



Spectral theory for repulsive and Stark Hamiltonians

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博 士 論 文

Spectral theory for repulsive and Stark
Hamiltonians
(斥力ハミルトニアン及びシュタルクハミルトニ
アンに対するスペクトル理論)

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Introduction

This thesis is written based on my papers [I1, I2] and the joint paper [AIIS] with Tadayoshi Adachi, Kenichi Ito and Erik Skibsted.

In this thesis we deal with the spherically symmetric repulsive Hamiltonians and the Stark Hamiltonians on the Euclidean space of any dimension. It is well-known that the spectrum of these Hamiltonians is purely absolutely continuous (cf. e.g. [BCHM, Sig, Ya]). In particular they have no eigenfunctions. The main purpose of this thesis is to prove Rellich's theorem for these Hamiltonians, which asserts the absence of generalized eigenfunctions in a certain weighted space called the Agmon-Hörmander space. The space is optimal in the sense that we can actually construct a generalized eigenfunction in any larger spaces. Since the space is larger than L^2 -space, it follows immediately from Rellich's theorem the absence of pure point spectrum of these Hamiltonians. Rellich's theorem is interesting in its own right and very important in obtaining other our results.

We also study the properties of the resolvent. We prove the limiting absorption principle bounds, the radiation condition bounds, the limiting absorption principle and the Sommerfeld uniqueness result. The limiting absorption principle bounds assert the locally uniform boundedness of the resolvent between the appropriate functional spaces related to Rellich's theorem, the Agmon-Hörmander type spaces. The locally uniform boundedness as operator between usual weighted L^2 -spaces is well-known. However, as far as the author knows, the locally uniform boundedness in the case of using the Agmon-Hörmander type spaces is not known. In particular the boundedness we obtained is a slightly stronger than the usual one. Limiting absorption principle bounds yield the absence of singular continuous spectrum of these Hamiltonians. Therefore by combining Rellich's theorem and the limiting absorption principle bounds, we obtain as a corollary that the spectrum of these Hamiltonians is purely absolutely continuous. In this thesis the limiting absorption principle is derived from the limiting absorption principle bounds and the radiation condition bounds. The limiting resolvents obtained by the limiting absorption principle belong to the same space of bounded operators as the space which the resolvent belong to. The Sommerfeld uniqueness result characterizes the limiting resolvents by the Helmholtz equation and the radiation condition.

In the proofs of our theorems, we mainly use a commutator argument. This argument is simple and elementary, and does not employ any advanced tools such as the Fourier analysis or the microlocal analysis. Our novelty here is a use of the conjugate operator associated with some radial flow, not with dilations or translations. Classical orbit of the particle is closely related to construction of the conjugate operator. We note that this construction is based on the Mourre theory, but we do not use the Mourre theory directly.

In Chapter 1, we deal with the spherically symmetric repulsive Hamiltonians. In Section 1.1 we introduce basic setting and state our main theorems. In Section 1.2

we introduce a commutator with some *weight inside* (see Subsection 1.2.2) and discuss its properties. Commutator estimate plays an important role to show our results. Rellich's theorem will be proved in Section 1.3. We divide the proof of Rellich's theorem into two steps: a priori super-exponential decay estimate and the absence of super-exponentially decaying eigenfunctions. As for the properties of the resolvent, we discuss in Section 1.4. The limiting absorption principle bounds is proved in Section 1.4. The radiation condition bounds, the limiting absorption principle and the Sommerfeld uniqueness result are also proved in the same section.

In Chapter 2, we deal with the Stark Hamiltonians. Stark potential corresponds to a constant electric field. Thus it is physically important subject to study. The structure of Chapter 2 is roughly the same as Chapter 1. Thus, arguments similar to Chapter 1, e.g. on domains of operators, will be omitted accordingly. We introduce basic setting and state main theorems in Section 2.1. In Section 2.2 we introduce a commutator with some weight inside and discuss its properties. We prove Rellich's theorem in Section 2.3. The proof of Rellich's theorem consists of two steps which are somewhat different from usual one: A priori super-*cubic*-exponential decay estimate and absence of super-*cubic*-exponentially decaying eigenfunctions. This will be related to the decay rate of the Airy function which is one of generalized eigenfunctions for one dimensional Stark Hamiltonians. The limiting absorption principle bounds is proved in Section 2.4. The radiation condition bounds, the limiting absorption principle bounds and the Sommerfeld uniqueness result are proved in Section 2.5 in this order.

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CHAPTER 1

Spherically symmetric repulsive Hamiltonians

This chapter is written based on the papers [I1, I2].

1.1. Setting and results

For any fixed $\epsilon \in (0, 2]$ we consider the spherically symmetric repulsive Schrödinger operator

$$H = -\frac{1}{2}\Delta - |x|^\epsilon + q; \quad -\Delta = p_j \delta^{jk} p_k, \quad p_j = -i\partial_{x_j},$$

on the Hilbert space $\mathcal{H} = L^2(\mathbb{R}^d)$. Here δ^{jk} is the Kronecker delta, and we use the Einstein summation convention (throughout the thesis we will use this convention), and q is a real-valued function that may grow slightly slower than $|x|^\epsilon$ at infinity. By the Faris-Lavine theorem (see [RS, II]) the operator H is essentially self-adjoint on $C_0^\infty(\mathbb{R}^d)$, and we denote the self-adjoint extension by the same letter. For the case of $\epsilon = 2$, the Hamiltonian H is called the inverted harmonic oscillator.

1.1.1. Basic setting. Choose $\chi \in C^\infty(\mathbb{R})$ such that

$$\chi(s) = \begin{cases} 1 & \text{for } s \leq 1, \\ 0 & \text{for } s \geq 2, \end{cases} \quad \partial_s \chi = \chi' \leq 0, \quad (1.1.1)$$

and set $r \in C^\infty(\mathbb{R}^d)$ and the associated differential operator ∂^r as

$$\begin{aligned} r(x) &= \chi(|x|) + |x|(1 - \chi(|x|)), \\ \partial^r &= (\partial_j r) \delta^{jk} \partial_k. \end{aligned}$$

Moreover we introduce the function $f \in C^\infty(\mathbb{R}^d)$ and the associated differential operator ∂^f as

$$\begin{aligned} f(x) &= \begin{cases} (r^{1-\epsilon/2} + 1)/(1 - \epsilon/2) & \text{for } \epsilon \neq 2, \\ \log r + 1 & \text{for } \epsilon = 2, \end{cases} \\ \partial^f &= (\partial_j f) \delta^{jk} \partial_k. \end{aligned} \quad (1.1.2)$$

We note that the function f is continuous with regard to ϵ and clearly the following properties hold:

$$r \geq 1, \quad f \geq 1, \quad \partial^f = r^{-\epsilon/2} \partial^r.$$

In this chapter we use the function f frequently. This is closely related to the classical orbits of the particles. In particular, it plays an important role for the case of $\epsilon = 2$. We are going to see the classical orbits in Subsection 1.1.3.

Condition 1.1.1. The perturbation q has a splitting by real-valued functions:

$$q = q_1 + q_2; \quad q_1 \in C^1(\mathbb{R}^d), \quad q_2 \in L^\infty(\mathbb{R}^d),$$

such that for some $\rho, C > 0$ the following bounds hold on $\mathbb{R}^d$.

$$|q_1| \leq \begin{cases} Cr^\epsilon f^{-\rho} & \text{for } \epsilon \neq 2, \\ Cr^{2-\rho} & \text{for } \epsilon = 2, \end{cases} \quad \partial^f q_1, |q_2| \leq \begin{cases} Cf^{-1-\rho}, & \text{for } \epsilon \neq 2, \\ Cr^{-\rho} & \text{for } \epsilon = 2. \end{cases}$$

We remark that we can deal with the perturbations which diverge to ∞ or $-\infty$ at infinity as q_1 due to the repulsive potential $-|x|^\epsilon$. As for the q_2 , we impose the short-range condition for $\epsilon \neq 2$.

We introduce the weighted Hilbert space $\mathcal{H}_s$ by

$$\mathcal{H}_s = f^{-s} \mathcal{H} \quad \text{for } s \in \mathbb{R}.$$

Note that we introduced the space $\mathcal{H}_s$ by using the function f , not r . This is also related to the classical orbit. We consider the characteristic functions

$$F_\nu = F(\{2^\nu \leq f < 2^{\nu+1}\}), \quad \nu \geq 0,$$

where $F(\Omega)$ denotes sharp characteristic function of a subset $\Omega \subseteq \mathbb{R}^d$. Define the Agmon-Hörmander spaces $\mathcal{B}$, $\mathcal{B}^*$ and $\mathcal{B}_0^*$ by

$$\begin{aligned} \mathcal{B} &= \{\psi \in L_{\text{loc}}^2(\mathbb{R}^d) \mid \|\psi\|_{\mathcal{B}} = \sum_{\nu \geq 0} 2^{\nu/2} \|F_\nu \psi\|_{\mathcal{H}} < \infty\}, \\ \mathcal{B}^* &= \{\psi \in L_{\text{loc}}^2(\mathbb{R}^d) \mid \|\psi\|_{\mathcal{B}^*} = \sup_{\nu \geq 0} 2^{-\nu/2} \|F_\nu \psi\|_{\mathcal{H}} < \infty\}, \\ \mathcal{B}_0^* &= \{\psi \in \mathcal{B}^* \mid \lim_{\nu \rightarrow \infty} 2^{-\nu/2} \|F_\nu \psi\|_{\mathcal{H}} = 0\}, \end{aligned} \tag{1.1.3}$$

respectively. By using the function spaces (1.1.3), which are somewhat different from the usual one, we can deal with the case of the inverted harmonic oscillator. We note that $\mathcal{B}_0^*$ coincides with the closure of $C_0^\infty(\mathbb{R}^d)$ in $\mathcal{B}^*$ and for any $s > 1/2$ the following inclusion relations hold:

$$\mathcal{H}_s \subsetneq \mathcal{B} \subsetneq \mathcal{H}_{1/2} \subsetneq \mathcal{H} \subsetneq \mathcal{H}_{-1/2} \subsetneq \mathcal{B}_0^* \subsetneq \mathcal{B}^* \subsetneq \mathcal{H}_{-s}. \tag{1.1.4}$$

We introduced the conjugate operator A as a maximal differential operator

$$A = \text{Re } p^f, \quad p^f = -i\partial^f, \tag{1.1.5}$$

with domain

$$\mathcal{D}(A) = \{\psi \in \mathcal{H} \mid A\psi \in \mathcal{H}\}.$$

The conjugate operator A has the following expressions:

$$A = \text{Re } p^f = (p^f)^* + \frac{i}{2}(\Delta f) = p^f - \frac{i}{2}(\Delta f). \tag{1.1.6}$$

We note that the conjugate operator A is a generator of some radial flow. In particular A become self-adjoint with the above domain $\mathcal{D}(A)$. These properties will be verified in Subsection 1.2.1.

By the definition of r , there exist $c > 0$ and $r_0 \geq 1$ such that

$$|\partial r| \geq c,$$

on $\{x \in \mathbb{R}^d \mid r(x) > r_0\}$. There we set

$$\eta = 1 - \chi(r/r_0), \quad \tilde{\eta} = \eta |\partial r|^{-2},$$

and introduce the tensor ℓ as follows.

$$\ell = \delta - \tilde{\eta}(\partial r) \otimes (\partial r).$$

Here we note that the tensor ℓ satisfy the bounds

$$0 \leq \xi_j \ell^{jk} \xi_k \leq |\xi|^2 = \xi_j \delta^{jk} \xi_k, \quad \forall \xi \in \mathbb{R}^d.$$

For notational simplicity we set

$$h = r^{-\epsilon/2-1} (\delta - (\partial r) \otimes (\partial r) + 2C f^{-1-2\rho} \delta).$$

Here we choose $C > 0$ large enough so that for any $\xi \in \mathbb{R}^d$

$$\xi_j h^{jk} \xi_k \geq \xi_j (r^{-\epsilon} f^{-1} \ell^{jk} + C r^{-\epsilon} f^{-2-2\rho} \delta^{jk}) \xi_k \geq 0.$$

For any subset $I \subseteq \mathbb{R}$ let us denote

$$I_{\pm} = \{z \in \mathbb{C} \mid \operatorname{Re} z \in I, \pm \operatorname{Im} z \in (0, 1)\},$$

respectively. We also use the notation $\langle T \rangle_{\psi} = \langle \psi, T\psi \rangle$ for a linear operator T .

1.1.2. Results. First we have Rellich's theorem of the following form.

Theorem 1.1.2. *Suppose Condition 1.1.1, and let $\lambda \in \mathbb{R}$. If a function $\phi \in \mathcal{B}_0^*$ satisfies that*

$$(H - \lambda)\phi = 0,$$

in the distributional sense, then $\phi = 0$ in $\mathbb{R}^d$.

By the inclusion relations (1.1.4), we have the absence of pure point spectrum of H as a corollary of Theorem 1.1.2.

Corollary 1.1.3. *Under Condition 1.1.1, the operator H has no pure point spectrum: $\sigma_{\text{pp}}(H) = \emptyset$.*

As we will see in Subsection 1.1.4 below we can actually construct a $\mathcal{B}^*$ -eigenfunction, and hence the function space $\mathcal{B}_0^*$ in Theorem 1.1.2 is optimal (cf. (1.1.4)).

Next we state some properties of the resolvent

$$R(z) = (H - z)^{-1}.$$

Theorem 1.1.4. *Suppose Condition 1.1.1 and let $I \subset \mathbb{R}$ be any compact interval. Then there exists $C > 0$ such that for any $\phi = R(z)\psi$ with $z \in I_{\pm}$ and $\psi \in \mathcal{B}$*

$$\|\phi\|_{\mathcal{B}^*} + \|p^f \phi\|_{\mathcal{B}^*} + \langle p_j h^{jk} p_k \rangle_{\phi}^{1/2} + \|r^{-\epsilon} p_j \delta^{jk} p_k \phi\|_{\mathcal{B}^*} \leq C \|\psi\|_{\mathcal{B}}. \quad (1.1.7)$$

It follows from the bounds (1.1.7) that the boundedness of the resolvent $R(z)$ as an operator from $\mathcal{B}$ to $\mathcal{B}^*$. This boundedness yields the following corollary.

Corollary 1.1.5. *Under Condition 1.1.1, the operator H has no singular continuous spectrum: $\sigma_{\text{sc}}(H) = \emptyset$.*

By Corollary 1.1.3 and Corollary 1.1.5, we have, under Condition 1.1.1, the spectrum of H is purely absolutely continuous. The limiting absorption principle does not immediately follow from Theorem 1.1.4. To obtain the limiting absorption principle we impose an additional condition and we state and prove the radiation condition bounds.

Condition 1.1.6. In addition to Condition 1.1.1, there exist $\tau, C > 0$ such that

$$|\partial^f q_1| \leq C f^{-1-\tau}, \quad |\ell^{\bullet k} r^{-\epsilon/2} \partial_k q_1| \leq C f^{-1-\tau}.$$

Now we choose a smooth decreasing function $r_\lambda \geq 1$ of $\lambda \in \mathbb{R}$ such that

$$\lambda - q_1 + r^\epsilon \geq 1 \quad \text{for } r \geq r_\lambda,$$

and set asymptotic complex phase a : For $z = \lambda \pm i\Gamma$ where $\lambda \in \mathbb{R}, \Gamma \in (0, 1)$

$$a = a_z = \eta_\lambda |\partial r| r^{-\epsilon/2} \sqrt{2(z - q_1 + r^\epsilon) \pm i \frac{\epsilon}{2} |\partial r|^2 r^{-\epsilon/2-1}}, \quad (1.1.8)$$

respectively, where $\eta_\lambda = 1 - \chi(r/r_\lambda)$. Here we choose the branch of square root as $\text{Re} \sqrt{w} > 0$ for $w \in \mathbb{C} \setminus (-\infty, 0]$. Let

$$\beta_c = \min\{\rho, \tau, \epsilon', 1 + \epsilon/2\}, \quad \epsilon' = \begin{cases} \epsilon/(1 - \epsilon/2) & \text{for } \epsilon \neq 2, \\ 2 & \text{for } \epsilon = 2. \end{cases}$$

Then we have the radiation condition bounds of the following form.

Theorem 1.1.7. *Suppose Condition 1.1.6, and let $I \subset \mathbb{R}$ be any compact interval. Then for all $\beta \in [0, \beta_c)$ there exists $C > 0$ such that for any $\phi = R(z)\psi$ with $\psi \in f^{-\beta}\mathcal{B}$ and $z \in I_\pm$*

$$\|f^\beta(A \mp a)\phi\|_{\mathcal{B}^*} + \langle p_i f^{2\beta} h^{ij} p_j \rangle_\phi^{1/2} \leq C \|f^\beta \psi\|_{\mathcal{B}}, \quad (1.1.9)$$

respectively.

By Theorem 1.1.4 and Theorem 1.1.7 we obtain the limiting absorption principle.

Corollary 1.1.8. *Suppose Condition 1.1.6 and let $I \subset \mathbb{R}$ be any compact interval. For any $s > 1/2$ and $\omega \in (0, \min\{s - 1/2, \beta_c\})$ there exists $C > 0$ such that for any $z, z' \in I_+$ or $z, z' \in I_-$*

$$\begin{aligned} \|R(z) - R(z')\|_{\mathcal{B}(\mathcal{H}_s, \mathcal{H}_{-s})} &\leq C |z - z'|^\omega, \\ \|r^{-\epsilon/2} p R(z) - r^{-\epsilon/2} p R(z')\|_{\mathcal{B}(\mathcal{H}_s, \mathcal{H}_{-s})} &\leq C |z - z'|^\omega. \end{aligned} \quad (1.1.10)$$

In particular, there exist uniform limits

$$\begin{aligned} R(\lambda \pm i0) &:= \lim_{I_\pm \ni z \rightarrow \lambda} R(z), \\ r^{-\epsilon/2} p R(\lambda \pm i0) &:= \lim_{I_\pm \ni z \rightarrow \lambda} r^{-\epsilon/2} p R(z), \end{aligned} \quad (1.1.11)$$

in the norm topology of $\mathcal{B}(\mathcal{H}_s, \mathcal{H}_{-s})$. Moreover, these limits belong to $\mathcal{B}(\mathcal{B}, \mathcal{B}^*)$.

Combining Theorem 1.1.7 and Corollary 1.1.8 we obtain the radiation condition bounds for real spectral parameters.

Corollary 1.1.9. *Suppose Condition 1.1.6, and let $I \subset \mathbb{R}$ be any compact interval. Then for all $\beta \in [0, \beta_c)$ there exists $C > 0$ such that for any $\phi = R(\lambda \pm i0)\psi$ with $\psi \in f^{-\beta}\mathcal{B}$ and $\lambda \in I$*

$$\|f^\beta(A \mp a)\phi\|_{\mathcal{B}^*} + \langle p_i f^{2\beta} h^{ij} p_j \rangle_\phi^{1/2} \leq C \|f^\beta \psi\|_{\mathcal{B}}, \quad (1.1.12)$$

respectively.

Finally we obtain the Sommerfeld uniqueness result.

Corollary 1.1.10. *Suppose Condition 1.1.6, and let $\lambda \in \mathbb{R}$, $\phi \in L_{\text{loc}}^2$ and $\psi \in f^{-\beta}\mathcal{B}$ with $\beta \in [0, \beta_c)$. Then $\phi = R(\lambda \pm i0)\psi$ holds if and only if both of the following conditions hold:*

- (i) $(H - \lambda)\phi = \psi$ in the distributional sense.
- (ii) $\phi \in f^\beta \mathcal{B}^*$ and $(A \mp a)\phi \in f^{-\beta} \mathcal{B}_0^*$.

As far as the author knows, there seem to be no literature on Rellich's theorem or the LAP boundedness for repulsive Hamiltonians so far, and our results are new. In particular, by setting the Agmon-Hörmander spaces using the function f of (1.1.2), even for the case of $\epsilon = 2$ we obtain the above results.

The spectral theory for repulsive Hamiltonians was also studied by [BCHM]. However they did not prove the absence of $\mathcal{B}_0^*$ -eigenfunctions or the limiting absorption principle bounds such as Theorem 1.1.4. Moreover as for the decay rate of perturbation at infinity, our conditions are considerably weaker for $\epsilon \neq 2$. In this sense, our results are stronger than theirs.

To prove the theorems and the corollaries we apply a new commutator argument with some weight inside invented recently by [IS]. A feature of this argument is a choice of the conjugate operator A . We choose A to be a generator of some radial flow, not of dilations or translations (see Section 1.2). Although by using a new conjugate operator we cannot bound the Mourre-type commutator from below by a uniform positive constant, we can bound it from below by a function which has a weaker positivity in the sense that it decays at infinity. However such a weaker positivity yields a stronger result such as Theorem 1.1.2 than the absence of L^2 -eigenfunctions of H . In [BCHM], to use Mourre theory they introduced a new conjugate operator by using the pseudo-differential operator, but we do not use Mourre theory or pseudo-differential operators. Due to this our argument is simpler than theirs.

In case $\epsilon = 0$, there has been an extensive amount of literature on spectral theory (e.g. [Ag, FH, FHH2O, H $\ddot{o}$, IJ, IS, Iso]). As for the case $\epsilon = 2$, Ishida studied inverse scattering problem in [Ishi] and borderline of the short-range condition in [Ishi2]. Moreover Finster and Isidro discussed the L^p -spectrum in [FI]. Our setting excludes $\epsilon > 2$, however, Matsumoto, Kakazu and Nagamine studied eigenvalue problems for $\epsilon > 2$ in [MKN]. Skibsted dealt with the limiting absorption principle bounds and the limiting absorption principle for attractive Hamiltonians in [Ski], whereas we considered the case of repulsive Hamiltonians. We also mention recent works related to the repulsive potentials. Josef studied in [J] the properties of spectrum of two-dimensional Pauli operator with repulsive potential. Lakaev studied in [L1, L2] eigenvalue problem for discrete Schrödinger operator with repulsive potential on the two-dimensional lattice $\mathbb{Z}^2$.

In Subsection 1.1.3 below, we see the classical orbit of the particles moving in repulsive electric field, and the suitability of the use of the function f . In Subsection 1.1.4, we verify existence of a generalized eigenfunction in $\mathcal{B}^*$. In Section 1.2, we introduce the conjugate operator A and show that A is the generator of a strongly continuous one-parameter unitary group of some radial flow. In addition, we introduce commutators with some weight inside and discuss its properties. In Section 1.3, we prove Theorem 1.1.2. The proof consists of two ingredients that are typical in such a topic: a priori super-exponential decay estimate and the absence of super-exponentially decaying eigenfunctions. In Subsection 1.4.1, we prove Theorem 1.1.4 by using a contradiction. In Subsection 1.4.2, we prove Theorem 1.1.7 and corollaries 1.1.8-1.1.10. In the proofs of these results commutator estimates play major role.

1.1.3. Classical orbit. In this subsection we consider the classical orbit on the non-perturbed spherically symmetric repulsive Hamiltonians

$$H = \frac{1}{2}p^2 - |x|^\epsilon.$$

Then the Hamilton equation is given by

$$\dot{x}(t) = p(t), \quad \dot{p}(t) = \epsilon|x(t)|^{\epsilon-2}x(t).$$

This yields the following equation:

$$\ddot{x}(t) = \epsilon|x(t)|^{\epsilon-2}x(t). \quad (1.1.13)$$

Here we denote the derivatives of x, p in time variable t by dots (it is well-known as the Newton's notation). Let us consider the solution to (1.1.13). If $\epsilon = 2$, we can compute explicitly and have

$$x(t) = \frac{1}{2} \left(x(0) + \frac{1}{\sqrt{2}} \dot{x}(0) \right) e^{\sqrt{2}t} + \frac{1}{2} \left(x(0) - \frac{1}{\sqrt{2}} \dot{x}(0) \right) e^{-\sqrt{2}t}.$$

Thus, in general, $x(t)$ grows exponentially as $t \rightarrow \infty$. As for the case of $0 < \epsilon < 2$, we consider the solution of the form

$$x(t) = t^s \omega,$$

where $\omega \in \mathbb{R}^d$ is constant. By substituting it into (1.1.13) we obtain $s = 1/(1 - \epsilon/2)$. Therefore it is expected that $x(t)$ has the order of $t^{1/(1-\epsilon/2)}$ as $t \rightarrow \infty$, in general. By these observations if we define the new position function

$$y(t) = \begin{cases} |x(t)|^{1-\epsilon/2} (x(t)/|x(t)|) & \text{for } 0 < \epsilon < 2, \\ \log |x(t)| (x(t)/|x(t)|) & \text{for } \epsilon = 2, \end{cases}$$

we have $|y(t)| = \mathcal{O}(t)$ as $t \rightarrow \infty$ similar to the case of $\epsilon = 0$. Hence it is natural to define the spaces $\mathcal{B}$ and $\mathcal{B}^*$ using the function f rather than r .

1.1.4. Existence of $\mathcal{B}^*$ -eigenfunctions. In this subsection, we show optimality of Theorem 1.1.2. To show that we construct a spherically symmetric solution $\psi(x) = \tilde{\psi}(r) \in \mathcal{B}^*$ of

$$\left(\frac{1}{2} \Delta + |x|^\epsilon \right) \psi = \left(\frac{1}{2} \frac{d^2}{dr^2} + \frac{d-1}{2r} \frac{d}{dr} + r^\epsilon \right) \tilde{\psi} = 0, \quad (1.1.14)$$

where $r = |x|$, $x \in \mathbb{R}^d$.

Recall the Bessel equation:

$$s^2 f''(s) + s f'(s) + (s^2 - \nu^2) f(s) = 0, \quad (1.1.15)$$

and the Bessel function of the first kind $J_\nu(s)$ which is one of the solutions of (1.1.15). In (1.1.15), we let $\nu = (d-2)/(\epsilon+2)$, and change variables by

$$s = \frac{2\sqrt{2}}{\epsilon+2} r^{\epsilon/2+1}, \quad f = r^{d/2-1} \tilde{\psi}.$$

Then we obtain the following equation:

$$\left(\frac{1}{2} \frac{d^2}{dr^2} + \frac{d-1}{2r} \frac{d}{dr} + r^\epsilon \right) \tilde{\psi} = 0.$$

Hence we obtain (1.1.14). Now, we show $\psi \in \mathcal{B}^*$. Since $\nu = (d-2)/(\epsilon+2)$, we can write by definition of the Bessel function

$$J_{(d-2)/(\epsilon+2)}(s) = \sum_{m=0}^{\infty} \frac{(-1)^m}{m! \Gamma(m + (d-2)/(\epsilon+2) + 1)} \left(\frac{s}{2} \right)^{2m + (d-2)/(\epsilon+2)}.$$

It is well-known that $J_{(d-2)/(\epsilon+2)}(s) = \mathcal{O}(s^{-1/2})$ as $s \rightarrow \infty$ (e.g. [K]). Hence we have

$$\begin{aligned}\psi(x) &= \tilde{\psi}(r) = r^{1-d/2} J_{(d-2)/(\epsilon+2)}\left(\frac{2\sqrt{2}}{\epsilon+2} r^{\epsilon/2+1}\right) \\ &= \mathcal{O}(r^{-d/2-\epsilon/4+1/2}), \quad r \rightarrow \infty.\end{aligned}\tag{1.1.16}$$

By the expression

$$\begin{aligned}\psi(x) &= |x|^{1-d/2} J_{(d-2)/(\epsilon+2)}\left(\frac{2\sqrt{2}}{\epsilon+2} |x|^{\epsilon/2+1}\right) \\ &= \sum_{m=0}^{\infty} \frac{(-1)^m}{m! \Gamma(m+(d-2)/(\epsilon+2)+1)} \left(\frac{\sqrt{2}}{\epsilon+2}\right)^{2m+(d-2)/(\epsilon+2)} |x|^{(\epsilon+2)m},\end{aligned}$$

we have

$$\psi \in C^2(\mathbb{R}^d).\tag{1.1.17}$$

(1.1.16) and (1.1.17) imply $\psi \in \mathcal{B}^*$. Therefore ψ is a $\mathcal{B}^*$ -eigenfunction for $H = -\frac{1}{2}\Delta - |x|^\epsilon$ with generalized eigenvalue $\lambda = 0$.

1.2. Preliminaries

In this section we are going to prepare some lemmas which are the basic tools to prove the our results stated in Section 1.1.

1.2.1. Unitary group and generator. Let

$$y : \mathbb{R} \times \mathbb{R}^d \rightarrow \mathbb{R}^d, \quad (t, x) \mapsto y(t, x) = \exp(t\partial f)(x),$$

be the maximal flow generated by the gradient vector field ∂f . Note that by definition it satisfies

$$\partial_t y^i(t, x) = (\partial f)^i(y(t, x)), \quad y(0, x) = x.$$

We define $T(t) : \mathcal{H} \rightarrow \mathcal{H}$, $t \in \mathbb{R}$, by

$$\begin{aligned}(T(t)\psi)(x) &= J(t, x)^{1/2} \psi(y(t, x)) \\ &= \exp\left(\int_0^t \frac{1}{2}(\Delta f)(y(s, x)) ds\right) \psi(y(t, x)),\end{aligned}\tag{1.2.1}$$

where $J(t, \cdot)$ is the Jacobian of the mapping $y(t, \cdot) : \mathbb{R}^d \rightarrow \mathbb{R}^d$. We can easily verify the equivalence of the two expressions in (1.2.1) by the following identity:

$$\begin{aligned}\partial_t [\log J(t, x)^2] &= 2J(t, x)^{-1} \partial_t J(t, x) \\ &= 2J(t, x)^{-1} (\Delta f)(y(t, x)) J(t, x) = 2(\Delta f)(y(t, x)).\end{aligned}$$

Now it follows by the upper expression of (1.2.1) that for any $\psi \in \mathcal{H}$

$$\|T(t)\psi\|^2 = \int_{\mathbb{R}^d} |\psi(y(t, x))|^2 J(t, x) dx = \|\psi\|^2.$$

and hence $T(t)$, $t \in \mathbb{R}$, forms a strongly continuous one-parameter unitary group.

Next we investigate the generator A_0 of group $T(t)$, $t \in \mathbb{R}$. By definition

$$\begin{aligned}\mathcal{D}(A_0) &= \{\psi \in \mathcal{H} \mid \lim_{t \rightarrow 0} (it)^{-1} (T(t)\psi - \psi) \text{ exists in } \mathcal{H}\}, \\ A_0\psi &= \lim_{t \rightarrow 0} (it)^{-1} (T(t)\psi - \psi) \quad \text{for } \psi \in \mathcal{D}(A_0).\end{aligned}$$

By the Stone theorem the generator A_0 is self-adjoint on $\mathcal{H}$. It is easy to verify that $C_0^\infty(\mathbb{R}^d) \subset \mathcal{D}(A_0)$, and that $T(t)$ preserves $C_0^\infty(\mathbb{R}^d)$. Hence by [RS, Theorem X.49] the space $C_0^\infty(\mathbb{R}^d)$ is a core for A_0 . It is also clear by definition that on $C_0^\infty(\mathbb{R}^d)$ the

generator A_0 and maximal differential operator A coincides, and therefore they are actually the identical operators:

$$\mathcal{D}(A_0) = \mathcal{D}(A), \quad A_0 = A = \operatorname{Re} p^f = \frac{1}{2} (p^f + (p^f)^*).$$

As for the domains $\mathcal{D}(H)$ and $\mathcal{D}(A)$ we have the following lemma.

Lemma 1.2.1. *Let $H^2(\mathbb{R}^d)$ be the Sobolev space of second order, and set*

$$H_{\text{comp}}^2(\mathbb{R}^d) = \{\psi \in H^2(\mathbb{R}^d) \mid \text{supp } \psi \text{ is compact}\}.$$

Then the following inclusion relations hold.

$$H_{\text{comp}}^2(\mathbb{R}^d) \subset \mathcal{D}(H) \subset \mathcal{D}(A). \quad (1.2.2)$$

Proof. First we prove $H_{\text{comp}}^2(\mathbb{R}^d) \subset \mathcal{D}(H)$. Let $\psi \in H_{\text{comp}}^2(\mathbb{R}^d)$ and $\text{supp } \psi \subseteq K$, and set

$$K_1 = \left\{ x \in \mathbb{R}^d \mid \inf_{y \in K} |x - y| \leq 1 \right\}.$$

Then there exists $\{\psi_n\} \subset C_0^\infty(\mathbb{R}^d)$ such that

$$\text{supp } \psi_n \subset K_1, \quad \|\psi_n - \psi\| + \|p^2(\psi_n - \psi)\| \rightarrow 0 \quad \text{as } n \rightarrow \infty. \quad (1.2.3)$$

By (1.2.3) and Condition 1.1.1 we can estimate as follows.

$$\|H\psi_n - H\psi\| + \|\psi_n - \psi\| \leq \frac{1}{2} \|p^2(\psi_n - \psi)\| + C_K \|\psi_n - \psi\| \rightarrow 0 \quad \text{as } n \rightarrow \infty.$$

This implies $\psi \in \mathcal{D}(H)$.

Now we prove $\mathcal{D}(H) \subset \mathcal{D}(A)$. Let us discuss similarly to [Sig]. Let $\psi \in \mathcal{D}(H)$. It suffices to show that $r^{-\epsilon/2}\psi \in \mathcal{D}(p^r) = \mathcal{D}((\partial r)^j p_j)$. We choose $\eta \in C_0^\infty(\mathbb{R}^d)$ such that for any multi-index α

$$0 \leq \eta \leq 1, \quad |\partial^\alpha \eta| \leq C_\alpha, \quad C_\alpha > 0,$$

and we let $\omega = \eta r^{-\epsilon/2}\psi$. Then we have $\omega \in \mathcal{D}(p^2)$ and obtain the following estimate:

$$\|(\partial r)^j p_j \omega\|^2 \leq C_1 \|p\omega\|^2 = 2C_1 \langle H \rangle_\omega + 2C_1 \langle |x|^\epsilon - q \rangle_\omega. \quad (1.2.4)$$

We estimate the first term of (1.2.4) by

$$\begin{aligned} 2C_1 \langle \omega, H\omega \rangle &= 2C_1 \langle \psi, \eta r^{-\epsilon} \eta H\psi \rangle + 2C_1 \operatorname{Re} \langle \psi, \eta r^{-\epsilon/2} [H, \eta r^{-\epsilon/2}] \psi \rangle \\ &\leq C_2 \|\psi\| \|H\psi\| + \|\partial \eta r^{-\epsilon/2} \psi\|^2 \\ &\leq C_3 \|\psi\|^2 + C_3 \|H\psi\|^2 < \infty. \end{aligned} \quad (1.2.5)$$

Using Condition 1.1.1 we can estimate the second term of (1.2.4) as

$$\langle |x|^\epsilon - q \rangle_\omega \leq C_4 \|\psi\|^2 < \infty. \quad (1.2.6)$$

Hence by substituting (1.2.5) and (1.2.6) into (1.2.4) we obtain

$$\|(\partial r)^j p_j \omega\| \leq C_5.$$

It provides

$$\|\eta (\partial r)^j p_j r^{-\epsilon/2} \psi\| \leq C_6.$$

Hence by Lebesgue's monotone convergence theorem we have

$$\|(\partial r)^j p_j r^{-\epsilon/2} \psi\| \leq C_6.$$

We are done. □

1.2.2. Commutators with weight inside. We consider commutators with a weight Θ inside:

$$[H, iA]_{\Theta} := i(H\Theta A - A\Theta H).$$

Let $\Theta = \Theta(r)$ be a non-negative smooth function with bounded derivatives, i.e. if we denote its derivatives in r by primes such as Θ' , then Θ satisfies

$$\Theta \geq 0, \quad |\Theta^{(k)}| \leq C_k, \quad k = 0, 1, 2, \dots \quad (1.2.7)$$

We first define the quadratic form $[H, iA]_{\Theta}$ on $C_0^\infty(\mathbb{R}^d)$, and then extend it to $\mathcal{D}(A)$ according to the following lemma.

Lemma 1.2.2. *Suppose Condition 1.1.1, and let Θ be a non-negative smooth function with bounded derivatives (1.2.7). Then, as quadratic forms on $C_0^\infty(\mathbb{R}^d)$,*

$$\begin{aligned} [H, iA]_{\Theta} &= p_j(\partial^2 f)^{jk}\Theta p_k + (p^f)^* r^{\epsilon/2} \Theta' p^f + \frac{1}{2} \operatorname{Re}((\Delta f)\Theta p_i \delta^{ij} p_j) \\ &\quad - \frac{1}{2} p_i (\Delta f)\Theta \delta^{ij} p_j - \frac{1}{2} \operatorname{Im}(r^{-\epsilon/2}(\partial|\partial r|^2)^j \Theta' p_j) - \operatorname{Im}(2q_2 \Theta p^f) \\ &\quad - \operatorname{Re}(r^{-\epsilon/2}|\partial r|^2 \Theta' H) + \epsilon(\partial f)^k |x|^{\epsilon-2} x_k \Theta + q_{\Theta} \\ &\quad - \frac{1}{4} r^{-\epsilon/2} |\partial r|^4 \Theta''; \\ q_{\Theta} &= -(\partial^f q_1)\Theta + q_2(\Delta f)\Theta + r^{-\epsilon/2} |\partial r|^2 q_2 \Theta' - \frac{\epsilon}{8} r^{-\epsilon/2-2} |\partial r|^4 \Theta' \\ &\quad + \frac{\epsilon}{8} r^{-1} |\partial r|^2 (\Delta f)\Theta' + \frac{\epsilon}{8} r^{-1} (\partial^f |\partial r|^2)\Theta' + \frac{\epsilon}{8} r^{-\epsilon/2-1} |\partial r|^4 \Theta'' \\ &\quad - \frac{1}{4} (\partial^f |\partial r|^2)\Theta'' - \frac{1}{4} |\partial r|^2 (\Delta f)\Theta''. \end{aligned} \quad (1.2.8)$$

In particular, if Θ has a compact support, by the Cauchy-Schwarz inequality $[H, iA]_{\Theta}$ restricted to $C_0^\infty(\mathbb{R}^d)$ extends to a bounded form on $\mathcal{D}(A)$. Here we regard $\mathcal{D}(A)$ as the Banach space with a graph norm.

Remark 1.2.3. Since we use a new conjugate operator (1.1.5), the first term on the right-hand side of (1.2.8) does not have positivity (cf. (1.3.3)). Therefore, we cannot bound the term below by zero. However, by using the positivity of the second and the eighth terms on the same side of (1.2.8) we can cancel the negativity from the first, and then positivity is recovered. Although this positivity decays at infinity, it provides a strong result which is the absence of $\mathcal{B}_0^*$ -eigenfunctions.

Proof of Lemma 1.2.2. By (1.1.6) we can compute as

$$\begin{aligned} [H, iA]_{\Theta} &= \frac{1}{2i} \{p_i p_k (\partial f)^k \Theta \delta^{ij} p_j - p_i \delta^{ij} \Theta (\partial f)^k p_k p_j\} + p_j (\partial^2 f)^{jk} \Theta p_k \\ &\quad + (p^f)^* r^{\epsilon/2} \Theta' p^f + \frac{1}{2} \operatorname{Re}((\Delta f)\Theta p_i \delta^{ij} p_j) + r^{-\epsilon/2} |\partial r|^2 \Theta' |x|^{\epsilon} \\ &\quad + \epsilon(\partial f)^k |x|^{\epsilon-2} x_k \Theta - r^{-\epsilon/2} |\partial r|^2 \Theta' q_1 - (\partial^f q_1)\Theta - \operatorname{Im}(2q_2 \Theta A) \\ &= p_j (\partial^2 f)^{jk} \Theta p_k + (p^f)^* r^{\epsilon/2} \Theta' p^f + \frac{1}{2} \operatorname{Re}((\Delta f)\Theta p_i \delta^{ij} p_j) \\ &\quad - \frac{1}{2} p_i (\Delta f)\Theta \delta^{ij} p_j - \frac{1}{2} p_i r^{-\epsilon/2} |\partial r|^2 \Theta' \delta^{ij} p_j + r^{-\epsilon/2} |\partial r|^2 \Theta' |x|^{\epsilon} \\ &\quad + \epsilon(\partial f)^k |x|^{\epsilon-2} x_k \Theta - r^{-\epsilon/2} |\partial r|^2 \Theta' q_1 - (\partial^f q_1)\Theta \\ &\quad - \operatorname{Im}(2q_2 \Theta p^f) + q_2(\Delta f)\Theta. \end{aligned} \quad (1.2.9)$$

We combine the fifth, sixth and eighth terms of (1.2.9) as follows.

$$\begin{aligned}
& -\frac{1}{2}p_i r^{-\epsilon/2} |\partial r|^2 \Theta' \delta^{ij} p_j + r^{-\epsilon/2} |\partial r|^2 \Theta' |x|^\epsilon - r^{-\epsilon/2} |\partial r|^2 \Theta' q_1 \\
& = -\operatorname{Re} \left(r^{-\epsilon/2} |\partial r|^2 \Theta' H \right) - \frac{1}{2} \operatorname{Im} \left(\left(\partial |\partial r|^2 r^{-\epsilon/2} \Theta' \right)^j p_j \right) + r^{-\epsilon/2} |\partial r|^2 q_2 \Theta' \\
& = -\operatorname{Re} \left(r^{-\epsilon/2} |\partial r|^2 \Theta' H \right) + r^{-\epsilon/2} |\partial r|^2 q_2 \Theta' - \frac{1}{2} \operatorname{Im} \left(r^{-\epsilon/2} \left(\partial |\partial r|^2 \right)^j \Theta' p_j \right) \\
& \quad + \frac{\epsilon}{8} r^{-1} \left(\partial^f |\partial r|^2 \right) \Theta' - \frac{\epsilon}{8} |\partial r|^4 r^{-\epsilon/2-2} \Theta' + \frac{\epsilon}{8} r^{-1} |\partial r|^2 (\Delta f) \Theta' \\
& \quad + \frac{\epsilon}{8} |\partial r|^4 r^{-\epsilon/2-1} \Theta'' - \frac{1}{4} \left(\partial^f |\partial r|^2 \right) \Theta'' - \frac{1}{4} r^{-\epsilon/2} |\partial r|^2 (\Delta f) \Theta'' \\
& \quad - \frac{1}{4} r^{-\epsilon/2} |\partial r|^4 \Theta'''.
\end{aligned} \tag{1.2.10}$$

By (1.2.9) and (1.2.10) we have the expression (1.2.8).

The boundedness of $[H, iA]_\Theta$ as a quadratic form on $\mathcal{D}(A)$ follows from (1.2.2), (1.2.8) and compactness of $\operatorname{supp} \Theta$. $\square$

On the other hand, throughout the thesis we shall use the notation

$$\operatorname{Im}(A\Theta H) = \frac{1}{2i}(A\Theta H - H\Theta A)$$

as a quadratic form defined on $\mathcal{D}(H)$, i.e. for $\psi \in \mathcal{D}(H)$

$$\langle \operatorname{Im}(A\Theta H) \rangle_\psi = \frac{1}{2i} (\langle A\psi, \Theta H\psi \rangle - \langle H\psi, \Theta A\psi \rangle).$$

Note that by the embedding (1.2.2) the above quadratic form is well-defined. Obviously the quadratic forms $[H, iA]_\Theta$ and $2 \operatorname{Im}(A\Theta H)$ coincide on $C_0^\infty(\mathbb{R}^d)$, and hence we obtain

$$[H, iA]_\Theta = 2 \operatorname{Im}(A\Theta H) \quad \text{on } \mathcal{D}(H), \tag{1.2.11}$$

if Θ is compactly supported. In fact, by the Faris-Lavine theorem for any $\psi \in \mathcal{D}(H)$ there exists $\{\psi_n\} \subset C_0^\infty(\mathbb{R}^d)$ such that

$$\|\psi - \psi_n\| + \|H(\psi - \psi_n)\| \rightarrow 0 \quad \text{as } n \rightarrow \infty.$$

Therefore we obtain

$$\langle [H, iA]_\Theta \rangle_\psi = \lim_{n \rightarrow \infty} \langle [H, iA]_\Theta \rangle_{\psi_n} = \lim_{n \rightarrow \infty} \langle 2 \operatorname{Im}(A\Theta H) \rangle_{\psi_n} = \langle 2 \operatorname{Im}(A\Theta H) \rangle_\psi.$$

1.3. Rellich's theorem

Throughout the section we suppose Condition 1.1.1. The proof of Theorem 1.1.2 consists of two steps, a priori super-exponential decay estimates and the absence of super-exponentially decaying eigenfunctions. Obviously, Theorem 1.1.2 follows immediately as a combination of the following propositions.

Proposition 1.3.1. *Let $\lambda \in \mathbb{R}$. If a function $\phi \in \mathcal{B}_0^*$ satisfies that*

$$(H - \lambda)\phi = 0,$$

in the distributional sense, then $e^{\alpha r} \phi \in \mathcal{B}_0^$ for any $\alpha \geq 0$.*

Proposition 1.3.2. *Let $\lambda \in \mathbb{R}$. If a function $\phi \in \mathcal{B}_0^*$ satisfies that*

- (1) $(H - \lambda)\phi = 0$ in the distributional sense,
- (2) $e^{\alpha r} \phi \in \mathcal{B}_0^*$ for any $\alpha \geq 0$,

then $\phi(x) = 0$ in $\mathbb{R}^d$.

We prove Propositions 1.3.1 and 1.3.2 in Subsections 1.3.1 and 1.3.2, respectively. The proofs are quite similar to each other, and both are dependent on commutator estimates with particular forms of weights inside.

Now, using the function $\chi \in C^\infty(\mathbb{R})$ of (1.1.1), we define $\chi_n, \bar{\chi}_n, \chi_{m,n} \in C^\infty(\mathbb{R}^d)$ for $n > m \geq 0$ by

$$\chi_n = \chi(f/2^n), \quad \bar{\chi}_n = 1 - \chi_n, \quad \chi_{m,n} = \bar{\chi}_m \chi_n,$$

and let us introduce the regularized weights

$$\Theta = \Theta_{m,n,\nu}^{\alpha,\delta} = \chi_{m,n} e^\theta; \quad n > m \geq 0, \quad (1.3.1)$$

with exponents

$$\theta = \theta_\nu^{\alpha,\delta} = 2\alpha \int_0^r (1 + s/2^\nu)^{-1-\delta} ds; \quad \alpha \geq 0, \delta > 0, \nu \geq 0.$$

We denote the derivatives of Θ and θ in r by primes, e.g.,

$$\theta' = 2\alpha\theta_0^{-1-\delta}, \quad \theta'' = -2(1+\delta)\alpha 2^{-\nu}\theta_0^{-2-\delta}; \quad \theta_0 = 1 + r/2^\nu.$$

In particular, since $2^{-\nu}\theta_0^{-1} \leq r^{-1}$, we have

$$|\theta^{(k)}| \leq C_{\delta,k} \alpha r^{1-k} \theta_0^{-1-\delta}; \quad k = 1, 2, \dots$$

Note that here we can use a slightly simpler exponent θ than that from [IS], and, accordingly, the proofs get slightly simpler. This is because our Hamiltonian has a repulsive property due to the potential term $-|x|^\epsilon$.

1.3.1. A priori super-exponential decay estimates. In this subsection we prove Proposition 1.3.1. In the proof, the following commutator estimate plays a major role.

Lemma 1.3.3. *Let $\lambda \in \mathbb{R}$, and fix any $\delta \in (0, \min\{1, \rho, \epsilon'\})$ and $\alpha > 0$. Then there exist $c, C > 0$ and $n_0 \geq 0$ such that uniformly in $n > m \geq n_0$ and $\nu \geq n_0$, as quadratic form on $\mathcal{D}(H)$,*

$$\begin{aligned} \operatorname{Im}(A\Theta(H - \lambda)) &\geq cf^{-1}\theta_0^{-\delta}\Theta - Cf^{-1}(\chi_{m-1,m+1}^2 + \chi_{n-1,n+1}^2)e^\theta \\ &\quad - \operatorname{Re}(\gamma(H - \lambda)), \end{aligned} \quad (1.3.2)$$

where $\gamma = \gamma_{n,m,\nu}$ is a certain function satisfying $\operatorname{supp} \gamma \subseteq \operatorname{supp} \chi_{m,n}$ and $|\gamma| \leq Ce^\theta$.

Proof. Let $\lambda \in \mathbb{R}$ and fix any $\delta \in (0, \min\{1, \rho, \epsilon'\})$ and $\alpha > 0$. We choose $n_0 \geq 1$ so that $r(x) = |x|$ on $\operatorname{supp} \Theta$. Then we have the following formulae (cf. (1.1.2)):

$$\begin{aligned} |\partial r|^2 &= 1, \quad (\partial^2 f)^{jk} = r^{-\epsilon/2-1} \delta^{jk} - \left(\frac{\epsilon}{2} + 1\right) r^{-\epsilon/2-1} (\partial r)^j (\partial r)^k, \\ (\partial r)^j &= x^j r^{-1}, \quad \Delta f = (d - \frac{\epsilon}{2} - 1) r^{-\epsilon/2-1}, \quad \Delta r = (d - 1) r^{-1}. \end{aligned} \quad (1.3.3)$$

In the latter half of the proof, we will retake $n_0 \geq 1$ larger, if necessary. By Lemma 1.2.2, (1.2.11), (1.3.3) and the Cauchy-Schwarz inequality we can estimate as

$$\begin{aligned} &2 \operatorname{Im}(A\Theta(H - \lambda)) \\ &\geq p_j r^{-\epsilon/2-1} \Theta \delta^{jk} p_k - \left(\frac{\epsilon}{2} + 1\right) p_j (\partial r)^j r^{-\epsilon/2-1} \Theta (\partial r)^k p_k + \epsilon r^{\epsilon/2-1} \Theta \\ &\quad - C_1 f^{-1-\min\{1, \rho, \epsilon'\}} \Theta + p_j (\partial r)^j r^{-\epsilon/2} \Theta' (\partial r)^k p_k - \frac{4d-3\epsilon-4}{8} r^{-\epsilon/2-1} \theta'^2 \Theta \\ &\quad - \frac{1}{4} r^{-\epsilon/2} \theta'^3 \Theta - \frac{3}{4} r^{-\epsilon/2} \theta' \theta'' \Theta - \operatorname{Re}(r^{-\epsilon/2} \Theta'(H - \lambda)) - C_1 \Theta \end{aligned}$$

$$\begin{aligned}
&\geq p_j r^{-\epsilon/2-1} \theta_0^{-\delta} \Theta \delta^{jk} p_k - \frac{\epsilon}{2} p_j (\partial r)^j r^{-\epsilon/2-1} \Theta (\partial r)^k p_k + \epsilon r^{\epsilon/2-1} \Theta \\
&\quad + p_j (\partial r)^j (\theta' - r^{-1} \theta_0^{-\delta}) r^{-\epsilon/2} \Theta (\partial r)^k p_k - C_1 f^{-1-\min\{1, \rho, \epsilon'\}} \Theta \\
&\quad - \frac{4d-3\epsilon-4}{8} r^{-\epsilon/2-1} \theta'^2 \Theta - \frac{1}{4} r^{-\epsilon/2} \theta'^3 \Theta - \frac{3}{4} r^{-\epsilon/2} \theta' \theta'' \Theta \\
&\quad - \operatorname{Re} (r^{-\epsilon/2} \Theta' (H - \lambda)) - C_1 Q.
\end{aligned} \tag{1.3.4}$$

Here we introduced for simplicity

$$\begin{aligned}
Q &= f^{-1-\min\{1, \rho, \epsilon'\}} \Theta + (|\chi'_{m,n}| + |\chi''_{m,n}| + |\chi'''_{m,n}|) e^\theta \\
&\quad + p_i \left(r^{-\epsilon} f^{-1-\min\{1, \rho, \epsilon'\}} \Theta + r^{-\epsilon} |\chi'_{m,n}| e^\theta \right) \delta^{ij} p_j.
\end{aligned} \tag{1.3.5}$$

Let us further compute and estimate the terms on the right-hand side of (1.3.4). Using a general identity holding for any $g \in C^\infty(\mathbb{R}^d)$:

$$p_i g \delta^{ij} p_j = \operatorname{Re} (g p_i \delta^{ij} p_j) + \frac{1}{2} (\Delta g), \tag{1.3.6}$$

we estimate the first term of (1.3.4) by

$$\begin{aligned}
p_j r^{-\epsilon/2-1} \theta_0^{-\delta} \Theta \delta^{jk} p_k &\geq 2 \operatorname{Re} (r^{-\epsilon/2-1} \theta_0^{-\delta} \Theta (H - \lambda)) + 2 r^{\epsilon/2-1} \theta_0^{-\delta} \Theta \\
&\quad + \frac{1}{2} r^{-\epsilon/2-1} \theta_0^{-\delta} \theta'^2 \Theta - C_2 Q.
\end{aligned} \tag{1.3.7}$$

Similarly, we estimate the second term of (1.3.4) by

$$\begin{aligned}
-\frac{\epsilon}{2} p_j (\partial r)^j r^{-\epsilon/2-1} \Theta (\partial r)^k p_k &\geq -\epsilon \operatorname{Re} (r^{-\epsilon/2-1} \Theta (H - \lambda)) - \epsilon r^{\epsilon/2-1} \Theta \\
&\quad - \frac{\epsilon}{4} r^{-\epsilon/2-1} \theta'^2 \Theta - C_3 Q.
\end{aligned} \tag{1.3.8}$$

We combine the fourth, seventh and eighth terms of (1.3.4) as

$$\begin{aligned}
&p_j (\partial r)^j (\theta' - r^{-1} \theta_0^{-\delta}) r^{-\epsilon/2} \Theta (\partial r)^k p_k - \frac{1}{4} r^{-\epsilon/2} \theta'^3 \Theta - \frac{3}{4} r^{-\epsilon/2} \theta' \theta'' \Theta \\
&= (p_j (\partial r)^j + \frac{i}{2} \theta') (\theta' - r^{-1} \theta_0^{-\delta}) r^{-\epsilon/2} \Theta ((\partial r)^k p_k - \frac{i}{2} \theta') \\
&\quad + \frac{i}{2} p_j (\partial r)^j (\theta'^2 - r^{-1} \theta_0^{-\delta} \theta') r^{-\epsilon/2} \Theta - \frac{i}{2} (\theta'^2 - r^{-1} \theta_0^{-\delta} \theta') r^{-\epsilon/2} \Theta (\partial r)^k p_k \\
&\quad - \frac{1}{4} (\theta'^3 - r^{-1} \theta_0^{-\delta} \theta'^2) r^{-\epsilon/2} \Theta - \frac{1}{4} r^{-\epsilon/2} \theta'^3 \Theta - \frac{3}{4} r^{-\epsilon/2} \theta' \theta'' \Theta \\
&\geq (p_j (\partial r)^j + \frac{i}{2} \theta') (\theta' - r^{-1} \theta_0^{-\delta}) r^{-\epsilon/2} \Theta ((\partial r)^k p_k - \frac{i}{2} \theta') \\
&\quad + \frac{2d-\epsilon-2}{4} r^{-\epsilon/2-1} \theta'^2 \Theta - \frac{1}{4} r^{-\epsilon/2-1} \theta_0^{-\delta} \theta'^2 \Theta + \frac{1}{4} r^{-\epsilon/2} \theta' \theta'' \Theta - C_4 Q.
\end{aligned} \tag{1.3.9}$$

Substitute (1.3.7), (1.3.8) and (1.3.9) into (1.3.4), and then it follows that

$$\begin{aligned}
&2 \operatorname{Im} (A \Theta (H - \lambda)) \\
&\geq 2 r^{\epsilon/2-1} \theta_0^{-\delta} \Theta - C_5 f^{-1-\min\{1, \rho, \epsilon'\}} \Theta - \frac{\epsilon}{8} r^{-\epsilon/2-1} \theta'^2 \Theta + \frac{1}{4} r^{-\epsilon/2-1} \theta_0^{-\delta} \theta'^2 \Theta \\
&\quad + \frac{1}{4} r^{-\epsilon/2} \theta' \theta'' \Theta + (p^r + \frac{i}{2} \theta') (\theta' - r^{-1} \theta_0^{-\delta}) r^{-\epsilon/2} \Theta ((p^r)^* - \frac{i}{2} \theta') \\
&\quad - \operatorname{Re} (\{r^{-\epsilon/2} \Theta' - 2 r^{-\epsilon/2-1} \theta_0^{-\delta} \Theta + \epsilon r^{-\epsilon/2-1} \Theta\} (H - \lambda)) - C_5 Q.
\end{aligned} \tag{1.3.10}$$

Using the formula (1.3.6) we rewrite and bound the remainder operator (1.3.5) as

$$\begin{aligned}
Q &\leq C_6 f^{-1-\min\{1, \rho, \epsilon'\}} \Theta + C_6 f^{-1} (\chi_{m-1, m+1}^2 + \chi_{n-1, n+1}^2) e^\theta \\
&\quad + 2 \operatorname{Re} \left(\left\{ r^{-\epsilon} f^{-1-\min\{1, \rho, \epsilon'\}} \Theta + r^{-\epsilon} |\chi'_{m,n}| e^\theta \right\} (H - \lambda) \right).
\end{aligned} \tag{1.3.11}$$

Hence we obtain by (1.3.10) and (1.3.11)

$$\begin{aligned} & \operatorname{Im}(A\Theta(H - \lambda)) \\ & \geq f^{-1}\theta_0^{-\delta}\Theta - C_7 f^{-1-\min\{1,\rho,\epsilon'\}}\Theta + \frac{1}{8}r^{-\epsilon/2}\theta'\theta''\Theta - \frac{\epsilon}{16}r^{-\epsilon/2-1}\theta'^2\Theta \\ & \quad + \frac{1}{8}r^{-\epsilon/2-1}\theta_0^{-\delta}\theta'^2\Theta - C_7 f^{-1}(\chi_{m-1,m+1}^2 + \chi_{n-1,n+1}^2)e^\theta - \operatorname{Re}(\gamma(H - \lambda)) \\ & \quad + \frac{1}{2}(p^r + \frac{i}{2}\theta')(\theta' - r^{-1}\theta_0^{-\delta})r^{-\epsilon/2}\Theta((p^r)^* - \frac{i}{2}\theta'), \end{aligned}$$

where

$$\gamma = \frac{1}{2}r^{-\epsilon/2}\Theta' - r^{-\epsilon/2-1}\theta_0^{-\delta}\Theta + \frac{1}{2}\epsilon r^{-\epsilon/2-1}\Theta + C_5 r^{-\epsilon} f^{-1-\min\{1,\rho,\epsilon'\}}\Theta + C_5 r^{-\epsilon} |\chi'_{m,n}|e^\theta.$$

Now we further restrict parameters. If we choose sufficiently large $n_0 \geq 1$, the term which combined the first term to the fifth term is bounded below uniformly in $n > m \geq n_0$ and $\nu \geq 0$ as

$$\begin{aligned} & \left(f^{-1}\theta_0^{-\delta} - C_7 f^{-1-\min\{1,\rho,\epsilon'\}} + \frac{1}{8}r^{-\epsilon/2}\theta'\theta'' - \frac{\epsilon}{16}r^{-\epsilon/2-1}\theta'^2 + \frac{1}{8}r^{-\epsilon/2-1}\theta_0^{-\delta}\theta'^2 \right) \Theta \\ & \geq c f^{-1}\theta_0^{-\delta}\Theta. \end{aligned}$$

Since

$$\theta' - r^{-1}\theta_0^{-\delta} = (2\alpha(1 + r/2^\nu)^{-1} - r^{-1})\theta_0^{-\delta},$$

by retaking $n_0 \geq 1$ larger, if necessary, the eighth term is non-negative for any $n > m \geq n_0$ and $\nu \geq n_0$. Hence the desired estimate follows. $\square$

Proof of Proposition 1.3.1. Let $\lambda \in \mathbb{R}$ and $\phi \in \mathcal{B}_0^*$ be as in the assertion, and fix any $\delta \in (0, \min\{1, \rho, \epsilon'\})$, $\alpha > 0$ and $n_0 \geq 0$ in agreement with Lemma 1.3.3. For any function ϕ obeying the assumptions of Proposition 1.3.1 we have $\chi_{m,n}\phi \in H_{\text{comp}}^2 \subset \mathcal{D}(H)$ for all $n > m \geq 0$. Note that we may assume $n_0 \geq 3$, so that for all $n > m \geq n_0$

$$\chi_{m-2,n+2}\phi \in \mathcal{D}(H).$$

We evaluate the inequality (1.3.2) in the state $\chi_{m-2,n+2}\phi \in \mathcal{D}(H)$, and then obtain for any $n > m \geq n_0$ and $\nu \geq n_0$

$$\|(f^{-1}\theta_0^{-\delta}\Theta)^{1/2}\phi\|^2 \leq C_m \|\chi_{m-1,m+1}\phi\|^2 + C_\nu 2^{-n} \|\chi_{n-1,n+1}\phi\|^2. \quad (1.3.12)$$

The second term on the right-hand side of (1.3.12) vanishes when $n \rightarrow \infty$ since $\phi \in \mathcal{B}_0^*$, and consequently by Lebesgue's monotone convergence theorem we have

$$\|(\bar{\chi}_m f^{-1}\theta_0^{-\delta}e^\theta)^{1/2}\phi\|^2 \leq C_m \|\chi_{m-1,m+1}\phi\|^2. \quad (1.3.13)$$

Next we let $\nu \rightarrow \infty$ in (1.3.13) invoking again Lebesgue's monotone convergence theorem, and then it follows that

$$f^{-1/2}e^{\alpha r}\phi \in \mathcal{H}.$$

Consequently this implies $e^{\alpha r}\phi \in \mathcal{B}_0^*$ for any $\alpha \geq 0$. Hence we are done. $\square$

1.3.2. Absence of super-exponentially decaying eigenfunctions. In this subsection we prove Proposition 1.3.2. Similarly to the proof of Proposition 1.3.1, commutator estimate plays important role. Due to the assumption (2) of Proposition 1.3.2, we use the simpler weight than that used in Lemma 1.3.3.

Lemma 1.3.4. *Let $\lambda \in \mathbb{R}$ and $\alpha_0 > 0$, and set $\Theta = \chi_{m,n}e^{2\alpha r}$. Then there exist $c, C > 0$ and $n_0 \geq 0$ such that uniformly in $\alpha > \alpha_0$ and $n > m \geq n_0$, as quadratic forms on $\mathcal{D}(H)$,*

$$\begin{aligned} \operatorname{Im}(A\Theta(H - \lambda)) &\geq c\alpha^2 r^{-\epsilon/2-1}\Theta - C\alpha^2 f^{-1}(\chi_{m-1,m+1}^2 + \chi_{n-1,n+1}^2) e^{2\alpha r} \\ &\quad - \operatorname{Re}(\gamma(H - \lambda)), \end{aligned} \quad (1.3.14)$$

where $\gamma = \gamma_{n,m}$ is a certain function satisfying $\operatorname{supp} \gamma \subseteq \operatorname{supp} \chi_{m,n}$ and $|\gamma| \leq C\alpha e^{2\alpha r}$.

Proof. Fix any $\lambda \in \mathbb{R}$. Choose $n_0 \geq 0$ large enough. Then by repeating similar argument of the proof of Lemma 1.3.3, we can estimate uniformly in $\alpha \geq 0$ and $n > m \geq n_0$ as

$$\begin{aligned} &2 \operatorname{Im}(A\Theta(H - \lambda)) \\ &\geq p_j r^{-\epsilon/2-1} \Theta \delta^{jk} p_k + p_j (\partial r)^j (2\alpha - \frac{\epsilon+2}{2} r^{-1}) r^{-\epsilon/2} \Theta (\partial r)^k p_k + \epsilon r^{\epsilon/2-1} \Theta \\ &\quad - C_1 f^{-1-\min\{1,\rho,\epsilon'\}} \Theta - \frac{4d-3\epsilon-4}{2} \alpha^2 r^{-\epsilon/2-1} \Theta - 2\alpha^3 r^{-\epsilon/2} \Theta \\ &\quad - \operatorname{Re}(r^{-\epsilon/2} \Theta'(H - \lambda)) - C_1 Q \\ &\geq (\epsilon + 2) r^{\epsilon/2-1} \Theta - C_2 f^{-1-\min\{1,\rho,\epsilon'\}} \Theta + \alpha^2 r^{-\epsilon/2-1} \Theta \\ &\quad + (p_j (\partial r)^j + i\alpha) (2\alpha - \frac{\epsilon+2}{2} r^{-1}) r^{-\epsilon/2} \Theta ((\partial r)^k p_k - i\alpha) \\ &\quad - \operatorname{Re}(\{r^{-\epsilon/2} \Theta' - 2r^{-\epsilon/2-1} \Theta\} (H - \lambda)) - C_2 Q \\ &\geq (\epsilon + 2) r^{\epsilon/2-1} \Theta - C_3 f^{-1-\min\{1,\rho,\epsilon'\}} \Theta + \alpha^2 r^{-\epsilon/2-1} \Theta - C_3 \alpha^2 r^{-\epsilon/2-1-\min\{1,\rho\}} \Theta \\ &\quad + (p_j (\partial r)^j + i\alpha) (2\alpha - \frac{\epsilon+2}{2} r^{-1}) r^{-\epsilon/2} \Theta ((\partial r)^k p_k - i\alpha) \\ &\quad - C_3 (1 + \alpha^2) f^{-1} (\chi_{m-1,m+1}^2 + \chi_{n-1,n+1}^2) e^{2\alpha r} - 2 \operatorname{Re}(\gamma(H - \lambda)), \end{aligned}$$

where

$$\begin{aligned} Q &= (1 + \alpha^2) r^{-\epsilon/2-1-\min\{1,\rho\}} \Theta + ((1 + \alpha^2) |\chi'_{m,n}| + (1 + \alpha) |\chi''_{m,n}| + |\chi'''_{m,n}|) e^{2\alpha r} \\ &\quad + p_i (r^{-\epsilon} f^{-1-\min\{1,\rho\}} \chi_{m,n} + r^{-\epsilon} |\chi'_{m,n}|) e^{2\alpha r} \delta^{ij} p_j, \\ \gamma &= \frac{1}{2} r^{-\epsilon/2} \Theta' - r^{-\epsilon/2-1} \Theta + C_2 r^{-\epsilon} f^{-1-\min\{1,\rho,\epsilon'\}} \Theta + C_2 r^{-\epsilon} |\chi'_{m,n}| e^{2\alpha r}. \end{aligned}$$

Now let us fix any $\alpha_0 > 0$ and choose sufficiently large $n_0 \geq 0$. Consequently we can easily verify the asserted inequality (1.3.14) uniformly in $\alpha > \alpha_0$ and $n > m \geq n_0$. Hence we are done. $\square$

Proof of Proposition 1.3.2. Let $\lambda \in \mathbb{R}$ and $\phi \in \mathcal{B}_0^*$ be as in the assertion. Fix any $\alpha_0 > 0$, and choose $n_0 \geq 0$ in agreement with Lemma 1.3.4. We may assume that $n_0 \geq 3$, so that for all $n > m \geq n_0$

$$\chi_{m-2,n+2}\phi \in \mathcal{D}(H).$$

Let us evaluate the inequality (1.3.14) in the state $\chi_{m-2,n+2}\phi \in \mathcal{D}(H)$. Then it follows that for any $\alpha > \alpha_0$ and $n > m \geq n_0$

$$\|r^{-\epsilon/4-1/2} \Theta^{1/2} \phi\|^2 \leq C_m \|\chi_{m-1,m+1} e^{\alpha r} \phi\|^2 + C_1 2^{-n} \|\chi_{n-1,n+1} e^{\alpha r} \phi\|^2. \quad (1.3.15)$$

The second term on the right-hand side of (1.3.15) vanishes when $n \rightarrow \infty$, and hence by Lebesgue's monotone convergence theorem we obtain

$$\|\bar{\chi}_m^{1/2} r^{-\epsilon/4-1/2} e^{\alpha r} \phi\|^2 \leq C_m \|\chi_{m-1,m+1} e^{\alpha r} \phi\|^2.$$

By (1.1.2) it hold that, on $\text{supp } \chi_{m-1, m+1}$,

$$r \leq c_m := \begin{cases} \{(1 - \epsilon/2)(2^{m+2} - 1) + 1\}^{1/(1-\epsilon/2)} & \text{for } \epsilon \neq 2, \\ e^{2^{m+2}-1} & \text{for } \epsilon = 2. \end{cases}$$

Therefore we have

$$\|\bar{\chi}_m^{1/2} r^{-\epsilon/4-1/2} e^{\alpha(r-c_m)} \phi\|^2 \leq C_m \|\chi_{m-1, m+1} \phi\|^2. \quad (1.3.16)$$

Now assume $\bar{\chi}_{m+2} \phi \neq 0$. Then the left-hand side of (1.3.16) grows exponentially as $\alpha \rightarrow \infty$ whereas the right-hand side remains bounded. This is a contradiction. Thus $\bar{\chi}_{m+2} \phi \equiv 0$. By invoking the unique continuation property for the second order elliptic operator H (cf. [Wo]) we conclude that $\phi \equiv 0$ globally on $\mathbb{R}^d$. $\square$

1.4. Resolvent properties

From this section, let us consider properties of the resolvent $R(z) = (H - z)^{-1}$. It is well-known that $R(z)$ belong to $\mathcal{B}(\mathcal{H}_s, \mathcal{H}_{-s})$ for $s > 1/2$ (cf. [BCHM]). On the other hand, Theorem 1.1.4 asserts $R(z)$ belong to $\mathcal{B}(\mathcal{B}, \mathcal{B}^*)$. Noting the inclusion relations (1.1.4), we see that (1.1.7) is stronger estimate than the well-known one. We prove Theorem 1.1.4 in Subsection 1.4.1. To obtain the limiting absorption principle we need to prove the radiation condition bounds stated as Theorem 1.1.7. By (1.1.9) we can infer that $R(z)\psi$, ψ in $\mathcal{B}$, oscillate like $\exp\{\pm i\sqrt{2}r^{1+\epsilon/2}/(1+\epsilon/2)\}$ at infinity. We note that the $\mathcal{B}^*$ -eigenfunction ψ constructed in Subsection 1.1.4 has the same behavior. We prove Theorem 1.1.7 and Corollaries 1.1.8-1.1.10 in Subsection 1.4.2. In the proofs of these results commutator estimates play major roles as with the previous section. However in this section we use the weights which depend on the function f , i.e. $\Theta = \Theta(f)$. We denote its derivatives in f by primes such as Θ' . Then we have the following lemma instead of Lemma 1.2.2.

Lemma 1.4.1. *Suppose Condition 1.1.1, and let Θ be a non-negative smooth function with bounded derivatives. Then, as quadratic forms on $C_0^\infty(\mathbb{R}^d)$,*

$$\begin{aligned} [H, iA]_\Theta &= p_j(\partial^2 f)^{jk} \Theta p_k + (p^f)^* \Theta' p^f + \frac{1}{2} \text{Re}((\Delta f) \Theta p_i \delta^{ij} p_j) \\ &\quad - \frac{1}{2} p_i (\Delta f) \Theta \delta^{ij} p_j - \frac{1}{2} \text{Im}((\partial |\partial f|^2)^j \Theta' p_j) - \text{Im}(2q_2 \Theta p^f) \\ &\quad - \text{Re}(|\partial f|^2 \Theta' H) + \epsilon (\partial f)^k |x|^{\epsilon-2} x_k \Theta - (\partial^f q_1) \Theta + q_2 (\Delta f) \Theta \\ &\quad + |\partial f|^2 q_2 \Theta' - \frac{1}{4} (\partial^f |\partial f|^2) \Theta'' - \frac{1}{4} |\partial f|^2 (\Delta f) \Theta'' - \frac{1}{4} |\partial f|^4 \Theta'''. \end{aligned}$$

In particular noting the formulae (1.3.3) and using the Cauchy-Schwarz inequality $[H, iA]_\Theta$ restricted to $C_0^\infty(\mathbb{R}^d)$ extends to a bounded form on $\mathcal{D}(A)$.

Since the proof of Lemma 1.4.1 is quite similar to the proof of Lemma 1.2.2, we omit the details. Note that in this case to extend $[H, iA]_\Theta$ to a bounded form on $\mathcal{D}(A)$ we do not need compactness of $\text{supp } \Theta$. By Lemma 1.4.1 clearly (1.2.11) holds.

1.4.1. Limiting absorption principle bounds. The goal of this subsection is to prove Theorem 1.1.4. The locally uniform LAP boundedness of $R(z)$ is interesting in its own right, and moreover, Theorem 1.1.4 is important to prove the other results. Before proving Theorem 1.1.4, we state and prove key estimates.

1.4.1.1. *Commutator estimate.* We introduce the regularized weight

$$\Theta = \Theta_\nu^\delta = \int_0^{f/2^\nu} (1+s)^{-1-\delta} ds = [1 - (1+f/2^\nu)^{-\delta}] / \delta; \quad \delta > 0, \nu \geq 0. \quad (1.4.1)$$

We can compute its derivatives in f as

$$\Theta' = (1+f/2^\nu)^{-1-\delta}/2^\nu, \quad \Theta'' = -(1+\delta)(1+f/2^\nu)^{-2-\delta}/2^{2\nu}. \quad (1.4.2)$$

Proposition 1.4.2. *Suppose Condition 1.1.1, let $I \subseteq \mathbb{R}$ be any compact interval, and fix any $\delta \in (0, \min\{1, \rho, \epsilon'\})$ in the definition (1.4.1) of Θ . Then there exist $C > 0$ and $n \geq 0$ such that for all $\phi = R(z)\psi$ with $z \in I_\pm$ and $\psi \in \mathcal{B}$ and for all $\nu \geq 0$*

$$\begin{aligned} & \|\Theta^{1/2}\phi\|^2 + \|\Theta^{1/2}A\phi\|^2 + \langle p_j h^{jk} \Theta p_k \rangle_\phi \\ & \leq C (\|\phi\|_{\mathcal{B}^*} \|\psi\|_{\mathcal{B}} + \|A\phi\|_{\mathcal{B}^*} \|\psi\|_{\mathcal{B}} + \|\chi_n \Theta^{1/2}\phi\|^2). \end{aligned} \quad (1.4.3)$$

We first note that Θ defined by (1.4.1) has the following properties.

Lemma 1.4.3. *Suppose Condition 1.1.1, and fix any $\delta > 0$ in the definition (1.4.1) of Θ . Then there exist $c, C, C_k > 0$, $k = 2, 3, \dots$, such that for any $k = 2, 3, \dots$ and uniformly in $\nu \geq 0$*

$$\begin{aligned} c/2^\nu & \leq \Theta \leq \min\{C, f/2^\nu\}, \\ c(\min\{2^\nu, f\})^\delta f^{-1-\delta} \Theta & \leq \Theta' \leq f^{-1} \Theta, \\ 0 & \leq (-1)^{k-1} \Theta^{(k)} \leq C_k f^{-k} \Theta. \end{aligned}$$

Proof. By the definition of Θ in (1.4.1) and expressions of its derivatives in (1.4.2), we can easily verify the asserted estimates except for the first estimate in the second line. As for the remainder estimate, by using the last estimate of the first line and noting the following inequality:

$$(\min\{2^\nu, f\})^\delta f^{-1-\delta} (\min\{2^\nu, f\}/2^\nu) ((1+f/2^\nu)^{1+\delta} 2^\nu) \leq C,$$

we can obtain it. $\square$

The following lemma is a key to prove Theorem 1.1.4.

Lemma 1.4.4. *Suppose Condition 1.1.1, let $I \subset \mathbb{R}$ be any compact interval, and fix any $\delta \in (0, \min\{1, \rho, \epsilon'\})$ in the definition (1.4.1) of Θ . Then there exist $c, C > 0$ and $n \geq 0$ such that uniformly in $z \in I_\pm$ and $\nu \geq 0$, as a quadratic forms on $\mathcal{D}(H)$,*

$$\begin{aligned} \text{Im}(A\Theta(H-z)) & \geq c\Theta' + cA\Theta'A + cp_j h^{jk} \Theta p_k - C\chi_n^2 \Theta \\ & \quad - \text{Re}(\gamma(H-z)), \end{aligned} \quad (1.4.4)$$

where $\gamma = \gamma_{z,\nu}$ is a uniformly bounded complex-valued function: $|\gamma| \leq C$.

Proof. Let I and δ be as in the assertion. Using Lemmas 1.4.1, 1.4.3, (1.2.11), (1.3.3), (1.3.6) and the Cauchy-Schwarz inequality, we can bound uniformly in $z =$

$\lambda \pm i\Gamma \in I_{\pm}$ and $\nu \geq 0$

$$\begin{aligned}
& \operatorname{Im}(A\Theta(H-z)) \\
& \geq \frac{1}{2}p_j r^{-\epsilon/2-1} \delta^{jk} \Theta p_k - \left(\frac{\epsilon}{4} + \frac{1}{2}\right) (p^r)^* r^{-\epsilon/2-1} \Theta p^r + \frac{1}{2}(p^f)^* \Theta' p^f \\
& \quad - \frac{1}{2} \operatorname{Re}(|\partial f|^2 \Theta'(H-z)) \mp \Gamma \operatorname{Re}(A\Theta) + \frac{\epsilon}{2} r^{\epsilon/2-1} \Theta - C_1 Q \\
& \geq \frac{1}{2}p_j r^{-\epsilon/2-1} (\delta^{jk} - (\partial r)^j (\partial r)^k + 2C f^{-1-\rho} \delta^{jk}) \Theta p_k + \frac{1}{2}(p^f)^* \Theta' p^f \\
& \quad - \operatorname{Re}\left(\left(\frac{1}{2}|\partial f|^2 \Theta' + \frac{\epsilon}{2} r^{-\epsilon/2-1} \Theta\right)(H-z)\right) \mp \Gamma \Theta^{1/2} A \Theta^{1/2} - C_2 Q \\
& \geq \frac{1}{4}p_j h^{jk} \Theta p_k + \frac{1}{2}\Theta' + \frac{1}{4}A\Theta' A \mp \Gamma \Theta^{1/2} A \Theta^{1/2} \\
& \quad - \operatorname{Re}\left(\left(\frac{1}{2}|\partial f|^2 \Theta' + \frac{\epsilon}{2} r^{-\epsilon/2-1} \Theta - \frac{1}{2} r^{-\epsilon} \Theta'\right)(H-z)\right) - C_3 Q,
\end{aligned} \tag{1.4.5}$$

where

$$Q = f^{-1-\min\{1, \rho, \epsilon'\}} \Theta + p_j r^{-\epsilon} f^{-1-\min\{1, \rho, \epsilon'\}} \Theta \delta^{jk} p_k.$$

As for the fourth term of (1.4.5), using Lemma 1.4.3 and the Cauchy-Schwarz inequality, and noting the general identity holding for any real functions $g, h \in C^1(\mathbb{R}^d)$:

$$h \operatorname{Re}(gp^2) h = \operatorname{Re}(h^2 gp^2) + \frac{1}{2}(\partial_j h) \delta^{jk} (\partial_k gh),$$

we can obtain the bound as follows.

$$\begin{aligned}
\mp \Gamma \Theta^{1/2} A \Theta^{1/2} &= \mp \Gamma \Theta^{1/2} (\operatorname{Re} p^f) \Theta^{1/2} \\
&\geq -C_4 \Gamma \Theta^{1/2} r^{-\epsilon/2} (H-\lambda) r^{-\epsilon/2} \Theta^{1/2} - C_5 \Gamma \\
&= -C_4 \Gamma \operatorname{Re}(\Theta^{1/2} r^{-\epsilon/2} (H-z) r^{-\epsilon/2} \Theta^{1/2}) \pm C_5 \operatorname{Im}(H-z) \\
&\geq -\operatorname{Re}((C_4 \Gamma r^{-\epsilon} \Theta \pm iC_5)(H-z)) - C_6 Q.
\end{aligned} \tag{1.4.6}$$

Therefore it follows that

$$\begin{aligned}
& \operatorname{Im}(A\Theta(H-z)) \\
& \geq \frac{1}{4}p_j h^{jk} \Theta p_k + \frac{1}{2}\Theta' + \frac{1}{4}A\Theta' A - C_7 Q \\
& \quad - \operatorname{Re}\left(\left(\frac{1}{2}(|\partial f|^2 - r^{-\epsilon}) \Theta' + \frac{\epsilon}{2} r^{-\epsilon/2-1} \Theta + C_4 \Gamma r^{-\epsilon} \Theta \pm iC_5\right)(H-z)\right).
\end{aligned} \tag{1.4.7}$$

By (1.3.6) and Lemma 1.4.3 we can combine and estimate the second and fourth terms of (1.4.7) as, for large $n \geq 0$,

$$\frac{1}{2}\Theta' - C_7 Q \geq \frac{1}{4}\Theta' - C_8 \chi_n^2 \Theta - 2C_7 \operatorname{Re}\left(r^{-\epsilon} f^{-1-\min\{1, \rho, \epsilon'\}} \Theta (H-z)\right). \tag{1.4.8}$$

Hence by substituting (1.4.8) into (1.4.7) and setting

$$\gamma = \frac{1}{2}(|\partial f|^2 - r^{-\epsilon}) \Theta' + \frac{\epsilon}{2} r^{-\epsilon/2-1} \Theta + C_4 \Gamma r^{-\epsilon} \Theta \pm iC_5 - 2C_7 r^{-\epsilon} f^{-1-\min\{1, \rho, \epsilon'\}} \Theta,$$

we can obtain the asserted inequality (1.4.4). $\square$

Proof of Proposition 1.4.2. By Lemma 1.4.4 the assertion follows immediately. $\square$

1.4.1.2. LAP boundedness. Now let us prove Theorem 1.1.4 by Proposition 1.4.2 and contradiction.

Proof of Theorem 1.1.4. Let $I \subset \mathbb{R}$ be any compact interval. For simplicity, we prove the assertion only for the upper sign.

Step 1. First we assume that for $C_1 > 0$ large enough

$$\|\phi\|_{\mathcal{B}^*} \leq C_1 \|\psi\|_{\mathcal{B}}. \tag{1.4.9}$$

Then the bound (1.1.7) holds. In fact, noting the identity

$$\frac{1}{2}r^{-\epsilon}p_j\delta^{jk}p_k\phi = r^{-\epsilon}\psi + r^{-\epsilon}(|x|^\epsilon - q + z)\phi \quad (1.4.10)$$

and Condition 1.1.1, the last term on the left-hand side of (1.1.7) satisfies

$$\|r^{-\epsilon}p_j\delta^{jk}p_k\phi\|_{\mathcal{B}^*} \leq C_2\|\psi\|_{\mathcal{B}}.$$

As for the second and third terms of (1.1.7) we can prove by using Proposition 1.4.2. Fix any $\delta \in (0, \min\{1, \rho, \epsilon'\})$. Then by Proposition 1.4.2 and (1.4.9) there exists $C_3 > 0$ such that for any $\phi = R(z)\psi$ with $z \in I_+$ and $\psi \in \mathcal{B}$ uniformly in $\epsilon_1 \in (0, 1)$ and $\nu \geq 0$

$$\|\Theta^{1/2}A\phi\|^2 + \langle p_j h^{jk} \Theta p_k \rangle_\phi \leq \epsilon_1 \|A\phi\|_{\mathcal{B}^*}^2 + \epsilon_1^{-1} C_3 \|\psi\|_{\mathcal{B}}^2. \quad (1.4.11)$$

For each $\nu \geq 0$ by restricting the integral region to $\{2^\nu \leq f < 2^{\nu+1}\}$, we can bound the first term on the left-hand side of (1.4.11) as

$$\|\Theta^{1/2}A\phi\|^2 \geq 3^{-(1+\delta)/2} 2^{-\nu} \|F_\nu A\phi\|^2. \quad (1.4.12)$$

As for the second term on the same side, by looking at the estimate (1.4.11) for any fixed $\nu \geq 0$, e.g. $\nu = 0$ and using Lemma 1.4.3, we have

$$\langle p_j h^{jk} \Theta p_k \rangle_\phi \geq c_1 \langle p_j h^{jk} p_k \rangle_\phi. \quad (1.4.13)$$

We substitute (1.4.12) and (1.4.13) into (1.4.11), and take the supremum in $\nu \geq 0$. Then we obtain the following inequality.

$$c_2 \|A\phi\|_{\mathcal{B}^*}^2 + c_2 \langle p_j h^{jk} p_k \rangle_\phi \leq 2\epsilon_1 \|A\phi\|_{\mathcal{B}^*}^2 + 2\epsilon_1^{-1} C_3 \|\psi\|_{\mathcal{B}}^2.$$

Therefore by letting $\epsilon_1 \in (0, c_2/2)$ and using (1.1.5) it follows that

$$\|p^f \phi\|_{\mathcal{B}^*} + \langle p_j h^{jk} p_k \rangle_\phi^{1/2} \leq C_4 \|\psi\|_{\mathcal{B}}.$$

Hence we have the bound (1.1.7).

Step 2. By the above argument to prove Theorem 1.1.4 it suffices to show that the bound (1.4.9) holds. Now we assume the opposite. Then there exist $z_k \in I_+$ and $\psi_k \in \mathcal{B}$ such that

$$\lim_{k \rightarrow \infty} \|\psi_k\|_{\mathcal{B}} = 0, \quad \|\phi_k\|_{\mathcal{B}^*} = 1; \quad \phi_k = R(z_k)\psi_k. \quad (1.4.14)$$

We note that the following estimate holds by (1.4.14).

$$\|r^{-\epsilon/2}p\phi_k\|_{\mathcal{B}^*} + \|r^{-\epsilon}p^2\phi_k\|_{\mathcal{B}^*} \leq C_4. \quad (1.4.15)$$

In fact, arguing similarly to Step 1, and using (1.4.14) and Proposition 1.4.2 we obtain

$$\|A\phi_k\|_{\mathcal{B}^*}^2 + \langle p_j h^{jk} p_k \rangle_{\phi_k} \leq C_5. \quad (1.4.16)$$

Then by Condition 1.1.1, (1.1.5), (1.4.10), (1.4.14) and (1.4.16) we have (1.4.15). Now, if necessary, choosing a subsequence and retaking $I \subset \mathbb{R}$ slightly larger we may assume that $z_k \in I_+$ converges to some $z \in I \cup I_+$. Since we have the bounds

$$\|\phi_k\|_{\mathcal{B}^*} \leq \|\phi_k\|_{\mathcal{H}} \leq \|R(z_k)\|_{\mathcal{B}(\mathcal{H})} \|\psi_k\|_{\mathcal{H}} \leq \|R(z_k)\|_{\mathcal{B}(\mathcal{H})} \|\psi_k\|_{\mathcal{B}},$$

if the limit z belongs to I_+ , by taking the limit $k \rightarrow \infty$ and using (1.4.14) we obtain contradict. Hence we have the limit

$$\lim_{k \rightarrow \infty} z_k = z = \lambda \in I. \quad (1.4.17)$$

Let $s > 1/2$. By choosing a further subsequence it is allowed to assume that ϕ_k converges weakly to some $\phi \in \mathcal{H}_{-s}$. Moreover, it holds that ϕ_k actually converges

strongly in $\mathcal{H}_{-s}$. To verify this let us fix $s' \in (1/2, s)$ and $g \in C_0^\infty(\mathbb{R})$ with $g = 1$ on a neighborhood of I , and decompose for any $n \geq 0$

$$\begin{aligned} f^{-s}\phi_k &= f^{-s}g(H)(\chi_n f^s)(f^{-s}\phi_k) + (f^{-s}g(H)f^s)(\bar{\chi}_n f^{s'-s})(f^{-s'}\phi_k) \\ &\quad + f^{-s}(1 - g(H))R(z_k)\psi_k. \end{aligned}$$

By (1.4.14) we see that the last term on the right-hand side converges to 0 in $\mathcal{H}$. Since $f^{-s}g(H)f^s$ is a bounded operator, by choosing $n \geq 0$ sufficiently large the second term can be arbitrarily small in $\mathcal{H}$. As we can see with ease $f^{-s}g(H)$ is a compact operator. Thus, for fixed $n \geq 0$ the first term converges strongly in $\mathcal{H}$. Therefore ϕ_k converges to ϕ in $\mathcal{H}_{-s}$, and then, by using (1.4.15) and (1.3.6) we can see that $r^{-\epsilon/2}p\phi_k$ converges to $r^{-\epsilon/2}p\phi$ in $\mathcal{H}_{-s}$. Hence we have

$$\lim_{k \rightarrow \infty} \phi_k = \phi \quad \text{in } \mathcal{H}_{-s}, \quad \lim_{k \rightarrow \infty} r^{-\epsilon/2}p\phi_k = r^{-\epsilon/2}p\phi \quad \text{in } \mathcal{H}_{-s}. \quad (1.4.18)$$

By (1.4.14), (1.4.17) and (1.4.18) it follows that

$$(H - \lambda)\phi = 0 \quad \text{in the distributional sense.} \quad (1.4.19)$$

In addition, we can obtain $\phi \in \mathcal{B}_0^*$. In fact, let us apply Proposition 1.4.2 with $\delta = 2s - 1 > 0$ to $\phi_k = R(z_k)\psi_k$. Then, taking the limit $k \rightarrow \infty$ and using (1.4.14), (1.4.15), (1.4.18) and Lemma 1.4.3, we have for all $\nu \geq 0$

$$\|\Theta^{1/2}\phi\| \leq \|\chi_n \Theta^{1/2}\phi\| \leq C_5 2^{-\nu/2} \|\chi_n f^{1/2}\phi\|. \quad (1.4.20)$$

By taking the limit $\nu \rightarrow \infty$ in (1.4.20), we can verify $\phi \in \mathcal{B}_0^*$. Therefore by (1.4.19) and Theorem 1.1.2, we have $\phi = 0$. On the other hand, similarly to Step 1 we have

$$1 = \|\phi_k\|_{\mathcal{B}^*}^2 \leq C_6 (\|\psi_k\|_{\mathcal{B}} + \|\chi_n \phi_k\|^2).$$

If it holds that $\phi = 0$, the right-hand side converges to 0 as $k \rightarrow \infty$. However, this is a contradiction. Hence the bound (1.4.9) holds. $\square$

1.4.2. Radiation condition. Our main purpose in this subsection is to prove the radiation condition bounds for complex spectral parameters, or Theorem 1.1.7. Throughout this subsection we suppose Condition 1.1.6, and prove the results only for the upper sign for simplicity.

1.4.2.1. Commutator estimate. We introduced the conjugate operator B as a maximal differential operator

$$B = \operatorname{Re} p^r = \frac{1}{2} (p^r + (p^r)^*),$$

with domain

$$\mathcal{D}(B) = \{\psi \in \mathcal{H} \mid B\psi \in \mathcal{H}\},$$

and set associated asymptotic complex phase b : For $z = \lambda + i\Gamma \in \mathbb{R} \cup \mathbb{R}_+$

$$b = b_z = \eta_\lambda |\partial r| \sqrt{2(z - q_1 + r^\epsilon)} + i \frac{\epsilon}{4} |\partial r|^2 r^{-1}. \quad (1.4.21)$$

We note that the operator B is self-adjoint on $\mathcal{D}(B)$ (cf. [IS]).

Lemma 1.4.5. *Let $I \subset \mathbb{R}$ be any compact interval. Then there exists $C > 0$ such that uniformly in $z \in I \cup I_+$*

$$\begin{aligned} |a| &\leq C, \quad |b| \leq Cr^{\epsilon/2}, \quad \operatorname{Im} a \geq \frac{\epsilon}{2} |\partial r|^2 r^{-\epsilon/2-1}, \quad \operatorname{Im} b \geq \frac{\epsilon}{4} |\partial r|^2 r^{-1}, \\ |\ell^{\bullet j} \partial_j a| &\leq Cr^{-\epsilon/2} f^{-1-\tau}, \quad |p^r b + b^2 - 2|\partial r|^2(z - q_1 + r^\epsilon)| \leq C f^{-1-\min\{\rho, \epsilon', \tau\}}. \end{aligned}$$

Proof. By the definitions of a and b (see (1.1.8), (1.4.21)) the first, second, third and fourth estimates clearly hold. Since we can write

$$\begin{aligned} \ell^{\bullet j} \partial_j a &= \ell^{\bullet j} (\partial_j \eta_\lambda |\partial r|) r^{-\epsilon/2} \sqrt{2(z - q_1 + r^\epsilon)} \\ &\quad - \frac{\epsilon}{2} (1 - \eta) (\partial r)^{\bullet} \eta_\lambda |\partial r| r^{-\epsilon/2-1} \sqrt{2(z - q_1 + r^\epsilon)} \\ &\quad - \eta_\lambda |\partial r| \left\{ \ell^{\bullet j} r^{-\epsilon/2} (\partial_j q_1) - \epsilon (1 - \eta) (\partial r)^{\bullet} r^{\epsilon/2-1} \right\} / \sqrt{2(z - q_1 + r^\epsilon)} \\ &\quad + i \frac{\epsilon}{2} \ell^{\bullet j} (\partial_j |\partial r|^2) r^{-\epsilon/2-1} - i \frac{\epsilon}{2} \left(\frac{\epsilon}{2} + 1 \right) (1 - \eta) (\partial r)^{\bullet} |\partial r|^2 r^{-\epsilon/2-2}, \end{aligned}$$

the fifth estimate follows by using Condition 1.1.6. By the following equation

$$\begin{aligned} p^r b + b^2 - 2|\partial r|^2(z - q_1 + r^\epsilon) &= (p^r \eta_\lambda |\partial r|) \sqrt{2(z - q_1 + r^\epsilon)} + i \eta_\lambda |\partial r| (\partial^r q_1) / \sqrt{2(z - q_1 + r^\epsilon)} \\ &\quad + i \epsilon \eta_\lambda |\partial r|^3 r^{-1} (z - q_1) / \sqrt{2(z - q_1 + r^\epsilon)} \\ &\quad - 2(1 - \eta_\lambda^2) |\partial r|^2 (z - q_1 + r^\epsilon) + \frac{\epsilon}{4} (\partial^r |\partial r|^2) r^{-1} - \left(\frac{\epsilon}{4} + \frac{\epsilon^2}{16} \right) |\partial r|^4 r^{-2}. \end{aligned}$$

we can see that the last estimate also holds. $\square$

Lemma 1.4.6. *Let $I \subset \mathbb{R}$ be any compact interval. Then there exist a complex-valued function q_3 and a constant $C > 0$ such that uniformly in $z \in I \cup I_+$, as a quadratic forms on $C_0^\infty(\mathbb{R}^d)$,*

$$H - z = \frac{1}{2} (B + b) \tilde{\eta} (B - b) + \frac{1}{2} p_j \ell^{jk} p_k + q_3; \quad |q_3| \leq C f^{-1-\min\{\rho, \epsilon', \tau\}}.$$

Proof. Noting the following expression:

$$B = p^r - \frac{i}{2} (\Delta r) = (p^r)^* + \frac{i}{2} (\Delta r),$$

we have

$$\begin{aligned} H - z &= \frac{1}{2} B \tilde{\eta} B + \frac{1}{2} p_j \ell^{jk} p_k - |x|^\epsilon + q_0 + q_1 + q_2 - z \\ &= \frac{1}{2} (B + b) \tilde{\eta} (B - b) + \frac{1}{2} p_j \ell^{jk} p_k - \frac{i}{2} (\partial^r \tilde{\eta}) b + \frac{1}{2} \tilde{\eta} (p^r b) + \frac{1}{2} \tilde{\eta} b^2 \\ &\quad - |x|^\epsilon + q_0 + q_1 + q_2 - z, \end{aligned}$$

where

$$q_0 = \frac{1}{4} (\partial^r \tilde{\eta}) (\Delta r) + \frac{1}{4} \tilde{\eta} (\partial^r \Delta r) + \frac{1}{8} \tilde{\eta} (\Delta r)^2.$$

Hence by setting

$$\begin{aligned} q_3 &= \frac{1}{2} \tilde{\eta} [(p^r b) + b^2 - 2|\partial r|^2(z - q_1 + r^\epsilon)] - (1 - \eta)(z - q_1 + r^\epsilon) \\ &\quad + (r^\epsilon - |x|^\epsilon) - \frac{i}{2} (\partial^r \tilde{\eta}) b + q_0 + q_2, \end{aligned}$$

and using Lemma 1.4.5, we have the assertion. $\square$

Also in this subsection, we use the regularized weight

$$\Theta = \Theta_\nu^\delta = \int_0^{f/2^\nu} (1 + s)^{-1-\delta} ds = [1 - (1 + f/2^\nu)^{-\delta}] / \delta; \quad \delta > 0, \quad \nu \geq 0,$$

which is the same weight as (1.4.1). We denote its derivatives in f by primes such as (1.4.2). To prove Theorem 1.1.7 the following lemma is a key.

Lemma 1.4.7. *Let $I \subset \mathbb{R}$ be any compact interval. Fix any $\delta \in (0, \min\{\rho, \epsilon', \tau\}]$ and $\beta \in (0, 1 + \epsilon/2)$. Then there exist $c, C > 0$ such that uniformly in $z \in I \cup I_+$ and $\nu \geq 0$, as quadratic forms on $\mathcal{D}(H)$*

$$\begin{aligned} \operatorname{Im} \left((A - a)^* \Theta^{2\beta} (H - z) \right) &\geq c(A - a)^* \Theta' \Theta^{2\beta-1} (A - a) + cp_j \Theta^{2\beta} h^{jk} p_k \\ &\quad - Cf^{-1-\min\{2\rho, 2\epsilon', 2\tau\}+2\delta} \Theta^{2\beta} - \operatorname{Re}(\gamma \Theta^{2\beta} (H - z)), \end{aligned}$$

where γ is a certain function satisfying $|\gamma| \leq Cf^{-1-\min\{2\rho, 2\epsilon', 2\tau\}+2\delta}$.

Proof. Let I, δ and β be as in the assertion. To prove the asserted inequality it suffices to compute as a quadratic forms on $C_0^\infty(\mathbb{R}^d)$. By Lemmas 1.4.3, 1.4.6 and the Cauchy-Schwarz inequality it follows that uniformly in $z \in I \cup I_+$ and $\nu \geq 0$

$$\begin{aligned} &\operatorname{Im} \left((A - a)^* \Theta^{2\beta} (H - z) \right) \\ &= \frac{1}{2} \operatorname{Im} \left((A - a)^* \Theta^{2\beta} (B + b) \tilde{\eta} r^{\epsilon/2} (A - a) \right) + \frac{1}{2} \operatorname{Im} \left(A \Theta^{2\beta} p_j \ell^{jk} p_k \right) \\ &\quad - \frac{1}{2} \operatorname{Im} \left(a^* \Theta^{2\beta} p_j \ell^{jk} p_k \right) + \operatorname{Im} \left((A - a)^* \Theta^{2\beta} q_3 \right) \\ &= \frac{1}{2} (A - a)^* \beta \Theta' \Theta^{2\beta-1} \eta (A - a) - \frac{1}{4} (A - a)^* \Theta^{2\beta} (\partial^r \tilde{\eta}) r^{\epsilon/2} (A - a) \\ &\quad - \frac{\epsilon}{8} (A - a)^* \Theta^{2\beta} \eta r^{\epsilon/2-1} (A - a) + \frac{1}{2} (A - a)^* \Theta^{2\beta} (\operatorname{Im} b) \tilde{\eta} r^{\epsilon/2} (A - a) \\ &\quad + \frac{1}{4} [p_j \Theta^{2\beta} \ell^{jk} p_k, iA] + \frac{1}{2} \operatorname{Re} \left(A(1 - \eta) (\Theta^{2\beta})' p^f \right) \\ &\quad - \frac{1}{2} \operatorname{Im} \left(a^* \Theta^{2\beta} p_j \ell^{jk} p_k \right) + \operatorname{Im} \left((A - a)^* \Theta^{2\beta} q_3 \right) \\ &\geq \frac{1}{2} (A - a)^* (\beta \Theta' - f^{-1-2\delta} \Theta) \Theta^{2\beta-1} (A - a) + \frac{1}{4} [p_j \Theta^{2\beta} \ell^{jk} p_k, iA] \\ &\quad - \frac{1}{2} \operatorname{Im} \left(a^* \Theta^{2\beta} p_j \ell^{jk} p_k \right) - C_1 Q, \end{aligned} \tag{1.4.22}$$

where

$$Q = f^{-1-\min\{2\rho, 2\epsilon', 2\tau\}+2\delta} \Theta^{2\beta} + p_j r^{-\epsilon} f^{-1-\min\{2\rho, 2\epsilon', 2\tau\}+2\delta} \Theta^{2\beta} \delta^{jk} p_k.$$

Let us further estimate each term of (1.4.22). By Lemma 1.4.3 we can estimate the first term of (1.4.22) as

$$\begin{aligned} &\frac{1}{2} (A - a)^* (\beta \Theta' - f^{-1-2\delta} \Theta) \Theta^{2\beta-1} (A - a) \\ &\geq c_1 (A - a)^* \Theta' \Theta^{2\beta-1} (A - a) - C_2 Q. \end{aligned} \tag{1.4.23}$$

To estimate the second term of (1.4.22) we use the following lemma used also in [IS].

Lemma 1.4.8. *Let $\tilde{g}^{ij} = \tilde{g}^{ji} \in C^\infty(\mathbb{R}^d)$ for $i, j = 1, 2, \dots, d$. Then, as a quadratic form on $C_0^\infty(\mathbb{R}^d)$,*

$$[p_i \tilde{g}^{ij} p_j, iA] = p_i \left\{ \tilde{g}^{ij} (\partial^2 f)_j^k + (\partial^2 f)_j^i \tilde{g}^{kj} - (\partial^f \tilde{g})^{ik} \right\} p_k - \operatorname{Im} (\tilde{g}^{jk} (\partial_k \Delta f) p_j).$$

We omit the proof of Lemma 1.4.8. We apply Lemma 1.4.8 with $\tilde{g} = \Theta^{2\beta} \ell$ to the second term of (1.4.22), and then we can bound as follows.

$$\begin{aligned} &\frac{1}{4} [p_j \Theta^{2\beta} \ell^{jk} p_k, iA] \\ &= \frac{1}{4} p_i \left\{ \Theta^{2\beta} \ell^{ij} (\partial^2 f)_j^k + (\partial^2 f)_j^i \Theta^{2\beta} \ell^{kj} - (\partial^f \Theta^{2\beta} \ell)^{ik} \right\} p_k \\ &\quad - \frac{1}{4} \operatorname{Im} (\Theta^{2\beta} \ell^{jk} (\partial_k \Delta f) p_j) \\ &\geq \frac{1}{2} p_j \{ h^{jk} \Theta - \beta r^{-\epsilon} \Theta' \ell^{jk} \} \Theta^{2\beta-1} p_k - C_2 Q. \end{aligned} \tag{1.4.24}$$

For any $\epsilon_1 \in (0, 1)$, we can estimate the third term of (1.4.22) by using Lemma 1.4.5 and the Cauchy-Schwarz inequality as

$$\begin{aligned} & -\frac{1}{2} \operatorname{Im} (a^* \Theta^{2\beta} p_j \ell^{jk} p_k) \\ &= -\frac{1}{2} \operatorname{Im} (p_j a^* \Theta^{2\beta} \ell^{jk} p_k + i(\partial_j a)^* \Theta^{2\beta} \ell^{jk} p_k + 2i\beta(1-\eta)a^* \Theta^{2\beta-1} \Theta' p^f) \quad (1.4.25) \\ &\geq \left(\frac{\epsilon}{4} - \epsilon_1\right) p_j r^{-\epsilon/2-1} \Theta^{2\beta} \ell^{jk} p_k - \epsilon_1^{-1} C_3 Q. \end{aligned}$$

By the bounds (1.4.22), (1.4.23), (1.4.24) and (1.4.25) we obtain

$$\begin{aligned} & \operatorname{Im} ((A-a)^* \Theta^{2\beta} (H-z)) \\ &\geq c_1 (A-a)^* \Theta' \Theta^{2\beta-1} (A-a) - \epsilon_1^{-1} C_4 Q \\ &\quad + \frac{1}{2} p_j \{h^{jk} \Theta + \left(\frac{\epsilon}{2} - 2\epsilon_1\right) r^{-\epsilon/2-1} \ell^{jk} \Theta - \beta r^{-\epsilon} \Theta' \ell^{jk}\} \Theta^{2\beta-1} p_k. \end{aligned} \quad (1.4.26)$$

We choose small ϵ_1 so that $0 < 2\epsilon_1 < 1 + \epsilon/2 - \beta$, and then we can bound the last term of (1.4.26) as

$$\frac{1}{2} p_j \{h^{jk} \Theta + \left(\frac{\epsilon}{2} - 2\epsilon_1\right) r^{-\epsilon/2-1} \ell^{jk} \Theta - \beta r^{-\epsilon} \Theta' \ell^{jk}\} \Theta^{2\beta-1} p_k \geq c_2 p_j \Theta^{2\beta} h^{jk} p_k. \quad (1.4.27)$$

Finally we estimate $-Q$ as

$$\begin{aligned} -Q &\geq -C_5 f^{-1-\min\{2\rho, 2\epsilon', 2\tau\}+2\delta} \Theta^{2\beta} \\ &\quad - 2 \operatorname{Re} \left(r^{-\epsilon} f^{-1-\min\{2\rho, 2\epsilon', 2\tau\}+2\delta} \Theta^{2\beta} (H-z) \right). \end{aligned} \quad (1.4.28)$$

By combining (1.4.26), (1.4.27) and (1.4.28), and setting

$$\gamma = 2\epsilon_1^{-1} C_4 r^{-\epsilon} f^{-1-\min\{2\rho, 2\epsilon', 2\tau\}+2\delta},$$

we obtain the asserted inequality. $\square$

1.4.2.2. RC bounds for complex spectral parameters.

Proof of Theorem 1.1.7. Let $I \subset \mathbb{R}$ be any compact interval. The assertion of the case of $\beta = 0$ is follows from Theorem 1.1.4. Hence we may let $\beta \in (0, \beta_c)$. We take any

$$\delta \in (0, \min\{\rho, \epsilon', \tau\} - \beta).$$

By Lemma 1.4.7, the Cauchy-Schwarz inequality and Theorem 1.1.4 there exists $C_1 > 0$ such that for any $\phi = R(z)\psi$ with $\psi \in f^{-\beta} \mathcal{B}$ and $z \in I_+$

$$\begin{aligned} & \|\Theta'^{1/2} \Theta^{\beta-1/2} (A-a)\phi\|^2 + \langle p_j \Theta^{2\beta} h^{jk} p_k \rangle_\phi \\ &\leq C_1 \left[\|\Theta^\beta (A-a)\phi\|_{\mathcal{B}^*} \|\Theta^\beta \psi\|_{\mathcal{B}} + \|f^{-1/2-\min\{\rho, \epsilon', \tau\}+\delta} \Theta^\beta \phi\|^2 \right. \\ &\quad \left. + \|f^{1/2-\min\{\rho, \epsilon', \tau\}+\delta} \Theta^\beta \psi\|^2 \right] \quad (1.4.29) \\ &\leq C_2 2^{-2\beta\nu} [\|f^\beta (A-a)\phi\|_{\mathcal{B}^*} \|f^\beta \psi\|_{\mathcal{B}} + \|f^\beta \psi\|_{\mathcal{B}}^2]. \end{aligned}$$

By commuting $R(z)$ and powers of f and using $\psi \in f^{-\beta} \mathcal{B}$, we obtain $f^\beta (A-a)\phi \in \mathcal{B}^*$ for each $z \in I_+$. Hence the quantity on the right-hand side of (1.4.29) is not infinite. Thus it follows that by (1.4.29)

$$\begin{aligned} & 2^{2\beta\nu} \|\Theta'^{1/2} \Theta^{\beta-1/2} (A-a)\phi\|^2 + 2^{2\beta\nu} \langle p_j \Theta^{2\beta} h^{jk} p_k \rangle_\phi \\ &\leq C_2 [\|f^\beta (A-a)\phi\|_{\mathcal{B}^*} \|f^\beta \psi\|_{\mathcal{B}} + \|f^\beta \psi\|_{\mathcal{B}}^2]. \end{aligned} \quad (1.4.30)$$

In the first term on the right-hand side of (1.4.30), noting (1.4.2) and taking the supremum in $\nu \geq 0$, we obtain

$$c_1 \|f^\beta(A - a)\phi\|_{\mathcal{B}^*}^2 \leq C_2 [\|f^\beta(A - a)\phi\|_{\mathcal{B}^*} \|f^\beta\psi\|_{\mathcal{B}} + \|f^\beta\psi\|_{\mathcal{B}}^2].$$

Therefore by the Cauchy-Schwarz inequality we have

$$\|f^\beta(A - a)\phi\|_{\mathcal{B}^*} \leq C_3 \|f^\beta\psi\|_{\mathcal{B}}. \quad (1.4.31)$$

As for the second term on the right-hand side of (1.4.30) we use (1.4.31), and take limit $\nu \rightarrow \infty$, and then by Lebesgue's monotone convergence theorem and the concavity of Θ we obtain

$$\langle p_j f^{2\beta} h^{jk} p_k \rangle_\phi \leq C_4 \|f^\beta\psi\|_{\mathcal{B}}^2. \quad (1.4.32)$$

By (1.4.31) and (1.4.32) the assertion follows. $\square$

1.4.2.3. Proof of Corollaries 1.1.8-1.1.10.

Proof of Corollary 1.1.8. Let $s > 1/2$ be as in the assertion, and then we fix any $\omega \in (0, \min\{\beta_c, s - 1/2\})$. First, we decompose for $n \geq 0$ and $z, z' \in I_+$ as

$$\begin{aligned} R(z) - R(z') &= \chi_n R(z) \chi_n - \chi_n R(z') \chi_n \\ &\quad + (R(z) - \chi_n R(z) \chi_n) - (R(z') - \chi_n R(z') \chi_n). \end{aligned} \quad (1.4.33)$$

By Theorem 1.1.4 we can estimate uniformly in $n \geq 0$ and $z, z' \in I_+$ as

$$\begin{aligned} &\|R(z) - \chi_n R(z) \chi_n\|_{\mathcal{B}(\mathcal{H}_s, \mathcal{H}_{-s})} \\ &\leq \|f^{-s} R(z) \bar{\chi}_n f^{-s}\|_{\mathcal{B}(\mathcal{H})} + \|f^{-s} \bar{\chi}_n R(z) \chi_n f^{-s}\|_{\mathcal{B}(\mathcal{H})} \\ &\leq C_1 2^{-(s-1/2)n}. \end{aligned} \quad (1.4.34)$$

Similarly, we obtain

$$\|R(z') - \chi_n R(z') \chi_n\|_{\mathcal{B}(\mathcal{H}_s, \mathcal{H}_{-s})} \leq C_2 2^{-(s-1/2)n}. \quad (1.4.35)$$

As for the first and second terms on the right-hand side of (1.4.33), noting the equation

$$i[H, \chi_n] = \operatorname{Re}(\chi'_n p^f) = \operatorname{Re}(\chi'_n A) \quad (1.4.36)$$

and the identity $\overline{a_z} = a_{z'}$, we can rewrite for $n > m$

$$\begin{aligned} &\chi_n R(z) \chi_n - \chi_n R(z') \chi_n \\ &= \chi_n R(z) \{ \chi_{n+1}(H - z') - (H - z) \chi_{n+1} \} R(z') \chi_n \\ &= \frac{i}{2} \chi_n R(z) \chi'_{n+1} (A - a_{z'}) R(z') \chi_n + \frac{i}{2} \chi_n R(z) (A + a_{\bar{z}})^* \chi'_{n+1} R(z') \chi_n \\ &\quad - \frac{i}{2} \chi_n R(z) (a_z - a_{z'}) \chi'_{n+1} R(z') \chi_n + (z - z') \chi_n R(z) \chi_{n+1} R(z') \chi_n \\ &= \frac{i}{2} \chi_n R(z) \chi'_{n+1} (A - a_{z'}) R(z') \chi_n + \frac{i}{2} \chi_n R(z) (A + a_{\bar{z}})^* \chi'_{n+1} R(z') \chi_n \\ &\quad - \frac{i}{2} \chi_n R(z) (a_z - a_{z'}) \chi'_{n+1} R(z') \chi_n + (z - z') \chi_n R(z) \chi_m R(z') \chi_n \\ &\quad - (z - z') \chi_n R(z) \chi_{m,n+1} (a_z + a_{z'})^{-1} (A - a_{z'}) R(z') \chi_n \\ &\quad + (z - z') \chi_n R(z) (A + a_{\bar{z}})^* \chi_{m,n+1} (a_z + a_{z'})^{-1} R(z') \chi_n \\ &\quad - (z - z') \chi_n R(z) [A, \chi_{m,n+1} (a_z + a_{z'})^{-1}] R(z') \chi_n. \end{aligned}$$

Here $m \in \mathbb{N}_0$ is chosen so that $(a_z + a_{z'})^{-1}$ is non-singular on $\operatorname{supp} \bar{\chi}_m$. Then by Theorems 1.1.4 and 1.1.7 we have uniformly in $n > m \geq 0$ and $z, z' \in I_+$

$$\|\chi_n R(z) \chi_n - \chi_n R(z') \chi_n\|_{\mathcal{B}(\mathcal{H}_s, \mathcal{H}_{-s})} \leq C_3 2^{-\omega n} + C_3 2^{(1-\omega)n} |z - z'|. \quad (1.4.37)$$

By (1.4.33), (1.4.34), (1.4.35) and (1.4.37), we obtain uniformly in large $n \in \mathbb{N}_0$ and $z, z' \in I_+$

$$\|R(z) - R(z')\|_{\mathcal{B}(\mathcal{H}_s, \mathcal{H}_{-s})} \leq C_4 2^{-\omega n} + C_3 2^{(1-\omega)n} |z - z'|.$$

Now, if $|z - z'| \leq 2^{-(m+1)}$, we choose $n \in \mathbb{N}_0$ with $2^n \leq |z - z'|^{-1} \leq 2^{n+1}$, and then we obtain uniformly in $z, z' \in I_+$

$$\|R(z) - R(z')\|_{\mathcal{B}(\mathcal{H}_s, \mathcal{H}_{-s})} \leq C_5 |z - z'|^\omega. \quad (1.4.38)$$

The same bound is trivial for $|z - z'| > 2^{-(m+1)}$. Hence the Hölder continuity (1.1.10) for $R(z)$ follows from (1.4.38). By using the Hölder continuity of $R(z)$ and (1.3.6), we can obtain also the Hölder continuity (1.1.10) for $r^{-\epsilon/2} p R(z)$.

The existence of the limits of (1.1.11) follows immediately from (1.1.10). By using (1.1.11) and Theorem 1.1.4 we see that the limits $R(\lambda \pm i0)$ and $r^{-\epsilon/2} p R(\lambda \pm i0)$ map into $\mathcal{B}^*$. By a density argument these limits are extended continuously to maps $\mathcal{B} \rightarrow \mathcal{B}^*$. $\square$

Proof of Corollary 1.1.9. Let $\beta \in [0, \beta_c)$ be as in the assertion. By Theorem 1.1.7 there exists $C > 0$ such that for any $\Gamma > 0$ and $n > 0$

$$\|\chi_n f^\beta (A - a) R(\lambda + i\Gamma) \psi\|_{\mathcal{B}^*} \leq C \|f^\beta \psi\|_{\mathcal{B}}, \quad \psi \in C_0^\infty(\mathbb{R}^d).$$

By taking the limit $\Gamma \downarrow 0$ and using Corollary 1.1.8 and a density argument, we obtain

$$\|\chi_n f^\beta (A - a) R(\lambda + i0) \psi\|_{\mathcal{B}^*} \leq C \|f^\beta \psi\|_{\mathcal{B}}, \quad \psi \in f^{-\beta} \mathcal{B}.$$

Finally by the Lebesgue's monotone convergence theorem we obtain

$$\|f^\beta (A - a) R(\lambda + i0) \psi\|_{\mathcal{B}^*} \leq C \|f^\beta \psi\|_{\mathcal{B}}, \quad \psi \in f^{-\beta} \mathcal{B}.$$

Similarly, we can obtain

$$\langle p_j f^{2\beta} h^{jk} p_k \rangle_{R(\lambda + i0)\psi}^{1/2} \leq C \|f^\beta \psi\|_{\mathcal{B}}.$$

Hence we are done. $\square$

Proof of Corollary 1.1.10. Let $\lambda \in \mathbb{R}$, $\phi \in \mathcal{H}_{\text{loc}}$ and $\psi \in f^{-\beta} \mathcal{B}$ with $\beta \in [0, \beta_c)$. First we let $\phi = R(\lambda + i0)\psi$. Then by Corollaries 1.1.8 and 1.1.9, noting $\beta < \beta_c$, we can verify that (i) and (ii) of the corollary hold. Conversely, we assume (i) and (ii) of the corollary, and let

$$\phi' = \phi - R(\lambda + i0)\psi.$$

Then by the above argument we have

$$(H - \lambda)\phi' = 0 \text{ in the distributional sense,} \quad (1.4.39)$$

and

$$\phi' \in f^\beta \mathcal{B}^*, \quad (A - a)\phi' \in f^{-\beta} \mathcal{B}_0^*. \quad (1.4.40)$$

In addition, we can see that $\phi' \in \mathcal{B}_0^*$. In fact, noting the identity, cf. (1.1.5) and (1.4.36),

$$2 \operatorname{Im}(\chi_\nu(H - \lambda)) = (\operatorname{Re} a) \chi'_\nu + \operatorname{Re}(\chi'_\nu(A - a))$$

with $a = a_\lambda$ and $\operatorname{Re} a > 0$ we have the bound

$$0 \leq \langle (\operatorname{Re} a) \bar{\chi}'_\nu \rangle_{\phi'} = \operatorname{Re} \langle \chi'_\nu(A - a) \rangle_{\phi'}. \quad (1.4.41)$$

In (1.4.41), we take the limit $\nu \rightarrow \infty$ and use (1.4.40), and then we obtain $\phi' \in \mathcal{B}_0^*$. By (1.4.39) and Theorem 1.1.2 it follows that $\phi' = 0$, i.e. $\phi = R(\lambda + i0)\psi$. $\square$

CHAPTER 2

Stark Hamiltonians

This chapter is written based on the joint paper [AIIS] with Adachi, Ito and Skibsted.

2.1. Setting and result

In this chapter we discuss spectral theory for a *perturbed* Stark Hamiltonian on the Euclidean space of dimension $d \in \mathbb{N} = \{1, 2, \dots\}$. Let us split the space variable of $\mathbb{R}^d$ as $(x, y) = (x, y_2, \dots, y_d) \in \mathbb{R} \times \mathbb{R}^{d-1}$, and apply the Stark field in the positive x -direction. The *free* Stark Hamiltonian is given by

$$H_0 = \frac{1}{2}(p_x^2 + p_y^2) - x = \frac{1}{2}(p_x^2 + p_{y_2}^2 + \dots + p_{y_d}^2) - x; \quad p = -i\partial.$$

We perturb it, and consider

$$H = H_0 + q.$$

Here q is a multiplication operator by a real-valued function $q \in L_{\text{loc}}^2(\mathbb{R}^d)$.

To formulate our condition on q fix a real-valued function $\chi \in C^\infty(\mathbb{R})$ such that

$$\chi(s) = \begin{cases} 1 & \text{for } s \leq 1, \\ 0 & \text{for } s \geq 2, \end{cases} \quad \partial_s \chi = \chi' \leq 0, \quad (2.1.1)$$

and we set

$$f = f(x, y) = \chi(r + x) + [1 - \chi(r + x)](r + x)^{1/2}; \quad r = (x^2 + y^2)^{1/2}.$$

Such choice of function f is stimulated by [HMS], but we can say that it is quite different from theirs. One particular difference is that the level surfaces of our f are paraboloid. We also define a differential operator ∂^f as

$$\partial^f = (\partial f)\partial = (\partial_x f)\partial_x + (\partial_y f)\partial_y. \quad (2.1.2)$$

Condition 2.1.1. There exist $\rho, C > 0$ and a splitting by real-valued functions:

$$q = q_1 + q_2; \quad q_1 \in C^1(\mathbb{R}^d), \quad q_2 \in L^\infty(\mathbb{R}^d),$$

such that for all $(x, y) \in \mathbb{R}^d$

$$|q_1(x, y)| \leq C f^{-\rho} r, \quad \partial^f q_1(x, y) \leq C f^{-1-\rho}, \quad |q_2(x, y)| \leq C f^{-1-\rho},$$

and for $x < 0$

$$|q_1(x, y)| \leq C(-x)^{1-\rho/2}.$$

Letting $F(S)$ be the characteristic function of a general subset $S \subseteq \mathbb{R}^d$, we set

$$F_n = F(\{(x, y) \in \mathbb{R}^d \mid 2^n \leq f(x, y) < 2^{n+1}\}) \quad \text{for } n \in \mathbb{N}_0 = \{0, 1, 2, \dots\}.$$

Define the Agmon-Hörmander spaces $\mathcal{B}$, $\mathcal{B}^*$ and $\mathcal{B}_0^*$ as

$$\mathcal{B} = \left\{ \psi \in L_{\text{loc}}^2(\mathbb{R}^d) \mid \sum_{n \in \mathbb{N}_0} 2^{n/2} \|F_n \psi\|_{L^2} < \infty \right\},$$

$$\mathcal{B}^* = \left\{ \psi \in L_{\text{loc}}^2(\mathbb{R}^d) \mid \sup_{n \in \mathbb{N}_0} 2^{-n/2} \|F_n \psi\|_{L^2} < \infty \right\},$$

$$\mathcal{B}_0^* = \left\{ \psi \in \mathcal{B}^* \mid \lim_{n \rightarrow \infty} 2^{-n/2} \|F_n \psi\|_{L^2} = 0 \right\},$$

respectively. Note that these are Banach spaces with respect to the norm

$$\|\psi\|_{\mathcal{B}} = \sum_{n \in \mathbb{N}_0} 2^{n/2} \|F_n \psi\|_{L^2},$$

$$\|\psi\|_{\mathcal{B}^*} = \|\psi\|_{\mathcal{B}_0^*} = \sup_{n \in \mathbb{N}_0} 2^{-n/2} \|F_n \psi\|_{L^2}.$$

We introduce the weighted Hilbert space L_s^2 by

$$L_s^2 = f^{-s} L^2 \quad \text{for } s \in \mathbb{R}.$$

We note that for any $s > 1/2$ the following inclusion relations hold:

$$L_s^2 \subsetneq \mathcal{B} \subsetneq L_{1/2}^2 \subsetneq L^2 \subsetneq L_{-1/2}^2 \subsetneq \mathcal{B}_0^* \subsetneq \mathcal{B}^* \subsetneq L_{-s}^2. \quad (2.1.3)$$

Using the function χ of (2.1.1), we define *smooth cut-off functions* $\chi_m, \bar{\chi}_m, \chi_{m,n} \in C^\infty(\mathbb{R}^d)$ for $n > m \geq 0$ as

$$\chi_m = \chi(f/2^m), \quad \bar{\chi}_m = 1 - \chi_m, \quad \chi_{m,n} = \bar{\chi}_m \chi_n = \chi_n - \chi_m. \quad (2.1.4)$$

We note that for any $g \in C_0^\infty(\mathbb{R}^d)$ and $n \in \mathbb{N}$ the following property holds under Condition 2.1.1:

$$g(H)\chi_n \text{ is compact operator.}$$

First we state Rellich's theorem of the following form.

Theorem 2.1.2. *Suppose Condition 2.1.1, and let $\lambda \in \mathbb{R}$. If $\phi \in L_{\text{loc}}^2(\mathbb{R}^d)$ satisfies*

- (1) $(H - \lambda)\phi = 0$ in the distributional sense,
- (2) $\bar{\chi}_{m_0}\phi \in \mathcal{B}_0^*$ for some $m_0 \in \mathbb{N}_0$,

then $\phi = 0$ on $\mathbb{R}^d$.

Let us denote the resolvent

$$R(z) = (H - z)^{-1},$$

and for any compact interval $I \subset \mathbb{R}$ we set

$$I_\pm = \{z \in \mathbb{C} \mid \text{Re } z \in I, \pm \text{Im } z \in (0, 1)\},$$

for simplicity. Moreover we introduce

$$h_{jk} = f^{-1} |\partial f|^2 \delta_{jk} - f^{-1} (\partial_j f) (\partial_k f).$$

We note that $\xi_j h_{jk} \xi_k$ is non-negative for any $\xi \in \mathbb{R}^d$. We use also this chapter the notation $\langle T \rangle_\psi = \langle \psi, T\psi \rangle$ for general linear operator T .

Theorem 2.1.3. *Assume Condition 2.1.1 and let $I \subset \mathbb{R}$ be any compact interval. Then there exists $C > 0$ such that for any $\psi \in \mathcal{B}$ and $z \in I_\pm$*

$$\|R(z)\psi\|_{\mathcal{B}^*} + \|p^f R(z)\psi\|_{\mathcal{B}^*} + \langle p_j h_{jk} p_k \rangle_{R(z)\psi} \leq C \|\psi\|_{\mathcal{B}}. \quad (2.1.5)$$

By the definition of f , there exist $c > 0$ and $f_0 \geq 1$ such that

$$|\partial f| \geq c \quad \text{on } \{(x, y) \in \mathbb{R}^d \mid f(x, y) > f_0\}.$$

We set

$$\eta = 1 - \chi(f/f_0), \quad \tilde{\eta} = \eta(2r|\partial f|^2)^{-1},$$

and introduce the notation

$$\ell_{jk} = \delta_{jk} - \eta|\partial f|^{-2}(\partial_j f)(\partial_k f).$$

Condition 2.1.4. In addition Condition 2.1.1, there exist $\tau, C > 0$ such that

$$|\partial^f q_1| \leq C f^{-1-\tau}, \quad |\ell_{\bullet j} r^{-1/2} \partial_j q_1| \leq C f^{-1-\tau}.$$

Now we choose a smooth cut-off function η_z which satisfies

$$\eta_z = \begin{cases} 1 & \text{for } \operatorname{Re} z - q_1 + r \geq 1, \\ 0 & \text{for } \operatorname{Re} z - q_1 + r \leq 1/2, \end{cases} \quad 0 \leq \eta_z \leq 1,$$

and set asymptotic complex phase $a_{\pm}$: For $z \in I_{\pm}$

$$a_{\pm} = \sqrt{2}\eta_z|\partial f|\sqrt{z - q_1 + r} \pm \frac{i}{2}r^{-1}(\partial f) \cdot (\partial r), \quad (2.1.6)$$

respectively. Here we choose the branch of square root so that $\operatorname{Re} \sqrt{\omega} > 0$ for $\omega \in \mathbb{C} \setminus (-\infty, 0]$. Let

$$\beta_c = \frac{1}{2} \min\{1, \rho, 2\tau\}.$$

We introduce the conjugate operator A :

$$A = [H, if] = \operatorname{Re} p^f = p^f - \frac{i}{2}(\Delta f); \quad p^f = -i\partial^f, \quad (2.1.7)$$

cf. (2.1.2). By similar argument to Section 1.2.1 of Chapter 1, we can verify that the conjugate operator A of (2.1.7) is self-adjoint with maximal domain. We omit the details. In the following, we do not discuss about the domain of operators, since we can verify by similar way to Chapter 1.

Theorem 2.1.5. Assume Condition 2.1.4 and let $I \subset \mathbb{R}$ be any compact interval. Then for all $\beta \in [0, \beta_c)$ there exists $C > 0$ such that for any $\psi \in f^{-\beta}\mathcal{B}$ and $z \in I_{\pm}$

$$\|f^{\beta}(A \mp a)R(z)\psi\|_{\mathcal{B}^*} + \langle p_j f^{2\beta} h_{jk} p_k \rangle_{R(z)\psi}^{1/2} \leq C \|f^{\beta}\psi\|_{\mathcal{B}},$$

respectively.

By Theorems 2.1.3 and 2.1.5 we obtain the limiting absorption principle of the following form.

Corollary 2.1.6. Assume Condition 2.1.4 and let $I \subset \mathbb{R}$ be any compact interval. For any $s > 1/2$ and $\omega \in (0, \min\{\beta_c, 2s - 1\})$ there exists $C > 0$ such that for any $z, z' \in I_+$ or $z, z' \in I_-$

$$\begin{aligned} \|R(z) - R(z')\|_{\mathcal{B}(L_s^2, L_{-s}^2)} &\leq C|z - z'|^{\omega}, \\ \| |\partial f| p R(z) - |\partial f| p R(z') \|_{\mathcal{B}(L_s^2, L_{-s}^2)} &\leq C|z - z'|^{\omega}. \end{aligned} \quad (2.1.8)$$

In particular, the operators $R(z)$ and $|\partial f| p R(z)$ has uniform limits as $I_{\pm} \ni z \rightarrow \lambda \in I$ in the norm topology of $\mathcal{B}(L_s^2, L_{-s}^2)$, say denoted by

$$\begin{aligned} R(\lambda \pm i0) &= \lim_{I_{\pm} \ni z \rightarrow \lambda} R(z), \\ |\partial f| p R(\lambda \pm i0) &= \lim_{I_{\pm} \ni z \rightarrow \lambda} |\partial f| p R(z), \end{aligned} \quad (2.1.9)$$

for $\lambda \in I$, respectively. Moreover, these limits belong to $\mathcal{B}(\mathcal{B}, \mathcal{B}^*)$.

Combining Theorem 2.1.5 and Corollary 2.1.6, we obtain the radiation condition bounds for real spectral parameters.

Corollary 2.1.7. *Assume Condition 2.1.4 and let $I \subset \mathbb{R}$ be any compact interval. Then for all $\beta \in [0, \beta_c)$ there exists $C > 0$ such that for any $\psi \in f^{-\beta}\mathcal{B}$ and $\lambda \in I$*

$$\|f^\beta(A \mp a)R(\lambda \pm i0)\psi\|_{\mathcal{B}^*} + \langle p_j f^{2\beta} h_{jk} p_k \rangle_{R(\lambda \pm i0)\psi}^{1/2} \leq C \|f^\beta \psi\|_{\mathcal{B}},$$

respectively.

Finally we obtain the Sommerfeld uniqueness result.

Corollary 2.1.8. *Assume Condition 2.1.4, and let $\lambda \in \mathbb{R}$, $\phi \in L_{\text{loc}}^2$ and $\psi \in f^{-\beta}\mathcal{B}$ with $\beta \in [0, \beta_c)$. Then $\phi = R(\lambda \pm i0)\psi$ holds if and only if both of the following condition hold:*

- (1) $(H - \lambda)\phi = \psi$ in the distributional sense.
- (2) $\phi \in f^\beta\mathcal{B}^*$ and $(A \mp a)\phi \in f^{-\beta}\mathcal{B}_0^*$.

The absence of eigenfunctions for Stark Hamiltonians is well-known (cf. e.g. [Sig]). However, as far as the author knows, the result such as Theorem 2.1.2 is not known, so far. As for the absence of singular continuous spectrum for Stark Hamiltonians is also well-known (cf. e.g. [Ad, Ya]). By using the Agmon-Hörmander spaces $\mathcal{B}$ and $\mathcal{B}^*$, we obtain slightly stronger estimate of the resolvent such as Theorem 2.1.3 rather than theirs.

In Section 2.2 we introduce a commutator with some weight inside. In the proofs of our results commutator estimate plays major roles. In Section 2.3 we prove Theorem 2.1.2. In Section 2.4 we prove Theorem 2.1.3. Theorem 2.1.5 and Corollaries 2.1.6-2.1.8 are proved in Section 2.5.

2.2. Conjugate operator

In this section we formally compute a *weighted commutator*

$$[H, iA]_\Theta := i(H\Theta A - A\Theta H).$$

The *weight function* $\Theta \in C^\infty(\mathbb{R}^d)$ varies depending on purposes, and will be specified later. For the moment we only assume that it is a function only of f , and that

$$\text{supp } \Theta \subseteq \{(x, y) \in \mathbb{R}^d \mid r + x \geq 2\}, \quad |\Theta^{(k)}| \leq C_k \text{ for all } k \in \mathbb{N}_0, \quad (2.2.1)$$

where $\Theta^{(k)}$ denotes the k -th derivative of Θ in f .

Note that on $\text{supp } \Theta$ we can compute derivatives of f as

$$\begin{aligned} \partial_x f &= \frac{1}{2} f r^{-1}, \\ \partial_y f &= \frac{1}{2} f^{-1} r^{-1} y, \\ \partial_x \partial_x f &= \frac{1}{2} f^{-1} r^{-1} - \frac{1}{4} f r^{-2} - \frac{1}{2} f^{-1} r^{-3} x^2, \\ \partial_{y_j} \partial_{y_k} f &= \frac{1}{2} f^{-1} r^{-1} \delta_{jk} - \frac{1}{4} f^{-3} r^{-2} y_j y_k - \frac{1}{2} f^{-1} r^{-3} y_j y_k, \\ \partial_x \partial_y f &= \partial_y \partial_x f = -\frac{1}{4} f^{-1} r^{-2} y - \frac{1}{2} f^{-1} r^{-3} x y. \end{aligned} \quad (2.2.2)$$

In particular, we have expressions

$$\begin{aligned}
\partial^f &= \frac{1}{2}fr^{-1}\partial_x + \frac{1}{2}f^{-1}r^{-1}y\partial_y, \\
|\partial f|^2 &= \frac{1}{2}r^{-1}, \\
\Delta f &= \frac{1}{2}(d-2)f^{-1}r^{-1}, \\
\partial_j\partial_k f &= f^{-1}|\partial f|^2\delta_{jk} - f^{-1}(\partial_j f)(\partial_k f) - f^{-1}|\partial f|^2(\partial_j r)(\partial_k r).
\end{aligned} \tag{2.2.3}$$

Lemma 2.2.1. *As quadratic forms on $C_c^\infty(\mathbb{R}^d)$,*

$$\begin{aligned}
[H, iA]_\Theta &= A\Theta' A + p_j f^{-1}(|\partial f|^2\delta_{jk} - (\partial_j f)(\partial_k f))\Theta p_k \\
&\quad + p_j f^{-1}|\partial f|^2(\delta_{jk} - (\partial_j r)(\partial_k r))\Theta p_k + \frac{1}{2}f^{-1}r^{-1}(r-x)\Theta \\
&\quad - \frac{1}{4}|\partial f|^4\Theta''' - \frac{1}{2}(\partial^f|\partial f|^2)\Theta'' - \frac{1}{2}f^{-1}|\partial f|^4\Theta'' + \tilde{q}_1\Theta' + \tilde{q}_0\Theta \\
&\quad - 2\operatorname{Im}(q_2\Theta p^f) - 2\operatorname{Re}(f^{-1}|\partial f|^2\Theta H) - \operatorname{Re}(|\partial f|^2\Theta' H)
\end{aligned}$$

with

$$\begin{aligned}
\tilde{q}_0 &= -\frac{1}{4}(\Delta^2 f) - \frac{1}{2}f^{-1}(\Delta|\partial f|^2) + \frac{1}{2}f^{-2}|\partial f|^2(\Delta f) + f^{-2}(\partial^f|\partial f|^2) - f^{-3}|\partial f|^4 \\
&\quad + 2f^{-1}|\partial f|^2q - (\partial^f q_1) + (\Delta f)q_2, \\
\tilde{q}_1 &= -\frac{1}{4}(\Delta|\partial f|^2) - \frac{1}{2}f^{-1}|\partial f|^2(\Delta f) - f^{-1}(\partial^f|\partial f|^2) + f^{-2}|\partial f|^4 + |\partial f|^2q_2.
\end{aligned}$$

In particular,

$$\tilde{q}_0 \geq -Cf^{-1-\min\{6,\rho\}}, \quad |\tilde{q}_1| \leq Cf^{-1-\min\{3,\rho\}}r^{-1}.$$

Proof. Using adjoint of the expression (2.1.7), we can compute

$$\begin{aligned}
[H, iA]_\Theta &= \operatorname{Im}((p^f)^*\Theta p^2) + 2\operatorname{Im}((p^f)^*\Theta(-x+q)) + \operatorname{Re}((\Delta f)\Theta H) \\
&= (p^f)^*\Theta' p^f + p_j(\partial_j\partial_k f)\Theta p_k - \frac{1}{2}p(\Delta f)\Theta p - \frac{1}{2}p|\partial f|^2\Theta' p \\
&\quad + (x-q_1)(\Delta f)\Theta + (x-q_1)|\partial f|^2\Theta' + (\partial_x f)\Theta - (\partial^f q_1)\Theta \\
&\quad - 2\operatorname{Im}(q_2\Theta p^f) + \operatorname{Re}((\Delta f)\Theta H).
\end{aligned} \tag{2.2.4}$$

We combine the third, fifth and tenth terms of (2.2.4) as

$$\begin{aligned}
&-\frac{1}{2}p(\Delta f)\Theta p + (x-q_1)(\Delta f)\Theta + \operatorname{Re}((\Delta f)\Theta H) \\
&= -\frac{1}{4}(\Delta(\Delta f)\Theta) + (\Delta f)q_2\Theta,
\end{aligned} \tag{2.2.5}$$

and, similarly, the fourth and sixth terms of (2.2.4) as

$$\begin{aligned}
&-\frac{1}{2}p|\partial f|^2\Theta' p + (x-q_1)|\partial f|^2\Theta' \\
&= -\frac{1}{4}(\Delta|\partial f|^2\Theta') + |\partial f|^2q_2\Theta' - \operatorname{Re}(|\partial f|^2\Theta' H).
\end{aligned} \tag{2.2.6}$$

In addition, let us add to the right-hand side of (2.2.4) the following “zero” term:

$$\begin{aligned}
0 &= pf^{-1}|\partial f|^2\Theta p - 2xf^{-1}|\partial f|^2\Theta - \frac{1}{2}(\Delta f^{-1}|\partial f|^2\Theta) \\
&\quad + 2f^{-1}|\partial f|^2q\Theta - 2\operatorname{Re}(f^{-1}|\partial f|^2\Theta H)
\end{aligned} \tag{2.2.7}$$

Then by (2.2.4), (2.2.5), (2.2.6) and (2.2.7) we obtain

$$\begin{aligned}
[H, iA]_{\Theta} &= (p^f)^* \Theta' p^f + p f^{-1} |\partial f|^2 \Theta p + p_j (\partial_j \partial_k f) \Theta p_k + (\partial_x f) \Theta \\
&\quad - 2x f^{-1} |\partial f|^2 \Theta - \frac{1}{4} (\Delta (\Delta f) \Theta) - \frac{1}{4} (\Delta |\partial f|^2 \Theta') - \frac{1}{2} (\Delta f^{-1} |\partial f|^2 \Theta) \\
&\quad + |\partial f|^2 q_2 \Theta' + 2 f^{-1} |\partial f|^2 q \Theta - (\partial^f q_1) \Theta + (\Delta f) q_2 \Theta \\
&\quad - 2 \operatorname{Im}(q_2 \Theta p^f) - 2 \operatorname{Re}(f^{-1} |\partial f|^2 \Theta H) - \operatorname{Re}(|\partial f|^2 \Theta' H).
\end{aligned} \tag{2.2.8}$$

Next, we expand the sixth to eighth terms of (2.2.8) as

$$\begin{aligned}
& - \frac{1}{4} (\Delta (\Delta f) \Theta) - \frac{1}{4} (\Delta |\partial f|^2 \Theta') - \frac{1}{2} (\Delta f^{-1} |\partial f|^2 \Theta) \\
&= - \frac{1}{4} |\partial f|^4 \Theta''' - \frac{1}{2} |\partial f|^2 (\Delta f) \Theta'' - \frac{1}{2} (\partial^f |\partial f|^2) \Theta'' - \frac{1}{2} f^{-1} |\partial f|^4 \Theta'' \\
&\quad - \frac{1}{2} (\partial^f \Delta f) \Theta' - \frac{1}{4} (\Delta |\partial f|^2) \Theta' - \frac{1}{4} (\Delta f)^2 \Theta' - \frac{1}{2} f^{-1} |\partial f|^2 (\Delta f) \Theta' \\
&\quad - f^{-1} (\partial^f |\partial f|^2) \Theta' + f^{-2} |\partial f|^4 \Theta' - \frac{1}{4} (\Delta^2 f) \Theta - \frac{1}{2} f^{-1} (\Delta |\partial f|^2) \Theta \\
&\quad + \frac{1}{2} f^{-2} |\partial f|^2 (\Delta f) \Theta + f^{-2} (\partial^f |\partial f|^2) \Theta - f^{-3} |\partial f|^4 \Theta.
\end{aligned} \tag{2.2.9}$$

Then the first term of (2.2.8) combined with the second, fifth and seventh terms of (2.2.9) makes the first term of asserted identity. Inserting expressions from (2.2.2) and (2.2.3) into the second to fourth terms of (2.2.8), we obtain the second to third terms of the asserted identity. The rest terms of (2.2.8) and (2.2.9) clearly correspond to the rest terms of the asserted identity. Hence we are done. $\square$

2.3. Proof of Rellich's theorem

In this section we prove Theorem 2.1.2. The proof reduces to the following two propositions.

Proposition 2.3.1. *If a function $\phi \in L_{\text{loc}}^2(\mathbb{R}^d)$ satisfies for some $m_0 \in \mathbb{N}_0$:*

- (1) $(H - \lambda)\phi = 0$ in the distributional sense,
- (2) $\bar{\chi}_{m_0} \phi \in \mathcal{B}_0^*$,

then $\bar{\chi}_{m_0} \exp(\alpha f^3) \phi \in \mathcal{B}_0^$ for any $\alpha \geq 0$.*

Proposition 2.3.2. *If a function $\phi \in L_{\text{loc}}^2(\mathbb{R}^d)$ satisfies for some $m_0 \in \mathbb{N}_0$:*

- (1) $(H - \lambda)\phi = 0$ in the distributional sense,
- (2) $\bar{\chi}_{m_0} \exp(\alpha f^3) \phi \in \mathcal{B}_0^*$ for any $\alpha \geq 0$,

then $\phi = 0$ on $\mathbb{R}^d$.

Propositions 2.3.1 and 2.3.2 will be proved in Sections 2.3.1 and 2.3.2, respectively.

2.3.1. A priori super-cubic-exponential decay estimate. Here we are going to prove Proposition 2.3.1. Choose a weight function Θ to be of the form

$$\begin{aligned}
\Theta &= \Theta_{m,n,\nu}^{\alpha,\beta,\delta} = \chi_{m,n} e^{\theta}; \\
\theta &= \theta_{\nu}^{\alpha,\beta,\delta} = 2\alpha f^3 + 6\beta \int_0^f s^2 (1 + s/2^\nu)^{-3-\delta} ds
\end{aligned} \tag{2.3.1}$$

with parameters $\alpha, \beta \geq 0$, $\delta > 0$ and $m, n, \nu \in \mathbb{N}$. Note that Θ actually satisfies (2.2.1) for large m . We denote the derivatives in f by primes as before. If we set

$$\theta_0 = 1 + f/2^\nu$$

for notational simplicity, then

$$\theta' = 6\alpha f^2 + 6\beta f^2 \theta_0^{-3-\delta}, \quad \theta'' = 12\alpha f + 12\beta f \theta_0^{-3-\delta} - 6\beta(3+\delta)2^{-\nu} f^2 \theta_0^{-4-\delta}, \dots$$

Noting that $2^{-\nu} \theta_0^{-1} \leq f^{-1}$, we have

$$|(\theta - 2\alpha f^3)^{(k)}| \leq C_{\delta,k} \beta f^{3-k} \theta_0^{-3-\delta} \quad \text{for } k = 1, 2, \dots$$

The following form inequality plays an essential role in the proof of Proposition 2.3.1.

Lemma 2.3.3. *Fix any $\alpha_0 \geq 0$ and $\delta \in (0, \min\{2, \rho\})$ in the definition (2.3.1) of Θ . Then there exist $\beta, c, C > 0$ and $n_0 \in \mathbb{N}$ such that uniformly in $\alpha \in [0, \alpha_0]$, $n > m \geq n_0$ and $\nu \geq n_0$, as quadratic forms on $\mathcal{D}(H)$,*

$$\begin{aligned} \operatorname{Im}(A\Theta(H - \lambda)) &\geq cf^{-1}\theta_0^{-\delta}\Theta - Cf^{-1}(\chi_{m-1,m+1}^2 + \chi_{n-1,n+1}^2)e^\theta \\ &\quad + \operatorname{Re}(\gamma(H - \lambda)), \end{aligned} \quad (2.3.2)$$

where $\gamma = \gamma_{m,n,\nu}$ is a certain function satisfying $\operatorname{supp} \gamma \subseteq \operatorname{supp} \chi_{m,n}$ and $|\gamma| \leq Ce^\theta$ uniformly in $n > m \geq n_0$ and $\nu \geq n_0$.

Proof. Fix any $\alpha_0 \geq 0$ and $\delta \in (0, \min\{2, \rho\})$. We will fix small $\beta \in (0, 1]$ and large $n_0 \in \mathbb{N}$ in the last step of the proof. For the moment all the estimates are uniform in $\alpha \in [0, \alpha_0]$, $\beta \in (0, 1]$, $n > m \geq n_0 > 0$ and $\nu \geq n_0$.

By Lemma 2.2.1 we can bound

$$\begin{aligned} &2\operatorname{Im}(A\Theta(H - \lambda)) \\ &= [H, iA]_\Theta + \lambda|\partial f|^2\Theta' \\ &\geq A\theta'\Theta A + p_j f^{-1}(|\partial f|^2\delta_{jk} - (\partial_j f)(\partial_k f))\Theta p_k + \frac{1}{2}f^{-1}r^{-1}(r - x)\Theta \\ &\quad - \frac{3}{4}|\partial f|^4\theta'\theta''\Theta - \frac{1}{4}|\partial f|^4\theta'^3\Theta - \frac{1}{2}(\partial^f|\partial f|^2)\theta'^2\Theta - \frac{1}{2}f^{-1}|\partial f|^4\theta'^2\Theta \\ &\quad - C_1Q - 2\operatorname{Re}(f^{-1}|\partial f|^2\Theta(H - \lambda)) - \operatorname{Re}(|\partial f|^2\Theta'(H - \lambda)) \\ &\geq (A + \frac{i}{2}|\partial f|^2\theta')\theta'\Theta(A - \frac{i}{2}|\partial f|^2\theta') + p_j f^{-1}(|\partial f|^2\delta_{jk} - (\partial_j f)(\partial_k f))\Theta p_k \\ &\quad + \frac{1}{2}f^{-1}r^{-1}(r - x)\theta_0^{-\delta}\Theta + \frac{1}{4}|\partial f|^4\theta'(\theta'' - 2f^{-1}\theta')\Theta \\ &\quad - C_2Q + \operatorname{Re}(\gamma_1(H - \lambda)). \end{aligned} \quad (2.3.3)$$

Here and below we gather *admissible* error terms in Q , which is of the form

$$\begin{aligned} Q &= f^{-4}|\chi_{m,n}'''|e^\theta + f^{-2}|\chi_{m,n}''|e^\theta + |\chi_{m,n}'|e^\theta + f^{-1-\min\{2,\rho\}}\Theta \\ &\quad + pr^{-1}|\chi_{m,n}'|e^\theta p + pf^{-1-\rho}r^{-1}\Theta p. \end{aligned}$$

Actually $-Q$ can be bounded below as

$$\begin{aligned} -Q &\geq -C_3f^{-1-\min\{2,\rho\}}\Theta - C_3f^{-1}(\chi_{m-1,m+1}^2 + \chi_{n-1,n+1}^2)e^\theta \\ &\quad - 2\operatorname{Re}(r^{-1}|\chi_{m,n}'|e^\theta(H - \lambda)) - 2\operatorname{Re}(f^{-1-\rho}r^{-1}\Theta(H - \lambda)), \end{aligned} \quad (2.3.4)$$

and we will see that this will be negligibly absorbed by other terms of (2.3.3).

Let us further combine and bound the first and second terms of (2.3.3) in the following manner. Choose $n_0 = n_{0,\beta}$ large enough depending on $\beta \in (0, 1]$, so that

$$\theta' \geq 6\beta(f\theta_0^{-1})^3 f^{-1}\theta_0^{-\delta} \geq 6\beta 2^{3(n_0-1)} f^{-1}\theta_0^{-\delta} \geq \frac{1}{2}f^{-1}\theta_0^{-\delta} \quad \text{on } \operatorname{supp} \Theta.$$

Then we have the first and second terms of (2.3.3) bounded as

$$\begin{aligned}
& (A + \frac{i}{2}|\partial f|^2\theta')\theta'\Theta(A - \frac{i}{2}|\partial f|^2\theta') + p_j f^{-1}(|\partial f|^2\delta_{jk} - (\partial_j f)(\partial_k f))\Theta p_k \\
& \geq \frac{1}{2}(A + \frac{i}{2}|\partial f|^2\theta')f^{-1}\theta_0^{-\delta}\Theta(A - \frac{i}{2}|\partial f|^2\theta') \\
& \quad + \frac{1}{2}p_j f^{-1}\theta_0^{-\delta}(|\partial f|^2\delta_{jk} - (\partial_j f)(\partial_k f))\Theta p_k \\
& \geq \frac{1}{2}(p^f)^* f^{-1}\theta_0^{-\delta}\Theta p^f - \frac{1}{8}f^{-1}|\partial f|^4\theta'^2\theta_0^{-\delta}\Theta \\
& \quad + \frac{1}{2}p_j f^{-1}\theta_0^{-\delta}(|\partial f|^2\delta_{jk} - (\partial_j f)(\partial_k f))\Theta p_k - C_4 Q \\
& \geq \frac{1}{2}p f^{-1}|\partial f|^2\theta_0^{-\delta}\Theta p - \frac{1}{8}f^{-1}|\partial f|^4\theta'^2\theta_0^{-\delta}\Theta - C_4 Q \\
& \geq f^{-1}|\partial f|^2 x \theta_0^{-\delta}\Theta + \frac{1}{8}f^{-1}|\partial f|^4\theta'^2\theta_0^{-\delta}\Theta - C_5 Q + \operatorname{Re}(f^{-1}|\partial f|^2\theta_0^{-\delta}\Theta(H - \lambda)).
\end{aligned} \tag{2.3.5}$$

On the other hand, it is clear that the fourth term of (2.3.3) is bounded as

$$\frac{1}{4}|\partial f|^4\theta'(\theta'' - 2f^{-1}\theta')\Theta \geq -C_6\beta f^{-1}\theta_0^{-\delta}\Theta. \tag{2.3.6}$$

Now by (2.3.3), (2.3.5), (2.3.6) and (2.3.4) we obtain

$$\begin{aligned}
2\operatorname{Im}(A\Theta(H - \lambda)) & \geq \frac{1}{2}(1 - C_7\beta)f^{-1}\theta_0^{-\delta}\Theta - C_7f^{-1-\min\{2,\rho\}}\Theta \\
& \quad - C_7f^{-1}(\chi_{m-1,m+1}^2 + \chi_{n-1,n+1}^2)e^\theta + \operatorname{Re}(\gamma_2(H - \lambda)).
\end{aligned}$$

By choosing $\beta \in (0, 1]$ small enough, and then $n_0 \in \mathbb{N}$ large enough we obtain the asserted inequality. $\square$

Proof of Proposition 2.3.1. Let $\phi \in L_{\text{loc}}^2(\mathbb{R}^d)$ and $m_0 \in \mathbb{N}_0$ satisfy the assumptions of the assertion, and set

$$\alpha_0 = \sup\{\alpha \geq 0 \mid \bar{\chi}_{m_0} \exp(\alpha f^3)\phi \in \mathcal{B}_0^*\}.$$

Assume $\alpha_0 < \infty$, and let us deduce a contradiction. Fix any $\delta \in (0, \min\{2, \rho\})$, and choose $\beta > 0$ and $n_0 \geq 0$ as in Lemma 2.3.3. We may let $n_0 \geq m_0 + 3$ without loss of generality. If $\alpha_0 = 0$, let $\alpha = 0$ so that we automatically have $\alpha + \beta > \alpha_0$. Otherwise, we choose $\alpha \in [0, \alpha_0)$ such that $\alpha + \beta > \alpha_0$. With such α and β we evaluate the inequality (2.3.2) on the state $\chi_{m-2,n+2}\phi \in \mathcal{D}(H)$, and then we obtain for any $n > m \geq n_0$ and $\nu \geq n_0$

$$\|(f^{-1}\theta_0^{-\delta}\Theta)^{1/2}\phi\|^2 \leq C_m\|\chi_{m-1,m+1}\phi\|^2 + C_\nu 2^{-n/2}\|\chi_{n-1,n+1}\exp(\alpha f^3)\phi\|^2. \tag{2.3.7}$$

The second term on the right-hand side of (2.3.7) vanishes in the limit $n \rightarrow \infty$ since $\bar{\chi}_{m_0} \exp(\alpha f^3)\phi \in \mathcal{B}_0^*$, and hence by Lebesgue's monotone convergence theorem

$$\|(\bar{\chi}_m f^{-1}\theta_0^{-\delta}e^\theta)^{1/2}\phi\|^2 \leq C_m\|\chi_{m-1,m+1}\phi\|^2. \tag{2.3.8}$$

Next we let $\nu \rightarrow \infty$ in (2.3.8) invoking again Lebesgue's monotone convergence theorem, and then it follows that

$$\bar{\chi}_m^{1/2} f^{-1/2} \exp((\alpha + \beta)f^3)\phi \in L^2(\mathbb{R}^d).$$

Consequently this implies $\bar{\chi}_m^{1/2} \exp(\kappa f^3)\phi \in \mathcal{B}_0^*$ for any $\kappa \in (0, \alpha + \beta)$. But this is a contradiction, since $\alpha + \beta > \alpha_0$. We are done. $\square$

2.3.2. Absence of super-cubic-exponentially decaying eigenfunctions.

Next we prove Proposition 2.3.2. In order to prove it we choose Θ to be

$$\Theta = \Theta_{m,n}^\alpha = \chi_{m,n} e^\theta; \quad \theta = \theta^\alpha = 2\alpha f^3$$

with parameters $\alpha \geq 1$ and $m, n \in \mathbb{N}$. We first prove the following form inequality similar to Lemma 2.3.3, however, focusing on different parameters.

Lemma 2.3.4. *There exist $c, C > 0$ and $n_0 \in \mathbb{N}$ such that uniformly in $\alpha \geq 1$ and $n > m \geq n_0$, as quadratic forms on $\mathcal{D}(H)$,*

$$\begin{aligned} \operatorname{Im}(A\Theta(H - \lambda)) &\geq c\alpha^2 f^3 r^{-2} \Theta - C\alpha^2 f^{-1} (\chi_{m-1,m+1}^2 + \chi_{n-1,n+1}^2) e^\theta \\ &\quad + \operatorname{Re}(\gamma(H - \lambda)), \end{aligned} \quad (2.3.9)$$

where $\gamma = \gamma_{m,n}^\alpha$ is a certain function satisfying $\operatorname{supp} \gamma \subseteq \operatorname{supp} \chi_{m,n}$ and $|\gamma| \leq C\alpha e^\theta$ uniformly in $\alpha \geq 1$ and $n > m \geq n_0$.

Proof. In this proof all the estimates are uniform in $\alpha \geq 1$ and $n > m \geq n_0$. We will retake $n_0 \in \mathbb{N}$ larger, if necessary, each time it appears below.

By Lemma 2.2.1 we bound

$$\begin{aligned} &2 \operatorname{Im}(A\Theta(H - \lambda)) \\ &= [H, iA]_\Theta + \lambda |\partial f|^2 \Theta' \\ &\geq A\theta' \Theta A + p_j f^{-1} (|\partial f|^2 \delta_{jk} - (\partial_j f)(\partial_k f)) \Theta p_k + \frac{1}{2} f^{-1} r^{-1} (r - x) \Theta \\ &\quad - \frac{3}{4} |\partial f|^4 \theta'' \Theta - \frac{1}{4} |\partial f|^4 \theta^3 \Theta - \frac{1}{2} (\partial^f |\partial f|^2) \theta^2 \Theta - \frac{1}{2} f^{-1} |\partial f|^4 \theta^2 \Theta \\ &\quad - C_1 Q - 2 \operatorname{Re}(f^{-1} |\partial f|^2 \Theta(H - \lambda)) - \operatorname{Re}(|\partial f|^2 \Theta'(H - \lambda)) \\ &\geq (A + \frac{i}{2} |\partial f|^2 \theta') \theta' \Theta (A - \frac{i}{2} |\partial f|^2 \theta') + p_j f^{-1} (|\partial f|^2 \delta_{jk} - (\partial_j f)(\partial_k f)) \Theta p_k \\ &\quad + \frac{1}{2} f^{-1} r^{-1} (r - x) \Theta - C_2 Q + \operatorname{Re}(\gamma_1(H - \lambda)), \end{aligned} \quad (2.3.10)$$

where Q consists of *admissible* error terms:

$$\begin{aligned} Q &= f^{-4} |\chi_{m,n}'''| e^\theta + \alpha f^{-2} |\chi_{m,n}''| e^\theta + \alpha^2 |\chi_{m,n}'| e^\theta + \alpha f^{1-\min\{2,\rho\}} r^{-1} \Theta \\ &\quad + f^{-1-\min\{6,\rho\}} \Theta + p r^{-1} |\chi_{m,n}'| e^\theta p + p f^{-1-\rho} r^{-1} \Theta p. \end{aligned}$$

Note that Q satisfies

$$\begin{aligned} -Q &\geq -C_3 \alpha^2 f^{3-\min\{2,\rho\}} r^{-2} \Theta - C_3 f^{-1-\min\{2,\rho\}} \Theta \\ &\quad - C_3 \alpha^2 f^{-1} (\chi_{m-1,m+1}^2 + \chi_{n-1,n+1}^2) e^\theta \\ &\quad - 2 \operatorname{Re}(r^{-1} |\chi_{m,n}'| e^\theta (H - \lambda)) - 2 \operatorname{Re}(f^{-1-\rho} r^{-1} \Theta(H - \lambda)). \end{aligned} \quad (2.3.11)$$

Let us combine and bound the first and second terms of (2.3.10) as

$$\begin{aligned} &(A + \frac{i}{2} |\partial f|^2 \theta') \theta' \Theta (A - \frac{i}{2} |\partial f|^2 \theta') + p_j f^{-1} (|\partial f|^2 \delta_{jk} - (\partial_j f)(\partial_k f)) \Theta p_k \\ &\geq \frac{1}{2} (A + \frac{i}{2} |\partial f|^2 \theta') f^{-1} \Theta (A - \frac{i}{2} |\partial f|^2 \theta') \\ &\quad + \frac{1}{2} p_j f^{-1} (|\partial f|^2 \delta_{jk} - (\partial_j f)(\partial_k f)) \Theta p_k \\ &\geq \frac{1}{2} p f^{-1} |\partial f|^2 \Theta p - \frac{1}{8} f^{-1} |\partial f|^4 \theta^2 \Theta - C_4 Q \\ &\geq \frac{1}{8} f^{-1} |\partial f|^4 \theta^2 \Theta + f^{-1} |\partial f|^2 x \Theta - C_5 Q + \operatorname{Re}(f^{-1} |\partial f|^2 \Theta(H - \lambda)). \end{aligned} \quad (2.3.12)$$

Now by (2.3.10), (2.3.11) and (2.3.12) we obtain

$$\begin{aligned} 2\operatorname{Im}(A\Theta(H - \lambda)) &\geq \left(\frac{9}{8} - C_6 f^{-\min\{2, \rho\}}\right) \alpha^2 f^3 r^{-2} \Theta + \left(\frac{1}{2} - C_6 f^{-\min\{2, \rho\}}\right) f^{-1} \Theta \\ &\quad - C_6 \alpha^2 f^{-1} (\chi_{m-1, m+1}^2 + \chi_{n-1, n+1}^2) e^\theta + \operatorname{Re}(\gamma_2(H - \lambda)). \end{aligned}$$

By letting $n_0 \in \mathbb{N}$ large enough we obtain the assertion. $\square$

Proof of Proposition 2.3.2. Let $\phi \in L_{\text{loc}}^2(\mathbb{R}^d)$ and $m_0 \in \mathbb{N}$ satisfy the assumptions of the assertion. Choose $n_0 \geq 0$ in agreement with Lemma 2.3.4. We may let $n_0 \geq m_0 + 3$. We evaluate the inequality (2.3.9) in the state $\chi_{m-2, n+2}\phi$, and then it follows that for any $\alpha \geq 1$ and $n > m \geq n_0$

$$\begin{aligned} \|f^{3/2} r^{-1} \chi_{m, n}^{1/2} \exp(\alpha f^3) \phi\|^2 &\leq C_m \|\chi_{m-1, m+1} \exp(\alpha f^3) \phi\|^2 \\ &\quad + C_1 2^{-n/2} \|\chi_{n-1, n+1} \exp(\alpha f^3) \phi\|^2. \end{aligned} \quad (2.3.13)$$

Since $\bar{\chi}_{m_0} \exp(\alpha f^3) \phi \in \mathcal{B}_0^*$ for any $\alpha > 0$, the second term on the right-hand side of (2.3.13) vanishes in the limit $n \rightarrow \infty$. Hence by the Lebesgue monotone convergence theorem we obtain

$$\|f^{3/2} r^{-1} \bar{\chi}_m^{1/2} \exp(\alpha f^3) \phi\|^2 \leq C_m \|\chi_{m-1, m+1} \exp(\alpha f^3) \phi\|^2,$$

or

$$\|f^{3/2} r^{-1} \bar{\chi}_m^{1/2} \exp[\alpha(f^3 - 2^{3(m+2)})] \phi\|^2 \leq C_m \|\chi_{m-1, m+1} \phi\|^2. \quad (2.3.14)$$

Now assume $\bar{\chi}_{m+2}\phi \not\equiv 0$. The left-hand side of (2.3.14) grows exponentially as $\alpha \rightarrow \infty$ whereas the right-hand side remains bounded. This is a contradiction. Thus $\bar{\chi}_{m+2}\phi \equiv 0$. By invoking the unique continuation property for the second order elliptic operator H , we obtain $\phi \equiv 0$ on $\mathbb{R}^d$. $\square$

2.4. Limiting absorption principle bounds

In this section we prove Theorem 2.1.3.

2.4.1. Commutator estimate. We introduce the weight

$$\Theta = \Theta_\nu^\delta = \int_0^{f/2^\nu} (1+s)^{-1-\delta} ds = [1 - (1 + f/2^\nu)^{-\delta}]/\delta, \quad (2.4.1)$$

where $\delta > 0$ and $\nu \in \mathbb{N}_0$. We denote its derivatives in f by primes such as Θ' , then we have

$$\Theta' = (1 + f/2^\nu)^{-1-\delta}/2^\nu, \quad \Theta'' = -(1 + \delta)(1 + f/2^\nu)^{-2-\delta}/2^{2\nu}. \quad (2.4.2)$$

We note that the weight function Θ of (2.4.1) has the following properties.

Lemma 2.4.1. *Assume Condition 2.1.1 and fix any $\delta > 0$ in the definition (2.4.1) of Θ . Then there exist $c, C, C_k > 0, k = 2, 3, \dots$, such that for any $k = 2, 3, \dots$ and uniformly in $\nu \in \mathbb{N}_0$*

$$\begin{aligned} c/2^\nu &\leq \Theta \leq \min\{C, f/2^\nu\}, \\ c(\min\{2^\nu, f\})^\delta f^{-1-\delta} \Theta &\leq \Theta' \leq f^{-1} \Theta, \\ 0 &\leq (-1)^{k-1} \Theta^{(k)} \leq C_k f^{-k} \Theta. \end{aligned}$$

Proof. Lemma 2.4.1 is same assertion as Lemma 1.4.3 of Chapter 1. Therefore we omit the details. $\square$

The following lemma is key to prove Theorem 2.1.3.

Lemma 2.4.2. *Assume Condition 2.1.1, let $I \subset \mathbb{R}$ be any compact interval and fix any $\delta \in (0, \min\{2, \rho\})$ in the definition (2.4.1) of Θ . Then there exist $c, C > 0$ and $n \geq 0$ such that uniformly in $z \in I_\pm$ and $\nu \in \mathbb{N}_0$, as a quadratic forms on $\mathcal{D}(H)$,*

$$\operatorname{Im}(A\Theta(H-z)) \geq c\Theta' + cA\Theta'A + cp_j h_{jk} \Theta p_k - C\chi_n^2 \Theta - \operatorname{Re}(\gamma(H-z)), \quad (2.4.3)$$

where $\gamma = \gamma_{z,\nu}$ is a uniformly bounded complex-valued function: $|\gamma| \leq C$.

Proof. Let I and δ be as in the assertion. By Lemmas 2.2.1, 2.4.1 and the Cauchy-Schwarz inequality we can bound uniformly in $z = \lambda \pm i\Gamma \in I_\pm$ and $\nu \in \mathbb{N}_0$

$$\begin{aligned} & 2\operatorname{Im}(A\Theta(H-z)) \\ & \geq A\Theta'A + p_j f^{-1}(|\partial f|^2 \delta_{jk} - (\partial_j f)(\partial_k f)) \Theta p_k + p_j f^{-1}|\partial f|^2 (\delta_{jk} - (\partial_j r)(\partial_k r)) \Theta p_k \\ & \quad - 2f^{-1}|\partial f|^2 x \Theta + (\partial_x f) \Theta - \frac{1}{4}|\partial f|^4 \Theta''' - \frac{1}{2}(\partial^f |\partial f|^2) \Theta'' - \frac{1}{2}f^{-1}|\partial f|^4 \Theta'' \\ & \quad + \tilde{q}_1 \Theta' + \tilde{q}_0 \Theta - 2\operatorname{Im}(q_2 \Theta p^f) - 2\operatorname{Re}(f^{-1}|\partial f|^2 \Theta(H-z)) \\ & \quad - \operatorname{Re}(|\partial f|^2 \Theta'(H-z)) \mp 2\Gamma \operatorname{Re}(A\Theta) - C_1 Q \\ & \geq \frac{1}{2}A\Theta'A + \frac{1}{2}p_j |\partial f|^2 \delta_{jk} \Theta' p_k + \frac{1}{2}p_j h_{jk} \Theta p_k + \frac{1}{2}r^{-1}(r-x)\Theta' \\ & \quad \mp 2\Gamma \Theta^{1/2} A \Theta^{1/2} - \operatorname{Re}(\gamma_1(H-z)) - C_2 Q, \end{aligned}$$

where

$$Q = f^{-1-\min\{2,\rho\}} \Theta + p_j r^{-1} f^{-1-\rho} \Theta \delta_{jk} p_k.$$

We can estimate the second term on the right-hand side by

$$\begin{aligned} \frac{1}{2}p_j |\partial f|^2 \delta_{jk} \Theta' p_k &= \operatorname{Re}(|\partial f|^2 \Theta'(H-z)) + x|\partial f|^2 \Theta' - (q-\lambda)|\partial f|^2 \Theta' \\ &\quad + \frac{1}{4}(\Delta|\partial f|^2 \Theta') \\ &\geq \operatorname{Re}(|\partial f|^2 \Theta'(H-z)) + x|\partial f|^2 \Theta' - C_3 Q. \end{aligned}$$

As for the fifth term, using the Cauchy-Schwarz inequality and Lemma 2.4.1, and noting the general identity holding for any real-valued functions $g, h \in C^1(\mathbb{R}^d)$:

$$h \operatorname{Re}(gp^2)h = \operatorname{Re}(h^2 gp^2) + \frac{1}{2}(\partial_j h) \delta_{jk} (\partial_k gh),$$

we can obtain the bound as follows.

$$\begin{aligned} \mp 2\Gamma \Theta^{1/2} A \Theta^{1/2} &= \mp 2\Gamma \Theta^{1/2} (\operatorname{Re} p^f) \Theta^{1/2} \\ &\geq -C_4 \Gamma \Theta^{1/2} f^{-1}(H-\lambda) f^{-1} \Theta^{1/2} - C_4 \Gamma \\ &= -C_4 \Gamma \operatorname{Re}(\Theta^{1/2} f^{-1/2}(H-z) f^{-1/2} \Theta^{1/2}) \pm C_4 \operatorname{Im}(H-z) \\ &\geq -\operatorname{Re}(\gamma_2(H-z)) - C_5 Q. \end{aligned}$$

Hence we have

$$\begin{aligned} 2\operatorname{Im}(A\Theta(H-z)) &\geq \frac{1}{2}A\Theta'A + \frac{1}{2}\Theta' + \frac{1}{2}p_j h_{jk} \Theta p_k - \operatorname{Re}(\gamma_1(H-z)) \\ &\quad - C_6 Q. \end{aligned} \quad (2.4.4)$$

We can combine and estimate the second and fifth terms of (2.4.4) as, for large $n \geq 0$,

$$\frac{1}{2}\Theta' - C_6 Q \geq \frac{1}{4}\Theta' - C_7 \chi_n^2 \Theta - 2C_6 \operatorname{Re}(r^{-1} f^{-1-\min\{2,\rho\}} \Theta(H-z)). \quad (2.4.5)$$

Hence by (2.4.4) and (2.4.5) the assertion follows. $\square$

By using Lemma 2.4.2 we have the following proposition.

Proposition 2.4.3. *Assume Condition 2.1.1, let $I \subset \mathbb{R}$ be any compact interval and fix any $\delta \in (0, \min\{2, \rho\})$ in the definition (2.4.1) of Θ . Then there exist $C > 0$ and $n \geq 0$ such that for all $\phi = R(z)\psi$ with $z \in I_\pm$ and $\psi \in \mathcal{B}$ and for all $\nu \in \mathbb{N}_0$*

$$\begin{aligned} & \|\Theta^{1/2}\phi\|^2 + \|\Theta^{1/2}A\phi\|^2 + \langle p_j h_{jk} \Theta p_k \rangle_\phi \\ & \leq C (\|\phi\|_{\mathcal{B}^*} \|\psi\|_{\mathcal{B}} + \|A\phi\|_{\mathcal{B}^*} \|\psi\|_{\mathcal{B}} + \|\chi_n \Theta^{1/2}\phi\|^2). \end{aligned}$$

For the proof of Theorem 2.1.3 we also use local compactness of the following form.

Proposition 2.4.4. *Assume Condition 2.1.1. Then for any $m \in \mathbb{N}_0$ and compact interval $I \subset \mathbb{R}$ the mapping*

$$\chi_m P_H(I): L^2 \rightarrow L^2$$

is compact, where $P_H(I)$ denotes the spectral projection for H onto I .

Proof. Fix any $m \in \mathbb{N}_0$ and any compact interval $I \subset \mathbb{R}$. We let $\{\psi_k\}_{k \in \mathbb{N}_0} \subset L^2$ be a bounded sequence, and set $\phi_k = \chi_m P_H(I)\psi_k$. First, using Condition 2.1.1, we have

$$\begin{aligned} \|\phi_k\|^2 + \|p\phi_k\|^2 & \leq \|\phi_k\|^2 + 2\langle H \rangle_{\phi_k} - 2\langle -x + q \rangle_{\phi_k} \\ & \leq C_1 \|\phi_k\|^2 + 2\langle H \rangle_{\phi_k} \\ & \leq C_2 \|\psi_k\|^2. \end{aligned}$$

Hence the sequence $\{\phi_k\}_{k \in \mathbb{N}_0}$ is bounded in the standard Sobolev space $H^1(\mathbb{R}^d)$. Then by Rellich's compact embedding theorem and the diagonal argument it suffices to show that

$$\lim_{\nu \rightarrow \infty} \sup_k \|\eta_\nu \phi_k\| = 0; \quad \eta_\nu(x, y) = 1 - \chi(-x/2^\nu), \quad (2.4.6)$$

see (2.1.1) for χ . We fix any $\epsilon > 0$. Next, use Condition 2.1.1 and then for any large $\nu \in \mathbb{N}_0$ independent of $k \in \mathbb{N}_0$ we have

$$\|\eta_\nu \phi_k\|^2 \leq \epsilon \langle -x + q \rangle_{\eta_\nu \phi_k} \leq \epsilon \langle H \rangle_{\eta_\nu \phi_k} \leq C_3 \epsilon \|\psi_k\|^2,$$

where $C_3 > 0$ does not depend on $\epsilon > 0$ or $k \in \mathbb{N}_0$. This verifies (2.4.6), and hence we are done. $\square$

2.4.2. Proof of LAP bounds. Now let us prove Theorem 2.1.3 by Proposition 2.4.3 and contradiction.

Proof of Theorem 2.1.3. Let $I \subset \mathbb{R}$ be any compact interval. For simplicity, we prove the assertion only for the upper sign.

Step 1. First we assume that for $C_1 > 0$ large enough

$$\|R(z)\psi\|_{\mathcal{B}^*} \leq C_1 \|\psi\|_{\mathcal{B}}. \quad (2.4.7)$$

Then the bound (2.1.5) holds. Let us verify this. Fix any $\delta \in (0, \min\{2, \rho\})$. Then by Proposition 2.4.3 and (2.4.7) there exists $C_2 > 0$ such that for any $\psi \in \mathcal{B}$ and $z \in I_+$ uniformly in $\epsilon_1 \in (0, 1)$ and $\nu \in \mathbb{N}_0$

$$\|\Theta^{1/2}AR(z)\psi\|^2 + \langle p_j h_{jk} \Theta p_k \rangle_{R(z)\psi} \leq \epsilon_1 \|AR(z)\psi\|_{\mathcal{B}^*}^2 + \epsilon_1^{-1} C_2 \|\psi\|_{\mathcal{B}}^2. \quad (2.4.8)$$

For each $\nu \geq 0$ by restricting the integral region to $\{2^\nu \leq f < 2^{\nu+1}\}$, we can bound the first term on the left-hand side of (2.4.8) as

$$\|\Theta^{1/2}AR(z)\psi\|^2 \geq 3^{-(1+\delta)} 2^{-\nu} \|F_\nu AR(z)\psi\|^2. \quad (2.4.9)$$

As for the second term on the same side, by looking at the estimate (2.4.8) for any fixed $\nu \in \mathbb{N}_0$, e.g. $\nu = 0$ and using Lemma 2.4.1, we have

$$\langle p_j h_{jk} \Theta p_k \rangle_{R(z)\psi} \geq c_1 \langle p_j h_{jk} p_k \rangle_{R(z)\psi}. \quad (2.4.10)$$

We substitute (2.4.9) and (2.4.10) into (2.4.8), and take the supremum in $\nu \in \mathbb{N}_0$. Then we obtain the following inequality.

$$c_2 \|AR(z)\psi\|_{\mathcal{B}^*}^2 + c_2 \langle p_j h_{jk} p_k \rangle_{R(z)\psi} \leq 2\epsilon_1 \|AR(z)\psi\|_{\mathcal{B}^*}^2 + 2\epsilon_1^{-1} C_2 \|\psi\|_{\mathcal{B}}^2.$$

Therefore by letting $\epsilon_1 \in (0, c_2/2)$ it follows that

$$\|p^f R(z)\psi\|_{\mathcal{B}^*} + \langle p_j h_{jk} p_k \rangle_{R(z)\psi}^{1/2} \leq C_3 \|\psi\|_{\mathcal{B}}.$$

Hence the bound (2.1.5) reduces to the single bound (2.4.7).

Step 2. In this step we prove (2.4.7). We assume the opposite. Then there exist $z_k \in I_+$ and $\psi_k \in \mathcal{B}$ such that

$$\lim_{k \rightarrow \infty} \|\psi_k\|_{\mathcal{B}} = 0, \quad \|R(z_k)\psi_k\|_{\mathcal{B}^*} = 1. \quad (2.4.11)$$

We can easily verify that the following estimate holds by (2.4.11).

$$\|\partial f |p R(z_k)\psi_k\|_{\mathcal{B}^*} \leq C_4.$$

Since $|z_k|$ is bounded uniformly in $k \geq 0$, by choosing a subsequence and retaking $I \subset \mathbb{R}$ slightly larger we may assume that $z_k \in I_+$ converges to some $z \in I \cup I_+$. If the limit z belongs to I_+ , (2.4.11) and the bounds

$$\|R(z_k)\psi_k\|_{\mathcal{B}^*} \leq \|R(z_k)\psi_k\|_{L^2} \leq \|R(z_k)\|_{\mathcal{B}(L^2)} \|\psi_k\|_{L^2} \leq \|R(z_k)\|_{\mathcal{B}(L^2)} \|\psi_k\|_{\mathcal{B}}$$

contradict by taking the limit $k \rightarrow \infty$. Hence we have the limit

$$\lim_{k \rightarrow \infty} z_k = z = \lambda \in I. \quad (2.4.12)$$

Let $s > 1/2$. By choosing a further subsequence we can assume that $R(z_k)\psi_k$ converges weakly to some $\phi \in L_{-s}^2$. Moreover, it holds that $R(z_k)\psi_k$ actually converges strongly in L_{-s}^2 . To verify this let us fix $s' \in (1/2, s)$ and $g \in C_c^\infty(\mathbb{R})$ with $g = 1$ on a neighborhood of I , and decompose for any $n \geq 0$

$$\begin{aligned} f^{-s} R(z_k)\psi_k &= f^{-s} g(H) (\chi_n f^s) (f^{-s} R(z_k)\psi_k) + (f^{-s} g(H) f^s) (\bar{\chi}_n f^{s'-s}) (f^{-s'} R(z_k)\psi_k) \\ &\quad + f^{-s} (1 - g(H)) R(z_k)\psi_k. \end{aligned}$$

By (2.4.11) we see that the last term on the right-hand side converges to 0 in L^2 . Since $f^{-s} g(H) f^s$ is a bounded operator, by choosing $n \geq 0$ sufficiently large the second term can be arbitrarily small in L^2 . Since $f^{-s} g(H) \chi_n$ is a compact operator by Proposition 2.4.4, for fixed $n \geq 0$ the first term converges strongly in L^2 . Therefore $R(z_k)\psi_k$ converges to ϕ in L_{-s}^2 :

$$\lim_{k \rightarrow \infty} R(z_k)\psi_k = \phi \text{ in } L_{-s}^2. \quad (2.4.13)$$

By (2.4.11), (2.4.12) and (2.4.13) it follows that

$$(H - \lambda)\phi = 0 \text{ in the distributional sense.} \quad (2.4.14)$$

In addition, we can obtain $\phi \in \mathcal{B}_0^*$. In fact, let us fix $\delta = 2s - 1 > 0$ and apply Proposition 2.4.3 to $R(z_k)\psi_k$. Then, taking the limit $k \rightarrow \infty$ and using (2.4.11), (2.4.13) and Lemma 2.4.1, we have for all $\nu \in \mathbb{N}_0$

$$\|\Theta^{1/2} \phi\| \leq \|\chi_n \Theta^{1/2} \phi\| \leq C_5 2^{-\nu/2} \|\chi_n f^{1/2} \phi\|. \quad (2.4.15)$$

By taking the limit $\nu \rightarrow \infty$ in (2.4.15), we can verify $\phi \in \mathcal{B}_0^*$. Therefore by (2.4.14) and Theorem 2.1.2, we have $\phi = 0$. On the other hand, similarly to Step 1 we have the bound

$$1 = \|R(z_k)\psi_k\|_{\mathcal{B}^*}^2 \leq C_6 (\|\psi_k\|_{\mathcal{B}} + \|\chi_n R(z_k)\psi_k\|^2).$$

If it holds that $\phi = 0$, the right-hand side converges to 0 as $k \rightarrow \infty$. However this is a contradiction. Hence we have (2.4.7). $\square$

2.5. Radiation condition bounds

Throughout this section we assume Condition 2.1.4. We introduce the operator B :

$$B = \frac{\sqrt{2}}{2} (r^{1/2}(\partial f) \cdot p + p \cdot (\partial f)r^{1/2}), \quad (2.5.1)$$

and set auxiliary complex phase $b_{\pm}$: For $z \in I_{\pm}$

$$b_{\pm} = 2\eta_z r^{1/2} |\partial f| \sqrt{z - q_1 + r} \pm \frac{i}{2\sqrt{2}} r^{-1/2} (\partial f) \cdot (\partial r), \quad (2.5.2)$$

respectively. Since

$$B = \sqrt{2} r^{1/2} A - i \frac{\sqrt{2}}{4} r^{-1/2} (\partial f) \cdot (\partial r) \quad \text{on } C_0^\infty(\mathbb{R}^d),$$

it holds that

$$B \mp b_{\pm} = \sqrt{2} r^{1/2} (A \mp a_{\pm}) \quad \text{on } C_0^\infty(\mathbb{R}^d),$$

respectively (cf. (2.1.6)).

2.5.1. Commutator estimate. In the following we consider only for the upper sign for simplicity.

Lemma 2.5.1. *Let $I \subset \mathbb{R}$ be any compact interval. Then there exist $C > 0$ and $m = m_I \in \mathbb{N}$ such that the following inequalities hold on $\{f \geq 2^m\}$ uniformly in $z \in I \cup I_+$*

$$\begin{aligned} |a| &\leq C, \quad |b| \leq C r^{1/2}, \quad \text{Im } a \geq \frac{1}{2} r^{-1} (\partial f) \cdot (\partial r), \quad \text{Im } b \geq \frac{1}{2\sqrt{2}} r^{-1/2} (\partial f) \cdot (\partial r), \\ |\ell_{\bullet j} \partial_j a| &\leq C r^{-1/2} f^{-1-\min\{2, \rho, \tau\}}, \\ |\sqrt{2} r^{1/2} (p^f b) + b^2 - 4r |\partial f|^2 (z - q_1 + r)| &\leq C f^{-1-\min\{2, \rho, \tau\}}. \end{aligned}$$

Proof. By the definition of a and b (see (2.1.6) and (2.5.2)), the estimates of the first line are clearly hold. Now we choose $m = m_I$ large enough so that $\eta_z = 1$ on $\{f \geq 2^m\}$. Then we can compute on $\{f \geq 2^m\}$

$$\begin{aligned} \ell_{\bullet j} \partial_j a &= \sqrt{2} \ell_{\bullet j} (\partial_j \eta_z) |\partial f| \sqrt{z - q_1 + r} + \sqrt{2} \eta_z \ell_{\bullet j} (\partial_j |\partial f|) \sqrt{z - q_1 + r} \\ &\quad + \sqrt{2} \eta_z \ell_{\bullet j} |\partial f| (\partial_j \sqrt{z - q_1 + r}) + \frac{i}{2} \ell_{\bullet j} (\partial_j r^{-1} (\partial f) \cdot (\partial r)) \\ &= -\frac{1}{2} \eta_z \ell_{\bullet j} r^{-3/2} (\partial_j r) \sqrt{z - q_1 + r} \\ &\quad + \frac{1}{2} \eta_z \ell_{\bullet j} r^{-1/2} (-(\partial_j q_1) + (\partial_j r)) / \sqrt{z - q_1 + r} + \frac{i}{4} \ell_{\bullet j} (\partial_j f r^{-2}) \\ &= -\frac{1}{2} \eta_z \ell_{\bullet j} r^{-3/2} (z - q_1) (\partial_j r) / \sqrt{z - q_1 + r} \\ &\quad - \frac{1}{2} \eta_z \ell_{\bullet j} r^{-1/2} (\partial_j q_1) / \sqrt{z - q_1 + r} + \frac{i}{4} \ell_{\bullet j} (\partial_j f r^{-2}). \end{aligned}$$

Thus by Condition 2.1.4 the fifth estimate holds. Similarly, the last estimate also holds on $\{f \geq 2^m\}$ by the following equality and Condition 2.1.4.

$$\begin{aligned} \sqrt{2} r^{1/2} (p^f b) + b^2 - 4r |\partial f|^2 (z - q_1 + r) \\ = 2\sqrt{2} r^{1/2} (p^f \eta_z r^{1/2} |\partial f|) \sqrt{z - q_1 + r} + i \sqrt{2} \eta_z r |\partial f| (\partial^f q_1) / \sqrt{z - q_1 + r} \end{aligned}$$

$$\begin{aligned}
& -i\sqrt{2}\eta_z r |\partial f| (\partial f) \cdot (\partial r) / \sqrt{z - q_1 + r} + \frac{1}{2} r^{1/2} (\partial^f r^{-1/2} (\partial f) \cdot (\partial r)) \\
& + 4\eta_z^2 r |\partial f|^2 (z - q_1 + r) + i\sqrt{2}\eta_z |\partial f| (\partial f) \cdot (\partial r) \sqrt{z - q_1 + r} \\
& - \frac{1}{8} r^{-1} \{(\partial f) \cdot (\partial r)\}^2 - 4r |\partial f|^2 (z - q_1 + r) \\
& = ir^{1/2} (\partial^f q_1) / \sqrt{z - q_1 + r} - \frac{i}{2} r^{-1/2} f / \sqrt{z - q_1 + r} \\
& + \frac{1}{4} r^{1/2} (\partial^f f r^{-3/2}) + \frac{i}{2} f r^{-3/2} \sqrt{z - q_1 + r} - \frac{1}{32} f^2 r^{-3} \\
& = ir^{1/2} (\partial^f q_1) / \sqrt{z - q_1 + r} + \frac{i}{2} f r^{-3/2} (z - q_1) / \sqrt{z - q_1 + r} \\
& + \frac{1}{4} r^{1/2} (\partial^f f r^{-3/2}) - \frac{1}{32} f^2 r^{-3}.
\end{aligned}$$

Hence we are done. $\square$

Now we rewrite $H - z$ by using the operator B of (2.5.1).

Lemma 2.5.2. *Let $I \subset \mathbb{R}$ be any compact interval. Then there exist a complex-valued function q_3 and constants $C > 0$ and $m = m_I \in \mathbb{N}$ such that uniformly in $z \in I \cup I_+$, as a quadratic form on $C_0^\infty(\mathbb{R}^d)$,*

$$\begin{aligned}
H - z &= \frac{1}{2}(B + b)\tilde{\eta}(B - b) + \frac{1}{2}p_j \ell_{jk} p_k + (r - x) + q_3, \\
|q_3| &\leq C f^{-1-\min\{2, \rho, \tau\}} \quad \text{on } \{f \geq 2^m\}.
\end{aligned}$$

Proof. We can compute

$$\begin{aligned}
H - z &= \frac{1}{2}B\tilde{\eta}B + \frac{1}{2}p_j \ell_{jk} p_k + \frac{1}{4}\tilde{\eta}(\Delta f)(\partial f) \cdot (\partial r) + \frac{1}{4}(\partial^f \tilde{\eta})(\partial f) \cdot (\partial r) \\
& + \frac{1}{4}\tilde{\eta}(\partial f) \cdot ((\partial^2 f) \cdot (\partial r)) + \frac{1}{4}\tilde{\eta}(\partial f) \cdot ((\partial f) \cdot (\partial^2 r)) \\
& - \frac{1}{16}\tilde{\eta}r^{-1}((\partial f) \cdot (\partial r))^2 - x + q - z \\
& = \frac{1}{2}B\tilde{\eta}B + \frac{1}{2}p_j \ell_{jk} p_k - x + q + q_0 - z \\
& = \frac{1}{2}(B + b)\tilde{\eta}(B - b) + \frac{1}{2}p_j \ell_{jk} p_k + \frac{\sqrt{2}}{2}r^{1/2}(p^f \tilde{\eta})b + \frac{\sqrt{2}}{2}r^{1/2}\tilde{\eta}(p^f b) \\
& + \frac{1}{2}\tilde{\eta}b^2 - x + q + q_0 - z \\
& = \frac{1}{2}(B + b)\tilde{\eta}(B - b) + \frac{1}{2}p_j \ell_{jk} p_k + (r - x) + q_3,
\end{aligned}$$

where

$$\begin{aligned}
q_0 &= \frac{1}{4}\tilde{\eta}(\Delta f)(\partial f) \cdot (\partial r) + \frac{1}{4}(\partial^f \tilde{\eta})(\partial f) \cdot (\partial r) + \frac{1}{4}\tilde{\eta}(\partial f) \cdot ((\partial^2 f) \cdot (\partial r)) \\
& + \frac{1}{4}\tilde{\eta}(\partial f) \cdot ((\partial f) \cdot (\partial^2 r)) - \frac{1}{16}\tilde{\eta}r^{-1}((\partial f) \cdot (\partial r))^2, \\
q_3 &= \frac{1}{2}\tilde{\eta} \left[\sqrt{2}r^{1/2}(p^f b) + b^2 - 4r|\partial f|^2(z - q_1 + r) \right] + \frac{\sqrt{2}}{2}r^{1/2}(p^f \tilde{\eta})b \\
& - (1 - \eta)(z - q_1 + r) + q_0 + q_2.
\end{aligned}$$

For $m \geq 0$ large enough it holds that $\eta = \tilde{\eta} = 1$. Then by using Lemma 2.5.1, we obtain the assertion. $\square$

We use also this section the same weight as (2.4.1)

$$\Theta = \Theta_\nu^\delta = \int_0^{f/2^\nu} (1 + s)^{-1-\delta} ds = [1 - (1 + f/2^\nu)^{-\delta}] / \delta; \quad \delta > 0, \quad \nu \in \mathbb{N}_0.$$

We denote its derivatives in f by primes such as (2.4.2).

Lemma 2.5.3. *Let $I \subset \mathbb{R}$ be any compact interval, and fix any $\beta \in (0, 1/2)$ and $\delta \in (0, \min\{1, \rho/2, \tau\})$. Then there exist $c, C > 0$ and $m = m_I \in \mathbb{N}$ such that uniformly in $z \in I_+$ and $n \in \mathbb{N}_0$, as quadratic forms on $\mathcal{D}(H)$*

$$\begin{aligned} & \operatorname{Im}((A - a)^* \bar{\chi}_m \Theta^{2\beta} (H - z)) \\ & \geq c(A - a)^* \bar{\chi}_m^2 \Theta' \Theta^{2\beta-1} (A - a) + cp_j \bar{\chi}_m \Theta^{2\beta} h_{jk} p_k \\ & \quad - C f^{-1-2\min\{1, \rho/2, \tau\}+2\delta} \Theta^{2\beta} - \operatorname{Re}(\gamma \Theta^{2\beta} (H - z)), \end{aligned}$$

where γ is a certain function satisfying $|\gamma| \leq C f^{-1-2\min\{1, \rho, \tau\}+2\delta}$.

Proof. Let I, δ and β be as in the assertion, and fix $m \in \mathbb{N}$ large enough so that $\eta_z = 1$ on $\operatorname{supp} \bar{\chi}_m$. To prove the asserted inequality it suffices to compute as a quadratic form on $C_0^\infty(\mathbb{R}^d)$. By Lemmas 2.4.1 and 2.5.2 and the Cauchy-Schwarz inequality it follows that uniformly in $z \in I_+$ and $n \in \mathbb{N}_0$

$$\begin{aligned} & 2 \operatorname{Im}((A - a)^* \bar{\chi}_m \Theta^{2\beta} (H - z)) \\ & = \sqrt{2}(A - a)^* \operatorname{Im}(\bar{\chi}_m \Theta^{2\beta} B \tilde{\eta} r^{1/2})(A - a) \\ & \quad + \sqrt{2}(A - a)^* \bar{\chi}_m \Theta^{2\beta} (\operatorname{Im} b) \tilde{\eta} r^{1/2} (A - a) \\ & \quad + \operatorname{Im}(A \bar{\chi}_m \Theta^{2\beta} p_j \ell_{jk} p_k) - \operatorname{Im}(a^* \bar{\chi}_m \Theta^{2\beta} p_j \ell_{jk} p_k) \\ & \quad + 2 \operatorname{Im}((A - a)^* \bar{\chi}_m \Theta^{2\beta} (r - x)) + 2 \operatorname{Im}((A - a)^* \bar{\chi}_m \Theta^{2\beta} q_3) \\ & \geq (A - a)^* \beta \bar{\chi}_m \Theta' \Theta^{2\beta-1} (A - a) + \operatorname{Im}(A p_j \bar{\chi}_m \Theta^{2\beta} \ell_{jk} p_k) - \operatorname{Im}(a^* \bar{\chi}_m \Theta^{2\beta} p_j \ell_{jk} p_k) \\ & \quad + 2 \operatorname{Im}((A - a)^* \bar{\chi}_m \Theta^{2\beta} (r - x)) - (A - a)^* f^{-1-2\delta} \bar{\chi}_m \Theta^{2\beta} (A - a) - C_1 Q \\ & \geq c_1 (A - a)^* \bar{\chi}_m^2 \Theta' \Theta^{2\beta-1} (A - a) + \frac{1}{2} [p_j \bar{\chi}_m \Theta^{2\beta} \ell_{jk} p_k, iA] \\ & \quad - \operatorname{Im}(a^* \bar{\chi}_m \Theta^{2\beta} p_j \ell_{jk} p_k) + 2 \operatorname{Im}((A - a)^* \bar{\chi}_m \Theta^{2\beta} (r - x)) - C_2 Q, \end{aligned} \tag{2.5.3}$$

where

$$Q = f^{-1-2\min\{1, \rho/2, \tau\}+2\delta} \Theta^{2\beta} + p_j \bar{\chi}_{m-1} r^{-1} f^{-1-2\min\{1, \rho, \tau\}+2\delta} \Theta^{2\beta} \delta_{jk} p_k.$$

We compute and estimate the second term of (2.5.3) as follows.

$$\begin{aligned} & \frac{1}{2} [p_j \bar{\chi}_m \Theta^{2\beta} \ell_{jk} p_k, iA] \\ & = \frac{1}{2} p_j \left[(\partial^2 f)_{ki} \bar{\chi}_m \Theta^{2\beta} \ell_{ij} + (\partial^2 f)_{ij} \bar{\chi}_m \Theta^{2\beta} \ell_{ik} - (\partial^f \bar{\chi}_m \Theta^{2\beta} \ell_{jk}) \right] p_k \\ & \quad - \frac{1}{2} \operatorname{Im}((\partial_j \Delta f) \bar{\chi}_m \Theta^{2\beta} \ell_{jk} p_k) \\ & = \frac{1}{2} p_j \left(f^{-1} |\partial f|^2 \delta_{ki} - f^{-1} (\partial_k f) (\partial_i f) - f^{-1} |\partial f|^2 (\partial_k r) (\partial_i r) \right) \bar{\chi}_m \Theta^{2\beta} \ell_{ij} p_k \\ & \quad + \frac{1}{2} p_j \left(f^{-1} |\partial f|^2 \delta_{ij} - f^{-1} (\partial_i f) (\partial_j f) - f^{-1} |\partial f|^2 (\partial_i r) (\partial_j r) \right) \bar{\chi}_m \Theta^{2\beta} \ell_{ik} p_k \\ & \quad - \frac{1}{2} p_j (\partial^f \bar{\chi}_m) \Theta^{2\beta} \ell_{jk} p_k - p_j \bar{\chi}_m \beta |\partial f|^2 \Theta' \Theta^{2\beta-1} \ell_{jk} p_k \\ & \quad - \frac{1}{2} p_j \bar{\chi}_m \Theta^{2\beta} (\partial^f \ell_{jk}) p_k - \frac{1}{2} \operatorname{Im}((\partial_j \Delta f) \bar{\chi}_m \Theta^{2\beta} \ell_{jk} p_k) \\ & \geq \frac{1}{2} p_j \left(f^{-1} |\partial f|^2 \ell_{jk} - f^{-1} |\partial f|^2 (\partial_j r) (\partial_k r) + f^{-1} (\partial f) \cdot (\partial r) (\partial_j f) (\partial_k r) \right) \bar{\chi}_m \Theta^{2\beta} p_k \\ & \quad + \frac{1}{2} p_j \left(f^{-1} |\partial f|^2 \ell_{jk} - f^{-1} |\partial f|^2 (\partial_j r) (\partial_k r) + f^{-1} (\partial f) \cdot (\partial r) (\partial_j r) (\partial_k f) \right) \bar{\chi}_m \Theta^{2\beta} p_k \\ & \quad - p_j \bar{\chi}_m \beta |\partial f|^2 \Theta' \Theta^{2\beta-1} \ell_{jk} p_k + p_j (\partial f) \cdot (\partial r) (\partial_j f) (\partial_k f) \bar{\chi}_m \Theta^{2\beta} p_k \\ & \quad + \frac{1}{2} p_j |\partial f|^{-2} (\partial_i f) (\partial_{ij}^2 f) (\partial_k f) \bar{\chi}_m \Theta^{2\beta} p_k \\ & \quad + \frac{1}{2} p_j |\partial f|^{-2} (\partial_j f) (\partial_i f) (\partial_{ik}^2 f) \bar{\chi}_m \Theta^{2\beta} p_k - C_3 Q \end{aligned}$$

$$\begin{aligned} &\geq p_j(h_{jk}\Theta - \beta|\partial f|^2\Theta'\ell_{jk})\bar{\chi}_m\Theta^{2\beta}p_k - p_jf^{-1}|\partial f|^2\bar{\chi}_m\Theta^{2\beta}\delta_{jk}p_k \\ &\quad + p_j(\partial f) \cdot (\partial r)(\partial_j f)(\partial_k f)\bar{\chi}_m\Theta^{2\beta}p_k - C_3Q. \end{aligned} \quad (2.5.4)$$

By Lemma 2.5.1, we can bound the third term of (2.5.3) as follows.

$$\begin{aligned} & - \operatorname{Im}(a^*\bar{\chi}_m\Theta^{2\beta}p_j\ell_{jk}p_k) \\ &= p_j(\operatorname{Im} a)\bar{\chi}_m\Theta^{2\beta}\ell_{jk}p_k - \operatorname{Re}((\partial_j a)^*\bar{\chi}_m\Theta^{2\beta}\ell_{jk}p_k) \\ &\geq p_j|\partial f|^2(\partial f) \cdot (\partial r)\bar{\chi}_m\Theta^{2\beta}\ell_{jk}p_k - \frac{1}{2}p_jf^{-1}|\partial f|^2\ell_{jk}\bar{\chi}_m\Theta^{2\beta}p_k - C_4Q. \end{aligned} \quad (2.5.5)$$

We combine the second and third terms of (2.5.4) and the first term of (2.5.5) and estimate by

$$\begin{aligned} &p_j(-f^{-1}|\partial f|^2\delta_{jk} + (\partial f) \cdot (\partial r)(\partial_j f)(\partial_k f) + |\partial f|^2(\partial f) \cdot (\partial r)\ell_{jk})\bar{\chi}_m\Theta^{2\beta}p_k \\ &= \frac{1}{2}p_j(fr^{-1} - 2f^{-1})|\partial f|^2\bar{\chi}_m\Theta^{2\beta}\delta_{jk}p_k \\ &= \operatorname{Re}((fr^{-1} - 2f^{-1})|\partial f|^2\bar{\chi}_m\Theta^{2\beta}\frac{1}{2}p^2) + \frac{1}{4}(\Delta(fr^{-1} - 2f^{-1})|\partial f|^2\bar{\chi}_m\Theta^{2\beta}) \\ &\geq (fr^{-1} - 2f^{-1})x|\partial f|^2\bar{\chi}_m\Theta^{2\beta} + \operatorname{Re}((fr^{-1} - 2f^{-1})|\partial f|^2\bar{\chi}_m\Theta^{2\beta}(H - z)) \\ &\quad - C_5Q. \end{aligned} \quad (2.5.6)$$

As for the fourth term of (2.5.3) we can estimate

$$\begin{aligned} &2\operatorname{Im}((A - a)^*\bar{\chi}_m\Theta^{2\beta}(r - x)) \\ &= 2\operatorname{Im}(A\bar{\chi}_m\Theta^{2\beta}(r - x)) + 2(\operatorname{Im} a)\bar{\chi}_m\Theta^{2\beta}(r - x) \\ &\geq -2\beta|\partial f|^2(r - x)\bar{\chi}_m\Theta'\Theta^{2\beta-1} + \frac{1}{2}fr^{-2}(r - x)\bar{\chi}_m\Theta^{2\beta} - C_6Q \\ &\geq -2\beta f^{-1}|\partial f|^2(r - x)\bar{\chi}_m\Theta^{2\beta} + fr^{-1}(r - x)|\partial f|^2\bar{\chi}_m\Theta^{2\beta} - C_6Q. \end{aligned} \quad (2.5.7)$$

By the bounds (2.5.3), (2.5.4), (2.5.5), (2.5.6) and (2.5.7), we have

$$\begin{aligned} &2\operatorname{Im}((A - a)^*\bar{\chi}_m\Theta^{2\beta}(H - z)) \\ &\geq c_1(A - a)^*\bar{\chi}_m^2\Theta'\Theta^{2\beta-1}(A - a) + p_j(\frac{1}{2}h_{jk}\Theta - \beta|\partial f|^2\Theta'\ell_{jk})\bar{\chi}_m\Theta^{2\beta-1}p_k \\ &\quad + (fr^{-1}x - 2f^{-1}x - 2\beta f^{-1}(r - x) + fr^{-1}(r - x))|\partial f|^2\bar{\chi}_m\Theta^{2\beta} \\ &\quad + \operatorname{Re}((fr^{-1} - 2f^{-1})|\partial f|^2\bar{\chi}_m\Theta^{2\beta}(H - z)) - C_7Q. \end{aligned} \quad (2.5.8)$$

Noting that $\beta \in (0, 1/2)$, we can estimate the second and the third terms by

$$\begin{aligned} &p_j(\frac{1}{2}h_{jk}\Theta - \beta|\partial f|^2\Theta'\ell_{jk})\bar{\chi}_m\Theta^{2\beta-1}p_k \geq c_2p_jh_{jk}\bar{\chi}_m\Theta^{2\beta}p_k, \\ &(fr^{-1}x - 2f^{-1}x - 2\beta f^{-1}(r - x) + fr^{-1}(r - x))|\partial f|^2\bar{\chi}_m\Theta^{2\beta} \\ &= (1 - 2\beta)f^{-1}(r - x)|\partial f|^2\bar{\chi}_m\Theta^{2\beta} \geq 0, \end{aligned} \quad (2.5.9)$$

respectively. Finally we bound the remainder term Q by

$$\begin{aligned} -Q &\geq -C_8f^{-1-2\min\{1, \rho/2, \tau\}+2\delta}\Theta^{2\beta} \\ &\quad - 2\operatorname{Re}(r^{-1}f^{-1-2\min\{1, \rho, \tau\}+2\delta}\Theta^{2\beta}(H - z)). \end{aligned} \quad (2.5.10)$$

By substituting (2.5.9) and (2.5.10) into (2.5.8), we have the assertion. $\square$

Proof of Theorem 2.1.5. Let $I \subset \mathbb{R}$ be any compact interval and fix $m = m_I$ large enough. If $\beta = 0$, the assertion is obvious by Theorem 2.1.3. Hence we may assume $\beta \in (0, \beta_c)$. We take any

$$\delta \in (0, \min\{1/2, \rho/2, \tau\} - \beta).$$

By Lemma 2.5.3, the Cauchy-Schwarz inequality and Theorem 2.1.3 there exist $C_1, C_2 > 0$ such that for any $R(z)\psi$ with $\psi \in f^{-\beta}\mathcal{B}$ and $z \in I_+$

$$\begin{aligned} & \|\bar{\chi}_m \Theta^{1/2} \Theta^{\beta-1/2} (A-a) R(z) \psi\|^2 + \langle p_j \bar{\chi}_m \Theta^{2\beta} h_{jk} p_k \rangle_{R(z)\psi} \\ & \leq C_1 \left[\|\bar{\chi}_m \Theta^\beta (A-a) R(z) \psi\|_{\mathcal{B}^*} \|\Theta^\beta \psi\|_{\mathcal{B}} + \|f^{-1/2-\min\{1/2, \rho/2, \tau\}+\delta} \Theta^\beta R(z) \psi\|^2 \right. \\ & \quad \left. + \|f^{1/2-\min\{1/2, \rho/2, \tau\}+\delta} \Theta^\beta \psi\|^2 \right] \\ & \leq C_2 2^{-2\nu\beta} \left[\|\bar{\chi}_m f^\beta (A-a) R(z) \psi\|_{\mathcal{B}^*} \|f^\beta \psi\|_{\mathcal{B}} + \|f^\beta \psi\|_{\mathcal{B}}^2 \right]. \end{aligned} \quad (2.5.11)$$

By commuting $R(z)$ and powers of f , we obtain $f^\beta (A-a) R(z) \psi \in \mathcal{B}^*$ for each $z \in I_+$. Hence the quantity on the right-hand side of (2.5.11) is not infinite. Then it follows that by (2.5.11)

$$\begin{aligned} & 2^{2\nu\beta} \|\bar{\chi}_m \Theta^{1/2} \Theta^{\beta-1/2} (A-a) R(z) \psi\|^2 + 2^{2\nu\beta} \langle p_j \bar{\chi}_m \Theta^{2\beta} h_{jk} p_k \rangle_{R(z)\psi} \\ & \leq C_2 \left[\|\bar{\chi}_m f^\beta (A-a) R(z) \psi\|_{\mathcal{B}^*} \|f^\beta \psi\|_{\mathcal{B}} + \|f^\beta \psi\|_{\mathcal{B}}^2 \right]. \end{aligned} \quad (2.5.12)$$

In the first term on the left-hand side of (2.5.12), we restrict the integral region to $\{2^\nu \leq f < 2^{\nu+1}\}$ and take the supremum in $\nu \in \mathbb{N}_0$, and then we obtain

$$c_1 \|\bar{\chi}_m f^\beta (A-a) R(z) \psi\|_{\mathcal{B}^*}^2 \leq C_2 \left[\|\bar{\chi}_m f^\beta (A-a) R(z) \psi\|_{\mathcal{B}^*} \|f^\beta \psi\|_{\mathcal{B}} + \|f^\beta \psi\|_{\mathcal{B}}^2 \right].$$

By the Cauchy-Schwarz inequality we have

$$\|\bar{\chi}_m f^\beta (A-a) R(z) \psi\|_{\mathcal{B}^*} \leq C_3 \|f^\beta \psi\|_{\mathcal{B}}. \quad (2.5.13)$$

As for the second term on the left-hand side of (2.5.12) we use (2.5.13) and take limit $\nu \rightarrow \infty$, and then by Lebesgue monotone convergence theorem and the concavity of Θ we obtain

$$\langle p_j \bar{\chi}_m f^{2\beta} h_{jk} p_k \rangle_{R(z)\psi} \leq C_4 \|f^\beta \psi\|_{\mathcal{B}}^2.$$

Therefore, by using Theorem 2.1.3, we have

$$\begin{aligned} & \|f^\beta (A-a) R(z) \psi\|_{\mathcal{B}^*} + \langle p_j f^{2\beta} h_{jk} p_k \rangle_{R(z)\psi} \\ & \leq \|\chi_m f^\beta (A-a) R(z) \psi\|_{\mathcal{B}^*} + \langle p_j \chi_m f^{2\beta} h_{jk} p_k \rangle_{R(z)\psi}^{1/2} \\ & \quad + \|\bar{\chi}_m f^\beta (A-a) R(z) \psi\|_{\mathcal{B}^*} + \langle p_j \bar{\chi}_m f^{2\beta} h_{jk} p_k \rangle_{R(z)\psi}^{1/2} \\ & \leq C \|f^\beta \psi\|_{\mathcal{B}}. \end{aligned}$$

Hence we are done. $\square$

2.5.2. Proof of limiting absorption principle.

Proof of Corollary 2.1.6. Let $s > 1/2$ be as in the assertion. Then we fix any $\omega \in (0, \min\{\beta_c, s - 1/2\})$. First, we decompose for $n \geq 0$ and $z, z' \in I_+$ as

$$\begin{aligned} R(z) - R(z') &= \chi_n R(z) \chi_n - \chi_n R(z') \chi_n \\ &+ (R(z) - \chi_n R(z) \chi_n) - (R(z') - \chi_n R(z') \chi_n). \end{aligned} \quad (2.5.14)$$

By Theorem 2.1.3 we can estimate uniformly in $m \geq 0$ and $z, z' \in I_+$ as

$$\begin{aligned} & \|R(z) - \chi_n R(z) \chi_n\|_{\mathcal{B}(L_s^2, L_{-s}^2)} \\ & \leq \|f^{-s} R(z) \bar{\chi}_n f^{-s}\|_{\mathcal{B}(L^2)} + \|f^{-s} \bar{\chi}_n R(z) \chi_n f^{-s}\|_{\mathcal{B}(L^2)} \\ & \leq C_1 2^{-(s-1/2)n}. \end{aligned} \quad (2.5.15)$$

Similarly, we obtain

$$\|R(z') - \chi_n R(z') \chi_n\|_{\mathcal{B}(L_s^2, L_{-s}^2)} \leq C_2 2^{-(s-1/2)n}. \quad (2.5.16)$$

As for the first and second terms on the right-hand side of (2.5.14), noting the equation

$$i[H, \chi_n] = \operatorname{Re}(\chi'_n p^f) = \operatorname{Re}(\chi'_n A) \quad (2.5.17)$$

and the identity $\overline{a_z} = a_{\bar{z}}$, we can rewrite as

$$\begin{aligned} & \chi_n R(z) \chi_n - \chi_n R(z') \chi_n \\ & = \chi_n R(z) \{ \chi_{n+1}(H - z') - (H - z) \chi_{n+1} \} R(z') \chi_n \\ & = \chi_n R(z) \{ (z - z') \chi_{n+1} + i \operatorname{Re}(\chi'_{n+1} A) \} R(z') \chi_n \\ & = \frac{i}{2} \chi_n R(z) \chi'_{n+1} (A - a_{z'}) R(z') \chi_n + \frac{i}{2} \chi_n R(z) (A + a_{\bar{z}})^* \chi'_{n+1} R(z') \chi_n \\ & \quad - \frac{i}{2} \chi_n R(z) (a_z - a_{z'}) \chi'_{n+1} R(z') \chi_n + (z - z') \chi_n R(z) \chi_{n+1} R(z') \chi_n \\ & = \frac{i}{2} \chi_n R(z) \chi'_{n+1} (A - a_{z'}) R(z') \chi_n + \frac{i}{2} \chi_n R(z) (A + a_{\bar{z}})^* \chi'_{n+1} R(z') \chi_n \\ & \quad - \frac{i}{2} \chi_n R(z) (a_z - a_{z'}) \chi'_{n+1} R(z') \chi_n + (z - z') \chi_n R(z) \chi_m R(z') \chi_n \\ & \quad - (z - z') \chi_n R(z) \chi_{m,n+1} (a_z + a_{z'})^{-1} (A - a_{z'}) R(z') \chi_n \\ & \quad + (z - z') \chi_n R(z) (A + a_{\bar{z}})^* \chi_{m,n+1} (a_z + a_{z'})^{-1} R(z') \chi_n \\ & \quad - (z - z') \chi_n R(z) [A, \chi_{m,n+1} (a_z + a_{z'})^{-1}] R(z') \chi_n. \end{aligned}$$

Here $m \in \mathbb{N}_0$ is chosen so that $(a_z + a_{z'})^{-1}$ is non-singular on $\operatorname{supp} \bar{\chi}_m$. Then by Theorems 2.1.3 and 2.1.5 we have uniformly in $n > m \geq 0$ and $z, z' \in I_+$

$$\|\chi_n R(z) \chi_n - \chi_n R(z') \chi_n\|_{\mathcal{B}(L_s^2, L_{-s}^2)} \leq C_3 2^{-\omega n} + C_3 2^{(1-\omega)n} |z - z'|. \quad (2.5.18)$$

By (2.5.14), (2.5.15), (2.5.16) and (2.5.18), we obtain uniformly in large $n \in \mathbb{N}_0$ and $z, z' \in I_+$

$$\|R(z) - R(z')\|_{\mathcal{B}(L_s^2, L_{-s}^2)} \leq C_4 2^{-\omega n} + C_3 2^{(1-\omega)n} |z - z'|.$$

Now, if $|z - z'| \leq 2^{-(m+1)}$, we choose $n \in \mathbb{N}_0$ so that $2^n \leq |z - z'|^{-1} \leq 2^{n+1}$, and then we obtain uniformly in $z, z' \in I_+$

$$\|R(z) - R(z')\|_{\mathcal{B}(L_s^2, L_{-s}^2)} \leq C_5 |z - z'|^\omega. \quad (2.5.19)$$

The same bound is trivial for $|z - z'| > 2^{-(m+1)}$. Hence the Hölder continuity (2.1.8) for $R(z)$ follows from (2.5.19). By using the Hölder continuity of $R(z)$, we can also obtain the Hölder continuity (2.1.8) for $|\partial f| p R(z)$.

The existence of the limits of (2.1.9) follows immediately from (2.1.8). By using (2.1.9), we see that the limits $R(\lambda \pm i0)$ and $|\partial f| p R(\lambda \pm i0)$ map into $\mathcal{B}^*$. Then by a density argument the limits extended continuously to maps $\mathcal{B} \rightarrow \mathcal{B}^*$. $\square$

Proof of Corollary 2.1.7. Let $\beta \in [0, \beta_c)$ be as in the assertion. By Theorem 2.1.5 there exists $C > 0$ such that for any $\Gamma > 0$ and $n \in \mathbb{N}$

$$\|\chi_n f^\beta (A - a) R(\lambda + i\Gamma) \psi\|_{\mathcal{B}^*} \leq C \|f^\beta \psi\|_{\mathcal{B}}, \quad \psi \in C_0^\infty(\mathbb{R}^d).$$

By taking the limit $\Gamma \downarrow 0$ and using Corollary 2.1.6 and a density argument, we obtain

$$\|\chi_n f^\beta (A - a) R(\lambda + i0) \psi\|_{\mathcal{B}^*} \leq C \|f^\beta \psi\|_{\mathcal{B}}, \quad \psi \in f^{-\beta} \mathcal{B}.$$

Finally by the Lebesgue's monotone convergence theorem we obtain

$$\|f^\beta (A - a) R(\lambda + i0) \psi\|_{\mathcal{B}^*} \leq C \|f^\beta \psi\|_{\mathcal{B}}, \quad \psi \in f^{-\beta} \mathcal{B}.$$

Similarly, we obtain

$$\langle p_j f^{2\beta} h_{jk} p_k \rangle_{R(\lambda + i0) \psi}^{1/2} \leq C \|f^\beta \psi\|_{\mathcal{B}}.$$

Hence we are done. $\square$

Proof of Corollary 2.1.8. Let $\lambda \in \mathbb{R}$, $\phi \in L_{\text{loc}}^2$ and $\psi \in f^{-\beta} \mathcal{B}$ with $\beta \in [0, \beta_c)$. First we let $\phi = R(\lambda + i0) \psi$. Then by Corollaries 2.1.6 and 2.1.7, noting $\beta < \beta_c$, we can verify that (1) and (2) of the corollary hold. Conversely, we assume (1) and (2) of the corollary, and let

$$\phi' = \phi - R(\lambda + i0) \psi.$$

Then by the above argument we have

$$(H - \lambda) \phi' = 0 \text{ in the distributional sense,} \quad (2.5.20)$$

and

$$\phi' \in f^\beta \mathcal{B}^*, \quad (A - a) \phi' \in f^{-\beta} \mathcal{B}_0^*. \quad (2.5.21)$$

In addition, we can see that $\phi' \in \mathcal{B}_0^*$. In fact, noting the identity, cf. (1.1.5) and (2.5.17),

$$2 \operatorname{Im}(\chi_\nu(H - \lambda)) = (\operatorname{Re} a) \chi'_\nu + \operatorname{Re}(\chi'_\nu(A - a))$$

with $a = a_\lambda$ and $\operatorname{Re} a > 0$ we have the bound

$$0 \leq \langle (\operatorname{Re} a) \bar{\chi}'_\nu \rangle_{\phi'} = \operatorname{Re} \langle \chi'_\nu(A - a) \rangle_{\phi'}. \quad (2.5.22)$$

In (2.5.22), we take the limit $\nu \rightarrow \infty$ and use (2.5.21), and then we obtain $\phi' \in \mathcal{B}_0^*$. By (2.5.20) and Theorem 2.1.2 it follows that $\phi' = 0$, i.e. $\phi = R(\lambda + i0) \psi$. $\square$

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