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博 士 論 文

令和元年 12 月

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経済学専攻

指導教員　宮川栄一

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Part I

Introduction

This doctoral thesis analyzes policies and institutions by applying game theory. The thesis is comprised by three part: (i) theoretical analysis of copyright infringement on internet, (ii) effect of severe punishment on crime rate and investigating accuracy: trade-off between crime deterrence and accuracy in court and (iii) a political alienation causes political divergence and its implication for turnout quorum.

The first part deals with copyright infringement on internet. Some kinds of goods such as book, music and software are uploaded without any permission of copyright holders and are downloaded illegally. Several papers provide reasons why some people upload illegally. For example, such a person uploads to gain "externality effect" (Takeyama, 1994). In this thesis, we focus on advertising revenue as the reason. Digital citizens alliance and JASRAC report that some people make illegal site to gain advertising revenue by uploading goods. As in Digital citizens alliance's report, it is estimated that these illegal sites received $ 209 million in 2014.

Piracy exists without any permission of copyright holders. Piracy is undesirable for the purpose of protecting copyright holders since consumers may not purchase a genuine product. In Japan, JASRAC and Japan Affiliate Organization (JAO) take measures against these illegal sites by a suspension of payment of advertising revenue and a prohibition to put advertisement on the web sites. In addition, in Japan, illegal uploading has been non-crime requiring a complaint from the victim for prosecution since 2018.

In this part, we investigate how a piracy affects on a producer's profit and analyze characteristics of the optimal punishment for a producer's profit.

We first investigate whether a piracy reduce a profit or not and we get two results. The first result is that, in large population, a piracy reduce a profit if advertising revenue per a download is sufficiently small and a valuation for the good is distributed uniformly. In large population, advertising revenue grows up even if advertising revenue per a download is low. Hence, some consumers upload the good to gain advertising revenue. Therefore, other consumers keep from purchasing a genuine product and download the good uploaded on internet.

The second result is that if the distribution of valuation for the good satisfies a specific condition, a piracy improves a profit. Specifically, for any price, if an expected advertising revenue at the price is greater than the price, then a piracy increases a volume of consumer who purchases the good. Hence, a piracy benefits a profit if this condition holds at a price that is the optimal price without piracy. This result implies that a suspension as mentioned above may reduce a producer's profit. Similarly, a policy that illegal uploading is treated as non-crime requiring a complaint from the victim for prosecution may reduce the profit.

Next, we analyze characteristics of the optimal punishment for a producer's profit. We investigate a characteristic of the optimal punishment for illegal uploading and find that the punishment should be 0 or a sufficiently high to eliminate an incentive of illegal uploading completely. This result implies that a halfway punishment may not be optimal for a producer's profit. We also investigate a characteristic of the optimal punishment for illegal downloading. We get a similar result under which some assumption is imposed. That is, a halfway punishment for illegal downloading may not be optimal for a producer's profit.

In the second part, we investigate how a severe punishment on crime affects on a crime rate and an accuracy in court. Posner (1973) mentions about the relation between a crime rate and an accuracy in court. However, the author does not provide an explicit theoretical model. As in the paper, three costs should be considered when we estimate a social loss of a crime. The first cost is one that occurs when we deter a crime. The second cost is a lose that is caused by a crime. The last cost is one that a person who does not commit a crime may be falsely accused in court. In addition, the author mentions that a fact that a prosecutor can choose whether to prosecute or not to prosecute affects on the last cost. In this thesis, we construct a theoretical model that describes a process which include a decision of committing a crime and a trial. We investigate effects of a severe punishment in the model.

We consider the model where a suspect, a prosecutor and a judge are engaged in. The suspect decides whether to commit a crime or not to commit a crime. The suspect is arrested with some probability that depends on a suspect's decision. After arresting the suspect, the prosecutor decides whether to investigate or not to investigate. We suppose that a prosecutor's ability of investigating is private information. This ability can be interpreted as an ability of proving. We assume that the prosecutor prefers a judgement of guilty to a judgement of innocent as in Posner (1973). If the prosecutor investigates, he gains a evidence. After gaining a evidence, the prosecutor decides whether to prosecute or not to prosecute. If the prosecutor prosecutes, then the judgement observes the evidence that is provided by the prosecutor and then the judgement adjudges the suspect guilty or innocent.

Several papers i.g. Becker (1968) and Ehrlich (1973) deal a prosecutor's decision and a punishment as an exogenous parameter. In our model, a prosecutor's decision is decided endogenously. From this, we get non monotonous results. The first result is that a severe punishment improves a crime rate and an accuracy in court when a crime rate is relatively low. That is, there is no trade-off between a crime deterrence and a prevention of false charge when a crime rate is relatively low. When a crime rate is relatively high, a severe punishment reduces a crime rate. However, an accuracy in court decreases by a severe punishment. That is, there is a trade-off between a crime deterrence and a prevention of false charge when a crime rate is relatively high. This result implies that we should take this trade-off into account when we consider a social loss of a crime.

In the third part, we analyze how a political alienation affect on policies that candidates choose in election and we argue an implication for turnout quorum. A political alienation is a feeling that a voter receives if a policy is far from his satisfaction. According to Adams et al. (2003, 2006) and Brown (2014), a voter abstains if he feels a political alienation. Adams et al. (2003) incorporates an abstention by a political alienation in a voting model. Precisely, a voter abstains if no policy gives a sufficient utility and a voter votes otherwise. In Adams et al. (2003), the authors does not investigate an equilibrium policy. In this part, we incorporate a political alienation in a probabilistic voting model and we analyze an effect of a political alienation.

In the model, two candidates compete in election. Voters are comprised by three segment, left-wing, right-wing and neutral. A policy that candidates choose is composed by two components, a public goods expenditure and an expenditure that benefit a specific group. We assume that a voter in neutral gains utility only from a public goods expenditure and other voters gains utility only from an expenditure for group that he belongs. Moreover, we assume that a voter in right-wing or left-wing feel a dissatisfaction if an opposite group is given an expenditure.

First, we investigate an equilibrium policy under which all the voter has to vote. The result is that there is an unique equilibrium and equilibrium policies converge to a policy that maximizes a social welfare.

Next, we investigate an equilibrium policy under which voters abstain by a political alienation. In this case, equilibrium policies diverge. Hence, a policy that maximizes a social welfare comes not to be chosen due to abstention by a political alienation. In the case without a political alienation, candidates choose their policies taking all voters into account since any voter comes to a poll. However, in the case with a political alienation, candidates choose their policies taking voters who is likely to come to a poll into account. As a result, equilibrium policies diverge. Moreover, at some parameter, candidates choose extreme policy and then all the voters in neutral abstain. In this case, an equilibrium policy that wins the election is very close to a policy that minimizes a social welfare. However, if we set a turnout quorum that is larger than amount of voters except for neutral, then at least we prevent such a policy being chosen. Hence, a turnout quorum prevent candidates from choosing extreme policies. This aspect of a turnout quorum does not appear in Hizen-Shinmyo (2009) and some other papers that study a turnout quorum.

Part II

Theoretical Analysis of Copyright Infringement on Internet

1 Introduction

Some kinds of goods are illegally uploaded on internet. For example, book, music, software and movies are typically goods that are uploaded. Since these goods are easy to copy, once these are uploaded, many people can share and download these goods. What does motivate people to upload these goods illegally? Several papers provide a motivation of this activity. *Net work externalities* that is the effect that a size of users increases a value of a product is one reason that people copy the goods (Conner and Rumelt (1991), Takeyama (1994) and so on). Another reason is to use the good within a group of users (Bakos et al (1999), Galbreth and Ghosh (2012), Jiang and Tian (2018)). In this paper, we focus on advertising revenue. Digital citizens alliance and JASRAC report that copyright infringement web sites made with the intention of gaining advertise revenue. As Digital citizens alliance report, an estimated total advertising revenue in 2014 reach to $ 209 million. Thus, advertising revenue is one of the main purpose to upload illegally.

Illegal uploading and downloading may be undesirable from the perspective of protecting creators because these activities prevent consumers from purchasing a genuine product. In Japan, JASRAC and Japan Affiliate Organization (JAO) take measures against these web sites by a suspension of payment of advertising revenue and a prohibition to put advertisement on the web sites. In addition, in Japan, illegal uploading has been non-crime requiring a complaint from the victim for prosecution since 2018.

In this paper, we first investigate how these activities affect on the producer's profit and compare it with an optimal profit under which there are no these activities i.e. an usual monopoly situation:

Prop. 1. If advertising revenue per a downloading is sufficiently small and a distribution of consumer's valuation for the good is an uniform distribution, then the optimal producer's profit is less than the optimal monopoly profit in large population.

Prop. 2. For any price that is in range of valuation for the good, if average advertising revenue at the price is larger than the price, then the producer's profit at the price is larger than the monopoly profit at the price.

Proposition 1 is a negative result and proposition 2 is a positive one.

In our model, consumers who upload the good illegally purchase the good and its incentive depends on advertising revenue. In large population, some consumer uploads the good even if advertising revenue per a downloading is small i.e. piracy occurs. In addition, large population implies that a probability that some consumer uploads the good is high. Then, most consumer choose to download the good rather than to buy the good under which the distribution of the valuation for the good is uniform. Consequently, the result that is mentioned by proposition 1 occurs.

The positive result does not arise from *product diffusions*[1] as in Givon et al. (1995) and . Prasad and Mahajan (2003). Too see the reason, consider the following rough example. Suppose that there is a consumer who face to decide whether to upload or to download under which the price is p and suppose that his valuation for the good is p. In addition, assume that there is no punishment for illegal uploading and downloading. If he downloads the good, he receives p. If he uploads the good, his utility is (the valuation − price + advertising revenue) i.e. the utility is advertising revenue. Assume that any other consumer uploads the good if the valuation is larger than p and download the good if the valuation is less than p. Thus, the former fraction is $1 - F(p)$ and the latter fraction is $F(p)$ where F is c.d.f. of the valuation. If the probability of getting one download is same across consumers who upload,

[1]Piracy enhance diffusion of the good. As a result, potential users become aware of the product and may buy the genuine product.

this probability is $\frac{F(p)}{1-F(p)}$. Denote advertising revenue per a downloading by γ, then average advertising revenue is $\frac{\gamma F(p)}{1-F(p)}$. If $\frac{\gamma F(p)}{1-F(p)} > p$, he decides to uploading. Hence, the valuation of the marginal type of consumer who purchases the good is lower than p. Thus, the producer's profit at the price is larger than the monopoly profit at the price.

Proposition 2 implies that the measure such as a suspension of payment of advertising revenue may not be desirable policy for creators. Moreover, the proposition implies that uniform punishment such as a policy that illegal uploading and downloading are treated as non-crime requiring a complaint from the victim for prosecution may also not be desirable policy.

As mentioned above, we claim that illegal uploading and downloading are treated as non-crime requiring a complaint from the victim for prosecution may also be not desirable policy. Hence, these activities may be treated as crime requiring a complaint from the victim for prosecution. If the producer takes a person who acts these illegal activities to court, then how degree a punishment for the illegal activities should be is an important question. We provide two results for this question and these two result suggest that the optimal punishment should be changed depending on a situation that the producer faces.

First one (proposition 3) is that a punishment for illegal uploading should be 0 or the one that eliminates an incentive of uploading the good completely from the aspect of the optimality for the producer's profit. In this model, consumers who upload the good illegally purchase the good and an increasing of punishment for uploading merely decreases an incentive for uploading. Consequently, an increasing of punishment for uploading causes a reduction of the profit. Therefore, the optimal producer's profit is achievable when the punishment is 0 or when the punishment satisfies that it eliminates an incentive of uploading the good completely.

Second one (proposition 4) is that the optimal punishment for illegal downloading should be in certain range that piracy occurs under which c.d.f of the valuation for the good is uniform. The effect of raising the punishment is not simple comparing with the effect of a punishment for illegal uploading. There are two effects of raising the punishment. One effect is an increasing of cost for downloading. The other one is a reduction of advertising revenue due to a decrease of consumers who download. If the latter exceeds the former, then the profit decreases as the punishment increases. The latter exceeds the former when the punishment is larger than the range and the c.d.f is uniform. Thus, the optimal producer's profit is achievable when the punishment is in the range or when the punishment satisfies that it eliminates an incentive of uploading the good completely.

2 Model

There are one producer and $n \geq 2$ consumers. We denote the producer by a and the set of consumer, $\{1, ..., n\}$, by N. Each consumer i has valuation θ_i for the good. The valuation is distributed over $[0, \overline{\theta}]$, where $\overline{\theta} > 0$. Its c.d.f is F and its density is $f > 0$ if $\theta \in (0, \overline{\theta})$. This valuation is a private information of consumer i, and these valuation is independently determined among consumers.

The producer chooses to produce the good or not to produce. If the producer chooses not to produce, there is no continuance after that. If the producer chooses to produce, the producer also decide price, p, of the good, where $p \geq 0$. Furthermore, it incurs a fixed cost $C \geq 0$. We assume that marginal cost is 0 since the good is easy to copy.

When the good is produced, consumers can choose to buy, to buy and to upload or to wait. We denote these actions b, b_u, w respectively. If a consumer chooses b, its utility is $\theta_i - p$. If a consumer chooses b_u, its utility is $\theta_i - p - c_u +$(advertising revenue), where $c_u \geq 0$. We define advertising revenue precisely later. If a consumer chooses w, its utility is 0. All the consumers decide these actions simultaneously.

After determination of their action, only consumers who choose w make a decision again. If some consumer chooses b_u at the first decision, the consumer can choose to buy, to download or not to do. If a consumer chooses to buy, its utility is $\theta_i - p$. If a consumer chooses to download, d, its utility is $\theta_i - c_d$, where $c_d \geq 0$ is cost to download. If a consumer does nothing, its utility is 0. $\varnothing$ indicate that a consumer chooses not to do. They can choose b or $\varnothing$ when no consumer chooses b_u at the first decision. Utility for choosing these actions is same as described above.

Here, we summarize the timing of events as follows:

Stage 0. $\theta_1, \ldots, \theta_n$ are determined by nature.

Stage 1. The producer chooses to produce the good or not. The producer decides p when the good is produced.

Stage 2. If the good is produced, all the consumers choose b, b_u or w simultaneously.

Stage 3. Only consumers who choose w at Stage 2,

(a) If some consumer chooses b_u at Stage 2, then they choose b, d or $\varnothing$,

(b) Otherwise, they choose b or $\varnothing$.

We need to define advertising revenue. We assume that advertising revenue is divided evenly among consumers who upload the good. Precisely, we define it as follows: Denoting advertising revenue by $e(N_d, N_u)$,

$$e(N_d, N_u) := \gamma \frac{N_d}{N_u},$$

where

$$N_d := |\{i \in N : i \text{ chose } w \text{ and } d.\}|,$$
$$N_u := |\{i \in N : i \text{ chose } b_u.\}|$$

and $\gamma > 0$ is payment per downloading.

2.1 Consumer's behavior at Stage 3.

We describe consumer's behavioral Strategy at Stage 3. An action profile at Stage 2 is denoted by $x \in \{b, b_u, w\}^n$ and we denote the behavioral strategy by g_i. The behavioral strategy g_i is a function that map from $[0,\overline{\theta}]\times\{b,b_u,w\}^n\times\mathbb{R}_+$ to $\{b,d,\varnothing\}$ such that $g_i(\theta,x,p)\in\{b,d,\varnothing\}$ if $x_i=b_u$ for some j and $g_i(\theta,x,p)\in\{b,\varnothing\}$ if $x_j\neq b_u$ for all j. Although there are various g_i, we provide some restriction on g_i. However, we restrict g_i in an intuitive way.

Here, we define an indicator function $\mathbb{1}$ as follows:

$$\mathbb{1}(x):=\begin{cases}1 & \text{if } x_i=b_u \text{ for some } i\in N,\\ 0 & \text{if } x_i\neq b_u \text{ for all } i\in N.\end{cases}$$

First, we assume that

$$\forall i\in N,\ \forall\theta\in[0,\overline{\theta}],\ \forall x,x'\in\{b,b_u,w\}^n,\ \mathbb{1}(x)=\mathbb{1}(x')\Rightarrow g_i(\theta,x,p)=g_i(\theta,x',p). \tag{1}$$

This assumption implies that the decision of consumer does not depend on which consumer uploads the good. Hence, virtually, an optimal action is determined by $\mathbb{1}(x)$ and θ_i.

First, we consider about an optimal action in the case that $\mathbb{1}(x)=0$. Feasible action for consumer is b or $\varnothing$ since the good is not uploaded. Therefore, g_i such that

$$\text{if } \mathbb{1}(x)=0, \text{then } g_i(\theta,x,p)=b \iff \theta\geq p \tag{2}$$

is an optimal behavior in this case.

Next, we consider about an optimal action in the case that $\mathbb{1}(x)=1$. In this case, an optimal action depends on θ_i, p and c_d. An optimal action depends on whether $p\leq c_d$ or not, since $p\leq c_d$ if and only $\theta_i-p\geq\theta_i-c_d$. In what follow, we consider two cases.

Case 1. $p\leq c_d$.

In this case, $\theta_i-p\geq\theta_i-c_d$. Hence, b is an optimal action for consumer if $\theta_i\geq p$.

Therefore, g_i such that

$$\text{if } \mathbb{1}(x) = 1 \text{ and } p \leq c_d, \text{then } g_i(\theta, x, p) = \begin{cases} b & \text{if } \theta_i \geq p, \\ \varnothing & \text{if } \theta_i < p \end{cases} \tag{3}$$

is an optimal behavior in the case.

Case 2. $p > c_d$.

In this case, $\theta_i - p < \theta_i - c_d$. Hence, d is an optimal action for consumer if $\theta_i \geq c_d$. Therefore, g_i such that

$$\text{if } \mathbb{1}(x) = 1 \text{ and } p > c_d, \text{then } g_i(\theta, x, p) = \begin{cases} d & \text{if } \theta_i \geq c_d, \\ \varnothing & \text{if } \theta_i < c_d \end{cases} \tag{4}$$

is an optimal behavior in the case.

We denote g_i that satisfies (1), (2), (3), (4) by g^*. Henceforth, we assume that all the consumers take an action according to g^* at Stage 3[2].

Assumption 1. *For any* $i \in N$, $g_i = g^*$.

2.2 Consumer's behavior at Stage 2.

In this section, we argue about a consumer's behavior at Stage 2. We denote the behavioral strategy by s_i and hence $s : \mathbb{R}_+ \times [0, \overline{\theta}] \to \{b, b_u, w\}$.

First, we consider the case that $p \leq c_d$. Suppose that no consumer uploads the good, i.e. $\mathbb{1}(x) = 0$. Then, no consumer downloads the good at Stage 3 since all the consumers behave according to g^*. Thus, $N_d = 0$. This leads fact that $e(0, N_u) = 0$. Therefore, utility when consumer chooses b is lager than or equal to utility when consumer chooses b_u, since $\theta_i - p \geq \theta_i - p - c_u$. If consumer chooses w and no consumer choose b_u, then $g^* \in \{b, \varnothing\}$. Hence, utility when consumer chooses w is

[2] A behavior of indifference type does not change the result.

$\max\{\theta_i - p, 0\}$. Therefore, a strategy profile that all the consumers implement s_i such that

$$s_i(p,\theta) = \begin{cases} b & \text{if } \theta_i \geq p, \\ w & \text{if } \theta_i < p \end{cases} \tag{5}$$

is an equilibrium in subgame after Stage 2 where $p \leq c_d$.

Next, we consider the case that $p > c_d$. In the above case, there is no consumer who chooses d when $\mathbb{1}(x) = 1$. Thus, at Stage 2, b and w are alternatives that are superior to b_u. In this case, a cost to download is lower than the price. Thus, some consumer may choose to w with expectation that other consumer uploads the good. That expectation depends on who uploads the good. That is, a situation after Stage 2 is changed by s_i. Here, we proceed an analysis with assumption that all the consumers implement a same behavioral strategy, that is, $s_i = s_j$ for all consumer $i, j \in N$. Henceforth, we omit a subscript and denote it by s. Here, we introduce the definition of cut-off strategy.

Definition 1. *A behavioral strategy s is a cut-off strategy if there exists $k \in [0, \overline{\theta}]$ such that $\theta \geq k \iff s(p,\theta) = b_u$.*

The definition implies that only consumers whose valuation satisfies that $\theta_i \geq k$ chooses b_u. Thus, given that symmetric and cut-off strategy, consumer i can compute a probability that some other consumer uploads the good and its probability is

$$\begin{aligned} \text{Prob}(\exists j \neq i \text{ s.t. } \theta_j \geq k) &:= P(k) \\ &= 1 - \text{Prob}(\forall j \neq i,\ \theta_j < k) \\ &= 1 - [F(k)]^{n-1}. \end{aligned}$$

Hence, we can compute expected utility for choosing an action. We denote it by $V(y, \theta; k)$, where $y \in \{b, b_u, w\}$. Obviously, $V(b, \theta; k) = \theta - p$. Hereafter, $V(b, \theta)$

indicates $V(b,\theta;k)$ since $V(b,\theta;k)$ does not depend on value of k. By assumption 1,

$$V(w,\theta;k) = \begin{cases} \theta - [P(k)c_d + (1-P(k))p] & \text{if } \theta \geq p, \\ P(k)(\theta - c_d) & \text{if } \theta \in [c_d, p), \\ 0 & \text{if } \theta < c_d. \end{cases}$$

Here, we introduce an assumption.

Assumption 2. *For any $i \in N$, if $V(b,\theta) = V(w,\theta;k)$ and $\theta < k$, then i chooses w.*

Before we compute $V(b_u,\theta;k)$, we must compute expected advertising revenue. Given assumption 2, some calculus shows that[3]

$$\mathbb{E}[e](k) = \gamma(F(k) - F(c_d)) \sum_{t=0}^{n-2} (F(k))^t.$$

Hence, $V(b_u,\theta;k) = \theta - p - c_u + \mathbb{E}[e](k)$. Since advertising revenue, $\mathbb{E}[e](k)$, increases as k increases, $\mathbb{E}[e](k)$ reaches the maximum value, $\gamma(n-1)(1-F(c_d))$, at $k = \overline{\theta}$. Therefore, no consumer chooses b_u if $\gamma(n-1)(1-F(c_d)) < c_u$. Hence, we deduce the following lemma.

Lemma 1. *Consider all the subgame after Stage 2 where $p > c_d$. If $\gamma(n-1)(1-F(c_d)) < c_u$, then strategy profile such that all consumers implement a same cut-off strategy is not an equilibrium in the subgame after Stage 2.*

By lemma 1, if there exists such an equilibrium, then $\gamma(n-1)(1-F(c_d)) \geq c_u$. However, converse is not true since no consumer may choose b_u if the price is too high. In what follows, we provide a condition that guarantees an existence of such an equilibrium.

Lemma 2. *Consider all the subgame after Stage 2 where $p > c_d$. Suppose that $\gamma(n-1)(1-F(c_d)) \geq c_u$. Strategy profile such that all consumers implement a*

[3]See appendix.

same cut-off strategy is an equilibrium in the subgame after Stage 2 if and only if $p \leq \overline{\theta} - c_u + \gamma(n-1)(1 - F(c_d))$.

Leaving the proof of lemma 2 to appendix, we descrive a sketch of the lemma.

Given that all the other consumer's cut-off is k, the consumer can compute a cut-off $\varphi(k)$ such that

$$\begin{cases} V(b_u, \theta; k) \geq V(w, \theta; k) & \text{if } \theta \geq \varphi(k), \\ V(b_u, \theta; k) \leq V(w, \theta; k) & \text{if } \theta < \varphi(k), \end{cases}$$

Furthermore, $\varphi(k)$ decreases as k increases. Hence, $\varphi(\overline{\theta}) \leq \varphi(k)$ for all k. The condition that $p \leq \overline{\theta} - c_u + \gamma(n-1)(1 - F(c_d))$ guarantees $\varphi(\overline{\theta}) \leq \overline{\theta} = k$. In other words, the consumer has an incentive to upload at least when advertising revenue reaches maximum. Thus, k decreases from $\overline{\theta}$, we can find that there exists k such that $\varphi(k) = k$ and $k \leq p$ or that there exists k such that $\varphi(k) = p$ and $k > p$ (Figure 1, 2). In addition, such a k is uniquely determined since $\varphi(k)$ strictly decrease as k increases.

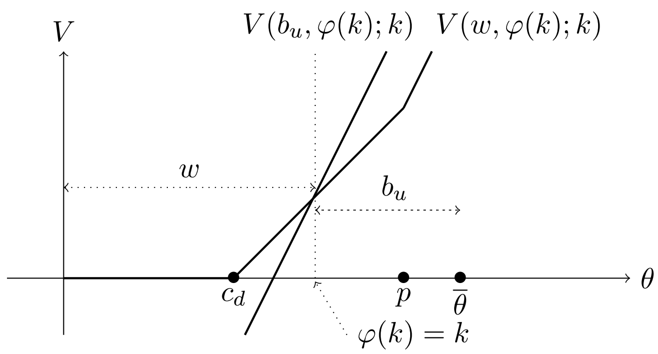

Figure 1:

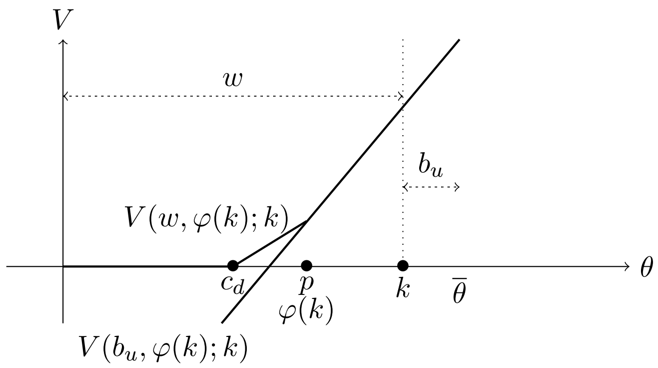

Figure 2:

We denote such a k by $k(p; \lambda)$ where $\lambda = (c_u, c_d, \gamma, n)$ and define s as

$$s(\theta, p) = \begin{cases} b_u & \text{if } \theta \geq k(p; \lambda), \\ w & \text{if } \theta < k(p; \lambda). \end{cases} \tag{6}$$

Then, a strategy profile such that all the consumers implement s is a symmetric equilibrium.

By lemma 2, if $p > \overline{\theta} - c_u + \gamma(n-1)(1 - F(c_d))$, then there does not exist a symmetric cut-off equilibrium. Thus, $p_h := \overline{\theta} - c_u + \gamma(n-1)(1 - F(c_d))$ is the upper bound for the existence of the equilibrium. We assume that all the consumers implement s such that (5) if $p > p_h$. Obviously, this action profile is an equilibrium if $p > p_h$.

We will proceed the analysis with the assumption that a behavioral strategy s satisfies (6) if $p \in (c_d, p_h]$ and satisfies (5) if $p \notin (c_d, p_h]$. Hereafter, we denote s that satisfies the above assumption by s^*. Before proceeding further discussion, we argue about properties of $k(p; \lambda)$.

By assumptions so far, a cut-off point, $k(p; \lambda)$, virtually determine the demand for the good. First, we investigate how the price affect $k(p; \lambda)$.

Pick any two prices p, p' such that $c_d < p < p' < p_h$. Then, the cost of uploading increases more than that of waiting. Hence, fixing $k(p;\lambda)$, the proportion of uploading is high so that b_u is inferior to w for a consumer whose valuation is $k(p;\lambda)$. Then, we need to increase $k(p;\lambda)$ so as to find $k(p';\lambda)$. By lemma 2, there must be $k(p';\lambda)$.

Lemma 3. *If $c_d < p < p' < p_h$, then $k(p;\lambda) < k(p';\lambda)$.*

Proof. See appendix. □

3 Producer's behavior

Producer can predict demand for the good, provided that each consumer implements s^*. In usual case, a probability that one consumer buys the good is $1 - F(p)$. In this case, the probability is determined by $k(p;\lambda)$. As previous section, there are two cases. One is $k(p;\lambda) < p$, the another is $k(p;\lambda) > p$. In the former case, the probability is $1 - F(k(p;\lambda))$ since consumer whose valuation is less than $k(p;\lambda)$ does not buy the good even if no one uploads the good. In the latter case, a consumer buys the good if $k(p;\lambda) \le \theta$ and also buys if $k(p;\lambda) > \theta \ge p$ and no one uploads the good. Thus, expected profit for producer is[4]

$$\Pi_a(p;\lambda) = \begin{cases} np[1 - F(k(p;\lambda))] & \text{if } k(p;\lambda) \le p, \\ np[1 - F(k(p;\lambda)) + F(k(p;\lambda))^{n-1}(F(k(p;\lambda)) - F(p))] & \text{if } k(p;\lambda) > p. \end{cases}$$

Note that $\Pi_a(p;\lambda)$ is less than the profit in monopoly case if $k(p;\lambda) > p$[5].

First, we state the following proposition.

Proposition 1. *Suppose that F is an uniform distribution, γ is sufficiently small and n is sufficiently large. Then, illegal uploading cause a reduction of producer's profit.*

Proof. Consider the case that $k = p < \overline{\theta}$. If $V(b_u, p; p) > V(w, p; p)$, then we deduce that $k(p;\lambda) < p$ since a proportion $1 - F(k(p;\lambda))$ is small so that a consumer whose

[4]The calculation is in appendix.

[5]$np[1 - F(k(p;\lambda)) + F(k(p;\lambda))^{n-1}(F(k(p;\lambda)) - F(p))] - np[1 - F(p)] < 0$.

valuation is $p - \delta < \theta < p$ switch his action from w to b_u. If $V(b_u, p; p) < V(w, p; p)$, then $k(p; \lambda) > p$. The difference is

$$\begin{aligned} h(p) &:= V(b_u, p; p) - V(w, p; p) \\ &= p - p - c_u + \mathbb{E}[e](k) - P(p)(p - c_d) \\ &= -c_u + P(p)(p - c_d)\left[\frac{\gamma}{\overline{\theta} - p} - 1\right]. \end{aligned}$$

The lowest γ that sustains the existence of cut-off equilibrium is $\frac{c_u}{(n-1)(1-F(c_d))}$. For sufficient large n, there exist a γ such that $\gamma > \frac{c_u}{(n-1)(1-F(c_d))}$ and $\frac{\gamma}{\overline{\theta}-p} - 1 < 0$. Thus, the difference satisfies that (i) $h(c_d) = -c_u \leq 0$, (ii) there exists sufficiently small $\varepsilon > 0$ such that $h(c_d + \varepsilon') < 0$ for all $0 < \varepsilon' < \varepsilon$ and (iii) $h(\overline{\theta}) = -c_u + \mathbb{E}[e](\overline{\theta}) > 0$. This implies that there exists $\hat{p} \in (0, \overline{\theta}]$ such that $h(p) < 0$ for all $p < \hat{p}$ and it depends on γ. In addition, $\hat{p}$ goes $\overline{\theta}$ as γ goes $\frac{c_u}{(n-1)(1-F(c_d))}$. Hence, there exists $\gamma < \frac{c_u}{(n-1)(1-F(c_d))}$ such that $h(p) < 0$ for all $p < \hat{p} \approx \overline{\theta}$. Therefore, $\Pi_a(p; \lambda)$ is less than the monopoly profit for $p < \hat{p} \approx \overline{\theta}$. □

Suppose that $\Pi_m^* > C > \Pi_a$ where Π_m^* is the maximal profit in monopoly case. Then, illegal uploading cause that the producer withdraws from market. Hence, consumer surplus disappear.

There are two ways that avoid such a result. First one is to raise c_d. If c_d rises to the monopoly price, then it guarantees the maximal monopoly profit. Second one is to raise c_u. An rising c_u offsets advertising revenue. By lemma 1, if the condition that $c_u > \gamma(n-1)(1 - F(c_d))$ holds, then c_u removes an incentive to upload.

In this model, we also have the result that illegal uploading may improve Π_a.

Proposition 2. *Suppose that $c_u = c_d = 0$. For all $\theta \in (0, \overline{\theta})$, if $F(\theta) > \frac{\theta}{\gamma+\theta}$, then $k(p; \lambda) < p$ if $p = \theta$.*

Proof. Pick any $\theta \in (0, \overline{\theta})$ and suppose that $F(\theta) > \frac{\theta}{\gamma+\theta}$ and $p = \theta$. Then, $F(p) >$

$\frac{p}{\gamma+p}$. We rearrange it and we have

$$P(p)\gamma\frac{F(p)}{1-F(p)} > P(p)p.$$

The left hand side is $V(b_u, p; p) = p - p - c_u + \mathbb{E}[e](p)$. On the other hand, the right hand side is $V(w, p; p) = P(p)(p - c_d)$. Hence, we get $V(b_u, p; p) > V(w, p; p)$. Therefore, $k(p; \lambda) < p$. □

Proposition 2 implies that $\Pi_a(p_m; \lambda)$ is lager than the profit in monopoly case if $F(p_m) > \frac{p_m}{\gamma+p_m}$. Therefore, $\max_p \Pi_a(p; \lambda) \geq \Pi_a(p_m; \lambda) > \Pi_m^*$. If $\max_p \Pi_a(p; \lambda) \geq C > \Pi_m^*$, the producer enter the market, but he does not enter the market nevertheless the situation is monopoly.

In the proof of proposition 2, we rearrange the condition $F(p_m) > \frac{p_m}{\gamma+p_m}$. We can also represent this condition as

$$\gamma\frac{F(p_m)}{1-F(p_m)} > p_m.$$

The left hand side is an average advertising revenue when a consumer whose valuation is less than p_m downloads and a consumer whose valuation is larger than p_m uploads. This condition implies that a net expected gain of uploading is larger than that of downloading for the marginal type p_m. As a consequence, the marginal type p_m buys the good. Thus, in the case that $c_u = c_d = 0$, the producer's profit $\max_p \Pi_a(p; \lambda)$ is higher than the monopoly profit if the average advertising revenue is higher than p_m.

3.1 Effect of c_u and c_d.

In this section, we consider how c_u and c_d impact on the producer's profit. To see it, we investigate how c_u and c_d affect on $k(p; \lambda)$.

If there exists a cut-off equilibrium, a condition that $V(b_u, k(p; \lambda); k(p; \lambda)) = V(w, k(p; \lambda); k(p; \lambda))$ holds. Suppose that $V(b_u, k(p; \lambda); k(p; \lambda)) > V(w, k(p; \lambda); k(p; \lambda))$

by a parameter changing from λ to λ'. Then, b_u is a better action than w for a consumer whose valuation is $k(p;\lambda)$. Hence, such a consumer changes his action from w to b_u. As a result, an equilibrium cut-off $k(p;\lambda')$ is less than $k(p;\lambda)$. Conversely, $k(p;\lambda')$ is larger than $k(p;\lambda)$ when $V(b_u, k(p;\lambda); k(p;\lambda)) < V(w, k(p;\lambda); k(p;\lambda))$ by a change of parameter. Therefore, we should compare $V(b_u, k(p;\lambda)$ and $V(w, k(p;\lambda)$ to see the effect.

First, we gain the following lemma.

Lemma 4. *If* $c'_u > c_u$, *then* $k(p;\lambda') > k(p;\lambda)$ *where* $\lambda' = (c'_u, c_d, \gamma, n)$, $\lambda = (c_u, c_d, \gamma, n)$.

Proof. If c_u increases to c'_u, only $V(b_u, \cdot; \cdot)$ goes down. Thus, $V(b_u, k(p;\lambda); k(p;\lambda))|_{c'_u} < V(w, k(p;\lambda); k(p;\lambda))|_{c'_u}$. That is, since increasing cost for uploading merely reduce utility for uploading, a proportion of uploading, i.e. $1 - k(p;\lambda)$ is too high that sustains the same proportion. Hence, $1 - k(p;\lambda)$ must decrease. Therefore, $1 - k(p;\lambda') < 1 - k(p;\lambda)$, that is, $k(p;\lambda') > k(p;\lambda)$. □

An upload cost c_u only affect on consumer's utility for uploading. Thus, an increasing of c_u leads an increasing of a cut-off point.

On the other hand, an effect of c_d is ambiguous. The ambiguity stems from the fact that c_d affects not only on utility for downloading, but also on utility for uploading. Utility for $k(p;\lambda)$ when he chooses w is

$$V(w, k(p;\lambda); k(p;\lambda)) = \begin{cases} k(p;\lambda) - (P(k(p;\lambda)c_d + (1 - P(k(p;\lambda))p) & \text{if } k(p;\lambda) > p, \\ P(k(p;\lambda) - c_d) & \text{if } k(p;\lambda) \leq p. \end{cases}$$

Thus, a marginal change for c_d is $-P(k(p;\lambda))$. Meanwhile, utility for $k(p;\lambda)$ when he chooses b_u is

$$V(b_u, k(p;\lambda); k(p;\lambda)) = k(p;\lambda) - p - c_u + \gamma P(k(p;\lambda)) \frac{F(k(p;\lambda)) - F(c_d)}{1 - F(k(p;\lambda))}.$$

Thus, a marginal change for c_d is $-\frac{\gamma P(k(p;\lambda)) f(c_d)}{1 - F(k(p;\lambda))}$. An increase of c_d reduce a fraction of consumers who downloads. As a result, marginal loss of advertising revenue is $\frac{\gamma P(k(p;\lambda))f(c_d)}{1-F(k(p;\lambda))}$. Hence, we get the following lemma.

Lemma 5. $\frac{dk(p;\lambda)}{dc_d} \gtreqless 0 \iff \frac{\gamma f(c_d)}{1-F(k(p;\lambda))} \gtreqless 1$.

A severe punishment for illegal uploading and illegal downloading can guarantee the producer the monopoly profit. In this model, (c_u, c_d) that satisfies $c_d \geq \overline{\theta}$ or $c_u \geq \gamma(n-1)(1-F(c_d))$ is such a punishment. When (c_u, c_d) satisfies $c_d < \overline{\theta}$ and $c_u < \gamma(n-1)(1-F(c_d))$, is there the optimal punishment for the producer' profit? First, we state the following proposition.

Proposition 3. *Suppose that* $c_d < \overline{\theta}$. *If* $c_u^* \in \arg\max_{c_u}[\max_p \Pi_a(p;\lambda)]$ *and* $\max_{c_u}[\max_p \Pi_a(p;\lambda)] > \Pi_m^*$, *then* $c_u^* = 0$.

Proof. Before proving the proposition, we introduce a notation and we simply write c_u instead of λ. We denote p such that $p \in \arg\max_{p'} \Pi_a(p'; c_u)]$ by $p(c_u)$.

Suppose that $0 < c_u^*$. By the supposition that $\max_{c_u}[\max_p \Pi_a(p;c_u)] = \Pi_a(p(c_u^*); c_u^*) > \Pi_m^*$, $k(p(c_u^*); c_u^*) < p(c_u^*)$ since $\Pi_a(p(c_u^*); c_u^*) < \Pi_m(p(c_u^*)) \leq \Pi_m^*$ if $k(p(c_u^*); c_u^*) \geq p(c_u^*)$.

By lemma 4, if $0 < c_u' < c_u^*$, then $k(p(c_u^*); c_u') < k(p(c_u^*); c_u^*)$. Hence, $\Pi_a(p(c_u^*); c_u') > \Pi_a(p(c_u^*); c_u^*)$. Therefore, $\Pi_a(p(c_u'); c_u') \geq \Pi_a(p(c_u^*); c_u') > \Pi_a(p(c_u^*); c_u^*)$, a contradiction. □

A consumer who buys the good is a consumer who uploads the good illegally at p such that $k(p;\lambda) \leq p$. Thus, an increase of upload cost decreases an incentive of buying the good. Therefore, the producer's profit at such a p decrease if a punishment for illegal uploading rises. In addition, the producer's profit is less than the monopoly profit at p such that $k(p;\lambda) > p$. Therefore, a producer optimal punishment for illegal uploading is 0 or one that eliminates an incentive of uploading completely. Proposition 3 implies that a halfway punishment for illegal uploading is inferior to these policies from the aspect of an optimality for producer's profit.

By proposition 3, we find out that a punishment for illegal uploading should be the most light one or the one that eliminates an incentive of uploading completely.

A punishment for illegal downloading can eliminate illegal uploading by setting c_d that eliminates an incentive of illegal downloading i.e. $c_d \geq \overline{\theta}$. Thus, a policy that implements c_d such that $c_d \geq \overline{\theta}$ can guarantee the producer the monopoly profit.

As in the proof of proposition 4, if $\Pi_a(k(p;\lambda)) \geq \Pi_m^*$, then $k(p;\lambda) \leq p$. If a cut-off point $k(p;\lambda)$ goes down as c_d varies, then the producer's profit at p is better off. By lemma 5, the producer's profit is better off reducing c_d if $\frac{\gamma f(c_d)}{1-F(k(p;\lambda))} > 1$ and increasing c_d if $\frac{\gamma f(c_d)}{1-F(k(p;\lambda))} < 1$. Thus, we do not conclude that c_d should be the most light one as c_u. However, we can narrow down a range that includes an producer optimal punishment for illegal downloading if a valuation for the good is distributed uniformly.

Proposition 4. *Suppose that $c_u = 0$ and F is uniform. If $c_d^* \in \arg\max_{c_d}[\max_p \Pi_a(p;\lambda)]$ and $\max_{c_d}[\max_p \Pi_a(p;\lambda)] > \Pi_m^*$, then $c_d^* \leq \max\{\overline{\theta} - \gamma, 0\}$.*

Proof. Similarly in the proof of proposition 3, we simply write c_d instead of λ and we denote p such that $p \in \arg\max_{p'} \Pi_a(p';c_d)]$ by $p(c_d)$.

We prove the proposition by contradiction. Suppose that $\max\{\overline{\theta} - \gamma, 0\} < c_d^*$. By the supposition that $\max_{c_d}[\max_p \Pi_a(p;c_d)] > \Pi_m^*$, there exists a consumer who uploads the good since the highest profit is Π_m^* if there is not such a consumer. For an existence of such a consumer, $p(c_d^*) > c_d^*$ and $c_d^* < \overline{\theta}$. Moreover, $k(p(c_d^*);c_d^*) < p(c_d^*)$ since $\Pi_a(p(c_d^*);c_d^*) < \Pi_m(p(c_d^*)) \leq \Pi_m^*$ if $k(p(c_d^*);c_d^*) \geq p(c_u^*)$.

Since $c_d^* > \max\{\overline{\theta} - \gamma, 0\}$ and $c_d^* < \overline{\theta}$, $\frac{\gamma}{\overline{\theta} - c_d^*} > 1$. Therefore, $\frac{\gamma}{\overline{\theta} - k(p(c_d^*);c_d^*)} > 1$ since $c_d^* < k(p(c_d^*);c_d^*)$. We can write $\frac{\gamma}{\overline{\theta} - k(p(c_d^*);c_d^*)} > 1$ as $\frac{\gamma f(c_d^*)}{1-F(k(p(c_d^*);c_d^*))} > 1$ since F is uniform. Thus, $\frac{dk(p(c_d^*);c_d^*)}{dc_d} > 0$ by lemma 5. Hence, $k(p(c_d^*);c_d^*)$ decreases as c_d^* slightly goes down. Therefore, $\frac{d\Pi_a(p(c_d^*);c_d^*)}{dc_d} > 0$, a contradiction. □

This proposition says that a punishment for illegal downloading should not be too much high when the producer's profit is strictly larger than the highest monopoly profit. The reason is that a marginal loss of average advertising revenue is larger than a marginal loss for downloading when an advertising revenue per downloading γ is larger than $\overline{\theta} - c_d$. That is, an increase of c_d implies a increase of cut-off point.

Therefore, the producer's profit decreases by a increase of c_d. In addition, the higher γ is, the lower the upper bound of c_d^* is. The upper bound becomes 0 if $\gamma \geq \overline{\theta}$.

4 Conclusion

In the model, a consumer who uploads the good is determined endogenously. The motivation to upload the good is advertising revenue. We investigate how copyright infringement affect the producer's profit and compare it with the monopoly profit and how a punishment for an illegal uploading and for an illegal downloading affect on the producer's profit.

First, we get the two results, proposition 1, 2. The former is that copyright infringement cause strict reduction of producer's profit in comparison with the monopoly profit if the valuation for the good is distributed uniformly and advertising revenue per a downloading is sufficiently small. The implication is that a punishment for illegal uploading and illegal downloading can improve the producer's profit under the situation. Furthermore, there exists fixed cost such that the producer exits the market by copyright infringement.

The latter provides a sufficient condition for that the producer's profit at p is strictly larger than the monopoly profit under a situation where there is no punishment and c.d.f satisfies the particular condition. The particular condition for c.d.f is that an average advertising revenue is larger than the price. Thus, the hightest producer's profit is strictly larger than the highest monopoly profit if the condition for c.d.f holds at the optimal price under monopoly. In addition, there exists fixed cost such that the producer enters the market by copyright infringement.

Proposition 3 and proposition 4 mention about the optimality of punishment for uploading and downloading respectively.

Proposition 3 implies that a punishment for illegal uploading should be 0 or the one that eliminates an incentive of uploading completely. Thus, a halfway punishment is not desirable from the aspect of an optimality for the producer's profit.

In contrast to proposition 3, there is a possibility that an optimal punishment for illegal downloading is a halfway. However, an optimal punishment for illegal downloading should be in particular range. The range has an upper bound and its upper bound decreases as an advertising revenue per download increases. The upper bound reaches 0 when an advertising revenue per download is too much high.

By proposition 1 ,2, whether piracy has a bad influence on the producer's profit depends on the situation that the producer faces. Thus, these two proposition implies that uniform punishment such as a policy that illegal uploading and downloading are treated as non-crime requiring a complaint from the victim for prosecution should not be desirable policy.

Proposition 3, 4 provide a necessary condition that the optimal punishment for illegal activities should satisfy.

Copyright infringement on internet has both a negative effect and a positive effect. Which effect occurs depends on the situation where the market faces. Thus, a policy should be changed depending on the situation is desirable.

5 Appendix

In this section, we provide omitted proofs and calculations.

5.1 Derive $\mathbb{E}[e](k)$

Given assumption 2, we find that consumer i chooses d if and only if $\theta_i \in [c_d, k)$. Therefore,

$$\begin{aligned}
\mathbb{E}[e](k) =&\gamma \mathbb{E}\left[\frac{N_d}{N_u}\right] \\
=&\gamma \sum_{m=0}^{n-1} \frac{1}{1+m}\binom{n-1}{m}(1-F(k))^m (F(k))^{n-1-m} \\
&\sum_{\ell=0}^{n-1-m} \ell \binom{n-1-m}{\ell}\left(1-\frac{F(c_d)}{F(k)}\right)^{\ell}\left(\frac{F(c_d)}{F(k)}\right)^{n-1-m-\ell} \\
=&\gamma \sum_{m=0}^{n-2} \frac{1}{1+m}\binom{n-1}{m}(1-F(k))^m (F(k))^{n-1-m}(n-1-m)\left(1-\frac{F(c_d)}{F(k)}\right) \\
=&\gamma\left(\frac{F(k)-F(c_d)}{1-F(k)}\right)\sum_{m=0}^{n-2}\frac{(n-1)!}{(n-2-m)!(m+1)!}(1-F(k))^{m+1}(F(k))^{n-2-m}.
\end{aligned}$$

We replace $m+1$ with z, then

$$\begin{aligned}
\mathbb{E}[e](k) =&\gamma\left(\frac{F(k)-F(c_d)}{1-F(k)}\right)\sum_{z=1}^{n-1}\frac{(n-1)!}{(n-1-z)!z!}(1-F(k))^{z}(F(k))^{n-1-z} \\
=&\gamma\left(\frac{F(k)-F(c_d)}{1-F(k)}\right)\left[1-(F(k))^{n-1}\right] \\
=&\gamma(F(k)-F(c_d))\sum_{t=0}^{n-2}(F(k))^t.
\end{aligned}$$

□

5.2 Proof of lemma 2

Fist, we prove that the strategy profile is not an equilibrium of the subgame after Stage 2 if $p > \overline{\theta} - c_u + \gamma(n-1)(1-F(c_d))$. Note that $\mathbb{E}[e](\overline{\theta}) = \gamma(n-1)(1-F(c_d))$ is the maximum value of $\mathbb{E}[e](k)$, since $\mathbb{E}[e](k)$ increases as k increases. Hence, $V(b_u, \theta; \overline{\theta}) \geq V(b_u, \theta; k)$ for all $k, \theta \in [0, \overline{\theta}]$. On the other hand, $V(w, \theta; \overline{\theta}) \leq V(w, \theta; k)$ for all $k, \theta \in [0, \overline{\theta}]$ since $P(k)$ decreases as k increases. Therefore, if $V(w, \theta; \overline{\theta}) > V(b_u, \theta; \overline{\theta})$ for all $\theta \in [0, \overline{\theta}]$, then no consumer chooses b_u. By $\gamma(n-1)(1-F(cd)) \geq c_u$, $V(w, \theta; \overline{\theta}) > V(b_u, \theta; \overline{\theta})$ if and only if $\theta < p + c_u - \gamma(n-1)(1-F(c_d))$ (Figure 3).

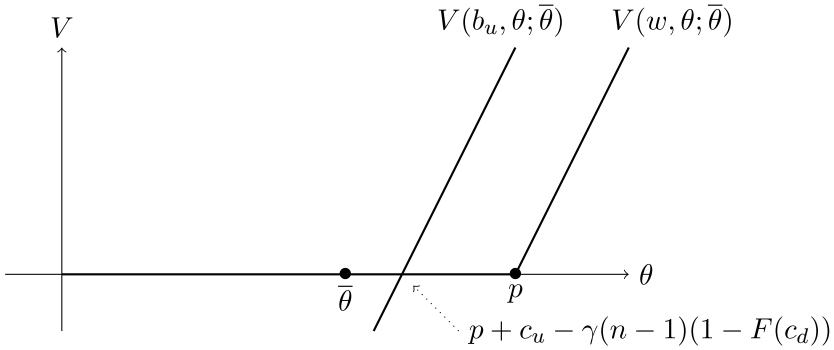

Figure 3:

Thus, we deduce that $p + c_u - \gamma(n-1)(1-F(c_d)) > \overline{\theta}$ if and only if $V(w, \theta; \overline{\theta}) > V(b_u, \theta; \overline{\theta})$ for all $\theta \in [0, \overline{\theta}]$. That is, if $p > \overline{\theta} - c_u + \gamma(n-1)(1-F(c_d))$, then no consumer chooses b_u.

Next, we show the converse of the above. By the above discussion, there must exist $\theta \in [0, \overline{\theta}]$ such that $V(w, \theta; \overline{\theta}) \geq V(b_u, \theta; \overline{\theta})$. Moreover, we obtain that $V(w, \theta; \overline{\theta}) \geq V(b_u, \theta; \overline{\theta})$ if and only if $\theta \geq p + c_u - \gamma(n-1)(1-F(c_d))$.

For confirming an existence of the equilibrium, we define $\varphi(k)$ on $k \in (0, \overline{\theta}]$ as

$$\varphi(k) := \max\{p + c_u - \mathbb{E}[e](k), \frac{p + c_u - \mathbb{E}[e](k) - P(k)c_d}{1 - P(k)}\}.$$

The former value in $\{p + c_u - \mathbb{E}[e](k), \frac{p + c_u - \mathbb{E}[e](k) - P(k)c_d}{1 - P(k)}\}$ is θ such that $V(b_u, \theta; k) = 0$

and the latter is θ such that $V(b_u, \theta; k) = P(k)(\theta - c_d)$. Therefore,

$$\begin{cases} \varphi(k) \leq c_d & \text{if } V(b_u, \varphi(k); k) = 0, \\ \varphi(k) \geq c_d & \text{if } V(b_u, \varphi(k); k) = P(k)(\theta - c_d) \end{cases}$$

as shown in Figure 4, 5.

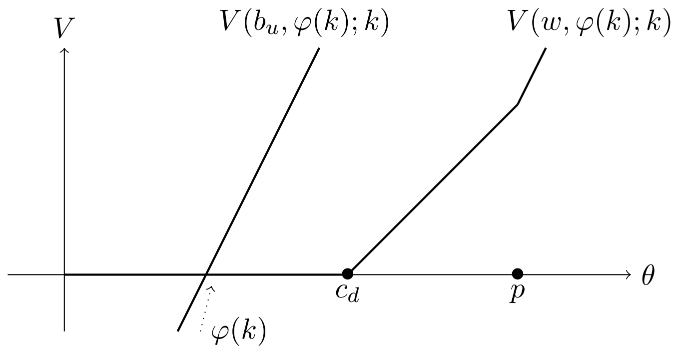

Figure 4:

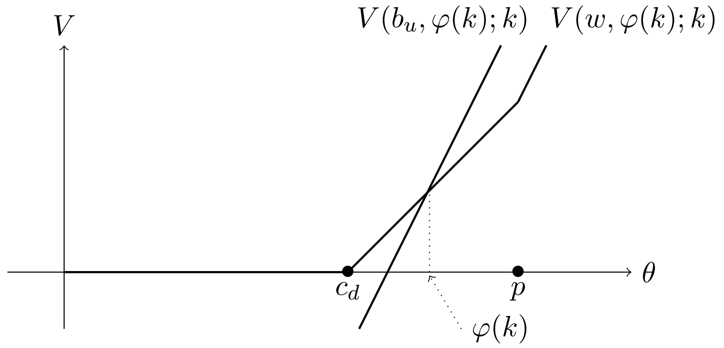

Figure 5:

By the definition of $\varphi(k)$, we can also find that $\varphi(k)$ strictly decrease as k increases. To confirm it, we do some calculses. First, consider the case that $V(b_u, \varphi(k); k) =$

0. Then,

$$\varphi(k) - p - c_u + \mathbb{E}[e](k) = 0.$$

Hence, $\varphi'(k) = -\frac{d\mathbb{E}[e](k)}{dk} < 0$ since $\mathbb{E}[e](k)$ increases as k increases. Next, we consider the another case, i.e. $V(b_u, \varphi(k); k) = P(k)(\varphi(k) - c_d)$. Same argument in above leads

$$\varphi'(k) = \frac{-\frac{d\mathbb{E}[e](k)}{dk} + P'(k)(\varphi(k) - c_d)}{1 - P(k)} < 0,$$

since $\varphi(k) \geq c_d$ in the case and $P(k)$ decreases as k increases.

By the supposition that $p > \overline{\theta} - c_u + \gamma(n-1)(1 - F(c_d))$, $\varphi(\overline{\theta}) = p + c_u - \gamma(n - 1)(1 - F(c_d)) \leq \overline{\theta}$ (Figure 6).

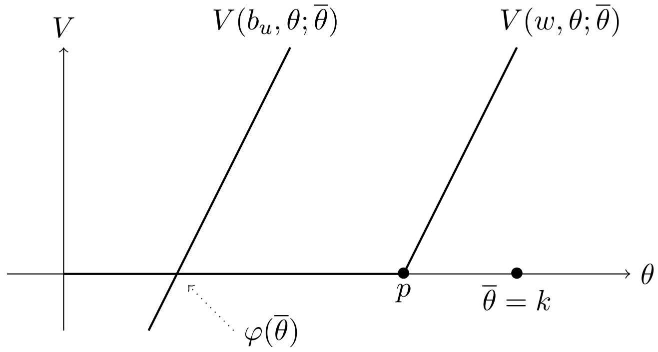

Figure 6:

Since $\varphi(k)$ strictly decreases, $\varphi(\overline{\theta})$ strictly increases as k decreases from $\overline{\theta}$. There are two possibility when k decreases from $\overline{\theta}$. One is that $\varphi(k) = k$ with $k \leq p$, the another is that $\varphi(k) = p$ with $k > p$ (Figure 7, 8). Such a k is also uniquely determined since $\varphi(k)$ strictly decreases as k increases. Moreover, $\varphi(k) = k \geq c_d$ since $V(b_u, k; \theta) < V(w, k; \theta)$ for all θ if $\varphi(k) = k < c_d$.

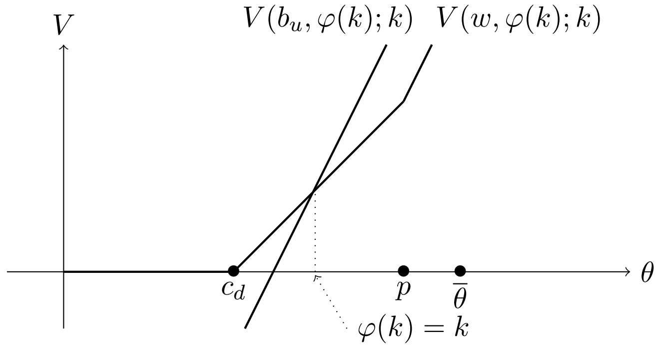

Figure 7:

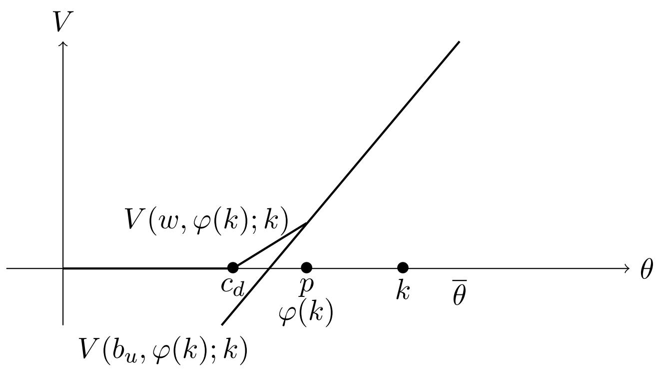

Figure 8:

In the both cases, a cut-off strategy s such that

$$s(\theta, p) = \begin{cases} b_u & \text{if } \theta \geq k, \\ w & \text{if } \theta < k, \end{cases} \tag{7}$$

is a best response when all the other consumers implement s such that (7). Therefore, there exists a symmetric equilibrium where all consumers implement a cut-off strategy such that (7). □

5.3 Proof of lemma 3

Pick any p, p' with $c_d < p < p' < p_h$. In the proof of lemma 2, there are two cases. One is that $k(p;\lambda) = \varphi(k(p;\lambda)) \leq p$ and the another one is that $k(p;\lambda) > \varphi(k(p;\lambda)) = p$. To avoid a confusion, we denote $k(p;\lambda)$ by $\hat{k}$.

Consider the former case. The price increases from p to p'. Then, $V(b_u, \hat{k}; \hat{k}) < V(w, \hat{k}; \hat{k})$. If $\varphi(\hat{k})$ exists, then $\varphi(\hat{k}) > \hat{k}$. It follows that $k(p';\lambda) > \hat{k} = k(p;\lambda)$ since φ decreases as k increases. On the other hand, if $\varphi(\hat{k})$ does not exist, then $V(b_u, \theta; \hat{k}) < V(w, \theta; \hat{k})$ for all $\theta \in [0, \overline{\theta}]$. By lemma 2, there must be $k(p';\lambda)$. If cut-off point $k(p';\lambda)$ is less than $\hat{k}$, then $V(b_u, \theta; k(p';\lambda)) < V(w, \theta; k(p';\lambda))$ for all $\theta \in [0, \overline{\theta}]$ since $\mathbb{E}[e](k)$ is an increasing function. Therefore, it must be that $k(p';\lambda) > \hat{k}$.

In the latter case, $V(b_u, \theta; \hat{k}) > V(w, \theta; \hat{k})$ for all θ if the price increases from p to p'. Thus, same argument as above leads that $k(p';\lambda) > \hat{k}$. □

5.4 The calculation of $\Pi_a(p;\lambda)$

In the case that $k(p;\lambda) \leq p$, $\Pi_a(p;\lambda)$ is simply an average of binomial distribution. Thus,

$$\begin{aligned}\Pi_a(p;\lambda) &= \sum_{i=0}^{n} pi \binom{n}{i} (1 - F(k(p;\lambda)))^i F(k(p;\lambda))^{n-i} \\ &= np[1 - F(k(p;\lambda))].\end{aligned}$$

In the another case,

$$\begin{aligned}\Pi_a(p;\lambda) &= \sum_{i=1}^{n} pi \binom{n}{i} (1 - F(k(p;\lambda)))^i F(k(p;\lambda))^{n-i} \\ &+ F(k(p;\lambda))^n \sum_{i=0}^{n} pj \binom{n}{i} \left(\frac{F(k(p;\lambda)) - F(p)}{F(k(p;\lambda))} \right)^i \left(\frac{F(p))}{F(k(p;\lambda))} \right)^{n-i} \\ &= np[1 - F(k(p;\lambda)) + F(k(p;\lambda))^{n-1}(F(k(p;\lambda)) - F(p))].\end{aligned}$$

□

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Part III

Effect of Severe Punishment on Crime Rate and Investigating Accuracy:Trade-off Between Crime Deterrence and Accuracy in Court

1 Introduction

Becker (1968) takes an economic approach to crime. The author considers the social loss from crime and argues an optimality condition of a punishment and a probability of conviction. In the paper, the author deal both a punishment and a conviction rate as a control variable.

On the other hand, Posner (1973) mentions that there is a possibility that an innocent person is convicted. Thus, a probability of convicting the innocent should also be taken into account in considering the loss. In addition, the author points out that a punishment may affect on a conviction rate. Assume that a prosecutor prefers a conviction to an acquittal and that an increase of punishment leads to a reduction of a crime rate. Then, a prosecutor does not prosecutes a suspect because a probability of getting a conviction decreases due to a reduction of a probability that a suspect commits a crime. Thus, whether a prosecutor prosecutes must be taken into account when we consider the loss.

In this paper, we study how an increase of a punishment for a crime affect on a crime rate and the probability that an innocent person is convicted though a prosecutor investigates.

An investigating authority is necessary for clearing away an ambiguity whether a suspect commits a crime. An innocent person may be convicted due to the ambiguity, although a prosecutor investigates. A prosecutor's ability of investigation is needed for that a judgement is not swung by the ambiguity. However, whether a prosecutor investigates depends on the ability and a crime rate, provided that a prosecutor prefers a conviction to an acquittal. To see why, note that a crime rate affect on the probability that a suspect commits a crime. Thus, a prosecutor who likely finds an advantageous evidence will investigate and a prosecutor who does not likely find an advantageous evidence will not investigate.

In our model, we introduce a prosecutor's ability of investigation and the ability is a private information. The set of type that investigates is determined by the posterior belief that suspect commits a crime. This posterior belief is high when a crime rate is high and is low when a crime rate is low. Provided that a prosecutor prefers a conviction to an acquittal, the lower the ability is, the lower the probability that an advantageous evidence is found is, when a crime rate is high. When a crime rate is low, the higher the ability is, the lower the probability that an advantageous evidence is found is.

The latter case in the above leads a notable result. Suppose that a increase of punishment decreases a crime rate. Then, it is more hard for a high ability type to find an advantageous evidence. Thus, a high ability type come not to investigate by the decrease of a crime rate. This implies that the precision in a trial decreases by the reduction of crime rate. Therefore, there is a trade-off between a crime deterrence and a reduction of a false charge.

2 Related literature

In our model, a suspect, a prosecutor and a judge are engaged in the model. First, the suspect determines whether to commit a crime. Second, the prosecutor determines whether to investigate and determines whether to prosecute the suspect after finding an evidence. Finally, the judge makes a judgement after observing the evidence if the prosecutor prosecutes.

There are several papers that study a procedure that includes the above three steps. Becker (1968) and Ehrlich (1973) includes the steps. However, as in the introduction, they deal second step as control variable. Thus, they does not mention the relationship between a crime deterrence and the prosecutor's decision of investigation and prosecution. Other papers (Png (1986), Craswell and Calfee (1986), Landeo, Nikitin and Baker (2006), Demougin and Fluet (2008), Kaplow (2011), Garoupa and Rizzolli (2012), and Landon and Mungan (2018)) mention about a precision in a trial and a crime rate. They deal a precision in a trial as exogenous variable, while we deal it as endogenous one. Finally, Bernard Talley and Welch (2000) provide the model in which a precision in a trial is determined endogenously. However, they does not study a relationship between a crime deterrence and a precision in a trial.

In our model, a precision in a trial is determined endogenously. The precision depends on the prosecutor's action. Katz (1988), Rubinfeld and Sappington (1987), Cooter and Rubinfeld (1989) and Emons and Fluet (2007) argue the accuracy in a trial. The accuracy is determined by how the prosecutor sends a evidence. Thus, a false charge stems from the lack of the information. In our model, we assume that an evidence is hard information. Thus, rather a false charge stems from the lack of a reliability of an evidence.

3 Model

Suspect, prosecutor and judge are involved in the model and they are denoted by s, p and j respectively. The suspect faces with decision whether to commit a crime (henceforth, g represents to commit a crime and i represents not to commit a crime). If he makes a decision to commit a crime, he receives utility, u_s. Otherwise, he receives 0. This utility, u_s, is a private information and distributed on $\mathbb{R}_+$ according to c.d.f F that has a p.d.f f. After making a decision, there is a probability that he is suspected by prosecutor Its probability is r_g if the decision is g and r_i if the decision is i. We assume that these probabilities are exogenous and $r_g > r_i > 0$. If he is not suspected, then the game is end.

After being suspected, the prosecutor chose to investigate the crime or not to investigate. Investigation incurs cost $c > 0$. The prosecutor can find a evidence

$x_p \in \{g, i\}$ if she investigates the crime. The probability that x_p is g or i depends on an ability of investigation, $\varepsilon \in (\frac{1}{2}, 1)$. The probability is as follows:

$$\text{Prob}(x_p = a : s \text{ chooses } a.) = \varepsilon \ \forall x_p \in \{g, i\}.$$

C.d.f of the ability is H and p.d.f is h. We assume that the evidence is a hard information. Thus, in the court, the judge can confirm the evidence that the prosecutor finds. If she does not investigate, then the game is end.

The prosecutor makes a decision whether to prosecute s after gaining the evidence. If she chose not to prosecute, then the game is end. If she prosecute s, then cost $k \geq 0$ occurs.

If trial takes place, the judge makes a judge after observing the evidence that the prosecutor shows. If the judge adjudge suspect guilty, then suspect takes utility, $-m < 0$, and the prosecutor takes utility, $v > 0$. However, prosecutor takes disutility t with probability $p_t \in [0, 1)$ when it becomes clear after the judgement that the suspect does not commit a crime. If the judge adjudge suspect innocent, then the suspect takes utility, 0, and the prosecutor takes utility, $-b > 0$. We assume that $v > k + c$ and $v > 0 > -b$. For simplicity, the judge has a criterion $\mu \in (0, 1)$. The judge adjudges the suspect guilty if posterior probability that the suspect commits a crime is μ and over and adjudges the suspect innocent otherwise.

Here, we summarize the timing of events:

1. Nature decides u_s and ε.

2. The suspect chooses g or i. If he is suspected, the game continues.

3. The prosecutor chose to investigate or not to investigate. If she chooses to investigate, she can find a evidence. After observing a evidence, she decides whether to prosecute. If she chooses to prosecute, the game continues.

4. After observing the evidence that the prosecutor shows, the judge makes a judgment.

3.1 Strategy

The strategy of the suspect is to chose whether to commit a crime after observing u_s. We denote it by $\sigma : \mathbb{R}_+ \to \Delta(X)$, where $X = \{g, i\}$. We simply denote the action by a.

The prosecutor has two behavioral strategies. One is to determine whether to investigate after observing ε and another one is whether to prosecute the suspect after observing ε and x_p. Denote to investigate and to prosecute by y and not to investigate and not to prosecute by n, the former is $\varphi_1 : (\frac{1}{2}, 1) \to \Delta(\{y, n\})$ and the latter is $\varphi_2 : (\frac{1}{2}, 1) \times X \to \Delta(\{y, n\})$. Thus, the strategy of the prosecutor is $\varphi = (\varphi_1, \varphi_2) = (\varphi_1, (\varphi_2(g), \varphi_2(i)))$.

We also denote the rule of judgement by $\psi : X \to \{1, 0\}$, where 1 indicates guilty and 0 indicates innocent. After stage 2, posterior probability that the suspect commits a crime is determined by the suspect's strategy. We denote it by $\hat{\mu}$. After observing a evidence that the prosecutor shows, the judge deduces a posterior probability that the suspect commits a crime. If judge can calculate it by Bayes rule, then the posterior probability, ν, is given by

$$\begin{aligned}\nu(x_p) &= \text{Prob}(a = g : x_p) \\ &= \frac{\int_{\varepsilon \in Y_{x_p}} \hat{\mu}\text{Prob}(x_p : \varepsilon, a = g)h(\varepsilon)d\varepsilon}{\int_{\varepsilon \in Y_{x_p}} \hat{\mu}\text{Prob}(x_p : \varepsilon, a = g)h(\varepsilon)d\varepsilon + \int_{\varepsilon \in Y_{x_p}} (1 - \hat{\mu})\text{Prob}(x_p : \varepsilon, a = i)h(\varepsilon)d\varepsilon}\end{aligned}$$

where $Y_{x_p} := \{\varepsilon : \varphi_1(\varepsilon) = \varphi_2(\varepsilon, x_p) = y\}$. Hence, the rule of judgement is

$$\psi(x_p) = \begin{cases} 1 & \text{if } \nu(x_p) \geq \mu \\ 0 & \text{if } \nu(x_p) < \mu. \end{cases} \tag{8}$$

4 Analysis

To derive the result, we first focus on a specific equilibrium such that

$$\exists u^* \in \mathbb{R},\ \sigma(u_s) = \begin{cases} g & \text{if } u_s \geq u^*, \\ i & \text{if } u_s < u^*, \end{cases} \tag{9}$$

$$\exists \varepsilon^* \in (\frac{1}{2}, 1),\ \varphi(\varepsilon) = \begin{cases} (y, (y, n)) & \text{if } \varepsilon \geq \varepsilon^*, \\ (n, \varphi_2(\varepsilon)) & \text{if } \varepsilon < \varepsilon^* \end{cases} \text{ or } \varphi(\varepsilon) = \begin{cases} (y, (y, n)) & \text{if } \varepsilon \leq \varepsilon^*, \\ (n, \varphi_2(\varepsilon)) & \text{if } \varepsilon > \varepsilon^* \end{cases} \tag{10}$$

$$\text{and } \nu(g) > \mu > \nu(i). \tag{11}$$

In this equilibrium, if the punishment for crime slightly increases, we can find a equilibrium that satisfies the above conditions. Specifically, there exists a trade-off between a crime deterrence and a prevention of false charge by the prosecutor when $\varphi(\varepsilon) = (y, (y, n))$ if $\varepsilon \leq \varepsilon^*$. That is, an increase of punishment for a crime induces an increase of false charge rate. In what follow, we assume that the equilibrium exists and we show the result.

4.1 Prosecutor

Suppose that there exists some type ε such that $\varphi_1(\varepsilon) = y$, equivalently $Z := \{\varepsilon : \varphi_1(\varepsilon) = y\} \neq \emptyset$. Expected utility that she prosecutes depends on the procedure after prosecuting. In the equilibrium, the judge adjudges suspect guilty if $\psi(g) = 1$ and innocent if $\psi(i) = 0$. Thus, the prosecutor does not prosecute when a evidence is i since the utility is $-b - k < 0$ if she prosecutes. The utility if she prosecutes when a evidence is g is

$$\text{Prob}(a = g : \varepsilon, g)v + \text{Prob}(a = i : \varepsilon, g)(v - p_t t) - k.$$

Thus, when a evidence is g, she prosecutes if

$$v-\frac{(1-\hat{\mu})(1-\varepsilon)}{\hat{\mu}\varepsilon+(1-\hat{\mu})(1-\varepsilon)}p_t t-k\geq 0. \tag{12}$$

In the equilibrium, this condition must holds for all type such that $\varphi_1(\varepsilon)=1$ because she switches her action from prosecuting to not prosecuting if this condition does not holds.

Now, we can calculate expected utility that she investigates as follows:

$$\begin{aligned}
w_p(y,\varepsilon) &= \mathbb{E}_{x_p}[u_p(\varphi_2(x_p,\varepsilon),\varepsilon)]-c\\
&= \text{Prob}(x_p=g:\hat{\mu},\varepsilon)(\text{Prob}(a=g:\varepsilon,g)v+\text{Prob}(a=i:\varepsilon,g)(v-p_t t)-k)-c\\
&= (\hat{\mu}\varepsilon+(1-\hat{\mu})(1-\varepsilon))\\
&\left[\frac{\hat{\mu}\varepsilon}{\hat{\mu}\varepsilon+(1-\hat{\mu})(1-\varepsilon)}v+\frac{(1-\hat{\mu})(1-\varepsilon)}{\hat{\mu}\varepsilon+(1-\hat{\mu})(1-\varepsilon)}(v-p_t t)-k\right]-c\\
&= (\hat{\mu}\varepsilon+(1-\hat{\mu})(1-\varepsilon))(v-k)-(1-\hat{\mu})(1-\varepsilon)p_t t-c.
\end{aligned}$$

Hence, she investigates if

$$\varepsilon[\hat{\mu}(v-k)-(1-\hat{\mu})(v-k-p_t t)]\geq c-(1-\hat{\mu})(v-k-p_t t). \tag{13}$$

Note that a set of type who investigates depends on $\hat{\mu}$. Suppose that there exists $\kappa(\hat{\mu})$ such that (13) holds with equality and $\kappa(\hat{\mu})\in(\frac{1}{2},1)$. Then, in the equilibrium, all type such that $\varepsilon\geq\kappa(\hat{\mu})$ investigates if $\hat{\mu}>\frac{v-k-p_t t}{2(v-k)-p_t t}$ and all type such that $\varepsilon\leq\kappa(\hat{\mu})$ investigates if $\hat{\mu}<\frac{v-k-p_t t}{2(v-k)-p_t t}$.

Lemma 6. *Suppose that there exists $\kappa(\hat{\mu})$ such that (13) holds with equality and $\kappa(\hat{\mu})\in(\frac{1}{2},1)$. A set of type who investigates is $Z=(\kappa(\hat{\mu}),1)$ if $\hat{\mu}>\frac{v-k-p_t t}{2(v-k)-p_t t}$ and $Z=(\frac{1}{2},\kappa(\hat{\mu}))$ if $\hat{\mu}<\frac{v-k-p_t t}{2(v-k)-p_t t}$.*

By lemma (6), we can compute $\nu(g)$ as follows:

$$\nu(g)=\frac{\int_{\varepsilon\in Z}\hat{\mu}\varepsilon h(\varepsilon)d\varepsilon}{\int_{\varepsilon\in Z}\hat{\mu}\varepsilon h(\varepsilon)d\varepsilon+\int_{\varepsilon\in Z}(1-\hat{\mu})(1-\varepsilon)h(\varepsilon)d\varepsilon}.$$

However, a path such that the prosecutor shows a evidence i is not on equilibrium path. Hence, we can set $\nu(i)$ arbitrary and define it as follows:

$$\nu(i) = \frac{\int_{\varepsilon \in Z} \hat{\mu}(1-\varepsilon)h(\varepsilon)d\varepsilon}{\int_{\varepsilon \in Z} \hat{\mu}(1-\varepsilon)h(\varepsilon)d\varepsilon + \int_{\varepsilon \in Z}(1-\hat{\mu})\varepsilon h(\varepsilon)d\varepsilon}.$$

Then, $\nu(g) > \hat{\mu} > \nu(i)$ since $\varepsilon > \frac{1}{2}$. The marginal type $\kappa(\hat{\mu})$ changes slightly when $\hat{\mu}$ slightly changes to μ'. Thus, if (11) holds, then (11) holds when $\hat{\mu}$ changes slightly. Hence, we can define $\kappa(\mu')$ and calculate its derivative. Given that $\hat{\mu} \neq \frac{v-k-p_t t}{2(v-k)-p_t t}$, we can calculate $\kappa'(\hat{\mu})$ since $\kappa(\hat{\mu}) \in (\frac{1}{2}, 1)$, and it is given by

$$\kappa'(\hat{\mu}) = \frac{(1-2\lambda)(v-k)-(1-\varepsilon)}{(2\hat{\mu}-1)(v-k)+(1-\hat{\mu})p_t t}.$$

Since the numerator is negative, the sign of $\kappa'(\hat{\mu})$ depends on the denominator. Therefore, we can deduce the following lemma:

Lemma 7. *Suppose that* $\hat{\mu} \neq \frac{v-k-p_t t}{2(v-k)-p_t t}$.

$$\kappa'(\hat{\mu}) \gtreqless 0 \iff \hat{\mu} \lesseqgtr \frac{v-k-p_t t}{2(v-k)-p_t t}.$$

This lemma says that whether an increase of $\hat{\mu}$ raises $\kappa(\mu)$ depends on a value of $\hat{\mu}$. Combining lemma 6 and 7, if $\hat{\mu} > \frac{v-k-p_t t}{2(v-k)-p_t t}$, an increase of $\hat{\mu}$ implies a decrease of $\kappa(\hat{\mu})$, that is, an average of accuracy ($\int_{\kappa(\hat{\mu})}^{1} \varepsilon h(\varepsilon : \varepsilon \in (\hat{\mu}, 1))d\varepsilon$) decreases. On the other hand, an average of accuracy increases if $\hat{\mu} < \frac{v-k-p_t t}{2(v-k)-p_t t}$. The marginal type $\kappa(\hat{\mu})$ is the lowest (highest) type when a crime rate is relatively high (low). An increase of a crime rate encourage the marginal type to enter the court since a probability of dropping a beneficial evidence (i.e. $x_p = g$). As a result, an average of accuracy decreases (increases).

Here, we consider a false charge by prosecutor. A false charge by prosecutor happens when the suspect does not commit a crime, the prosecutor prosecutes and judgement is guilty. We define a false charge rate e as a average probability of false charge. Precisely, in this equilibrium, it is given by

$$e := \frac{\int_Z (1-\varepsilon) h(\varepsilon) d\varepsilon}{\int_Z h(\varepsilon) d\varepsilon}.$$

A term $1-\varepsilon$ is a probability that the suspect is adjudged guilty, and $\frac{h(\varepsilon)}{\int_Z h(\varepsilon)d\varepsilon}$ is a density function conditional on that type ε invastgates. Consider the case that $\hat{\mu} < \frac{v-k-p_t t}{2(v-k)-p_t t}$. Then, by lemma (6),

$$e = \frac{\int_{\frac{1}{2}}^{\kappa(\hat{\mu})} (1-\varepsilon) h(\varepsilon) d\varepsilon}{\int_{\frac{1}{2}}^{\kappa(\hat{\mu})} h(\varepsilon) d\varepsilon}.$$

Differentiate the above with respect to $\hat{\mu}$, then

$$e' = -\kappa'(\hat{\mu}) \left(\frac{h(\kappa(\hat{\mu})) \left(\int_{\frac{1}{2}}^{\kappa(\hat{\mu})} (\kappa(\hat{\mu}) - \varepsilon) h(\varepsilon) d\varepsilon \right)}{\left(\int_{\frac{1}{2}}^{\kappa(\hat{\mu})} h(\varepsilon) d\varepsilon \right)^2} \right).$$

By lemma 7, $e' < 0$. In other case that $\hat{\mu} > \frac{1}{2}$, we can also get the following:

$$\frac{\partial e}{\partial \hat{\mu}} = -\kappa'(\hat{\mu}) \left(\frac{-h(\kappa(\hat{\mu})) \left(\int_{\kappa(\hat{\mu})}^{1} (\kappa(\hat{\mu}) - \varepsilon) h(\varepsilon) d\varepsilon \right)}{\left(\int_{\kappa(\hat{\mu})}^{1} h(\varepsilon) d\varepsilon \right)^2} \right) > 0.$$

Lemma 8. *Suppose that* $\hat{\mu} \neq \frac{v-k-p_t t}{2(v-k)-p_t t}$.

$$e' \gtrless 0 \iff \hat{\mu} \gtrless \frac{v-k-p_t t}{2(v-k)-p_t t}.$$

By this lemma, a crime deterrence induces decrease (increase) of false charge rate if a crime rate is relatively high (low). Thus, There exists a trade-off between a crime deterrence and a prevention of false charge by prosecutor when a crime rate is relatively low.

4.2 Suspect

We now consider a behavior of the suspect. In this section, we investigate how an increase of punishment for a crime impact on a crime rate.

If he commit a crime, then he gains

$$u_s(g, u_s) = u_s - r_g \text{Prob(A judgement is guilty.} : a = g)m.$$

On the other hand, utility for not committing a crime is

$$u_s(i, u_s) = -r_i \text{Prob(A judgement is guilty.} : a = i)m.$$

The difference between these two utility is

$$u_s - m(r_g \text{Prob(A judgement is guilty.} : a = g) - r_i \text{Prob(A judgement is guilty.} : a = i)).$$

Hence, for any equilibrium, σ satisfies

$$\sigma(u_s) = \begin{cases} g & \text{if } u_s > u^*, \\ i & \text{if } u_s < u^* \end{cases}$$

and $\sigma(u^*)$ is arbitrary. Thus, condition (9) always holds. In this equilibrium, an event that a judgement is guilty happens when $\varepsilon \in Z$ and the prosecutor drops g. Therefore,

$$\text{Prob(A judgement is guilty.} : a = g) = \int_Z \varepsilon h(\varepsilon) d\varepsilon,$$
$$\text{Prob(A judgement is guilty.} : a = i)) = \int_Z (1 - \varepsilon) h(\varepsilon) d\varepsilon.$$

Then, the equilibrium condition for the suspect is

$$\begin{aligned} u^* &= m(r_g \int_Z \varepsilon h(\varepsilon) d\varepsilon - r_i \int_Z (1-\varepsilon) h(\varepsilon) d\varepsilon) \\ &= m \int_Z [(r_g + r_i)\varepsilon - r_i] h(\varepsilon) d\varepsilon. \end{aligned}$$

The right hand side of the above equation is positive since $\varepsilon > \frac{1}{2}$ and $r_g > r_i$. The following lemma shows that in this equilibrium the increase of punishment for a crime decreases a crime rate i.e. u^* rises.

Lemma 9. *In this equilibrium, u^* increases as m increases.*

Proof. We prove the lemma by contradiction. Suppose that m increases but u^* decreases. In the equilibrium, posterior probability that the suspect commits a crime is given by

$$\hat{\mu}(u^*) = \frac{r_g(1 - F(u^*))}{r_g(1 - F(u^*)) + r_i F(u^*)}.$$

Thus, the supposition implies an icrease of $\hat{\mu}(u^*)$.

Here, the right hand side of an equilibrium condition for the suspect is

$$\text{R.H.S} = \begin{cases} m \int_{\kappa(\hat{\mu}(u^*))}^{1} ((r_g + r_i)\varepsilon - r_i) h(\varepsilon) d\varepsilon & \text{if } \kappa(\hat{\mu}(u^*)) > \frac{1}{2}, \\ m \int_{\frac{1}{2}}^{\kappa(\hat{\mu}(u^*))} ((r_g + r_i)\varepsilon - r_i) h(\varepsilon) d\varepsilon & \text{if } \kappa(\hat{\mu}(u^*)) < \frac{1}{2}. \end{cases}$$

By lemma 7, the R.H.S increases in each case, a contradiction. □

By this lemma, we also get the fact that an increase of m implies a decrease of $\hat{\mu}(u^*)$. Hence, combining the fact and lemma 6, 7, 8, 9, we get the following proposition.

Proposition 5. *If $\hat{\mu} > \frac{1}{2}$, then an increase of punishment for a crime decrease both a crime rate and a false charge. If $\hat{\mu} < \frac{1}{2}$, then an increase of punishment for a crime decrease crime rate but increases a false charge.*

5 Existence

We so far analyze properties of an equilibrium that satisfies (9), (10) and (11). Then, the properties leads proposition 5. However, all the arguments proceed with supposition that such a equilibrium exists. In this section, we check an existence of the equilibrium by finding a parameters that supports the existence.

We denote a taple of the parameters $(r_g, r_i, m, c, k, v, b, \mu)$ by λ and its set by Λ. We state the following proposition:

Proposition 6. *There exists $\lambda, \lambda' \in \Lambda$ such that*

(i) λ supports an existence of the equilibrium where $Z = (\kappa(\hat{\mu}(u^*)), 1)$,

(ii) λ' supports an existence of the equilibrium where $Z = (\frac{1}{2}, \kappa(\hat{\mu}(u^*)))$.

Proof. Pick any r_g, r_i, b such that $r_g > r_i > 0$ and $b > 0$. First, we prove (i). Pick any $p_t t$ and any v, k with $v - k > 0$. Moreover, pick any $\hat{\mu}$ such that $\hat{\mu} > \frac{v-k-p_t t}{2(v-k)-p_t t}$[6]. Therefore, there exist ε' such that for all $\varepsilon \geq \varepsilon'$, (12) holds since (12) holds strictly for $\varepsilon = 1$. Then we set c so as to satisfy

$$\varepsilon'[\hat{\mu}(v-k) - (1-\hat{\mu})(v-k-p_t t)] + (1-\hat{\mu})(v-k-p_t t) = c.$$

Note that $\varepsilon[\hat{\mu}(v-k) - (1-\hat{\mu})(v-k-p_t t)] + (1-\hat{\mu})(v-k-p_t t) > c$ if $\varepsilon > \varepsilon'$ and $\varepsilon[\hat{\mu}(v-k) - (1-\hat{\mu})(v-k-p_t t)] + (1-\hat{\mu})(v-k-p_t t) < c$ other wise. That is, all type with $\varepsilon \geq \varepsilon'$ investigates and prosecutes and all type with $\varepsilon < \varepsilon'$ does not investigate . Hence, condition (10) holds.

As mentioned previous section, the posterior probability ν satisfies $\nu(g) > \hat{\mu} > \nu(i)$. Thus, we can find μ with $\nu(g) > \mu > \nu(i)$. Therefore, condition (11) holds.

If we can find m such that $\hat{\mu} = \frac{r_g(1-F(u^*))}{r_g(1-F(u^*)) + r_i F(u^*)}$, then we complete the proof.

Note that the cut off point u^* is defined by

$$u^* = m \int_Z [(r_g + r_i)\varepsilon - r_i] h(\varepsilon) d\varepsilon.$$

[6] $\frac{v-k-p_t t}{2(v-k)-p_t t}$ is at most $\frac{1}{2}$.

Thus, u^* can takes any positive value since $\int_Z[(r_g + r_i)\varepsilon - r_i]h(\varepsilon)d\varepsilon > 0$. Therefore, we can find m such that $\hat{\mu} = \frac{r_g(1-F(u^*))}{r_g(1-F(u^*))+r_iF(u^*)}$ since $\frac{r_g(1-F(u^*))}{r_g(1-F(u^*))+r_iF(u^*)} > \hat{\mu}$ for sufficiently small u^* and $\frac{r_g(1-F(u^*))}{r_g(1-F(u^*))+r_iF(u^*)} < \hat{\mu}$ for sufficiently large u^*.

Next, we show (ii). Pick any $v, k, p_t t$ with $v - p_t t - k > 0$. Then, we can find $\hat{\mu}$ such that $\hat{\mu} < \frac{v-k-p_t t}{2(v-k)-p_t t}$. Hence, (12) holds for any ε. Pick any ε'' and set c so as to satisfy that for all type with $\varepsilon \leq \varepsilon''$ (13) holds and for all type with $\varepsilon > \varepsilon''$ (13) does not hold. Then, condition (10) holds.

The rest of the proof can be proven analogously to (i). □

6 Conclusion and Discussion

We study the model that deal the decision of committing a crime, investigating and prosecuting, while some literature does not mention the relationship between these actions and others deal the latter two decisions exogenously.

We investigate how the increase of punishment for a crime affect on a crime rate and a false charge rate. When we consider the specific type of equilibrium, the result is that a punishment decreases a crime rate and decreases (increases) a false charge rate and if a crime rate is relatively high (low) and such an equilibrium exists. Thus, we find that the trade-off between a crime deterrence and an avoidance of a false charge.

To obtain this result, the assumption that the prosecutor prefer a conviction to an acquittal is key. Consider the case that the judge adjudges the suspect guilty when the prosecutor finds an advantageous evidence and show it in the court, and the judge adjudges the suspect innocent when the prosecutor finds a disadvantageous evidence and show it in the court. Then, the decision of investigating is like a lottery. If a crime rate is relatively high, then the lower the ability is, the lower a probability that the prosecutor finds a good evidence is. Otherwise, then the higher the ability is, the lower a probability that the prosecutor finds a good evidence is. Thus, the marginal type who prosecutes is the lowest (highest) type when a crime rate is relatively high (low). Therefore, an decrease of a crime rate implies an exit of the marginal type since a decrease of a crime rate lower the probability that a good evidence is found.

We merely state the property the equilibrium and check the existence of the equilibrium. There are some points that further research is needed. The first is that we assume that the prosecutor prefer a conviction to an acquittal. There may be some prosecutor who wants to clarify the true state. The second is that exogenous variables may change and it depends on a kind of a crime. When a crime causes a tiny impact, then v may be small.

Finally, a crime deterrence and a avoidance of a false charge is an important purpose. We show that there exists a trade-off between the two purpose. When we consider a mechanism that deal whole process for a crime, then the trade-off should be taken into account.

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Part IV

A Political Alienation Causes Political Divergence and its Implication for Turnout Quorum

1 Introduction

In large election, a probability that one vote affects on a policy outcome is very low. Practically, voter turnout is not degenerate. Some paper provide a reason for why people vote.

Ricker and Ordeshook (1968), Strom (1975) and Hillman (2010) provide some motivation of voting. Political alienation is one of them. It is that people feel policy which is posted by politician being distant from their ideal one or not being attractive. As a consequence, people vote if he does not feel alienation and abstain from voting otherwise.

There are several studies that investigate how political alienation affects on policy outcome, turnout and policy choice by politician. Adams and Merrill (2003), Adams et al. (2006) and Brown (2014) incorporate an alienation in utility function. They treat an alienation as a threshold of going to vote. According to their definition, a voter's behavior is as follow: there are two candidates R, L and they set policy r, ℓ respectively.

- If $u(r) > u(\ell)$ and $u(r) \geq \theta$, then he votes for R.
- If $u(\ell) > u(r)$ and $u(\ell) \geq \theta$, then he votes for L.
- If $\max\{u(r), u(\ell)\} < \theta$, he abstains from voting.

Parameter θ represents a alienation threshold. It works as participation constraint. In this paper, we use this utility function and incorporate it in a probabilistic voting model.

We study how alienation affect on candidates policy choice and argue about a turnout quorum. In the model, two candidate are engaged in an electoral competition and there are continuum voter. The voters are separated three distinction. One is the set of centrist who does not care about political position. The others are the set of right-wing and the set of left-wing. Voters except for the centrist only care about political position. In usual spacial model, each candidate choose there political position. In our setting, two candidates choose the allocation of budget. The first component is an expenditure for public goods and the second is a compensation that benefit only one specific group, i.e. right-wing or left-wing. We deal this compensation as a political position.

As mentioned above, the centrist prefers a candidate whose policy gives the highest expenditure for public goods to the other candidate. We assume that the utility function for extremist (right-wing, left-wing) is concave.

First, we consider the case that there is no alienation constraint i.e. all the voter cast their vote. We show that there is an unique pure strategy Nash equilibrium in which the outcome policy is the most favorable policy for the centrist. In this equilibrium, the two candidates choose same policy. By the concavity of utility function for the extremist, the result implies that the outcome is a welfare maximizing policy.

In the benchmark case (no alienation), the political convergence always occur. However, in the election with a political alienation, the outcome diverge. To see this, suppose that two candidates choose extreme policy and there are located distant each other. In the benchmark case, if the candidate slightly moves his policy toward centrist, then the gain of vote share from distant extremist outweighs the loss of vote share from near extremist. As a result, the policy convergence occurs. However, with alienation, the opposite side voter who votes for other side candidate abstains. Therefore, the candidates does not care about a lose of vote share from other side. Thus, a policy divergence occurs. Moreover, a concavity and this fact implies that policy outcome is worse off than the outcome in the benchmark case. With alienation, a concavity promotes abstention. The reason is that in probabilistic voting setting there exist some voters who vote for the opposite side candidate rather than the near candidate. This result is different from Kamada and Kojima (2014). They rather find that a concavity leads a political convergence.

The divergence equilibrium that is found under which alienation exists has a property that the weaker the marginal propensity for the political position at the near distance, the more extreme the outcome becomes. That is, there is a Nash equilibrium that social welfare is worse than the status quo (in this paper, it means that there is no expenditure for public goods and political position is center). In addition, if the above is the case, then the centrist voters turnout rate is almost 0 and the total vote share is divided by extremist. However, this is the case, then we impose turnout quorum at the volume that is larger than the total amount of extremist. Therefore, a turnout quorum works as restriction that keeps politician from choosing an extreme policy.

2 Model

There are three segments of continuum voters. These are denoted by $j = -1, 0, 1$ respectively and the set of indices is denoted by $J = \{1-, 0, 1\}$. The size of each segments is given by

$$\begin{cases} k & \text{if } j = -1, 1, \\ 1 - 2k & \text{if } j = 0, \end{cases}$$

where $k \in (0, \frac{1}{2})$. We define $S : J \to [0, 1]$ as a function that maps a segment into its size. There are two candidates, L and R. They are engaged in an electoral competition in which they provide a policy that reforms a status quo. The policy space that the candidates can choose is composed by two component. One is a expenditure for public goods, g. The other one is a political position, x. We assume that the policy space is

$$\{(g, x) \in [0,1] \times [-1,1] : g + |x| = 1\},$$

and the status quo is $(0, 0)$. This set up means that candidates choose an allocation of budget and its total amount is 1. Thus, $|x| = 1 - g$ is interpreted as a budget for a specific group. Moreover, we assume that the strategy set of the candidate $c \in \{L, R\} = C$ is

$$X_c = \begin{cases} \{(g_R, x_R) \in [0,1] \times [-1,1] : g + |x| = 1,\ x_R \geq 0\}, \\ \{(g_L, x_L) \in [0,1] \times [-1,1] : g + |x| = 1,\ x_L \leq 0\}. \end{cases}$$

Hence, $g_R = 1 - x_R$ and $g_L = 1 + x_L$. Henceforth, we write $(g_L g_R)$ as a pair of strategy.

The procedure is conducted as follows:

1. Each candidates chooses their policy, g_R, g_L.
2. After observing (g_R, g_L), each voters chooses to vote for L, vote for R or abstain.
3. The candidate wins an election if a vote that he gains is larger than a vote that his opponent gains. If a vote share is even, then their winning probability is $\frac{1}{2}$.
4. The policy that wins the election is implemented.

I We assume that a cost of voting is 0. In this setting, one vote can not change the outcome. We argue a voter's behavior latter.

We assume that each candidates is an office-motivated candidate. Thus, they prefer a winning in competition to a losing and the utility is given by

$$\begin{cases} V & \text{if he wins,} \\ 0 & \text{if he loses,} \end{cases}$$

where $V > 0$.

As mentioned above, we argue a voter's behavior here. In usual voting setting where voters are continuum, they cast their vote to a candidate whose policy gives higher utility than the other's policy. We modify this to incorporate a political alienation.

First, we set up the voter's utility. Voters in each segment have different preference. The voters in segment 0 have the following utility function:

$$u_0(g_c, x_c : \sigma_{0,c}) = g_c + \sigma_{0,c},$$

where $\sigma_{0,c}$ is a bias toward candidate c. The bias toward c $\sigma_{0,c}$ is uniformly distributed on $[0, 1]$ and independent across candidates. Thus, a 0's voter only cares about a expenditure for public goods. The utility function of voters in segment $j = -1, 1$ is given by

$$u_j(g_c, x_c : \sigma_{j,c}) = \lambda - \alpha(|j - x_c|) + \sigma_{j,c},$$

where

$$\alpha(|j - x_c|) = \begin{cases} h|j - x_c| + \ell - h & \text{if } |j - x_c| \geq 1, \\ \ell|j - x_c| & \text{if } |j - x_c| < 1 \end{cases}$$

and $h > \ell > 1$. The biases are independent across candidates and segments and uniformly distributed on $[0, 1]$. This setting implies that $-1's$ voters and $1's$ voters are left-wing voter and right-wing voter respectively. In addition, this setting implies that the utility function is concave.

The voter's behavior is defined as follows:

- If $u_j(g_c, x_c) > u_j(g_{c'}, x_{c'})$ and $u_j(g_c, x_c) \geq \theta$, then he votes for candidate c.
- If $u_j(g_c, x_c) = u_j(g_{c'}, x_{c'})$ and $u_j(g_c, x_c) \geq \theta$, then he votes for each candidates with same probability.
- If $\max\{u_j(g_c, x_c), u_j(g_{c'}, x_{c'})\} < \theta$, then he abstains from voting.

The parameter θ capture a feeling of political alienation. A voter abstains from voting if the policies does not gives a sufficient benefit. The alienation behaves as a participation constraint.

3 Analysis

In this chapter, we analyze the effect of abstention stemming from a political alienation and argue about a turnout quorum. In addition, we focus on a pure strategy Nash equilibrium (henceforth, we simply say it NE).

3.1 Benchmark

We first argue about the case where there is no alienation, that is, there is no participation constraint.

Consider the case that $g_L \neq g_R$ and without loss of generality $g_L > g_R$. We denote the set of candidate c's voting share in each segments at (g_L, g_R) by $P_{c,j}(g_L, g_R)$. Then,

$$\begin{aligned} P_{R,-1}(g_L, g_R) &= \{(\sigma_{-1,L}, \sigma_{-1,R}) \in [0,1]^2 : u_{-1}(g_R, x_R : \sigma_{-1,R}) > u_{-1}(g_L, x_L : \sigma_{-1,L}).\} \\ &= \{(\sigma_{-1,L}, \sigma_{-1,R}) \in [0,1]^2 : \sigma_{-1,L} < \sigma_{-1,R} + \ell(g_L - 1) - h(1 - g_R).\}. \end{aligned}$$

Similarly,

$$P_{L,1}(g_L, g_R) = \{(\sigma_{1,L}, \sigma_{1,R}) \in [0,1]^2 : \sigma_{1,L} > \sigma_{1,R} + \ell(1 - g_R) + h(1 - g_L).\}.$$

In this case, g_L can be represented as $g_L = g_R + \varepsilon$ where $0 < \varepsilon < 1$. Hence, the voting share can be calculated as

$$\begin{aligned} P_{R,-1}(g_L, g_R) &= \{(\sigma_{-1,L}, \sigma_{-1,R}) \in [0,1]^2 : \sigma_{-1,L} + (1 - g_R)(\ell + h) - \varepsilon\ell < \sigma_{-1,R}.\}, \\ P_{L,1}(g_L, g_R) &= \{(\sigma_{1,L}, \sigma_{1,R}) \in [0,1]^2 : \sigma_{1,L} > \sigma_{1,R} + (1 - g_R)(\ell + h) - \varepsilon h.\}. \end{aligned}$$

Note that $(1-g_R)(\ell+h)-\varepsilon\ell > (1-g_R)(\ell+h)-\varepsilon h$ since $g_R < 1$ and $h > \ell$. This implies that the R's vote share in -1 is weakly smaller than the L's voting share in 1 since $S(1) = S(-1)$. Therefore, the sum of R's voting share in -1 and 1 is weakly less than k. Finally, the R's voting share in 0 is strictly less than $\frac{1}{2}S(0)$ since $g_L > g_R$. Therefore, the total R's voting share is strictly less than $\frac{1}{2}$. Thus, he loses in the election. However, if R switches his policy from g_R to g_L, he can gain a half amount of votes and the winning probability becomes $\frac{1}{2}$. Hence, every strategy profile (g_L, g_R) such that $g_L \neq g_R$ is not NE.

Every NE must satisfy that $g_L = g_R$. We consider the case that $g_L = g_R \neq 1$. In this case, a voting share is equally divided. Thus, their winning probability is $\frac{1}{2}$. We claim that a strategy pair such that $g_L = g_R \neq 1$ is not NE. Note that

$$P_{R,-1}(g_L, g_R) = \{(\sigma_{-1,L}, \sigma_{-1,R}) \in [0,1]^2 : \sigma_{-1,L} + \ell(1-g_L) + h(1-g_R) < \sigma_{-1,R}.\},$$
$$P_{L,1}(g_L, g_R) = \{(\sigma_{1,L}, \sigma_{1,R}) \in [0,1]^2 : \sigma_{1,L} > \sigma_{1,R} + \ell(1-g_R) + h(1-g_L).\}.$$

and $\ell(1-g_R)+h(1-g_L) = \ell(1-g_L)+h(1-g_R)$ since $g_L = g_R$. Hence, the candidate R's vote share in segment -1 is equal to the candidate L's voting share in segment 1. To show the claim, we consider the two cases, (i)$\ell(1-g_L)+h(1-g_R) > 1$ and (ii) $\ell(1-g_L)+h(1-g_R) \leq 1$.

In case (i), for each candidates the voting share in segment $-1, 1$ does not change if he slightly increases g_R to g'_R since $\ell(1-g'_R)+h(1-g_L) > 1$ and $\ell(1-g_L)+h(1-g'_R) > 1$ hold. On the other hand his voting share in 0 increases. Hence, his winning probability changes from $\frac{1}{2}$ to 1. Thus, $g_L = g_R \neq 1$ can not be NE.

In case (ii), we calculate the total R's voting share and it is

$$\underbrace{(1-2k)\frac{1}{2}[1-(1-g_R+g_L)^2]}_{\text{vote share in 0}} + \underbrace{\frac{1}{2}k(1-\ell(1-g_L)-h(1-g_R))^2}_{\text{vote share in -1}} + \underbrace{\frac{1}{2}k[1-(1-\ell(1-g_R)-h(1-g_L))^2]}_{\text{vote share in 1}}.$$

Differentiate it with respect to g_R and evaluate it at $g_L = g_R$, then

$$(1-2k) + k(h-\ell)(1-\ell(1-g_L) - h(1-g_R)) > 0.$$

Thus, the candidate R can raise his winning probability from $\frac{1}{2}$ to 1. Therefore, $g_L = g_R \neq 1$ can not be NE.

Only one left is $g_L = g_R = 1$. This is the unique NE. To see this,

$$P_{R,-1}(g_L, g_R) = \{(\sigma_{-1,L}, \sigma_{-1,R}) \in [0,1]^2 : \sigma_{-1,L} + h(1-g_R) < \sigma_{-1,R}.\},$$
$$P_{L,1}(g_L, g_R) = \{(\sigma_{1,L}, \sigma_{1,R}) \in [0,1]^2 : \sigma_{1,L} > \sigma_{1,R} + \ell(1-g_R).\},$$

provided that $g_L = 1$. Since $h > \ell$, the R's voting share in segment -1 is weakly less than the L' voting share in segment 1 for any $g_R < 1$. In addition, the R's voting share is strictly less than a half for any $g_R < 1$. Therefore, there is no incentive to change his policy. The same argument holds when we consider L's deviation.

Proposition 7. *The unique NE is* $(1,1)$ *without political alienation.*

This result is derived by a concavity of utility function of $j = -1, 1$. The left-wing voter's marginal propensity for political position x_R is h and the other one is ℓ. Thus, when L increases expenditure for public goods, an increase of opposite side supporter outweighs a decrease of own side supporter. Similarly, R increases his expenditure for public goods. This process lead candidates policies toward center.

The concavity and voter 0's utility function implies that $g = 1$ is a welfare maximizing outcome. Therefore, a welfare maximizing outcome is achievable under which there is no political alienation. Moreover, a policy divergence does not occur.

3.2 Election with political alienation

In this section, we consider the case with political alienation. There is a possibility that no one comes the election. If this is the case, we assume that the status quo is implemented.

In the election with political alienation, a participation constraint comes out. The analysis becomes more complicated one. Thus, we do not seek every equilibrium. However, we investigate the consequence of alienation in the specific situation and discuss about it.

We first assume that

$$\theta = 1, \tag{14}$$

$$\theta - \lambda + \ell = 1. \tag{15}$$

Assumption (14) is one that makes the analysis simple. The second implies the follows. As I mentioned above, the voter in segment $j = -1, 1$ has the following utility function:

$$u_j(g_c, x_c : \sigma_{j,c}) = \lambda - \alpha(|j - x_c|) + \sigma_{j,c},$$

where

$$\alpha(|j - x_c|) = \begin{cases} h|j - x_c| + \ell - h & \text{if } |j - x_c| \geq 1, \\ \ell|j - x_c| & \text{if } |j - x_c| < 1. \end{cases}$$

Consider the voter in 1, then

$$u_1(g_c, x_c : \sigma_{1,c}) = \begin{cases} \lambda - \ell(1 - x_R) + \sigma_{1,R}, \\ \lambda - h(1 - x_L) - \ell + h + \sigma_{1,L}. \end{cases}$$

Replase x_R and x_L with $1 - g_R$ and $g_L - 1$, then

$$u_1(g_c, x_c : \sigma_{1,c}) = \begin{cases} \lambda - \ell g_R + \sigma_{1,R}, \\ \lambda - h(2 - g_L) - \ell + h + \sigma_{1,L}. \end{cases}$$

Thus, if he votes for R, $u_1(g_R : \sigma_{1,R}) \geq \theta$ must hold. This condition is

$$\lambda - \ell g_R + \sigma_{1,R} \geq \theta.$$

That is, $\sigma_{1,R} \geq \theta - \lambda + \ell g_R$. By assumptions (14), (15), the voter who votes for R is negligible if $g_R = 1$.

On the other hand, a necessary condition that he votes for L is $\lambda - h(2 - g_L) - \ell + h + \sigma_{1,L} \geq \theta$ i.e. $\sigma_{1,L} \geq \theta - \lambda + h(1 - g_L) + \ell$. Therefore, (14) and (15) implies that the voter who votes for L is negligible for all $g_L \in [0, 1]$. Thus, all the vote share in 1 is R's vote share. Same argument also holds for voters in -1.

Here, we show that polarization occur under which some conditions hold. To show the claim, we first find one of the best response when the opposite policy is fixed at 0 and impose a condition in that process. Then, for L the opposite policy is also best response for that policy. Without loss of generality, we fix L's policy at 0.

By the assumptions, there is no voter in -1 who votes for R. Thus, R only cares about vote share in 0 and 1.

First, we consider the R' vote share in 0. Since $g_L = 0$, there is no voter in 0 who votes for L. When R choose g_R, the vote share is given by

$$P_{R,0}(0, g_R) = \{(\sigma_{0,L}, \sigma_{0,R}) : \sigma_{0,R} > 1 - g_R\}.$$

We can calculate its amount as $(1 - 2k)g_R$. On the other hand, the vote share in 1 is given by

$$P_{R,1}(0, g_R) = \{(\sigma_{1,L}, \sigma_{1,R}) : \sigma_{1,R} > 1 - \lambda + \ell g_R\}.$$

Thus, its amount is $k(\max\{1, \lambda - \ell g_R\})$. Therefore, the total amount of the vote share is

$$(1 - 2k)g_R + k(\min\{1, \lambda - \ell g_R\}).$$

Differentiate it with respect to $g_R > \frac{\lambda-1}{\ell}$, then it is $1 - k(2+\ell)$. Here, we impose the following assumption:

$$k > \frac{1}{2+\ell}. \tag{16}$$

By assumption (16), the total amount increase as long as $1 > \lambda - \ell g_R$ i.e. $g_R > \frac{\lambda-1}{\ell}$.

If $g_R < \frac{\lambda-1}{\ell}$, then derivative of the total amount is $(1-2k) > 0$. Hence, $g^* := g_R = \frac{\lambda-1}{\ell}$ is one of the best response for $g_L = 0$.

Next, consider L's behavior. His vote share is occupied only by all the voters in -1 while R's vote share is comprised by every voters in 1 and a part of voters in 0. Thus, he loses in the election. However, if he chooses g^*, then his winning probability is $\frac{1}{2}$. If he decrease g_L, the vote share in 0 decreases while the vote share in -1 is constant. Thus, he does not decrease g_L. His vote share at the strategy pair is

$$P_{L,0}(g^*, g^*) = \{(\sigma_{0,L}\sigma_{0,R}) : \sigma_{0,L} > \sigma_{0,R},\ \sigma_{0,L} > 1 - g^*\},$$

and the amount is given by

$$\underbrace{(1-2k)(1+1-g^*)g^*\frac{1}{2}}_{\text{L's vote share in 0.}} + \underbrace{k(\lambda - \ell g^*)}_{\text{L's vote share in } -1.}.$$

If L increases g^* to $g^* + \varepsilon$, then his vote share becomes

$$(1-2k)(1+1-g^*)g^*\frac{1}{2} + (1-2k)\varepsilon + k(\lambda - \ell(g^* + \varepsilon)).$$

Hence, his vote share changes by $(1-2k)\varepsilon - \ell k\varepsilon$. However, assumption (16) implies that $(1-2k)\varepsilon - \ell k\varepsilon < 0$. Thus, he loses his vote share by increasing g_L. Hence, g^* is the best response for g^*. By assumption (14), (15), $\lambda = \ell$. Hence, $g^* = \frac{\ell-1}{\ell}$

Proposition 8. *Assumption (14), (15), (16) holds. Then, $(\frac{\ell-1}{\ell}, \frac{\ell-1}{\ell})$ is NE in election with political alienation, i.e divergence occurs.*

A political alienation induces a policy divergence. Thus, a welfare is worse off than the situation where there is no alienation. Too see why, in the benchmark case candidates choose their policy taking the voter who is distant into account. However, with political alienation, these voter abstains from voting. Thus, candidates loses their attention for these voters.

4 Discussion and Conclusion

The previous section, we show that a political alienation induces political divergence. A concavity of the utility function implies that policy outcome is not a welfare maximizing policy. The equilibrium policy is $\frac{\ell-1}{\ell}$ and this outcome goes to 0 as ℓ goes to 1 but keeping (16). This implies that the policy outcome comes to be the most extreme one and a 0'voter turnout is almost 0. Therefore, when we set turnout quorum that is larger than the total amount of extremist, at least the turnout quorum makes this outcome not being implemented. Thus, a turnout quorum works as stopper that keeps politician from choosing extreme policy. This point of view for a turn out quorum is not in Hizen and Shimmyo (2009), Aguiar-Conraria et al. (2010) and so on. These papers does not consider the choice of candidates and suppose yes-no voting.

We so far consider the model where two candidates are engaged in the electoral competition. There are three type of voter left-wing, centrist and right-wing. The centrist voter only cares about the amount of expenditure for the public goods. The left-wing interest and the right-wing interest are opposed each other. By the probabilistic voting setting, some of voters who is extremist votes for the opposite side policy. This fact and the assumption that each extremists has a concave utility function induce that there is a unique NE which achieves a welfare maximizing outcome.

In the case where a political alienation arises, the policy convergence result as in the case where there is no alienation is broken. The policy divergence occurs. By the concavity, the policy divergence means that the outcome is worse off than the outcome with no alienation. The reason why this happens is that each candidate becomes not to care about the supporter who is in the opposite segment due to a alienation.

The NE that is found in the case where a alienation exists has a property. The lower the marginal propensity for the political distance is, the more extreme the out come is. Moreover, it can be almost the worst outcome. At the same time, a voter turnout in the centrists goes to 0. Hence, setting turnout quorum larger than the size of extremists before the election, at least the worst outcome can be avoided.

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