \frac{1}{2m_j} D_{j,\perp}^2 - \frac{1}{2} \Delta_{X_{\parallel}^{C_l}} + V^{C_l} \right) - \frac{1}{2} \Delta_{X_{a,\parallel}} \\
&= \sum_{l=1}^{\#(a)} \mathrm{Id} \otimes \cdots \otimes \mathrm{Id} \otimes \tilde{H}^{C_l}(0) \otimes \mathrm{Id} \otimes \cdots \otimes \mathrm{Id} \\
&\quad + \mathrm{Id} \otimes \cdots \otimes \mathrm{Id} \otimes \left(-\frac{1}{2} \Delta_{X_{a,\parallel}} \right)
\end{aligned}
$$

on $L^2(\boldsymbol{R}^{2\times N} \times X_{\parallel}^{\mathrm{cm}}) \cong L^2(\boldsymbol{R}^{2\times\#(C_1)} \times X_{\parallel}^{C_1}) \otimes \cdots \otimes L^2(\boldsymbol{R}^{2\times\#(C_{\#(a)})} \times X_{\parallel}^{C_{\#(a)}}) \otimes L^2(X_{a,\parallel})$. Here we would like to consider the sum of all the innercluster Hamiltonians except $\tilde{H}^{C_{\#(a)}}(0)$, which is the part of $\tilde{H}_a(0)$ corresponding to the neutral clusters and denoted by $\tilde{H}_a^{\mathrm{n}}$ and defined by

$$
\tilde{H}_a^{\mathrm{n}} = \sum_{l=1}^{\#(a)-1} \mathrm{Id} \otimes \cdots \otimes \mathrm{Id} \otimes \tilde{H}^{C_l}(0) \otimes \mathrm{Id} \otimes \cdots \otimes \mathrm{Id}
$$

on $L^2(\boldsymbol{R}^{2\times\#(C_1)} \times X_{\parallel}^{C_1}) \otimes \cdots \otimes L^2(\boldsymbol{R}^{2\times\#(C_{\#(a)-1})} \times X_{\parallel}^{C_{\#(a)-1}})$, in the center-of-mass frame: Firstly, we will equip $\boldsymbol{R}^{2\times\#(C_l)}$, $l = 1, \ldots, \#(a) - 1$, with the metric

$$
\langle \eta, \tilde{\eta} \rangle = \sum_{k=1}^{\#(C_l)} m_{c_l(k)} x_{c_l(k),\perp} \cdot \tilde{x}_{c_l(k),\perp}
$$

for $\eta = (x_{c_l(1),\perp}, \ldots, x_{c_l(\#(C_l)),\perp})$, $\tilde{\eta} = (\tilde{x}_{c_l(1),\perp}, \ldots, \tilde{x}_{c_l(\#(C_l)),\perp}) \in \boldsymbol{R}^{2\times\#(C_l)}$, and define two subspaces $X_{\perp}^{C_l}$ and $X_{C_l,\perp}$ of $\boldsymbol{R}^{2\times\#(C_l)}$ by

$$
\begin{aligned}
X_{\perp}^{C_l} &= \left\{ (x_{c_l(1),\perp}, \ldots, x_{c_l(\#(C_l)),\perp}) \in \boldsymbol{R}^{2\times\#(C_l)} \;\middle|\; \sum_{k=1}^{\#(C_l)} m_{c_l(k)} x_{c_l(k),\perp} = 0 \right\}, \\
X_{C_l,\perp} &= \boldsymbol{R}^{2\times\#(C_l)} \ominus X_{\perp}^{C_l}.
\end{aligned}
$$

Secondly, we put $X^{C_l} = X_\perp^{C_l} \times X_\parallel^{C_l}$ and $X^{a,\mathrm{n}} = X^{C_1} \times \cdots \times X^{C_{\#(a)-1}}$, and define two subspaces $X_\perp^{a,\mathrm{n}}$ and $X_{a,\mathrm{n},\perp}$ of $\boldsymbol{R}^{2\times(N-\#(C_{\#(a)}))}$ by

$$X_\perp^{a,\mathrm{n}} = X_\perp^{C_1} \times \cdots \times X_\perp^{C_{\#(a)-1}}, \quad X_{a,\mathrm{n},\perp} = \boldsymbol{R}^{2\times(N-\#(C_{\#(a)}))} \ominus X_\perp^{a,\mathrm{n}},$$

which are equipped with the metric $\langle\, , \,\rangle$. They represents the configuration space of the inner structures of the neutral clusters in a and that of the positions of the centers of mass of the neutral clusters in a, respectively. Then $\tilde{H}^{C_l}(0)$, $l < \#(a)$, is represented as

$$\begin{aligned}\tilde{H}^{C_l}(0) &= -\frac{1}{2}\Delta_{X_\perp^{C_l}} - \frac{1}{2}\Delta_{X_{C_l,\perp}} - \frac{1}{2}\Delta_{X_\parallel^{C_l}} + V^{C_l} \\ &= -\frac{1}{2}\Delta_{X^{C_l}} + V^{C_l} - \frac{1}{2}\Delta_{X_{C_l,\perp}}\end{aligned}$$

and $\tilde{H}_a^{\mathrm{n}}$ can be decomposed as

$$\begin{aligned}\tilde{H}_a^{\mathrm{n}} &= \sum_{l=1}^{\#(a)-1} \mathrm{Id} \otimes \cdots \otimes \mathrm{Id} \otimes \left(-\frac{1}{2}\Delta_{X^{C_l}} + V^{C_l} - \frac{1}{2}\Delta_{X_{C_l,\perp}}\right) \otimes \mathrm{Id} \otimes \cdots \otimes \mathrm{Id} \\ &= \left(-\frac{1}{2}\Delta_{X^{a,\mathrm{n}}} + (V^a - V^{C_{\#(a)}})\right) \otimes \mathrm{Id} + \mathrm{Id} \otimes \left(-\frac{1}{2}\Delta_{X_{a,\mathrm{n},\perp}}\right) \\ &= \tilde{H}^{a,\mathrm{n}} \otimes \mathrm{Id} + \mathrm{Id} \otimes \left(-\frac{1}{2}\Delta_{X_{a,\mathrm{n},\perp}}\right)\end{aligned} \tag{2.3.7}$$

on $L^2(\boldsymbol{R}^{2\times\#(C_1)} \times X_\parallel^{C_1}) \otimes \cdots \otimes L^2(\boldsymbol{R}^{2\times\#(C_{\#(a)-1})} \times X_\parallel^{C_{\#(a)-1}}) \cong L^2(X^{a,\mathrm{n}}) \otimes L^2(X_{a,\mathrm{n},\perp})$, where

$$\tilde{H}^{a,\mathrm{n}} = -\frac{1}{2}\Delta_{X^{a,\mathrm{n}}} + (V^a - V^{C_{\#(a)}})$$

is an $(N - \#(C_{\#(a)}))$-body Schrödinger operator without external electromagnetic fields in the center-of-mass frame. Thus we have

$$\begin{aligned}\tilde{H}_a(0) = \tilde{H}^{a,\mathrm{n}} &\otimes \mathrm{Id} \otimes \mathrm{Id} \otimes \mathrm{Id} + \mathrm{Id} \otimes \tilde{H}^{C_{\#(a)}}(0) \otimes \mathrm{Id} \otimes \mathrm{Id} \\ &+ \mathrm{Id} \otimes \mathrm{Id} \otimes \left(-\frac{1}{2}\Delta_{X_{a,\mathrm{n},\perp}}\right) \otimes \mathrm{Id} + \mathrm{Id} \otimes \mathrm{Id} \otimes \mathrm{Id} \otimes \left(-\frac{1}{2}\Delta_{X_{a,\parallel}}\right)\end{aligned} \tag{2.3.8}$$

on $L^2(\boldsymbol{R}^{2\times N} \times X_\parallel^{\mathrm{cm}}) \cong L^2(X^{a,\mathrm{n}}) \otimes L^2(\boldsymbol{R}^{2\times\#(C_{\#(a)})}) \otimes L^2(X_{a,\mathrm{n},\perp}) \otimes L^2(X_{a,\parallel})$. We put

$$\Pi^a(0) := P_{\mathrm{pp}}(\tilde{H}^{a,\mathrm{n}}) \otimes P_{\mathrm{pp}}(\tilde{H}^{C_{\#(a)}}(0)) \otimes \mathrm{Id} \otimes \mathrm{Id} \tag{2.3.9}$$

on $L^2(\boldsymbol{R}^{2\times N} \times X_{\|}^{\rm cm}) \cong L^2(X^{a,{\rm n}}) \otimes L^2(\boldsymbol{R}^{2\times \#(C_{\#(a)})}) \otimes L^2(X_{a,{\rm n},\perp}) \otimes L^2(X_{a,\|})$, where $P_{\rm pp}(\tilde{H}^{a,{\rm n}})$ and $P_{\rm pp}(\tilde{H}^{C_{\#(a)}}(0))$ are the eigenprojections for $\tilde{H}^{a,{\rm n}}$ and $\tilde{H}^{C_{\#(a)}}(0)$, respectively. $\Pi^a(0)$ is called the channel identification operator associated to a. If $a = a_{\max}$, then $\Pi^{a_{\max}}(0) = P_{\rm pp}(\tilde{H}(0))$ holds; while, if $a = a_{\min}$, then $X^{C_l} = \{(0,0,0)\}$, $(l = 1,\dots,N-1)$, and $\tilde{H}^{C_N}(0) = D_{N,\perp}^2/(2m_N)$ so $\Pi^{a_{\min}}(0) = {\rm Id}$ holds. Here we introduce the wave operators

$$\begin{aligned} W_a^{\pm}(E) &= \operatorname*{s\text{-}lim}_{t\to\pm\infty} e^{it\tilde{H}(E)} e^{-it\tilde{H}_a(E)} \Pi^a(E), \\ \Pi^a(E) &= \tilde{\mathscr{T}}(0)\Pi^a(0)\tilde{\mathscr{T}}(0)^* = e^{iM\alpha\cdot x_{\rm cm}} \Pi^a(0) e^{-iM\alpha\cdot x_{\rm cm}}, \end{aligned} \tag{2.3.10}$$

for $a \in \mathscr{A}$. $W_{a_{\max}}^{\pm}(E)$ is identified with $\Pi^{a_{\max}}(E)$. Then one can obtain the following result of the asymptotic completeness for $\tilde{H}(E)$:

Theorem 2.3.1. *Assume V satisfies $(V1)_{3,{\rm SR}}$. Then the wave operators $W_a^{\pm}(E)$, $a \in \mathscr{A}$, all exist, and are asymptotically complete:*

$$L^2(\boldsymbol{R}^{2\times N} \times X_{\|}^{\rm cm}) = \sum_{a\in\mathscr{A}} \oplus \operatorname{Ran} W_a^{\pm}(E). \tag{2.3.11}$$

This theorem with $N = 2$ was already obtained by Kiyose [20] (where $d = 2$ was supposed for the sake of simplicity). The asymptotic completeness (2.3.11) is equivalent to that the time evolution $e^{-it\tilde{H}(E)}\psi$ of any scattering state $\psi \in L^2_{\rm ac}(\tilde{H}(E)) = L^2(\boldsymbol{R}^{2\times N} \times X_{\|}^{\rm cm})$ is asymptotically represented as

$$e^{-it\tilde{H}(E)}\psi = \sum_{a\in\mathscr{A}} e^{-it\tilde{H}_a(E)} \Pi^a(E) \psi_a^{\pm} + o(1) \quad \text{as} \quad t \to \pm\infty \tag{2.3.12}$$

with some $\psi_a^{\pm} \in L^2(\boldsymbol{R}^{2\times N} \times X_{\|}^{\rm cm})$. In particular, $e^{-it\tilde{H}_{a_{\max}}(E)} \Pi^{a_{\max}}(E) \psi_{a_{\max}}^{\pm}$ implies that all the particles in the system move with the velocity α in forming a certain bound state. In fact, the guiding center of the N-th particle, which is the only charged one, drifts with the velocity α. This result can be obtained immediately by using Theorem 2.1.1 and the following result due to Adachi [1] and [2]:

Theorem 2.3.2 (Adachi [1] [2], 2001, 2002)**.** *Assume V satisfies $(V1)_{3,{\rm SR}}$. Then the wave operators*

$$W_a^{\pm}(0) = \operatorname*{s\text{-}lim}_{t\to\pm\infty} e^{it\tilde{H}(0)} e^{-it\tilde{H}_a(0)} \Pi^a(0), \quad a \in \mathscr{A}, \tag{2.3.13}$$

all exist, and are asymptotically complete:

$$L^2(\boldsymbol{R}^{2\times N} \times X_{\|}^{\rm cm}) = \sum_{a\in\mathscr{A}} \oplus \operatorname{Ran} W_a^{\pm}(0). \tag{2.3.14}$$

In fact, Theorem 2.1.1 yields

$$W_a^{\pm}(E) = \tilde{\mathscr{T}}(0) W_a^{\pm}(0) \tilde{\mathscr{T}}(0)^* = e^{iM\alpha\cdot x_{\mathrm{cm}}} W_a^{\pm}(0) e^{-iM\alpha\cdot x_{\mathrm{cm}}} \tag{2.3.15}$$

for $a \in \mathscr{A}$. Hence, Theorem 2.3.1 can be obtained by virtue of Theorem 2.3.2.

In the case where $N = 2$, the way of reducing the motion of the center of mass in the direction parallel to the magnetic field is different and explicit: we introduce the bijective transformation of the positions $\zeta = (x_{1,\|}, x_{2,\|}) \in \boldsymbol{R}^2$ into the relative position and the position of the center of mass

$$J\zeta = \left(x_{1,\|} - x_{2,\|}, \frac{m_1 x_{1,\|} + m_2 x_{2,\|}}{M} \right).$$

Then, for any $\tilde{\varphi} \in \mathscr{S}(\boldsymbol{R}^2_{(z,Z)})$, using the chain rule we write that

$$\begin{aligned} \partial_{x_{1,\|}}(\tilde{\varphi} \circ J) &= (\partial_z \tilde{\varphi}) \circ J + \frac{m_1}{M}(\partial_Z \tilde{\varphi}) \circ J, \\ \partial_{x_{2,\|}}(\tilde{\varphi} \circ J) &= -(\partial_z \tilde{\varphi}) \circ J + \frac{m_2}{M}(\partial_Z \tilde{\varphi}) \circ J, \end{aligned}$$

and its solution with respect to $\partial_z \tilde{\varphi}$ and $\partial_Z \tilde{\varphi}$

$$\begin{aligned} (\partial_z \tilde{\varphi}) \circ J &= \mu \left(\frac{\partial_{x_{1,\|}}}{m_1} - \frac{\partial_{x_{2,\|}}}{m_2} \right) (\tilde{\varphi} \circ J), \\ (\partial_Z \tilde{\varphi}) \circ J &= (\partial_{x_{1,\|}} + \partial_{x_{2,\|}})(\tilde{\varphi} \circ J), \end{aligned}$$

where μ is the reduced mass $m_1 m_2/(m_1 + m_2)$. Here we introduce the composition operator J^* defined by the pullback by J:

$$J^* \tilde{u} = \tilde{u} \circ J, \quad \tilde{u} \in L^2(\boldsymbol{R}^2_{(z,Z)}),$$

which is well-defined and unitary because $\det J = 1$. From the above equations we have

$$\begin{aligned} p_{1,\|} J^* &= J^* \left(p_z + \frac{m_1}{M} p_Z \right), & p_{2,\|} J^* &= J^* \left(-p_z + \frac{m_2}{M} p_Z \right), \\ J^* p_z &= \mu \left(\frac{p_{1,\|}}{m_1} - \frac{p_{2,\|}}{m_2} \right) J^*, & J^* p_Z &= (p_{1,\|} + p_{2,\|}) J^*, \end{aligned}$$

where $p_{1,\|} = -i\partial_{x_{1,\|}}$, $p_{2,\|} = -i\partial_{x_{2,\|}}$, $p_z = -i\partial_z$ (relative momentum) and $p_Z = -i\partial_Z$ (center-of-mass momentum). Calculating the transformation of the free Hamiltonian in the direction parallel to the magnetic field, we have

$$\left(\frac{p_{1,\|}^2}{2m_1} + \frac{p_{2,\|}^2}{2m_2} \right) J^* = J^* \left(\frac{1}{2m_1} \left(p_z + \frac{m_1}{M} p_Z \right)^2 + \frac{1}{2m_2} \left(-p_z + \frac{m_2}{M} p_Z \right)^2 \right)$$

$$
\begin{aligned}
&= J^* \left(\left(\frac{1}{2m_1} + \frac{1}{2m_2} \right) p_z^2 + \frac{p_z p_Z}{M} - \frac{p_z p_Z}{M} + \frac{m_1 + m_2}{2M^2} p_Z^2 \right) \\
&= J^* \left(\frac{p_z^2}{2\mu} + \frac{p_Z^2}{2M} \right),
\end{aligned}
$$

and therefore the total free Hamiltonian $\tilde{H}_0(\boldsymbol{E}_\perp)$ is transformed as

$$
\begin{aligned}
\tilde{H}_0(\boldsymbol{E}_\perp)(\mathrm{Id} \otimes J^*) = &\left(\left(\frac{1}{2m_1} p_{1,\perp}^2 + \left(\frac{1}{2m_2} D_{2,\perp}^2 - q_2 E \cdot x_{2,\perp} \right) \right) \otimes \mathrm{Id} \right) (\mathrm{Id} \otimes J^*) \\
&+ \left(\mathrm{Id} \otimes \left(\frac{p_{1,\parallel}^2}{2m_1} + \frac{p_{2,\parallel}^2}{2m_2} \right) \right) (\mathrm{Id} \otimes J^*) \\
= &(\mathrm{Id} \otimes J^*) \left(\left(\frac{1}{2m_1} p_{1,\perp}^2 + \left(\frac{1}{2m_2} D_{2,\perp}^2 - q_2 E \cdot x_{2,\perp} \right) \right) \otimes \mathrm{Id} \right) \\
&+ (\mathrm{Id} \otimes J^*) \left(\mathrm{Id} \otimes \left(\frac{p_z^2}{2\mu} + \frac{p_Z^2}{2M} \right) \right) \\
= &(\mathrm{Id} \otimes J^*) \left(\tilde{H}_0(E) \otimes \mathrm{Id}_{L^2(\boldsymbol{R})} + \mathrm{Id}_{L^2(\boldsymbol{R}^{2\times 2} \times \boldsymbol{R})} \otimes \frac{p_Z^2}{2M} \right)
\end{aligned}
$$

on $L^2(\boldsymbol{R}^{3\times 2}) \cong L^2(\boldsymbol{R}_{x_\perp}^{2\times 2}) \otimes L^2(\boldsymbol{R}_{(z,Z)}^2) \cong L^2(\boldsymbol{R}_{x_\perp}^{2\times 2} \times \boldsymbol{R}_z) \otimes L^2(\boldsymbol{R}_Z)$, where the reduced free Hamiltonian $\tilde{H}_0(E)$ is represented by

$$
\tilde{H}_0(E) = \tilde{H}_{a_{\min}}(E) = \frac{1}{2m_1} p_{1,\perp}^2 + \left(\frac{1}{2m_2} D_{2,\perp}^2 - q_2 E \cdot x_{2,\perp} \right) + \frac{1}{2\mu} p_z^2
$$

on $L^2(\boldsymbol{R}_{x_\perp}^{2\times 2} \times \boldsymbol{R}_z)$. On the other hand, as for the potential, for any $\varphi_1 \in \mathscr{S}(\boldsymbol{R}_{x_\perp}^{2\times 2})$, $\tilde{\varphi}_2 \in \mathscr{S}(\boldsymbol{R}_z)$ and $\tilde{\varphi}_3 \in \mathscr{S}(\boldsymbol{R}_Z)$,

$$
\begin{aligned}
&V(\mathrm{Id} \otimes J^*)(\varphi_1 \otimes (\tilde{\varphi}_2 \otimes \tilde{\varphi}_3))(x) \\
&\quad = V_{12}(x_1 - x_2) \varphi_1(x_\perp)(\tilde{\varphi}_2 \otimes \tilde{\varphi}_3)(J x_\parallel) \\
&\quad = V_{12}(x_{1,\perp} - x_{2,\perp}, x_{1,3} - x_{2,3}) \varphi_1(x_\perp) \tilde{\varphi}_2(x_{1,3} - x_{2,3}) \tilde{\varphi}_3 \left(\frac{m_1 x_{1,3} + m_2 x_{2,3}}{M} \right) \\
&\quad = (\tilde{V}(\varphi_1 \otimes \tilde{\varphi}_2))(x_\perp, x_{1,3} - x_{2,3}) \tilde{\varphi}_3 \left(\frac{m_1 x_{1,3} + m_2 x_{2,3}}{M} \right) \\
&\quad = ((\tilde{V}(\varphi_1 \otimes \tilde{\varphi}_2)) \otimes \tilde{\varphi}_3)(x_\perp, J x_\parallel) \\
&\quad = (\mathrm{Id} \otimes J^*)(\tilde{V} \otimes \mathrm{Id}_{L^2(\boldsymbol{R})})((\varphi_1 \otimes \tilde{\varphi}_2) \otimes \tilde{\varphi}_3)(x),
\end{aligned}
$$

where $\tilde{V}$ is the operator of multiplication by

$$
V_{12}(x_{1,\perp} - x_{2,\perp}, z)
$$

on $L^2(\boldsymbol{R}_{x_\perp}^{2\times 2} \times \boldsymbol{R}_z)$ and in the last member we used the fact that

$$(\mathrm{Id} \otimes J^*)\tilde{u}(x) = \tilde{u}(x_\perp, Jx_\parallel), \quad \text{for all } \tilde{u} \in \mathscr{S}(\boldsymbol{R}_{x_\perp}^{2\times 2}) \otimes \mathscr{S}(\boldsymbol{R}_{(z,Z)}^2),$$

which extends to the equality for all $\tilde{u} \in L^2(\boldsymbol{R}_{x_\perp}^{2\times 2}) \otimes L^2(\boldsymbol{R}_{(z,Z)}^2)$ because the operator in the right member $\tilde{u}(x) \mapsto \tilde{u}(x_\perp, Jx_\parallel)$ is also bounded (unitary); hence we have

$$V(\mathrm{Id} \otimes J^*) = (\mathrm{Id} \otimes J^*)(\tilde{V} \otimes \mathrm{Id}_{L^2(\boldsymbol{R})})$$

on $L^2(\boldsymbol{R}^{3\times 2}) \cong L^2(\boldsymbol{R}_{x_\perp}^{2\times 2}) \otimes L^2(\boldsymbol{R}_{(z,Z)}^2) \cong L^2(\boldsymbol{R}_{x_\perp}^{2\times 2} \times \boldsymbol{R}_z) \otimes L^2(\boldsymbol{R}_Z)$. Therefore the total Hamiltonian $\tilde{H}(\boldsymbol{E}_\perp)$ is transformed as

$$\tilde{H}(\boldsymbol{E}_\perp)(\mathrm{Id} \otimes J^*) = (\mathrm{Id} \otimes J^*) \left(\tilde{H}(E) \otimes \mathrm{Id}_{L^2(\boldsymbol{R})} + \mathrm{Id}_{L^2(\boldsymbol{R}^{2\times 2} \times \boldsymbol{R})} \otimes \frac{p_Z^2}{2M} \right)$$

on $L^2(\boldsymbol{R}^{3\times 2}) \cong L^2(\boldsymbol{R}_{x_\perp}^{2\times 2}) \otimes L^2(\boldsymbol{R}_{(z,Z)}^2) \cong L^2(\boldsymbol{R}_{x_\perp}^{2\times 2} \times \boldsymbol{R}_z) \otimes L^2(\boldsymbol{R}_Z)$, where the reduced Hamiltonian $\tilde{H}(E)$ on $L^2(\boldsymbol{R}_{x_\perp}^{2\times 2} \times \boldsymbol{R}_z)$ is represented by

$$\tilde{H}(E) = \tilde{H}_0(E) + \tilde{V}.$$

Then, one can also obtain the result of the asymptotic completeness for $\tilde{H}(E)$ with some long-range potential $V = V_{12}$ by virtue of the result of [1]:

Theorem 2.3.3. *Suppose $N = 2$. Assume $V = V_{12} = I^{a_{\min}}$ belongs to $C^\infty(\boldsymbol{R}^3; \boldsymbol{R})$, and satisfies the decaying condition*

$$|\partial_r^\beta V(r)| \leq C_\beta \langle r \rangle^{-\rho - |\beta|} \tag{2.3.16}$$

with some $1/2 < \rho \leq 1$. Then the modified wave operators

$$W_{a_{\min},D}^\pm(E) = \operatorname*{s-lim}_{t\to\pm\infty} e^{it\tilde{H}(E)} e^{-it\tilde{H}_{a_{\min}}(E)} e^{-i\int_0^t V(sp_{1,\perp}/m_1 - s\alpha_\perp, sp_z/\mu)\,ds}, \tag{2.3.17}$$

exist, and are asymptotically complete:

$$L^2(\boldsymbol{R}^{2\times 2} \times \boldsymbol{R}) = \mathrm{Ran}\, W_{a_{\min},D}^\pm(E) \oplus \mathrm{Ran}\, \Pi^{a_{\max}}(E). \tag{2.3.18}$$

By virtue of the result of [1], the modified wave operators

$$W_{a_{\min},D}^\pm(0) = \operatorname*{s-lim}_{t\to\pm\infty} e^{it\tilde{H}(0)} e^{-it\tilde{H}_{a_{\min}}(0)} e^{-i\int_0^t V(sp_{1,\perp}/m_1, sp_z/\mu)\,ds}, \tag{2.3.19}$$

exist, and are asymptotically complete:

$$L^2(\boldsymbol{R}^{2\times 2} \times \boldsymbol{R}) = \mathrm{Ran}\, W_{a_{\min},D}^\pm(0) \oplus \mathrm{Ran}\, \Pi^{a_{\max}}(0). \tag{2.3.20}$$

Since Theorem 2.1.1 yields

$$W^{\pm}_{a_{\min},D}(E) = \tilde{\mathscr{T}}(0)W^{\pm}_{a_{\min},D}(0)\tilde{\mathscr{T}}(0)^* = e^{iM\alpha\cdot x_{\rm cm}}W^{\pm}_{a_{\min},D}(0)e^{-iM\alpha\cdot x_{\rm cm}}, \tag{2.3.21}$$

Theorem 2.3.3 can be shown in the same way as above (see also Kiyose [20]).

Also in the case where the space dimension d is not three but two, the asymptotic completeness is true, because the results due to Adachi is still valid (see [1, REMARK 1.3], and [2, pp. 205–207]). This fact is in strong connection with the classical picture that the charged particles are bounded in the direction perpendicular to the magnetic field, but the neutral particles are not so. In this case, we consider the problem of the asymptotic completeness for an N-body quantum system in the plane $\boldsymbol{R}^2$ to which constant electric and magnetic fields are impressed. Then the total Hamiltonian $\tilde{H}_\perp(E)$ on $L^2(\boldsymbol{R}^{2\times N})$ is represented as

$$\begin{aligned} &\tilde{H}_\perp(E) = \tilde{H}_{0,\perp}(E) + V, \\ &\tilde{H}_{0,\perp}(E) = \sum_{j=1}^{N-1}\frac{1}{2m_j}p_{j,\perp}^2 + \left(\frac{1}{2m_N}D_{N,\perp}^2 - q_N E\cdot x_{N,\perp}\right) \end{aligned} \tag{2.3.22}$$

with $q_N \neq 0$, and the wave operators are defined by

$$\begin{aligned} &W_a^{\pm}(E) = \operatorname*{s-lim}_{t\to\pm\infty} e^{it\tilde{H}_\perp(E)}e^{-it\tilde{H}_{a,\perp}(E)}\Pi^a(E), \\ &\Pi^a(E) = \tilde{\mathscr{T}}_\perp(0)\Pi^a(0)\tilde{\mathscr{T}}_\perp(0)^* = e^{iM\alpha_\perp\cdot x_{{\rm cm},\perp}}\Pi^a(0)e^{-iM\alpha_\perp\cdot x_{{\rm cm},\perp}}, \end{aligned} \tag{2.3.23}$$

for $a \in \mathscr{A}$, where the formulations of the Hamiltonians can be done similarly:

$$\begin{aligned} &\tilde{H}_{a,\perp}(E) = \tilde{H}_{0,\perp}(E) + V^a, \quad V^a = \sum_{(j,k)\subset a} V_{jk}(x_{j,\perp} - x_{k,\perp}), \\ &\tilde{H}_\perp^{C_l}(0) = \sum_{j\in C_l}\frac{1}{2m_j}D_{j,\perp}^2 + V^{C_l}, \quad V^{C_l} = \sum_{\{j,k\}\subset C_l} V_{jk}(x_{j,\perp} - x_{k,\perp}), \\ &\qquad \text{on } L^2(\boldsymbol{R}^{2\times\#(C_l)}), \\ &\tilde{H}_\perp^{a,\rm n} = -\frac{1}{2}\Delta_{X_\perp^{a,\rm n}} + (V^a - V^{C_{\#(a)}}), \\ &\qquad \text{on } L^2(\boldsymbol{R}^{2\times(N-\#(C_{\#(a)}))}) \cong L^2(X_\perp^{a,\rm n}) \otimes L^2(X_{a,{\rm n},\perp}), \\ &\Pi^a(0) := P_{\rm pp}(\tilde{H}_\perp^{a,\rm n}) \otimes P_{\rm pp}(\tilde{H}_\perp^{C_{\#(a)}}(0)) \otimes {\rm Id}, \\ &\qquad \text{on } L^2(\boldsymbol{R}^{2\times N}) \cong L^2(X_\perp^{a,\rm n}) \otimes L^2(\boldsymbol{R}^{2\times\#(C_{\#(a)})}) \otimes L^2(X_{a,{\rm n},\perp}). \end{aligned} \tag{2.3.24}$$

Then one can obtain the following result of the asymptotic completeness for $\tilde{H}_\perp(E)$:

Theorem 2.3.4. *Assume V satisfies $(V1)_{2,\mathrm{SR}}$. Then the wave operators $W_a^{\pm}(E)$, $a \in \mathscr{A}$, all exist, and are asymptotically complete:*

$$L^2(\boldsymbol{R}^{2\times N}) = \sum_{a\in\mathscr{A}} \oplus \operatorname{Ran} W_a^{\pm}(E). \tag{2.3.25}$$

This result can be obtained immediately by using Theorem 2.1.2 and the following result due to Adachi [1] and [2]:

Theorem 2.3.5 (Adachi [1] [2], 2001, 2002)**.** *Assume V satisfies $(V1)_{2,\mathrm{SR}}$. Then the wave operators*

$$W_a^{\pm}(0) = \operatorname*{s-lim}_{t\to\pm\infty} e^{it\tilde{H}_\perp(0)} e^{-it\tilde{H}_{a,\perp}(0)} \Pi^a(0), \quad a \in \mathscr{A}, \tag{2.3.26}$$

all exist, and are asymptotically complete:

$$L^2(\boldsymbol{R}^{2\times N}) = \sum_{a\in\mathscr{A}} \oplus \operatorname{Ran} W_a^{\pm}(0). \tag{2.3.27}$$

In fact, Theorem 2.1.2 yields

$$W_a^{\pm}(E) = \tilde{\mathscr{T}}_\perp(0) W_a^{\pm}(0) \tilde{\mathscr{T}}_\perp(0)^* = e^{iM\alpha_\perp\cdot x_{\mathrm{cm},\perp}} W_a^{\pm}(0) e^{-iM\alpha_\perp\cdot x_{\mathrm{cm},\perp}} \tag{2.3.28}$$

for $a \in \mathscr{A}$. Hence, Theorem 2.3.4 can be obtained by virtue of Theorem 2.3.5.

In the case where $N = 2$, one can also obtain the result of the asymptotic completeness for $\tilde{H}_\perp(E)$ with some long-range potential $V = V_{12}$ by virtue of the result of [1]:

Theorem 2.3.6. *Suppose $N = 2$. Assume $V = V_{12} = I^{a_{\min}}$ belongs to $C^\infty(\boldsymbol{R}^2; \boldsymbol{R})$, and satisfies the decaying condition*

$$|\partial_r^\beta V(r)| \le C_\beta \langle r \rangle^{-\rho-|\beta|} \tag{2.3.29}$$

with some $1/2 < \rho \le 1$. Then the modified wave operators

$$W_{a_{\min},D}^{\pm}(E) = \operatorname*{s-lim}_{t\to\pm\infty} e^{it\tilde{H}_\perp(E)} e^{-it\tilde{H}_{a_{\min},\perp}(E)} e^{-i\int_0^t V(sp_{1,\perp}/m_1 - s\alpha_\perp)\,ds}, \tag{2.3.30}$$

exist, and are asymptotically complete:

$$L^2(\boldsymbol{R}^{2\times 2}) = \operatorname{Ran} W_{a_{\min},D}^{\pm}(E) \oplus \operatorname{Ran} \Pi^{a_{\max}}(E). \tag{2.3.31}$$

By virtue of the result of [1], the modified wave operators

$$W_{a_{\min},D}^{\pm}(0) = \operatorname*{s-lim}_{t\to\pm\infty} e^{it\tilde{H}_\perp(0)} e^{-it\tilde{H}_{a_{\min},\perp}(0)} e^{-i\int_0^t V(sp_{1,\perp}/m_1)\,ds}, \tag{2.3.32}$$

exist, and are asymptotically complete:

$$L^2(\boldsymbol{R}^{2\times 2}) = \operatorname{Ran} W^{\pm}_{a_{\min},D}(0) \oplus \operatorname{Ran} \Pi^{a_{\max}}(0). \tag{2.3.33}$$

Since Theorem 2.1.2 yields

$$W^{\pm}_{a_{\min},D}(E) = \tilde{\mathscr{T}}_{\perp}(0) W^{\pm}_{a_{\min},D}(0) \tilde{\mathscr{T}}_{\perp}(0)^* = e^{iM\alpha_{\perp}\cdot x_{\mathrm{cm},\perp}} W^{\pm}_{a_{\min},D}(0) e^{-iM\alpha_{\perp}\cdot x_{\mathrm{cm},\perp}}, \tag{2.3.34}$$

Theorem 2.3.6 can be shown in the same way as above. The Dollard type modifier $e^{-i\int_0^t V(sp_{1,\perp}/m_1 - s\alpha_{\perp})\,ds}$ in the definition of $W^{\pm}_{a_{\min},D}(E)$ seems quite natural, by taking account of that the guiding center of the second particle, which is the only charged one, drifts with the velocity $\alpha_{\perp}$. Here we note that when $N = 1$, the corresponding long-range scattering problem has not been solved yet, as far as we know (see Adachi-Kawamoto [4]). Unlike in the case where $N \geq 2$, in general, V does not commute with the conjugate operator $\tilde{A} = q_1 E \cdot p_{1,\perp}$ (cf. (2.3.3)). For reference, the problem of the asymptotic completeness for $\tilde{H}_{\perp}(0)$ with $N \geq 3$ and long-range interactions has not been solved yet, as far as we know. But, maybe one can show the asymptotic completeness under the additional assumption on smooth V_{jk}'s

$$|\partial_r^{\beta} V_{jk}(r)| \leq C_{\beta} \langle r \rangle^{-\rho-|\beta|} \tag{2.3.35}$$

with some $\sqrt{3}-1 < \rho \leq 1$, by using the arguments of Dereziński [10] and Gérard-Łaba [16]. $\sqrt{3}-1$ is called the so-called Enss number. Hence we may expect that a natural extension of Theorem 2.3.3 to the case where $N \geq 3$ is obtained.

On the other hand, Gérard and Łaba [17] showed that if the system is strongly charged, that is, all the proper subsystems are charged (in particular no neutral particles are present) and the motions of the particles are restricted to the plane $\boldsymbol{R}^2$ perpendicular to the magnetic field $\boldsymbol{B}$ and $\boldsymbol{E} = 0$, there shall exist no scattering state. In considering the scattering problem on the plane $\boldsymbol{R}^2$ perpendicular to the magnetic field $\boldsymbol{B}$, it is crucial whether the neutral clusters or particles are present or not.

2.4 Concluding remarks

We have considered the case where the homogeneous electric field is independent of t only. Here we will make some remarks on the case where the electric field is strictly dependent on t.

In order to make the point at issue clear, we suppose that the space dimension d is two, that the time-dependent electric field $E(t) = (E_1(t), E_2(t)) \in C(\boldsymbol{R}; \boldsymbol{R}^2)$ lies in the plane $\boldsymbol{R}^2$, and that the system under consideration has at least one charged particle. Then the free Hamiltonian $\tilde{H}_{0,\perp}(E(t))$ of the system is defined by

$$\tilde{H}_{0,\perp}(E(t)) = \sum_{j=1}^{N} \left(\frac{1}{2m_j} D_{j,\perp}^2 - q_j E(t) \cdot x_{j,\perp} \right) \tag{2.4.1}$$

on $L^2(\boldsymbol{R}^{2\times N})$. We denote by $\tilde{U}_{0,\perp}(t,s)$ the propagator generated by $\tilde{H}_{0,\perp}(E(t))$. By using the results of Adachi-Kawamoto [4], one can obtain the following Avron-Herbst type formula for $\tilde{U}_{0,\perp}(t,0)$ immediately:

$$\tilde{U}_{0,\perp}(t,0) = \tilde{\mathscr{T}}_{1,\perp}(t) e^{-it\tilde{H}_{0,\perp}(0)} \tilde{\mathscr{T}}_{1,\perp}(0)^* \tag{2.4.2}$$

with

$$\begin{aligned}
&\tilde{\mathscr{T}}_{1,\perp}(t) = \prod_{j=1}^{N} \mathscr{T}_{j,\perp}(t), \quad \mathscr{T}_{j,\perp}(t) = e^{-im_j a_{j,\perp}(t)} e^{im_j b_{j,\perp}(t)\cdot x_{j,\perp}} e^{-ic_{j,\perp}(t)\cdot k_{j,\perp}}, \\
&b_{j,\perp}(t)^{\mathrm{T}} = \frac{\omega_j}{B} \int_0^t \hat{R}(-\omega_j(t-s)) E(s)^{\mathrm{T}}\, ds, \quad c_{j,\perp}(t) = \int_0^t b_{j,\perp}(s)\, ds, \\
&a_{j,\perp}(t) = \int_0^t \left\{ \frac{1}{2} b_{j,\perp}(s)^2 + \frac{\omega_j}{B} b_{j,\perp}(s) \cdot A(c_{j,\perp}(s)) \right\} ds, \\
&\omega_j = \frac{q_j B}{m_j}, \quad \hat{R}(\varphi) = \begin{pmatrix} \cos\varphi & -\sin\varphi \\ \sin\varphi & \cos\varphi \end{pmatrix}.
\end{aligned} \tag{2.4.3}$$

$|\omega_j|$ is called the Larmor frequency of the j-th particle. ω_j/B is equal to the specific charge q_j/m_j. The Avron-Herbst type formula (2.4.2) with $N = 1$ was already obtained in [4]. The differential equations for $a_{j,\perp}(t)$, $b_{j,\perp}(t)$ and $c_{j,\perp}(t)$ are given as

$$\begin{aligned}
&\dot{b}_{j,\perp}(t) + \frac{2\omega_j}{B} A(b_{j,\perp}(t)) = \frac{\omega_j}{B} E(t), \quad b_{j,\perp}(0) = 0, \\
&\dot{c}_{j,\perp}(t) = b_{j,\perp}(t), \quad c_{j,\perp}(0) = 0, \\
&\dot{a}_{j,\perp}(t) = \frac{1}{2} b_{j,\perp}(t)^2 + \frac{\omega_j}{B} b_{j,\perp}(t) \cdot A(c_{j,\perp}(t)), \quad a_{j,\perp}(0) = 0
\end{aligned} \tag{2.4.4}$$

(see [4]). Now we introduce the total Hamiltonian $\tilde{H}_{\perp}(E(t))$ of the system is defined by

$$\tilde{H}_{\perp}(E(t)) = \tilde{H}_{0,\perp}(E(t)) + V, \quad V = \sum_{1\le j<k\le N} V_{jk}(x_{j,\perp} - x_{k,\perp}) \tag{2.4.5}$$

on $L^2(\boldsymbol{R}^{2\times N})$, and denote by $\tilde{U}_{\perp}(t,s)$ the propagator generated by $\tilde{H}_{\perp}(E(t))$. Then the following Avron-Herbst type formula for $\tilde{U}_{\perp}(t,0)$ can be obtained by virtue of (2.4.2):

Theorem 2.4.1. *Denote by $\bar{U}_\perp(t,s)$ the propagator generated by the time-dependent Hamiltonian*

$$\begin{aligned} &\bar{H}_\perp(t) = \tilde{H}_{0,\perp}(0) + V(t), \\ &V(t) = \sum_{1\le j<k\le N} V_{jk}((x_{j,\perp} + c_{j,\perp}(t)) - (x_{k,\perp} + c_{k,\perp}(t))) \end{aligned} \tag{2.4.6}$$

on $L^2(\boldsymbol{R}^{2\times N})$. Suppose that both $\tilde{U}_\perp(t,s)$ and $\bar{U}_\perp(t,s)$ exist uniquely. Then the Avron-Herbst type formula

$$\tilde{U}_\perp(t,0) = \tilde{\mathscr{T}}_{1,\perp}(t)\bar{U}_\perp(t,0)\tilde{\mathscr{T}}_{1,\perp}(0)^* \tag{2.4.7}$$

holds.

We note $\tilde{\mathscr{T}}_{1,\perp}(0) = \mathrm{Id}$, because $\mathscr{T}_{j,\perp}(0) = \mathrm{Id}$ for any j. By definition, if the specific charges q_j/m_j and q_k/m_k are different from each other, then $b_{j,\perp}(t) \neq b_{k,\perp}(t)$ and $c_{j,\perp}(t) \neq c_{k,\perp}(t)$ in general, because $\omega_j \neq \omega_k$. Hence $V(t)$ is time-dependent generally. Because of this, it seems hard to get useful propagation properties of $\bar{U}_\perp(t,0)$. To overcome this difficulty is an issue in the future. However, if all the specific charges are the same, then since Q/M is equal to that specific charge as is remarked above, $QB/M = \omega_1 = \cdots = \omega_N$, $b_{1,\perp}(t) = \cdots = b_{N,\perp}(t)$ and $c_{1,\perp}(t) = \cdots = c_{N,\perp}(t)$ hold. Therefore $V(t)$ is time-independent. Hence we have the following corollary:

Corollary 2.4.2. *Suppose that all the specific charges of the system are the same. Then the Avron-Herbst type formula*

$$\begin{aligned} &\tilde{U}_\perp(t,0) = \tilde{\mathscr{T}}_{1,\perp}(t)e^{-it\bar{H}_\perp(0)}\tilde{\mathscr{T}}_{1,\perp}(0)^*, \\ &\tilde{\mathscr{T}}_{1,\perp}(t) = e^{-iMa_{\mathrm{total},\perp}(t)}e^{iMb_{\mathrm{total},\perp}(t)\cdot x_{\mathrm{cm},\perp}}e^{-ic_{\mathrm{total},\perp}(t)\cdot k_{\mathrm{total},\perp}} \end{aligned} \tag{2.4.8}$$

holds with

$$\begin{aligned} &b_{\mathrm{total},\perp}(t)^{\mathrm{T}} = \frac{\Omega}{B}\int_0^t \hat{R}(-\Omega(t-s))E(s)^{\mathrm{T}}\,ds, \\ &c_{\mathrm{total},\perp}(t) = \int_0^t b_{\mathrm{total},\perp}(s)\,ds, \\ &a_{\mathrm{total},\perp}(t) = \int_0^t \left\{\frac{1}{2}b_{\mathrm{total},\perp}(s)^2 + \frac{\Omega}{B}b_{\mathrm{total},\perp}(s)\cdot A(c_{\mathrm{total},\perp}(s))\right\}ds, \\ &\Omega = \frac{QB}{M} = \omega_1 = \cdots = \omega_N. \end{aligned} \tag{2.4.9}$$

In this case, the unique existence of $\tilde{U}_\perp(t,s)$ can be guaranteed by the self-adjointness of $\tilde{H}_\perp(0)$, in virtue of (2.4.8). (2.4.8) can be also obtained directly as in [4]. We will give an outline of the proof. Put

$$\hat{U}_\perp(t) := e^{-iMa_{\text{total},\perp}(t)} e^{iMb_{\text{total},\perp}(t)\cdot x_{\text{cm},\perp}} e^{-ic_{\text{total},\perp}(t)\cdot k_{\text{total},\perp}} e^{-it\tilde{H}_\perp(0)}. \tag{2.4.10}$$

By differentiating (2.4.10) in t formally, one can obtain

$$\begin{aligned} &i\dot{\hat{U}}_\perp(t) \\ &= e^{-iMa_{\text{total},\perp}(t)} e^{iMb_{\text{total},\perp}(t)\cdot x_{\text{cm},\perp}} e^{-ic_{\text{total},\perp}(t)\cdot k_{\text{total},\perp}} \tilde{H}_\perp(0) e^{-it\tilde{H}_\perp(0)} \\ &\quad + e^{-iMa_{\text{total},\perp}(t)} e^{iMb_{\text{total},\perp}(t)\cdot x_{\text{cm},\perp}} e^{-ic_{\text{total},\perp}(t)\cdot QA(x_{\text{cc},\perp})} \\ &\qquad\qquad \times (\dot{c}_{\text{total},\perp}(t)\cdot p_{\text{total},\perp}) e^{-ic_{\text{total},\perp}(t)\cdot p_{\text{total},\perp}} e^{-it\tilde{H}_\perp(0)} \\ &\quad + (M\dot{a}_{\text{total},\perp}(t) - M\dot{b}_{\text{total},\perp}(t)\cdot x_{\text{cm},\perp} + \dot{c}_{\text{total},\perp}(t)\cdot QA(x_{\text{cc},\perp}))\hat{U}_\perp(t). \end{aligned}$$

Here we used

$$e^{-ic_{\text{total},\perp}(t)\cdot k_{\text{total},\perp}} = e^{-ic_{\text{total},\perp}(t)\cdot QA(x_{\text{cc},\perp})} e^{-ic_{\text{total},\perp}(t)\cdot p_{\text{total},\perp}}.$$

Since $\tilde{H}_\perp(0)$ commutes with $e^{-ic_{\text{total},\perp}(t)\cdot k_{\text{total},\perp}}$,

$$e^{-ic_{\text{total},\perp}(t)\cdot QA(x_{\text{cc},\perp})} p_{\text{total},\perp} e^{ic_{\text{total},\perp}(t)\cdot QA(x_{\text{cc},\perp})} = p_{\text{total},\perp} - QA(c_{\text{total},\perp}(t)),$$

writing that

$$\begin{aligned} &e^{iMb_{\text{total},\perp}(t)\cdot x_{\text{cm},\perp}} \tilde{H}_{0,\perp}(0) \\ &= \sum_{j=1}^N \frac{1}{2m_j}(p_{j,\perp} - m_j b_{\text{total},\perp}(t) - q_j A(x_{j,\perp}))^2 e^{iMb_{\text{total},\perp}(t)\cdot x_{\text{cm},\perp}} \end{aligned}$$

and

$$\begin{aligned} &e^{iMb_{\text{total},\perp}(t)\cdot x_{\text{cm},\perp}} (\dot{c}_{\text{total},\perp}(t)\cdot(p_{\text{total},\perp} - QA(c_{\text{total},\perp}(t)))) \\ &= \dot{c}_{\text{total},\perp}(t)\cdot(p_{\text{total},\perp} - Mb_{\text{total},\perp}(t) - QA(c_{\text{total},\perp}(t))) e^{iMb_{\text{total},\perp}(t)\cdot x_{\text{cm},\perp}}, \end{aligned}$$

we have

$$i\dot{\hat{U}}_\perp(t) = \hat{H}_\perp(t)\hat{U}_\perp(t)$$

with

$$\begin{aligned} \hat{H}_\perp(t) = &\sum_{j=1}^N \frac{1}{2m_j}(p_{j,\perp} - m_j b_{\text{total},\perp}(t) - q_j A(x_{j,\perp}))^2 + V \\ &\quad + \dot{c}_{\text{total},\perp}(t)\cdot(p_{\text{total},\perp} - Mb_{\text{total},\perp}(t) - QA(c_{\text{total},\perp}(t))) \end{aligned}$$

$$+ M\dot{a}_{\text{total},\perp}(t) - M\dot{b}_{\text{total},\perp}(t)\cdot x_{\text{cm},\perp} + \dot{c}_{\text{total},\perp}(t)\cdot QA(x_{\text{cc},\perp}).$$

Using

$$D_{\text{total},\perp} = p_{\text{total},\perp} - QA(x_{\text{cc},\perp}),$$

we can expand the first term as

$$\begin{aligned}
&\sum_{j=1}^{N} \frac{1}{2m_j}(D_{j,\perp} - m_j b_{\text{total},\perp}(t))^2 \\
&= \tilde{H}_{0,\perp}(0) - b_{\text{total},\perp}(t)\cdot(p_{\text{total},\perp} - QA(x_{\text{cc},\perp})) + \frac{M}{2}b_{\text{total},\perp}(t)^2,
\end{aligned}$$

and so we have

$$\begin{aligned}
\hat{H}_\perp(t) = \tilde{H}_\perp(0) &+ (-b_{\text{total},\perp}(t) + \dot{c}_{\text{total},\perp}(t))\cdot(p_{\text{total},\perp} - QA(x_{\text{cc},\perp})) \\
&+ 2\dot{c}_{\text{total},\perp}(t)\cdot QA(x_{\text{cc},\perp}) - M\dot{b}_{\text{total},\perp}(t)\cdot x_{\text{cm},\perp} \\
&+ M\dot{a}_{\text{total},\perp}(t) - \dot{c}_{\text{total},\perp}(t)\cdot(Mb_{\text{total},\perp}(t) + QA(c_{\text{total},\perp}(t))) \\
&+ \frac{M}{2}b_{\text{total},\perp}(t)^2.
\end{aligned}$$

If we take $c_{\text{total},\perp}(t)$ as

$$\dot{c}_{\text{total},\perp}(t) = b_{\text{total},\perp}(t), \quad c_{\text{total},\perp}(0) = 0,$$

then we have

$$\begin{aligned}
\hat{H}_\perp(t) = \tilde{H}_\perp(0) &- 2QA(b_{\text{total},\perp}(t))\cdot x_{\text{cc},\perp} - M\dot{b}_{\text{total},\perp}(t)\cdot x_{\text{cm},\perp} \\
&+ M\dot{a}_{\text{total},\perp}(t) - \frac{M}{2}b_{\text{total},\perp}(t)^2 - Qb_{\text{total},\perp}(t)\cdot A(c_{\text{total},\perp}(t)).
\end{aligned}$$

Here we used $b_{\text{total},\perp}(t)\cdot A(x_{\text{cc},\perp}) = -A(b_{\text{total},\perp}(t))\cdot x_{\text{cc},\perp}$. Moreover, if we take $a_{\text{total},\perp}(t)$ as

$$\dot{a}_{\text{total},\perp}(t) = \frac{1}{2}b_{\text{total},\perp}(t)^2 + \frac{Q}{M}b_{\text{total},\perp}(t)\cdot A(c_{\text{total},\perp}(t)), \quad a_{\text{total},\perp}(0) = 0,$$

then we have

$$\hat{H}_\perp(t) = \tilde{H}_\perp(0) - (2QA(b_{\text{total},\perp}(t))\cdot x_{\text{cc},\perp} + M\dot{b}_{\text{total},\perp}(t)\cdot x_{\text{cm},\perp}).$$

If $N = 1$, then $q_1/Q = m_1/M = 1$, so $x_{\text{cc},\perp} = x_{\text{cm},\perp}$ holds automatically (cf. [4]). On the other hand, if $N \geq 2$, then $x_{\text{cc},\perp} \neq x_{\text{cm},\perp}$ in general, except in the case where all the specific charges are the same. Since $x_{\text{cc},\perp} = x_{\text{cm},\perp}$ by assumption, if we take $b_{\text{total},\perp}(t)$ as

$$2QA(b_{\text{total},\perp}(t)) + M\dot{b}_{\text{total},\perp}(t) = QE(t), \quad b_{\text{total},\perp}(0) = 0,$$

then

$$\hat{H}_\perp(t) = \sum_{j=1}^{N} \frac{1}{2m_j} D_{j,\perp}^2 + V - E(t) \cdot \sum_{j=1}^{N} q_j x_{j,\perp} = \tilde{H}_\perp(E(t))$$

holds. This yields $\hat{U}_\perp(t) = \tilde{U}_\perp(t,0)$. $b_{\mathrm{total},\perp}(t)$, $c_{\mathrm{total},\perp}(t)$ and $a_{\mathrm{total},\perp}(t)$ are given by (2.4.9) as in [4]; that is, putting

$$\tilde{b}_{\mathrm{total},\perp}(t)^{\mathrm{T}} = \hat{R}(\Omega t) b_{\mathrm{total},\perp}(t)^{\mathrm{T}},$$

we find $b_{\mathrm{total},\perp}(t)$ from

$$\dot{\tilde{b}}_{\mathrm{total},\perp}(t)^{\mathrm{T}} = \frac{\Omega}{B} \hat{R}(\Omega t) E(t)^{\mathrm{T}}.$$

Now, as $E(t)$ under consideration, we take the rotating electric field

$$E_{\nu,\theta}(t) = E_0(\cos(\nu t + \theta), \sin(\nu t + \theta))$$

with $E_0 > 0$, $\nu \in \boldsymbol{R}$ and $\theta \in [0, 2\pi)$. Fix $j \in \{N_{\mathrm{n}} + 1, \ldots, N\}$, that is, $j \in \{1, \ldots, N\}$ such that $q_j \neq 0$. If $\nu = 0$, that is, $E(t) \equiv E_0(\cos\theta, \sin\theta)$, then we have

$$b_{j,\perp}(s)^{\mathrm{T}} = \alpha_\perp^{\mathrm{T}} + \frac{E_0}{B} \hat{R}\left(\omega_j s - \frac{\pi}{2}\right) (\cos\theta, \sin\theta)^{\mathrm{T}},$$

and so

$$\begin{aligned} c_{j,\perp}(t) - t\alpha_\perp &= \int_0^t (b_{j,\perp}(s) - \alpha_\perp) ds \\ &= \left(\frac{E_0}{B\omega_j} (\hat{R}(\omega_j t - \pi) - \hat{R}(-\pi))(\cos\theta, \sin\theta)^{\mathrm{T}} \right)^{\mathrm{T}} \end{aligned}$$

is bounded in t, where

$$\alpha_\perp = \frac{E_0}{B}(\sin\theta, -\cos\theta)$$

is the drift velocity. If $\nu = -\omega_j$, then we have

$$b_{j,\perp}(s) = \frac{\omega_j E_0}{B} s(\cos(-\omega_j s + \theta), \sin(-\omega_j s + \theta)),$$

and so the integration by parts

$$\begin{aligned} c_{j,\perp}(t) &= \int_0^t b_{j,\perp}(s) ds \\ &= -t\alpha_{j,\perp}(t) + \frac{E_0}{B\omega_j}((\cos(-\omega_j t + \theta), \sin(-\omega_j t + \theta)) - (\cos\theta, \sin\theta)) \end{aligned}$$

shows that $c_{j,\perp}(t) - (-t\alpha_{j,\perp}(t))$ is bounded in t, where

$$\alpha_{j,\perp}(t) = \frac{E_0}{B}(\sin(-\omega_j t + \theta), -\cos(-\omega_j t + \theta))$$

is the instantaneous drift velocity. Here we note that

$$|\alpha_\perp| = |\alpha_{j,\perp}(t)| = \frac{E_0}{B}.$$

Hence, in both two cases, the growth order of $|c_{j,\perp}(t)|$ is $O(|t|)$ as $|t| \to \infty$. The case where $\nu = -\omega_j$ is closely related to the so-called cyclotron resonance. On the other hand, if $\nu \neq 0$ and $\nu \neq -\omega_j$, then we have

$$b_{j,\perp}(s) = \frac{\omega_j E_0}{(\nu + \omega_j)B}(\sin(\nu s+\theta) - \sin(-\omega_j s+\theta), -\cos(\nu s+\theta) + \cos(-\omega_j s+\theta)),$$

so

$$\begin{aligned} c_{j,\perp}(t) = \frac{\omega_j E_0}{(\nu + \omega_j)B} \bigg(&-\frac{1}{\nu}((\cos(\nu t + \theta), \sin(\nu t + \theta)) - (\cos\theta, \sin\theta)) \\ &- \frac{1}{\omega_j}((\cos(-\omega_j t + \theta), \sin(-\omega_j t + \theta)) - (\cos\theta, \sin\theta)) \bigg) \end{aligned}$$

is bounded in t. These results are due to [4]. Here we suppose that $N = 2$, that the first particle is charged, and that the specific charge of the second particle is different from that of the first one. Let $\nu = -\omega_1$. Then, by virtue of the above results, we see that the growth order of $|c_{1,\perp}(t) - c_{2,\perp}(t)|$ is $O(|t|)$ as $|t| \to \infty$, which implies the possibility of the existence of scattering states for the system. Roughly speaking, by virtue of the effect of the cyclotron resonance, the separation of these two particles may occur. In fact, Sato [32] showed the existence of (modified) wave operators under some appropriate assumption on V_{12}, because some useful propagation properties of the free propagator $\tilde{U}_{0,\perp}(t, 0)$ can be obtained by using (2.4.2) and the argument of [4]. On the other hand, the problem of the asymptotic completeness for such a system has not been solved yet. To get some useful propagation properties of $\tilde{U}_\perp(t, 0)$ is an issue in the future.

Chapter 3

On the Mourre estimates for Floquet Hamiltonians

3.1 Introduction

In this chapter, we consider the following time-dependent Schrödinger equation

$$i\partial_t u(t) = H(t)u(t), \quad t \in \boldsymbol{R}, \tag{3.1.1}$$

$$H(t) = H_0 + V(t), \quad H_0 = \frac{1}{2}p^2 \quad \text{on } \mathscr{H} := L^2(\boldsymbol{R}^d), \tag{3.1.2}$$

where $p = -i\nabla_x$, and $V(t)$ is the multiplication operator by the real-valued function $V(t,x)$ on $\boldsymbol{R} \times \boldsymbol{R}^d$ which is periodic in t with a period $T > 0$:

$$V(t+T,x) = V(t,x), \quad (t,x) \in \boldsymbol{R} \times \boldsymbol{R}^d. \tag{3.1.3}$$

Under some suitable conditions on V, the existence and uniqueness of the unitary propagator $U(t,s)$ generated by $H(t)$ can be guaranteed (see e.g. Yajima [38]). In the study of the asymptotic behavior of $U(t,s)\phi$, $\phi \in \mathscr{H}$, as $t \to \pm\infty$, we will frequently utilize the so-called Floquet Hamiltonian K associated with $H(t)$: Let $\boldsymbol{T} = \boldsymbol{R}/(T\boldsymbol{Z})$ be the torus. Set $\mathscr{K} := L^2(\boldsymbol{T};\mathscr{H}) \cong L^2(\boldsymbol{T}) \otimes L^2(\boldsymbol{R}^d)$, and introduce a strongly continuous one-parameter unitary group $\{\hat{U}(\sigma)\}_{\sigma\in\boldsymbol{R}}$ on $\mathscr{K}$ given by

$$(\hat{U}(\sigma)\Phi)(t) = U(t,t-\sigma)\Phi(t-\sigma) \tag{3.1.4}$$

for $\Phi \in \mathscr{K}$. By virtue of Stone's theorem, $\hat{U}(\sigma)$ is written as

$$\hat{U}(\sigma) = e^{-i\sigma K} \tag{3.1.5}$$

with a unique self-adjoint operator K on $\mathscr{K}$. K is called the Floquet Hamiltonian associated with $H(t)$, and is equal to the natural self-adjoint realization of $-i\partial_t +$

$H(t)$. Here we denote by D_t the operator $-i\partial_t$ with domain $AC(\boldsymbol{T})$, which is the space of absolutely continuous functions on $\boldsymbol{T}$ with their derivatives being square integrable (following the notation in Reed-Simon [30]). As is well-known, D_t is self-adjoint on $L^2(\boldsymbol{T})$, and its spectrum $\sigma(D_t)$ is equal to $\mathscr{T} := \omega\boldsymbol{Z}$ with $\omega := 2\pi/T$. Hence the real part of the resolvent set of D_t, $\rho(D_t) \cap \boldsymbol{R}$, is equal to $\boldsymbol{R} \setminus \mathscr{T}$, which can be decomposed as

$$\boldsymbol{R} \setminus \mathscr{T} = \bigcup_{n \in \boldsymbol{Z}} I_n, \quad I_n := (n\omega, (n+1)\omega).$$

In [39], Yokoyama introduced the self-adjoint operator

$$\tilde{A}_1 = \frac{1}{2}\{x \cdot p(1+p^2)^{-1} + (1+p^2)^{-1} p \cdot x\} \tag{3.1.6}$$

on $\mathscr{K}$ as a conjugate operator for K. Roughly speaking, $\tilde{A}_1$ is defined by multiplying the generator of dilations

$$\hat{A}_0 = \frac{1}{2}(x \cdot p + p \cdot x) \tag{3.1.7}$$

and the resolvent $(1+p^2)^{-1} = \langle p \rangle^{-2}$ of p^2. He established the following Mourre estimate under some suitable conditions on V: Let $\lambda_0 \in \boldsymbol{R} \setminus \mathscr{T}$ and $0 < \delta < \mathrm{dist}(\lambda_0, \mathscr{T})$. Put $d_1(\lambda) := \mathrm{dist}(\lambda, \mathscr{T} \cap (-\infty, \lambda])$. Then, for any real-valued $f_\delta \in C_0^\infty(\boldsymbol{R})$ supported in $[-\delta, \delta]$, the Mourre estimate

$$f_\delta(K-\lambda_0) i[K, \tilde{A}_1] f_\delta(K-\lambda_0) \geq \frac{2(d_1(\lambda_0)-\delta)}{1+2(d_1(\lambda_0)-\delta)} f_\delta(K-\lambda_0)^2 + C_{1,\lambda_0,f_\delta} \tag{3.1.8}$$

holds with some compact operator C_{1,λ_0,f_δ} on $\mathscr{K}$. (3.1.8) which we have given above is slightly better than the estimate obtained in [39]

$$f_\delta(K-\lambda_0) i[K, \tilde{A}_1] f_\delta(K-\lambda_0) \geq \frac{2(\mathrm{dist}(\lambda_0, \mathscr{T})-\delta)}{1+2(\mathrm{dist}(\lambda_0, \mathscr{T})-\delta)} f_\delta(K-\lambda_0)^2 + C'_{1,\lambda_0,f_\delta}$$

with some compact operator $C'_{1,\lambda_0,f_\delta}$ on $\mathscr{K}$, since $\mathrm{dist}(\lambda_0, \mathscr{T})$ is less than or equal to $d_1(\lambda_0)$. Then the standard Mourre theory (see e.g. Cycon-Froese-Kirsch-Simon [9], Amrein-Boutet de Monvel-Georgescu [3] and so on) yields the following spectral properties of K which are important in the scattering theory: The eigenvalues of K in $\boldsymbol{R} \setminus \mathscr{T}$ are of finite multiplicity, and can accumulate only at $\mathscr{T}$. $\mathscr{T} \cup \sigma_{\mathrm{pp}}(K)$ is a countable closed set. Moreover, the limiting absorption principle for K holds: Let $s > 1/2$, and I be a compact interval in $\boldsymbol{R} \setminus (\mathscr{T} \cup \sigma_{\mathrm{pp}}(K))$. Then, for instance, one has

$$\sup_{\substack{\mathrm{Re}\, z \in I \\ \mathrm{Im}\, z \neq 0}} \|\langle \tilde{A}_1 \rangle^{-s} (K-z)^{-1} \langle \tilde{A}_1 \rangle^{-s}\|_{\boldsymbol{B}(\mathscr{K})} < \infty. \tag{3.1.9}$$

Here $\langle x \rangle = \sqrt{1+|x|^2}$.

In this chapter, we will propose an alternative conjugate operator for K at a non-threshold energy λ_0: Let $\lambda_0 \in \boldsymbol{R} \setminus \mathscr{T}$. Then there exists a unique $n_{\lambda_0} \in \boldsymbol{Z}$ such that $\lambda_0 \in I_{n_{\lambda_0}}$. Take δ as $0 < \delta < \operatorname{dist}(\lambda_0, \mathscr{T})$. Since $\lambda_0 - \delta \in I_{n_{\lambda_0}}$, it is obvious that $\lambda_0 - \delta \in \boldsymbol{R} \setminus \mathscr{T} = \rho(D_t) \cap \boldsymbol{R}$. Then, for the sake of obtaining the Mourre estimate for K at λ_0, we introduce the self-adjoint operator

$$A_{\lambda_0,\delta} = (\lambda_0 - \delta - D_t)^{-1} \otimes \hat{A}_0 \tag{3.1.10}$$

on $\mathscr{K} \cong L^2(\boldsymbol{T}) \otimes L^2(\boldsymbol{R}^d)$, by multiplying $\hat{A}_0$ and the resolvent $(\lambda_0 - \delta - D_t)^{-1}$ of D_t instead of $\langle p \rangle^{-2}$. Here we note that $(\lambda_0 - \delta - D_t)^{-1}$ is bounded and self-adjoint. One of the basic properties of $A_{\lambda_0,\delta}$ is that

$$\begin{aligned}
i[K_0, A_{\lambda_0,\delta}] &= i[D_t \otimes \mathrm{Id}, (\lambda_0 - \delta - D_t)^{-1} \otimes \hat{A}_0] \\
&\qquad + i[\mathrm{Id} \otimes H_0, (\lambda_0 - \delta - D_t)^{-1} \otimes \hat{A}_0] \\
&= i[D_t, (\lambda_0 - \delta - D_t)^{-1}] \otimes \hat{A}_0 \\
&\qquad + (\lambda_0 - \delta - D_t)^{-1} \otimes i[H_0, \hat{A}_0] \\
&= (\lambda_0 - \delta - D_t)^{-1} \otimes 2H_0 \\
&= (\lambda_0 - \delta - D_t)^{-1}\{2(K_0 - D_t)\}, \\
i[i[K_0, A_{\lambda_0,\delta}], A_{\lambda_0,\delta}] &= 2i[(\lambda_0 - \delta - D_t)^{-1} \otimes H_0, (\lambda_0 - \delta - D_t)^{-1} \otimes \hat{A}_0] \\
&= (\lambda_0 - \delta - D_t)^{-2} \otimes 2i[H_0, \hat{A}_0] \\
&= (\lambda_0 - \delta - D_t)^{-2} \otimes 4H_0 \\
&= (\lambda_0 - \delta - D_t)^{-2}\{4(K_0 - D_t)\}
\end{aligned}$$

hold, where $K_0 = D_t + H_0$ is the free Floquet Hamiltonian and $i[H_0, \hat{A}_0] = 2H_0$. This yields the fact that

$$\begin{aligned}
i[K_0, A_{\lambda_0,\delta}]\langle K_0 \rangle^{-1} &= 2\{(\lambda_0 - \delta - D_t)^{-1} K_0 \langle K_0 \rangle^{-1} \\
&\qquad - (\lambda_0 - \delta - D_t)^{-1} D_t \langle K_0 \rangle^{-1}\} \\
i[i[K_0, A_{\lambda_0,\delta}], A_{\lambda_0,\delta}]\langle K_0 \rangle^{-1} &= 4\{(\lambda_0 - \delta - D_t)^{-2} K_0 \langle K_0 \rangle^{-1} \\
&\qquad - (\lambda_0 - \delta - D_t)^{-2} D_t \langle K_0 \rangle^{-1}\}
\end{aligned}$$

are bounded.

Next we impose the following condition (V) on V under consideration:

(V) $V(t,x)$ is a real-valued function on $\boldsymbol{R} \times \boldsymbol{R}^d$, is T-periodic in t, and is decomposed into the sum of $V^{\mathrm{sing}}(t,x)$ and $V^{\mathrm{reg}}(t,x)$, which are also T-periodic in t. If $d < 3$, then $V^{\mathrm{sing}} = 0$. If $d \geq 3$, then $V^{\mathrm{sing}}(t,\cdot)$ belongs to $C(\boldsymbol{R}, L^{q_0}(\boldsymbol{R}^d))$ with some $\infty > q_0 > d$, and $\operatorname{supp} V^{\mathrm{sing}}(t,\cdot)$'s are included in a common compact subset of $\boldsymbol{R}^d$. $(\partial_t V^{\mathrm{sing}})(t,\cdot)$ and $|(\nabla V^{\mathrm{sing}})(t,\cdot)|$ belong to $C(\boldsymbol{R}, L^{q_1}(\boldsymbol{R}^d))$ with some $\infty > q_1 > d/2$, where if $d=3$, then we define q_1 by $1/q_1 = 1/(2q_0)+1/2$ $(< 2/d)$. On the other hand, $V^{\mathrm{reg}}(t,x)$ belongs to $C^2(\boldsymbol{R}\times\boldsymbol{R}^d)$, and satisfies the decaying conditions

$$\sup_{t\in\boldsymbol{R}} |(\partial_t^k \partial_x^\alpha V^{\mathrm{reg}})(t,x)| \le C\langle x\rangle^{-\rho-(k+|\alpha|)}, \quad k+|\alpha|\le 2 \tag{3.1.11}$$

with some $\rho > 0$.

Under the condition (V), the existence and uniqueness of the unitary propagator $U(t,s)$ generated by $H(t)$ can be guaranteed by the results of Yajima [38]. Actually, for any compact interval I, $V = V^{\mathrm{sing}} + V^{\mathrm{reg}} \in C(I, L^{q_0}(\boldsymbol{R}^d)) + C(I, L^\infty(\boldsymbol{R}^d)) \subset L^2(I, L^{q_0}(\boldsymbol{R}^d)) + L^\beta(I, L^\infty(\boldsymbol{R}^d))$ holds, with $1/2 < 1 - d/(2q_0)$ and $\beta > 1$ being any number, so Yajima's Assumption (A.1) is satisfied; moreover, for any $1\le q_2\le q_1$ and $\alpha_1 \ge 1$, $\partial_t V^{\mathrm{sing}} \in C(I, L^{q_2}(\boldsymbol{R}^d)) \subset L^{\alpha_1}(I, L^{q_2}(\boldsymbol{R}^d))$ holds; to see this, since $\operatorname{supp} V^{\mathrm{sing}}(t,\cdot)$'s are included in some common compact set C, taking some $\psi_C \in C_0^\infty(\boldsymbol{R}^d)$ such that $\psi_C(x) = 1$ on C, we see that

$$\begin{aligned}\int_{\boldsymbol{R}}\int_{\boldsymbol{R}^d} \partial_t V^{\mathrm{sing}}(t,x)\varphi(t,x)dxdt &= -\int_{\boldsymbol{R}}\int_{\boldsymbol{R}^d} V^{\mathrm{sing}}(t,x)\psi_C(x)\partial_t\varphi(t,x)dxdt \\ &= -\int_{\boldsymbol{R}}\int_{\boldsymbol{R}^d} V^{\mathrm{sing}}(t,x)\partial_t(\psi_C(x)\varphi(t,x))dxdt \\ &= \int_{\boldsymbol{R}}\int_{\boldsymbol{R}^d} \partial_t V^{\mathrm{sing}}(t,x)\psi_C(x)\varphi(t,x)dxdt\end{aligned}$$

for all $\varphi \in C_0^\infty(\boldsymbol{R}\times\boldsymbol{R}^d)$ and

$$\partial_t V^{\mathrm{sing}}(t,x) = \partial_t V^{\mathrm{sing}}(t,x)\psi_C(x)$$

for a.e. (t,x), but this holds for all t and for a.e. x because both members are continuous in t, so by Hölder's inequality,

$$\|\partial_t V^{\mathrm{sing}}(t,\cdot) - \partial_t V^{\mathrm{sing}}(t_0,\cdot)\|_{q_2} \le \|\partial_t V^{\mathrm{sing}}(t,\cdot) - \partial_t V^{\mathrm{sing}}(t_0,\cdot)\|_{q_1}\|\psi_C\|_{q_3}$$

for all $t, t_0 \in \boldsymbol{R}$ with $1/q_2 = 1/q_1 + 1/q_3$; therefore we can take q_2 such that $1/q_2 = 1/(2q_0)+2/d > 1/q_1$ if $d\ge 5$ ($1/(2q_0)+1/2 > 1/q_1$ holds automatically if $d=4$) and Yajima's Assumption (A.2) is also satisfied. It can be also guaranteed that

$$\langle K_0\rangle^{-1/2} i[V, A_{\lambda_0,\delta}]\langle K_0\rangle^{-1}, \quad \langle K_0\rangle^{-1} i[i[V, A_{\lambda_0,\delta}], A_{\lambda_0,\delta}]\langle K_0\rangle^{-1}$$

are bounded. To see this, we first note that

$$i[V, A_{\lambda_0,\delta}] = (\lambda_0-\delta-D_t)^{-1} i[V,\hat{A}_0] + i[V,(\lambda_0-\delta-D_t)^{-1}]\hat{A}_0$$

$$= -(\lambda_0-\delta-D_t)^{-1}((x\cdot\nabla)V)-(\lambda_0-\delta-D_t)^{-1}(\partial_t V)(\lambda_0-\delta-D_t)^{-1}\hat{A}_0,$$

where $i[V,\hat{A}_0]=-((x\cdot\nabla)V)$ and

$$\begin{aligned}i[V,(\lambda_0-\delta-D_t)^{-1}] &= (\lambda_0-\delta-D_t)^{-1}i[\lambda_0-\delta-D_t,V](\lambda_0-\delta-D_t)^{-1}\\ &= -(\lambda_0-\delta-D_t)^{-1}(\partial_t V)(\lambda_0-\delta-D_t)^{-1}.\end{aligned}$$

Then, as for the regular part V^{reg} of V, it follows from

$$\begin{aligned}&i[i[V^{\mathrm{reg}},A_{\lambda_0,\delta}],A_{\lambda_0,\delta}] = -i[(\lambda_0-\delta-D_t)^{-1}((x\cdot\nabla)V^{\mathrm{reg}}),A_{\lambda_0,\delta}]\\ &\quad -i[(\lambda_0-\delta-D_t)^{-1}(\partial_t V^{\mathrm{reg}})(\lambda_0-\delta-D_t)^{-1}\hat{A}_0,A_{\lambda_0,\delta}]\\ &= -(\lambda_0-\delta-D_t)^{-1}i[((x\cdot\nabla)V^{\mathrm{reg}}),A_{\lambda_0,\delta}]\\ &\quad -(\lambda_0-\delta-D_t)^{-1}i[(\partial_t V^{\mathrm{reg}}),A_{\lambda_0,\delta}](\lambda_0-\delta-D_t)^{-1}\hat{A}_0\\ &= (\lambda_0-\delta-D_t)^{-2}((x\cdot\nabla)^2V^{\mathrm{reg}})\\ &\quad +2(\lambda_0-\delta-D_t)^{-2}(\partial_t(x\cdot\nabla)V^{\mathrm{reg}})(\lambda_0-\delta-D_t)^{-1}\hat{A}_0\\ &\quad +(\lambda_0-\delta-D_t)^{-2}(\partial_t^2V^{\mathrm{reg}})(\lambda_0-\delta-D_t)^{-2}\hat{A}_0^2\end{aligned}$$

that

$$\begin{aligned}&i[V^{\mathrm{reg}},A_{\lambda_0,\delta}]\langle K_0\rangle^{-1} = -(\lambda_0-\delta-D_t)^{-1}((x\cdot\nabla)V^{\mathrm{reg}})\langle K_0\rangle^{-1}\\ &-(\lambda_0-\delta-D_t)^{-1}\langle x\rangle(\partial_t V^{\mathrm{reg}})\langle x\rangle^{-1}\hat{A}_0\langle p\rangle^{-1}(\lambda_0-\delta-D_t)^{-1}\langle p\rangle\langle K_0\rangle^{-1}\end{aligned}$$

and

$$\begin{aligned}&i[i[V^{\mathrm{reg}},A_{\lambda_0,\delta}],A_{\lambda_0,\delta}]\langle K_0\rangle^{-1}\\ &= (\lambda_0-\delta-D_t)^{-2}\left((x\cdot\nabla)V^{\mathrm{reg}}+\sum_{j=1}^{d}\sum_{k=1}^{d}x_jx_k(\partial_j\partial_kV^{\mathrm{reg}})\right)\langle K_0\rangle^{-1}\\ &+2(\lambda_0-\delta-D_t)^{-2}\langle x\rangle(\partial_t(x\cdot\nabla)V^{\mathrm{reg}})\langle x\rangle^{-1}\hat{A}_0\langle p\rangle^{-1}(\lambda_0-\delta-D_t)^{-1}\langle p\rangle\langle K_0\rangle^{-1}\\ &+(\lambda_0-\delta-D_t)^{-2}\langle x\rangle^2(\partial_t^2V^{\mathrm{reg}})\langle x\rangle^{-2}\hat{A}_0^2\langle p\rangle^{-2}(\lambda_0-\delta-D_t)^{-2}\langle p\rangle^2\langle K_0\rangle^{-1}\end{aligned}$$

are bounded. Here we used the fact that

$$\langle D_t\rangle^{-1/2}\langle p\rangle\langle K_0\rangle^{-1},\quad \langle D_t\rangle^{-1}\langle p\rangle^2\langle K_0\rangle^{-1}$$

are bounded, which can be shown in the same way as in the case of Stark Hamiltonians (see e.g. Simon [31]). Moreover, we see that $\langle K_0\rangle^{-1}i[V^{\mathrm{reg}},A_{\lambda_0,\delta}]\langle K_0\rangle^{-1}$ is compact, by virtue of the local compactness property of K_0. On the other hand, as for the singular part V^{sing} of V, by using the fact that

$$\langle p\rangle^{-1}((x\cdot\nabla)V^{\mathrm{sing}}(t))\langle p\rangle^{-1},\quad \langle p\rangle^{-1}(\partial_t V^{\mathrm{sing}}(t))\langle p\rangle^{-1}$$

are bounded in $L^2(\boldsymbol{R}^d)$, one can show firstly that $\langle K_0\rangle^{-1/2} i[V^{\mathrm{sing}}, A_{\lambda_0,\delta}]\langle K_0\rangle^{-1}$ is bounded. Moreover, we see that $\langle K_0\rangle^{-1} i[V^{\mathrm{sing}}, A_{\lambda_0,\delta}]\langle K_0\rangle^{-1}$ is compact. And, by identifying $i[i[V^{\mathrm{sing}}, A_{\lambda_0,\delta}], A_{\lambda_0,\delta}]$ with

$$i\{i[V^{\mathrm{sing}}, A_{\lambda_0,\delta}]A_{\lambda_0,\delta} - A_{\lambda_0,\delta} i[V^{\mathrm{sing}}, A_{\lambda_0,\delta}]\},$$

one can show that $\langle K_0\rangle^{-1} i[i[V^{\mathrm{sing}}, A_{\lambda_0,\delta}], A_{\lambda_0,\delta}]\langle K_0\rangle^{-1}$ is also bounded. Here we note that in [39], it was assumed that $V^{\mathrm{reg}} \in C^\infty(\boldsymbol{R}\times\boldsymbol{R}^d)$, because the pseudodifferential calculus was needed. Our conjugate operator $A_{\lambda_0,\delta}$ can relax the smoothness condition on V^{reg} considerably.

Then some of the main results of this paper are as follows:

Theorem 3.1.1. *Assume V satisfies (V). Let $\lambda_0 \in \boldsymbol{R}\setminus\mathscr{T}$. Take δ as $0<\delta<\mathrm{dist}(\lambda_0,\mathscr{T})$. Define $A_{\lambda_0,\delta}$ by* (3.1.10). *Then:*
(1) *For any real-valued $f_\delta \in C_0^\infty(\boldsymbol{R})$ supported in $[-\delta,\delta]$,*

$$f_\delta(K-\lambda_0) i[K, A_{\lambda_0,\delta}] f_\delta(K-\lambda_0) \geq 2 f_\delta(K-\lambda_0)^2 + C_{\lambda_0,f_\delta} \tag{3.1.12}$$

holds with some compact operator C_{λ_0,f_δ} on $\mathscr{K}$. It follows from this that $\sigma_{\mathrm{pp}}(K)\cap[\lambda_0-\delta/2,\lambda_0+\delta/2]$ is finite, and the eigenvalues of K in $[\lambda_0-\delta/2,\lambda_0+\delta/2]$ are of finite multiplicity.
(2) *In addition, assume $\lambda_0 \notin \sigma_{\mathrm{pp}}(K)$. Let $0<\varepsilon<2$. Take $\delta>0$ so small that $[\lambda_0-2\delta,\lambda_0+2\delta]\subset\boldsymbol{R}\setminus(\mathscr{T}\cup\sigma_{\mathrm{pp}}(K))$ and*

$$f_{2\delta}(K-\lambda_0) i[K, A_{\lambda_0,2\delta}] f_{2\delta}(K-\lambda_0) \geq (2-\varepsilon) f_{2\delta}(K-\lambda_0)^2 \tag{3.1.13}$$

holds. Suppose $s>1/2$. Then

$$\sup_{\substack{\mathrm{Re}\, z\in[\lambda_0-\delta,\lambda_0+\delta]\\ \mathrm{Im}\, z\neq 0}} \|\langle A_{\lambda_0,2\delta}\rangle^{-s}(K-z)^{-1}\langle A_{\lambda_0,2\delta}\rangle^{-s}\|_{\boldsymbol{B}(\mathscr{K})} < \infty \tag{3.1.14}$$

holds. Moreover, $\langle A_{\lambda_0,2\delta}\rangle^{-s}(K-z)^{-1}\langle A_{\lambda_0,2\delta}\rangle^{-s}$ is a $\boldsymbol{B}(\mathscr{K})$-valued $\theta(s)$-Hölder continuous function on $z\in S_{\lambda_0,\delta,\pm}$, where

$$\theta(s) = \frac{\min\{s-1/2,\rho\}}{\min\{s-1/2,\rho\}+1},$$
$$S_{\lambda_0,\delta,\pm} = \bigl\{\zeta\in\boldsymbol{C} \bigm| \mathrm{Re}\,\zeta\in[\lambda_0-\delta,\lambda_0+\delta],\ 0<\pm\mathrm{Im}\,\zeta\leq 1\bigr\}.$$

And, there exist the norm limits

$$\langle A_{\lambda_0,2\delta}\rangle^{-s}(K-(\lambda\pm i0))^{-1}\langle A_{\lambda_0,2\delta}\rangle^{-s} = \lim_{\epsilon\to+0}\langle A_{\lambda_0,2\delta}\rangle^{-s}(K-(\lambda\pm i\epsilon))^{-1}\langle A_{\lambda_0,2\delta}\rangle^{-s}$$

in $\boldsymbol{B}(\mathscr{K})$ for any $\lambda\in[\lambda_0-\delta,\lambda_0+\delta]$. $\langle A_{\lambda_0,2\delta}\rangle^{-s}(K-(\lambda\pm i0))^{-1}\langle A_{\lambda_0,2\delta}\rangle^{-s}$ are also $\theta(s)$-Hölder continuous in λ.

Corollary 3.1.2. *Assume V satisfies (V). Then:*
(1) *The eigenvalues of K in $\boldsymbol{R} \setminus \mathscr{T}$ can accumulate only at $\mathscr{T}$. Moreover, $\mathscr{T} \cup \sigma_{\mathrm{pp}}(K)$ is a countable closed set.*
(2) *Let I be a compact interval in $\boldsymbol{R} \setminus (\mathscr{T} \cup \sigma_{\mathrm{pp}}(K))$. Suppose $1/2 < s \leq 1$. Then*

$$\sup_{\substack{\operatorname{Re} z \in I \\ \operatorname{Im} z \neq 0}} \|\langle x \rangle^{-s} (K - z)^{-1} \langle x \rangle^{-s}\|_{\boldsymbol{B}(\mathscr{K})} < \infty \tag{3.1.15}$$

holds. Moreover, $\langle x \rangle^{-s}(K-z)^{-1}\langle x \rangle^{-s}$ is a $\boldsymbol{B}(\mathscr{K})$-valued $\theta(s)$-Hölder continuous function on $z \in S_{I,\pm}$, where

$$S_{I,\pm} = \{\zeta \in \boldsymbol{C} \mid \operatorname{Re} \zeta \in I,\ 0 < \pm \operatorname{Im} \zeta \leq 1\}.$$

And, there exist the norm limits

$$\langle x \rangle^{-s}(K - (\lambda \pm i0))^{-1}\langle x \rangle^{-s} = \lim_{\epsilon \to +0} \langle x \rangle^{-s}(K - (\lambda \pm i\epsilon))^{-1}\langle x \rangle^{-s}$$

in $\boldsymbol{B}(\mathscr{K})$ for $\lambda \in I$. $\langle x \rangle^{-s}(K - (\lambda \pm i0))^{-1}\langle x \rangle^{-s}$ are also $\theta(s)$-Hölder continuous in λ.

In order to obtain Corollary 3.1.2, we use the argument due to Perry-Sigal-Simon [29], and the boundedness of

$$A_{\lambda_0, 2\delta}(K - \lambda_0 - i)^{-1}\langle x \rangle^{-1},$$

which follows from that $\langle D_t \rangle^{-1}(K - \lambda_0 - i)^{-1}\langle p \rangle^2$ is bounded. By virtue of this, one can show that

$$A_{\lambda_0, 2\delta}(K - \lambda_0 - i)^{-1}\langle p \rangle \langle x \rangle^{-1}, \quad A_{\lambda_0, 2\delta}(K - \lambda_0 - i)^{-1}\langle D_t \rangle^{1/2}\langle x \rangle^{-1}$$

are also bounded. Then one may expect that the limiting absorption principle

$$\sup_{\substack{\operatorname{Re} z \in I \\ \operatorname{Im} z \neq 0}} \|\langle x \rangle^{-s} \mathcal{D}^s (K - z)^{-1} \mathcal{D}^s \langle x \rangle^{-s}\|_{\boldsymbol{B}(\mathscr{K})} < \infty$$

will also hold, where the unbounded 'weight' $\mathcal{D} = \langle p \rangle + \langle D_t \rangle^{1/2}$ is equivalent to the 'weight' $\mathscr{D}^{1/2} = (\langle p \rangle^4 + \langle D_t \rangle^2)^{1/4}$, which was introduced in Kuwabara-Yajima [22] for the sake of obtaining a refined limiting absorption principle for K. But we have not proved this yet, unfortunately. It is caused by the unboundedness of

$$(K - \lambda_0 - i)^{-1}\langle p \rangle \langle x \rangle^{-1}, \quad (K - \lambda_0 - i)^{-1}\langle D_t \rangle^{1/2}\langle x \rangle^{-1}.$$

Instead of the above limiting absorption principle, one can obtain

$$\sup_{\substack{\operatorname{Re} z \in I \\ \operatorname{Im} z \neq 0}} \|\langle D_t \rangle^{-s/2}\langle x \rangle^{-s}\langle p \rangle^s (K - z)^{-1}\langle p \rangle^s \langle x \rangle^{-s}\langle D_t \rangle^{-s/2}\|_{\boldsymbol{B}(\mathscr{K})} < \infty \tag{3.1.16}$$

from (3.1.14) immediately. As for the N-body Floquet Hamiltonians, a refined limiting absorption principle for K

$$\sup_{\substack{\mathrm{Re}\, z \in I \\ \mathrm{Im}\, z \neq 0}} \| \langle x \rangle^{-s} \langle p \rangle^{r} (K - z)^{-1} \langle p \rangle^{r} \langle x \rangle^{-s} \|_{\boldsymbol{B}(\mathscr{K})} < \infty$$

with $0 \leq r < 1/2 < s \leq 1$ was obtained in Møller-Skibsted [27]. They used an extended Mourre theory due to Skibsted [33], and took a 'conjugate operator' for K in the theory as $\hat{A}_0$. However, we would like to stick to find a candidate of a conjugate operator for K not in an extended but in the standard Mourre theory, because it seems much easier to obtain some useful propagation estimates for K by applying the standard one.

The plan of this chapter is as follows: In §3.2, we will give the proof of Theorem 3.1.1, in particular, (3.1.12). In §3.3, as an application of our results, we will deal with the problem of the asymptotic completeness for the so-called AC Stark Hamiltonians in the short-range case, although the result was already obtained in [37] and [39]. In §3.4, we will make some remarks on the extension to the many body case.

3.2 Proof of Theorem 3.1.1

In this section, we prove Theorem 3.1.1. Here we will give the proof of the Mourre estimate (3.1.12) only, because the other results can be shown directly by the standard Mourre theory.

As is well-known, $AC(\boldsymbol{T}) \otimes C_0^\infty(\boldsymbol{R}^d)$ is a core for K_0, and $D_t \otimes \mathrm{Id} + \mathrm{Id} \otimes H_0$ defined on $AC(\boldsymbol{T}) \otimes C_0^\infty(\boldsymbol{R}^d)$ is essentially self-adjoint and its closure is equal to K_0. If V satisfies the condition (V), then K is self-adjoint with the domain $\mathscr{D}(K_0)$, and $D_t \otimes \mathrm{Id} + \mathrm{Id} \otimes H(t)$ defined on $AC(\boldsymbol{T}) \otimes C_0^\infty(\boldsymbol{R}^d)$ is essentially self-adjoint and its closure is equal to K.

Now we will show

$$\sup_{|\sigma| \leq 1} \| K_0 e^{i\sigma A_{\lambda_0,\delta}} (K_0 + i)^{-1} \|_{\mathscr{B}(\mathscr{K})} < \infty \tag{3.2.1}$$

with $\lambda_0 \in \boldsymbol{R} \setminus \mathscr{T}$ and $0 < \delta < \mathrm{dist}(\lambda_0, \mathscr{T})$. First of all, we note that the direct integral decomposition of $(K_0 + i)^{-1}$ can be given by

$$(K_0 + i)^{-1} = \bigoplus_{k \in \boldsymbol{Z}} (k\omega + H_0 + i)^{-1}, \tag{3.2.2}$$

and that $e^{i\sigma A_{\lambda_0,\delta}} (K_0 + i)^{-1} e^{-i\sigma A_{\lambda_0,\delta}}$ with $|\sigma| \leq 1$ can be represented as

$$e^{i\sigma A_{\lambda_0,\delta}} (K_0 + i)^{-1} e^{-i\sigma A_{\lambda_0,\delta}} = \bigoplus_{k \in \boldsymbol{Z}} (k\omega + e^{-2\sigma/(\lambda_0 - \delta - k\omega)} H_0 + i)^{-1}. \tag{3.2.3}$$

For the sake of estimating $\|(k\omega + H_0)(k\omega + e^{-2\sigma/(\lambda_0-\delta-k\omega)}H_0 + i)^{-1}\|_{\mathscr{B}(\mathscr{H})}$, we will introduce the function

$$\eta_\sigma(\kappa, \tau) = \frac{(\tau+\kappa)^2}{(\tau + e^{-2\sigma/(\lambda_0-\delta-\tau)}\kappa)^2 + 1}$$

on $[0, \infty) \times \boldsymbol{R}$. Here we note

$$(\partial_\kappa \eta_\sigma)(\kappa, \tau) = \frac{2(\tau+\kappa)\{(\tau + e^{-2\sigma/(\lambda_0-\delta-\tau)}\kappa)(1 - e^{-2\sigma/(\lambda_0-\delta-\tau)})\tau + 1\}}{\{(\tau + e^{-2\sigma/(\lambda_0-\delta-\tau)}\kappa)^2 + 1\}^2},$$

and, since for any $k \in \boldsymbol{Z}$, $0 < \mathrm{dist}(\lambda_0, \mathscr{T}) - \delta \le |\lambda_0 - k\omega| - \delta$ holds,

$$\frac{1}{\lambda_0 - k\omega - \delta} \le \frac{1}{|\lambda_0 - k\omega| - \delta} \le \frac{1}{\mathrm{dist}(\lambda_0, \mathscr{T}) - \delta}, \quad k \in \boldsymbol{Z}.$$

Firstly we consider the case where $\tau = k\omega$ with $k \in \boldsymbol{Z} \cap (0, \infty)$. Suppose $1 - e^{-2\sigma/(\lambda_0-\delta-k\omega)} \ge 0$. Since $(\partial_\kappa \eta_\sigma)(\kappa, k\omega) > 0$ on $[0, \infty)$,

$$\eta_\sigma(\kappa, k\omega) \le \lim_{\tilde{\kappa}\to\infty} \eta_\sigma(\tilde{\kappa}, k\omega) = e^{4\sigma/(\lambda_0-\delta-k\omega)} \le e^{4/(\mathrm{dist}(\lambda_0,\mathscr{T})-\delta)}$$

holds. Suppose $1 - e^{-2\sigma/(\lambda_0-\delta-k\omega)} < 0$. $(\partial_\kappa \eta)(\kappa, k\omega)$ has two zero points $-k\omega < 0$ and

$$\kappa_{0,\sigma}(k\omega) = \frac{e^{2\sigma/(\lambda_0-\delta-k\omega)}\{1 - (e^{-2\sigma/(\lambda_0-\delta-k\omega)} - 1)(k\omega)^2\}}{(e^{-2\sigma/(\lambda_0-\delta-k\omega)} - 1)k\omega};$$

if $\kappa_{0,\sigma}(k\omega)$ belongs to $[0, \infty)$, then $\eta_\sigma(\kappa, k\omega)$ takes the maximum at $\kappa_{0,\sigma}(k\omega)$, so

$$\begin{aligned}
\eta_\sigma(\kappa, k\omega) &\le \eta_\sigma(\kappa_{0,\sigma}(k\omega), k\omega) \\
&= \frac{\{e^{2\sigma/(\lambda_0-\delta-k\omega)} + (e^{2\sigma/(\lambda_0-\delta-k\omega)} + e^{-2\sigma/(\lambda_0-\delta-k\omega)} - 2)(k\omega)^2\}^2}{1 + (e^{-2\sigma/(\lambda_0-\delta-k\omega)} - 1)^2(k\omega)^2} \\
&= e^{4\sigma/(\lambda_0-\delta-k\omega)}\{1 + (e^{-2\sigma/(\lambda_0-\delta-k\omega)} - 1)^2(k\omega)^2\} \\
&= e^{4\sigma/(\lambda_0-\delta-k\omega)} + 4e^{2\sigma/(\lambda_0-\delta-k\omega)}\sinh^2(\sigma/(\lambda_0-\delta-k\omega))(k\omega)^2 \\
&\le e^{4/(\mathrm{dist}(\lambda_0,\mathscr{T})-\delta)} + 4e^{2/(\mathrm{dist}(\lambda_0,\mathscr{T})-\delta)}\sinh^2(1/(\lambda_0-\delta-k\omega))(k\omega)^2 \\
&\le e^{4/(\mathrm{dist}(\lambda_0,\mathscr{T})-\delta)} + 4M_{1,\lambda_0,\delta}e^{2/(\mathrm{dist}(\lambda_0,\mathscr{T})-\delta)}
\end{aligned}$$

with

$$M_{1,\lambda_0,\delta} = \sup_{k\in\boldsymbol{Z}}\{\sinh^2(1/(\lambda_0-\delta-k\omega))(k\omega)^2\} < \infty.$$

Here we used

$$\lim_{k\to\pm\infty} \sinh^2(1/(\lambda_0-\delta-k\omega))(k\omega)^2 = 1.$$

On the other hand, if $\kappa_{0,\sigma}(k\omega)$ does not belong to $[0,\infty)$, then $\eta_\sigma(\kappa,k\omega)$ is monotone decreasing, so

$$\eta_\sigma(\kappa,k\omega) \leq \eta_\sigma(0,k\omega) = \frac{(k\omega)^2}{(k\omega)^2+1} \leq 1 = e^0 < e^{4/(\mathrm{dist}(\lambda_0,\mathscr{T})-\delta)}$$

holds. Secondly in the case where $k=0$ and $\tau = k\omega = 0$,

$$\eta_\sigma(\kappa,k\omega) = \frac{\kappa^2}{e^{-4\sigma/(\lambda_0-\delta)}\kappa^2+1} < e^{4\sigma/(\lambda_0-\delta)} = e^{4\sigma/(\lambda_0-\delta-k\omega)} \leq e^{4/(\mathrm{dist}(\lambda_0,\mathscr{T})-\delta)}.$$

Lastly we consider the case where $\tau = k\omega$ with $k \in \boldsymbol{Z}\cap(-\infty,0)$. Suppose $1-e^{-2\sigma/(\lambda_0-\delta-k\omega)} > 0$. Since $(e^{-2\sigma/(\lambda_0-\delta-k\omega)}-1)(k\omega)^2 < 0 < 1$, $\kappa_{0,\sigma}(k\omega) > 0$ holds, so $\eta_\sigma(\kappa,k\omega)$ takes the maximum at 0 or $-k\omega$ or $\kappa_{0,\sigma}(k\omega)$, and

$$\begin{aligned}
\eta_\sigma(\kappa,k\omega) \leq\ & \max\{\eta_\sigma(0,k\omega), \eta_\sigma(-k\omega,k\omega), \eta_\sigma(\kappa_{0,\sigma}(k\omega),k\omega)\} \\
\leq\ & \max\{e^{4/(\mathrm{dist}(\lambda_0,\mathscr{T})-\delta)}, 0, e^{4/(\mathrm{dist}(\lambda_0,\mathscr{T})-\delta)} + 4M_{1,\lambda_0,\delta}e^{2/(\mathrm{dist}(\lambda_0,\mathscr{T})-\delta)}\} \\
=\ & e^{4/(\mathrm{dist}(\lambda_0,\mathscr{T})-\delta)} + 4M_{1,\lambda_0,\delta}e^{2/(\mathrm{dist}(\lambda_0,\mathscr{T})-\delta)} =: M_{2,\lambda_0,\delta}^2
\end{aligned}$$

holds. Suppose $1-e^{-2\sigma/(\lambda_0-\delta-k\omega)} = 0$. The sign of $(\partial_\kappa\eta)(\kappa,k\omega)$ depends on the linear function $\kappa + k\omega$, so $\eta_\sigma(\kappa,k\omega)$ takes the maximum at 0 or ∞, and

$$\eta_\sigma(\kappa,k\omega) \leq \max\{\eta_\sigma(0,k\omega), \lim_{\tilde{\kappa}\to\infty}\eta_\sigma(\tilde{\kappa},k\omega)\} \leq e^{4/(\mathrm{dist}(\lambda_0,\mathscr{T})-\delta)}$$

holds. Suppose $1-e^{-2\sigma/(\lambda_0-\delta-k\omega)} < 0$. If $\kappa_{0,\sigma}(k\omega)$ belongs to $[0,\infty)$, then $\eta_\sigma(\kappa,k\omega)$ takes the maximum at 0 or $-k\omega$ or $\kappa_{0,\sigma}(k\omega)$, and $\eta_\sigma(\kappa,k\omega) \leq M_{2,\lambda_0,\delta}^2$ holds. On the other hand, if $\kappa_{0,\sigma}(k\omega)$ does not belong to $[0,\infty)$, then $\eta_\sigma(\kappa,k\omega)$ takes the maximum at 0 or ∞, and $\eta_\sigma(\kappa,k\omega) \leq e^{4/(\mathrm{dist}(\lambda_0,\mathscr{T})-\delta)}$ holds. Finally we have

$$\eta_\sigma(\kappa,k\omega) \leq M_{2,\lambda_0,\delta}^2, \quad \kappa \in [0,\infty),$$

for any $k \in \boldsymbol{Z}$, which yields

$$\begin{aligned}
& \|(k\omega+H_0)(k\omega+e^{-2\sigma/(\lambda_0-\delta-k\omega)}H_0+i)^{-1}\varphi\|_{\mathscr{H}}^2 \\
=\ & \int_0^\infty |(k\omega+\kappa)(k\omega+e^{-2\sigma/(\lambda_0-\delta-k\omega)}\kappa+i)^{-1}|^2 d\|E_{H_0}(\kappa)\varphi\|_{\mathscr{H}}^2 \\
=\ & \int_0^\infty \eta_\sigma(\kappa,k\omega)d\|E_{H_0}(\kappa)\varphi\|_{\mathscr{H}}^2 \\
\leq\ & M_{2,\lambda_0,\delta}^2\|\varphi\|_{\mathscr{H}}^2, \quad \varphi \in \mathscr{H},
\end{aligned}$$

where E_{H_0} denotes the spectral measure of H_0, and

$$\sup_{k\in\boldsymbol{Z}} \|(k\omega+H_0)(k\omega+e^{-2\sigma/(\lambda_0-\delta-k\omega)}H_0+i)^{-1}\|_{\mathscr{B}(\mathscr{H})} \leq M_{2,\lambda_0,\delta} \qquad (3.2.4)$$

This implies (3.2.1) because of

$$\begin{aligned}&\sup_{|\sigma|\le 1}\|K_0e^{i\sigma A_{\lambda_0,\delta}}(K_0+i)^{-1}\|_{\mathscr{B}(\mathscr{K})}\\ &=\sup_{|\sigma|\le 1}\|K_0e^{i\sigma A_{\lambda_0,\delta}}(K_0+i)^{-1}e^{-i\sigma A_{\lambda_0,\delta}}\|_{\mathscr{B}(\mathscr{K})}\le M_{2,\lambda_0,\delta}.\end{aligned}$$

Thus we also have

$$\sup_{|\sigma|\le 1}\|Ke^{i\sigma A_{\lambda_0,\delta}}(K+i)^{-1}\|_{\mathscr{B}(\mathscr{K})}<\infty. \tag{3.2.5}$$

By including the relations between K and $A_{\lambda_0,\delta}$ mentioned in §3.1, eventually we have completed checking the required conditions on $A_{\lambda_0,\delta}$ as a conjugate operator for K in the standard Mourre theory.

Now we will show Theorem 3.1.1, in particular, the Mourre estimate (3.1.12). Take a unique $n_{\lambda_0}\in\boldsymbol{Z}$ such that $\lambda_0\in I_{n_{\lambda_0}}$. Let $f_\delta\in C_0^\infty(\boldsymbol{R})$ be real-valued, and be supported in $[-\delta,\delta]$. Under the condition (V), $f_\delta(K-\lambda_0)-f_\delta(K_0-\lambda_0)$ is compact. Since $i[K_0,A_{\lambda_0,\delta}]\langle K_0\rangle^{-1}$ is bounded, and $\langle K_0\rangle^{-1}i[V,A_{\lambda_0,\delta}]\langle K_0\rangle^{-1}$ is compact as mentioned in §3.1, we have

$$\begin{aligned}&f_\delta(K-\lambda_0)i[K,A_{\lambda_0,\delta}]f_\delta(K-\lambda_0)\\ &=f_\delta(K-\lambda_0)i[K_0,A_{\lambda_0,\delta}]f_\delta(K-\lambda_0)\\ &\qquad+f_\delta(K-\lambda_0)i[V,A_{\lambda_0,\delta}]f_\delta(K-\lambda_0)\\ &=f_\delta(K_0-\lambda_0)i[K_0,A_{\lambda_0,\delta}]f_\delta(K_0-\lambda_0)+C'_{\lambda_0,f_\delta}\end{aligned}\tag{3.2.6}$$

with some compact operator C'_{λ_0,f_δ} on $\mathscr{K}$. $f_\delta(K_0-\lambda_0)i[K_0,A_{\lambda_0,\delta}]f_\delta(K_0-\lambda_0)$ can be decomposed into the direct integral

$$\bigoplus_{k\in\boldsymbol{Z}}\frac{2}{\lambda_0-\delta-k\omega}H_0f_\delta(H_0-(\lambda_0-k\omega))^2.$$

Suppose $\lambda_0-k\omega<0$, that is, $k\ge n_{\lambda_0}+1$. Then $f_\delta(H_0-(\lambda_0-k\omega))=0$ holds because of $H_0=p^2/2$ and

$$\kappa-(\lambda_0-k\omega)\ge k\omega-\lambda_0\ge \operatorname{dist}(\lambda_0,\mathscr{T})>\delta,\quad \kappa\in[0,\infty).$$

Suppose $\lambda_0-k\omega>0$, that is, $k\le n_{\lambda_0}$. Then considering $\kappa\in[0,\infty)$ such that $\kappa-(\lambda_0-k\omega)\ge-\delta$, one can obtain

$$H_0f_\delta(H_0-(\lambda_0-k\omega))^2\ge(\lambda_0-k\omega-\delta)f_\delta(H_0-(\lambda_0-k\omega))^2$$

easily. Thus we have

$$f_\delta(K_0-\lambda_0)i[K_0,A_{\lambda_0,\delta}]f_\delta(K_0-\lambda_0)$$

$$= \bigoplus_{k \le n_{\lambda_0}} \frac{2}{\lambda_0 - \delta - k\omega} H_0 f_\delta(H_0 - (\lambda_0 - k\omega))^2$$

$$\ge \bigoplus_{k \le n_{\lambda_0}} 2 f_\delta(H_0 - (\lambda_0 - k\omega))^2 = 2 f_\delta(K_0 - \lambda_0)^2.$$

By combining this and (3.2.6), and using that $f_\delta(K-\lambda_0) - f_\delta(K_0 - \lambda_0)$ is compact again, we obtain the Mourre estimate (3.1.12)

$$f_\delta(K - \lambda_0) i[K, A_{\lambda_0,\delta}] f_\delta(K - \lambda_0) \ge 2 f_\delta(K - \lambda_0)^2 + C_{\lambda_0, f_\delta}$$

with some compact operator C_{λ_0, f_δ} on $\mathscr{K}$.

3.3 Application

As an application of our results, we consider a scattering problem for the so-called AC Stark Hamiltonians.

We consider a system of one particle moving in a given time-periodic electric field $E(t) \in \boldsymbol{R}^d$. Suppose that $E(t)$ belongs to $C^0(\boldsymbol{R}; \boldsymbol{R}^d)$, and T-periodic, that is, $E(t+T) = E(t)$ for any $t \in \boldsymbol{R}$. Moreover, the mean E_{m} of $E(t)$ in time is zero, that is,

$$E_{\mathrm{m}} := \frac{1}{T} \int_0^T E(t)\, dt = 0.$$

A typical example of such $E(t)$'s is $E_0 \cos(\omega t)$ with non-zero $E_0 \in \boldsymbol{R}^d$ and $\omega = 2\pi/T$, which was considered in Kitada-Yajima [23]. As for the case where $E_{\mathrm{m}} \ne 0$, see Møller [26] and Adachi-Kimura-Shimizu [7]. Then the Hamiltonian $\hat{H}(t)$ for the system is given by

$$\hat{H}(t) = \hat{H}_0(t) + V(x), \quad \hat{H}_0(t) = \frac{1}{2} p^2 - E(t) \cdot x$$

on $L^2(\boldsymbol{R}^d)$. $\hat{H}_0(t)$ is called the free AC Stark Hamiltonian, and $\hat{H}(t)$ is called an AC Stark Hamiltonian. We denote by $\hat{U}_0(t,s)$ and $\hat{U}(t,s)$ the unitary propagators generated by $\hat{H}_0(t)$ and $\hat{H}(t)$, respectively. Now, as in [26], we define $\boldsymbol{R}^d$-valued T-periodic functions $b_0(t)$, $b(t)$ and $c(t)$ on $\boldsymbol{R}$ by

$$b_0(t) := \int_0^t E(s)\, ds, \quad b_{0,\mathrm{m}} := \frac{1}{T} \int_0^T b_0(s)\, ds,$$

$$b(t) := b_0(t) - b_{0,\mathrm{m}}, \quad c(t) := \int_0^t b(s)\, ds.$$

$b_0(t)$ is an auxiliary one for the sake of making $c(t)$ T-periodic. Here we introduce the time-dependent Hamiltonian

$$H(t) = H_0 + V(x + c(t)), \quad H_0 = \frac{1}{2}p^2$$

on $\mathscr{H} = L^2(\boldsymbol{R}^d)$. We denote by $U(t,s)$ the unitary propagator generated by $H(t)$. As is well-known, the following Avron-Herbst formula holds:

$$\hat{U}_0(t,s) = \mathscr{T}(t)e^{-i(t-s)H_0}\mathscr{T}(s)^*, \quad \hat{U}(t,s) = \mathscr{T}(t)U(t,s)\mathscr{T}(s)^* \tag{3.3.1}$$

with

$$\mathscr{T}(t) = e^{-ia(t)}e^{ib(t)\cdot x}e^{-ic(t)\cdot p}, \quad a(t) = \int_0^t \frac{1}{2}|b(s)|^2\, ds.$$

This formula with $E(t) = E_0\cos(\omega t)$ was first proved in [23]. Now we will consider the problem of the asymptotic completeness of the wave operators

$$\hat{W}^{\pm} = \operatorname*{s-lim}_{t\to\infty} \hat{U}(t,0)^*\hat{U}_0(t,0) \tag{3.3.2}$$

for short-range V. The asymptotic completeness of $\hat{W}^{\pm}$ is formulated as

$$\mathrm{Ran}\,(\hat{W}^{\pm}) = L^2_{\mathrm{c}}(\hat{U}(T,0)), \tag{3.3.3}$$

where $L^2_{\mathrm{c}}(\hat{U}(T,0))$ is the continuous spectral subspace of the Floquet operator $\hat{U}(T,0)$. We impose the following short-range condition $(V)_{\mathrm{SR}}$ on V:

$(V)_{\mathrm{SR}}$ $V(x)$ is a real-valued function on $\boldsymbol{R}^d$, and is decomposed into the sum of $\hat{V}^{\mathrm{sing}}(x)$ and $\hat{V}^{\mathrm{SR}}(x)$. If $d < 3$, then $\hat{V}^{\mathrm{sing}} = 0$. If $d \geq 3$, then $\hat{V}^{\mathrm{sing}}$ belongs to $L^{q_0}(\boldsymbol{R}^d)$ with some $\infty > q_0 > d$, and is compactly supported. $|(\nabla\hat{V}^{\mathrm{sing}})|$ belongs to $L^{q_1}(\boldsymbol{R}^d)$ with some $\infty > q_1 > d/2$, where if $d = 3$, then we define q_1 by $1/q_1 = 1/(2q_0) + 1/2$ $(< 2/d)$. $\hat{V}^{\mathrm{SR}}(x)$ belongs to $C^2(\boldsymbol{R}^d)$, and satisfies the decaying conditions

$$|(\partial_x^\alpha \hat{V}^{\mathrm{SR}})(x)| \leq C\langle x\rangle^{-\rho_{\mathrm{SR}}-|\alpha|}, \quad |\alpha| \leq 2 \tag{3.3.4}$$

with some $\rho_{\mathrm{SR}} > 1$.

Here we note that the singular part $\hat{V}^{\mathrm{sing}}$ of V satisfies the same condition posed in [7], but the short-range part $\hat{V}^{\mathrm{SR}}$ of V has to satisfy the condition which is stronger than the one posed in [26] and [7]. It is caused by that the mean of $E(t)$ in time is not non-zero but zero. Basically we have in mind the very typical singularity of the type $|x|^{-\gamma}$ as the singular part $\hat{V}^{\mathrm{sing}}$; if $\hat{V}^{\mathrm{sing}}$ has such a singularity, γ must satisfy $-\gamma q_0 + d > 0$, so $\gamma < d/q_0 < 1$ and thus, unfortunately, we cannot allow Coulomb type singularity; conversely, if $\gamma < 1$, for any $d < q_0 < d/\gamma$, $|x|^{-\gamma}$

belongs to $L^{q_0}(B(0,1))$, and for any $d/2 < q_1 < d/(\gamma+1)$, $(-\gamma-1)q_1+d > 0$, so $|\nabla|x|^{-\gamma}| = |-\gamma|x|^{-\gamma-1}x/|x|| = \gamma|x|^{-\gamma-1}$ belongs to $L^{q_1}(B(0,1))$ (in addition, if $d = 3$, for any $d < q_0 < d/\gamma$, $2(\gamma+1)/d - 1 < \gamma/d < 1/q_0$, so $1/q_1 = (1/q_0+1)/2 > (\gamma+1)/d$ is satisfied automatically). Under the condition $(V)_{\mathrm{SR}}$, $V(x+c(t)) = \hat{V}^{\mathrm{sing}}(x+c(t)) + \hat{V}^{\mathrm{SR}}(x+c(t))$ satisfies the condition (V) with $\rho = \rho_{\mathrm{SR}} - 1 > 0$, which means that $\hat{V}^{\mathrm{SR}}$ cannot be replaced by long-range potentials. Here we note $\partial_t(V(x+c(t))) = b(t)\cdot(\nabla V)(x+c(t))$ and $\partial_t^2(\hat{V}^{\mathrm{SR}}(x+c(t))) = E(t)\cdot(\nabla\hat{V}^{\mathrm{SR}})(x+c(t)) + \sum_{j=1}^d\sum_{k=1}^d b_j(t)b_k(t)(\partial_j\partial_k\hat{V}^{\mathrm{SR}})(x+c(t))$. Actually, since $c(t)$ is T-periodic in t, $\hat{V}^{\mathrm{sing}}(x+c(t))$ and $\hat{V}^{\mathrm{SR}}(x+c(t))$ are T-periodic in t; if $d \geq 3$, since there is some $C > 0$ such that $\mathrm{supp}\hat{V}^{\mathrm{sing}} \subset B[0,C]$, $\mathrm{supp}\hat{V}^{\mathrm{sing}}(\cdot + c(t))$'s are included in the common compact set $B[0, C + \max_{0\leq s\leq T}|c(s)|]$; since $\hat{V}^{\mathrm{sing}} \in L^{q_0}(\boldsymbol{R}^d)$, there is some $\varphi \in \mathscr{S}(\boldsymbol{R}^d)$ such that $\|\hat{V}^{\mathrm{sing}} - \varphi\|_{q_0} < \varepsilon$ for any $\varepsilon > 0$ and

$$\begin{aligned}
&\|\hat{V}^{\mathrm{sing}}(\cdot + c(t)) - \hat{V}^{\mathrm{sing}}(\cdot + c(t_0))\|_{q_0}\\
&\leq \|\hat{V}^{\mathrm{sing}}(\cdot + c(t)) - \varphi(\cdot + c(t))\|_{q_0} + \|\varphi(\cdot + c(t)) - \varphi(\cdot + c(t_0))\|_{q_0}\\
&\quad + \|\varphi(\cdot + c(t_0)) - \hat{V}^{\mathrm{sing}}(\cdot + c(t_0))\|_{q_0}\\
&= 2\|\hat{V}^{\mathrm{sing}} - \varphi\|_{q_0} + \left\|\int_{t_0}^t b(s)\cdot(\nabla\varphi)(\cdot + c(s))ds\right\|_{q_0}\\
&< 2\varepsilon + \max_{0<s<T}|b(s)|\||\nabla\varphi|\|_{q_0}|t - t_0|
\end{aligned}$$

for all $t, t_0 \in \boldsymbol{R}$, so we can choose $|t - t_0|$ so small that the second term of the last member is less than ε; therefore $\hat{V}^{\mathrm{sing}}(\cdot + c(t))$ belongs to $C(\boldsymbol{R}, L^{q_0}(\boldsymbol{R}^d))$; $(\partial_j\hat{V}^{\mathrm{sing}})(\cdot + c(t)) \in C(\boldsymbol{R}_t, L^{q_1}(\boldsymbol{R}^d))$, $(j = 1, \ldots, d)$, can be proved in the same way, so $\partial_t(\hat{V}^{\mathrm{sing}}(\cdot + c(t))) = b(t)\cdot(\nabla\hat{V}^{\mathrm{sing}})(\cdot + c(t))$ and $|(\nabla\hat{V}^{\mathrm{sing}})(\cdot + c(t))|$ also belong to $C(\boldsymbol{R}, L^{q_1}(\boldsymbol{R}^d))$; on the other hand, since $c(t) \in C^2(\boldsymbol{R}_t)$, $\hat{V}^{\mathrm{SR}}(\cdot + c(t))$ belongs to $C^2(\boldsymbol{R}\times\boldsymbol{R}^d)$ and

$$\begin{aligned}
|\partial_x^\alpha\hat{V}^{\mathrm{SR}}(x+c(t))| &\lesssim \langle x+c(t)\rangle^{-\rho_{\mathrm{SR}}-|\alpha|} \lesssim \max_{0\leq s\leq T}\langle c(s)\rangle^{\rho_{\mathrm{SR}}+|\alpha|}\langle x\rangle^{-\rho_{\mathrm{SR}}-|\alpha|}\\
&\quad (\lesssim \langle x\rangle^{-(\rho_{\mathrm{SR}}-1)-|\alpha|}), \quad |\alpha| \leq 2,\\
|\partial_t\partial_x^\alpha\hat{V}^{\mathrm{SR}}(x+c(t))| &= |b(t)\cdot(\nabla(\partial^\alpha\hat{V}^{\mathrm{SR}}))(x+c(t))|\\
&\lesssim |b(t)|\langle x+c(t)\rangle^{-\rho_{\mathrm{SR}}-(1+|\alpha|)}\\
&\lesssim \max_{0\leq s\leq T}|b(s)|\langle c(s)\rangle^{\rho_{\mathrm{SR}}+(1+|\alpha|)}\langle x\rangle^{-\rho_{\mathrm{SR}}-(1+|\alpha|)}\\
&\quad (\lesssim \langle x\rangle^{-(\rho_{\mathrm{SR}}-1)-(1+|\alpha|)}), \quad |\alpha| \leq 1,\\
|\partial_t^2(\hat{V}^{\mathrm{SR}}(x+c(t)))| &\leq |E(t)\cdot(\nabla\hat{V}^{\mathrm{SR}})(x+c(t))|\\
&\quad + \sum_{j=1}^d\sum_{k=1}^d|b_j(t)b_k(t)(\partial_j\partial_k\hat{V}^{\mathrm{SR}})(x+c(t))|
\end{aligned}$$

$$
\begin{aligned}
&\lesssim |E(t)|\langle x+c(t)\rangle^{-\rho_{\mathrm{SR}}-1} \\
&\quad + \sum_{j=1}^{d}\sum_{k=1}^{d} |b_j(t)b_k(t)|\langle x+c(t)\rangle^{-\rho_{\mathrm{SR}}-2} \\
&\lesssim \max_{0\le s\le T} |E(s)|\langle c(s)\rangle^{\rho_{\mathrm{SR}}+1}\langle x\rangle^{-\rho_{\mathrm{SR}}-1} \\
&\quad + \sum_{j=1}^{d}\sum_{k=1}^{d} \max_{0\le s\le T} |b_j(s)b_k(s)|\langle c(s)\rangle^{\rho_{\mathrm{SR}}+2}\langle x\rangle^{-\rho_{\mathrm{SR}}-2} \\
&\lesssim \langle x\rangle^{-(\rho_{\mathrm{SR}}-1)-2}
\end{aligned}
$$

by Peetre's inequality. Now we also introduce the wave operators

$$
W^{\pm} = \operatorname*{s-lim}_{t\to\infty} U(t,0)^* e^{-itH_0}. \tag{3.3.5}
$$

Then it is obvious that the relation between $\hat{W}^{\pm}$ and $W^{\pm}$

$$
\hat{W}^{\pm} = \mathscr{T}(0) W^{\pm} \mathscr{T}(0)^*
$$

holds. We note $\mathscr{T}(0) = e^{-ib_{0,\mathrm{m}}\cdot x}$. Thus the problem of the asymptotic completeness of $\hat{W}^{\pm}$ can be reduced to that of $W^{\pm}$

$$
\mathrm{Ran}\,(W^{\pm}) = \mathscr{H}_{\mathrm{c}}(U(T,0)), \tag{3.3.6}
$$

where $\mathscr{H}_{\mathrm{c}}(U(T,0))$ is the continuous spectral subspace of the Floquet operator $U(T,0)$. Here we used

$$
\begin{aligned}
L^2_{\mathrm{c}}(\hat{U}(T,0)) &= L^2_{\mathrm{c}}(\mathscr{T}(0)\mathscr{T}(T)^*\hat{U}(T,0)) \\
&= L^2_{\mathrm{c}}(\mathscr{T}(0)U(T,0)\mathscr{T}(0)^*) = \mathscr{T}(0)\mathscr{H}_{\mathrm{c}}(U(T,0)),
\end{aligned}
$$

because $b_0(T)=0$, $b(T) = -b_{0,\mathrm{m}}$, $c(T)=0$, so $\mathscr{T}(0)\mathscr{T}(T)^* = e^{ia(T)}$ is a scalar.

As is well-known, in the proof of the asymptotic completeness of $W^{\pm}$, the so-called Howland-Yajima method plays an important role: Introduce the Floquet Hamiltonians K_0 and K on $\mathscr{K} = L^2(\boldsymbol{T};\mathscr{H})$, associated with H_0 and $H(t)$, respectively, and the wave operators

$$
\mathscr{W}^{\pm}(K,K_0) = \operatorname*{s-lim}_{\sigma\to\pm\infty} e^{i\sigma K}e^{-i\sigma K_0}, \tag{3.3.7}
$$

where $\boldsymbol{T} = \boldsymbol{R}/(T\boldsymbol{Z})$ is the torus. After the existence of $W^{\pm}$ has been guaranteed, the asymptotic completeness of $\mathscr{W}^{\pm}(K,K_0)$ yields that of $W^{\pm}$. This is the essence of the Howland-Yajima method. We refer to Yajima [37, §4] for that proof.

If $\hat{V}^{\text{sing}} = 0$, then we have only to use the limiting absorption principle (3.1.15) in order to show the asymptotic completeness of $\mathscr{W}^{\pm}(K, K_0)$. In fact, (3.1.15) yields the local K-smoothness of $\langle x\rangle^{-s}$ with $s > 1/2$: if I is any compact interval in $\boldsymbol{R} \setminus (\mathscr{T} \cup \sigma_{\text{pp}}(K))$,

$$\int_{-\infty}^{\infty} \|\langle x\rangle^{-s} E_K(I) e^{-i\sigma K}\Phi\|_{\mathscr{K}}^2 \, d\sigma \le 2\pi \|\langle x\rangle^{-s} E_K(I)\|_K^2 \|\Phi\|_{\mathscr{K}}^2, \quad \Phi \in \mathscr{K} \tag{3.3.8}$$

by Kato's smoothness theory [18], where E_K denotes the spectral measure of K and $\|\langle x\rangle^{-s} E_K(I)\|_K$ is the smallest number M for which

$$\int_{-\infty}^{\infty} (\|\langle x\rangle^{-s} E_K(I)(K - \lambda - i\varepsilon)^{-1}\Phi\|_{\mathscr{K}}^2 + \|\langle x\rangle^{-s} E_K(I)(K - \lambda + i\varepsilon))^{-1}\Phi\|_{\mathscr{K}}^2) d\lambda$$
$$\le 4\pi^2 M^2 \|\Phi\|_{\mathscr{K}}^2, \quad \Phi \in \mathscr{K}, \varepsilon > 0$$

is true (see also Reed and Simon [30, Theorem XIII.30]). We denote by T_1, T_2 respectively the bounded operators of multiplication by

$$|\hat{V}^{\text{SR}}(x + c(t))|^{1/2}, \quad (\text{sign}\hat{V}^{\text{SR}}(x + c(t)))|\hat{V}^{\text{SR}}(x + c(t))|^{1/2}$$

defined in $\mathscr{K}$. Then $K - K_0 = T_2^* T_1$ holds in the sense that

$$(\Phi_1, K\Phi_2)_{\mathscr{K}} - (K_0\Phi_1, \Phi_2)_{\mathscr{K}} = (T_1\Phi_1, T_2\Phi_2)_{\mathscr{K}}, \quad \Phi_1 \in \mathscr{D}(K_0), \Phi_2 \in \mathscr{D}(K). \tag{3.3.9}$$

To see this, we take $a_1, a_2 \in AC(\boldsymbol{T})$ and $\varphi_1, \varphi_2 \in C_0^\infty(\boldsymbol{R}^d)$, and let

$$\Phi_1(t) = a_1(t)\varphi_1, \Phi_2(t) = a_2(t)\varphi_2, \quad t \in \boldsymbol{T}.$$

Since $\Phi_1 \in \mathscr{D}(K_0), \Phi_2 \in \mathscr{D}(K)$,

$$-i \left. \frac{d}{d\sigma}(e^{-i\sigma K_0}\Phi_1, e^{-i\sigma K}\Phi_2)_{\mathscr{K}} \right|_{\sigma=0} = (\Phi_1, K\Phi_2)_{\mathscr{K}} - (K_0\Phi_1, \Phi_2)_{\mathscr{K}}.$$

On the other hand, from the definitions of K_0 and K and the translation invariance of dt,

$$\begin{aligned}(e^{-i\sigma K_0}\Phi_1, e^{-i\sigma K}\Phi_2)_{\mathscr{K}} &= \int_{\boldsymbol{T}} (e^{-i(t-(t-\sigma))H_0}\Phi_1(t-\sigma), U(t, t-\sigma)\Phi_2(t-\sigma))_{\mathscr{H}} dt \\ &= \int_{\boldsymbol{T}} (e^{-i\sigma H_0}\Phi_1(t), U(t+\sigma, t)\Phi_2(t))_{\mathscr{H}} dt \\ &= \int_{\boldsymbol{T}} a_1(t)\overline{a_2(t)}(e^{-i\sigma H_0}\varphi_1, U(t+\sigma, t)\varphi_2)_{\mathscr{H}} dt.\end{aligned}$$

The integrand of the last member is continuously differentiable in σ, because $\varphi_1, \varphi_2 \in H^2(\boldsymbol{R}^d)$ and so satisfy the hypothesis of Yajima [38, Theorem 1.3].

Since a_1 and a_2 are continuous on the compact space $\boldsymbol{T}$ and dt is the finite measure,

$$\begin{aligned}\sup_{\sigma\in\boldsymbol{R}} |a_1(t)\overline{a_2(t)}(e^{-i\sigma H_0}\varphi_1, U(t+\sigma,t)\varphi_2)_{\mathscr{H}}| &\le \max_{\tau\in\boldsymbol{T}} |a_1(\tau)||a_2(\tau)|\|\varphi_1\|_{\mathscr{H}}\|\varphi_2\|_{\mathscr{H}} \\ &\in L^1_t(\boldsymbol{T}).\end{aligned}$$

Moreover, since $i\partial_t U(t,s)\varphi_2 = H(t)U(t,s)\varphi_2$ is continuous in (t,s) jointly, for any compact set C in $\boldsymbol{R}$,

$$\begin{aligned}&\sup_{\sigma\in C}\left|a_1(t)\overline{a_2(t)}\frac{\partial}{\partial\sigma}(e^{-i\sigma H_0}\varphi_1, U(t+\sigma,t)\varphi_2)_{\mathscr{H}}\right| \\ &\le \max_{\tau\in\boldsymbol{T}}|a_1(\tau)||a_2(\tau)|\sup_{\sigma\in C}|(e^{-i\sigma H_0}\varphi_1, -iH(t+\sigma)U(t+\sigma,t)\varphi_2)_{\mathscr{H}} \\ &\quad + (-ie^{-i\sigma H_0}H_0\varphi_1, U(t+\sigma,t)\varphi_2)_{\mathscr{H}}| \\ &\le \max_{\tau\in\boldsymbol{T}}|a_1(\tau)||a_2(\tau)|(\|\varphi_1\|_{\mathscr{H}}\max_{(t',s')\in([0,T]+C)\times[0,T]}\|H(t')U(t',s')\varphi_2\|_{\mathscr{H}} \\ &\quad + \|H_0\varphi_1\|_{\mathscr{H}}\|\varphi_2\|_{\mathscr{H}}) \\ &\in L^1_t(\boldsymbol{T}).\end{aligned}$$

Therefore that integral is continuously differentiable in σ and

$$\begin{aligned}&-i\frac{d}{d\sigma}\int_{\boldsymbol{T}} a_1(t)\overline{a_2(t)}(e^{-i\sigma H_0}\varphi_1, U(t+\sigma,t)\varphi_2)_{\mathscr{H}}dt\bigg|_{\sigma=0} \\ &= \int_{\boldsymbol{T}} a_1(t)\overline{a_2(t)}((\varphi_1, H(t)\varphi_2)_{\mathscr{H}} - (H_0\varphi_1,\varphi_2)_{\mathscr{H}})dt \\ &= \int_{\boldsymbol{T}}(V(x+c(t))\Phi_1(t),\Phi_2(t))_{\mathscr{H}}dt = (T_1\Phi_1, T_2\Phi_2)_{\mathscr{K}}.\end{aligned}$$

Since these pure tensor products Φ_1, Φ_2 generate $AC(\boldsymbol{T})\otimes C_0^\infty(\boldsymbol{R}^d)$ and it is a core for K_0 and K, (3.3.9) holds for all $\Phi_1,\Phi_2\in AC(\boldsymbol{T})\otimes C_0^\infty(\boldsymbol{R}^d)$, and for general $\Phi_1\in\mathscr{D}(K_0),\Phi_2\in\mathscr{D}(K)$, there exist $\Phi_{1,j},\Phi_{2,j}\in AC(\boldsymbol{T})\otimes C_0^\infty(\boldsymbol{R}^d)$, $j\in\boldsymbol{N}$, such that $\Phi_{1,j}\to\Phi_1$, $K_0\Phi_{1,j}\to K_0\Phi_1$, $\Phi_{2,j}\to\Phi_2$, $K\Phi_{2,j}\to K\Phi_2$, $(j\to\infty)$; these $\Phi_{1,j}$, $\Phi_{2,j}$ satisfy (3.3.9) and T_1 and T_2 are bounded, so Φ_1, Φ_2 satisfy (3.3.9). Taking $s=\rho_{\mathrm{SR}}/2>1/2$, we see that

$$\begin{aligned}|\hat{V}^{\mathrm{SR}}(x+c(t))|^{1/2}\langle x\rangle^s &\lesssim \langle x+c(t)\rangle^{-\rho_{\mathrm{SR}}/2}\langle x\rangle^{\rho_{\mathrm{SR}}/2} \\ &\lesssim \langle x\rangle^{-\rho_{\mathrm{SR}}/2}\langle c(t)\rangle^{\rho_{\mathrm{SR}}/2}\langle x\rangle^{\rho_{\mathrm{SR}}/2} \\ &\le \max_{0\le s\le T}\langle c(s)\rangle^{\rho_{\mathrm{SR}}/2}<\infty, \quad x\in\boldsymbol{R}^d, t\in\boldsymbol{R}\end{aligned}$$

by Peetre's inequality, so T_2 is K-smooth on I. Since the K_0-smoothness on I of $\langle x\rangle^{-s}$ with $s>1/2$ can be also obtained by replacing V by 0, T_1 is K_0-smooth on

I. Then the existence of local wave operators on I°

$$\operatorname*{s-lim}_{\sigma\to\pm\infty} e^{i\sigma K} e^{-i\sigma K_0} E_{K_0}(I^\circ), \quad \operatorname*{s-lim}_{\sigma\to\pm\infty} e^{i\sigma K_0} e^{-i\sigma K} E_K(I^\circ)$$

follows by Lavine's local smoothness theorem [24, Theorem 2.3] , where E_{K_0} denotes the spectral measure of K_0. Since $\mathscr{T} \cup \sigma_{\mathrm{pp}}(K)$ is a countable closed set, there is a sequence of compact intervals I_j in $\boldsymbol{R}\setminus(\mathscr{T}\cup\sigma_{\mathrm{pp}}(K))$, $j \in \boldsymbol{N}$, such that $\bigcup_{j=1}^\infty I_j^\circ = \boldsymbol{R} \setminus (\mathscr{T} \cup \sigma_{\mathrm{pp}}(K))$. Gluing those local wave operators on I_j° together yields the existence of $\mathscr{W}^\pm(K, K_0)$ and the adjoint wave operators

$$\mathscr{W}^\pm(K_0, K) = \operatorname*{s-lim}_{\sigma\to\pm\infty} e^{i\sigma K_0} e^{-i\sigma K} P_{\mathrm{c}}(K) \tag{3.3.10}$$

immediately, where $P_{\mathrm{c}}(K)$ is the spectral projection onto the continuous spectral subspace $\mathscr{K}_{\mathrm{c}}(K)$ of K. To see this, we first write the unitary operator of multiplication by e^{-itH_0} as $\mathscr{U}$ and note that

$$\mathscr{K}_{\mathrm{ac}}(K_0) = \mathscr{U}\langle L^2(\boldsymbol{T}, \mathscr{H}_{\mathrm{ac}}(e^{-iTH_0}), dt)\rangle = \mathscr{K},$$

and so the spectrum of K_0 is absolutely continuous (see Yajima [37, §4]). Then since $(\bigcup_{j=1}^\infty I_j^\circ)^c = \mathscr{T} \cup \sigma_{\mathrm{pp}}(K)$ has Lebesgue measure 0,

$$E_{K_0}\left(\bigcup_{j=1}^N I_j^\circ\right) \xrightarrow{\mathrm{s}} E_{K_0}\left(\bigcup_{j=1}^\infty I_j^\circ\right) = 1 - E_{K_0}\left(\left(\bigcup_{j=1}^\infty I_j^\circ\right)^c\right) = 1, \quad (N \to \infty).$$

On the other hand, since $\mathscr{T} \cup \sigma_{\mathrm{pp}}(K)$ is a countable set, for all $u \in \mathscr{K}_{\mathrm{c}}(K)$,

$$E_K\left(\bigcup_{j=1}^N I_j^\circ\right) u \to u - E_K(\mathscr{T} \cup \sigma_{\mathrm{pp}}(K))u = u, \quad (N \to \infty).$$

Let $u \in \mathscr{K}_{\mathrm{c}}(K)$ and $\varepsilon > 0$. If N is sufficiently large, $\|E_K(\bigcup_{j=1}^N I_j^\circ)u - u\|_{\mathscr{K}} < \varepsilon$ holds. Since $\bigcup_{j=1}^N I_j^\circ$ is a finite union of the open intervals, it can be decomposed into disjoint open intervals: that is, there are some compact intervals J_j in $\boldsymbol{R}\setminus(\mathscr{T}\cup \sigma_{\mathrm{pp}}(K))$, $(j = 1, \dots, N')$, such that J_j°'s are disjoint and $\bigcup_{j=1}^N I_j^\circ = \bigcup_{j=1}^{N'} J_j^\circ$. Hence the strong limit

$$\lim_{\sigma\to\pm\infty} e^{i\sigma K_0} e^{-i\sigma K} E_K\left(\bigcup_{j=1}^N I_j^\circ\right) u = \sum_{j=1}^{N'} \lim_{\sigma\to\pm\infty} e^{i\sigma K_0} e^{-i\sigma K} E_K(J_j^\circ) u$$

exists, so for sufficiently large σ, τ (or $-\sigma, -\tau$) $\in \boldsymbol{R}$,

$$\left\| e^{i\sigma K_0} e^{-i\sigma K} E_K\left(\bigcup_{j=1}^N I_j^\circ\right) u - e^{i\tau K_0} e^{-i\tau K} E_K\left(\bigcup_{j=1}^N I_j^\circ\right) u \right\|_{\mathscr{K}} < \varepsilon$$

holds. Therefore by the uniform boundedness of $e^{i\sigma K_0}e^{-i\sigma K}$,

$$
\begin{aligned}
&\|e^{i\sigma K_0}e^{-i\sigma K}u - e^{i\tau K_0}e^{-i\tau K}u\|_{\mathscr{K}} \\
\le &\left\| e^{i\sigma K_0}e^{-i\sigma K}u - e^{i\sigma K_0}e^{-i\sigma K}E_K\left(\bigcup_{j=1}^N I_j^\circ\right)u\right\|_{\mathscr{K}} \\
&+ \left\| e^{i\sigma K_0}e^{-i\sigma K}E_K\left(\bigcup_{j=1}^N I_j^\circ\right)u - e^{i\tau K_0}e^{-i\tau K}E_K\left(\bigcup_{j=1}^N I_j^\circ\right)u\right\|_{\mathscr{K}} \\
&+ \left\| e^{i\tau K_0}e^{-i\tau K}E_K\left(\bigcup_{j=1}^N I_j^\circ\right)u - e^{i\tau K_0}e^{-i\tau K}u\right\|_{\mathscr{K}} \\
\le &\, 2\left\| u - E_K\left(\bigcup_{j=1}^N I_j^\circ\right)u\right\|_{\mathscr{K}} + \varepsilon < 3\varepsilon,
\end{aligned}
$$

so

$$\mathscr{W}^\pm(K_0, K) = \operatorname*{s-lim}_{\sigma\to\pm\infty} e^{i\sigma K_0}e^{-i\sigma K}P_{\mathrm{c}}(K)$$

exists. Inverting K_0 and K, the existence of

$$\mathscr{W}^\pm(K, K_0) = \operatorname*{s-lim}_{\sigma\to\pm\infty} e^{i\sigma K}e^{-i\sigma K_0}$$

can be proved in the same way. Thus one can obtain the asymptotic completeness of $\mathscr{W}^\pm$, as is well-known. If $\hat{V}^{\mathrm{sing}} \neq 0$, then we have to avoid the matter caused by its singularity in the proof of the existence of both $\mathscr{W}^\pm(K, K_0)$ and $\mathscr{W}^\pm(K_0, K)$. To this end, we will use the so-called minimal velocity estimate like

$$\int_1^\infty \left\| F\left(\frac{|x|}{\sigma} \le \sqrt{(2-3\varepsilon)(d_2(\lambda_0)-2\delta)}\right) e^{-i\sigma K} f_{2\delta}(K-\lambda_0)\Phi\right\|_{\mathscr{K}}^2 \frac{d\sigma}{\sigma} \le C\|\Phi\|_{\mathscr{K}}^2 \tag{3.3.11}$$

with sufficiently small $\varepsilon > 0$, which follows from

$$\int_1^\infty \left\| F\left(2-4\varepsilon \le \frac{A_{\lambda_0,2\delta}}{\sigma} \le 2-2\varepsilon\right) e^{-i\sigma K} f_{2\delta}(K-\lambda_0)\Phi\right\|_{\mathscr{K}}^2 \frac{d\sigma}{\sigma} \le C\|\Phi\|_{\mathscr{K}}^2. \tag{3.3.12}$$

These propagation estimates can be proved in the same way as in Sigal-Soffer [35], by virtue of the Mourre estimate (3.1.13). Here $F(x \in \Omega)$ denotes the characteristic function of the set of Ω, and

$$d_2(\lambda) = \mathrm{dist}(\lambda, \mathscr{T} \cup \sigma_{\mathrm{pp}}(K)).$$

If $d_2(\lambda_0)$ in (3.3.11) could be replaced by

$$\mathrm{dist}(\lambda_0, (\mathscr{T} \cup \sigma_{\mathrm{pp}}(K)) \cap (-\infty, \lambda_0]),$$

then (3.3.11) would become more natural and refined.

In the long-range case, it seems necessary to obtain some refined propagation estimates for $\hat{U}(t,s)$ or $U(t,s)$. Unfortunately, we have not done it yet. The result on the asymptotic completeness was already obtained in Kitada-Yajima [23] via the Enss method. As for the case where $E_{\mathrm{m}} \neq 0$, see Adachi [2] and Adachi-Kimura-Shimizu [7].

3.4 Concluding remarks

Although we consider the one body case only in this paper, here we will make some remarks on the many body case.

We consider a system of N particles moving in a given T-periodic electric field in $\boldsymbol{R}^d$. In the center-of-mass frame, the total Hamiltonian $\hat{H}(t)$ is given as

$$\hat{H}(t) = -\frac{1}{2}\Delta_X - \langle E(t), x\rangle + V, \quad V = \sum_{1\le j<k\le N} V_{jk}(x_j - x_k)$$

on $L^2(X)$, where X is the configuration space for the system under consideration in the center-of-mass frame with a certain suitable metric $\langle\cdot,\cdot\rangle$, $x \in X$, Δ_X is the Laplace-Beltrami operator on X, $E(t) \in C^0(\boldsymbol{R}; X)$ is T-periodic, and V_{jk}'s are pair interactions. If $N = 2$, then $\hat{H}(t)$ is essentially the same as that in §3.3. Hence we suppose $N \ge 3$. We denote by $\hat{U}(t,s)$ the propagator generated by $\hat{H}(t)$, and put

$$E_{\mathrm{m}} := \frac{1}{T}\int_0^T E(s)\,ds \in X.$$

As in Møller [26] and Adachi [1], we define X-valued T-periodic functions $b_0(t)$, $b(t)$ and $c(t)$ on $\boldsymbol{R}$ by

$$b_0(t) := \int_0^t (E(s) - E_{\mathrm{m}})\,ds, \quad b_{0,\mathrm{m}} := \frac{1}{T}\int_0^T b_0(s)\,ds,$$
$$b(t) := b_0(t) - b_{0,\mathrm{m}}, \quad c(t) := \int_0^t b(s)\,ds,$$

and introduce the time-dependent Hamiltonian

$$H(t) = H_0 + V(x + c(t)), \quad H_0 = -\frac{1}{2}\Delta_X - \langle E_{\mathrm{m}}, x\rangle$$

on $L^2(X)$. If $E_{\mathrm{m}} \neq 0$, then H_0 is called the free N-body Stark Hamiltonian. We denote by $U(t,s)$ the unitary propagator generated by $H(t)$. As is well-known, the following Avron-Herbst formula holds:

$$\hat{U}_0(t,s) = \mathscr{T}(t)e^{-i(t-s)H_0}\mathscr{T}(s)^*, \quad \hat{U}(t,s) = \mathscr{T}(t)U(t,s)\mathscr{T}(s)^* \qquad (3.4.1)$$

with

$$\mathscr{T}(t) = e^{-ia(t)} e^{ib(t)\cdot x} e^{-ic(t)\cdot p}, \quad a(t) = \int_0^t \left(\frac{1}{2}|b(s)|^2 - \langle E_{\mathrm{m}}, c(s) \rangle \right) ds,$$

where $|b(s)|^2 = \langle b(s), b(s) \rangle$.

When $E_{\mathrm{m}} \neq 0$, in [1] and [2], Adachi already obtained the result of the asymptotic completeness for the system under consideration, both in the short-range and the long-range cases, by introducing the Floquet Hamiltonian K associated with $\hat{H}(t)$. As for this K,

$$A = \left\langle \frac{E_{\mathrm{m}}}{|E_{\mathrm{m}}|}, -i\nabla_X \right\rangle$$

is a conjugate operator for K in the standard Mourre theory, where $-i\nabla_X$ is the velocity operator on X. Here we emphasize that in the case where $N = 2$, in [26], Møller proposed this operator as a conjugate operator for K before [1]. Roughly speaking, the conjugate operator due to Møller possesses its natural extension to N-body systems. On the other hand, when $E_{\mathrm{m}} = 0$, any candidates of a conjugate operator for K in the standard Mourre theory have not been found yet, except in the case where $N = 2$. As mentioned above, in the case where $N = 2$, Yokoyama proposed a conjugate operator $\tilde{A}_1$ for K in [39]. Unfortunately, $\tilde{A}_1$ seems not have any natural extension to N-body systems. It is caused by the 'factor' $(1 + p^2)^{-1}$ of $\tilde{A}_1$ (see [27] for the detail). Hence, in [27], Møller and Skibsted took $\hat{A}_0$ as a conjugate operator for K in an extended Mourre theory, as mentioned in §3.1. As for the study of the asymptotic completeness for three-body AC Stark Hamiltonians via the Faddeev method, see Korotyaev [21] and Nakamura [28].

Our aim of this chapter is to replace the factor $(1 + p^2)^{-1}$ by some other appropriate one in order to let a conjugate operator possess its extension to N-body systems. However, we have not accomplished this aim yet, unfortunately. We have to deal with the term like

$$-(\lambda_0 - \delta - D_t)^{-1} \langle b(t), (\nabla_X V)(x + c(t)) \rangle (\lambda_0 - \delta - D_t)^{-1} \hat{A}_0 \tag{3.4.2}$$

in $i[V(x+c(t)), A_{\lambda_0,\delta}]$ skillfully, in the proof of the Mourre estimate for K, where $\hat{A}_0$ is the generator of dilations on X. It is caused by that $|(\nabla_X V)(x + c(t))|$ does not vanish as $|x| \to \infty$, if $N \geq 3$. These are the issues in the future. Finally we note that if $V(x+c(t))$ is time-independent, one can obtain the Mourre estimate for K by taking $(\lambda_0 - \delta - D_t)^{-1} \hat{A}_0$ as a conjugate operator in the standard Mourre theory, even if $N \geq 3$. Hence we have a faint expectation that the factor $(\lambda_0 - \delta - D_t)^{-1}$ will overcome the matter mentioned above.

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