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Orbital exponential sums for some quadratic and cubic prehomogeneous vector spaces

(いくつかの二次と三次の概均質ベクトル空間における軌道指数和)

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Contents

ORBITAL EXPONENTIAL SUMS FOR SOME QUADRATIC AND CUBIC PREHOMOGENEOUS VECTOR SPACES

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Abstract. Let (G, V) be a prehomogeneous vector space over a finite field of odd characteristic. Taniguchi and Thorne [5] developed a method to calculate explicit formulas of the Fourier transforms of any G-invariant functions over V. By means of their method, we calculate the Fourier transform of any G-invariant function for several "quadratic" and "cubic" prehomogeneous vector spaces, parametrizing quadratic and cubic fields.

1. Introduction

Let K be a field and $\overline{K}$ be its algebraic closure. Let V be a finite dimensional representation of a reductive algebraic group G defined over K. When there exists a Zariski open $G(\overline{K})$-orbit in $V(\overline{K})$, we refer to the pair (G, V) as a prehomogeneous vector space. Let us consider a prehomogeneous vector space (G, V) defined over a finite field. Let p be an odd prime, and let $\mathbb{F}_q$ be a finite field of order $q = p^n$. Let V^* be the dual space of V. For a function $\phi : V(\mathbb{F}_q) \to \mathbb{C}$, its Fourier transform $\widehat{\phi} : V^*(\mathbb{F}_q) \to \mathbb{C}$ is defined as follows:

$$\widehat{\phi}(y) := |V(\mathbb{F}_q)|^{-1} \sum_{x \in V(\mathbb{F}_q)} \phi(x) \exp\left(\frac{2\pi i \mathrm{Tr}_{\mathbb{F}_q/\mathbb{F}_p}([x, y])}{p}\right). \tag{1}$$

Here, $[x, y] = y(x) \in \mathbb{F}_q$ is the canonical pairing of $V(\mathbb{F}_q)$ and $V^*(\mathbb{F}_q)$, and $\mathrm{Tr}_{\mathbb{F}_q/\mathbb{F}_p} : \mathbb{F}_q \to \mathbb{F}_p$ is the trace map.

The purpose of this paper is to determine an explicit formula for the Fourier transform of any $G(\mathbb{F}_q)$-invariant function ϕ for certain prehomogeneous vector spaces. Taniguchi and Thorne [5] developed a general method to compute this type of Fourier transform and applied it to obtain explicit formulas for the following prehomogeneous vector spaces (G, V) over $\mathbb{F}_q$:

- $V = \mathrm{Sym}^3(2)$, the space of binary cubic forms; $G = \mathrm{GL}_2$,
- $V = \mathrm{Sym}^2(2)$, the space of binary quadratic forms; $G = \mathrm{GL}_1 \times \mathrm{GL}_2$,
- $V = \mathrm{Sym}^2(3)$, the space of ternary quadratic forms; $G = \mathrm{GL}_1 \times \mathrm{GL}_3$,
- $V = 2 \otimes \mathrm{Sym}^2(2)$, the space of pairs of binary quadratic forms; $G = \mathrm{GL}_2 \times \mathrm{GL}_2$,
- $V = 2 \otimes \mathrm{Sym}^2(3)$, the space of pairs of ternary quadratic forms; $G = \mathrm{GL}_2 \times \mathrm{GL}_3$.

There are many prehomogeneous vector spaces for which the Fourier transform is not yet calculated. In this paper, we study the following nine more prehomogeneous vector spaces over $\mathbb{F}_q$:

- $V = 2 \otimes 2 \otimes 2$, the space of pairs of 2-by-2matrices; $G = \mathrm{GL}_2 \times \mathrm{GL}_2 \times \mathrm{GL}_2$,
- $V = 2 \otimes 2 \otimes 3$, the space of triplets of 2-by-2matrices; $G = \mathrm{GL}_2 \times \mathrm{GL}_2 \times \mathrm{GL}_3$,
- $V = 2 \otimes 2 \otimes 4$, the space of quadruples of 2-by-2matrices; $G = \mathrm{GL}_2 \times \mathrm{GL}_2 \times \mathrm{GL}_4$,
- $V = 2 \otimes \mathrm{H}_2(\mathbb{F}_{q^2})$, the space of pairs of Hermitian matrices of order 2; $G = \mathrm{GL}_2 \times \mathrm{GL}_2(\mathbb{F}_{q^2})$,
- $V = 2 \otimes \wedge^2(4)$, the space of pairs of alternating matrices of order 4; $G = \mathrm{GL}_2 \times \mathrm{GL}_4$,
- V is the space of binary tri-Hermitian forms over $\mathbb{F}_{q^3}$; $G = \mathrm{GL}_1 \times \mathrm{GL}_2(\mathbb{F}_{q^3})$,
- $V = 2 \otimes 3 \otimes 3$, the space of pairs of 3-by-3matrices; $G = \mathrm{GL}_2 \times \mathrm{GL}_3 \times \mathrm{GL}_3$,
- $V = 2 \otimes \mathrm{H}_3(\mathbb{F}_{q^2})$, the space of pairs of Hermitian matrices of order 3; $G = \mathrm{GL}_2 \times \mathrm{GL}_3(\mathbb{F}_{q^2})$,
- $V = 2 \otimes \wedge^2(6)$, the space of pairs of alternating matrices of order 6; $G = \mathrm{GL}_2 \times \mathrm{GL}_6$.

Our main theorem is as follows:

Theorem 1.1. *Let (G, V) be the prehomogeneous vector space in the above. We have an explicit formula for the Fourier transform $\widehat{e_i}$ of any indicator function e_i of $G(\mathbb{F}_q)$-orbit $\mathcal{O}_i$ in $V(\mathbb{F}_q)$.*

For the concrete formulas, we refer to Theorems 4.3, 5.3, 6.3, 7.3, 8.3, 9.3, 10.3, 11.3, and 13.3, respectively. As a consequence, we have the Fourier transform formula $\widehat{\Psi}$ of the indicator function Ψ of the singular set of each space (see Corollaries 4.4, 5.4, 6.4, 7.4, 8.4, 9.410.4, 11.4, and 13.4). As observed in [5], we obtain better than the square root cancellation for $\widehat{\Psi}$ in each space.

For the prehomogeneous vector spaces $2 \otimes 2 \otimes 2$, $2 \otimes \mathrm{H}_2(\mathbb{F}_{q^2})$, $2 \otimes \wedge^2(4)$ and the space of binary tri-Hermitian forms over $\mathbb{F}_{q^3}$, the set of nonsingular orbits naturally correspond to the set of the isomorphism classes of the separable algebras over $\mathbb{F}_q$ of degree 2. In this sense we say that these are quadratic cases. For the prehomogeneous vector spaces $2 \otimes 3 \otimes 3$, $2 \otimes \mathrm{H}_3(\mathbb{F}_{q^2})$ and $2 \otimes \wedge^2(6)$, the set of nonsingular orbits naturally correspond to the set of the isomorphism classes of the separable algebras over $\mathbb{F}_q$ of degree 3. In this sense we say that these are cubic cases. A classification of reduced irreducible prehomogeneous vector spaces over $\mathbb{C}$ was given by Sato and Kimura [3]. The prehomogeneous vector spaces we study in this paper may be defined over an arbitrary field and in particular over $\mathbb{C}$. If we consider them over $\mathbb{C}$, $2 \otimes 2 \otimes 3$ is a castling transform of a trivial prehomogeneous vector space $2 \otimes 2$, and $2 \otimes 2 \otimes 4$ is another trivial prehomogeneous vector space. $2 \otimes 2 \otimes 3$ and $2 \otimes 2 \otimes 4$ are neither quadratic nor cubic case, but we can calculate the Fourier transforms for them by means of the results of $2 \otimes 2 \otimes 2$.

These explicit formulas of the Fourier transforms, or upper bounds, have applications in counting problems for prehomogeneous vector spaces. See [1], [4], [7] for example. Furtehrmore, these concrete results may be used to study other prehomogenous vector spaces of higher degree. We hope our results in this paper have applications in these and other directions.

The composition of this paper is as follows. In Section 2, we recall Taniguchi-Thorne's method of calculating the Fourier transform and see a simple example of the calculation with the prehomogeneous vector space $(\mathrm{GL}_2, \mathrm{M}_2(\mathbb{F}_q))$. In Section 3, we see some results for the preparation for the calculation of Fourier transform. In Section 3.1, we recall the orbit decomposition of the prehomogeneous vector spaces $(\mathrm{GL}_1 \times \mathrm{GL}_n, \mathrm{Sym}^2(n))$ for $n = 2, 3, 4$ over $\mathbb{F}_q$. In Section 3.2, we recall the orbit decomposition of the prehomogeneous vector spaces $(\mathrm{GL}_2, \mathbb{F}_q \otimes \mathrm{Sym}^3(2))$. In Section 3.3, we recall the number of matrices of each rank and the order of the general linear group and the special linear group. In Section 3.4, we prove a proposition about a relationship of orbits of the prehomogeneous vector spaces $(\mathrm{GL}_2 \times \mathrm{GL}_2 \times \mathrm{GL}_n, 2 \otimes 2 \otimes n)$ for $n \in \mathbb{Z}_{\geq 1}$ and consider the cardinality of the intersection of their orbits and certain subspaces.

In Chapters 1 and 2, we look into the orbit decomposition and calculate the Fourier transforms for the prehomogeneous vector spaces above by turns. Each section but Section 12 consists of three subsections: In the first subsection, we look into the orbit decomposition of each space. In the second subsection, we choose appropriate subspaces and count the cardinality of the intersection of each subspace and each orbit. In the third subsection, we obtain the explicit formula of the Fourier transforms. In Section 12, we look into the orbit decomposition of $(\mathrm{GL}_2 \times \mathrm{GL}_5, 2 \otimes \wedge^2(5))$ for preparetion for some calculations in Section 13.

In Section 14, we see a method of verification of the calculation for the Fourier transform and some remarks which we observe from the result of the calculation.

We use the following notation throughout this paper:

- p is an odd prime and $\mathbb{F}_q$ is a finite field of order q with characteristic p, and $\mathbb{F}_{q^n}$ be the n-th extension field of $\mathbb{F}_q$.
- For a matrix A, we write its transpose as A^T.
- For a field K, let $\mathrm{M}(i, j)(K)$ be the set of all i-by-j matrices over K.
- For a field K, let $\mathrm{M}_n(K)$ be the set of all n-by-n matrices over K.
- Let $O_{i,j}$ be the i-by-j zero matrix.
- Let I_n be the identity matrix of order n.
- For $a \in \mathbb{F}_{q^2}$, let $\overline{a}$ be the conjugate of a over $\mathbb{F}_q$, and for a matrix $A = (a_{ij})$ over $\mathbb{F}_{q^2}$ let $\overline{A} = (\overline{a_{ij}})$.

2. Calculation method of Fourier transform

Let V be a finite dimensional vector space over $\mathbb{F}_q$ with a finite group G linearly acting on V. Suppose the pair (G, V) satisfies the following condition.

Assumption 2.1. *There exist an automorphism $\iota : G \ni g \mapsto g^\iota \in G$ of order 2 and a bilinear form $\beta : V \times V \to \mathbb{F}_q$ such that*

$$\beta(gx, g^\iota y) = \beta(x, y) \quad (x, y \in V, g \in G).$$

Then we can identify the dual space V^* with V by the linear isomorphism $V \ni x \mapsto \beta(x, \cdot) \in V^*$ (see [5] for detail). We reformulate the definition of Fourier transform only in terms of V. For $\phi : V \to \mathbb{C}$, we define its Fourier transform $\widehat{\phi} : V \to \mathbb{C}$ as follows:

$$\widehat{\phi}(y) := |V|^{-1} \sum_{x \in V} \phi(x) \exp\left(\frac{2\pi i \mathrm{Tr}_{\mathbb{F}_q/\mathbb{F}_p}(\beta(x,y))}{p}\right). \tag{2}$$

Here $\mathrm{Tr}_{\mathbb{F}_q/\mathbb{F}_p} : \mathbb{F}_q \to \mathbb{F}_p$ is the trace map. Let $\mathcal{F}_V^G$ be the set of all G-invariant maps from V to $\mathbb{C}$, i.e.,

$$\mathcal{F}_V^G := \{\phi : V \to \mathbb{C} \mid \phi(gx) = \phi(x) \quad (g \in G, x \in V)\}.$$

Note that $\mathcal{F}_V^G$ is a finite dimensional vector space over $\mathbb{C}$. We easily see that if ϕ is a G-invariant function, $\widehat{\phi}$ is also G-invariant. In fact, the Fourier transform map $\mathcal{F}_V^G \ni \phi \mapsto \widehat{\phi} \in \mathcal{F}_V^G$ is a linear isomorphism. Let $\mathcal{O}_i (1 \leq i \leq r)$ be the all distinct G-orbits in V, and for each i let e_i be the indicator function of $\mathcal{O}_i$. The functions $e_1, \ldots, e_r$ form a basis of $\mathcal{F}_V^G$. Thus we only have to calculate the Fourier transform of $e_1, \ldots, e_r$ to calculate that of all $\phi \in \mathcal{F}_V^G$. We use the following proposition for our calculation.

Proposition 2.2. [5, Proposition 6] *Let W be a subspace of V, and let $W^\perp := \{y \in V \mid \forall x \in W, \beta(x,y) = 0\}$. Then*

$$\sum_{i=1}^{r} \frac{|\mathcal{O}_i \cap W|}{|\mathcal{O}_i|} \widehat{e_i} = \frac{|W|}{|V|} \sum_{i=1}^{r} \frac{|\mathcal{O}_i \cap W^\perp|}{|\mathcal{O}_i|} e_i.$$

In this paper, we call $W^\perp$ orthogonal complement of W. By Proposition 2.2, when we choose one subspace of V, we obtain one equation of linear combinations of $\widehat{e_i}$ and e_i. Therefore if we choose r different subspaces and the corresponding equations are linearly independent, we obtain an explicit formula of the form $(\widehat{e_1}, ..., \widehat{e_n}) = (e_1, ..., e_n)M$ with a r-by-r matrix M. In each section from Section 10 we calculate the matrix M.

Example 2.3. Now we will look at a simple example for demonstration. Let $G = \mathrm{GL}_2 \times \mathrm{GL}_2$ and $V = \mathrm{M}_2(\mathbb{F}_q)$. G acts on V by

$$G \times V \ni ((g_1, g_2), x) \mapsto g_1 x g_2^T \in V.$$

We define an automorphism ι on G by

$$\iota : G \ni (g_1, g_2) \mapsto ((g_1^{-1})^T, (g_2^{-1})^T) \in G$$

and a bilinear form β on V by

$$\beta : V \times V \ni (x, y) \mapsto \mathrm{Tr}(xy^T) \in \mathbb{F}_q.$$

We can easily confirm that these ι and β satisfy Assumption 2.1.

Elements $x, y \in V$ are G-invariant if and only if x and y move each other by elementary operation. Therefore the orbit decomposition is given as follows:

Orbit name	Representative	Cardinality
$\mathcal{O}_1$	$\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$	1
$\mathcal{O}_2$	$\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$	$(q-1)(q+1)^2$
$\mathcal{O}_3$	$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$	$(q-1)^2 q(q+1)$

Choose three subspaces $\{0\}, W_0 = \left\{ \begin{bmatrix} a & b \\ 0 & 0 \end{bmatrix} \in V \,\middle|\, a, b \in \mathbb{F}_q \right\}$ and V. We have $\{0\}^\perp = V$, $V^\perp = \{0\}$ and $W_0^\perp = \left\{ \begin{bmatrix} 0 & 0 \\ a & b \end{bmatrix} \in V \,\middle|\, a, b \in \mathbb{F}_q \right\}$. W_0 and $W_0^\perp$ are different but since $g \in G$ such that $g \cdot W_0 = W_0^\perp$ exists, we have $|W_0^\perp \cap \mathcal{O}_i| = |gW_0 \cap \mathcal{O}_i| = |gW_0 \cap g\mathcal{O}_i| = |g(W_0 \cap \mathcal{O}_i)| = |W_0 \cap \mathcal{O}_i|$ for $i = 1, 2, 3$. Thus when we count the cardinalities of the intersections of the orbits and the subspaces, we can identify W_0 and $W_0^\perp$. In this sense we write $W_0^\perp = W_0$.

The cardinalities of the intersections are given as follows:

	$\{0\}$	W_0	V
$\mathcal{O}_1$	1	1	1
$\mathcal{O}_2$	0	$(q-1)(q+1)$	$(q-1)(q+1)^2$
$\mathcal{O}_3$	0	0	$(q-1)^2q(q+1)$

By Proposition 2.2, we obtain the following 3 equations.

$$\begin{aligned}\widehat{e}_1 &= \frac{1}{q^4}(e_1+e_2+e_3),\\ \widehat{e}_1 + \frac{1}{q+1}\widehat{e}_2 &= \frac{1}{q^2}e_1 + \frac{1}{q^2(q+1)}e_2,\\ \widehat{e}_1+\widehat{e}_2+\widehat{e}_3 &= e_1.\end{aligned}$$

So we obtain

$$\begin{bmatrix}\widehat{e}_1\\ \widehat{e}_2\\ \widehat{e}_3\end{bmatrix} = \frac{1}{q^4}\begin{bmatrix}1 & 1 & 1\\ (q-1)(q+1)^2 & q^2-q-1 & -q-1\\ (q-1)^2q(q+1) & -(q-1)q & q\end{bmatrix}\begin{bmatrix}e_1\\ e_2\\ e_3\end{bmatrix}.$$

Remark 2.4. *In what follows, when two subspaces W and W' of a prehomogeneous vector space (G,V) satisfy the condition that there exists $g\in G$ such that $gW=W'$, we identify the two and write $W=W'$.*

3. Preliminaries

In this section, we review and summarize some basic results which we use in later sections.

3.1. **The space of quadratic forms.** Let $\mathrm{Sym}^2(\mathbb{F}_q^n)$ be the vector space of n variable quadratic forms over $\mathbb{F}_q$. We write an element of $\mathrm{Sym}^2(\mathbb{F}_q^n)$ as $x(u_1,...,u_n)$ where $u_1,...,u_n$ are the variables. The group $\mathrm{GL}_1(\mathbb{F}_q)\times\mathrm{GL}_n(\mathbb{F}_q)$ acts on $\mathrm{Sym}^2(\mathbb{F}_q^n)$ by

$$(\mathrm{GL}_1(\mathbb{F}_q)\times\mathrm{GL}_n(\mathbb{F}_q))\times\mathrm{Sym}^2(\mathbb{F}_q^n)\ni((g_1,g_2),x(u_1,...,u_n))\mapsto g_1\cdot x((u_1,...,u_n)g_2^T)\in\mathrm{Sym}^2(\mathbb{F}_q^n).$$

We recall the orbit decomposition with respect to this action. In this paper we use the cases $n=2,3,4$. The orbit decomposition of $(\mathrm{GL}_1(\mathbb{F}_q)\times\mathrm{GL}_n(\mathbb{F}_q),\mathrm{Sym}^2(\mathbb{F}_q^n))$ is given as follows:

- $n=2$

Orbit name	Representative	rank
$O_{\langle\langle 0\rangle\rangle}$	0	0
$O_{\langle\langle 1\rangle\rangle}$	u_1^2	1
$O_{\langle\langle 2r\rangle\rangle}$	u_1u_2	2
$O_{\langle\langle 2i\rangle\rangle}$	$u_1^2+\mu_1u_1u_2+\mu_0u_2^2$	2

- $n=3$

Orbit name	Representative	rank
$O_{\langle\langle 0\rangle\rangle}$	0	0
$O_{\langle\langle 1\rangle\rangle}$	u_1^2	1
$O_{\langle\langle 2r\rangle\rangle}$	u_1u_2	2
$O_{\langle\langle 2i\rangle\rangle}$	$u_1^2+\mu_1u_1u_2+\mu_0u_2^2$	2
$O_{\langle\langle 3\rangle\rangle}$	$u_1^2+u_2^2+u_3^2$	3

- $n=4$

Orbit name	Representative	rank
$O_{\langle\langle 0\rangle\rangle}$	0	0
$O_{\langle\langle 1\rangle\rangle}$	u_1^2	1
$O_{\langle\langle 2r\rangle\rangle}$	u_1u_2	2
$O_{\langle\langle 2i\rangle\rangle}$	$u_1^2+\mu_1u_1u_2+\mu_0u_2^2$	2
$O_{\langle\langle 3\rangle\rangle}$	$u_1^2+u_2^2+u_3^2$	3
$O_{\langle\langle 4r\rangle\rangle}$	$u_1^2+u_2^2+u_3^2-u_4^2$	4
$O_{\langle\langle 4i\rangle\rangle}$	$u_1^2+u_2^2+u_3^2-\lambda u_4^2$	4

Here, the word "rank" means the rank of the symmetric matrix corresponding to the quadratic form and $u_1^2+\mu_1u_1u_2+\mu_0u_2^2 \in \mathrm{Sym}^2(\mathbb{F}_q^2)$ is an arbitrary irreducible polynomial and λ is an arbitrary quadratic non-residue in $\mathbb{F}_q$.

3.2. **Orbit decomposition of** $(\mathrm{GL}_2, \mathrm{Sym}^3(2))$**.** Let $V := \mathrm{Sym}^3(\mathbb{F}_q^2)$ be the vector space of 2 variable cubic polynomials over $\mathbb{F}_q$. We write an element of $\mathrm{Sym}^3(\mathbb{F}_q^2)$ as $x(u,v)$ where u, v are the variables. The group $\mathrm{GL}_2(\mathbb{F}_q)$ acts on V by

$$\mathrm{GL}_2(\mathbb{F}_q) \times \mathrm{Sym}^3(\mathbb{F}_q^2) \ni (g, x(u,v)) \mapsto x((u,v)g^T) \in \mathrm{Sym}^3(\mathbb{F}_q^2).$$

The orbit decomposition of this action is as follows:

Orbit name	Representative
$O_{\langle\langle 0\rangle\rangle}$	0
$O_{\langle\langle 1^3\rangle\rangle}$	u^3
$O_{\langle\langle 1^21\rangle\rangle}$	u^2v
$O_{\langle\langle 111\rangle\rangle}$	$uv(u-v)$
$O_{\langle\langle 12\rangle\rangle}$	$u(u^2+\mu_1uv+\mu_0v^2)$
$O_{\langle\langle 3\rangle\rangle}$	$u^3+\nu_2u^2v+\nu_1uv^2+\nu_0v^3$

Here, $u^2+\mu_1uv+\mu_0v^2 \in \mathrm{Sym}^2(\mathbb{F}_q^2)$ and $u^3+\nu_2u^2v+\nu_1uv^2+\nu_0v^3 \in \mathrm{Sym}^3(\mathbb{F}_q^2)$ are arbitrary irreducible polynomials in degrees 2 and 3, respectively.

3.3. **The cardinality of certain sets of matrices.** We introduce the following notation.

$$|(n_1,n_2),m| := |\{M \in \mathrm{M}(n_1,n_2)(\mathbb{F}_q) | \mathrm{rank}(M) = m\}| = \frac{\prod_{i=0}^{m-1}(q^{n_2-i}-1)\prod_{j=0}^{m-1}(q^{n_1}-q^j)}{\prod_{k=1}^{m}(q^k-1)},$$

$$|n,m| := |(n,n),m| = \prod_{i=0}^{m-1} \frac{(q^{n-i}-1)(q^n-q^i)}{q^{m-i}-1},$$

$$\mathrm{gl}_0 := 1,$$

$$\mathrm{gl}_n := |\mathrm{GL}_n(\mathbb{F}_q)| = \prod_{i=0}^{n-1}(q^n-q^i),$$

$$\mathrm{sl}_n := |\mathrm{SL}_n(\mathbb{F}_q)| = |\mathrm{GL}_n(\mathbb{F}_q)|/(q-1).$$

3.4. **Orbit correspondence and their cardinalities.** We identify $V_n = \mathbb{F}_q^2 \otimes \mathbb{F}_q^2 \otimes \mathbb{F}_q^n$ as the vector space of n-tuples of square matrices of order 2. For $x = (X_1, ..., X_n) \in V_n$, let $r_1(x)$ be the dimension of the subspace of $\mathrm{M}_2(\mathbb{F}_q)$ generated by $X_1, ..., X_n$. Let $G_n = \mathrm{GL}_2 \times \mathrm{GL}_2 \times \mathrm{GL}_n$. G_n acts on V_n by

$$G_n \times V_n \ni ((g_1,g_2,g_3),(X_1,...,X_n)) \mapsto (g_1Xg_2^T, ..., g_1X_ng_2^T)g_3^T \in V_n.$$

For $n < k$, we consider the embeddings

$$f_n^k : V_n \ni x \mapsto (\underbrace{0,...,0}_{k-n}, x) \in V_k$$

and

$$h_n^k : G_n \ni (g_1,g_2,g_3) \mapsto (g_1, g_2, \begin{pmatrix} I_{k-n} & 0 \\ 0 & g_3 \end{pmatrix}) \in G_k$$

where I_{k-n} is the identity matrix of order $k-n$. For all $m < n < k$, we have

$$f_m^k = f_n^k \circ f_m^n$$

and

$$h_m^k = h_n^k \circ h_m^n.$$

In addition, for all $x \in V_n$ and $g \in G_n$,

$$f_n^k(gx) = h_n^k(g) f_n^k(x)$$

holds. By these embeddings, we regard V_n as a subspace of V_k and regard G_n as a subgroup of G_k. Then we have the following proposition.

Proposition 3.1. *Let $x, y \in V_n \subset V_k$. x and y are G_n-equivalent if and only if x and y are G_k-equivalent.*

[Proof]

If x and y are G_n-equivalent, we easily see x and y are G_k-equivalent. We consider its conversion.

Assume x and y are G_k-equivalent. When $r_1(x) = m \leq n$, we can let $x = (0, ..., 0, X_1, ..., X_m)$ such that $X_1, ..., X_m$ are linearly independent, by the action of G_n. Since $r_1(y)$ is also m, we also may assume $y = (0, ..., 0, Y_1, ..., Y_m)$ such that $Y_1, ..., Y_m$ are linearly independent. Let $(g_1, g_2, \begin{pmatrix} g_{1,1} & \cdots & g_{1,k} \\ \vdots & \ddots & \vdots \\ g_{k,1} & \cdots & g_{k,k} \end{pmatrix})x = y$. Then we have

$$(g_1(\sum_{j=k-m+1}^{k} g_{1,j}X_{j-k+m})g_2^T, ..., g_1(\sum_{j=k-m+1}^{k} g_{k,j}X_{j-k+m})g_2^T) = (0, ..., 0, Y_1, ..., Y_m),$$

i.e.,

$$g_1(\sum_{j=1}^{m} g_{i,j+k-m}X_j)g_2^T = 0 \ \text{ where } 1 \leq i \leq k-m, \tag{3}$$

$$g_1(\sum_{j=1}^{m} g_{i,j+k-m}X_j)g_2^T = Y_i \ \text{ where } k-m+1 \leq i \leq k. \tag{4}$$

Since $X_1, ..., X_m$ are linearly independent, we obtain $g_{i,j} = 0$ where $1 \leq i \leq k-m$ and $k-m+1 \leq j \leq k$ by (3). It follows that $g_3' := (g_{i,j})_{k-m+1 \leq i \leq k, k-m+1 \leq j \leq k} \in \mathrm{GL}_m$. By (4), we obtain

$$(g_1, g_2, \begin{pmatrix} I_{k-m} & 0 \\ 0 & g_3' \end{pmatrix}) \cdot (0, ..., 0, X_1, ..., X_m) = (0, ..., 0, Y_1, ..., Y_m).$$

□

Next we consider particular subspaces in V_n and V_k. Let U_1 be an arbitrary subspace of V_1, and we let $U_n := U_1 \otimes \mathbb{F}_q^n \subset V_n$. For $n < k$, we regard U_n as a subspace of U_k by the embedding f_n^k. We consider a relation between $|(G_n x) \cap U_n|$ and $|(G_k x) \cap U_k|$.

Proposition 3.2. *For $x \in V_n$, let $r_1(x) = m \leq n$. Then we have*

$$|(G_k x) \cap U_k| = \frac{\prod_{i=1}^{m}(q^k - q^i)}{\prod_{i=1}^{m}(q^n - q^i)} \cdot |(G_n x) \cap U_n|.$$

[Proof]

Since the case $m = 0$ is obvious, we assume $m \geq 1$. Let $x \in U_n$ and $r_1(x) = m$, we have $x \sim (0, ..., 0, X_1, ..., X_m) \in U_n$ such that $X_1, ..., X_m$ are linearly independent, by the action of G_n. Therefore we assume $x = (0, ..., 0, X_1, ..., X_m)$. Let $\mathrm{Stab}_n(x) := \{g \in G_n \mid gx = x\}$ and $G_n(x, U_n) := \{g \in G_n \mid gx \in U_n\}$. Then $\mathrm{Stab}_n(x)$ is a subgroup of G and we have

$$|(G_n x) \cap U_n| = |G_n(x, U_n)|/|\mathrm{Stab}_n(x)|.$$

By

$$\begin{aligned} |G_n(x, U_n)| &= \mathrm{gl}_n \cdot |\{(g_1, g_2) \in \mathrm{GL}_2 \times \mathrm{GL}_2 \mid g_1 X_i g_2^T \in U_1 (1 \leq i \leq m)\}| \\ &= \frac{\mathrm{gl}_n}{\mathrm{gl}_m} \cdot |G_m(x, U_m)| \end{aligned}$$

and

$$|\mathrm{Stab}_n(x)| = q^{m(n-m)} \mathrm{gl}_{n-m} \cdot |\mathrm{Stab}_m(x)|,$$

we have

$$\begin{aligned} |(G_n x) \cap U_n| &= \frac{\mathrm{gl}_n}{q^{m(n-m)} \mathrm{gl}_{n-m} \mathrm{gl}_m} \cdot |(G_m x) \cap U_m| \\ &= \frac{\prod_{i=1}^{m}(q^n - q^i)}{\mathrm{gl}_m} \cdot |(G_m x) \cap U_m|. \end{aligned}$$

Therefore we obtain

$$\begin{aligned}
|(G_k x) \cap U_k| &= \frac{\prod_{i=1}^{m}(q^k - q^i)}{\mathrm{gl}_m} \cdot |(G_m x) \cap U_m| \\
&= \frac{\prod_{i=1}^{m}(q^k - q^i)}{\prod_{i=1}^{m}(q^n - q^i)} \cdot \frac{\prod_{i=1}^{m}(q^n - q^i)}{\mathrm{gl}_m} \cdot |(G_m x) \cap U_m| \\
&= \frac{\prod_{i=1}^{m}(q^k - q^i)}{\prod_{i=1}^{m}(q^n - q^i)} \cdot |(G_n x) \cap U_n|.
\end{aligned}$$

□

CHAPTER 1

Quadratic cases

4. $2 \otimes 2 \otimes 2$

Let $V = \mathbb{F}_q^2 \otimes \mathbb{F}_q^2 \otimes \mathbb{F}_q^2$ and $G = G_1 \times G_2 \times G_3 = \mathrm{GL}_2 \times \mathrm{GL}_2 \times \mathrm{GL}_2$. We write $x \in V$ as $x = (A, B)$ where A and B are 2-by-2 matrices, and write $g \in G$ as $g = (g_1, g_2, g_3)$ where $g_1, g_2, g_3 \in \mathrm{GL}_2$. G acts on V by

$$gx = (g_1 A g_2^T, g_1 B g_2^T) g_3^T.$$

Define a bilinear form β of V as

$$\beta((A_1, B_1), (A_2, B_2)) = \mathrm{Tr}(A_1 A_2^T + B_1 B_2^T).$$

In addition, define an automorphism ι of G as

$$(g_1, g_2, g_3)^\iota = ((g_1^T)^{-1}, (g_2^T)^{-1}, (g_3^T)^{-1}).$$

By an easy computation, we see that these β and ι satisfy Assumption 2.1.

4.1. **Orbit decomposition.**

For $x = (A, B) = (\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}, \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix}) \in V$, we define

$$r_1(x) := \mathrm{rank}(\begin{bmatrix} a_{11} & a_{12} & a_{21} & a_{22} \\ b_{11} & b_{12} & b_{21} & b_{22} \end{bmatrix}),$$
$$r_2(x) := \mathrm{rank}(\begin{bmatrix} a_{11} & a_{12} & b_{11} & b_{12} \\ a_{21} & a_{22} & b_{21} & b_{22} \end{bmatrix}),$$
$$r_3(x) := \mathrm{rank}(\begin{bmatrix} a_{11} & a_{21} & b_{11} & b_{21} \\ a_{12} & a_{22} & b_{12} & b_{22} \end{bmatrix}).$$

$r_1(x)$, $r_2(x)$, $r_3(x)$ are invariants of the orbits. We also define

$$\mathrm{det}_x(u, v) := \det(uA + vB) \in \mathrm{Sym}^2(\mathbb{F}_q^2) \text{ where } u, v \text{ are variables,}$$
$$\mathrm{T}(x) := \langle\!\langle \alpha \rangle\!\rangle \text{ if and only if } \mathrm{det}_x(u, v) \in O_{\langle\!\langle \alpha \rangle\!\rangle} \text{ in } \mathrm{Sym}^2(\mathbb{F}_q^2).$$

Note that we introduced the representation $(\mathrm{GL}_1(\mathbb{F}_q) \times \mathrm{GL}_2(\mathbb{F}_q), \mathrm{Sym}^2(\mathbb{F}_q^2))$ in Section 3.1. For $x \in V$ and $g = (g_1, g_2, g_3) \in G$, we have

$$\mathrm{det}_{gx}(u, v) = \det(g_1 g_2) \mathrm{det}_x((u, v) g_3).$$

Therefore $\mathrm{T}(x)$ is also an invariant of the orbits.

Proposition 4.1. *V consists of 8 G-orbits in all.*

Orbit name	Representative	$r_1(x)$	$r_2(x)$	$r_3(x)$	$\mathrm{T}(x)$	Cardinality
$\mathcal{O}_1$	$(\begin{bmatrix}0&0\\0&0\end{bmatrix},\begin{bmatrix}0&0\\0&0\end{bmatrix})$	0	0	0	$\langle\!\langle 0\rangle\!\rangle$	1
$\mathcal{O}_2$	$(\begin{bmatrix}0&0\\0&0\end{bmatrix},\begin{bmatrix}0&0\\0&1\end{bmatrix})$	1	1	1	$\langle\!\langle 0\rangle\!\rangle$	$[1,0,3]$
$\mathcal{O}_3$	$(\begin{bmatrix}0&0\\0&0\end{bmatrix},\begin{bmatrix}1&0\\0&1\end{bmatrix})$	1	2	2	$\langle\!\langle 1\rangle\!\rangle$	$[2,1,2]$
$\mathcal{O}_4$	$(\begin{bmatrix}0&0\\1&0\end{bmatrix},\begin{bmatrix}0&0\\0&1\end{bmatrix})$	2	1	2	$\langle\!\langle 0\rangle\!\rangle$	$[2,1,2]$
$\mathcal{O}_5$	$(\begin{bmatrix}0&1\\0&0\end{bmatrix},\begin{bmatrix}0&0\\0&1\end{bmatrix})$	2	2	1	$\langle\!\langle 0\rangle\!\rangle$	$[2,1,2]$
$\mathcal{O}_6$	$(\begin{bmatrix}0&0\\0&1\end{bmatrix},\begin{bmatrix}0&1\\1&0\end{bmatrix})$	2	2	2	$\langle\!\langle 1\rangle\!\rangle$	$[3,1,3]$
$\mathcal{O}_7$	$(\begin{bmatrix}1&0\\0&0\end{bmatrix},\begin{bmatrix}0&0\\0&1\end{bmatrix})$	2	2	2	$\langle\!\langle 2r\rangle\!\rangle$	$\frac{1}{2}[2,3,3]$
$\mathcal{O}_8$	$(\begin{bmatrix}1&0\\0&1\end{bmatrix},\begin{bmatrix}0&-1\\\mu_0&\mu_1\end{bmatrix})$	2	2	2	$\langle\!\langle 2i\rangle\!\rangle$	$\frac{1}{2}[4,3,1]$

Here $[a,b,c]=(q-1)^a q^b (q+1)^c$ and μ_1, μ_0 are elements of $\mathbb{F}_q$ such that $X^2+\mu_1 X+\mu_0\in\mathbb{F}_q[X]$ is irreducible.

[Proof]

The invariants $r_1(x)$, $r_2(x)$, $r_3(x)$ and $\mathrm{T}(x)$ for the 8 elements in the "Representative" column of the table are easily calculated. Since they do not coincide, these 8 elements belong to different orbits. Let $\mathcal{O}_i$ be the orbit of each element.

First we prove $V=\bigcup_{i=1}^{8}\mathcal{O}_i$. Let $x\in V$. Let $(r_1(x),r_2(x),r_3(x))\neq(2,2,2)$. When $r_1(x)=0$, since $x=0$ we have $x\in\mathcal{O}_1$. When $r_1(x)\geq 1$, we have $r_2(x)\geq 1$. When $(r_1(x),r_2(x))=(1,1)$, we have $x\sim(0,B)\sim(\begin{bmatrix}0&0\\0&0\end{bmatrix},\begin{bmatrix}0&0\\0&1\end{bmatrix})$ by the action of G and thus $x\in\mathcal{O}_2$. When $(r_1(x),r_2(x))=(1,2)$, $x\sim(0,B)\sim(\begin{bmatrix}0&0\\0&0\end{bmatrix},\begin{bmatrix}1&0\\0&1\end{bmatrix})$. When $(r_1(x),r_2(x))=(2,1)$, $x\sim(\begin{bmatrix}0&0\\a_{21}&b_{22}\end{bmatrix},\begin{bmatrix}0&0\\b_{21}&b_{22}\end{bmatrix})\sim(\begin{bmatrix}0&0\\1&0\end{bmatrix},\begin{bmatrix}0&0\\0&1\end{bmatrix})$. When $(r_1(x),r_2(x),r_3(x))=(2,2,1)$, $x\sim(\begin{bmatrix}0&a_{21}\\0&a_{22}\end{bmatrix},\begin{bmatrix}0&b_{12}\\0&b_{22}\end{bmatrix})\sim(\begin{bmatrix}0&1\\0&0\end{bmatrix},\begin{bmatrix}0&0\\0&1\end{bmatrix})$. In the case $(r_1(x),r_2(x),r_3(x))=(2,2,2)$, we have $x\sim(\begin{bmatrix}1&0\\0&a_{22}\end{bmatrix},\begin{bmatrix}0&b_{12}\\b_{21}&b_{22}\end{bmatrix})$ by the action of G. We have $\det_x(u,v)\sim a_{22}u^2+b_{22}uv-b_{12}b_{21}v^2$. If $\det_x(u,v)=0$, we have $a_{22}=b_{22}=b_{12}b_{21}=0$, which contradicts to $(r_1(x),r_2(x),r_3(x))=(2,2,2)$. It follows that $\mathrm{T}(x)=\langle\!\langle 1\rangle\!\rangle$, $\langle\!\langle 2r\rangle\!\rangle$ or $\langle\!\langle 2i\rangle\!\rangle$. When $\mathrm{T}(x)=\langle\!\langle 1\rangle\!\rangle$, there exists a root of $\det_x(u,v)$ which belongs to $\mathbb{P}^1(\mathbb{F}_q)$. Thus we can let $\mathrm{rank}(A)=1$ by the action of G_3. Therefore $x\sim(\begin{bmatrix}1&0\\0&0\end{bmatrix},\begin{bmatrix}0&b_{12}\\b_{21}&b_{22}\end{bmatrix})$ and $\det_x(u,v)\sim b_{22}uv-b_{12}b_{21}v^2$. Since $\mathrm{T}(x)=\langle\!\langle 1\rangle\!\rangle$, we have $b_{22}=0$ and $b_{12}b_{21}\neq 0$. Thus we have $x\sim(\begin{bmatrix}1&0\\0&0\end{bmatrix},\begin{bmatrix}0&1\\1&0\end{bmatrix})$ by the action of G. When $\mathrm{T}(x)=\langle\!\langle 2r\rangle\!\rangle$, we have $x\sim(\begin{bmatrix}1&0\\0&0\end{bmatrix},\begin{bmatrix}0&b_{12}\\b_{21}&b_{22}\end{bmatrix})$ in the same way as in the case of $\mathrm{T}(x)=\langle\!\langle 1\rangle\!\rangle$. Since $b_{22}\neq 0$ in this case, we have $x\sim(\begin{bmatrix}1&0\\0&0\end{bmatrix},\begin{bmatrix}0&0\\0&b_{22}\end{bmatrix})\sim(\begin{bmatrix}1&0\\0&0\end{bmatrix},\begin{bmatrix}0&0\\0&1\end{bmatrix})$. When $\mathrm{T}(x)=\langle\!\langle 2i\rangle\!\rangle$, $\det_x(u,v)\in\mathrm{Sym}^2(\mathbb{F}_q^2)$ is irreducible. By the fact stated in Section 3.1, irreducible polynomials in $\mathrm{Sym}^2(\mathbb{F}_q^2)$ belong to the same $\mathrm{GL}_1\times\mathrm{GL}_2$-orbit. This fact and the surjectivity of the map $G\ni(g_1,g_2,g_3)\mapsto(\det(g_1g_2),g_3^T)\in\mathrm{GL}_1\times\mathrm{GL}_2$ means that we can move $\det_x(u,v)$ to an arbitrary irreducible polynomial by the action of G. Therefore we assume $\det_x(u,v)=u^2+\mu_1uv+\mu_0v^2=(u-\gamma v)(u-\overline{\gamma}v)$ where $\gamma\in\mathbb{F}_{q^2}\setminus\mathbb{F}_q$. We can move x to $y:=(\begin{bmatrix}a_{11}&a_{12}\\a_{21}&a_{22}\end{bmatrix},\begin{bmatrix}b_{11}&b_{12}\\b_{21}&b_{22}\end{bmatrix})$ with $a_{11}\neq 0$ by the action of G_1 and G_2. Thus there exists a pair $(r,s)\in\mathbb{F}_q^2\setminus\{0\}$ such that

$$a_{11}r+b_{11}s=0. \tag{5}$$

Besides, for such r and s, there exist a pair $(p,q)\in\mathbb{F}_q^2\setminus\{0\}$ such that

$$-r+s\gamma=p\gamma-q\gamma^2, \tag{6}$$

since $\{\gamma,\gamma^2\}$ is a basis of $\mathbb{F}_{q^2}$ as a vector space over $\mathbb{F}_q$. The equation (6) is equivalent to

$$-\frac{r-s\gamma}{p-q\gamma}=\gamma. \tag{7}$$

If $(p,q)/\!\!/(r,s)$, we have $\gamma\in\mathbb{F}_q$, which contradicts to the assumption. Therefore $\begin{pmatrix}p&q\\r&s\end{pmatrix}\in\mathrm{GL}_2$. Let $g:=(1,1,\begin{pmatrix}p&q\\r&s\end{pmatrix})$. By (5), the $(1,1)$-entry of the first matrix of gy is nonzero and the $(1,1)$-entry of the second matrix of gy is 0. In addition, by (7), $\det_{gy}(u,v)=((p-q\gamma)u+(r-s\gamma)v)((p-q\overline{\gamma})u+(r-s\overline{\gamma})v)$ is a nonzero scalar multiple of $u^2+\mu_1uv+\mu_0v^2$. Since the rank of the matrix $uA+vB$ is 2 for all $(u,v)\in\mathbb{P}^1(\mathbb{F}_q)$, we have $x\sim y'=(\begin{bmatrix}1&0\\0&1\end{bmatrix},\begin{bmatrix}0&-1\\b'_{21}&b'_{22}\end{bmatrix})$ by the action of G_1 and G_2. Therefore $\det_{y'}(u,v)=g'(u^2+\mu_1uv+\mu_0v^2)$ for certain $g'\in\mathrm{GL}_1$. On the other hand, $\det_{y'}(u,v)=u^2+b'_{22}uv+b'_{21}v^2$. Therefore we have $g'=1$ and $b'_{22}=\mu_1$, $b'_{21}=\mu_0$.

Next we count the cardinality of each orbit. $|\mathcal{O}_1|,|\mathcal{O}_2|$, and $|\mathcal{O}_3|$ can be calculated by means of the cardinality of each orbit in Example 2.3 and Proposition 3.2. $|\mathcal{O}_4|$ can be calculated in the same way as $|\mathcal{O}_3|$ by regarding $x\in V$ as the pair $(\begin{bmatrix}a_{11}&a_{12}\\b_{11}&b_{12}\end{bmatrix},\begin{bmatrix}a_{21}&a_{22}\\b_{21}&b_{22}\end{bmatrix})$. For the count of $|\mathcal{O}_5|$, we regard $x\in V$ as $(\begin{bmatrix}a_{11}&a_{21}\\b_{11}&b_{21}\end{bmatrix},\begin{bmatrix}a_{12}&a_{22}\\b_{12}&b_{22}\end{bmatrix})$ and calculate it in the same way as $|\mathcal{O}_3|$ and $|\mathcal{O}_4|$. Next we count $|\mathcal{O}_6|$. Let $x_6=(\begin{bmatrix}0&0\\0&1\end{bmatrix},\begin{bmatrix}0&1\\1&0\end{bmatrix})$, and $\mathrm{Stab}(x_6):=\{g\in G\mid gx_6=x_6\}$. Let $g=(g_1,g_2,g_3)=(\begin{pmatrix}p_1&q_1\\r_1&s_1\end{pmatrix},\begin{pmatrix}p_2&q_2\\r_2&s_2\end{pmatrix},\begin{pmatrix}p_3&q_3\\r_3&s_3\end{pmatrix})\in\mathrm{Stab}(x_6)$. Then we have $(g_1\begin{bmatrix}0&q_3\\q_3&p_3\end{bmatrix}g_2^T,g_1\begin{bmatrix}0&s_3\\s_3&r_3\end{bmatrix}g_2^T)=(\begin{bmatrix}0&0\\0&1\end{bmatrix},\begin{bmatrix}0&1\\1&0\end{bmatrix})$. By comparing the rank of the first entry, we have $q_3=0$, and $p_3s_3\neq0$. It follows that $p_3\begin{bmatrix}q_1q_2&q_1s_2\\s_1q_2&s_1s_2\end{bmatrix}=\begin{bmatrix}0&0\\0&1\end{bmatrix}$, and therefore $q_1=q_2=0$, and $s_1s_2p_3\neq0$. Thus we have

$$g_1\begin{bmatrix}0&s_3\\s_3&r_3\end{bmatrix}g_2^T=\begin{bmatrix}0&p_1s_2s_3\\s_1p_2s_3&s_1s_2r_3+s_1r_2s_3+r_1s_2s_3\end{bmatrix}=\begin{bmatrix}0&1\\1&0\end{bmatrix},$$

and

$$\begin{aligned}\mathrm{Stab}(x_6)&=\left\{(\begin{pmatrix}(s_2s_3)^{-1}&0\\r_1&s_1\end{pmatrix},\begin{pmatrix}(s_3s_1)^{-1}&0\\r_2&s_2\end{pmatrix},\begin{pmatrix}(s_1s_2)^{-1}&0\\r_3&s_3\end{pmatrix})\in G\;\middle|\;r_3=-s_3\left(\tfrac{r_1}{s_1}+\tfrac{r_2}{s_2}\right)\right\}\\&\cong(\mathrm{GL}_1)^3\ltimes\mathbb{F}_q^2.\end{aligned}$$

Therefore $|\mathrm{Stab}(x_6)|=(q-1)^3q^2$ and we obtain $|\mathcal{O}_6|=|G|/|\mathrm{Stab}(x_6)|=(\mathrm{gl}_2)^3/(q-1)^3q^2=(q-1)^3q(q+1)^3$. Next we count $|\mathcal{O}_7|$. Let $x_7:=(\begin{bmatrix}1&0\\0&0\end{bmatrix},\begin{bmatrix}0&0\\0&1\end{bmatrix})$ and $\mathrm{Stab}(x_7):=\{g\in G\mid gx_7=x_7\}$. Let $g=(g_1,g_2,g_3)=(\begin{pmatrix}p_1&q_1\\r_1&s_1\end{pmatrix},\begin{pmatrix}p_2&q_2\\r_2&s_2\end{pmatrix},\begin{pmatrix}p_3&q_3\\r_3&s_3\end{pmatrix})\in\mathrm{Stab}(x_7)$. We have

$$(g_1\begin{bmatrix}p_3&0\\0&q_3\end{bmatrix}g_2^T,g_1\begin{bmatrix}r_3&0\\0&s_3\end{bmatrix}g_2^T)=(\begin{bmatrix}1&0\\0&0\end{bmatrix},\begin{bmatrix}0&0\\0&1\end{bmatrix}).$$

By comparing the rank of each entry, we obtain the following propositions:

(8) If $p_3\neq0$, then $q_3=r_3=0$ and $s_3\neq0$.

(9) If $p_3=0$, then $q_3r_3\neq0$ and $s_3=0$.

First we assume the case (8). Then we have

$$(p_3\begin{bmatrix}p_1p_2&p_1r_2\\r_1p_2&r_1r_2\end{bmatrix},s_3\begin{bmatrix}q_1q_2&q_1s_2\\s_1q_2&s_1s_2\end{bmatrix})=(\begin{bmatrix}1&0\\0&0\end{bmatrix},\begin{bmatrix}0&0\\0&1\end{bmatrix}),$$

and therefore

$$q_1 = q_2 = r_1 = r_2 = 0,\ p_3 = (p_1p_2)^{-1},\text{ and } s_3 = (s_1s_2)^{-1}. \tag{10}$$

Next we assume the case (9). Then we have

$$(q_3\begin{bmatrix} q_1q_2 & q_1s_2 \\ s_1q_2 & s_1s_2\end{bmatrix}, r_3\begin{bmatrix} p_1p_2 & p_1r_2 \\ r_1p_2 & r_1r_2\end{bmatrix}) = (\begin{bmatrix} 1 & 0 \\ 0 & 0\end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1\end{bmatrix}),$$

and therefore

$$p_1 = p_2 = s_1 = s_2 = 0,\ q_3 = (q_1q_2)^{-1},\text{ and } r_3 = (r_1r_2)^{-1}. \tag{11}$$

By (10) and (11), we obtain

$$\begin{aligned}\mathrm{Stab}(x_7) &= \langle (\begin{pmatrix} 0 & 1 \\ 1 & 0\end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 1 & 0\end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 1 & 0\end{pmatrix})\rangle \ltimes \left\{(\begin{pmatrix} p_1 & 0 \\ 0 & s_1\end{pmatrix}, \begin{pmatrix} p_2 & 0 \\ 0 & s_2\end{pmatrix}, \begin{pmatrix} (p_1p_2)^{-1} & 0 \\ 0 & (s_1s_2)^{-1}\end{pmatrix}) \in G\right\} \\ &\cong \mathbb{Z}/2\mathbb{Z} \ltimes (\mathrm{GL}_1)^4.\end{aligned}$$

It follows that $|\mathcal{O}_7| = |G|/|\mathrm{Stab}(x_7)| = (\mathrm{gl}_2)^3/2(q-1)^4 = (q-1)^2q^3(q+1)^3/2$. In the end, $|\mathcal{O}_8| = q^8 - \sum_{i=1}^{7} |\mathcal{O}_i| = (q-1)^4q^3(q+1)/2$. □

4.2. **The intersection between the orbits and the subspaces.** The subspaces we choose to calculate the Fourier transform are as follows:

$$W_1 = 0, W_2 = (\begin{bmatrix} 0 & 0 \\ 0 & *\end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & *\end{bmatrix}), W_3 = (\begin{bmatrix} 0 & 0 \\ 0 & 0\end{bmatrix}, \begin{bmatrix} * & * \\ * & *\end{bmatrix}), W_4 = (\begin{bmatrix} 0 & 0 \\ * & *\end{bmatrix}, \begin{bmatrix} 0 & 0 \\ * & *\end{bmatrix}),$$

$$W_5 = (\begin{bmatrix} 0 & * \\ 0 & *\end{bmatrix}, \begin{bmatrix} 0 & * \\ 0 & *\end{bmatrix}), W_6 = (\begin{bmatrix} 0 & 0 \\ 0 & *\end{bmatrix}, \begin{bmatrix} 0 & * \\ * & *\end{bmatrix}), W_7 = (\begin{bmatrix} 0 & * \\ * & *\end{bmatrix}, \begin{bmatrix} 0 & * \\ * & *\end{bmatrix}) \text{ and } W_8 = V.$$

Here, the notations mean, for example,

$$(\begin{bmatrix} 0 & 0 \\ 0 & 0\end{bmatrix}, \begin{bmatrix} * & * \\ * & *\end{bmatrix}) = \left\{(\begin{bmatrix} 0 & 0 \\ 0 & 0\end{bmatrix}, \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22}\end{bmatrix}) \in V \,\middle|\, b_{11}, b_{12}, b_{21}, b_{22} \in \mathbb{F}_q\right\}.$$

Orthogonal complements of them are as follows (See Remark 2.4 for the convention for some of these equalities):

$$W_1^\perp = W_8, W_2^\perp = W_7, W_3^\perp = W_3, W_4^\perp = W_4, W_5^\perp = W_5, W_6^\perp = W_6, W_7^\perp = W_2 \text{ and } W_8^\perp = W_1.$$

Proposition 4.2. *The cardinalities $|\mathcal{O}_i \cap W_j|$ for the orbit $\mathcal{O}_i$ and the subspace W_j are given as follows:*

	W_1	W_2	W_3	W_4	W_5	W_6	W_7	W_8
$\mathcal{O}_1$	1	1	1	1	1	1	1	1
$\mathcal{O}_2$	0	$[1,0,1]$	$[1,0,2]$	$[1,0,2]$	$[1,0,2]$	$[1,0,0](3q+1)$	$[1,0,1](2q+1)$	$[1,0,3]$
$\mathcal{O}_3$	0	0	$[2,1,1]$	0	0	$[2,1,0]$	$[2,1,1]$	$[2,1,2]$
$\mathcal{O}_4$	0	0	0	$[2,1,1]$	0	$[2,1,0]$	$[2,1,1]$	$[2,1,2]$
$\mathcal{O}_5$	0	0	0	0	$[2,1,1]$	$[2,1,0]$	$[2,1,1]$	$[2,1,2]$
$\mathcal{O}_6$	0	0	0	0	0	$[3,1,0]$	$[3,1,1]$	$[3,1,3]$
$\mathcal{O}_7$	0	0	0	0	0	0	$[2,3,1]$	$\frac{1}{2}[2,3,3]$
$\mathcal{O}_8$	0	0	0	0	0	0	0	$\frac{1}{2}[4,3,1]$

Here $[a,b,c] = (q-1)^a q^b (q+1)^c$ and

[Proof]

For W_1, W_2, W_3, W_4 and W_5 we can calculate by means of Example 2.3 and Proposition 3.2. Let $x \in W_6$ be $(\begin{bmatrix} 0 & 0 \\ 0 & a_{22}\end{bmatrix}, \begin{bmatrix} 0 & b_{12} \\ b_{21} & b_{22}\end{bmatrix})$. When $a_{22}b_{12}b_{21} \neq 0$, we have $x \in \mathcal{O}_6$. The number of such x is $(q-1)^3q$. When $a_{22} = 0$ and $b_{12}b_{21} \neq 0$, we have $x \in \mathcal{O}_3$. When $b_{12} = 0$ and $a_{22}b_{21} \neq 0$, we have $x \in \mathcal{O}_4$. When $b_{21} = 0$ and $a_{22}b_{12} \neq 0$, we have $x \in \mathcal{O}_5$. For each of these cases, there are $(q-1)^2q$ of such x. The remaining elements all belong to $\mathcal{O}_1$ or $\mathcal{O}_2$. Let $x \in W_7$ be $(\begin{bmatrix} 0 & a_{12} \\ a_{21} & a_{22}\end{bmatrix}, \begin{bmatrix} 0 & b_{12} \\ b_{21} & b_{22}\end{bmatrix})$. When $(a_{12}, b_{12}) \not\parallel (a_{21}, b_{21})$, we have $x \in \mathcal{O}_7$. The number of such x is $q^2\mathrm{gl}_2$. When $(a_{12}, b_{12}) \neq 0$, $(a_{21}, b_{21}) \neq 0$, $(a_{12}, b_{12}) \parallel (a_{21}, b_{21})$ and $(a_{12}, b_{12}) \not\parallel (a_{22}, b_{22})$, we have $x \in \mathcal{O}_6$. The number of such x is $(q-1)\mathrm{gl}_2$. When $(a_{12}, b_{12}) \neq 0$, $(a_{21}, b_{21}) \neq 0$, $(a_{12}, b_{12}) \parallel (a_{21}, b_{21})$ and $(a_{12}, b_{12}) \parallel (a_{22}, b_{22})$, we have $x \in \mathcal{O}_3$. The number of such x

is$(q^2-1)(q-1)q$. When $(a_{12},b_{12})=0$ and $(a_{21},b_{21})\not\parallel(a_{22},b_{22})$, we have $x\in\mathcal{O}_4$. The number of such x is gl_2. When $(a_{21},b_{21})=0$ and $(a_{12},b_{12})\not\parallel(a_{22},b_{22})$, we have $x\in\mathcal{O}_5$. The number of such x is gl_2. The remaining elements all belong to $\mathcal{O}_1$ or $\mathcal{O}_2$. □

4.3. **Fourier transform.** By applying these results to Proposition 2.2, we obtain an explicit formula for the Fourier transform.

Theorem 4.3. *The representation matrix M of the Fourier transform on $\mathcal{F}_V^G$ with respect to the basis $e_1,\dots,e_8$ is given as follows:*

$$\frac{1}{q^8}\begin{bmatrix} 1 & [1,0,3] & [2,1,2] & [2,1,2] & [2,1,2] & [3,1,3] & \frac12[2,3,3] & \frac12[4,3,1] \\ 1 & c_1 & [1,1,0]b_1 & [1,1,0]b_1 & [1,1,0]b_1 & -[2,1,0]a_1 & \frac12[1,3,0]b_2 & -\frac12[3,3,0] \\ 1 & [0,0,1]b_1 & qc_2 & -[1,1,1] & -[1,1,1] & [1,1,1] & -\frac12[1,3,1] & \frac12[2,3,0] \\ 1 & [0,0,1]b_1 & -[1,1,1] & qc_2 & -[1,1,1] & [1,1,1] & -\frac12[1,3,1] & \frac12[2,3,0] \\ 1 & [0,0,1]b_1 & -[1,1,1] & -[1,1,1] & qc_2 & [1,1,1] & -\frac12[1,3,1] & \frac12[2,3,0] \\ 1 & -a_1 & q & q & q & qc_3 & -\frac12[1,3,0] & -\frac12[1,3,0] \\ 1 & b_2 & -[1,1,0] & -[1,1,0] & -[1,1,0] & -[2,1,0] & q^3 & 0 \\ 1 & -[0,0,2] & [0,1,1] & [0,1,1] & [0,1,1] & -[0,1,2] & 0 & q^3 \end{bmatrix}.$$

Here $[a,b,c]=(q-1)^aq^b(q+1)^c$, $a_1=2q+1$, $b_1=q^2-q-1$, $b_2=q^2-2q-1$, $c_1=2q^3-2q-1$, $c_2=q^3-q^2+1$ and $c_3=q^3-q^2-1$.

We used PARI/GP [8] to calculate the matrix from Proposition 4.2.

By Theorem 4.3, we can calculate the Fourier transform of the indicator function Ψ of the singular set $S=\{x\in V\mid \mathrm{Disc}(\det_x(u,v))=0\}=\bigcup_{i=1}^{6}\mathcal{O}_i$, i.e., $\Psi=\sum_{i=1}^{6}e_i$.

Corollary 4.4. *The Fourier transform of Ψ is given as follows:*

$$\widehat{\Psi}(x)=\begin{cases} q^{-1}+q^{-4}-q^{-5} & x=0,\\ q^{-4}-q^{-5} & x\neq 0, \mathrm{Disc}(\det_x(u,v))=0,\\ -q^{-5} & \mathrm{Disc}(\det_x(u,v))\neq 0.\end{cases}$$

In particular, we have the following L_1-norm bound of $\widehat{\Psi}$:

$$\sum_{x\in V}|\widehat{\Psi}(x)|=O(q^3).$$

5. $2\otimes 2\otimes 3$

Let $V=\mathbb{F}_q^2\otimes\mathbb{F}_q^2\otimes\mathbb{F}_q^3$ and $G=G_1\times G_2\times G_3=\mathrm{GL}_2\times\mathrm{GL}_2\times\mathrm{GL}_3$. We write $x\in V$ as $x=(A,B,C)$ where A,B and C are 2-by-2 matrices, and write $g\in G$ as $g=(g_1,g_2,g_3)$ where $g_1,g_2\in\mathrm{GL}_2$ and $g_3\in\mathrm{GL}_3$. We define the action of G on V by

$$gx=(g_1Ag_2^T,g_1Bg_2^T,g_1Cg_2^T)g_3^T.$$

Define a bilinear form β of V as

$$\beta((A_1,B_1,C_1),(A_2,B_2,C_2))=\mathrm{Tr}(A_1A_2^T+B_1B_2^T+C_1C_2^T).$$

In addition, define an automorphism ι of G as

$$(g_1,g_2,g_3)^\iota=((g_1^T)^{-1},(g_2^T)^{-1},(g_3^T)^{-1}).$$

These β and ι satisfy Assumption 2.1.

5.1. **Orbit decomposition.** For $x = (A, B, C) = (\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}, \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix}, \begin{bmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \end{bmatrix}) \in V$, we define

$$r_1(x) := \mathrm{rank}(\begin{bmatrix} a_{11} & a_{12} & a_{21} & a_{22} \\ b_{11} & b_{12} & b_{21} & b_{22} \\ c_{11} & c_{12} & c_{21} & c_{22} \end{bmatrix}),$$
$$r_2(x) := \mathrm{rank}(\begin{bmatrix} a_{11} & a_{12} & b_{11} & b_{12} & c_{11} & c_{12} \\ a_{21} & a_{22} & b_{21} & b_{22} & c_{21} & c_{22} \end{bmatrix}),$$
$$r_3(x) := \mathrm{rank}(\begin{bmatrix} a_{11} & a_{21} & b_{11} & b_{21} & c_{11} & c_{21} \\ a_{12} & a_{22} & b_{12} & b_{22} & c_{12} & c_{22} \end{bmatrix}),$$
$$\det_x(u_1, u_2, u_3) := \det(u_1 A + u_2 B + u_3 C) \in \mathrm{Sym}^2(\mathbb{F}_q^3) \text{ where } u_1, u_2, u_3 \text{ are variables,}$$
$$\mathrm{T}(x) := \langle\!\langle \alpha \rangle\!\rangle \text{ if and only if } \det_x(u_1, u_2, u_3) \in O_{\langle\!\langle \alpha \rangle\!\rangle} \text{ in } \mathrm{Sym}^2(\mathbb{F}_q^3).$$

Note that we introduced the representation $(\mathrm{GL}_1(\mathbb{F}_q) \times \mathrm{GL}_3(\mathbb{F}_q), \mathrm{Sym}^2(\mathbb{F}_q^3))$ in Section 3.1. For $x \in V$ and $g = (g_1, g_2, g_3) \in G$, we have

$$\det_{gx}(u_1, u_2, u_3) = \det(g_1 g_2)\det_x((u_1, u_2, u_3)g_3).$$

Proposition 5.1. *V consists of 10 G-orbits in all.*

Orbit name	Representative	$r_1(x)$	$r_2(x)$	$r_3(x)$	$\mathrm{T}(x)$	Cardinality
$\mathcal{O}_1$	$(\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix})$	0	0	0	$\langle\!\langle 0 \rangle\!\rangle$	1
$\mathcal{O}_2$	$(\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix})$	1	1	1	$\langle\!\langle 0 \rangle\!\rangle$	$[1, 0, 2, 1]$
$\mathcal{O}_3$	$(\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix})$	1	2	2	$\langle\!\langle 1 \rangle\!\rangle$	$[2, 1, 1, 1]$
$\mathcal{O}_4$	$(\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix})$	2	1	2	$\langle\!\langle 0 \rangle\!\rangle$	$[2, 1, 2, 1]$
$\mathcal{O}_5$	$(\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix})$	2	2	1	$\langle\!\langle 0 \rangle\!\rangle$	$[2, 1, 2, 1]$
$\mathcal{O}_6$	$(\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix})$	2	2	2	$\langle\!\langle 1 \rangle\!\rangle$	$[3, 1, 3, 1]$
$\mathcal{O}_7$	$(\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix})$	2	2	2	$\langle\!\langle 2r \rangle\!\rangle$	$\frac{1}{2}[2, 3, 3, 1]$
$\mathcal{O}_8$	$(\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & -1 \\ \mu_0 & \mu_1 \end{bmatrix})$	2	2	2	$\langle\!\langle 2i \rangle\!\rangle$	$\frac{1}{2}[4, 3, 1, 1]$
$\mathcal{O}_9$	$(\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix})$	3	2	2	$\langle\!\langle 2r \rangle\!\rangle$	$[3, 3, 3, 1]$
$\mathcal{O}_{10}$	$(\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix})$	3	2	2	$\langle\!\langle 3 \rangle\!\rangle$	$[4, 4, 2, 1]$

Here $[a, b, c, d] = (q-1)^a q^b (q+1)^c (q^2+q+1)^d$, and μ_1, μ_0 are elements of $\mathbb{F}_q$ such that $X^2 + \mu_1 X + \mu_0 \in \mathbb{F}_q[X]$ is irreducible.

[Proof]

The invariants $r_1(x)$, $r_2(x)$, $r_3(x)$ and $\mathrm{T}(x)$ for the 10 elements in the "Representative" column of the table are easily calculated. Since they do not coincide, these 10 elements belong to different orbits. Let $\mathcal{O}_i$ be the orbit of each element.

First we prove that $\bigcup_{i=1}^{10} \mathcal{O}_i = V$. Let $x \in V$. When $r_1(x) \leq 2$, we have $x \sim (0, B, C)(B, C \in \mathrm{M}_2(\mathbb{F}_q))$. Therefore by Propositions 3.1 and 4.1 we see that

$$\{x \in V | r_1(x) \leq 2\} = \bigcup_{i=1}^{8} \mathcal{O}_i.$$

When $r_1(x) = 3$ we have $x \sim (\begin{bmatrix} 1 & 0 \\ 0 & a_{22} \end{bmatrix}, \begin{bmatrix} 0 & b_{12} \\ b_{21} & b_{22} \end{bmatrix}, \begin{bmatrix} 0 & c_{12} \\ c_{21} & c_{22} \end{bmatrix})$. If $b_{12} = c_{12} = 0$, then we have $x \sim (\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix})$. If $b_{12} \neq 0$, we have $x \sim (\begin{bmatrix} 1 & 0 \\ 0 & a_{22} \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ b_{21} & b_{22} \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ c_{21} & c_{22} \end{bmatrix})$. If $c_{12} \neq 0$, we have $x\sim(\begin{bmatrix}1 & 0\\ 0 & a_{22}\end{bmatrix},\begin{bmatrix}0 & 0\\ b_{21} & b_{22}\end{bmatrix},\begin{bmatrix}0 & 1\\ c_{21} & c_{22}\end{bmatrix})$. In any case, the orbit of x contains an element $(A,B,C)\in V$ where at least one of A,B,C is of rank 1. Therefore we have $x\sim(\begin{bmatrix}1 & 0\\ 0 & 0\end{bmatrix},\begin{bmatrix}0 & b_{12}\\ b_{21} & b_{22}\end{bmatrix},\begin{bmatrix}0 & c_{12}\\ c_{21} & c_{22}\end{bmatrix})$. Thus $\det_x(u_1,u_2,u_3)\sim u_1(b_{22}u_2+c_{22}u_3)-(b_{12}u_2+c_{12}u_3)(b_{21}u_2+c_{21}u_3)$. If $\det_x(u_1,u_2,u_3)$ is reducible, we see that $(b_{22},c_{22})/\!\!/(b_{21},c_{21})$ or $(b_{22},c_{22})/\!\!/(b_{12},c_{12})$.

- When $(b_{22},c_{22})=0$, by $(b_{22},c_{22})\not\parallel(b_{21},c_{21})$ we have $x\sim(\begin{bmatrix}1 & 0\\ 0 & 0\end{bmatrix},\begin{bmatrix}0 & 1\\ 0 & 0\end{bmatrix},\begin{bmatrix}0 & 0\\ 1 & 0\end{bmatrix})\in\mathcal{O}_9$.
- When $(b_{22},c_{22})\neq 0$, we can assume $(b_{21},c_{21})=t(b_{22},c_{22})$ where $t\in\mathbb{F}_q$ without loss of generality.

  It follows that $x\sim(\begin{bmatrix}1 & 0\\ 0 & 0\end{bmatrix},\begin{bmatrix}0 & 0\\ 0 & 1\end{bmatrix},\begin{bmatrix}0 & 0\\ 1 & 0\end{bmatrix})\in\mathcal{O}_9$.

If $\det_x(u_1,u_2,u_3)$ is irreducible, then we have $(b_{22},c_{22})\not\parallel(b_{21},c_{21})$ and $(b_{22},c_{22})\not\parallel(b_{12},c_{12})$. It follows that $x\sim(\begin{bmatrix}1 & 0\\ 0 & 0\end{bmatrix},\begin{bmatrix}0 & 0\\ 0 & 1\end{bmatrix},\begin{bmatrix}0 & 1\\ 1 & 0\end{bmatrix})\in\mathcal{O}_{10}$.

Next we count the number of the orbits. $|\mathcal{O}_1|,...,|\mathcal{O}_8|$ can be calculated by Propositions 3.2 and 4.1. For the count of $|\mathcal{O}_9|$, we calculate the order of the stabilizer subgroup of $x_9:=(\begin{bmatrix}0 & 0\\ 1 & 0\end{bmatrix},\begin{bmatrix}0 & 1\\ 0 & 0\end{bmatrix},\begin{bmatrix}0 & 0\\ 0 & 1\end{bmatrix})$ and $\mathrm{Stab}(x_9):=\{g\in G\mid gx_9=x_9$. Let $g=(g_1,g_2,g_3)=(\begin{pmatrix}p_1 & q_1\\ r_1 & s_1\end{pmatrix},\begin{pmatrix}p_2 & q_2\\ r_2 & s_2\end{pmatrix},\begin{pmatrix}g_{11} & g_{12} & g_{13}\\ g_{21} & g_{22} & g_{23}\\ g_{31} & g_{32} & g_{33}\end{pmatrix})\in\mathrm{Stab}(x_9)$. We have

$$(1,1,g_3)\cdot(\begin{bmatrix}q_1p_2 & q_1r_2\\ s_1p_2 & s_1r_2\end{bmatrix},\begin{bmatrix}p_1q_2 & p_1s_2\\ r_1q_2 & r_1s_2\end{bmatrix},\begin{bmatrix}q_1q_2 & q_1s_2\\ s_1q_2 & s_1s_2\end{bmatrix})=(\begin{bmatrix}0 & 0\\ 1 & 0\end{bmatrix},\begin{bmatrix}0 & 1\\ 0 & 0\end{bmatrix},\begin{bmatrix}0 & 0\\ 0 & 1\end{bmatrix}).$$

By comparing the $(1,1)$-entry of each matrix, we obtain $\begin{bmatrix}q_1p_2 & p_1q_2 & q_1q_2\end{bmatrix}g_3=\begin{bmatrix}0 & 0 & 0\end{bmatrix}$, and therefore $q_1=q_2=0$. Thus

$$\begin{aligned}&(1,1,g_3)\cdot(\begin{bmatrix}0 & 0\\ s_1p_2 & s_1r_2\end{bmatrix},\begin{bmatrix}0 & p_1s_2\\ 0 & r_1s_2\end{bmatrix},\begin{bmatrix}0 & 0\\ 0 & s_1s_2\end{bmatrix})\\ =&(\begin{bmatrix}0 & p_1s_2g_{12}\\ s_1p_2g_{11} & s_1r_2g_{11}+r_1s_2g_{12}+s_1s_2g_{13}\end{bmatrix},\begin{bmatrix}0 & p_1s_2g_{22}\\ s_1p_2g_{21} & s_1r_2g_{21}+r_1s_2g_{22}+s_1s_2g_{23}\end{bmatrix},\\ &\begin{bmatrix}0 & p_1s_2g_{32}\\ s_1p_2g_{31} & s_1r_2g_{31}+r_1s_2g_{32}+s_1s_2g_{33}\end{bmatrix})\\ =&(\begin{bmatrix}0 & 0\\ 1 & 0\end{bmatrix},\begin{bmatrix}0 & 1\\ 0 & 0\end{bmatrix},\begin{bmatrix}0 & 0\\ 0 & 1\end{bmatrix}).\end{aligned}$$

Therefore we obtain $s_1p_2g_{11}=p_1s_2g_{22}=1$ and $g_{12}=g_{21}=g_{31}=g_{32}=0$. It follows that

$$(\begin{bmatrix}0 & 0\\ s_1p_2g_{11} & s_1r_2g_{11}+s_1s_2g_{13}\end{bmatrix},\begin{bmatrix}0 & p_1s_2g_{22}\\ 0 & r_1s_2g_{22}+s_1s_2g_{23}\end{bmatrix},\begin{bmatrix}0 & 0\\ 0 & s_1s_2g_{33}\end{bmatrix})=(\begin{bmatrix}0 & 0\\ 1 & 0\end{bmatrix},\begin{bmatrix}0 & 1\\ 0 & 0\end{bmatrix},\begin{bmatrix}0 & 0\\ 0 & 1\end{bmatrix}).$$

Thus we obtain $g_{11}=(s_1p_2)^{-1}$, $g_{22}=(p_1s_2)^{-1}$, $g_{33}=(s_1s_2)^{-1}$, $g_{13}=-\frac{r_2g_{11}}{s_2}$, and $g_{23}=-\frac{r_2g_{11}}{s_2}$. Therefore

$$\begin{aligned}\mathrm{Stab}(x_9)=&\left\{(\begin{pmatrix}p_1 & 0\\ r_1 & s_1\end{pmatrix},\begin{pmatrix}p_2 & 0\\ r_2 & s_2\end{pmatrix},\begin{pmatrix}(s_1p_2)^{-1} & 0 & -\frac{r_2}{s_1p_2s_2}\\ 0 & (p_1s_2)^{-1} & -\frac{r_2}{s_1p_2s_2}\\ 0 & 0 & (s_1s_2)^{-1}\end{pmatrix})\in G\right\}\\ \cong&((\mathrm{GL}_1)^2\ltimes\mathbb{F}_q)\times((\mathrm{GL}_1)^2\ltimes\mathbb{F}_q).\end{aligned}$$

Thus we obtain $|\mathrm{Stab}(x)|=(q-1)^4q^2$, and $|\mathcal{O}_9|=|G|/|\mathrm{Stab}(x)|=(q-1)^3q^3(q+1)^3(q^2+q+1)$. Lastly, we obtain $|\mathcal{O}_{10}|=q^{12}-\sum_{i=1}^{9}|\mathcal{O}_i|=(q-1)^4q^4(q+1)^2(q^2+q+1)$. □

5.2. **The intersection between the orbits and the subspaces.** The subspaces we choose to calculate the Fourier transform are as follows:

$$W_1=0, W_2=(\begin{bmatrix}0&0\\0&*\end{bmatrix},\begin{bmatrix}0&0\\0&*\end{bmatrix},\begin{bmatrix}0&0\\0&*\end{bmatrix}), W_3=(\begin{bmatrix}0&0\\0&0\end{bmatrix},\begin{bmatrix}0&0\\0&0\end{bmatrix},\begin{bmatrix}*&*\\*&*\end{bmatrix}), W_4=(\begin{bmatrix}0&0\\*&*\end{bmatrix},\begin{bmatrix}0&0\\*&*\end{bmatrix},\begin{bmatrix}0&0\\*&*\end{bmatrix}),$$

$$W_5=(\begin{bmatrix}0&*\\0&*\end{bmatrix},\begin{bmatrix}0&*\\0&*\end{bmatrix},\begin{bmatrix}0&*\\0&*\end{bmatrix}),\ W_6=(\begin{bmatrix}0&0\\0&*\end{bmatrix},\begin{bmatrix}0&0\\0&*\end{bmatrix},\begin{bmatrix}0&*\\*&*\end{bmatrix}),\ W_7=(\begin{bmatrix}*&0\\0&*\end{bmatrix},\begin{bmatrix}*&0\\0&*\end{bmatrix},\begin{bmatrix}*&0\\0&*\end{bmatrix}),$$

$$W_8=(\begin{bmatrix}0&0\\0&0\end{bmatrix},\begin{bmatrix}*&*\\*&*\end{bmatrix},\begin{bmatrix}*&*\\*&*\end{bmatrix}),\ W_9=(\begin{bmatrix}0&*\\*&*\end{bmatrix},\begin{bmatrix}0&*\\*&*\end{bmatrix},\begin{bmatrix}0&*\\*&*\end{bmatrix}) \text{ and } W_{10}=V.$$

Orthogonal complements of them are as follows:

$$W_1^\perp=W_{10},\ W_2^\perp=W_9,\ W_3^\perp=W_8,\ W_4^\perp=W_4,\ W_5^\perp=W_5,\ W_6^\perp=(\begin{bmatrix}0&0\\0&*\end{bmatrix},\begin{bmatrix}0&*\\*&*\end{bmatrix},\begin{bmatrix}0&*\\*&*\end{bmatrix}),$$

$$W_7^\perp=W_7,\ W_8^\perp=W_3,\ W_9^\perp=W_2 \text{ and } W_{10}^\perp=W_1.$$

Proposition 5.2. *The cardinalities $|\mathcal{O}_i\cap W_j|$ for the orbit $\mathcal{O}_i$ and the subspace W_j are given as follows:*

	W_1	W_2	W_3	W_4	W_5	W_6
$\mathcal{O}_1$	1	1	1	1	1	1
$\mathcal{O}_2$	0	$[1,0,0,1]$	$[1,0,2,0]$	$[1,0,1,1]$	$[1,0,1,1]$	$[1,0,0,0]b_1$
$\mathcal{O}_3$	0	0	$[2,1,1,0]$	0	0	$[2,1,0,0]$
$\mathcal{O}_4$	0	0	0	$[2,1,1,1]$	0	$[2,1,1,0]$
$\mathcal{O}_5$	0	0	0	0	$[2,1,1,1]$	$[2,1,1,0]$
$\mathcal{O}_6$	0	0	0	0	0	$[3,1,1,0]$
$\mathcal{O}_7$	0	0	0	0	0	0
$\mathcal{O}_8$	0	0	0	0	0	0
$\mathcal{O}_9$	0	0	0	0	0	0
$\mathcal{O}_{10}$	0	0	0	0	0	0

	W_7	W_8	W_9	W_{10}	$W_6^\perp$
$\mathcal{O}_1$	1	1	1	1	1
$\mathcal{O}_2$	$2[1,0,0,1]$	$[1,0,3,0]$	$[1,0,0,1]a_1$	$[1,0,2,1]$	$[1,0,0,0]b_2$
$\mathcal{O}_3$	$[2,0,0,1]$	$[2,1,2,0]$	$[2,1,0,1]$	$[2,1,1,1]$	$[2,1,1,0]$
$\mathcal{O}_4$	0	$[2,1,2,0]$	$[2,1,1,1]$	$[2,1,2,1]$	$[2,1,2,0]$
$\mathcal{O}_5$	0	$[2,1,2,0]$	$[2,1,1,1]$	$[2,1,2,1]$	$[2,1,2,0]$
$\mathcal{O}_6$	0	$[3,1,3,0]$	$[3,1,1,1]$	$[3,1,3,1]$	$[3,1,2,0]$
$\mathcal{O}_7$	$[2,1,1,1]$	$\frac{1}{2}[2,3,3,0]$	$[2,3,1,1]$	$\frac{1}{2}[2,3,3,1]$	$[2,3,1,0]$
$\mathcal{O}_8$	0	$\frac{1}{2}[4,3,1,0]$	0	$\frac{1}{2}[4,3,1,1]$	0
$\mathcal{O}_9$	0	0	$[3,3,1,1]$	$[3,3,3,1]$	$[3,3,1,0]$
$\mathcal{O}_{10}$	0	0	0	$[4,4,2,1]$	0

Here $[a,b,c,d]=(q-1)^a q^b (q+1)^c (q^2+q+1)^d$, $a_1=2q+1$, $b_1=q^2+3q+1$ and $b_2=3q^2+3q+1$.

[Proof]

We can calculate for all subspaces but W_6 and $W_6^\perp$ by means of Propositions 3.2, 4.1 and 4.2. Let $x\in W_6$ be $(\begin{bmatrix}0&0\\0&a_{22}\end{bmatrix},\begin{bmatrix}0&0\\0&b_{22}\end{bmatrix},\begin{bmatrix}0&c_{12}\\c_{21}&c_{22}\end{bmatrix})$. For the case $a_{22}=0$, we calculated in the proof of Proposition 4.2. We only count the case $a_{22}\neq 0$ and add up the two result. When $c_{12}c_{21}\neq 0$, we have $x\in\mathcal{O}_6$. The number of such elements is $(q-1)^3q^2$. When $c_{12}=0$ and $c_{21}\neq 0$, we have $x\in\mathcal{O}_4$. When $c_{21}=0$ and $c_{12}\neq 0$, we have $x\in\mathcal{O}_5$. For both cases, there are $(q-1)^2q^2$ of such x. The remaining elements all belong to $\mathcal{O}_2$. Next, let $x\in W_6^\perp$ be $(\begin{bmatrix}0&0\\0&a_{22}\end{bmatrix},\begin{bmatrix}0&b_{12}\\b_{21}&b_{22}\end{bmatrix},\begin{bmatrix}0&c_{12}\\c_{21}&c_{22}\end{bmatrix})$. Again, we only count the case $a_{22}\neq 0$. When $(b_{12},c_{12})\not\parallel(b_{21},c_{21})$, we have $x\in\mathcal{O}_9$. The number of such elements is $(q-1)q^2\mathrm{gl}_2$. When $(b_{12},c_{12})\neq 0$ and $(b_{21},c_{21})\neq 0$ and $(b_{12},c_{12})\parallel(b_{21},c_{21})$, we have $x\in\mathcal{O}_6$. The number of such elements is $(q-1)^2q^2(q^2-1)$. When $(b_{12},c_{12})=0$ and $(b_{21},c_{21})\neq 0$, we have $x\in\mathcal{O}_4$. When $(b_{12},c_{12})\neq 0$ and $(b_{21},c_{21})=0$, we have $x\in\mathcal{O}_5$. For both cases, there are $(q^2-1)(q-1)q^2$ of such x. The remaining elements all belong to $\mathcal{O}_2$. □

5.3. **Fourier transform.**

Theorem 5.3. *The representation matrix M of the Fourier transform on $\mathcal{F}_V^G$ with respect to the basis $e_1, \dots, e_{10}$ is given as follows:*

$$\frac{1}{q^{12}}\left[\begin{array}{cccccc} 1 & [1,0,2,1] & [2,1,1,1] & [2,1,2,1] & [2,1,2,1] & [3,1,3,1] \\ 1 & d_1 & [1,1,0,0]c_1 & [1,1,1,0]c_1 & [1,1,1,0]c_1 & [2,1,1,0]c_2 \\ 1 & [0,0,1,0]c_1 & qd_2 & -[1,1,2,0] & -[1,1,2,0] & [1,1,2,0]c_4 \\ 1 & [0,0,1,0]c_1 & -[1,1,1,0] & qe_1 & -[1,1,2,0] & -[1,1,1,0]b_1 \\ 1 & [0,0,1,0]c_1 & -[1,1,1,0] & -[1,1,2,0] & qe_1 & -[1,1,1,0]b_1 \\ 1 & c_2 & qc_4 & -qb_1 & -qb_1 & qe_2 \\ 1 & c_3 & [1,1,0,0]b_1 & -[1,1,0,0]a_1 & -[1,1,0,0]a_1 & -[2,1,0,0]b_3 \\ 1 & -[0,0,2,0] & [0,1,1,0]b_2 & [0,1,1,0] & [0,1,1,0] & -[0,1,2,0]b_2 \\ 1 & c_2 & -[1,1,1,0] & -qb_1 & -qb_1 & [1,1,0,0]a_1 \\ 1 & -[0,0,2,0] & q & [0,1,1,0] & [0,1,1,0] & -[0,1,1,0] \end{array}\right.$$
$$\left.\begin{array}{cccc} \frac{1}{2}[2,3,3,1] & \frac{1}{2}[4,3,1,1] & [3,3,3,1] & [4,4,2,1] \\ \frac{1}{2}[1,3,1,0]c_3 & -\frac{1}{2}[3,3,1,0] & [2,3,1,0]c_2 & -[3,4,2,0] \\ \frac{1}{2}[1,3,2,0]b_1 & \frac{1}{2}[2,3,1,0]b_2 & -[2,3,3,0] & [2,4,1,0] \\ -\frac{1}{2}[1,3,1,0]a_1 & \frac{1}{2}[2,3,0,0] & -[1,3,1,0]b_1 & [2,4,1,0] \\ -\frac{1}{2}[1,3,1,0]a_1 & \frac{1}{2}[2,3,0,0] & -[1,3,1,0]b_1 & [2,4,1,0] \\ -\frac{1}{2}[1,3,0,0]b_3 & -\frac{1}{2}[1,3,0,0]b_2 & [1,3,0,0]a_1 & -[1,4,0,0] \\ \frac{1}{2}q^3c_5 & -\frac{1}{2}[3,3,0,0] & -[1,3,0,0]b_4 & [2,4,0,0] \\ -\frac{1}{2}[1,3,2,0] & \frac{1}{2}q^3c_6 & [1,3,2,0] & -[1,4,1,0] \\ -\frac{1}{2}q^3b_4 & \frac{1}{2}[2,3,0,0] & q^3b_1 & -[1,4,0,0] \\ \frac{1}{2}[0,3,1,0] & -\frac{1}{2}[1,3,0,0] & -[0,3,1,0] & q^4 \end{array}\right].$$

Here $[a,b,c,d] = (q-1)^a q^b (q+1)^c (q^2+q+1)^d$ *and*

$$\begin{array}{lll} a_1 = 2q+1, & c_1 = q^3 - q - 1, & d_1 = 2q^4 + q^3 - q^2 - 2q - 1, \\ b_1 = q^2 - q - 1, & c_2 = q^3 - q^2 - 2q - 1, & d_2 = q^4 - q^2 + 1, \\ b_2 = q^2 - q + 1, & c_3 = 2q^3 - q^2 - 2q - 1, & e_1 = q^5 - q^3 - q^2 + q + 1, \\ b_3 = q^2 + 3q + 1, & c_4 = q^3 - q^2 + 1, & e_2 = q^5 - q^4 - q^3 + 2q^2 - q - 1. \\ b_4 = q^2 - 2q - 1, & c_5 = q^3 - 3q^2 + 3q + 1, & \\ & c_6 = q^3 + q^2 - q + 1, & \end{array}$$

We used PARI/GP [8] to calculate the matrix from Proposition 5.2.

Corollary 5.4. *The indicator function of singular set of V is $\Psi = \sum_{i=1}^{9} e_i$. Its Fourier transform $\widehat{\Psi}$ is given as follows:*

$$\widehat{\Psi}(x) = \begin{cases} q^{-1} + 2q^{-2} - q^{-3} - 2q^{-4} - q^{-5} + 2q^{-6} + q^{-7} - q^{-8} & x \in \mathcal{O}_1, \\ q^{-3} - q^{-4} - 2q^{-5} + 2q^{-6} + q^{-7} - q^{-8} & x \in \mathcal{O}_2, \\ -q^{-5} + q^{-6} + q^{-7} - q^{-8} & x \in \mathcal{O}_3, \mathcal{O}_4, \mathcal{O}_5, \\ -q^{-6} + 2q^{-7} - q^{-8} & x \in \mathcal{O}_7, \\ q^{-6} - q^{-8} & x \in \mathcal{O}_8, \\ q^{-7} - q^{-8} & x \in \mathcal{O}_6, \mathcal{O}_9, \\ -q^{-8} & x \in \mathcal{O}_{10}. \end{cases}$$

In particular, we have the following L_1-norm bound of $\widehat{\Psi}$:

$$\sum_{x \in V} |\widehat{\Psi}(x)| = O(q^4).$$

6. $2 \otimes 2 \otimes 4$

Let $V = \mathbb{F}_q^2 \otimes \mathbb{F}_q^2 \otimes \mathbb{F}_q^4$ and $G = G_1 \times G_2 \times G_3 = \mathrm{GL}_2 \times \mathrm{GL}_2 \times \mathrm{GL}_4$. We write $x \in V$ as $x = (A, B, C, D)$ where A, B, C and D are 2-by-2 matrices, and write $g \in G$ as $g = (g_1, g_2, g_3)$ where $g_1, g_2 \in \mathrm{GL}_2$ and $g_3 \in \mathrm{GL}_4$. The action of G on V is defined by

$$gx = (g_1 A g_2^T, g_1 B g_2^T, g_1 C g_2^T, g_1 D g_2^T) g_3^T.$$

Define a bilinear form β of V as

$$\beta((A_1, B_1, C_1, D_1), (A_2, B_2, C_2, D_2)) = \mathrm{Tr}(A_1 A_2^T + B_1 B_2^T + C_1 C_2^T + D_1 D_2^T).$$

In addition, define an automorphism ι of G as

$$(g_1, g_2, g_3)^\iota = ((g_1^T)^{-1}, (g_2^T)^{-1}, (g_3^T)^{-1}).$$

These β and ι satisfy Assumption 2.1.

6.1. **Orbit decomposition.** For $x = (A, B, C, D) = (\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}, \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix}, \begin{bmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \end{bmatrix}, \begin{bmatrix} d_{11} & d_{12} \\ d_{21} & d_{22} \end{bmatrix}) \in V$, we define

$$r_1(x) := \mathrm{rank}(\begin{bmatrix} a_{11} & a_{12} & a_{21} & a_{22} \\ b_{11} & b_{12} & b_{21} & b_{22} \\ c_{11} & c_{12} & c_{21} & c_{22} \\ d_{11} & d_{12} & d_{21} & d_{22} \end{bmatrix}),$$

$$r_2(x) := \mathrm{rank}(\begin{bmatrix} a_{11} & a_{12} & b_{11} & b_{12} & c_{11} & c_{12} & d_{11} & d_{12} \\ a_{21} & a_{22} & b_{21} & b_{22} & c_{21} & c_{22} & d_{21} & d_{22} \end{bmatrix}),$$

$$r_3(x) := \mathrm{rank}(\begin{bmatrix} a_{11} & a_{21} & b_{11} & b_{21} & c_{11} & c_{21} & d_{11} & d_{21} \\ a_{12} & a_{22} & b_{12} & b_{22} & c_{12} & c_{22} & d_{12} & d_{22} \end{bmatrix}),$$

$$\det{}_x(u_1, u_2, u_3, u_4) := \det(u_1 A + u_2 B + u_3 C + u_4 D) \in \mathrm{Sym}^2(\mathbb{F}_q^4) \text{ where } u_1, u_2, u_3, u_4 \text{ are variables,}$$

$$\mathrm{T}(x) := \langle\!\langle \alpha \rangle\!\rangle \text{ if and only if } \det{}_x(u, v) \in O_{\langle\!\langle \alpha \rangle\!\rangle} \text{ in } \mathrm{Sym}^2(\mathbb{F}_q^4).$$

Note that we introduced the representation $(\mathrm{GL}_1(\mathbb{F}_q) \times \mathrm{GL}_4(\mathbb{F}_q), \mathrm{Sym}^2(\mathbb{F}_q^4))$ in Section 3.1. For $x \in V$ and $g = (g_1, g_2, g_3) \in G$, we have

$$\det{}_{gx}(u_1, u_2, u_3, u_4) = \det(g_1 g_2) \det{}_x((u_1, u_2, u_3, u_4) g_3).$$

Proposition 6.1. *V consists of 11 G-orbits in all.*

Orbit name	Representative	$r_1(x)$	$r_2(x)$	$r_3(x)$	$\mathrm{T}(x)$	Cardinality
$\mathcal{O}_1$	$(\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix})$	0	0	0	$\langle\!\langle 0 \rangle\!\rangle$	1
$\mathcal{O}_2$	$(\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix})$	1	1	1	$\langle\!\langle 0 \rangle\!\rangle$	$[1, 0, 3, 0, 1]$
$\mathcal{O}_3$	$(\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix})$	1	2	2	$\langle\!\langle 1 \rangle\!\rangle$	$[2, 1, 2, 0, 1]$
$\mathcal{O}_4$	$(\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix})$	2	1	2	$\langle\!\langle 0 \rangle\!\rangle$	$[2, 1, 2, 1, 1]$
$\mathcal{O}_5$	$(\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix})$	2	2	1	$\langle\!\langle 0 \rangle\!\rangle$	$[2, 1, 2, 1, 1]$
$\mathcal{O}_6$	$(\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix})$	2	2	2	$\langle\!\langle 1 \rangle\!\rangle$	$[3, 1, 3, 1, 1]$
$\mathcal{O}_7$	$(\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix})$	2	2	2	$\langle\!\langle 2r \rangle\!\rangle$	$\frac{1}{2}[2, 3, 3, 1, 1]$
$\mathcal{O}_8$	$(\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & -1 \\ \mu_0 & \mu_1 \end{bmatrix})$	2	2	2	$\langle\!\langle 2i \rangle\!\rangle$	$\frac{1}{2}[4, 3, 1, 1, 1]$
$\mathcal{O}_9$	$(\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix})$	3	2	2	$\langle\!\langle 2r \rangle\!\rangle$	$[3, 3, 4, 1, 1]$
$\mathcal{O}_{10}$	$(\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix})$	3	2	2	$\langle\!\langle 3 \rangle\!\rangle$	$[4, 4, 3, 1, 1]$
$\mathcal{O}_{11}$	$(\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix})$	4	2	2	$\langle\!\langle 4i \rangle\!\rangle$	$[4, 6, 2, 1, 1]$

Here $[a, b, c, d, e] = (q-1)^a q^b (q+1)^c (q^2+q+1)^d (q^2+1)^e$, and μ_1, μ_0 are elements of $\mathbb{F}_q$ such that $X^2 + \mu_1 X + \mu_0 \in \mathbb{F}_q[X]$ is irreducible.

[Proof]

First we consider the orbit decomposition. When $r_1(x) \leq 3$, we have $x \sim (0, B, C, D)$ where $B, C, D \in \mathrm{M}_2(\mathbb{F}_q)$. Therefore by Propositions 3.1 and 5.1, we have

$$\{x \in V | r_1(x) \leq 3\} = \bigcup_{i=1}^{10} \mathcal{O}_i.$$

When $r_1(x) = 4$, it is easy to show that $x \sim (\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix})$ by the action of GL_4. Therefore the number of orbits is 11.

$|\mathcal{O}_1|, \cdots, |\mathcal{O}_{10}|$ can be calculated by means of Propositions 3.2 and 5.1. By subtracting these numbers from $|V| = q^{16}$, we obtain $|\mathcal{O}_{11}|$. Alternatively, we may say that $|\mathcal{O}_{11}|$ coincides with gl_4. □

6.2. **The intersection between the orbits and the subspaces.** The subspaces we choose to calculate the Fourier transform are as follows:

$$W_1 = 0,\ W_2 = (\begin{bmatrix} 0 & 0 \\ 0 & * \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & * \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & * \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & * \end{bmatrix}),\ W_3 = (\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} * & * \\ * & * \end{bmatrix}),$$

$$W_4 = (\begin{bmatrix} 0 & 0 \\ * & * \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ * & * \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ * & * \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ * & * \end{bmatrix}),\ W_5 = (\begin{bmatrix} 0 & * \\ 0 & * \end{bmatrix}, \begin{bmatrix} 0 & * \\ 0 & * \end{bmatrix}, \begin{bmatrix} 0 & * \\ 0 & * \end{bmatrix}, \begin{bmatrix} 0 & * \\ 0 & * \end{bmatrix}),$$

$$W_6 = (\begin{bmatrix} 0 & 0 \\ 0 & * \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & * \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & * \end{bmatrix}, \begin{bmatrix} 0 & * \\ * & * \end{bmatrix}),\ W_7 = (\begin{bmatrix} * & 0 \\ 0 & * \end{bmatrix}, \begin{bmatrix} * & 0 \\ 0 & * \end{bmatrix}, \begin{bmatrix} * & 0 \\ 0 & * \end{bmatrix}, \begin{bmatrix} * & 0 \\ 0 & * \end{bmatrix}),$$

$$W_8 = (\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} * & * \\ * & * \end{bmatrix}, \begin{bmatrix} * & * \\ * & * \end{bmatrix}),\ W_9 = (\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} * & * \\ * & * \end{bmatrix}, \begin{bmatrix} * & * \\ * & * \end{bmatrix}, \begin{bmatrix} * & * \\ * & * \end{bmatrix}),$$

$$W_{10} = (\begin{bmatrix} 0 & * \\ * & * \end{bmatrix}, \begin{bmatrix} 0 & * \\ * & * \end{bmatrix}, \begin{bmatrix} 0 & * \\ * & * \end{bmatrix}, \begin{bmatrix} 0 & * \\ * & * \end{bmatrix}) \text{ and } W_{11} = V.$$

Orthogonal complements of them are as follows:

$$W_1^\perp = W_{11}, W_2^\perp = W_{10}, W_3^\perp = W_9, W_4^\perp = W_4, W_5^\perp = W_5, W_6^\perp = (\begin{bmatrix} 0 & 0 \\ 0 & * \end{bmatrix}, \begin{bmatrix} 0 & * \\ * & * \end{bmatrix}, \begin{bmatrix} 0 & * \\ * & * \end{bmatrix}, \begin{bmatrix} 0 & * \\ * & * \end{bmatrix}),$$

$$W_7^\perp = W_7, W_8^\perp = W_8, W_9^\perp = W_3, W_{10}^\perp = W_2 \text{ and } W_{11}^\perp = W_1.$$

Proposition 6.2. *The cardinalities $|\mathcal{O}_i \cap W_j|$ for the orbit $\mathcal{O}_i$ and the subspace W_j are given as follows:*

	W_1	W_2	W_3	W_4	W_5	W_6	W_7
$\mathcal{O}_1$	1	1	1	1	1	1	1
$\mathcal{O}_2$	0	$[1,0,1,0,1]$	$[1,0,2,0,0]$	$[1,0,2,0,1]$	$[1,0,2,0,1]$	$(q-1)c_1$	$2[1,0,1,0,1]$
$\mathcal{O}_3$	0	0	$[2,1,1,0,0]$	0	0	$[2,1,0,0,0]$	$[2,0,1,0,1]$
$\mathcal{O}_4$	0	0	0	$[2,1,1,1,1]$	0	$[2,1,0,1,0]$	0
$\mathcal{O}_5$	0	0	0	0	$[2,1,1,1,1]$	$[2,1,0,1,0]$	0
$\mathcal{O}_6$	0	0	0	0	0	$[3,1,0,1,0]$	0
$\mathcal{O}_7$	0	0	0	0	0	0	$[2,1,1,1,1]$
$\mathcal{O}_8$	0	0	0	0	0	0	0
$\mathcal{O}_9$	0	0	0	0	0	0	0
$\mathcal{O}_{10}$	0	0	0	0	0	0	0
$\mathcal{O}_{11}$	0	0	0	0	0	0	0

	W_8	W_9	W_{10}	W_{11}	$W_6^{\perp}$
$\mathcal{O}_1$	1	1	1	1	1
$\mathcal{O}_2$	$[1,0,3,0,0]$	$[1,0,1,0,1]a_1$	$[1,0,2,1,0]$	$[1,0,3,0,1]$	$(q-1)c_2$
$\mathcal{O}_3$	$[2,1,2,0,0]$	$[2,1,1,0,1]$	$[2,1,1,1,0]$	$[2,1,2,0,1]$	$[2,1,0,1,0]$
$\mathcal{O}_4$	$[2,1,2,0,0]$	$[2,1,1,1,1]$	$[2,1,2,1,0]$	$[2,1,2,1,1]$	$[2,1,0,2,0]$
$\mathcal{O}_5$	$[2,1,2,0,0]$	$[2,1,1,1,1]$	$[2,1,2,1,0]$	$[2,1,2,1,1]$	$[2,1,0,2,0]$
$\mathcal{O}_6$	$[3,1,3,0,0]$	$[3,1,1,1,1]$	$[3,1,3,1,0]$	$[3,1,3,1,1]$	$[3,1,0,2,0]$
$\mathcal{O}_7$	$\frac{1}{2}[2,3,3,0,0]$	$[2,3,1,1,1]$	$\frac{1}{2}[2,3,3,1,0]$	$\frac{1}{2}[2,3,3,1,1]$	$[2,3,1,1,0]$
$\mathcal{O}_8$	$\frac{1}{2}[4,3,1,0,0]$	0	$\frac{1}{2}[4,3,1,1,0]$	$\frac{1}{2}[4,3,1,1,1]$	0
$\mathcal{O}_9$	0	$[3,3,2,1,1]$	$[3,3,3,1,0]$	$[3,3,4,1,1]$	$[3,3,2,1,0]$
$\mathcal{O}_{10}$	0	0	$[4,4,2,1,0]$	$[4,4,3,1,1]$	0
$\mathcal{O}_{11}$	0	0	0	$[4,6,2,1,1]$	0

Here $[a,b,c,d,e]=(q-1)^a q^b (q+1)^c (q^2+q+1)^d (q^2+1)^e$, $a_1=2q+1$, $c_1=q^3+q^2+3q+1$ *and* $c_2=3q^3+3q^2+3q+1$.

[Proof]

We can calculate for all subspaces but W_6 and $W_6^{\perp}$ by means of Propositions 3.2 and 5.1. For W_6, we can easily calculate by considering only the case $a_{22}\neq 0$ as in the proof of Proposition 5.2. Write $x\in W_6^{\perp}$ as
$\left(\begin{bmatrix}0&0\\0&a_{22}\end{bmatrix},\begin{bmatrix}0&b_{12}\\b_{21}&b_{22}\end{bmatrix},\begin{bmatrix}0&c_{12}\\c_{21}&c_{22}\end{bmatrix},\begin{bmatrix}0&d_{12}\\d_{21}&d_{22}\end{bmatrix}\right)$. We already calculated for the case $a_{22}=0$ in the proof of Proposition 5.2. Thus we only count the case $a_{22}\neq 0$. When $(b_{12},c_{12},d_{12})\not\parallel(b_{21},c_{21},d_{21})$, we have $x\in\mathcal{O}_9$. The number of such elements is $(q-1)q^3|(3,2),2|$. When $(b_{12},c_{12},d_{12})\neq 0$ and $(b_{21},c_{21},d_{21})\neq 0$ and $(b_{12},c_{12},d_{12})/\!\!/(b_{21},c_{21},d_{21})$, we have $x\in\mathcal{O}_6$. The number of such elements is $(q^3-1)(q-1)^2q^3$. When $(b_{12},c_{12},d_{12})=0$ and $(b_{21},c_{21},d_{21})\neq 0$, we have $x\in\mathcal{O}_4$. When $(b_{12},c_{12},d_{12})\neq 0$ and $(b_{21},c_{21},d_{21})=0$, we have $x\in\mathcal{O}_5$. The numbers of elements are both $(q^3-1)q^3(q-1)$. The remaining elements all belong to $\mathcal{O}_2$. □

6.3. Fourier transform.

Theorem 6.3. *The representation matrix M of the Fourier transform on $\mathcal{F}_V^G$ with respect to the basis $e_1,\dots,e_{11}$ is given as follows:*

$$\frac{1}{q^{16}}\left[\begin{array}{cccccc}
1 & [1,0,3,0,1] & [2,1,2,0,1] & [2,1,2,1,1] & [2,1,2,1,1] & [3,1,3,1,1]\\
1 & e_1 & [1,1,0,0,0]d_1 & [1,1,0,1,0]d_1 & [1,1,0,1,0]d_1 & [2,1,0,2,0]b_1\\
1 & (q+1)d_1 & qe_2 & -[1,1,1,1,0] & -[1,1,1,1,0] & [1,1,1,1,0]d_3\\
1 & (q+1)d_1 & -[1,1,1,0,0] & qg_1 & -[1,1,1,1,0] & -[1,1,1,0,0]c_1\\
1 & (q+1)d_1 & -[1,1,1,0,0] & -[1,1,1,1,0] & qg_1 & -[1,1,1,0,0]c_1\\
1 & [0,0,0,1,0]b_1 & qd_3 & -qc_1 & -qc_1 & qg_2\\
1 & d_2 & [1,1,0,0,0]c_1 & -[1,1,0,0,0]b_2 & -[1,1,0,0,0]b_2 & -[2,1,0,0,0]c_4\\
1 & -(q+1)^2 & [0,1,1,0,0]c_2 & [0,1,1,0,0] & [0,1,1,0,0] & -[0,1,2,0,0]c_2\\
1 & [0,0,0,1,0]b_1 & -[1,1,1,0,0] & -qc_1 & -qc_1 & -[1,1,0,0,0]c_5\\
1 & -(q+1)^2 & qc_3 & [0,1,1,0,0] & [0,1,1,0,0] & -[0,1,1,0,0]c_3\\
1 & -(q+1)^2 & -[1,1,1,0,0] & [0,1,1,0,0] & [0,1,1,0,0] & [1,1,2,0,0]
\end{array}\right.$$

$$\left.\begin{array}{ccccc}
\frac{1}{2}[2,3,3,1,1] & \frac{1}{2}[4,3,1,1,1] & [3,3,4,1,1] & [4,4,3,1,1] & [4,6,2,1,1]\\
\frac{1}{2}[1,3,0,1,0]d_2 & -\frac{1}{2}[3,3,0,1,0] & [2,3,1,2,0]b_1 & -[3,4,2,1,0] & -[3,6,1,1,0]\\
\frac{1}{2}[1,3,1,1,0]c_1 & \frac{1}{2}[2,3,0,1,0]c_2 & -[2,3,3,1,0] & [2,4,1,1,0]c_3 & -[3,6,1,1,0]\\
-\frac{1}{2}[1,3,1,0,0]b_2 & \frac{1}{2}[2,3,0,0,0] & -[1,3,2,0,0]c_1 & [2,4,2,0,0] & [2,6,1,0,0]\\
-\frac{1}{2}[1,3,1,0,0]b_2 & \frac{1}{2}[2,3,0,0,0] & -[1,3,2,0,0]c_1 & [2,4,2,0,0] & [2,6,1,0,0]\\
-\frac{1}{2}[1,3,0,0,0]c_4 & -\frac{1}{2}[1,3,0,0,0]c_2 & -[1,3,1,0,0]c_5 & -[1,4,1,0,0]c_3 & [2,6,1,0,0]\\
\frac{1}{2}q^3e_3 & -\frac{1}{2}[3,3,1,0,0] & -[1,3,1,0,0]c_6 & -[2,4,1,0,0]b_1 & [2,6,1,0,0]\\
-\frac{1}{2}[1,3,3,0,0] & \frac{1}{2}q^3e_4 & [1,3,3,0,0] & -[1,4,2,0,0]b_3 & [2,6,1,0,0]\\
-\frac{1}{2}q^3c_6 & \frac{1}{2}[2,3,0,0,0] & q^3c_3 & [1,4,0,0,0]b_1 & -[1,6,0,0,0]\\
-\frac{1}{2}[0,3,1,0,0]b_1 & -\frac{1}{2}[1,3,0,0,0]b_3 & [0,3,1,0,0]b_1 & q^4c_3 & -[1,6,0,0,0]\\
\frac{1}{2}[0,3,2,0,0] & \frac{1}{2}[2,3,0,0,0] & -[0,3,2,0,0] & -[1,4,1,0,0] & q^6
\end{array}\right].$$

Here $[a,b,c,d,e]=(q-1)^a q^b (q+1)^c (q^2+q+1)^d (q^2+1)^e$ *and*

$$\begin{array}{lll}
b_1=q^2-q-1, & c_1=q^3-q-1, & e_1=2q^5+q^4-q^2-2q-1,\\
b_2=2q^2+2q+1, & c_2=q^3-q+1, & e_2=q^5-q^2+1,\\
b_3=q^2-q+1, & c_3=q^3-q^2+1, & e_3=2q^5-q^4-4q^3+2q^2+2q+1,\\
d_1=q^4-q-1, & c_4=q^3+4q^2+3q+1, & e_4=2q^5-q^4+2q^2-2q+1,\\
d_2=2q^4-q^2-2q-1, & c_5=q^3-q^2-2q-1, & g_1=q^7-q^4-q^3+q+1,\\
d_3=q^4-q^2+1, & c_6=2q^3-q^2-2q-1, & g_2=q^7-q^5-2q^4+2q^3+q^2-q-1.\\
 & c_7=2q^3-2q-1, &
\end{array}$$

We used PARI/GP [8] to calculate the matrix from Proposition 6.2.

Corollary 6.4. *The indicator function of singular set of* V *is* $\Psi=\sum_{i=1}^{10} e_i$. *Its Fourier transform* $\widehat{\Psi}$ *is given as follows:*

$$\widehat{\Psi}(x)=\begin{cases} q^{-1}+q^{-2}-2q^{-5}+q^{-8}+q^{-9}-q^{-10} & r_1(x)=0,\\ q^{-4}-q^{-5}-q^{-6}+q^{-8}+q^{-9}-q^{-10} & r_1(x)=1,\\ -q^{-7}+q^{-8}+q^{-9}-q^{-10} & r_1(x)=2,\\ q^{-9}-q^{-10} & r_1(x)=3,\\ -q^{-10} & r_1(x)=4.\end{cases}$$

In particular, we have the following L_1*-norm bound of* $\widehat{\Psi}$*:*

$$\sum_{x\in V}|\widehat{\Psi}(x)|=O(q^6).$$

7. $2\otimes \mathrm{H}_2(\mathbb{F}_{q^2})$

For $a\in\mathbb{F}_{q^2}$, let $\overline{a}$ be the conjugate of a over $\mathbb{F}_q$. Let $\mathrm{H}_2(\mathbb{F}_{q^2})$ be the set of Hermitian matrices of order 2, i.e.,

$$\mathrm{H}_2(\mathbb{F}_{q^2}):=\left\{A=\begin{bmatrix}a_{11}&a_{12}\\a_{21}&a_{22}\end{bmatrix}\in \mathrm{M}_2(\mathbb{F}_{q^2})\,\middle|\, a_{11},a_{22}\in\mathbb{F}_q, a_{21}=\overline{a_{12}}\right\}.$$

Let $V=\mathbb{F}_q^2\otimes \mathrm{H}_2(\mathbb{F}_{q^2})$ and $G=G_1\times G_2=\mathrm{GL}_2(\mathbb{F}_q)\times \mathrm{GL}_2(\mathbb{F}_{q^2})$. We write $x\in V$ as $x=(A,B)$ where $A,B\in \mathrm{H}_2(\mathbb{F}_{q^2})$, and write $g\in G$ as $g=(g_1,g_2)$ where $g_1\in\mathrm{GL}_2(\mathbb{F}_q)$ and $g_2\in \mathrm{GL}_2(\mathbb{F}_{q^2})$. The action of G on V is defined by

$$gx=(g_2A\overline{g_2^T},g_2B\overline{g_2^T})g_1^T.$$

Here, for a matrix h, $\overline{h}$ is the matrix whose (i,j)-entry is the conjugate over $\mathbb{F}_q$ of the (i,j)-entry of h. Define a bilinear form β of V as

$$\beta((A_1,B_1),(A_2,B_2))=\mathrm{Tr}(A_1A_2^T+B_1B_2^T).$$

In addition, define an automorphism ι of G as

$$(g_1,g_2)^\iota=((g_1^T)^{-1},(g_2^T)^{-1}).$$

These β and ι satisfy Assumption 2.1.

7.1. **Orbit decomposition.**

For $x=(A,B)=(\begin{bmatrix}a_{11}&a_{12}\\\overline{a_{12}}&a_{22}\end{bmatrix},\begin{bmatrix}b_{11}&b_{12}\\\overline{b_{12}}&b_{22}\end{bmatrix})\in V$, we define

$$r_1(x):=\mathrm{rank}(\begin{bmatrix}a_{11}&a_{12}&\overline{a_{12}}&a_{22}\\b_{11}&b_{12}&\overline{b_{12}}&b_{22}\end{bmatrix}),$$

$$r_2(x):=\mathrm{rank}(\begin{bmatrix}a_{11}&a_{12}&b_{11}&b_{12}\\\overline{a_{12}}&a_{22}&\overline{b_{12}}&b_{22}\end{bmatrix}),$$

$$\det{}_x(u,v):=\det(uA+vB)\in\mathrm{Sym}^2(\mathbb{F}_q^2)\text{ where }u,v\text{ are variables,}$$

$$\mathrm{T}(x):=\langle\!\langle\alpha\rangle\!\rangle\text{ if and only if }\det{}_x(u,v)\in O_{\langle\!\langle\alpha\rangle\!\rangle}\text{ in }\mathrm{Sym}^2(\mathbb{F}_q^2).$$

For $x\in V$ and $g=(g_1,g_2)\in G$, we have

$$\det{}_{gx}(u,v)=\mathrm{N}_2(\det(g_2))\det{}_x((u,v)g_1),$$

where $\mathrm{N}_2:\mathbb{F}_{q^2}\ni z\mapsto z\overline{z}\in\mathbb{F}_q$ is the norm map.

Proposition 7.1. *V consists of 6 G-orbits in all.*

Orbit name	Representative	$r_1(x)$	$r_2(x)$	$\mathrm{T}(x)$	Cardinality
$\mathcal{O}_1$	$(\begin{bmatrix}0&0\\0&0\end{bmatrix},\begin{bmatrix}0&0\\0&0\end{bmatrix})$	0	0	$\langle\langle 0\rangle\rangle$	1
$\mathcal{O}_2$	$(\begin{bmatrix}0&0\\0&0\end{bmatrix},\begin{bmatrix}0&0\\0&1\end{bmatrix})$	1	1	$\langle\langle 0\rangle\rangle$	$[1,0,1,1]$
$\mathcal{O}_3$	$(\begin{bmatrix}0&0\\0&0\end{bmatrix},\begin{bmatrix}1&0\\0&1\end{bmatrix})$	1	2	$\langle\langle 1\rangle\rangle$	$[1,1,1,1]$
$\mathcal{O}_4$	$(\begin{bmatrix}0&0\\0&1\end{bmatrix},\begin{bmatrix}0&1\\1&0\end{bmatrix})$	2	2	$\langle\langle 1\rangle\rangle$	$[2,1,2,1]$
$\mathcal{O}_5$	$(\begin{bmatrix}1&0\\0&0\end{bmatrix},\begin{bmatrix}0&0\\0&1\end{bmatrix})$	2	2	$\langle\langle 2r\rangle\rangle$	$\frac{1}{2}[2,3,1,1]$
$\mathcal{O}_6$	$(\begin{bmatrix}1&0\\0&1\end{bmatrix},\begin{bmatrix}0&\mu_0\\\overline{\mu_0}&\mu_1\end{bmatrix})$	2	2	$\langle\langle 2i\rangle\rangle$	$\frac{1}{2}[2,3,1,1]$

Here $[a,b,c,d]=(q-1)^a q^b (q+1)^c (q^2+1)^d$, and $\mu_1 \in \mathbb{F}_q$, $\mu_0 \in \mathbb{F}_{q^2}$ are elements such that $X^2+\mu_1 X - \mathrm{N}_2(\mu_0) \in \mathbb{F}_q[X]$ is irreducible.

Note that there exist such μ_1 and μ_0 of the lowest row of the table because of the surjectivity of the norm map N_2.
[Proof]

The invariants $r_1(x)$, $r_2(x)$ and $\mathrm{T}(x)$ for the 6 elements in the "Representative" column of the table are easily calculated. Since they do not coincide, these 6 elements belong to different orbits. Let $\mathcal{O}_i$ be the orbit of each element.

First we consider the orbit decomposition. When $r_1(x)=0$, we easily see that $x \in \mathcal{O}_1$. When $r_1(x)=1$, we have $x\sim(0,B)$. If $\mathrm{rank}(B)=1$ we have $x\in\mathcal{O}_2$, and if $\mathrm{rank}(B)=2$ we have $x\in\mathcal{O}_3$. When $r_1(x)=2$, we have $\mathrm{T}(x)=\langle\langle 1\rangle\rangle$, $\langle\langle 2r\rangle\rangle$ or $\langle\langle 2i\rangle\rangle$. If $\mathrm{T}(x)=\langle\langle 1\rangle\rangle$ or $\langle\langle 2r\rangle\rangle$, we have $x\sim(\begin{bmatrix}1&0\\0&0\end{bmatrix},\begin{bmatrix}0&b_{12}\\\overline{b_{12}}&b_{22}\end{bmatrix})$. Since $\det_x(u,v)\sim v(b_{22}u-\mathrm{N}_2(b_{12})v)$, we have $x\in\mathcal{O}_4$ if $b_{22}=0$ and $x\in\mathcal{O}_5$ if $b_{22}\neq 0$. If $\mathrm{T}(x)=\langle\langle 2i\rangle\rangle$, we have $\det_x(u,v)\sim u^2+\mu_1 uv-\mathrm{N}_2(\mu_0)v^2=(u-\gamma v)(u-\overline{\gamma}v)$ where $\gamma\in\mathbb{F}_{q^2}\setminus\mathbb{F}_q$ by Section 3.1 and the surjectivity of the map $G\ni(g_1,g_2)\mapsto(\mathrm{N}_2(\det(g_2)),g_1^T)\in\mathrm{GL}_1(\mathbb{F}_q)\times\mathrm{GL}_2(\mathbb{F}_q)$. Therefore we assume $\det_x(u,v)=u^2+\mu_1 uv-\mathrm{N}_2(\mu_0)v^2$. We can move x to $y:=(\begin{bmatrix}a_{11}&a_{12}\\\overline{a_{12}}&a_{22}\end{bmatrix},\begin{bmatrix}b_{11}&b_{12}\\\overline{b_{12}}&b_{22}\end{bmatrix})$ with $a_{11}\neq 0$ by the action of G_2. As we saw in the proof of Proposition 4.1, there exists $\begin{pmatrix}p&q\\r&s\end{pmatrix}\in\mathrm{GL}_2(\mathbb{F}_q)$ such that p,q,r,s satisfy the equation (5) and (7). Let $g:=(\begin{pmatrix}p&q\\r&s\end{pmatrix},1)\in G$. Then the $(1,1)$-entry of the first matrix of gy is nonzero and the $(1,1)$-entry of the second matrix of gy is zero. Thus we can move x to $y':=(\begin{bmatrix}1&0\\0&1\end{bmatrix},\begin{bmatrix}0&b'_{12}\\\overline{b'_{12}}&b'_{22}\end{bmatrix})$ such that $\det_{y'}(u,v)=g'(u^2+\mu_1 uv-\mathrm{N}_2(\mu_0)v^2)$ for certain $g'\in\mathrm{GL}_1(\mathbb{F}_q)$. On the other hand, $\det_{y'}(u,v)=u^2+b'_{22}uv-\mathrm{N}_2(b'_{12})v^2$. Therefore we have $g'=1$ and $b'_{22}=\mu_1$, $\mathrm{N}_2(b'_{12})=\mathrm{N}_2(\mu_0)$. Since $\mathrm{N}_2(\frac{\mu_0}{b'_{12}})=1$ we obtain $(1,\begin{pmatrix}\frac{\mu_0}{b'_{12}}&0\\0&1\end{pmatrix})y'=(\begin{bmatrix}1&0\\0&1\end{bmatrix},\begin{bmatrix}0&\mu_0\\\overline{\mu_0}&\mu_1\end{bmatrix})$.

Next we consider the cardinality of the each orbit. We use the following facts for the calculation:

$$|\{M\in\mathrm{H}_2(\mathbb{F}_q)\mid \mathrm{rank}(M)=1\}|=(q-1)(q^2+1),$$
$$|\{M\in\mathrm{H}_2(\mathbb{F}_q)\mid \mathrm{rank}(M)=2\}|=(q-1)q(q^2+1).$$

It is clear that $|\mathcal{O}_1|=1$. In addition, we easily see that $|\mathcal{O}_2|=(q+1)\cdot|\{M\in\mathrm{H}_2(\mathbb{F}_q)\mid\mathrm{rank}(M)=1\}|$ and $|\mathcal{O}_3|=(q+1)\cdot|\{M\in\mathrm{H}_2(\mathbb{F}_q)\mid\mathrm{rank}(M)=2\}|$. Next we count $|\mathcal{O}_4|$. Let $x_4=(\begin{bmatrix}0&0\\0&1\end{bmatrix},\begin{bmatrix}0&1\\1&0\end{bmatrix})$, and $\mathrm{Stab}(x_4):=\{g\in G\mid gx_4=x_4\}$. Let $g=(g_1,g_2)=(\begin{pmatrix}p_1&q_1\\r_1&s_1\end{pmatrix},\begin{pmatrix}p_2&q_2\\r_2&s_2\end{pmatrix})\in\mathrm{Stab}(x_4)$. We have $(g_2\begin{bmatrix}0&q_1\\q_1&p_1\end{bmatrix}\overline{g_2^T},g_2\begin{bmatrix}0&s_1\\s_1&r_1\end{bmatrix}\overline{g_2^T})=(\begin{bmatrix}0&0\\0&1\end{bmatrix},\begin{bmatrix}0&1\\1&0\end{bmatrix})$. By comparing the rank of the first entry, we have $q_1 = 0$, and $p_1 s_1 \neq 0$. It follows that $p_1 \begin{bmatrix} \mathrm{N}_2(q_2) & q_2\overline{s_2} \\ s_2\overline{q_2} & \mathrm{N}_2(s_2) \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$, and therefore $q_2 = 0$ and $p_1 \mathrm{N}_2(s_2) = 1$. Thus we have $g_2 \begin{bmatrix} 0 & s_1 \\ s_1 & r_1 \end{bmatrix} \overline{g_2^T} = \begin{bmatrix} 0 & s_1 p_2 \overline{s_2} \\ s_1 s_2 \overline{p_2} & r_1 \mathrm{N}_2(s_2) + s_1 \mathrm{Tr}_2(s_2\overline{r_2}) \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$, and therefore

$$\begin{aligned}\mathrm{Stab}(x_4) &= \left\{ (\begin{pmatrix} (\mathrm{N}_2(s_2))^{-1} & 0 \\ r_1 & s_1 \end{pmatrix}, \begin{pmatrix} (s_1\overline{s_2})^{-1} & 0 \\ r_2 & s_2 \end{pmatrix}) \in G \,\middle|\, r_1 = -\frac{s_1 \mathrm{Tr}_2(s_2\overline{r_2})}{\mathrm{N}_2(s_2)} \right\} \\ &\cong (\mathrm{GL}_1(\mathbb{F}_q) \times \mathrm{GL}_1(\mathbb{F}_{q^2})) \ltimes \mathbb{F}_{q^2}.\end{aligned}$$

Thus we obtain $|\mathrm{Stab}(x_4)| = (q-1)^2 q^2 (q+1)$, and $|\mathcal{O}_4| = |G|/(q-1)^2 q^2 (q+1) = (q-1)^2 q (q+1)^2 (q^2+1)$.

Next we count $|\mathcal{O}_5|$. Let $x_5 := (\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix})$ and $\mathrm{Stab}(x_5) := \{g \in G \mid g x_5 = x_5\}$. Let $g = (g_1, g_2, g_3) = (\begin{pmatrix} p_1 & q_1 \\ r_1 & s_1 \end{pmatrix}, \begin{pmatrix} p_2 & q_2 \\ r_2 & s_2 \end{pmatrix}) \in \mathrm{Stab}(x_5)$. We have

$$(g_2 \begin{bmatrix} p_1 & 0 \\ 0 & q_1 \end{bmatrix} \overline{g_2^T}, g_2 \begin{bmatrix} r_1 & 0 \\ 0 & s_1 \end{bmatrix} \overline{g_2^T}) = (\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}).$$

By comparing the rank of each entry, we obtain the following propositions:

$$\text{If } p_1 \neq 0, \text{ then } q_1 = r_1 = 0 \text{ and } s_1 \neq 0. \tag{12}$$

$$\text{If } p_1 = 0, \text{ then } q_1 r_1 \neq 0 \text{ and } s_1 = 0. \tag{13}$$

First we assume the case (12). Then we have

$$(p_1 \begin{bmatrix} \mathrm{N}_2(p_2) & p_2\overline{r_2} \\ r_2\overline{p_2} & \mathrm{N}_2(r_2) \end{bmatrix}, s_1 \begin{bmatrix} \mathrm{N}_2(q_2) & q_2\overline{s_2} \\ s_2\overline{q_2} & \mathrm{N}_2(s_2) \end{bmatrix}) = (\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}),$$

and therefore

$$q_2 = r_2 = 0,\ p_1 = \mathrm{N}_2(p_2)^{-1}, \text{ and } s_1 = \mathrm{N}_2(s_2)^{-1}. \tag{14}$$

Next we assume the case (13). Then we have

$$(q_1 \begin{bmatrix} \mathrm{N}_2(q_2) & q_2\overline{s_2} \\ s_2\overline{q_2} & \mathrm{N}_2(s_2) \end{bmatrix}, r_1 \begin{bmatrix} \mathrm{N}_2(p_2) & p_2\overline{r_2} \\ r_2\overline{p_2} & \mathrm{N}_2(r_2) \end{bmatrix}) = (\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}),$$

and therefore

$$p_2 = s_2 = 0,\ q_1 = \mathrm{N}_2(q_2)^{-1}, \text{ and } r_1 = \mathrm{N}_2(r_2)^{-1}. \tag{15}$$

By (14) and (15), we obtain

$$\begin{aligned}\mathrm{Stab}(x_5) &= \langle (\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}) \rangle \ltimes \left\{ (\begin{pmatrix} \mathrm{N}_2(p_2)^{-1} & 0 \\ 0 & \mathrm{N}_2(s_2)^{-1} \end{pmatrix}, \begin{pmatrix} p_2 & 0 \\ 0 & s_2 \end{pmatrix}) \in G \right\} \\ &\cong \mathbb{Z}/2\mathbb{Z} \ltimes \mathrm{GL}_1(\mathbb{F}_{q^2})^2.\end{aligned}$$

Thus we obtain $|\mathrm{Stab}(x_5)| = 2(q-1)^2(q+1)^2$, and $|\mathcal{O}_5| = |G|/2(q-1)^2(q+1)^2 = (q-1)^2 q^3 (q+1)(q^2+1)/2$. $|\mathcal{O}_6|$ can be calculated by subtracting $|\mathcal{O}_1|, \cdots, |\mathcal{O}_5|$ from $|V| = q^8$. □

7.2. **The intersection between the orbits and the subspaces.** The subspaces we choose to calculate the Fourier transform are as follows:

$$W_1 = 0,\ W_2 = (\begin{bmatrix} 0 & 0 \\ 0 & * \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & * \end{bmatrix}),\ W_3 = (\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} * & * \\ * & * \end{bmatrix}),\ W_4 = (\begin{bmatrix} 0 & 0 \\ 0 & * \end{bmatrix}, \begin{bmatrix} 0 & * \\ * & * \end{bmatrix}),$$
$$W_5 = (\begin{bmatrix} * & 0 \\ 0 & * \end{bmatrix}, \begin{bmatrix} * & 0 \\ 0 & * \end{bmatrix}) \text{ and } W_6 = V.$$

Orthogonal complements of them are as follows:

$$W_1^\perp = W_6, W_2^\perp = (\begin{bmatrix} 0 & * \\ * & * \end{bmatrix}, \begin{bmatrix} 0 & * \\ * & * \end{bmatrix}), W_3^\perp = W_3, W_4^\perp = W_4, W_5^\perp = (\begin{bmatrix} 0 & * \\ * & 0 \end{bmatrix}, \begin{bmatrix} 0 & * \\ * & 0 \end{bmatrix}) \text{ and } W_6^\perp = W_1.$$

Proposition 7.2. *The cardinalities* $|\mathcal{O}_i \cap W_j|$ *for the orbit* $\mathcal{O}_i$ *and the subspace* W_j *are given as follows:*

	W_1	W_2	W_3	W_4	W_5	W_6	$W_2^\perp$	$W_5^\perp$
$\mathcal{O}_1$	1	1	1	1	1	1	1	1
$\mathcal{O}_2$	0	$[1,0,1,0]$	$[1,0,0,1]$	$[1,0,1,0]$	$2[1,0,1,0]$	$[1,0,1,1]$	$[1,0,1,0]$	0
$\mathcal{O}_3$	0	0	$[1,1,0,1]$	$[1,1,1,0]$	$[2,0,1,0]$	$[1,1,1,1]$	$[1,1,2,0]$	$[1,0,2,0]$
$\mathcal{O}_4$	0	0	0	$[2,1,1,0]$	0	$[2,1,2,1]$	$[2,1,2,0]$	0
$\mathcal{O}_5$	0	0	0	0	$[2,1,1,0]$	$\frac{1}{2}[2,3,1,1]$	0	0
$\mathcal{O}_6$	0	0	0	0	0	$\frac{1}{2}[2,3,1,1]$	$[2,3,1,0]$	$[2,1,1,0]$

Here $[a,b,c,d] = (q-1)^a q^b (q+1)^c (q^2+1)^d$.

[Proof]

We only consider the cases of $W_2^\perp$ and $W_5^\perp$, since the rest cases are easy. We write $x \in W_2^\perp$ as $x = (A,B) = (\begin{bmatrix} 0 & a_{12} \\ \overline{a_{12}} & a_{22} \end{bmatrix}, \begin{bmatrix} 0 & b_{12} \\ \overline{b_{12}} & b_{22} \end{bmatrix})$. We consider the case $(a_{12}, b_{12}) \neq 0$. We have $\det_x(u,v) = \mathrm{N}_2(a_{12})u^2 + (a_{12}\overline{b_{12}} + \overline{a_{12}}b_{12})uv + \mathrm{N}_2(b_{12})v^2$ and the discriminant $\mathrm{Disc}(\det_x(u,v))$ is $(a_{12}\overline{b_{12}} - \overline{a_{12}}b_{12})^2$. Thus we have

$$a_{12} \text{ and } b_{12} \text{ are parallel over } \mathbb{F}_q \text{ if and only if } a_{12}\overline{b_{12}} = \overline{a_{12}}b_{12}, \tag{16}$$

$$a_{12} \text{ and } b_{12} \text{ are not parallel over } \mathbb{F}_q \text{ if and only if } a_{12}\overline{b_{12}} - \overline{a_{12}}b_{12} \notin \mathbb{F}_q. \tag{17}$$

When a_{12} and b_{12} are not parallel over $\mathbb{F}_q$, we have $x \in \mathcal{O}_6$. The number of such elements is $q^2 \mathrm{gl}_2$. When a_{12} and b_{12} are parallel over $\mathbb{F}_q$ and $(a_{12}, b_{12}) \not\parallel (a_{22}, b_{22})$, we have $x \in \mathcal{O}_4$. The number of such elements is $q(q^2-1)(q^2-q)$. When $(a_{22}, b_{22}) \neq 0$ and $A /\!/ B$ over $\mathbb{F}_q$, We have $x \in \mathcal{O}_3$. The number of such elements is $(q^2-1)q(q+1)$. Remaining elements belong to $\mathcal{O}_1$ or $\mathcal{O}_2$. We write $x \in W_5^\perp$ as $x = (\begin{bmatrix} 0 & a_{12} \\ \overline{a_{12}} & 0 \end{bmatrix}, \begin{bmatrix} 0 & b_{12} \\ \overline{b_{12}} & 0 \end{bmatrix})$. We only consider the case $(a_{12}, b_{12}) \neq 0$. By (17), we find that when a_{12} and b_{12} are not parallel over $\mathbb{F}_q$, we have $x \in \mathcal{O}_6$. The number of such elements is gl_2. By (16), when a_{12} and b_{22} are parallel over $\mathbb{F}_q$, we have $x \in \mathcal{O}_3$. The number of such elements is $(q^2-1)(q+1)$. □

7.3. Fourier transform.

Theorem 7.3. *The representation matrix* M *of the Fourier transform on* $\mathcal{F}_V^G$ *with respect to the basis* $e_1, \ldots, e_6$ *is given as follows:*

$$\frac{1}{q^8}\begin{bmatrix} 1 & [1,0,1,1] & [1,1,1,1] & [2,1,2,1] & \frac{1}{2}[2,3,1,1] & \frac{1}{2}[2,3,1,1] \\ 1 & -1 & [1,1,0,0]b_1 & -[1,1,1,0] & -\frac{1}{2}[1,3,0,1] & \frac{1}{2}[2,3,1,0] \\ 1 & [1,0,0,0]b_1 & qc_1 & -[1,1,1,0] & \frac{1}{2}[2,3,0,0] & -\frac{1}{2}[1,3,1,0] \\ 1 & -1 & -q & qc_2 & -\frac{1}{2}[1,3,0,0] & -\frac{1}{2}[1,3,0,0] \\ 1 & -[0,0,0,1] & [1,1,0,0] & -[1,1,1,0] & q^3 & 0 \\ 1 & [1,0,1,0] & -[0,1,1,0] & -[1,1,1,0] & 0 & q^3 \end{bmatrix}.$$

Here $[a,b,c,d] = (q-1)^a q^b (q+1)^c (q^2+1)^d$, $b_1 = q^2+q+1$, $c_1 = q^3-q^2-1$ *and* $c_2 = q^3-q^2+1$.

We used PARI/GP [8] to calculate the matrix from Proposition 7.2.

Corollary 7.4. *The indicator function of singular set of* V *is* $\Psi = \sum_{i=1}^{4} e_i$. *Its Fourier transform* $\widehat{\Psi}$ *is given as follows:*

$$\widehat{\Psi}(x) = \begin{cases} q^{-1} + q^{-4} - q^{-5} & x = 0, \\ q^{-4} - q^{-5} & x \neq 0, \mathrm{Disc}(\det_x(u,v)) = 0, \\ -q^{-5} & \mathrm{Disc}(\det_x(u,v)) \neq 0. \end{cases}$$

In particular, we have the following L_1*-norm bound of* $\widehat{\Psi}$*:*

$$\sum_{x \in V} |\widehat{\Psi}(x)| = O(q^3).$$

8. $2 \otimes \wedge^2(4)$

Let $\wedge^2(\mathbb{F}_q^4)$ be the set of all alternating matrices of order 4 over $\mathbb{F}_q$. We write $A \in \wedge^2(\mathbb{F}_q^4)$ as

$$A = \begin{bmatrix} 0 & a_{12} & a_{13} & a_{14} \\ -a_{12} & 0 & a_{23} & a_{24} \\ -a_{13} & -a_{23} & 0 & a_{34} \\ -a_{14} & -a_{24} & -a_{34} & 0 \end{bmatrix} \text{ where } a_{ij} \in \mathbb{F}_q.$$

The Pfaffian of A is defined by

$$\mathrm{Pfaff}(A) = a_{12}a_{34} - a_{13}a_{24} + a_{14}a_{23}.$$

Let $V = \mathbb{F}_q^2 \otimes \wedge^2(\mathbb{F}_q^4)$ and $G = G_1 \times G_2 = \mathrm{GL}_2 \times \mathrm{GL}_4$. We write $x \in V$ as $x = (A, B)$ where $A, B \in \wedge^2(\mathbb{F}_q^4)$, and write $g \in G$ as $g = (g_1, g_2)$ where $g_1 \in \mathrm{GL}_2$ and $g_2 \in \mathrm{GL}_4$. The action of G on V is defined by

$$gx = (g_2 A g_2^T, g_2 B g_2^T) g_1^T.$$

Define a bilinear form β of V as

$$\beta((A_1, B_1), (A_2, B_2)) = \mathrm{Tr}(A_1 A_2^T + B_1 B_2^T).$$

In addition, define an automorphism ι of G as

$$(g_1, g_2)^\iota = ((g_1^T)^{-1}, (g_2^T)^{-1}).$$

These β and ι satisfy Assumption 2.1.

8.1. **Orbit decomposition.** For $x = (A, B) = (\begin{bmatrix} 0 & a_{12} & a_{13} & a_{14} \\ -a_{12} & 0 & a_{23} & a_{24} \\ -a_{13} & -a_{23} & 0 & a_{34} \\ -a_{14} & -a_{24} & -a_{34} & 0 \end{bmatrix}, \begin{bmatrix} 0 & b_{12} & b_{13} & b_{14} \\ -b_{12} & 0 & b_{23} & b_{24} \\ -b_{13} & -b_{23} & 0 & b_{34} \\ -b_{14} & -b_{24} & -b_{34} & 0 \end{bmatrix}) \in V$, we define

$$r_1(x) := \dim(\langle A, B\rangle_{\mathbb{F}_q}), \text{ i.e., the dimension of the subspace of } \wedge^2(\mathbb{F}_q^4) \text{ generated by } A \text{ and } B,$$

$$r_2(x) := \mathrm{rank}(\begin{bmatrix} 0 & a_{12} & a_{13} & a_{14} & 0 & b_{12} & b_{13} & b_{14} \\ -a_{12} & 0 & a_{23} & a_{24} & -b_{12} & 0 & b_{23} & b_{24} \\ -a_{13} & -a_{23} & 0 & a_{34} & -b_{13} & -b_{23} & 0 & b_{34} \\ -a_{14} & -a_{24} & -a_{34} & 0 & -b_{14} & -b_{24} & -b_{34} & 0 \end{bmatrix}),$$

$$\mathrm{Pf}_x(u, v) := \mathrm{Pfaff}(uA + vB) \in \mathrm{Sym}^2(\mathbb{F}_q^2) \text{ where } u, v \text{ are variables},$$

$$\mathrm{T}(x) := \langle\!\langle \alpha \rangle\!\rangle \text{ if and only if } \mathrm{Pf}_x(u, v) \in O_{\langle\!\langle \alpha \rangle\!\rangle} \text{ in } \mathrm{Sym}^2(\mathbb{F}_q^2).$$

For $x \in V$ and $g = (g_1, g_2) \in G$, we have

$$\mathrm{Pf}_{gx}(u, v) = \det(g_2)\mathrm{Pf}_x((u, v)g_1).$$

Since alternating matrix is determined by its upper triangular part, we write $(\begin{bmatrix} 0 & a_{12} & a_{13} & a_{14} \\ -a_{12} & 0 & a_{23} & a_{24} \\ -a_{13} & -a_{23} & 0 & a_{34} \\ -a_{14} & -a_{24} & -a_{34} & 0 \end{bmatrix}, \begin{bmatrix} 0 & b_{12} & b_{13} & b_{14} \\ -b_{12} & 0 & b_{23} & b_{24} \\ -b_{13} & -b_{23} & 0 & b_{34} \\ -b_{14} & -b_{24} & -b_{34} & 0 \end{bmatrix})$ as $\begin{bmatrix} a_{12} & a_{13} & a_{14} & a_{23} & a_{24} & a_{34} \\ b_{12} & b_{13} & b_{14} & b_{23} & b_{24} & b_{34} \end{bmatrix}$.

Proposition 8.1. *V consists of 7 G-orbits in all.*

Orbit name	Representative	$r_1(x)$	$r_2(x)$	$\mathrm{T}(x)$	Cardinality
$\mathcal{O}_1$	$\begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$	0	0	$\langle\langle 0 \rangle\rangle$	1
$\mathcal{O}_2$	$\begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$	1	2	$\langle\langle 0 \rangle\rangle$	$[1,0,1,1,1]$
$\mathcal{O}_3$	$\begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$	1	4	$\langle\langle 1 \rangle\rangle$	$[2,2,1,1,0]$
$\mathcal{O}_4$	$\begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix}$	2	3	$\langle\langle 0 \rangle\rangle$	$[2,1,2,1,1]$
$\mathcal{O}_5$	$\begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & -1 & 0 & 0 \end{bmatrix}$	2	4	$\langle\langle 1 \rangle\rangle$	$[3,2,2,1,1]$
$\mathcal{O}_6$	$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$	2	4	$\langle\langle 2r \rangle\rangle$	$\frac{1}{2}[2,5,1,1,1]$
$\mathcal{O}_7$	$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & \mu_0 & 0 & \mu_1 \end{bmatrix}$	2	4	$\langle\langle 2i \rangle\rangle$	$\frac{1}{2}[4,5,1,1,0]$

Here $[a,b,c,d,e]=(q-1)^a q^b (q+1)^c (q^2+q+1)^d (q^2+1)^e$, and μ_1, μ_0 are elements of $\in \mathbb{F}_q$ such that $X^2+\mu_1 X+\mu_0 \in \mathbb{F}_q[X]$ is irreducible.

[Proof]

The invariants $r_1(x)$, $r_2(x)$ and $\mathrm{T}(x)$ for the 7 elements in the "Representative" column of the table are easily calculated. Since they do not coincide, these 7 elements belong to different orbits. Let $\mathcal{O}_i$ be the orbit of each element.

First we prove that $V=\bigcup_{i=1}^{7}\mathcal{O}_i$. Let $x\in V$. When $r_1(x)=0$ we easily see that $x\in\mathcal{O}_1$. When $(r_1(x),r_2(x))=(1,2)$, we have $x\sim(0,B)\sim\begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$ by the action of G. It follows that $x\in\mathcal{O}_2$. When $(r_1(x),r_2(x))=(1,4)$, we have $x\sim(0,B)\sim\begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$. It follows that $x\in\mathcal{O}_3$. When $r_1(x)=2$, we have $r_2(x)\geq 3$. If $(r_1(x),r_2(x))=(2,3)$, we have $x\sim \begin{bmatrix} 0 & 0 & 0 & a_{23} & a_{24} & a_{34} \\ 0 & 0 & 0 & b_{23} & b_{24} & b_{34} \end{bmatrix}\sim\begin{bmatrix} 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$. It follows that $x\in\mathcal{O}_4$. When $(r_1(x),r_2(x))=(2,4)$, we have $x\sim\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & a_{34} \\ 0 & b_{13} & b_{14} & b_{23} & b_{24} & b_{34} \end{bmatrix}$. Thus we have $\mathrm{Pf}_x(u,v)\sim a_{34}u^2+b_{34}uv-\begin{vmatrix} b_{13} & b_{14} \\ b_{23} & b_{24} \end{vmatrix}v^2$. If $\mathrm{T}(x)=\langle\langle 0\rangle\rangle$, we have $a_{34}=b_{34}=\begin{vmatrix} b_{13} & b_{14} \\ b_{23} & b_{24} \end{vmatrix}=0$, which contradicts to the assumption $r_2(x)=4$. It follows that $\mathrm{T}(x)=\langle\langle 1\rangle\rangle,\langle\langle 2r\rangle\rangle$ or $\langle\langle 2i\rangle\rangle$. If $\mathrm{T}(x)=\langle\langle 1\rangle\rangle$, we can let $\mathrm{rank}(A)=2$ and therefore $x\sim\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & b_{13} & b_{14} & b_{23} & b_{24} & b_{34} \end{bmatrix}$. Thus we have $\mathrm{Pf}_x(u,v)\sim b_{34}uv-\begin{vmatrix} b_{13} & b_{14} \\ b_{23} & b_{24} \end{vmatrix}v^2$. Since $\mathrm{T}(x)=\langle\langle 1\rangle\rangle$, we have $b_{34}=0$ and $\begin{vmatrix} b_{13} & b_{14} \\ b_{23} & b_{14} \end{vmatrix}\neq 0$. It follows that $x\sim\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & -1 & 0 & 0 \end{bmatrix}$. If $\mathrm{T}(x)=\langle\langle 2r\rangle\rangle$, we have $x\sim\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & b_{13} & b_{14} & b_{23} & b_{24} & b_{34} \end{bmatrix}$ as in the case of $\mathrm{T}(x)=\langle\langle 1\rangle\rangle$. Since in this case we have $b_{34}\neq 0$, we have $x\sim\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$. If $\mathrm{T}(x)=\langle\langle 2i\rangle\rangle$, we have $\mathrm{Pf}_x(u,v)\sim u^2+\mu_1 uv+\mu_0 v^2=(u-\gamma v)(u-\overline{\gamma}v)$ where $\gamma\in\mathbb{F}_{q^2}\setminus\mathbb{F}_q$ by Section 3.1 and the surjectivity of the map $G\ni(g_1,g_2)\mapsto(\det(g_2),g_1^T)\in\mathrm{GL}_1(\mathbb{F}_q)\times\mathrm{GL}_2(\mathbb{F}_q)$. Therefore we assume $\mathrm{Pf}_x(u,v)=u^2+\mu_1 uv+\mu_0 v^2$. We can move x to $y:=\begin{bmatrix} a_{12} & a_{13} & a_{14} & a_{23} & a_{24} & a_{34} \\ b_{12} & b_{13} & b_{14} & b_{23} & b_{24} & b_{34} \end{bmatrix}$ with $a_{12}\neq 0$ by the action of G_2. As we saw in the proof of Proposition 4.1, there exists $\begin{pmatrix} p & q \\ r & s \end{pmatrix}\in\mathrm{GL}_2$ such that p,q,r,s satisfy the equation (7) and

$$a_{12}r+b_{12}s=0. \tag{18}$$

Let $g := (\begin{pmatrix} p & q \\ r & s \end{pmatrix}, 1) \in G$. Then the $(1,2)$-entry of the first matrix of gy is nonzero and the $(1,2)$-entry of the second matrix of gy is zero. Thus we can move x to $y' := \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 1 \\ 0 & b'_{13} & b'_{14} & b'_{23} & b'_{24} & b'_{34} \end{bmatrix}$ such that $\mathrm{Pf}_{y'}(u,v) = g'(u^2 + \mu_1 uv + \mu_0 v^2)$ for certain $g' \in \mathrm{GL}_1$. On the other hand, $\mathrm{Pf}_{y'}(u,v) = u^2 + b'_{34}uv - \begin{vmatrix} b'_{13} & b'_{14} \\ b'_{23} & b'_{24} \end{vmatrix} v^2$. Therefore we have $g' = 1$ and $b'_{34} = \mu_1$, $\begin{vmatrix} b'_{13} & b'_{14} \\ b'_{23} & b'_{24} \end{vmatrix} = -\mu_0$. Thus there exists $h \in \mathrm{SL}_2$ such that $h \begin{bmatrix} b'_{13} & b'_{14} \\ b'_{23} & b'_{24} \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ \mu_0 & 0 \end{bmatrix}$. Thus we obtain $(1, \begin{pmatrix} h & 0 \\ 0 & I_2 \end{pmatrix})y' = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & \mu_0 & 0 & \mu_1 \end{bmatrix}$.

Next we calculate the cardinality of each orbit. We use the following facts:

$$\begin{aligned}
|\mathrm{Sp}_2(\mathbb{F}_q)| &= (q-1)q(q+1), \\
|\mathrm{Sp}_4(\mathbb{F}_q)| &= (q-1)^2 q^4 (q+1)^2 (q^2+1), \\
|\{M \in \wedge^2(\mathbb{F}_q^4) \mid \mathrm{rank} M = 2\}| &= (q-1)(q^2+q+1)(q^2+1), \\
|\{M \in \wedge^2(\mathbb{F}_q^4) \mid \mathrm{rank} M = 4\}| &= (q-1)^2 q^2 (q^2+q+1).
\end{aligned}$$

It is obvious that $|\mathcal{O}_1| = 1$. We easily see that $|\mathcal{O}_2| = (q+1)|\{M \in \wedge^2(\mathbb{F}_q^4) \mid \mathrm{rank} M = 2\}|$ and $|\mathcal{O}_3| = (q+1)|\{M \in \wedge^2(\mathbb{F}_q^4) \mid \mathrm{rank} M = 4\}|$. To count $|\mathcal{O}_4|$, we calculate the order of the stabilizer subgroup $\mathrm{Stab}(x_4)$ of $x_4 := (A,B) = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix}$. In other words, we count the number of

$$g = (g_1, g_2^{-1}) = (\begin{pmatrix} p & r \\ q & s \end{pmatrix}, \begin{pmatrix} g_{11} & g_{12} & g_{13} & g_{14} \\ g_{21} & g_{22} & g_{23} & g_{24} \\ g_{31} & g_{32} & g_{33} & g_{34} \\ g_{41} & g_{42} & g_{43} & g_{44} \end{pmatrix}^{-1}) \in G \text{ such that}$$

$$(pA + qB, rA + sB) = (g_2 A g_2^T, g_2 B g_2^T). \tag{19}$$

By (19), we have

$$\begin{vmatrix} g_{13} & g_{14} \\ g_{23} & g_{24} \end{vmatrix} = 0, \begin{vmatrix} g_{13} & g_{14} \\ g_{33} & g_{34} \end{vmatrix} = 0, \begin{vmatrix} g_{13} & g_{14} \\ g_{43} & g_{44} \end{vmatrix} = 0, \begin{vmatrix} g_{23} & g_{24} \\ g_{33} & g_{34} \end{vmatrix} = 0, \begin{vmatrix} g_{12} & g_{14} \\ g_{22} & g_{24} \end{vmatrix} = 0, \begin{vmatrix} g_{12} & g_{14} \\ g_{32} & g_{34} \end{vmatrix} = 0, \begin{vmatrix} g_{12} & g_{14} \\ g_{42} & g_{44} \end{vmatrix} = 0,$$

$$\begin{vmatrix} g_{22} & g_{24} \\ g_{32} & g_{34} \end{vmatrix} = 0, \begin{vmatrix} g_{23} & g_{24} \\ g_{43} & g_{44} \end{vmatrix} = q, \begin{vmatrix} g_{33} & g_{34} \\ g_{43} & g_{44} \end{vmatrix} = p, \begin{vmatrix} g_{22} & g_{24} \\ g_{42} & g_{44} \end{vmatrix} = s \text{ and } \begin{vmatrix} g_{32} & g_{34} \\ g_{42} & g_{44} \end{vmatrix} = r.$$

If $q \neq 0$, we have $\begin{vmatrix} g_{23} & g_{24} \\ g_{43} & g_{44} \end{vmatrix} \neq 0$. Since $\begin{vmatrix} g_{13} & g_{14} \\ g_{23} & g_{24} \end{vmatrix} = 0$ and $\begin{vmatrix} g_{13} & g_{14} \\ g_{43} & g_{44} \end{vmatrix} = 0$, we obtain $g_{13} = g_{14} = 0$. If $q = 0$, we have $\begin{vmatrix} g_{33} & g_{34} \\ g_{43} & g_{44} \end{vmatrix} \neq 0$. Since $\begin{vmatrix} g_{13} & g_{14} \\ g_{33} & g_{34} \end{vmatrix} = 0$ and $\begin{vmatrix} g_{13} & g_{14} \\ g_{43} & g_{44} \end{vmatrix} = 0$, we also obtain $g_{13} = g_{14} = 0$. Furthermore, we similarly obtain $g_{12} = g_{14} = 0$ by $\begin{vmatrix} g_{12} & g_{14} \\ g_{22} & g_{24} \end{vmatrix} = 0$, $\begin{vmatrix} g_{12} & g_{14} \\ g_{32} & g_{34} \end{vmatrix} = 0$, $\begin{vmatrix} g_{12} & g_{14} \\ g_{42} & g_{44} \end{vmatrix} = 0$, $\begin{vmatrix} g_{22} & g_{24} \\ g_{42} & g_{44} \end{vmatrix} = s$ and $\begin{vmatrix} g_{32} & g_{34} \\ g_{42} & g_{44} \end{vmatrix} = r$. Thus we have $g_{12} = g_{13} = g_{14} = 0$. If $(g_{24}, g_{34}) \neq 0$, then we have $(g_{22}, g_{23}, g_{24}) /\!/ (g_{32}, g_{33}, g_{34})$ by $\begin{vmatrix} g_{23} & g_{24} \\ g_{33} & g_{34} \end{vmatrix} = \begin{vmatrix} g_{22} & g_{24} \\ g_{32} & g_{34} \end{vmatrix} = 0$, which contradicts to $g_2 \in \mathrm{GL}_4$. It also follows that $g_{24}=g_{34}=0$. Hence we have $g_{44}\begin{bmatrix} g_{22} & g_{23} \\ g_{32} & g_{33}\end{bmatrix}=\begin{bmatrix} s & q \\ r & p\end{bmatrix}$. Therefore we obtain

$$\begin{aligned}\mathrm{Stab}(x_4) &= \left\{ (g_{44}\begin{pmatrix} g_{23} & g_{22} \\ g_{33} & g_{33}\end{pmatrix}, \begin{pmatrix} g_{11} & 0 & 0 & 0 \\ g_{21} & g_{22} & g_{23} & 0 \\ g_{31} & g_{32} & g_{33} & 0 \\ g_{41} & g_{42} & g_{43} & g_{44}\end{pmatrix}^{-1}) \in G \right\} \\ &= \left\{ (g_{44}^{-1}\begin{pmatrix} g_{23} & g_{22} \\ g_{33} & g_{33}\end{pmatrix}^{-1}, \begin{pmatrix} g_{11} & 0 & 0 & 0 \\ g_{21} & g_{22} & g_{23} & 0 \\ g_{31} & g_{32} & g_{33} & 0 \\ g_{41} & g_{42} & g_{43} & g_{44}\end{pmatrix}) \in G \right\} \\ &\cong ((\mathrm{GL}_1)^2 \times \mathrm{GL}_2) \ltimes \mathbb{F}_q^5,\end{aligned}$$

and $|\mathrm{Stab}(x_4)|=(q-1)^2q^5\mathrm{gl}_2$. By this result, we can calculate $|\mathcal{O}_4|$. Next we count $|\mathcal{O}_5|$. Let $J=\begin{bmatrix} 0 & 1 \\ -1 & 0\end{bmatrix}$ and $O=\begin{bmatrix} 0 & 0 \\ 0 & 0\end{bmatrix}$. Let $x_5:=(\begin{bmatrix} O & O \\ O & J\end{bmatrix}, \begin{bmatrix} O & J \\ J & O\end{bmatrix})$, and $\mathrm{Stab}(x_5):=\{g\in G \mid gx_5=x_5\}$. Let $g=(g_1,g_2)=(\begin{pmatrix} p & q \\ r & s\end{pmatrix}, \begin{pmatrix} G_{11} & G_{12} \\ G_{21} & G_{22}\end{pmatrix})\in \mathrm{Stab}(x_5)$, where $G_{ij}\in \mathrm{M}_2(\mathbb{F}_q)$. We have $(g_2\begin{bmatrix} O & qJ \\ qJ & pJ\end{bmatrix}g_2^T, g_2\begin{bmatrix} O & sJ \\ sJ & rJ\end{bmatrix}g_2^T)=(\begin{bmatrix} O & O \\ O & J\end{bmatrix}, \begin{bmatrix} O & J \\ J & O\end{bmatrix})$. By comparing the rank of the first entry, we have $q=0$, and $ps\neq 0$. It follows that $p\begin{bmatrix} |G_{12}|J & G_{12}JG_{22}^T \\ G_{22}JG_{12}^T & |G_{22}|J\end{bmatrix}=\begin{bmatrix} O & O \\ O & J\end{bmatrix}$, and therefore $G_{12}=O$ and $|G_{22}|=p^{-1}$. Thus we have

$$g_2\begin{bmatrix} O & qJ \\ qJ & pJ\end{bmatrix}g_2^T=\begin{bmatrix} 0 & sG_{11}JG_{22}^T \\ sG_{22}JG_{11}^T & sG_{22}JG_{21}^T+sG_{21}JG_{22}^T+r|G_{22}|J\end{bmatrix}=\begin{bmatrix} O & J \\ J & O\end{bmatrix},$$

and therefore

$$\begin{aligned}\mathrm{Stab}(x_5) &= \left\{ (\begin{pmatrix} |G_{22}|^{-1} & 0 \\ r & s\end{pmatrix}, \begin{pmatrix} s^{-1}G_{22} & O \\ G_{21} & G_{22}\end{pmatrix}) \in G \;\middle|\; sG_{22}JG_{21}^T+sG_{21}JG_{22}^T+r|G_{22}|J=O \right\} \\ &\cong (\mathrm{GL}_1\times \mathrm{GL}_2)\ltimes \mathrm{M}_2(\mathbb{F}_q).\end{aligned}$$

Thus we obtain $|\mathrm{Stab}(x_5)|=(q-1)q^4\mathrm{gl}_2$, and we can calculate $|\mathcal{O}_5|$. Next we count $|\mathcal{O}_6|$. Let $x_6:=(\begin{bmatrix} J & O \\ O & O\end{bmatrix}, \begin{bmatrix} O & O \\ O & J\end{bmatrix})$, and $\mathrm{Stab}(x_6):=\{g\in G \mid gx_6=x_6\}$. Let $g=(g_1,g_2)=(\begin{pmatrix} p & q \\ r & s\end{pmatrix}, \begin{pmatrix} G_{11} & G_{12} \\ G_{21} & G_{22}\end{pmatrix})\in \mathrm{Stab}(x_6)$, where $G_{ij}\in \mathrm{M}_2(\mathbb{F}_q)$. We have

$$(g_2\begin{bmatrix} pJ & O \\ O & qJ\end{bmatrix}\overline{g_2^T}, g_2\begin{bmatrix} rJ & O \\ O & sJ\end{bmatrix}\overline{g_2^T})=(\begin{bmatrix} J & O \\ O & O\end{bmatrix}, \begin{bmatrix} O & O \\ O & J\end{bmatrix}).$$

By comparing the rank of each entry, we obtain the following propositions:

$$\text{If } p\neq 0, \text{ then } q=r=0 \text{ and } s\neq 0. \tag{20}$$

$$\text{If } p=0, \text{ then } qr\neq 0 \text{ and } s=0. \tag{21}$$

First we assume the case (20). Then we have

$$(p\begin{bmatrix} |G_{11}|J & G_{11}JG_{21}^T \\ G_{21}JG_{11}^T & |G_{21}|J\end{bmatrix}, s\begin{bmatrix} |G_{12}|J & G_{12}JG_{22}^T \\ G_{22}JG_{12}^T & |G_{22}|J\end{bmatrix})=(\begin{bmatrix} J & O \\ O & O\end{bmatrix}, \begin{bmatrix} O & O \\ O & J\end{bmatrix}),$$

and therefore

$$G_{12}=G_{21}=O,\ p=|G_{11}|^{-1}, \text{ and } s=|G_{22}|^{-1}. \tag{22}$$

Next we assume the case (21). Then we have

$$(q\begin{bmatrix} |G_{12}|J & G_{12}JG_{22}^T \\ G_{22}JG_{12}^T & |G_{22}|J\end{bmatrix}, r\begin{bmatrix} |G_{11}|J & G_{11}JG_{21}^T \\ G_{21}JG_{11}^T & |G_{21}|J\end{bmatrix})=(\begin{bmatrix} J & O \\ O & O\end{bmatrix}, \begin{bmatrix} O & O \\ O & J\end{bmatrix}),$$

and therefore

$$G_{11}=G_{22}=O,\ q=|G_{12}|^{-1}, \text{ and } r=|G_{21}|^{-1}. \tag{23}$$

By (22) and (23), we obtain

$$\begin{aligned}\mathrm{Stab}(x_6) &= \langle (\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}, \begin{pmatrix}O & I_2\\ I_2 & O\end{pmatrix})\rangle \ltimes \left\{ (\begin{pmatrix}|G_{11}|^{-1} & 0\\ 0 & |G_{22}^{-1}|\end{pmatrix}, \begin{pmatrix}G_{11} & 0\\ 0 & G_{22}\end{pmatrix}) \in G \right\}\\ &\cong \mathbb{Z}/2\mathbb{Z} \ltimes (\mathrm{GL}_2)^2.\end{aligned}$$

Thus we obtain $|\mathrm{Stab}(x_6)| = 2\mathrm{gl}_2^2$ and we can calculate $|\mathcal{O}_6|$. Lastly we obtain $|\mathcal{O}_7|$ by subtracting the sum of $|\mathcal{O}_1|, ..., |\mathcal{O}_6|$ from $|V| = q^{12}$. $\square$

8.2. **The intersection between the orbits and the subspaces.** The subspaces we choose to calculate the Fourier transform are as follows:

$$W_1 = 0, W_2 = \begin{bmatrix} * & 0 & 0 & 0 & 0 & * \\ * & 0 & 0 & 0 & 0 & * \end{bmatrix}, W_3 = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ * & * & * & * & * & * \end{bmatrix}, W_4 = \begin{bmatrix} * & * & 0 & * & 0 & 0 \\ * & * & 0 & * & 0 & 0 \end{bmatrix},$$

$$W_5 = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & * \\ 0 & * & * & * & * & * \end{bmatrix}, W_6 = \begin{bmatrix} 0 & * & * & * & * & 0 \\ 0 & * & * & * & * & 0 \end{bmatrix} \text{ and } W_7 = V.$$

Orthogonal complements of them are as follows:

$$W_1^\perp = W_7, W_2^\perp = W_6, W_3^\perp = W_3, W_4^\perp = \begin{bmatrix} 0 & 0 & * & 0 & * & * \\ 0 & 0 & * & 0 & * & * \end{bmatrix}, W_5^\perp = W_5, W_6^\perp = W_2 \text{ and } W_7^\perp = W_1.$$

Proposition 8.2. *The cardinalities $|\mathcal{O}_i \cap W_j|$ for the orbit $\mathcal{O}_i$ and the subspace W_j are given as follows:*

	W_1	W_2	W_3	W_4	W_5	W_6	W_7	$W_4^\perp$
$\mathcal{O}_1$	1	1	1	1	1	1	1	1
$\mathcal{O}_2$	0	$2[1,0,1,0,0]$	$[1,0,0,1,1]$	$[1,0,1,1,0]$	$[1,0,1,1,0]$	$[1,0,3,0,0]$	$[1,0,1,1,1]$	$[1,0,1,1,0]$
$\mathcal{O}_3$	0	$[2,0,1,0,0]$	$[2,2,0,1]$	0	$[2,2,1,0,0]$	$[2,1,2,0,0]$	$[2,2,1,1,0]$	0
$\mathcal{O}_4$	0	0	0	$[2,1,1,1,0]$	$[2,1,2,0,0]$	$2[2,1,2,0,0]$	$[2,1,2,1,1]$	$[2,1,1,1,0]$
$\mathcal{O}_5$	0	0	0	0	$[3,2,1,0,0]$	$[3,1,3,0,0]$	$[3,2,2,1,1]$	0
$\mathcal{O}_6$	0	$[2,1,1,0,0]$	0	0	0	$\frac{1}{2}[2,3,3,0,0]$	$\frac{1}{2}[2,5,1,1,1]$	0
$\mathcal{O}_7$	0	0	0	0	0	$\frac{1}{2}[4,3,1,0,0]$	$\frac{1}{2}[4,5,1,1,0]$	0

Here $[a,b,c,d,e] = (q-1)^a q^b (q+1)^c (q^2+q+1)^d (q^2+1)^e$.

[Proof]

We only consider the cases of W_4, W_5 and W_6, since the rest cases are easy. First we calculate for W_4. Restrict the representation of G on V to the subgroup $H := \left\{ (g_1, \begin{pmatrix} g_2 & 0 \\ 0 & 1 \end{pmatrix}) \in G \,\middle|\, g_2 \in \mathrm{GL}_3 \right\}$. Then H acts on W_4. We can choose three elements 0, $x = (A, 0) = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$ and $y = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \end{bmatrix}$ as complete representatives of W with this action of H. We count the cardinalities of these orbits. For x, we calculate the order of the stabilizer subgroup $\mathrm{Stab}(x)$ of x in H. We have

$$\mathrm{Stab}(x) = \left\{ (g_1, g_2^{-1}) = (\begin{pmatrix} p & q \\ r & s \end{pmatrix}, \begin{pmatrix} G_{11} & G_{12} \\ G_{21} & G_{22} \end{pmatrix}^{-1}) \in H \,\middle|\, \begin{array}{l} G_{ij} \in \mathrm{M}_2(\mathbb{F}_q), \\ (g_2 A g_2^T, 0) = (pA, rA) \end{array} \right\}.$$

We see $r = 0$, $G_{21} = O$ and $|G_{11}| = p$ by calculation. Therefore $\mathrm{Stab}(x) \cong ((\mathrm{GL}_1)^2 \times \mathrm{GL}_2) \ltimes \mathbb{F}_q^3$, and its order is $(q-1)^2 q^3 \mathrm{gl}_2$ and the cardinality is $|H|/(q-1)^2 q^3 \mathrm{gl}_2 = (q-1)(q+1)(q^2+q+1)$. It follows that $|Hy| = q^6 - |Hx| - 1 = (q-1)^2 q(q+1)(q^2+q+1)$. In view of $0 \in \mathcal{O}_1$, $x \in \mathcal{O}_2$ and $y \in \mathcal{O}_4$, we obtain $|W_4 \cap \mathcal{O}_2| = |Hx|$ and $|W_4 \cap \mathcal{O}_4| = |Hy|$. Next we consider W_5. We write $x \in W_5$ as $x = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & a_{34} \\ 0 & b_{13} & b_{14} & b_{23} & b_{24} & b_{34} \end{bmatrix}$. When $a_{34} = 0$ and $\begin{bmatrix} b_{13} & b_{14} \\ b_{23} & b_{24} \end{bmatrix} = 0$ and $b_{34} \neq 0$, we have $x \in \mathcal{O}_2$. The number of such elements is $q-1$. When $a_{34} = 0$ and $\mathrm{rank}(\begin{bmatrix} b_{13} & b_{14} \\ b_{23} & b_{24} \end{bmatrix}) = 1$, we have $x \in \mathcal{O}_2$. The number of such elements is $q|2,1|$. When $a_{34} = 0$ and $\mathrm{rank}(\begin{bmatrix} b_{13} & b_{14} \\ b_{23} & b_{24} \end{bmatrix}) = 2$, we have $x \in \mathcal{O}_3$. The number of such elements is $q\mathrm{gl}_2$. When $a_{34} \neq 0$ and $\begin{bmatrix} b_{13} & b_{14} \\ b_{23} & b_{24} \end{bmatrix} = 0$, we have $x \in \mathcal{O}_2$. The number of such elements is $(q-1)q$. When $a_{34}\neq 0$ and $\operatorname{rank}(\begin{bmatrix} b_{13} & b_{14} \\ b_{23} & b_{24}\end{bmatrix})=1$, we have $x\in\mathcal{O}_4$. The number of such elements is $(q-1)q|2,1|$. When $a_{34}\neq 0$ and $\operatorname{rank}(\begin{bmatrix} b_{13} & b_{14} \\ b_{23} & b_{24}\end{bmatrix})=2$, we have $x\in\mathcal{O}_5$. The number of such elements is $(q-1)q\mathrm{gl}_2$. Lastly we consider W_6. Restrict the representation of G on V to the subgroup $H=\left\{(g_1,\begin{pmatrix} g_2 & 0 \\ 0 & h_2\end{pmatrix})\in G \,\middle|\, g_2,h_2\in \mathrm{GL}_2\right\}$. Then H acts on W_6. Identify $(\begin{bmatrix} 0 & a_{13} & a_{14} & a_{23} & a_{24} & 0 \\ 0 & b_{13} & b_{14} & b_{23} & b_{24} & 0\end{bmatrix})\in W_6$ as the pair of the 2-by-2 matrices $(\begin{bmatrix} a_{13} & a_{14} \\ a_{23} & a_{24}\end{bmatrix},\begin{bmatrix} b_{13} & b_{14} \\ b_{23} & b_{24}\end{bmatrix})$. The action of H on W_6 is identical to the action of $\mathrm{GL}_2\times\mathrm{GL}_2\times\mathrm{GL}_2$ on $2\otimes 2\otimes 2$ which we considered in Section 4. By this identification, $2\otimes 2\otimes 2$ can be embedded in V. Let σ be this embedding. We easily see that $\sigma(\mathcal{O}_1)\subset\mathcal{O}_1$, $\sigma(\mathcal{O}_2)\subset\mathcal{O}_2$, $\sigma(\mathcal{O}_3)\subset\mathcal{O}_3$, $\sigma(\mathcal{O}_4)\subset\mathcal{O}_4$, $\sigma(\mathcal{O}_5)\subset\mathcal{O}_4$, $\sigma(\mathcal{O}_6)\subset\mathcal{O}_5$, $\sigma(\mathcal{O}_7)\subset\mathcal{O}_6$ and $\sigma(\mathcal{O}_8)\subset\mathcal{O}_7$. The cardinalities can be calculated by these results. □

8.3. **Fourier transform.**

Theorem 8.3. *The representation matrix M of the Fourier transform on $\mathcal{F}_V^G$ with respect to the basis $e_1,\dots,e_7$ is given as follows:*

$$\frac{1}{q^{12}}\begin{bmatrix}
1 & [1,0,1,1,1] & [2,2,1,1,0] & [2,1,2,1,1] & [3,2,2,1,1] & \frac{1}{2}[2,5,1,1,1] & \frac{1}{2}[4,5,1,1,0] \\
1 & e_1 & [1,2,0,0,0]c_1 & [1,1,2,0,0]c_2 & -[2,2,1,1,0] & \frac{1}{2}[1,5,0,0,0]c_3 & -\frac{1}{2}[3,5,1,0,0] \\
1 & [0,0,0,0,1]c_1 & q^2d_1 & -[1,1,2,0,1] & [1,2,1,0,1] & -\frac{1}{2}[1,5,0,0,1] & \frac{1}{2}[2,5,1,0,0] \\
1 & [0,0,1,0,0]c_2 & -[1,2,1,0,0] & qe_2 & [1,2,1,0,0] & -\frac{1}{2}[1,5,1,0,0] & \frac{1}{2}[2,5,0,0,0] \\
1 & -[0,0,0,1,0] & q^2 & [0,1,1,0,0] & q^2d_2 & -\frac{1}{2}[1,5,0,0,0] & -\frac{1}{2}[1,5,0,0,0] \\
1 & c_3 & -[1,2,0,0,0] & -[1,1,2,0,0] & -[2,2,1,0,0] & q^5 & 0 \\
1 & -[0,0,1,0,1] & [0,2,1,0,0] & [0,1,1,0,1] & -[0,2,1,0,1] & 0 & q^5
\end{bmatrix}.$$

Here $[a,b,c,d,e]=(q-1)^aq^b(q+1)^c(q^2+q+1)^d(q^2+1)^e$, $c_1=q^3-q-1$, $c_2=q^3-q^2-1$, $c_3=q^3-q^2-q-1$, $d_1=q^4-q^3+1$, $e_1=q^5+q^4-q^2-q-1$ and $e_2=q^5-q^4-q^3+q+1$.

We used PARI/GP [8] to calculate the matrix from Proposition 8.2.

Corollary 8.4. *The indicator function of singular set of V is $\Psi=\sum_{i=1}^{5}e_i$. Its Fourier transform $\widehat{\Psi}$ is given as follows:*

$$\widehat{\Psi}(x)=\begin{cases} q^{-1}+q^{-6}-q^{-7} & x=0, \\ q^{-6}-q^{-7} & x\neq 0, \mathrm{Disc}(\mathrm{Pf}_x(u,v))=0, \\ -q^{-7} & \mathrm{Disc}(\mathrm{Pf}_x(u,v))\neq 0. \end{cases}$$

In particular, we have the following L_1-norm bound of $\widehat{\Psi}$:

$$\sum_{x\in V}|\widehat{\Psi}(x)|=O(q^5).$$

9. The space of binary tri-Hermitian forms

Fix a non-identity element σ in $\mathrm{Gal}(\mathbb{F}_{q^3}/\mathbb{F}_q)$. For $z\in\mathbb{F}_{q^3}$, we write $\sigma(z)$ and $\sigma^2(z)$ as z' and z'', respectively. Define the trace map and the norm map as follows:

$$\mathrm{Tr}_3:\mathbb{F}_{q^3}\ni z\mapsto z+z'+z''\in\mathbb{F}_q,$$

$$\mathrm{N}_3:\mathbb{F}_{q^3}\ni z\mapsto zz'z''\in\mathbb{F}_q.$$

Both maps are surjective. Tr_3 is a $\mathbb{F}_q$- linear map. $\mathrm{N}_3|_{\mathbb{F}_{q^3}^\times}:\mathbb{F}_{q^3}^\times\to\mathbb{F}_q^\times$ is a surjective group homomorphism.

Let

$$V:=\left\{x=(A,B)=(\begin{bmatrix} a_1 & a_2'' \\ a_2' & a_3\end{bmatrix},\begin{bmatrix} a_2 & a_3' \\ a_3'' & a_4\end{bmatrix})\,\middle|\, a_1,a_4\in\mathbb{F}_q \text{ and } a_2,a_3\in\mathbb{F}_{q^3}\right\}.$$

V is an 8 dimensional vector space over $\mathbb{F}_q$. Let $G = G_1 \times G_2 = \mathrm{GL}_1(\mathbb{F}_q) \times \mathrm{GL}_2(\mathbb{F}_{q^3})$. We write $g \in G$ as $g = (g_1, g_2)$ where $g_1 \in \mathrm{GL}_1(\mathbb{F}_q)$ and $g_2 \in \mathrm{GL}_2(\mathbb{F}_{q^3})$. For a matrix $h = (h_{ij}) \in \mathrm{GL}_2(\mathbb{F}_{q^3})$, we define $h' = (h'_{ij})$ and $h'' = (h''_{ij})$. Then the action of G on V is defined by

$$gx = g_1(g_2 A(g_2^T)', g_2 B(g_2^T)')(g_2^T)''.$$

Define a bilinear form β of V as

$$\beta((A_1, B_1), (A_2, B_2)) = \mathrm{Tr}(A_1 A_2^T + B_1 B_2^T).$$

In addition, define an automorphism ι of G as

$$(g_1, g_2)^\iota = ((g_1)^{-1}, (g_2^T)^{-1}).$$

These β and ι satisfy Assumption 2.1.

9.1. **Orbit decomposition.** By substituting $\mathbb{F}_{q^3}$ for $\mathbb{F}_q$ in Section 3.1, we obtain the following orbit decomposition of $(\mathrm{GL}_1(\mathbb{F}_{q^3}) \times \mathrm{GL}_2(\mathbb{F}_{q^3}), \mathrm{Sym}^2(\mathbb{F}_{q^3}^2))$:

Orbit name	Representative	rank
$O_{\langle\langle 0\rangle\rangle}$	0	0
$O_{\langle\langle 1\rangle\rangle}$	u^2	1
$O_{\langle\langle 2r\rangle\rangle}$	uv	2
$O_{\langle\langle 2i\rangle\rangle}$	$u^2 + \mu_1 uv + \mu_0 v^2$	2

Here, $u^2 + \mu_1 uv + \mu_0 v^2 \in \mathrm{Sym}^2(\mathbb{F}_{q^3}^2)$ is an arbitrary irreducible polynomial. For $x = (A, B) \in V$, we define

$$r_1(x) := \dim(\langle A, B\rangle_{\mathbb{F}_{q^3}}),\ \text{i.e., the dimension of the subspace of } \mathrm{M}_2(\mathbb{F}_{q^3}) \text{ generated by } A \text{ and } B,$$

$$\det{}_x(u, v) := \det(uA + vB) \in \mathrm{Sym}^2(\mathbb{F}_{q^3}^2) \text{ where } u, v \text{ are variables},$$

$$\mathrm{T}(x) := \langle\langle \alpha\rangle\rangle \text{ if and only if } \det{}_x(u, v) \in O_{\langle\langle \alpha\rangle\rangle} \text{ in } \mathrm{Sym}^2(\mathbb{F}_{q^3}^2).$$

For $x \in V$ and $g = (g_1, g_2) \in G$, we have

$$\det{}_{gx}(u, v) = g_1^2 \det(g_2 g_2') \det{}_x((u, v) g_2'').$$

Proposition 9.1. *V consists of 5 G-orbits in all.*

Orbit name	Representative	$r_1(x)$	$\mathrm{T}(x)$	Cardinality
$\mathcal{O}_1$	$(\begin{bmatrix}0&0\\0&0\end{bmatrix}, \begin{bmatrix}0&0\\0&0\end{bmatrix})$	0	$\langle\langle 0\rangle\rangle$	1
$\mathcal{O}_2$	$(\begin{bmatrix}0&0\\0&0\end{bmatrix}, \begin{bmatrix}0&0\\0&1\end{bmatrix})$	1	$\langle\langle 0\rangle\rangle$	$[1,0,1,0,1]$
$\mathcal{O}_3$	$(\begin{bmatrix}0&0\\0&1\end{bmatrix}, \begin{bmatrix}0&1\\1&0\end{bmatrix})$	2	$\langle\langle 1\rangle\rangle$	$[1,1,1,1,1]$
$\mathcal{O}_4$	$(\begin{bmatrix}1&0\\0&0\end{bmatrix}, \begin{bmatrix}0&0\\0&1\end{bmatrix})$	2	$\langle\langle 2r\rangle\rangle$	$\frac{1}{2}[2,3,1,0,1]$
$\mathcal{O}_5$	$(\begin{bmatrix}2&\mu_1\\\mu_1&\mu_1^2-2\mu_0\end{bmatrix}, \begin{bmatrix}\mu_1&\mu_1^2-2\mu_0\\\mu_1^2-2\mu_0&\mu_1^3-3\mu_1\mu_0\end{bmatrix})$	2	$\langle\langle 2i\rangle\rangle$	$\frac{1}{2}[2,3,1,1,0]$

Here $[a, b, c, d, e] = (q-1)^a q^b (q+1)^c (q^2+q+1)^d (q^2-q+1)^e$, and $\mu_1, \mu_0 \in \mathbb{F}_q$ are elements such that $X^2 + \mu_1 X + \mu_0 \in \mathbb{F}_q[X]$ is irreducible.

[Proof]

The invariants $r_1(x)$ and $\mathrm{T}(x)$ for the 5 elements in the "Representative" column of the table are easily calculated. Since they do not coincide, these 5 elements belong to different orbits. Let

$$x_1 = (\begin{bmatrix}0&0\\0&0\end{bmatrix}, \begin{bmatrix}0&0\\0&0\end{bmatrix}), x_2 = (\begin{bmatrix}0&0\\0&0\end{bmatrix}, \begin{bmatrix}0&0\\0&1\end{bmatrix}), x_3 = (\begin{bmatrix}1&0\\0&0\end{bmatrix}, \begin{bmatrix}0&1\\1&0\end{bmatrix}), x_4 = (\begin{bmatrix}1&0\\0&0\end{bmatrix}, \begin{bmatrix}0&0\\0&1\end{bmatrix}),$$

$$x_5 = (\begin{bmatrix}2&\mu_1\\\mu_1&\mu_1^2-2\mu_0\end{bmatrix}, \begin{bmatrix}\mu_1&\mu_1^2-2\mu_0\\\mu_1^2-2\mu_0&\mu_1^3-3\mu_1\mu_0\end{bmatrix}), \text{ and } \mathcal{O}_i = Gx_i.$$

For $x \in V$, let $\mathrm{Stab}(x) = \{g \in G \mid gx = x\}$.

In Section 9, we substitute $\mathbb{F}_{q^3}$ for $\mathbb{F}_q$ in the notation of Section 3.3, i.e.,

$$|(n_1,n_2),m| := |\{M \in \mathrm{M}(n_1,n_2)(\mathbb{F}_{q^3}) | \mathrm{rank}(M)=m\}| = \frac{\prod_{i=0}^{m-1}(q^{3(n_2-i)}-1)\prod_{j=0}^{m-1}(q^{3n_1}-q^{3j})}{\prod_{k=1}^{m}(q^{3k}-1)},$$

$$|n,m| := |(n,n),m| = \prod_{i=0}^{m-1}\frac{(q^{3(n-i)}-1)(q^{3n}-q^{3i})}{q^{3(m-i)}-1},$$

$$\mathrm{gl}_n := |\mathrm{GL}_n(\mathbb{F}_{q^3})| = \prod_{i=0}^{n-1}(q^{3n}-q^{3i}),$$

$$\mathrm{sl}_n := |\mathrm{SL}_n(\mathbb{F}_{q^3})| = \mathrm{gl}_n/(q^3-1).$$

We count $|\mathcal{O}_i|$. Clearly $|\mathcal{O}_1|=1$. To calculate $|\mathcal{O}_2|$, we count the order of $\mathrm{Stab}(x_2)$. Let $g=(g_1,g_2)=(g_1,\begin{pmatrix} p_2 & q_2 \\ r_2 & s_2\end{pmatrix}) \in \mathrm{Stab}(x_2)$. Then

$$(\begin{bmatrix} \mathrm{N}_3(q_2) & q_2q_2''s_2' \\ q_2'q_2''s_2 & q_2''s_2s_2' \end{bmatrix}, \begin{bmatrix} q_2q_2's_2'' & q_2s_2's_2'' \\ q_2's_2s_2'' & \mathrm{N}_3(s_2) \end{bmatrix}) = (\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & g_1^{-1} \end{bmatrix})$$

holds. Thus we have $q_2=0$ and $\mathrm{N}_3(s_2)=g_1^{-1}$. By the surjectivity of N_3, we obtain $\mathrm{Stab}(x_2) \cong (\mathrm{GL}_1(\mathbb{F}_{q^3}))^2 \ltimes \mathbb{F}_{q^3}$, and $|\mathrm{Stab}(x_2)| = q^3(q^3-1)^2$. Therefore $|\mathcal{O}_2| = |G|/|\mathrm{Stab}(x_2)| = (q-1)\mathrm{gl}_2/q^3(q^3-1)^2 = (q-1)(q+1)(q^2-q+1)$. Next we count $|\mathcal{O}_3|$. If $g \in \mathrm{Stab}(x_3)$,

$$(g_2\begin{bmatrix} 0 & q_2'' \\ q_2'' & p_2'' \end{bmatrix} g_2', g_2\begin{bmatrix} 0 & s_2'' \\ s_2'' & r_2'' \end{bmatrix} g_2') = (\begin{bmatrix} 0 & 0 \\ 0 & g_1^{-1} \end{bmatrix}, \begin{bmatrix} 0 & g_1^{-1} \\ g_1^{-1} & 0 \end{bmatrix})$$

holds. By comparing the rank of the first entry, we have $q_2=0$. Therefore we have $p_2''s_2s_2' = g_1^{-1}$ and $\mathrm{Tr}_3(r_2s_2's_2'')=0$. It follows that

$$\mathrm{Stab}(x_3) = \left\{ (g_1, \begin{pmatrix} (g_1s_2's_2'')^{-1} & 0 \\ r_2 & s_2 \end{pmatrix}) \in G \,\middle|\, \mathrm{Tr}_3(r_2s_2's_2'')=0 \right\} \cong (\mathrm{GL}_1(\mathbb{F}_q)\times \mathrm{GL}_1(\mathbb{F}_{q^3})) \ltimes \mathrm{Ker}(\mathrm{Tr}_3),$$

and $|\mathrm{Stab}(x_3)| = q^2(q-1)(q^3-1)$. Therefore we obtain $|\mathcal{O}_3| = q(q^6-1)$. Next we calculate $|\mathcal{O}_4|$ and $|\mathcal{O}_5|$. Kable and Yukie [2, Proposition (3,9), (3.12), Theorem (3.13)] proved the following facts:

$$\mathrm{Stab}(x_4) \cong \mathbb{Z}/2\mathbb{Z} \times \{(g_1,g_2) \in \mathrm{GL}_1(\mathbb{F}_{q^3}) \times \mathrm{GL}_1(\mathbb{F}_{q^3}) | \mathrm{N}_3(g_1)=\mathrm{N}_3(g_2)\}, \tag{24}$$

$$\mathrm{Stab}(x_5) \cong \mathbb{Z}/2\mathbb{Z} \times \{g \in \mathrm{GL}_1(\mathbb{F}_{q^6}) | g\delta^2(g)\delta^4(g) \in \mathbb{F}_q^\times\}. \tag{25}$$

Here δ is an element of $\mathrm{Gal}(\mathbb{F}_{q^6}/\mathbb{F}_q)$ such that $\langle\delta\rangle = \mathrm{Gal}(\mathbb{F}_{q^6}/\mathbb{F}_q)$. In [2], Kable and Yukie assume that V is defined over infinite field, but the method to determine the structures for $\mathrm{Stab}(x_4)$ and $\mathrm{Stab}(x_5)$ holds for the $\mathbb{F}_q$. By applying (24) and (25), we obtain $|\mathrm{Stab}(x_4)| = 2(q-1)(q^2+q+1)^2$ and $|\mathrm{Stab}(x_5)| = 2(q^3-1)(q^2-q+1)$. Thus we have $|\mathcal{O}_4| = \frac{1}{2}q^3(q-1)^2(q+1)(q^2-q+1)$ and $|\mathcal{O}_5| = \frac{1}{2}q^3(q-1)^2(q+1)(q^2+q+1)$. Lastly, since $\sum_{i=1}^5 |\mathcal{O}_i| = q^8 = |V|$, we have $\bigcup_{i=1}^5 \mathcal{O}_i = V$. □

9.2. **The intersection between the orbits and the subspaces.** The subspaces we choose to calculate the Fourier transform are as follows:

$$W_1=0,\ W_2=(\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & * \end{bmatrix}),\ W_3=(\begin{bmatrix} 0 & 0 \\ 0 & * \end{bmatrix}, \begin{bmatrix} 0 & * \\ * & * \end{bmatrix}),\ W_4=(\begin{bmatrix} * & 0 \\ 0 & * \end{bmatrix}, \begin{bmatrix} 0 & * \\ * & 0 \end{bmatrix}) \text{ and } W_5=V.$$

Orthogonal complements of them are as follows:

$$W_1^\perp = W_5, W_2^\perp = (\begin{bmatrix} 0 & * \\ * & * \end{bmatrix}, \begin{bmatrix} * & * \\ * & * \end{bmatrix}), W_3^\perp = W_3, W_4^\perp = W_4 \text{ and } W_5^\perp = W_1.$$

Proposition 9.2. *The cardinalities $|\mathcal{O}_i \cap W_j|$ for the orbit $\mathcal{O}_i$ and the subspace W_j are given as follows:*

	W_1	W_2	W_3	W_4	W_5	$W_2^\perp$
$\mathcal{O}_1$	1	1	1	1	1	1
$\mathcal{O}_2$	0	$[1,0,0,0,0]$	$[1,0,0,0,0]$	$[1,0,0,0,0]$	$[1,0,1,0,1]$	$[1,0,0,0,0]$
$\mathcal{O}_3$	0	0	$[1,1,0,1,0]$	$[1,0,0,1,0]$	$[1,1,1,1,1]$	$[1,1,0,1,0]b_2$
$\mathcal{O}_4$	0	0	0	$\frac{1}{2}[2,0,0,1,0]$	$\frac{1}{2}[2,3,1,0,1]$	$\frac{1}{2}[2,3,0,1,0]$
$\mathcal{O}_5$	0	0	0	$\frac{1}{2}[2,0,0,1,0]$	$\frac{1}{2}[2,3,1,1,0]$	$\frac{1}{2}[2,3,0,1,0]$

Here $[a,b,c,d,e] = (q-1)^a q^b (q+1)^c (q^2+q+1)^d (q^2-q+1)^e$ and $b_2 = q^2+1$.

[Proof]

We only consider the case of $W_2^{\perp}$, since the rest cases are easy. We easily see that $|\mathcal{O}_1 \cap W_2^{\perp}| = 1$ and $|\mathcal{O}_2 \cap W_2^{\perp}| = (q-1)$. For $1 \le i \le 5$ and $W \subset V$, let $G(i, W) = \{g \in G \mid gx_i \in W\}$. Then we have $|\mathcal{O}_i \cap W| = |G(i, W)|/|\mathrm{Stab}(x_i)|$. Let $g = (g_1, \begin{pmatrix} p_2 & q_2 \\ r_2 & s_2 \end{pmatrix})$ and assume $g \in G(3, W_2^{\perp})$. Then we have

$$\begin{bmatrix} p_2 & q_2 \end{bmatrix} \begin{bmatrix} 0 & q_2'' \\ q_2'' & p_2'' \end{bmatrix} \begin{bmatrix} p_2' \\ q_2' \end{bmatrix} = \mathrm{Tr}_3(p_2 q_2' q_2'') = 0.$$

Therefore

$$\begin{aligned} |G(3, W_2^{\perp})| &= |\{g \in G(3, W_2^{\perp}) \mid q_2 = 0\}| + |\{g \in G(3, W_2^{\perp}) \mid q_2 \neq 0\}| \\ &= q^3(q-1)\mathrm{gl}_1^2 + q^2(q-1)(q^6 - q^3)\mathrm{gl}_1 \\ &= q^3(q-1)^3(q^2+q+1)^2(q^2+1), \end{aligned}$$

and we obtain $|\mathcal{O}_3 \cap W_2^{\perp}| = q(q-1)(q^2+q+1)(q^2+1)$. Next, assume $g \in G(4, W_2^{\perp})$. Then we have $\mathrm{N}_3(p_2) + \mathrm{N}_3(q_2) = 0$. If $p_2 = 0$, we have $q_2 = 0$, which contradicts to $\begin{pmatrix} p_2 & q_2 \\ r_2 & s_2 \end{pmatrix} \in \mathrm{GL}_2(\mathbb{F}_{q^3})$. Thus we have $p_2 \neq 0$, and $G(4, W_2^{\perp}) = (q-1)\mathrm{gl}_1^2(q^6 - q^3)/(q-1)$. Therefore we obtain $|\mathcal{O}_4 \cap W_2^{\perp}| = \frac{1}{2}q^3(q-1)^2(q^2+q+1)$. Lastly, $|\mathcal{O}_5 \cap W_2^{\perp}| = q^7 - \sum_{i=1}^4 |\mathcal{O}_i \cap W_2^{\perp}| = \frac{1}{2}q^3(q-1)^2(q^2+q+1)$. □

9.3. **Fourier transform.**

Theorem 9.3. *The representation matrix M of the Fourier transform on $\mathcal{F}_V^G$ with respect to the basis $e_1, ..., e_5$ is given as follows:*

$$\frac{1}{q^8}\begin{bmatrix} 1 & [1,0,1,0,1] & [1,1,1,1,1] & \frac{1}{2}[2,3,1,0,1] & \frac{1}{2}[2,3,1,1,0] \\ 1 & -c_1 & [1,1,0,1,0] & \frac{1}{2}[1,3,0,0,0]b_1 & -\frac{1}{2}[1,3,0,1,0] \\ 1 & [1,0,0,0,0] & qc_2 & -\frac{1}{2}[1,3,0,0,0] & -\frac{1}{2}[1,3,0,0,0] \\ 1 & b_1 & -[0,1,0,1,0] & q^3 & 0 \\ 1 & -[0,0,0,0,1] & -[0,1,0,0,1] & 0 & q^3 \end{bmatrix}.$$

Here $[a,b,c,d,e] = (q-1)^a q^b (q+1)^c (q^2+q+1)^d (q^2-q+1)^e$, $b_1 = q^2+q-1$, $c_1 = q^3-q+1$ and $c_2 = q^3 - q^2 - 1$.

We used PARI/GP [8] to calculate the matrix from Proposition 9.2.

Corollary 9.4. *The indicator function of singular set of V is $\Psi = \sum_{i=1}^{3} e_i$. Its Fourier transform $\widehat{\Psi}$ is given as follows:*

$$\widehat{\Psi}(x) = \begin{cases} q^{-1} + q^{-4} - q^{-5} & x = 0, \\ q^{-4} - q^{-5} & x \neq 0, \mathrm{Disc}(\det_x(u,v)) = 0, \\ -q^{-5} & \mathrm{Disc}(\det_x(u,v)) \neq 0. \end{cases}$$

In particular, we have the following L_1-norm bound of $\widehat{\Psi}$:

$$\sum_{x \in V} |\widehat{\Psi}(x)| = O(q^3).$$

CHAPTER 2

Cubic cases

10. $\mathbb{F}_q^2 \otimes \mathbb{F}_q^3 \otimes \mathbb{F}_q^3$

Let $V = \mathbb{F}_q^2 \otimes \mathbb{F}_q^3 \otimes \mathbb{F}_q^3$ and $G = G_1 \times G_2 \times G_3 = \mathrm{GL}_2 \times \mathrm{GL}_3 \times \mathrm{GL}_3$. We write $x \in V$ as $x = (A, B)$ where A and B are 3-by-3 matrices, and write $g \in G$ as $g = (g_1, g_2, g_3)$ where $g_1 \in \mathrm{GL}_2$ and $g_2, g_3 \in \mathrm{GL}_3$. G acts on V by

$$gx = (g_2 A g_3^T, g_2 B g_3^T) g_1^T.$$

Define a bilinear form β of V as

$$\beta((A_1, B_1), (A_2, B_2)) = \mathrm{Tr}(A_1 A_2^T + B_1 B_2^T).$$

In addition, define an automorphism ι of G as

$$(g_1, g_2, g_3)^\iota = ((g_1^T)^{-1}, (g_2^T)^{-1}, (g_3^T)^{-1}).$$

By an easy computation, we see that these β and ι satisfy Assumption 2.1.

10.1. **Orbit decomposition.** For $x = (A, B) = (\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}, \begin{bmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \\ b_{31} & b_{32} & b_{33} \end{bmatrix})$, we define

$$\begin{aligned}
r_1(x) &:= \dim(\langle A, B \rangle_{\mathbb{F}_q}),\ \text{i.e., the dimension of the subspace of } \mathrm{M}_3(\mathbb{F}_q) \text{ generated by } A \text{ and } B, \\
r_2(x) &:= \mathrm{rank}(\begin{bmatrix} a_{11} & a_{12} & a_{13} & b_{11} & b_{12} & b_{13} \\ a_{21} & a_{22} & a_{23} & b_{21} & b_{22} & b_{23} \\ a_{31} & a_{32} & a_{33} & b_{31} & b_{32} & b_{33} \end{bmatrix}), \\
r_3(x) &:= \mathrm{rank}(\begin{bmatrix} a_{11} & a_{21} & a_{31} & b_{11} & b_{21} & b_{31} \\ a_{12} & a_{22} & a_{32} & b_{12} & b_{22} & b_{32} \\ a_{13} & a_{23} & a_{33} & b_{13} & b_{23} & b_{33} \end{bmatrix}), \\
\mathrm{mi}(x) &:= \min\{\mathrm{rank}(rA + sB) | (r, s) \in \mathbb{F}_q^2 \backslash \{(0,0)\}\}, \\
\mathrm{ma}(x) &:= \max\{\mathrm{rank}(rA + sB) | (r, s) \in \mathbb{F}_q^2 \backslash \{(0,0)\}\}.
\end{aligned}$$

$r_1(x)$, $r_2(x)$, $r_3(x)$, $\mathrm{mi}(x)$ and $\mathrm{ma}(x)$ are invariants of the orbits. We also define

$$\mathrm{det}_x(u, v) := \det(uA + vB) \in \mathrm{Sym}^3(\mathbb{F}_q^2) \text{ where } u, v \text{ are variables,}$$
$$\mathrm{T}(x) := \langle\!\langle \alpha \rangle\!\rangle \text{ if and only if } \mathrm{det}_x(u, v) \in O_{\langle\!\langle \alpha \rangle\!\rangle} \text{ in } \mathrm{Sym}^3(\mathbb{F}_q^2).$$

Note that we introduced the representation $(\mathrm{GL}_1(\mathbb{F}_q) \times \mathrm{GL}_2(\mathbb{F}_q), \mathrm{Sym}^3(\mathbb{F}_q^2))$ in Section 3.2. For $x \in V$ and $g = (g_1, g_2, g_3) \in G$, we have

$$\mathrm{det}_{gx}(u, v) = \det(g_2 g_3) \mathrm{det}_x((u, v) g_1).$$

Therefore $\mathrm{T}(x)$ is also an invariant of the orbits.

Proposition 10.1. *V consists of 21 G-orbits in all.*

Orbit name	Representative	$r_1(x)$	$r_2(x)$	$r_3(x)$	$\mathrm{T}(x)$	$\mathrm{mi}(x)$	$\mathrm{ma}(x)$	Cardinality
$\mathcal{O}_1$	$(\begin{bmatrix}0&0&0\\0&0&0\\0&0&0\end{bmatrix},\begin{bmatrix}0&0&0\\0&0&0\\0&0&0\end{bmatrix})$	0	0	0	$\langle\langle 0\rangle\rangle$	0	0	1
$\mathcal{O}_2$	$(\begin{bmatrix}0&0&0\\0&0&0\\0&0&0\end{bmatrix},\begin{bmatrix}0&0&0\\0&0&0\\0&0&1\end{bmatrix})$	1	1	1	$\langle\langle 0\rangle\rangle$	0	1	[1012]
$\mathcal{O}_3$	$(\begin{bmatrix}0&0&0\\0&0&0\\0&0&0\end{bmatrix},\begin{bmatrix}0&0&0\\0&1&0\\0&0&1\end{bmatrix})$	1	2	2	$\langle\langle 0\rangle\rangle$	0	2	[2122]
$\mathcal{O}_4$	$(\begin{bmatrix}0&0&0\\0&0&0\\0&0&0\end{bmatrix},\begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix})$	1	3	3	$\langle\langle 1^3\rangle\rangle$	0	3	[3321]
$\mathcal{O}_5$	$(\begin{bmatrix}0&0&0\\0&0&0\\0&1&0\end{bmatrix},\begin{bmatrix}0&0&0\\0&0&0\\0&0&1\end{bmatrix})$	2	1	2	$\langle\langle 0\rangle\rangle$	1	1	[2112]
$\mathcal{O}_6$	$(\begin{bmatrix}0&0&0\\0&0&1\\0&0&0\end{bmatrix},\begin{bmatrix}0&0&0\\0&0&0\\0&0&1\end{bmatrix})$	2	2	1	$\langle\langle 0\rangle\rangle$	1	1	[2112]
$\mathcal{O}_7$	$(\begin{bmatrix}0&0&0\\0&0&1\\0&1&0\end{bmatrix},\begin{bmatrix}0&0&0\\0&0&0\\0&0&1\end{bmatrix})$	2	2	2	$\langle\langle 0\rangle\rangle$	1	2	[3132]
$\mathcal{O}_8$	$(\begin{bmatrix}0&0&0\\0&1&0\\0&0&0\end{bmatrix},\begin{bmatrix}0&0&0\\0&0&0\\0&0&1\end{bmatrix})$	2	2	2	$\langle\langle 0\rangle\rangle$	1	2	$\frac{1}{2}$[2332]
$\mathcal{O}_9$	$(\begin{bmatrix}0&0&0\\0&1&0\\0&0&1\end{bmatrix},\begin{bmatrix}0&0&0\\0&0&-1\\0&\mu_0&\mu_1\end{bmatrix})$	2	2	2	$\langle\langle 0\rangle\rangle$	2	2	$\frac{1}{2}$[4312]
$\mathcal{O}_{10}$	$(\begin{bmatrix}0&0&0\\1&0&0\\0&1&0\end{bmatrix},\begin{bmatrix}0&0&0\\0&0&0\\0&0&1\end{bmatrix})$	2	2	3	$\langle\langle 0\rangle\rangle$	1	2	[3332]
$\mathcal{O}_{11}$	$(\begin{bmatrix}0&0&0\\0&0&1\\1&0&0\end{bmatrix},\begin{bmatrix}0&0&0\\0&1&0\\0&0&1\end{bmatrix})$	2	2	3	$\langle\langle 0\rangle\rangle$	2	2	[4422]
$\mathcal{O}_{12}$	$(\begin{bmatrix}0&1&0\\0&0&1\\0&0&0\end{bmatrix},\begin{bmatrix}0&0&0\\0&0&0\\0&0&1\end{bmatrix})$	2	3	2	$\langle\langle 0\rangle\rangle$	1	2	[3332]
$\mathcal{O}_{13}$	$(\begin{bmatrix}0&1&0\\0&0&1\\0&0&0\end{bmatrix},\begin{bmatrix}0&0&1\\0&0&0\\0&1&0\end{bmatrix})$	2	3	2	$\langle\langle 0\rangle\rangle$	2	2	[4422]
$\mathcal{O}_{14}$	$(\begin{bmatrix}0&0&1\\0&0&0\\1&0&0\end{bmatrix},\begin{bmatrix}0&0&0\\0&0&1\\0&1&0\end{bmatrix})$	2	3	3	$\langle\langle 0\rangle\rangle$	2	2	[4422]

Orbit name	Representative	$r_1(x)$	$r_2(x)$	$r_3(x)$	T(x)	mi(x)	ma(x)	Cardinality
$\mathcal{O}_{15}$	$(\begin{bmatrix}0&0&0\\0&0&0\\0&0&1\end{bmatrix},\begin{bmatrix}0&0&1\\0&1&0\\1&0&0\end{bmatrix})$	2	3	3	$\langle\!\langle 1^3\rangle\!\rangle$	1	3	[4332]
$\mathcal{O}_{16}$	$(\begin{bmatrix}0&0&0\\0&0&1\\0&1&0\end{bmatrix},\begin{bmatrix}0&0&1\\0&1&0\\1&0&0\end{bmatrix})$	2	3	3	$\langle\!\langle 1^3\rangle\!\rangle$	2	3	[5432]
$\mathcal{O}_{17}$	$(\begin{bmatrix}1&0&0\\0&1&0\\0&0&0\end{bmatrix},\begin{bmatrix}0&0&0\\0&0&0\\0&0&1\end{bmatrix})$	2	3	3	$\langle\!\langle 1^21\rangle\!\rangle$	1	3	[3622]
$\mathcal{O}_{18}$	$(\begin{bmatrix}1&0&0\\0&1&0\\0&0&0\end{bmatrix},\begin{bmatrix}0&0&0\\0&0&1\\0&1&0\end{bmatrix})$	2	3	3	$\langle\!\langle 1^21\rangle\!\rangle$	2	3	[4632]
$\mathcal{O}_{19}$	$(\begin{bmatrix}1&0&0\\0&1&0\\0&0&0\end{bmatrix},\begin{bmatrix}0&0&0\\0&1&0\\0&0&1\end{bmatrix})$	2	3	3	$\langle\!\langle 111\rangle\!\rangle$	2	3	$\frac{1}{6}$[4732]
$\mathcal{O}_{20}$	$(\begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix},\begin{bmatrix}0&0&0\\0&0&-1\\0&\mu_0&\mu_1\end{bmatrix})$	2	3	3	$\langle\!\langle 12\rangle\!\rangle$	2	3	$\frac{1}{2}$[5722]
$\mathcal{O}_{21}$	$(\begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix},\begin{bmatrix}\nu_2&0&-1\\-1&0&0\\\nu_1&\nu_0&0\end{bmatrix})$	2	3	3	$\langle\!\langle 3\rangle\!\rangle$	3	3	$\frac{1}{3}$[6731]

Here, we put $[abcd] = (q-1)^a q^b (q+1)^c (q^2+q+1)^d$ *and* $\mu_1,\ \mu_0,\ \nu_2,\ \nu_1,\ \nu_0 \in \mathbb{F}_q$ *are elements such that* $X^2+\mu_1 X+\mu_0, X^3+\nu_2X^2+\nu_1X+\nu_0 \in \mathbb{F}_q[X]$ *are irreducible.*

[Proof] We count the cardinalities of the orbits of the 21 elements in the "Representative" column of the table. We refer to these elements as $x_1, ..., x_{21}$ in order from the top, and let $\mathcal{O}_i$ be the orbit of x_i. First, we count the cardinalities for the cases of $r_2(x) \le 2$, by using the result of $(G_1, V_1) = (\mathrm{GL}_2\times\mathrm{GL}_2\times\mathrm{GL}_3, 2\otimes 2\otimes 3)$ (see Proposition 5.1). We regard $x \in V_1$ as a pair (A, B) of 2-by-3 matrices A and B. We identify V_1 as the subspace of V by the embedding

$$V_1 \ni (A,B) \mapsto (\begin{bmatrix}O_{1,3}\\A\end{bmatrix}, \begin{bmatrix}O_{1,3}\\B\end{bmatrix}) \in V,$$

and identify G_1 as the subgroup of G by the embedding

$$G_1 \ni (g_1,g_2,g_3) \mapsto (g_1, \begin{pmatrix}1 & \\ & g_2\end{pmatrix}, g_3) \in G.$$

We consider the induced map $G_1 \setminus V_1 \to G \setminus V$. This map is injective. For $x \in V_1$, we have

$$|Gx| = \frac{|G|\mathrm{gl}_{2-r_2(x)}}{q^{r_2(x)}|G_1|\mathrm{gl}_{3-r_2(x)}}|G_1x|.$$

Therefore we obtain $|\mathcal{O}_i|$ for $1 \le i \le 11$, $i \ne 4$. Next we count the cardinalities for the cases of $(r_1(x), r_2(x), r_3(x)) = (2,3,2)$. We identify V_1 as the subspace of V by the embedding

$$V_1 \ni (A,B) \mapsto (\begin{bmatrix}O_{3,1} & A^T\end{bmatrix}, \begin{bmatrix}O_{3,1} & B^T\end{bmatrix}) \in V,$$

and identify G_1 as the subgroup of G by the embedding

$$G_1 \ni (g_1,g_2,g_3) \mapsto (g_1, g_2, \begin{pmatrix}1 & \\ & g_3\end{pmatrix}) \in G.$$

We consider the induced injective map $G_1 \setminus V_1 \to G \setminus V$. For $x \in V_1$, we have

$$|Gx| = \frac{|G|}{q^2|G_1|\mathrm{gl}_1}|G_1x|.$$

Therefore we obtain $|\mathcal{O}_i|$ for $i = 12, 13$. Next we calculate the rest cardinalities. Let $\mathrm{Stab}(x_i)$ be the group of stabilizers of x_i;

$$\mathrm{Stab}(x_i) := \{g \in G \mid gx_i = x_i\}\ (1 \le i \le 21).$$

Since $|\mathcal{O}_i| = |G|/|\mathrm{Stab}(x_i)|$, it is enough to count $|\mathrm{Stab}(x_i)|$. The structure and the order of $\mathrm{Stab}(x_i)$ for $i = 4, 14 \le i \le 21$ is summarized as follows:

x_i	$\mathrm{Stab}(x_i) \cong$	$\lvert\mathrm{Stab}(x_i)\rvert$
x_4	$((\mathrm{GL}_1)^2 \times \mathrm{GL}_3) \ltimes \mathbb{F}_q$	$q(q-1)^2 \cdot \mathrm{gl}_3$
x_{14}	$((\mathrm{GL}_2) \times (\mathrm{GL}_1)^2) \ltimes \mathbb{F}_q^2$	$q^2(q-1)^2 \cdot \mathrm{gl}_2$
x_{15}	$(\mathrm{GL}_1)^4 \ltimes \mathbb{F}_q^4$	$q^4(q-1)^4$
x_{16}	$(\mathrm{GL}_1)^3 \ltimes \mathbb{F}_q^3$	$q^3(q-1)^3$
x_{17}	$(\mathrm{GL}_1)^3 \times \mathrm{GL}_2$	$(q-1)^3 \cdot \mathrm{gl}_2$
x_{18}	$(\mathrm{GL}_1)^4 \ltimes \mathbb{F}_q$	$q(q-1)^4$
x_{19}	$\mathfrak{S}_3 \ltimes ((\mathrm{GL}_1)^4)$	$6(q-1)^4$
x_{20}	$\mathbb{Z}/2\mathbb{Z} \ltimes ((\mathrm{GL}_1)^2 \times \mathrm{GL}_1(\mathbb{F}_{q^2}))$	$2(q-1)^3(q+1)$
x_{21}	$\mathbb{Z}/3\mathbb{Z} \ltimes (\mathrm{GL}_1 \times \mathrm{GL}_1(\mathbb{F}_{q^3}))$	$3(q-1)^2(q^2+q+1)$

First we consider $\mathrm{Stab}(x_4)$. Let $g = ((g_{1ij})_{1\le i,j\le 2}, g_2, g_3) \in \mathrm{Stab}(x_4)$. We have $g_{112} = 0$, and $g_{122}g_2g_3^T = I_3$. Therefore $\mathrm{Stab}(x_4) = \left\{ (\begin{pmatrix} g_{111} & 0 \\ g_{121} & g_{122} \end{pmatrix}, g_2, (g_{122}g_2^T)^{-1}) \in G \right\} \cong ((\mathrm{GL}_1)^2 \times \mathrm{GL}_3) \ltimes \mathbb{F}_q$. Next we consider $\mathrm{Stab}(x_{14})$. Let $g = (g_1, g_2, g_3) = ((g_{1ij})_{1\le i,j\le 2}, (g_{2ij})_{1\le i,j\le 3}, (g_{3ij})_{1\le i,j\le 3}) \in \mathrm{Stab}(x_{14})$. We have $g_{213} = g_{223} = g_{313} = g_{323} = 0$, $g_{333}\begin{pmatrix} g_{211} & g_{212} \\ g_{221} & g_{222} \end{pmatrix} = g_{233}\begin{pmatrix} g_{311} & g_{312} \\ g_{321} & g_{322} \end{pmatrix} = (g_1^T)^{-1}$ and $g_{231}g_{333} + g_{233}g_{331} = g_{232}g_{333} + g_{233}g_{332} = 0$. Therefore $\mathrm{Stab}(x_{14}) \cong ((\mathrm{GL}_2) \times (\mathrm{GL}_1)^2) \ltimes \mathbb{F}_q^2$. Next we consider $\mathrm{Stab}(x_{15})$. Let $g = (g_1^{-1}, g_2, g_3) = ((g_{1ij})^{-1}_{1\le i,j\le 2}, (g_{2ij})_{1\le i,j\le 3}, (g_{3ij})_{1\le i,j\le 3}) \in \mathrm{Stab}(x_{15})$. We have

$$\begin{cases} g_{112} = g_{212} = g_{213} = g_{223} = g_{312} = g_{313} = g_{323} = 0, \\ g_{233}h_{333} = g_{111} \neq 0, \\ g_{211}g_{333} = g_{222}g_{322} = g_{233}g_{311} = g_{122} \neq 0, \\ g_{221}g_{333} + g_{222}g_{332} = g_{232}g_{322} + g_{233}g_{321} = 0, \\ g_{231}g_{333} + g_{232}g_{332} + g_{233}g_{331} = g_{121}. \end{cases}$$

Therefore $\mathrm{Stab}(x_{15}) \cong (\mathrm{GL}_1)^4 \ltimes \mathbb{F}_q^4$. Next we consider $\mathrm{Stab}(x_{16})$. Let $((g_{1ij})_{1\le i,j\le 2}, (g_{2ij})_{1\le i,j\le 3}, (g_{3ij})_{1\le i,j\le 3}) \in \mathrm{Stab}(x_{16})$. We have

$$\begin{cases} g_{112} = g_{212} = g_{213} = g_{223} = g_{312} = g_{313} = g_{323} = 0, \\ g_{222}h_{333} = g_{233}g_{322} = g_{111} \neq 0, \\ g_{232}g_{333} + g_{233}g_{332} = 0, \\ g_{211}g_{333} = g_{222}g_{322} = g_{233}g_{311} = g_{122} \neq 0, \\ g_{221}g_{333} + g_{222}g_{332} = g_{232}g_{322} + g_{233}g_{321} = g_{121}, \\ g_{231}g_{333} + g_{232}g_{332} + g_{233}g_{331} = 0. \end{cases}$$

Therefore $\mathrm{Stab}(x_{16}) \cong (\mathrm{GL}_1)^3 \ltimes \mathbb{F}_q^3$. Next we consider $\mathrm{Stab}(x_{17})$. Let $(g_1, g_2, g_3) \in \mathrm{Stab}(x_{17})$. By comparing the rank of two entries, we see that g_1 must be diagonal. Now it is easy to see that $\mathrm{Stab}(x_{17}) \cong (\mathrm{GL}_1)^3 \times \mathrm{GL}_2$. Next we consider $\mathrm{Stab}(x_{18})$. Let $g = (g_1, g_2, g_3) = ((g_{1ij})_{1\le i,j\le 2}, (g_{2ij})_{1\le i,j\le 3}, (g_{3ij})_{1\le i,j\le 3}) \in \mathrm{Stab}(x_{18})$. By comparing the rank of two entries, we see that g_1 is diagonal or anti-diagonal. If g_1 is anti-diagonal, then $g_{121} = 0$. It contradicts to the assumption. Thus we see that g_1 is diagonal. Furthermore, we have $g_{212} = g_{213} = g_{221} = g_{231} = g_{332} = g_{312} = g_{313} = g_{321} = g_{331} = g_{332} = 0$, $g_{211}g_{311} = g_{222}g_{322} = g_{111}$, $g_{222}g_{333} = g_{233}g_{322} = g_{122}$ and $g_{223}g_{322} + g_{222}g_{323} = 0$. Therefore $\mathrm{Stab}(x_{18}) \cong (\mathrm{GL}_1)^4 \ltimes \mathbb{F}_q$. For $i = 19, 20, 21$, the structure of $\mathrm{Stab}(x_i)$ is determined by Wright and Yukie [9, Proposition 3.2, 3.7]. (It is assumed that V is defined over an infinite field in [9], but the method to determine the structures of $\mathrm{Stab}(x_{19})$, $\mathrm{Stab}(x_{20})$, and $\mathrm{Stab}(x_{21})$ holds for the $\mathbb{F}_q$.)

Lastly, since $\sum_{i=1}^{21} |\mathcal{O}_i| = q^{18} = |V|$, we have $\bigcup_{i=1}^{21} \mathcal{O}_i = V$. □

10.2. **The intersection between the orbits and the subspaces.** The subspaces we choose to calculate the Fourier transform are as follows:

$$W_1 = (\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}), W_2 = (\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ * & * & * \end{bmatrix}), W_3 = (\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 \\ * & * & * \\ * & * & * \end{bmatrix}),$$

$$W_4 = (\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} * & * & * \\ * & * & * \\ * & * & * \end{bmatrix}), W_5 = (\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ * & * & * \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ * & * & * \end{bmatrix}), W_6 = (\begin{bmatrix} 0 & 0 & * \\ 0 & 0 & * \\ 0 & 0 & * \end{bmatrix}, \begin{bmatrix} 0 & 0 & * \\ 0 & 0 & * \\ 0 & 0 & * \end{bmatrix}),$$

$$W_7 = (\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & * \end{bmatrix}, \begin{bmatrix} 0 & 0 & * \\ 0 & 0 & * \\ * & * & * \end{bmatrix}), W_8 = (\begin{bmatrix} 0 & 0 & 0 \\ * & * & * \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ * & * & * \end{bmatrix}), W_9 = (\begin{bmatrix} 0 & 0 & 0 \\ 0 & * & * \\ 0 & * & * \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 \\ 0 & * & * \\ 0 & * & * \end{bmatrix}),$$

$$W_{10} = (\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ * & * & * \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 \\ * & * & * \\ * & * & * \end{bmatrix}), W_{11} = (\begin{bmatrix} 0 & 0 & 0 \\ * & * & * \\ * & * & * \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 \\ * & * & * \\ * & * & * \end{bmatrix}), W_{12} = (\begin{bmatrix} 0 & 0 & * \\ 0 & 0 & * \\ 0 & 0 & * \end{bmatrix}, \begin{bmatrix} 0 & * & * \\ 0 & * & * \\ 0 & * & * \end{bmatrix}),$$

$$W_{13} = (\begin{bmatrix} 0 & * & * \\ 0 & * & * \\ 0 & * & * \end{bmatrix}, \begin{bmatrix} 0 & * & * \\ 0 & * & * \\ 0 & * & * \end{bmatrix}), W_{14} = (\begin{bmatrix} 0 & 0 & * \\ 0 & 0 & * \\ * & * & * \end{bmatrix}, \begin{bmatrix} 0 & 0 & * \\ 0 & 0 & * \\ * & * & * \end{bmatrix}), W_{15} = (\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & * \\ 0 & 0 & * \end{bmatrix}, \begin{bmatrix} 0 & 0 & * \\ * & * & * \\ * & * & * \end{bmatrix}),$$

$$W_{16} = (\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & * \\ 0 & * & * \end{bmatrix}, \begin{bmatrix} 0 & 0 & * \\ 0 & * & * \\ * & * & * \end{bmatrix}), W_{17} = (\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ * & * & * \end{bmatrix}, \begin{bmatrix} * & * & * \\ * & * & * \\ * & * & * \end{bmatrix}), W_{18} = (\begin{bmatrix} * & * & * \\ * & * & * \\ * & * & 0 \end{bmatrix}, \begin{bmatrix} * & * & 0 \\ * & * & 0 \\ 0 & 0 & 0 \end{bmatrix}),$$

$$W_{19} = (\begin{bmatrix} 0 & 0 & 0 \\ * & * & * \\ * & * & * \end{bmatrix}, \begin{bmatrix} * & * & * \\ 0 & 0 & 0 \\ * & * & * \end{bmatrix}), W_{20} = (\begin{bmatrix} 0 & 0 & 0 \\ * & * & * \\ * & * & * \end{bmatrix}, \begin{bmatrix} * & * & * \\ * & * & * \\ * & * & * \end{bmatrix}), W_{21} = (\begin{bmatrix} * & * & * \\ * & * & * \\ * & * & * \end{bmatrix}, \begin{bmatrix} * & * & * \\ * & * & * \\ * & * & * \end{bmatrix}).$$

Here, the notations mean, for example,

$$(\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & * \end{bmatrix}, \begin{bmatrix} 0 & 0 & * \\ 0 & 0 & * \\ * & * & * \end{bmatrix}) = \left\{ (\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & a_{33} \end{bmatrix}, \begin{bmatrix} 0 & 0 & b_{13} \\ 0 & 0 & b_{23} \\ b_{31} & b_{32} & b_{33} \end{bmatrix}) \in V \;\middle|\; a_{33}, b_{13}, b_{23}, b_{31}, b_{32}, b_{33} \in \mathbb{F}_q \right\}.$$

The orthogonal complements of these subspaces are as follows: $W_1^\perp = W_{21}$, $W_2^\perp = W_{20}$, $W_3^\perp = W_{17}$, $W_4^\perp = W_4$, $W_5^\perp = W_{11}$, $W_6^\perp = W_{13}$, $W_7^\perp = W_{18}$, $W_8^\perp = W_{19}$, $W_9^\perp = W_{14}$, $W_{10}^\perp = W_{10}$, $W_{12}^\perp = W_{12}$, $W_{15}^\perp = (\begin{bmatrix} * & * & * \\ * & * & 0 \\ * & * & 0 \end{bmatrix}, \begin{bmatrix} * & * & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix})$, $W_{16}^\perp = W_{16}$. (See Remark 2.4 for the convention for some of these equalities).

Proposition 10.2. *The cardinalities $|\mathcal{O}_i \cap W_j|$ for the orbit $\mathcal{O}_i$ and the subspace W_j are given as follows:*

	W_1	W_2	W_3	W_4	W_5	W_6	W_7	W_8	W_9	W_{10}	W_{11}	W_{12}
$\mathcal{O}_1$	1	1	1	1	1	1	1	1	1	1	1	1
$\mathcal{O}_2$	0	[1001]	[1011]	[1002]	[1011]	[1011]	$[1010]a_1$	2[1001]	[1030]	$[1001]a_1$	[1021]	$[1001]a_1$
$\mathcal{O}_3$	0	0	[2111]	[2112]	0	0	[2120]	0	[2120]	[2111]	[2121]	[2111]
$\mathcal{O}_4$	0	0	0	[3311]	0	0	0	0	0	0	0	0
$\mathcal{O}_5$	0	0	0	0	[2111]	0	[2110]	0	[2120]	[2111]	[2121]	[2101]
$\mathcal{O}_6$	0	0	0	0	0	[2111]	[2110]	[2001]	[2120]	[2101]	[2111]	[2111]
$\mathcal{O}_7$	0	0	0	0	0	0	[3120]	0	[3130]	[3111]	[3131]	[3111]
$\mathcal{O}_8$	0	0	0	0	0	0	0	[2111]	$\frac{1}{2}[2330]$	[2311]	$\frac{1}{2}[2331]$	[2311]
$\mathcal{O}_9$	0	0	0	0	0	0	0	0	$\frac{1}{2}[4310]$	0	$\frac{1}{2}[4311]$	0
$\mathcal{O}_{10}$	0	0	0	0	0	0	0	0	0	[3311]	[3331]	0
$\mathcal{O}_{11}$	0	0	0	0	0	0	0	0	0	0	[4421]	0
$\mathcal{O}_{12}$	0	0	0	0	0	0	0	0	0	0	0	[3311]
$\mathcal{O}_{13}$	0	0	0	0	0	0	0	0	0	0	0	0
$\mathcal{O}_{14}$	0	0	0	0	0	0	0	0	0	0	0	0
$\mathcal{O}_{15}$	0	0	0	0	0	0	0	0	0	0	0	0
$\mathcal{O}_{16}$	0	0	0	0	0	0	0	0	0	0	0	0
$\mathcal{O}_{17}$	0	0	0	0	0	0	0	0	0	0	0	0
$\mathcal{O}_{18}$	0	0	0	0	0	0	0	0	0	0	0	0
$\mathcal{O}_{19}$	0	0	0	0	0	0	0	0	0	0	0	0
$\mathcal{O}_{20}$	0	0	0	0	0	0	0	0	0	0	0	0
$\mathcal{O}_{21}$	0	0	0	0	0	0	0	0	0	0	0	0

W_{13}	W_{14}	W_{15}	W_{16}	W_{17}	W_{18}	W_{19}	W_{20}	W_{21}	$W_{15}^{\perp}$
1	1	1	1	1	1	1	1	1	1
[1021]	$[1010]b_1$	$[1000]c_1$	$[1000]b_2$	[1021]	$[1010]b_5$	$[1001]a_2$	$[1001]b_1$	[1012]	$[1000]c_1$
[2121]	[2130]	$[2110]b_1$	$[2110]a_2$	[2112]	[2140]	2[2111]	[2131]	[2122]	$[2110]b_1$
0	0	[3310]	[3300]	[3311]	[3320]	0	[3311]	[3321]	[3310]
[2111]	[2111]	[2120]	$[2100]b_3$	[2111]	[2130]	[2111]	[2121]	[2112]	[2120]
[2121]	[2111]	[2120]	$[2100]b_3$	[2111]	[2130]	$[2101]a_3$	[2121]	[2112]	[2120]
[3131]	[3130]	[3130]	$[3110]a_4$	[3121]	$[3120]b_3$	2[3111]	$[3121]a_1$	[3132]	[3130]
$\frac{1}{2}[2331]$	[2330]	[2320]	$\frac{1}{2}[2300]b_4$	[2321]	$\frac{1}{2}[2320]a_5$	3[2311]	$\frac{1}{2}[2321]a_2$	$\frac{1}{2}[2332]$	[2320]
$\frac{1}{2}[4311]$	0	0	$\frac{1}{2}[4300]$	0	$\frac{1}{2}[4310]$	0	$\frac{1}{2}[4311]$	$\frac{1}{2}[4312]$	0
0	[3330]	[3320]	$[3300]a_4$	[3321]	$[3320]a_2$	2[3311]	$[3321]a_1$	[3332]	[3320]
0	0	0	[4400]	0	[4420]	0	[4421]	[4422]	0
[3331]	[3330]	[3320]	$[3300]a_4$	[3321]	$[3320]a_2$	$[3211]a_2$	$[3321]a_1$	[3332]	[3320]
[4421]	0	0	[4400]	0	[4420]	[4311]	[4421]	[4422]	0
0	[4420]	0	[4400]	0	[4420]	[4311]	[4421]	[4422]	0
0	0	[4320]	$[4300]a_1$	[4321]	$[4320]a_1$	0	[4331]	[4332]	[4320]
0	0	0	[5400]	0	[5420]	0	[5421]	[5432]	0
0	0	0	0	[3611]	[3620]	2[3411]	$[3611]a_3$	[3622]	0
0	0	0	0	0	[4620]	2[4411]	2[4621]	[4632]	0
0	0	0	0	0	0	[4511]	$\frac{1}{2}[4721]$	$\frac{1}{6}[4732]$	0
0	0	0	0	0	0	0	$\frac{1}{2}[5711]$	$\frac{1}{2}[5722]$	0
0	0	0	0	0	0	0	0	$\frac{1}{3}[6731]$	0

Here, we put $[abcd] = (q-1)^a q^b (q+1)^c (q^2+q+1)^d$ *and*

$$\begin{array}{lll} a_1 = 2q+1 & b_1 = 2q^2+2q+1 & c_1 = q^3+4q^2+3q+1 \\ a_2 = 3q+1 & b_2 = 5q^2+3q+1 & \\ a_3 = q+2 & b_3 = q^2+3q+1 & \\ a_4 = 4q+1 & b_4 = b_7 = q^2+8q+1 & \\ a_5 = 5q+1 & b_5 = 3q^2+2q+1 & \end{array}$$

[Proof]

We obviously have $|\mathcal{O}_1 \cap W_j| = 1$ for all j. We obtain the cardinalities $|\mathcal{O}_i \cap W_j|$ for $1 \le j \le 13$, $j \ne 4, 7$ and $1 \le i \le 21$ from Proposition 5.2. For $j = 4$, we easily obtain the cardinalities. For $j = 21$, we already calculated the cardinalities. We calculate the rest cardinalities. For $1 \le i, j \le 21$, let $G(i,j) = \{g \in G \mid gx_i \in W_j\}$. We have

$$|\mathcal{O}_i \cap W_j| = |G(i,j)|/|\mathrm{Stab}(x_i)| = |G(i,j)| \cdot |\mathcal{O}_i|/|G|.$$

Thus when it is difficult to count $|\mathcal{O}_i \cap W_j|$ directly, we count $|G(i,j)|$.

We consider W_7. We write an element $x \in W_7$ as $x = (\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & a_{33} \end{bmatrix}, \begin{bmatrix} 0 & 0 & b_{13} \\ 0 & 0 & b_{23} \\ b_{31} & b_{32} & b_{33} \end{bmatrix})$. Let $W_7^0 = \{x \in W_7 \mid b_{31} = 0\}$ and $W_7^1 = \{x \in W_7 \mid b_{31} \ne 0\}$. Then $|\mathcal{O}_i \cap W_7| = |\mathcal{O}_i \cap W_7^0| + |\mathcal{O}_i \cap W_7^1|$. We already counted the cardinalities $|\mathcal{O}_i \cap W_7^0|$ in the proof of Proposition 5.2. Thus we count $|\mathcal{O}_i \cap W_7^1|$. If $a_{33} = b_{13} = b_{23} = 0$, then $x \in \mathcal{O}_2$. If $a_{33} = 0$ and $(b_{13}, b_{23}) \ne (0,0)$, then $x \in \mathcal{O}_3$. If $a_{33} \ne 0$ and $b_{13} = b_{23} = 0$, then $x \in \mathcal{O}_5$. If $a_{33} \ne 0$ and $(b_{13}, b_{23}) \ne (0,0)$, then $x \in \mathcal{O}_7$. Thus we obtain $|\mathcal{O}_2 \cap W_7^1| = q^2(q-1)$, $|\mathcal{O}_3 \cap W_7^1| = q^2(q^2-1)(q-1)$, $|\mathcal{O}_5 \cap W_7^1| = q^2(q-1)^2$ and $|\mathcal{O}_7 \cap W_7^1| = q^2(q^2-1)(q-1)^2$.

Next we consider W_{14}. We write an element $x \in W_{14}$ as $x = (\begin{bmatrix} 0 & 0 & a_{13} \\ 0 & 0 & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}, \begin{bmatrix} 0 & 0 & b_{13} \\ 0 & 0 & b_{23} \\ b_{31} & b_{32} & b_{33} \end{bmatrix})$. Note that $W_5, W_6 \subset W_{14}$. Let $W_{14}^0 = W_5 \cap W_6$ and $W_{14}^1 = W_{14} \setminus (W_5 \cup W_6)$. Then $|\mathcal{O}_i \cap W_{14}| = |\mathcal{O}_i \cap W_{14}^1| + |\mathcal{O}_i \cap W_5| + |\mathcal{O}_i \cap W_6| - |\mathcal{O}_i \cap W_{14}^0|$. We easily see that $|\mathcal{O}_2 \cap W_{14}^0| = q^2 - 1$. Thus we count $|\mathcal{O}_i \cap W_{14}^1|$. If $(a_{13}, a_{23}, a_{33}, a_{31}, a_{32}) /\!\!/ (b_{13}, b_{23}, b_{33}, b_{31}, b_{32})$, we have $x \in \mathcal{O}_3$. If $(a_{13}, a_{23}, a_{31}, a_{32}) /\!\!/ (b_{13}, b_{23}, b_{31}, b_{32})$ and $(a_{13}, a_{23}, a_{33}, a_{31}, a_{32}) \not{/\!\!/} (b_{13}, b_{23}, b_{33}, b_{31}, b_{32})$, we have $x \in \mathcal{O}_7$. If $\mathrm{rank}(\begin{bmatrix} a_{13} & b_{13} \\ a_{23} & b_{23} \end{bmatrix}) = 1$, $\mathrm{rank}(\begin{bmatrix} a_{31} & b_{31} \\ a_{32} & b_{32} \end{bmatrix}) = 1$ and $\mathrm{rank}(\begin{bmatrix} a_{13} & b_{13} & a_{31} & b_{31} \\ a_{23} & b_{23} & a_{32} & b_{32} \end{bmatrix}) = 2$, then $x \in \mathcal{O}_8$. If $\mathrm{rank}(\begin{bmatrix} a_{13} & b_{13} \\ a_{23} & b_{23} \end{bmatrix}) = 1$ and $\mathrm{rank}(\begin{bmatrix} a_{31} & b_{31} \\ a_{32} & b_{32} \end{bmatrix}) = 2$, then $x \in \mathcal{O}_{10}$. If $\mathrm{rank}(\begin{bmatrix} a_{13} & b_{13} \\ a_{23} & b_{23} \end{bmatrix}) = 2$ and $\mathrm{rank}(\begin{bmatrix} a_{31} & b_{31} \\ a_{32} & b_{32} \end{bmatrix}) = 1$, then $x \in \mathcal{O}_{12}$. If $\mathrm{rank}(\begin{bmatrix} a_{13} & b_{13} \\ a_{23} & b_{23} \end{bmatrix}) = 2$ and $\mathrm{rank}(\begin{bmatrix} a_{31} & b_{31} \\ a_{32} & b_{32} \end{bmatrix}) = 2$, then $x \in \mathcal{O}_{14}$. Thus we obtain $|\mathcal{O}_3 \cap W_{14}^1| = q(q^2-1)^2(q+1)$, $|\mathcal{O}_7 \cap W_{14}^1| = (q^2-1)^2(q+1)(q^2-q)$, $|\mathcal{O}_8 \cap W_{14}^1| = q^2(q+1)^2 \cdot \mathrm{gl}_2$, $|\mathcal{O}_{10} \cap W_{14}^1| = q^2 \cdot |2,1| \cdot \mathrm{gl}_2$, $|\mathcal{O}_{12} \cap W_{14}^1| = q^2 \cdot |2,1| \cdot \mathrm{gl}_2$, and $|\mathcal{O}_{14} \cap W_{14}^1| = q^2 \cdot \mathrm{gl}_2^2$.

Next we consider W_{15} and $W_{15}^{\perp}$. For $X \subset V$, let $X^T = \{(A^T, B^T) \in V \mid (A,B) \in X\}$. We see $W_{15}^{\perp} = W_{15}^T$. We easily see that $\mathcal{O}_i^T = \mathcal{O}_i$ for $i \neq 5,6,10,11,12,13$, $\mathcal{O}_5^T = \mathcal{O}_6$, $\mathcal{O}_{10}^T = \mathcal{O}_{12}$ and $\mathcal{O}_{11}^T = \mathcal{O}_{13}$. Furthermore, we have $(\mathcal{O}_i^T) \cap (W_{15}^{\perp}) = (\mathcal{O}_i \cap W_{15})^T$. Thus we only calculate for W_{15}. We write $x \in W_{15}$ as $x = (A,B) = (\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & a_{23} \\ 0 & 0 & a_{33} \end{bmatrix}, \begin{bmatrix} 0 & 0 & b_{13} \\ b_{21} & b_{22} & b_{23} \\ b_{31} & b_{32} & b_{33} \end{bmatrix})$. Let

$$r_{15}(x) := (\mathrm{rank}(\begin{bmatrix} b_{13} \end{bmatrix}), \mathrm{rank}(\begin{bmatrix} b_{21} & b_{22} \\ b_{31} & b_{32} \end{bmatrix}), \mathrm{rank}(\begin{bmatrix} a_{23} \\ a_{33} \end{bmatrix})).$$

$r_{15}(x)$ and some additional conditions determine the orbits to which x belongs:

$r_{15}(x)$	additional condition	x is in
$(0,0,0)$	$(b_{23}, b_{33}) \neq (0,0)$	$\mathcal{O}_2$
(0,0,1)	$\mathrm{rank}(\begin{bmatrix} a_{23} & b_{23} \\ a_{33} & b_{33} \end{bmatrix}) = 1$	$\mathcal{O}_2$
	$\mathrm{rank}(\begin{bmatrix} a_{23} & b_{23} \\ a_{33} & b_{33} \end{bmatrix}) = 2$	$\mathcal{O}_6$
(0,1,0)	$\mathrm{rank}(\begin{bmatrix} b_{21} & b_{22} & b_{23} \\ b_{31} & b_{32} & b_{33} \end{bmatrix}) = 1$	$\mathcal{O}_2$
	$\mathrm{rank}(\begin{bmatrix} b_{21} & b_{22} & b_{23} \\ b_{31} & b_{32} & b_{33} \end{bmatrix}) = 2$	$\mathcal{O}_3$
(0,1,1)	$\mathrm{rank}(\begin{bmatrix} a_{23} & b_{21} & b_{22} & b_{23} \\ a_{33} & b_{31} & b_{32} & b_{33} \end{bmatrix}) = 1$	$\mathcal{O}_5$
	$\mathrm{rank}(\begin{bmatrix} b_{21} & b_{22} & a_{23} \\ b_{31} & b_{32} & a_{33} \end{bmatrix}) = 1$, $\mathrm{rank}(\begin{bmatrix} a_{23} & b_{21} & b_{22} & b_{23} \\ a_{33} & b_{31} & b_{32} & b_{33} \end{bmatrix}) = 2$	$\mathcal{O}_7$
$(0,1,1)$	$\mathrm{rank}(\begin{bmatrix} b_{21} & b_{22} & a_{23} \\ b_{31} & b_{32} & a_{33} \end{bmatrix}) = 2$	$\mathcal{O}_8$
$(0,2,0)$	-	$\mathcal{O}_3$
$(0,2,1)$	-	$\mathcal{O}_{10}$
$(0,3,0)$	-	$\mathcal{O}_4$
$(1,0,0)$	-	$\mathcal{O}_2$
$(1,0,1)$	-	$\mathcal{O}_6$
$(1,1,0)$	-	$\mathcal{O}_3$
(1,1,1)	$\mathrm{rank}(\begin{bmatrix} b_{21} & b_{22} & a_{23} \\ b_{31} & b_{32} & a_{33} \end{bmatrix}) = 1$	$\mathcal{O}_7$
	$\mathrm{rank}(\begin{bmatrix} b_{21} & b_{22} & a_{23} \\ b_{31} & b_{32} & a_{33} \end{bmatrix}) = 2$	$\mathcal{O}_{12}$
$(1,2,1)$	-	$\mathcal{O}_{15}$

Thus we obtain the cardinalities $|\mathcal{O}_i \cap W_{15}|$.

Next we calculate for W_{16}. Let

$$W_{16}^0 = (\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & * \\ 0 & * & * \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 \\ 0 & * & * \\ 0 & * & * \end{bmatrix}),$$

$$W_{16}^1 = \left\{ (\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & a_{23} \\ 0 & a_{32} & a_{33} \end{bmatrix}, \begin{bmatrix} 0 & 0 & b_{13} \\ 0 & b_{22} & b_{23} \\ 0 & b_{32} & b_{33} \end{bmatrix}) \in V \;\middle|\; b_{13} \neq 0 \right\},$$

$$W_{16}^2 = \left\{ (\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & a_{23} \\ 0 & a_{32} & a_{33} \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 \\ 0 & b_{22} & b_{23} \\ b_{31} & b_{32} & b_{33} \end{bmatrix}) \in V \;\middle|\; b_{31} \neq 0 \right\},$$

$$W_{16}^3 = \left\{ (\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & a_{23} \\ 0 & a_{32} & a_{33} \end{bmatrix}, \begin{bmatrix} 0 & 0 & b_{13} \\ 0 & b_{22} & b_{23} \\ b_{31} & b_{32} & b_{33} \end{bmatrix}) \in V \;\middle|\; b_{13}b_{31} \neq 0 \right\}.$$

We have $|\mathcal{O}_i \cap W_{16}| = |\mathcal{O}_i \cap W_{16}^0| + |\mathcal{O}_i \cap W_{16}^1| + |\mathcal{O}_i \cap W_{16}^2| + |\mathcal{O}_i \cap W_{16}^3|$. Furthermore, we see $W_{16}^2 = (W_{16}^2)^T$. Thus we only calculate for W_{16}^0, W_{16}^1 and W_{16}^3. To calculate $|\mathcal{O}_i \cap W_{16}^0|$, we use the Fourier transform for $2 \otimes 2 \otimes 2$. We refer to $\mathcal{O}_i$, W_j, $W_j^\perp$ and M in Section 4 as $\mathcal{O}_i^{222}$, W_j^{222}, ${W_j^\perp}^{222}$ and M^{222}, respectively. Let $W = (\begin{bmatrix} 0 & * \\ * & * \end{bmatrix}, \begin{bmatrix} * & * \\ * & * \end{bmatrix}) \in 2 \otimes 2 \otimes 2$. The orthogonal complement of W is $W^\perp = W_2^{222}$. We have $|\mathcal{O}_i \cap W_{16}^0| = |cO_i^{222} \cap W|$ for $i = 1, 2, 3$ and $|\mathcal{O}_i \cap W_{16}^0| = |\mathcal{O}_{i-1}^{222} \cap W|$ for $5 \le i \le 9$. By Proposition 2.2, we have

$$\begin{bmatrix} |\mathcal{O}_1^{222} \cap W| \\ \vdots \\ |\mathcal{O}_8^{222} \cap W| \end{bmatrix} = \frac{|V|}{|W_2^{222}|} \begin{bmatrix} |\mathcal{O}_1^{222}| & & 0 \\ & \ddots & \\ 0 & & |\mathcal{O}_8^{222}| \end{bmatrix} M^{222} \begin{bmatrix} |\mathcal{O}_1^{222}| & & 0 \\ & \ddots & \\ 0 & & |\mathcal{O}_8^{222}| \end{bmatrix}^{-1} \begin{bmatrix} |\mathcal{O}_1^{222} \cap W_2^{222}| \\ \vdots \\ |\mathcal{O}_8^{222} \cap W_2^{222}| \end{bmatrix}.$$

The matrix M^{222} is explicitly determined in Theorem 4.3. We have $\begin{bmatrix} |\mathcal{O}_1^{222} \cap W_2^{222}| & \cdots & |\mathcal{O}_8^{222} \cap W_2^{222}| \end{bmatrix}^T = \begin{bmatrix} 1 & (q-1) & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}^T$. Thus we obtain the cardinalities $|\mathcal{O}_i \cap W_{16}^0|$; $|\mathcal{O}_2 \cap W_{16}^0| = (q-1)(3q^2+3q+1)$, $|\mathcal{O}_3 \cap W_{16}^0| = q(q-1)^2(2q+1)$, $|\mathcal{O}_5 \cap W_{16}^0| = q(q-1)^2(2q+1)$, $|\mathcal{O}_6 \cap W_{16}^0| = q(q-1)^2(2q+1)$, $|\mathcal{O}_7 \cap W_{16}^0| = q(q-1)^3(q^2+3q+1)$, $|\mathcal{O}_8 \cap W_{16}^0| = q^3(q-1)^2(q^2+4q+1)$, $|\mathcal{O}_9 \cap W_{16}^0| = q^3(q-1)^4$. Next we calculate $|\mathcal{O}_i \cap W_{16}^1|$. Let

$$W_{16}^4 = \left\{ (\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & a_{23} \\ 0 & 0 & a_{33} \end{bmatrix}, \begin{bmatrix} 0 & 0 & b_{13} \\ 0 & b_{22} & b_{23} \\ 0 & b_{32} & b_{33} \end{bmatrix}) \in V \;\middle|\; b_{13} \neq 0 \right\},$$

$$W_{16}^5 = \left\{ (\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & a_{23} \\ 0 & a_{32} & a_{33} \end{bmatrix}, \begin{bmatrix} 0 & 0 & b_{13} \\ 0 & b_{22} & b_{23} \\ 0 & b_{32} & b_{33} \end{bmatrix}) \in V \;\middle|\; a_{32}b_{13} \neq 0 \right\}.$$

We have $|\mathcal{O}_i \cap W_{16}^1| = |\mathcal{O}_i \cap W_{16}^4| + |\mathcal{O}_i \cap W_{16}^5|$. We count $|\mathcal{O}_i \cap W_{16}^4|$. We write $x \in W_{16}^4$ as $x = (\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & a_{23} \\ 0 & 0 & a_{33} \end{bmatrix}, \begin{bmatrix} 0 & 0 & b_{13} \\ 0 & b_{22} & b_{23} \\ 0 & b_{32} & b_{33} \end{bmatrix})$. If $a_{23} = a_{33} = b_{22} = b_{32} = 0$, we have $x \in \mathcal{O}_2$. If $\mathrm{rank}(\begin{bmatrix} a_{23} & b_{22} \\ a_{33} & b_{32} \end{bmatrix}) = 1$ and $a_{23} = a_{33} = 0$, we have $x \in \mathcal{O}_3$. If $\mathrm{rank}(\begin{bmatrix} a_{23} & b_{22} \\ a_{33} & b_{32} \end{bmatrix}) = 1$ and $b_{22} = b_{32} = 0$, we have $x \in \mathcal{O}_6$. If $\mathrm{rank}(\begin{bmatrix} a_{23} & b_{22} \\ a_{33} & b_{32} \end{bmatrix}) = 1$, $(a_{23}, a_{33}) \neq 0$ and $(b_{22}, b_{32}) \neq 0$, then we have $x \in \mathcal{O}_7$. If $\mathrm{rank}(\begin{bmatrix} a_{23} & b_{22} \\ a_{33} & b_{32} \end{bmatrix}) = 2$, we have $x \in \mathcal{O}_{12}$. Thus we obtain $|\mathcal{O}_2 \cap W_{16}^4| = q^2(q-1)$, $|\mathcal{O}_3 \cap W_{16}^4| = q^2(q-1)(q^2-1)$, $|\mathcal{O}_6 \cap W_{16}^4| = q^2(q-1)(q^2-1)$, $|\mathcal{O}_7 \cap W_{16}^4| = q^2(q-1)^3(q+1)$, $|\mathcal{O}_{12} \cap W_{16}^4| = q^2(q-1)\mathrm{gl}_2$. Next we count $|\mathcal{O}_i \cap W_{16}^5|$. We write $x \in W_{16}^5$ as $x = (\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & a_{23} \\ 0 & a_{32} & a_{33} \end{bmatrix}, \begin{bmatrix} 0 & 0 & b_{13} \\ 0 & b_{22} & b_{23} \\ 0 & b_{32} & b_{33} \end{bmatrix})$. If $a_{23} = b_{22} = 0$, we have $x \in \mathcal{O}_8$. If $a_{23} \neq 0$ and $b_{22} = 0$, we have $x \in \mathcal{O}_{12}$. If $a_{23} = 0$ and $b_{22} \neq 0$, we have $x \in \mathcal{O}_{12}$. If $a_{23}b_{22} \neq 0$, we have $x \in \mathcal{O}_{13}$. Thus we obtain $|\mathcal{O}_8 \cap W_{16}^5| = q^4(q-1)^2$, $|\mathcal{O}_{12} \cap W_{16}^5| = 2q^4(q-1)^3$, $|\mathcal{O}_{13} \cap W_{16}^5| = q^4(q-1)^4$. Next we count

$|\mathcal{O}_i \cap W_{16}^3|$. We write $x \in W_{16}^3$ as $x = (\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & a_{23} \\ 0 & a_{32} & a_{33} \end{bmatrix}, \begin{bmatrix} 0 & 0 & b_{13} \\ 0 & b_{22} & b_{23} \\ b_{31} & b_{32} & b_{33} \end{bmatrix})$. If $b_{22} = a_{23} = a_{32} = a_{33} = 0$, we have $x \in \mathcal{O}_3$. If $b_{22} = a_{23} = a_{32} = 0$ and $a_{33} \neq 0$, we have $x \in \mathcal{O}_7$. If $b_{22} = a_{23} = 0$ and $a_{32} \neq 0$, we have $x \in \mathcal{O}_{10}$. If $b_{22} = a_{32} = 0$ and $a_{23} \neq 0$, we have $x \in \mathcal{O}_{12}$. If $b_{22} = 0$ and $a_{23}a_{32} \neq 0$, we have $x \in \mathcal{O}_{14}$. If $b_{22} \neq 0$ and $\mathrm{rank}(\begin{bmatrix} 0 & a_{23} \\ a_{32} & a_{33} \end{bmatrix}) = 0$, we have $x \in \mathcal{O}_4$. If $b_{22} \neq 0$ and $\mathrm{rank}(\begin{bmatrix} 0 & a_{23} \\ a_{32} & a_{33} \end{bmatrix}) = 1$, we have $x \in \mathcal{O}_{15}$. If $b_{22} \neq 0$ and $\mathrm{rank}(\begin{bmatrix} 0 & a_{23} \\ a_{32} & a_{33} \end{bmatrix}) = 2$, we have $x \in \mathcal{O}_{16}$. Thus we obtain $|\mathcal{O}_3 \cap W_{16}^3| = q^3(q-1)^2$, $|\mathcal{O}_7 \cap W_{16}^3| = q^3(q-1)^3$, $|\mathcal{O}_{10} \cap W_{16}^3| = q^4(q-1)^3$, $|\mathcal{O}_{12} \cap W_{16}^3| = q^4(q-1)^3$, $|\mathcal{O}_{14} \cap W_{16}^3| = q^4(q-1)^4$, $|\mathcal{O}_4 \cap W_{16}^3| = q^3(q-1)^3$, $|\mathcal{O}_{15} \cap W_{16}^3| = q^3(q-1)^4(2q+1)$, $|\mathcal{O}_{16} \cap W_{16}^3| = q^4(q-1)^5$.

Next we consider W_{17}. We write $x \in W_{17}$ as $x = (A, B) = (\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ a_{31} & a_{32} & a_{33} \end{bmatrix}, \begin{bmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \\ b_{31} & b_{32} & b_{33} \end{bmatrix})$, and let $v_1 = \begin{bmatrix} a_{31} & a_{32} & a_{33} \end{bmatrix}$, $v_2 = \begin{bmatrix} b_{11} & b_{12} & b_{13} \end{bmatrix}$, $v_3 = \begin{bmatrix} b_{21} & b_{22} & b_{23} \end{bmatrix}$ and $v_4 = \begin{bmatrix} b_{31} & b_{32} & b_{33} \end{bmatrix}$. Let $W_{17}^0 = \{x \in W_{17} \mid v_1 \neq 0\}$. We have $|\mathcal{O}_i \cap W_{17}| = |\mathcal{O}_i \cap W_4| + |\mathcal{O}_i \cap W_{17}^0|$. Let $x \in W_{17}^0$. Let

$$r_{17}^0 := (\mathrm{rank}(\begin{bmatrix} v_2^T & v_3^T \end{bmatrix}), \mathrm{rank}(\begin{bmatrix} v_1^T & v_2^T & v_3^T \end{bmatrix}), \mathrm{rank}(\begin{bmatrix} v_1^T & v_2^T & v_3^T & v_4^T \end{bmatrix})).$$

$r_{17}^0(x)$ determines the orbits to which x belongs:

$r_{17}^0(x)$	x is in
$(0,1,1)$	$\mathcal{O}_2$
$(0,1,2)$	$\mathcal{O}_5$
$(1,1,1)$	$\mathcal{O}_6$
$(1,1,2)$	$\mathcal{O}_7$
$(1,2,2)$	$\mathcal{O}_8$
$(1,2,3)$	$\mathcal{O}_{10}$
$(2,2,2)$	$\mathcal{O}_{12}$
$(2,2,3)$	$\mathcal{O}_{15}$
$(2,3,3)$	$\mathcal{O}_{17}$

Thus we obtain the cardinalities $|\mathcal{O}_i \cap W_{17}^0|$.

Next we consider W_{18}. We write $x \in W_{18}$ as $x = (A, B) = (\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & 0 \end{bmatrix}, \begin{bmatrix} b_{11} & b_{12} & 0 \\ b_{21} & b_{22} & 0 \\ 0 & 0 & 0 \end{bmatrix})$. First we count $|\mathcal{O}_i \cap W_{18}|$ for $i = 2, 3, 4$. Let $W_{18}^0 = \{x \in W_{18} \mid B = 0\}$ and $W_{18}^1 = \{x \in W_{18} \mid B \neq 0\}$. We see $|\mathcal{O}_i \cap W_{18}| = |\mathcal{O}_i \cap W_{18}^0| + |\mathcal{O}_i \cap W_{18}^1|$. Let $\mathrm{M}_3^{33}(\mathbb{F}_q) = \left\{ A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \in \mathrm{M}_3(\mathbb{F}_q) \;\middle|\; a_{33} = 0 \right\}$. We have $|\mathcal{O}_2 \cap W_{18}^0| = |\{A \in \mathrm{M}_3^{33}(\mathbb{F}_q) \mid \mathrm{rank}(A) = 1\}| = |(2,3),1| + (q^2-1)q^2$, $|\mathcal{O}_3 \cap W_{18}^0| = |\{A \in \mathrm{M}_3^{33}(\mathbb{F}_q) \mid \mathrm{rank}(A) = 2\}| = |(2,3),2| + q \cdot \mathrm{gl}_2 + q^2 \cdot \mathrm{gl}_2$ and $|\mathcal{O}_4 \cap W_{18}^0| = |\{A \in \mathrm{M}_3^{33}(\mathbb{F}_q) \mid \mathrm{rank}(A) = 3\}| = (q^2-1)(q^3-q)(q^3-q^2)$. On the other hand, we see $|\mathcal{O}_2 \cap W_{18}^1| = q \cdot |2,1|$. $|\mathcal{O}_3 \cap W_{18}^1| = q \cdot \mathrm{gl}_2$, and $|\mathcal{O}_4 \cap W_{18}^1| = 0$. Next we count $|G(i,18)|$ for $5 \leq i \leq 18$. Let $g = (g_1, g_2, g_3) = ((g_{1ij})_{1 \leq i,j \leq 2}, (g_{2ij})_{1 \leq i,j \leq 3}, (g_{3ij})_{1 \leq i,j \leq 3}) \in G$ and we consider when $g \in G(i,18)$. The action of g_1 gives a linear change of A and B. The action of g_2 means the same elementary row operation of A and B. The action of g_3 means the same elementary column operation of A and B. Since there actions are all invertible, $g \cdot (A, B) \in W_{18}$ holds if and only if

$$\begin{cases} \begin{bmatrix} \begin{bmatrix} g_{231} & g_{232} & g_{233} \end{bmatrix} A \begin{bmatrix} g_{331} & g_{332} & g_{333} \end{bmatrix}^T \\ \begin{bmatrix} g_{231} & g_{232} & g_{233} \end{bmatrix} B \begin{bmatrix} g_{331} & g_{332} & g_{333} \end{bmatrix}^T \end{bmatrix} & = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \\ (g_{121}A + g_{122}B) \begin{bmatrix} g_{331} & g_{332} & g_{333} \end{bmatrix}^T & = \begin{bmatrix} 0 & 0 & 0 \end{bmatrix}^T, \\ \begin{bmatrix} g_{331} & g_{332} & g_{333} \end{bmatrix} (g_{121}A + g_{122}B) & = \begin{bmatrix} 0 & 0 & 0 \end{bmatrix}. \end{cases}$$

Let us count $|G(5,18)|$. Let $g=(g_1,g_2,g_3)=((g_{1ij})_{1\le i,j\le 2},(g_{2ij})_{1\le i,j\le 3},(g_{3ij})_{1\le i,j\le 3})\in G(5,18)$. We have

$$\begin{cases} \begin{bmatrix} g_{233}g_{332} & g_{233}g_{333}\end{bmatrix} & = & \begin{bmatrix}0 & 0\end{bmatrix}, \\ \begin{bmatrix}0 & 0 & g_{121}g_{332}+g_{122}g_{333}\end{bmatrix} & = & \begin{bmatrix}0 & 0 & 0\end{bmatrix}, \\ \begin{bmatrix}0 & g_{121}g_{233} & g_{122}g_{233}\end{bmatrix} & = & \begin{bmatrix}0 & 0 & 0\end{bmatrix}. \end{cases}$$

It follows that $g_{233}=0$ and $g_{121}g_{332}+g_{122}g_{333}=0$. Thus we obtain $|G(5,18)|=\mathrm{gl}_2\cdot(q^2-1)(q^3-q)(q^3-q^2)\cdot(q^2-1)(q^3-q)(q^3-q^2)$. The counts of the cardinalities $|G(i,18)|$ for $6\le i\le 18$ are carried out in the same way, and we omit the detail.

Next we consider W_{19}. We write $x\in W_{19}$ as $x=(\begin{bmatrix}0&0&0\\a_{21}&a_{22}&a_{23}\\a_{31}&a_{32}&a_{33}\end{bmatrix},\begin{bmatrix}b_{11}&b_{12}&b_{13}\\0&0&0\\b_{31}&b_{32}&b_{33}\end{bmatrix})$. Let $W_{19}^1=\{x\in W_{19}\mid v_1\ne 0, v_3\ne 0\}$. We have $|\mathcal{O}_i\cap W_{19}|=2|\mathcal{O}_i\cap W_{10}|-|\mathcal{O}_i\cap W_5|+|\mathcal{O}_i\cap W_{19}^1|$. We count $|\mathcal{O}_i\cap W_{19}^1|$. Let $x\in W_{19}^1$, $v_1=\begin{bmatrix}a_{21}&a_{22}&a_{23}\end{bmatrix}$, $v_2=\begin{bmatrix}a_{31}&a_{32}&a_{33}\end{bmatrix}$, $v_3=\begin{bmatrix}b_{11}&b_{12}&b_{13}\end{bmatrix}$, $v_4=\begin{bmatrix}b_{31}&b_{32}&b_{33}\end{bmatrix}$. Let

$$r_{19}(x):=(\mathrm{rank}(\begin{bmatrix}v_3^T & v_4^T\end{bmatrix}),\mathrm{rank}(\begin{bmatrix}v_3^T & v_1^T & v_4^T\end{bmatrix}),\mathrm{rank}(\begin{bmatrix}v_1^T & v_2^T\end{bmatrix}),\mathrm{rank}(\begin{bmatrix}v_3^T & v_1^T & v_2^T\end{bmatrix})).$$

$r_{19}(x)$ and some additional conditions determine the orbits to which x belongs:

$r_{19}(x)$	x is in
$(0,0,1,1)$	$\mathcal{O}_2$
$(0,1,1,1)$	$\mathcal{O}_2$
$(0,1,2,2)$	$\mathcal{O}_3$
$(1,1,0,0)$	$\mathcal{O}_2$
$(1,1,0,1)$	$\mathcal{O}_2$
$(1,1,1,1)$	$\mathcal{O}_6$
$(1,1,2,2)$	$\mathcal{O}_{12}$
$(1,2,1,2)$	$\mathcal{O}_8$
$(1,2,2,2)$	$\mathcal{O}_{12}$
$(1,2,2,3)$	$\mathcal{O}_{17}$
$(2,2,0,1)$	$\mathcal{O}_3$
$(2,2,1,1)$	$\mathcal{O}_{12}$
$(2,2,1,2)$	$\mathcal{O}_{12}$
$(2,2,2,2)$	$\mathcal{O}_{12}$ or $\mathcal{O}_{13}$ or $\mathcal{O}_{14}$
$(2,2,2,3)$	$\mathcal{O}_{18}$
$(2,3,1,2)$	$\mathcal{O}_{17}$
$(2,3,2,2)$	$\mathcal{O}_{18}$
$(2,3,2,3)$	$\mathcal{O}_{19}$

Consider the case $r_{19}(x)=(2,2,2,2)$. When $v_1/\!\!/v_3$ and $\mathrm{rank}(\begin{bmatrix}v_1^T & v_2^T & v_4^T\end{bmatrix})=2$, we have $x\in\mathcal{O}_{12}$. When $v_1/\!\!/v_3$ and $\mathrm{rank}(\begin{bmatrix}v_1^T & v_2^T & v_4^T\end{bmatrix})=3$, then we have $x\in\mathcal{O}_{14}$. When $v_1\not\!/\!\!/v_3$, we have $x\in\mathcal{O}_{13}$. Thus we obtain the cardinalities $|\mathcal{O}_i\cap W_{19}^1|$.

Lastly, we consider W_{20}. First, we easily see that

$$|\mathcal{O}_2\cap W_{20}|=q|(2,3),1|+|3,1|,$$
$$|\mathcal{O}_3\cap W_{20}|=q|(2,3),2|+|3,2|,$$
$$|\mathcal{O}_4\cap W_{20}|=\mathrm{gl}_3.$$

Next we count $|G(i,20)|$ for $5\le i\le 20$. Let $g=(g_1,g_2,g_3)=((g_{1ij})_{1\le i,j\le 2},(g_{2ij})_{1\le i,j\le 3},(g_{3ij})_{1\le i,j\le 3})\in G$ and we consider when $g\in G(i,20)$. $g\cdot(A,B)\in W_{20}$ holds if and only if

$$\begin{bmatrix}g_{211} & g_{212} & g_{213}\end{bmatrix}(g_{111}A+g_{112}B)=\begin{bmatrix}0&0&0\end{bmatrix}.$$

Let us count $|G(5,20)|$. Let $g=(g_1,g_2,g_3)=((g_{1ij})_{1\le i,j\le 2},(g_{2ij})_{1\le i,j\le 3},(g_{3ij})_{1\le i,j\le 3})\in G(5,20)$. We have

$$\begin{bmatrix}g_{211} & g_{212} & g_{213}\end{bmatrix}\begin{bmatrix}0&0&0\\0&0&0\\0&g_{111}&g_{112}\end{bmatrix}g_3^T=\begin{bmatrix}0&0&0\end{bmatrix},$$

if and only if $g_{213} = 0$. Thus we obtain $|G(5, 20)| = \mathrm{gl}_2 \cdot \mathrm{gl}_3 \cdot (q^2 - 1)(q^3 - q)(q^3 - q^2)$. The counts of the cardinalities $|G(i, 20)|$ for $6 \le i \le 20$ are carried out in the same way, and we omit the detail. $\square$

10.3. **Fourier transform.**

Theorem 10.3. *The representation matrix M of the Fourier transform on $\mathcal{F}_V^G$ with respect to the basis $e_1, ..., e_{21}$ is given as follows:*

$$
q^{-18}\left[\begin{matrix}
1 & [1012] & [2122] & [3321] & [2112] & [2112] & [3132] & \tfrac{1}{2}[2332] & \tfrac{1}{2}[4312] & [3332] & [4422] \\
1 & e_1 & [1120]d_1 & [2310]c_1 & [1110]d_1 & [1110]d_1 & [2120]d_2 & \tfrac{1}{2}[1320]d_3 & \tfrac{1}{2}[3310]c_2 & [2320]d_2 & [3420]c_2 \\
1 & [0010]d_1 & qg_1 & [1300]e_2 & [1110]c_2 & [1110]c_2 & [1110]e_3 & \tfrac{1}{2}[1310]d_4 & -\tfrac{1}{2}[2300]c_1 & [1310]e_3 & -[2410]c_1 \\
1 & [0001]c_1 & [0101]e_2 & q^3f_1 & -[1111] & -[1111] & -[1111]b_1 & -\tfrac{1}{2}[1311]a_1 & \tfrac{1}{2}[2301] & -[1311]b_1 & [2411] \\
1 & [0010]d_1 & [1120]c_2 & -[2320] & qf_2 & [1110]c_2 & [1120]e_4 & \tfrac{1}{2}[1320]d_5 & \tfrac{1}{2}[2310]d_6 & [1320]e_4 & [2420]d_6 \\
1 & [1000]d_1 & [1120]c_2 & -[2320] & [1110]c_2 & qf_2 & [1120]e_4 & \tfrac{1}{2}[1320]d_5 & \tfrac{1}{2}[2310]d_6 & -[2320]b_2 & -[2410]c_1 \\
1 & d_2 & qe_3 & -[1300]b_1 & qe_3 & qe_3 & qg_2 & \tfrac{1}{2}[1300]d_7 & -\tfrac{1}{2}[1300]d_8 & [1300]e_5 & -[1400]d_9 \\
1 & d_3 & [1100]d_4 & -[2300]a_1 & [1100]d_5 & [1100]d_5 & [2100]d_7 & \tfrac{1}{2}q^3e_6 & -\tfrac{1}{2}[3320] & [1300]e_7 & -[2400]c_3 \\
1 & [0010]c_2 & -[0110]c_1 & [1310] & [0110]d_6 & [0110]d_6 & -[0120]d_8 & -\tfrac{1}{2}[1340] & \tfrac{1}{2}q^3e_8 & [1330] & -[1410] \\
1 & d_2 & qe_3 & -[1300]b_1 & qe_4 & -[1100]b_2 & [1100]e_5 & \tfrac{1}{2}q^3e_7 & \tfrac{1}{2}[2310] & q^3f_3 & -[1400]d_9 \\
1 & [0010]c_2 & -[0110]c_1 & [1310] & [0110]d_6 & -qc_1 & -[0110]d_9 & -\tfrac{1}{2}[0310]c_3 & -\tfrac{1}{2}[1300] & -[0310]d_9 & q^4d_8 \\
1 & d_2 & qe_3 & -[1300]b_1 & -[1100]b_2 & qe_4 & [1100]e_5 & \tfrac{1}{2}q^3e_7 & \tfrac{1}{2}[2310] & -[1310]c_4 & [2420] \\
1 & [0010]c_2 & -[0110]c_1 & [1310] & -qc_1 & [0110]d_6 & -[0110]d_9 & -\tfrac{1}{2}[0310]c_3 & -\tfrac{1}{2}[1300] & [1330] & -[1410] \\
1 & [0010]c_2 & -[0110]c_1 & [1310] & -qc_1 & -qc_1 & [1110]d_{10} & \tfrac{1}{2}0310]d_{11} & \tfrac{1}{2}[2310]b_3 & -[0310]d_9 & -[1410] \\
1 & [0010]c_2 & qe_9 & -q^3c_5 & -qc_1 & -qc_1 & qf_4 & -\tfrac{1}{2}[0310]c_3 & -\tfrac{1}{2}[1300] & q^3c_3 & -[1410] \\
1 & -b_2 & [0120] & -q^3 & [0110] & [0110] & qb_1c_6 & -\tfrac{1}{2}[0310]b_4 & -\tfrac{1}{2}[1300]b_3 & -q^3a_1 & q^4 \\
1 & d_5 & qb_1c_1 & -[2310] & -[1120] & -[1120] & -[1110]c_7 & -\tfrac{1}{2}[0310]b_5 & \tfrac{1}{2}[2300] & -[1310]b_4 & [2410] \\
1 & c_8 & -[0110]b_1 & [1300] & -qb_1 & -qb_1 & qe_{10} & -\tfrac{1}{2}q^3c_9 & -\tfrac{1}{2}[1300]b_6 & q^3b_7 & -[1400] \\
1 & c_{10} & -[1110]a_1 & q^3a_2 & -[1100]a_1 & -[1100]a_1 & -[2110]a_3 & \tfrac{1}{2}q^3c_{11} & -\tfrac{1}{2}[3300] & -[1300]b_8 & [2400] \\
1 & -b_2 & [0120] & -q^3 & [0110] & [0110] & -[0110]c_{12} & -\tfrac{1}{2}[0310]b_4 & \tfrac{1}{2}q^3c_{13} & [0310]b_1 & -[1410] \\
1 & -[0011] & [0111] & -[0310] & [0101] & [0101] & -[0111] & \tfrac{1}{2}[0311] & -\tfrac{1}{2}[1301] & -[0311] & [0401]
\end{matrix}\right.
$$

$$
\left.\begin{matrix}
[3332] & [4422] & [4422] & [4332] & [5432] & [3622] & [4632] & \tfrac{1}{6}[4732] & \tfrac{1}{2}[5722] & \tfrac{1}{3}[6731] \\
[2320]d_2 & [3420]c_2 & [3420]c_2 & [3330]c_2 & -[4420]b_2 & [2610]d_5 & [3620]c_8 & \tfrac{1}{6}[3720]c_{10} & -\tfrac{1}{2}[4710]b_2 & -\tfrac{1}{3}[5730] \\
[1310]e_3 & -[2410]c_1 & -[2410]c_1 & [2310]e_9 & [3430] & [1600]b_1c_1 & -[2620]b_1 & -\tfrac{1}{6}[3720]a_1 & \tfrac{1}{2}[3720] & \tfrac{1}{3}[4720] \\
-[1311]b_1 & [2411] & [2411] & -[1311]c_5 & -[2411] & -[2611] & [2611] & \tfrac{1}{6}[1711]a_2 & -\tfrac{1}{2}[2701] & -\tfrac{1}{3}[3720] \\
-[2320]b_2 & -[2410]c_1 & -[2410]c_1 & -[2320]c_1 & [3430] & -[2630] & -[2620]b_1 & -\tfrac{1}{6}[3720]a_1 & \tfrac{1}{2}[3720] & \tfrac{1}{3}[4720] \\
[1320]e_4 & [2420]d_6 & -[2410]c_1 & -[2320]c_1 & [3430] & -[2630] & -[2620]b_1 & -\tfrac{1}{6}[3720]a_1 & \tfrac{1}{2}[3720] & \tfrac{1}{3}[4720] \\
[1300]e_5 & -[1400]d_9 & [2400]d_{10} & [1300]f_4 & [2400]b_1c_6 & -[1600]c_7 & [1600]e_{10} & -\tfrac{1}{6}[3710]a_3 & -\tfrac{1}{2}[2700]c_{12} & -\tfrac{1}{3}[3710] \\
[1300]e_7 & -[2400]c_3 & [2400]d_{11} & -[2310]c_3 & -[3410]b_4 & -[1600]b_5 & -[2600]c_9 & \tfrac{1}{6}[2700]c_{11} & -\tfrac{1}{2}[3700]b_4 & \tfrac{1}{3}[4710] \\
[1330] & -[1410] & [2420]b_3 & -[1320] & -[2420]b_3 & [1610] & -[1620]b_6 & -\tfrac{1}{6}[3720] & \tfrac{1}{2}[1710]c_{13} & -\tfrac{1}{3}[3720] \\
-[1310]c_4 & [2420] & -[1400]d_9 & [1300]c_3 & -[2400]a_1 & -[1600]b_4 & [1600]b_7 & -\tfrac{1}{6}[2700]b_8 & \tfrac{1}{2}[2700]b_1 & -\tfrac{1}{3}[3710] \\
[1330] & -[1410] & -[1410] & -[1320] & [1410] & [1610] & -[1610] & \tfrac{1}{6}[2710] & -\tfrac{1}{2}[2710] & \tfrac{1}{3}[2710] \\
q^3f_3 & -[1400]d_9 & -[1400]d_9 & [1300]c_3 & -[2400]a_1 & -[1600]b_4 & [1600]b_7 & -\tfrac{1}{6}[2700]b_8 & \tfrac{1}{2}[2700]b_1 & -\tfrac{1}{3}[3710] \\
-[0310]d_9 & q^4d_8 & -[1410] & -[1320] & [1410] & [1610] & -[1610] & \tfrac{1}{6}[2710] & -\tfrac{1}{2}[2710] & \tfrac{1}{3}[2710] \\
-[0310]d_9 & -[1410] & q^4d_8 & -[1320] & [1410] & [1610] & -[1610] & \tfrac{1}{6}[2710] & -\tfrac{1}{2}[2710] & \tfrac{1}{3}[2710] \\
q^3c_3 & -[1410] & -[1410] & q^3f_5 & -[1400]b_1b_3 & -[1600]b_1 & -[1600]c_{12} & \tfrac{1}{6}[1700]a_1a_2 & \tfrac{1}{2}[1700] & \tfrac{1}{3}[2710] \\
-q^3a_1 & q^4 & q^4 & -q^3b_1b_3 & q^4e_{11} & [1600] & -[1600]b_3 & \tfrac{1}{6}[1700]a_2 & \tfrac{1}{2}[1700] & -\tfrac{1}{3}[1710] \\
-[1310]b_4 & [2410] & [2410] & -[2310]b_1 & [3410] & q^6b_9 & [1610]a_4 & -\tfrac{1}{2}[1710] & -\tfrac{1}{2}[2700] & 0 \\
q^3b_7 & -[1400] & -[1400] & -[1300]c_{12} & -[2400]b_3 & q^6a_4 & q^6b_{10} & -\tfrac{1}{2}[1700] & \tfrac{1}{2}[1700] & 0 \\
-[1300]b_8 & [2400] & [2400] & [1300]a_1a_2 & [2400]a_2 & -3q^6 & -3[1600] & q^7 & 0 & 0 \\
[0310]b_1 & -[1410] & -[1410] & [0310] & [1410] & -q^6 & [0610] & 0 & -q^7 & 0 \\
-[0311] & [0401] & [0401] & [0311] & -[0411] & 0 & 0 & 0 & 0 & q^7
\end{matrix}\right]
$$

Here, we put $[abcd] = (q-1)^a q^b (q+1)^c (q^2+q+1)^d$ and

$$
\begin{array}{lll}
a_1 = 2q+1 & c_1 = q^3-q-1 & e_1 = 2q^5+2q^4-2q^2-2q-1 \\
a_2 = 2q-1 & c_2 = q^3-q^2-q-1 & e_2 = q^5-q^3-q^2+q+1 \\
a_3 = 3q+1 & c_3 = 2q^3-2q-1 & e_3 = q^5-2q^4-2q^3+q^2+2q+1 \\
a_4 = q-2 & c_4 = 2q^3-q^2-2q-1 & e_4 = q^5-2q^3+q+1 \\
b_1 = q^2-q-1 & c_5 = q^3-q^2+1 & e_5 = q^5-2q^4-q^3+3q^2+3q+1 \\
b_2 = 2q^2+2q+1 & c_6 = q^3+q+1 & e_6 = 5q^5-7q^4-4q^3+4q^2+3q+1 \\
b_3 = q^2+1 & c_7 = q^3+q^2-2q-1 & e_7 = 2q^5-4q^4-3q^3+3q^2+3q+1 \\
b_4 = q^2-2q-1 & c_8 = q^3-2q^2-2q-1 & e_8 = q^5+q^4-q+1 \\
b_5 = 2q^2-3q-1 & c_9 = q^3-4q-1 & e_9 = q^5-q^4-q^3+q^2+2q+1 \\
b_6 = q^2-q+1 & c_{10} = 2q^3-2q^2-2q-1 & e_{10} = q^5-2q^4+q^3+2q^2-2q-1 \\
b_7 = 2q^2-2q-1 & c_{11} = q^3-q^2+5q+1 & e_{11} = q^5-q^4+q^3-q^2-1 \\
b_8 = q^2-3q-1 & c_{12} = q^3-q^2+q+1 & f_1 = q^6-q^5-q^4+q^2-1 \\
b_9 = q^2-2 & c_{13} = q^3+q^2-q+1 & f_2 = q^6+q^5-q^4-2q^3+q+1 \\
b_{10} = q^2-2q+2 & d_1 = q^4+q^3-q^2-q-1 & f_3 = q^6-3q^5+4q^3-2q-1 \\
 & d_2 = 2q^4-2q^2-2q-1 & f_4 = q^6-q^5+2q^3-2q-1 \\
 & d_3 = 3q^4-2q^2-2q-1 & f_5 = q^6-2q^5+q^4-q^2+q+1 \\
 & d_4 = 2q^4-q^3-4q^2-3q-1 & g_1 = q^7+q^6-3q^4-2q^3+q^2+2q+1 \\
 & d_5 = q^4+q^3-2q^2-2q-1 & g_2 = q^7-4q^5+q^4+4q^3-2q-1 \\
 & d_6 = q^4-q^3+1 & \\
 & d_7 = q^4-4q^3-7q^2-4q-1 & \\
 & d_8 = q^4-q^2+1 & \\
 & d_9 = q^4-q^3-q^2+q+1 & \\
 & d_{10} = q^4+q^2+2q+1 & \\
 & d_{11} = q^4-2q^3+2q+1 &
\end{array}
$$

By Theorem 10.3, we can calculate the Fourier transform of the indicator function Ψ of the singular set $S = \{x \in V \mid \mathrm{D}(x) = 0\} = \bigcup_{i=1}^{18} \mathcal{O}_i$, i.e., $\Psi = \sum_{i=1}^{18} e_i$.

Corollary 10.4. *The Fourier transform of Ψ is given as follows:*

$$
\widehat{\Psi}(x) = \begin{cases}
q^{-1}+2q^{-2}-q^{-3}-2q^{-4}-q^{-5}+2q^{-6}+q^{-7}-q^{-8} & x \in \mathcal{O}_1, \\
q^{-4}-2q^{-5}+2q^{-7}-q^{-8} & x \in \mathcal{O}_2, \\
0 & x \in \mathcal{O}_3, \mathcal{O}_5, \mathcal{O}_6, \mathcal{O}_{10}, \mathcal{O}_{11}, \mathcal{O}_{12}, \mathcal{O}_{13}, \mathcal{O}_{14}, \mathcal{O}_{16}, \mathcal{O}_{18}, \\
-q^{-8}+q^{-9} & x \in \mathcal{O}_4, \mathcal{O}_{15}, \\
q^{-6}-2q^{-7}+q^{-8} & x \in \mathcal{O}_7, \\
-q^{-7}+2q^{-8}-q^{-9} & x \in \mathcal{O}_8, \\
-q^{-7}+q^{-9} & x \in \mathcal{O}_9, \\
q^{-9}-q^{-10} & x \in \mathcal{O}_{17}, \\
-q^{-11} & x \in \mathcal{O}_{19}, \mathcal{O}_{21}, \\
q^{-11} & x \in \mathcal{O}_{20}.
\end{cases}
$$

In particular we have the following L_1-norm bound of $\widehat{\Psi}$:

$$\sum_{x \in V} |\widehat{\Psi}(x)| = O(q^7).$$

11. $\mathbb{F}_q^2 \otimes \mathrm{H}_3(\mathbb{F}_q)$

Define the trace map and the norm map as follows:

$$\mathrm{Tr}_2 : \mathbb{F}_{q^2} \ni z \mapsto z + \overline{z} \in \mathbb{F}_q,$$

$$\mathrm{N}_2 : \mathbb{F}_{q^2} \ni z \mapsto z\overline{z} \in \mathbb{F}_q.$$

Both maps are surjective. Tr_2 is a $\mathbb{F}_q$- linear map. $\mathrm{N}_2|_{\mathbb{F}_{q^2}^\times} : \mathbb{F}_{q^2}^\times \to \mathbb{F}_q^\times$ is a surjective group homomorphism. Let $\mathrm{H}_n(\mathbb{F}_{q^2})$ be the set of Hermitian matrices of order n. We consider $\mathrm{H}_3(\mathbb{F}_{q^2})$, i.e.,

$$\mathrm{H}_3(\mathbb{F}_{q^2}) := \left\{ A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ \overline{a_{12}} & a_{22} & a_{23} \\ \overline{a_{13}} & \overline{a_{23}} & a_{33} \end{bmatrix} \in \mathrm{M}_3(\mathbb{F}_{q^2}) \,\middle|\, a_{ii} \in \mathbb{F}_q, a_{ij} \in \mathbb{F}_{q^2} (1 \le i < j \le 3) \right\}.$$

Let $V = \mathbb{F}_q^2 \otimes \mathrm{H}_3(\mathbb{F}_{q^2})$ and $G = G_1 \times G_2 = \mathrm{GL}_2(\mathbb{F}_q) \times \mathrm{GL}_3(\mathbb{F}_{q^2})$. We write $x \in V$ as $x = (A, B)$ where $A, B \in \mathrm{H}_3(\mathbb{F}_{q^2})$, and write $g \in G$ as $g = (g_1, g_2)$ where $g_1 \in \mathrm{GL}_2(\mathbb{F}_q)$ and $g_2 \in \mathrm{GL}_3(\mathbb{F}_{q^2})$. The action of G on V is defined by

$$gx = (g_2 A \overline{g_2^T}, g_2 B \overline{g_2^T}) g_1^T.$$

Here, for a matrix h, $\overline{h}$ is the matrix whose (i, j)-entry is the conjugate over $\mathbb{F}_q$ of the (i, j)-entry of h. Define a bilinear form β of V as

$$\beta((A_1, B_1), (A_2, B_2)) = \mathrm{Tr}(A_1 A_2^T + B_1 B_2^T).$$

In addition, define an automorphism ι of G as

$$(g_1, g_2)^\iota = ((g_1^T)^{-1}, (g_2^T)^{-1}).$$

These β and ι satisfy Assumption 2.1.

11.1. **Orbit decomposition.** For $x = (A, B) = \left(\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ \overline{a_{12}} & a_{22} & a_{23} \\ \overline{a_{13}} & \overline{a_{23}} & a_{33} \end{bmatrix}, \begin{bmatrix} b_{11} & b_{12} & b_{13} \\ \overline{b_{12}} & b_{22} & b_{23} \\ \overline{b_{13}} & \overline{b_{23}} & b_{33} \end{bmatrix} \right)$, we define

$$r_1(x) := \dim(\langle A, B \rangle_{\mathbb{F}_q}), \text{ i.e., the } \mathbb{F}_q\text{-dimension of the subspace of } \mathrm{H}_3(\mathbb{F}_{q^2}) \text{ generated by } A \text{ and } B,$$

$$r_2(x) := \mathrm{rank}\left(\begin{bmatrix} a_{11} & a_{12} & a_{13} & b_{11} & b_{12} & b_{13} \\ \overline{a_{12}} & a_{22} & a_{23} & \overline{b_{12}} & b_{22} & b_{23} \\ \overline{a_{13}} & \overline{a_{23}} & a_{33} & \overline{b_{13}} & \overline{b_{23}} & b_{33} \end{bmatrix} \right),$$

$$\mathrm{mi}(x) := \min\{\mathrm{rank}(rA + sB) | (r, s) \in \mathbb{F}_q^2 \setminus \{(0, 0)\}\},$$

$$\mathrm{ma}(x) := \max\{\mathrm{rank}(rA + sB) | (r, s) \in \mathbb{F}_q^2 \setminus \{(0, 0)\}\}.$$

$r_1(x)$, $r_2(x)$, $\mathrm{mi}(x)$ and $\mathrm{ma}(x)$ are invariants of the orbits. We also define

$$\det_x(u, v) := \det(uA + vB) \in \mathrm{Sym}^3(\mathbb{F}_q^2) \text{ where } u, v \text{ are variables,}$$

$$\mathrm{T}(x) := \langle\!\langle \alpha \rangle\!\rangle \text{ if and only if } \det_x(u, v) \in O_{\langle\!\langle \alpha \rangle\!\rangle} \text{ in } \mathrm{Sym}^3(\mathbb{F}_q^2).$$

For $x \in V$ and $g = (g_1, g_2, g_3) \in G$, we have

$$\det_{gx}(u, v) = \mathrm{N}_2(\det(g_2)) \det_x((u, v) g_1).$$

Therefore $\mathrm{T}(x)$ is also an invariant of the orbits.

Proposition 11.1. *V consists of 15 G-orbits in all.*

Orbit name	Representative	$r_1(x)$	$r_2(x)$	$\mathrm{T}(x)$	$\mathrm{mi}(x)$	$\mathrm{ma}(x)$	Cardinality
$\mathcal{O}_1$	$\left(\begin{bmatrix}0&0&0\\0&0&0\\0&0&0\end{bmatrix},\begin{bmatrix}0&0&0\\0&0&0\\0&0&0\end{bmatrix}\right)$	0	0	$\langle\!\langle 0\rangle\!\rangle$	0	0	1
$\mathcal{O}_2$	$\left(\begin{bmatrix}0&0&0\\0&0&0\\0&0&0\end{bmatrix},\begin{bmatrix}0&0&0\\0&0&0\\0&0&1\end{bmatrix}\right)$	1	1	$\langle\!\langle 0\rangle\!\rangle$	0	1	[101101]
$\mathcal{O}_3$	$\left(\begin{bmatrix}0&0&0\\0&0&0\\0&0&0\end{bmatrix},\begin{bmatrix}0&0&0\\0&1&0\\0&0&1\end{bmatrix}\right)$	1	2	$\langle\!\langle 0\rangle\!\rangle$	0	2	[111111]
$\mathcal{O}_4$	$\left(\begin{bmatrix}0&0&0\\0&0&0\\0&0&0\end{bmatrix},\begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix}\right)$	1	3	$\langle\!\langle 1^3\rangle\!\rangle$	0	3	[231110]
$\mathcal{O}_5$	$\left(\begin{bmatrix}0&0&0\\0&0&1\\0&1&0\end{bmatrix},\begin{bmatrix}0&0&0\\0&0&0\\0&0&1\end{bmatrix}\right)$	2	2	$\langle\!\langle 1^3\rangle\!\rangle$	1	2	[212111]
$\mathcal{O}_6$	$\left(\begin{bmatrix}0&0&0\\0&1&0\\0&0&0\end{bmatrix},\begin{bmatrix}0&0&0\\0&0&0\\0&0&1\end{bmatrix}\right)$	2	2	$\langle\!\langle 0\rangle\!\rangle$	1	2	$\frac{1}{2}$[231111]
$\mathcal{O}_7$	$\left(\begin{bmatrix}0&0&0\\0&1&0\\0&0&1\end{bmatrix},\begin{bmatrix}0&0&0\\0&0&\mu_0\\0&\overline{\mu_0}&\mu_1\end{bmatrix}\right)$	2	2	$\langle\!\langle 0\rangle\!\rangle$	2	2	$\frac{1}{2}$[231111]
$\mathcal{O}_8$	$\left(\begin{bmatrix}0&0&1\\0&0&0\\1&0&0\end{bmatrix},\begin{bmatrix}0&0&0\\0&0&1\\0&1&0\end{bmatrix}\right)$	2	3	$\langle\!\langle 0\rangle\!\rangle$	2	2	[242111]
$\mathcal{O}_9$	$\left(\begin{bmatrix}0&0&0\\0&0&0\\0&0&1\end{bmatrix},\begin{bmatrix}0&0&1\\0&1&0\\1&0&0\end{bmatrix}\right)$	2	3	$\langle\!\langle 1^3\rangle\!\rangle$	1	3	[332111]
$\mathcal{O}_{10}$	$\left(\begin{bmatrix}0&0&0\\0&0&1\\0&1&0\end{bmatrix},\begin{bmatrix}0&0&1\\0&1&0\\1&0&0\end{bmatrix}\right)$	2	3	$\langle\!\langle 1^3\rangle\!\rangle$	2	3	[343111]
$\mathcal{O}_{11}$	$\left(\begin{bmatrix}1&0&0\\0&1&0\\0&0&0\end{bmatrix},\begin{bmatrix}0&0&0\\0&0&0\\0&0&1\end{bmatrix}\right)$	2	3	$\langle\!\langle 1^21\rangle\!\rangle$	1	3	[261111]
$\mathcal{O}_{12}$	$\left(\begin{bmatrix}1&0&0\\0&1&0\\0&0&0\end{bmatrix},\begin{bmatrix}0&0&0\\0&0&1\\0&1&0\end{bmatrix}\right)$	2	3	$\langle\!\langle 1^21\rangle\!\rangle$	2	3	[362111]
$\mathcal{O}_{13}$	$\left(\begin{bmatrix}1&0&0\\0&1&0\\0&0&0\end{bmatrix},\begin{bmatrix}0&0&0\\0&1&0\\0&0&1\end{bmatrix}\right)$	2	3	$\langle\!\langle 111\rangle\!\rangle$	2	3	$\frac{1}{6}$[471111]
$\mathcal{O}_{14}$	$\left(\begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix},\begin{bmatrix}0&0&0\\0&0&\mu_0\\0&\overline{\mu_0}&\mu_1\end{bmatrix}\right)$	2	3	$\langle\!\langle 12\rangle\!\rangle$	2	3	$\frac{1}{2}$[372111]
$\mathcal{O}_{15}$	$\left(\begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix},\begin{bmatrix}0&\nu_0&\nu_1\\\overline{\nu_0}&0&0\\\overline{\nu_1}&0&\nu_2\end{bmatrix}\right)$	2	3	$\langle\!\langle 3\rangle\!\rangle$	3	3	$\frac{1}{3}$[473011]

Here $[abcdef] := (q-1)^a q^b (q+1)^c (q^2-q+1)^d (q^2+1)^e (q^2+q+1)^f$ *and* $\mu_1,\ \nu_2 \in \mathbb{F}_q$ *and* $\mu_0,\ \nu_1,\ \nu_0 \in \mathbb{F}_{q^2}$ *are elements such that* $X^2+\mu_1X-N(\mu_0), X^3+\nu_2X^2-(N(\nu_0)+N(\nu_1))X+\nu_2N(\nu_0) \in \mathbb{F}_q[X]$ *are irreducible. Since* N_2 *is surjective, there exist such* $\mu_1,\mu_0,\nu_2,\nu_1,\nu_0$.

[Proof]

We count the cardinalities of the orbits of the 15 elements in the "Representative" column of the table. We refer to these elements as $x_1,...,x_{15}$ in order from the top, and let $\mathcal{O}_i$ be the orbit of x_i. First, we count the cardinalities for the cases of $r_2(x) \leq 2$, by using the result of $(G_1,V_1) =$

$(\mathrm{GL}_2(\mathbb{F}_q)\times\mathrm{GL}_2(\mathbb{F}_{q^2}), 2\otimes\mathrm{H}_2(\mathbb{F}_{q^2}))$, the space of pairs of binary Hermitian matrices (see Proposition 7.1). We identify V_1 as the subspace of V by the embedding

$$V_1 \ni (A,B)\mapsto (\begin{bmatrix}0 & O_{1,2}\\ O_{2,1} & A\end{bmatrix}, \begin{bmatrix}0 & O_{1,2}\\ O_{2,1} & B\end{bmatrix})\in V,$$

and identify G_1 as the subgroup of G by the embedding

$$G_1\ni (g_1,g_2,g_3)\mapsto (g_1, \begin{pmatrix}1 & \\ & g_2\end{pmatrix}, g_3)\in G.$$

We consider the induced injective map $G_1\setminus V_1\to G\setminus V$. For $x\in V_1$, we have

$$|Gx| = \frac{|G|\mathrm{gl}_{2-r_2(x)}}{q^{r_2(x)}|G_1|\mathrm{gl}_{3-r_2(x)}}|G_1x|.$$

Therefore we obtain $|\mathcal{O}_i|$ for $1\le i\le 7, x\ne 4$. Next we calculate the rest cardinalities. Let $\mathrm{Stab}(x_i)$ be the group of stabilizers of x_i. Since $|\mathcal{O}_i| = |G|/|\mathrm{Stab}(x_i)|$, it is enough to count $|\mathrm{Stab}(x_i)|$. Let $\mathrm{U}_n(\mathbb{F}_{q^2}) = \{g\in \mathrm{GL}_n(\mathbb{F}_{q^2}) \mid g\overline{g}^T = I_n\}$ and $\mathrm{GU}_n(\mathbb{F}_{q^2}) = \{g\in\mathrm{GL}_n(\mathbb{F}_{q^2}) \mid \exists t\in\mathbb{F}_q^\times, g\overline{g}^T = tI_n\}$. We use the following fact for the calculation:

$$|\mathrm{U}_n(\mathbb{F}_{q^2})| = q^{\frac{n(n-1)}{2}}\prod_{i=1}^n (q^i-(-1)^i),$$

$$|\mathrm{GU}_n(\mathbb{F}_{q^2})| = (q-1)|\mathrm{U}_n(\mathbb{F}_{q^2})| = (q-1)q^{\frac{n(n-1)}{2}}\prod_{i=1}^n (q^i-(-1)^i).$$

Let $\mathrm{H}(n,m) := \{x\in\mathrm{H}_n(\mathbb{F}_{q^2}) \mid \mathrm{rank}(x) = m\}$. We have

$$|\mathrm{H}(n,m)| := \frac{\mathrm{gl}_n(\mathbb{F}_{q^2})}{q^{2m(n-m)}\cdot|\mathrm{U}_m(\mathbb{F}_{q^2})|\cdot \mathrm{gl}_{n-m}(\mathbb{F}_{q^2})}.$$

The structure and the order of $\mathrm{Stab}(x_i)$ for $i=4, 8\le i\le 15$ is summarized as follows:

x_i	$\mathrm{Stab}(x_i)\cong$	$\lvert\mathrm{Stab}(x_i)\rvert$
x_4	$(\mathrm{GL}_1(\mathbb{F}_q)\times\mathrm{GU}_3(\mathbb{F}_{q^2}))\ltimes\mathbb{F}_q\ltimes\mathbb{F}_q$	$q(q-1)\cdot\lvert\mathrm{GU}_3(\mathbb{F}_{q^2})\rvert$
x_8	$(\mathrm{GL}_2(\mathbb{F}_q)\times\mathrm{GL}_1(\mathbb{F}_{q^2}))\ltimes\mathbb{F}_q^2$	$q^2(q^2-1)\cdot\mathrm{gl}_2$
x_9	$(\mathrm{GL}_1(\mathbb{F}_{q^2}))^2\ltimes\mathbb{F}_{q^2}^2$	$q^4(q^2-1)^2$
x_{10}	$(\mathrm{GL}_1(\mathbb{F}_q)\times\mathrm{GL}_1(\mathbb{F}_{q^2}))\ltimes\mathbb{F}_q^3$	$q^3(q-1)(q^2-1)$
x_{11}	$\mathrm{GU}_2(\mathbb{F}_{q^2})\times\mathrm{GL}_1(\mathbb{F}_{q^2})$	$(q^2-1)\cdot\lvert\mathrm{GU}_2(\mathbb{F}_{q^2})\rvert$
x_{12}	$(\mathrm{GL}_1(\mathbb{F}_q)\times\mathrm{GL}_1(\mathbb{F}_{q^2})\times\mathrm{Ker}(\mathrm{N}_2\lvert_{\mathbb{F}_{q^2}\setminus\{0\}}))\ltimes\mathbb{F}_q$	$q(q-1)(q+1)(q^2-1)$
x_{13}	$\mathfrak{S}_3\ltimes(\{(g_1,g_2,g_3)\in(\mathrm{GL}_1(\mathbb{F}_{q^2}))^3 \mid \mathrm{N}_2(g_1)=\mathrm{N}_2(g_2)=\mathrm{N}_2(g_3)\})$	$6(q^2-1)^3/(q-1)^2$
x_{14}	$\mathbb{Z}/2\mathbb{Z}\ltimes((\mathrm{GL}_1(\mathbb{F}_{q^2}))^2)$	$2(q^2-1)^2$
x_{15}	$\mathbb{Z}/3\mathbb{Z}\ltimes(\{g\in\mathrm{GL}_1(\mathbb{F}_{q^6}) \mid \mathrm{N}_{6/3}(g)\in\mathrm{GL}_1(\mathbb{F}_{q^2})\})$	$3(q^6-1)/(q^2+q+1)$

First we consider $\mathrm{Stab}(x_4)$. Let $g=(g_1,g_2) = ((g_{ij})_{1\le i,j\le 2}, (h_{ij})_{1\le i,j\le 3})\in\mathrm{Stab}(x_4)$. Comparing the rank of two entries, we see that $g_{12}=0$. Therefore we have $g_{22}g_2\overline{g_2}^T = I_3$. Thus we obtain $\mathrm{Stab}(x_4)\cong(\mathrm{GL}_1(\mathbb{F}_q)\times\mathrm{GU}_3(\mathbb{F}_{q^2}))\ltimes\mathbb{F}_q$. Next we consider $\mathrm{Stab}(x_8)$. Let $g=(g_1,g_2) = ((g_{ij})^{-1}_{1\le i,j\le 2}, (h_{ij})_{1\le i,j\le 3})\in\mathrm{Stab}(x_8)$. We have $h_{13}=h_{23}=0$, $\overline{h_{33}}\begin{bmatrix}h_{11} & h_{21}\\ h_{12} & h_{22}\end{bmatrix} = \begin{bmatrix}g_{11} & g_{12}\\ g_{21} & g_{22}\end{bmatrix}$ and $\mathrm{Tr}_2(h_{33}\overline{h_{31}}) = \mathrm{Tr}_2(h_{33}\overline{h_{32}}) = 0$. Thus we obtain $\mathrm{Stab}(x_8)\cong(\mathrm{GL}_2(\mathbb{F}_q)\times\mathrm{GL}_1(\mathbb{F}_{q^2}))\ltimes\mathbb{F}_q^2$. Next we consider $\mathrm{Stab}(x_9)$. Let $g=(g_1,g_2) = ((g_{ij})^{-1}_{1\le i,j\le 2}, (h_{ij})_{1\le i,j\le 3})\in\mathrm{Stab}(x_9)$. We have $g_{12}=h_{12}=h_{13}=h_{23}=0$, $\mathrm{N}_2(h_{33}) = g_{11}$, $\mathrm{N}_2(h_{22}) = h_{33}\overline{h_{11}} = g_{22}$, $h_{32}\overline{h_{22}}+h_{33}\overline{h_{21}} = 0$ and $\mathrm{Tr}_2(h_{31}\overline{h_{33}})+\mathrm{N}_2(h_{32}) = g_{21}$. Thus we obtain $\mathrm{Stab}(x_9)\cong(\mathrm{GL}_1(\mathbb{F}_{q^2}))^2\ltimes\mathbb{F}_{q^2}^2$. Next we consider $\mathrm{Stab}(x_{10})$. Let $g=(g_1,g_2) = ((g_{ij})^{-1}_{1\le i,j\le 2}, (h_{ij})_{1\le i,j\le 3})\in\mathrm{Stab}(x_{10})$. We have $g_{12}=h_{12}=h_{13}=h_{23}=0$, $h_{33}\overline{h_{22}} = g_{11}$, $\mathrm{N}_2(h_{22}) = h_{33}\overline{h_{11}} = g_{22}$, $h_{32}\overline{h_{22}}+h_{33}\overline{h_{21}} = g_{21}$ and $\mathrm{Tr}_2(h_{31}\overline{h_{33}})+\mathrm{N}_2(h_{32}) = 0$. Thus we obtain $\mathrm{Stab}(x_{10})\cong(\mathrm{GL}_1(\mathbb{F}_q)\times\mathrm{GL}_1(\mathbb{F}_{q^2}))\ltimes\mathbb{F}_q^3$. Next we consider $\mathrm{Stab}(x_{11})$. Let $g=(g_1,g_2) = ((g_{ij})_{1\le i,j\le 2}, (h_{ij})_{1\le i,j\le 3})\in\mathrm{Stab}(x_{11})$. We have $g_{12}=g_{21}=h_{13}=h_{23}=h_{31}=h_{32}=0$, $g_{11}\begin{bmatrix}h_{11} & h_{12}\\ h_{21} & h_{22}\end{bmatrix}\begin{bmatrix}\overline{h_{11}} & \overline{h_{21}}\\ \overline{h_{12}} & \overline{h_{22}}\end{bmatrix} = I_2$ and $g_{22}\mathrm{N}_2(h_{33}) = 1$. Thus we obtain $\mathrm{Stab}(x_{11})\cong\mathrm{GU}_2(\mathbb{F}_{q^2})\times\mathrm{GL}_1(\mathbb{F}_{q^2})$. Next we consider $\mathrm{Stab}(x_{12})$. Let $g=(g_1,g_2) = ((g_{ij})^{-1}_{1\le i,j\le 2}, (h_{ij})_{1\le i,j\le 3})\in\mathrm{Stab}(x_{12})$. We have $g_{12}=g_{21}=h_{12}=h_{13}=h_{21}=h_{31}=h_{32}=0$,

$\mathrm{N}_2(h_{11}) = \mathrm{N}_2(h_{22}) = g_{11}$, $h_{33}\overline{h_{22}} = g_{22}$ and $\mathrm{Tr}_2(h_{22}\overline{h_{23}}) = 0$. Thus we obtain $\mathrm{Stab}(x_{12}) \cong (\mathrm{GL}_1(\mathbb{F}_q) \times \mathrm{GL}_1(\mathbb{F}_{q^2}) \times \mathrm{Ker}(\mathrm{N}_2|_{\mathbb{F}_{q^2}\setminus\{0\}})) \ltimes \mathbb{F}_q$. For $i = 13, 14, 15$, the structure of $\mathrm{Stab}_i(x_i)$ is determined by Kable and Yukie [2, Proposition (4.3), (4.8)]. Here, $\mathrm{N}_{6/3} : \mathbb{F}_{q^6} \to \mathbb{F}_{q^3}$ is the norm map. (It is assumed that V is defined over an infinite field in [2], but the method to determine the structures of $\mathrm{Stab}(x_{13})$, $\mathrm{Stab}(x_{14})$, and $\mathrm{Stab}(x_{15})$ holds for the $\mathbb{F}_{q^2}$.)

Lastly, since $\sum_{i=1}^{15} |\mathcal{O}_i| = q^{18} = |V|$, we have $\bigcup_{i=1}^{15} \mathcal{O}_i = V$. □

11.2. **The intersection between the orbits and the subspaces.** The subspaces we choose to calculate the Fourier transform are as follows:

$$W_1 = (\begin{bmatrix}0&0&0\\0&0&0\\0&0&0\end{bmatrix}, \begin{bmatrix}0&0&0\\0&0&0\\0&0&0\end{bmatrix}), W_2 = (\begin{bmatrix}0&0&0\\0&0&0\\0&0&*\end{bmatrix}, \begin{bmatrix}0&0&0\\0&0&0\\0&0&*\end{bmatrix}), W_3 = (\begin{bmatrix}0&0&0\\0&0&0\\0&0&0\end{bmatrix}, \begin{bmatrix}0&0&*\\0&0&*\\*&*&*\end{bmatrix}),$$

$$W_4 = (\begin{bmatrix}0&0&0\\0&0&0\\0&0&0\end{bmatrix}, \begin{bmatrix}*&*&*\\*&*&*\\*&*&*\end{bmatrix}), W_5 = (\begin{bmatrix}0&0&0\\0&0&0\\0&0&*\end{bmatrix}, \begin{bmatrix}0&0&*\\0&0&*\\*&*&*\end{bmatrix}), W_6 = (\begin{bmatrix}0&0&0\\0&*&*\\0&*&*\end{bmatrix}, \begin{bmatrix}0&0&0\\0&*&*\\0&*&*\end{bmatrix}),$$

$$W_7 = (\begin{bmatrix}0&0&0\\0&0&*\\0&*&*\end{bmatrix}, \begin{bmatrix}0&0&0\\0&0&*\\0&*&*\end{bmatrix}), W_8 = (\begin{bmatrix}0&0&*\\0&0&*\\*&*&*\end{bmatrix}, \begin{bmatrix}0&0&*\\0&0&*\\*&*&*\end{bmatrix}), W_9 = (\begin{bmatrix}0&0&0\\0&0&0\\0&0&*\end{bmatrix}, \begin{bmatrix}0&0&*\\0&*&*\\*&*&*\end{bmatrix}),$$

$$W_{10} = (\begin{bmatrix}0&0&0\\0&0&*\\0&*&*\end{bmatrix}, \begin{bmatrix}0&0&*\\0&*&*\\*&*&*\end{bmatrix}), W_{11} = (\begin{bmatrix}0&0&0\\0&0&0\\0&0&*\end{bmatrix}, \begin{bmatrix}0&*&*\\*&*&*\\*&*&*\end{bmatrix}), W_{12} = (\begin{bmatrix}0&0&0\\0&0&*\\0&*&*\end{bmatrix}, \begin{bmatrix}0&*&*\\*&*&*\\*&*&*\end{bmatrix}),$$

$$W_{13} = (\begin{bmatrix}0&0&0\\0&*&*\\0&*&*\end{bmatrix}, \begin{bmatrix}*&*&*\\*&*&*\\*&*&*\end{bmatrix}), W_{14} = (\begin{bmatrix}0&0&*\\0&*&*\\*&*&*\end{bmatrix}, \begin{bmatrix}0&0&*\\0&*&*\\*&*&*\end{bmatrix}) \text{ and } W_{15} = V.$$

The orthogonal complements of these subspaces are as follows:

$$W_1^{\perp} = W_{15}, W_2^{\perp} = (\begin{bmatrix}0&*&*\\*&*&*\\*&*&*\end{bmatrix}, \begin{bmatrix}0&*&*\\*&*&*\\*&*&*\end{bmatrix}), W_3^{\perp} = W_{13}, W_4^{\perp} = W_4, W_5^{\perp} = (\begin{bmatrix}0&0&0\\0&*&*\\0&*&*\end{bmatrix}, \begin{bmatrix}0&*&*\\*&*&*\\*&*&*\end{bmatrix}),$$

$W_7^{\perp} = W_{14}$, $W_6^{\perp} = W_8$, $W_9^{\perp} = W_{12}$, $W_{10}^{\perp} = W_{10}$ and $W_{11}^{\perp} = W_{11}$.

Proposition 11.2. *The cardinalities $|\mathcal{O}_i \cap W_j|$ for the orbit $\mathcal{O}_i$ and the subspace W_j are given as follows:*

	W_1	W_2	W_3	W_4	W_5	W_6	W_7	W_8	W_9
$\mathcal{O}_1$	1	1	1	1	1	1	1	1	1
$\mathcal{O}_2$	0	[101000]	[100000]	[100101]	[101000]	[101010]	[101000]	[101000]	[100001]
$\mathcal{O}_3$	0	0	[111010]	[110111]	[111010]	[111010]	[112000]	[112010]	$[110000]c_1$
$\mathcal{O}_4$	0	0	0	[230011]	0	0	0	0	[231000]
$\mathcal{O}_5$	0	0	0	0	[211010]	[212010]	[212000]	[212010]	[211010]
$\mathcal{O}_6$	0	0	0	0	0	$\frac{1}{2}[231010]$	0	0	[230000]
$\mathcal{O}_7$	0	0	0	0	0	$\frac{1}{2}[231010]$	[231000]	[231010]	0
$\mathcal{O}_8$	0	0	0	0	0	0	0	[242010]	0
$\mathcal{O}_9$	0	0	0	0	0	0	0	0	[331000]
$\mathcal{O}_{10}$	0	0	0	0	0	0	0	0	0
$\mathcal{O}_{11}$	0	0	0	0	0	0	0	0	0
$\mathcal{O}_{12}$	0	0	0	0	0	0	0	0	0
$\mathcal{O}_{13}$	0	0	0	0	0	0	0	0	0
$\mathcal{O}_{14}$	0	0	0	0	0	0	0	0	0
$\mathcal{O}_{15}$	0	0	0	0	0	0	0	0	0

W_{10}	W_{11}	W_{12}	W_{13}	W_{14}	W_{15}	$W_2^{\perp}$	$W_5^{\perp}$
1	1	1	1	1	1	1	1
[100001]	[100001]	[10001]	$[100000]d_1$	[101010]	[101101]	[101010]	[101010]
$[110000]c_2$	$[110010]c_3$	$[110000]f_1$	$[110010]d_2$	$[111000]c_1$	[111111]	$[111010]c_3$	[111020]
[231000]	[231010]	[231010]	[230011]	[232000]	[231011]	[232010]	[231010]
$[211000]b_2$	[211010]	$[211000]b_2$	[211011]	$[212000]b_3$	[212111]	[212020]	[211011]
$\frac{1}{2}[230010]$	[230000]	$\frac{1}{2}[230010]$	$\frac{1}{2}[230010]b_2$	$\frac{1}{2}[231010]$	$\frac{1}{2}[231111]$	$\frac{1}{2}[231010]$	$\frac{1}{2}[231010]$
$\frac{1}{2}[232000]$	0	$\frac{1}{2}[232000]$	$\frac{1}{2}[231010]$	$\frac{1}{2}[231000]b_4$	$\frac{1}{2}[231111]$	$\frac{1}{2}[231010]b_3$	$\frac{1}{2}[231010]$
[242000]	0	[243000]	[242010]	[242010]	[242111]	$[242010]b_3$	[242010]
[331000]	[331000]	[331000]	[331020]	[332000]	[332111]	[332010]	[331010]
[342000]	0	[343000]	[342010]	[343000]	[343111]	[343020]	[342010]
0	[261000]	[261000]	$[260010]b_1$	[262000]	[261111]	[262010]	[261010]
0	0	[362000]	2[361010]	[362000]	[362111]	[362011]	[361010]
0	0	0	$\frac{1}{2}[470010]$	0	$\frac{1}{6}[471111]$	$\frac{1}{6}[473010]$	0
0	0	0	$\frac{1}{2}[371010]$	[372000]	$\frac{1}{2}[372111]$	$\frac{1}{2}[372020]$	0
0	0	0	0	0	$\frac{1}{3}[473011]$	$\frac{1}{3}[473010]$	0

Here, we put $[abcdef] = (q-1)^a q^b (q+1)^c (q^2-q+1)^d (q^2+1)^e (q^2+q+1)^f$ and

$$
\begin{array}{lll}
b_1 = q^2+2 & c_1 = q^3+2q^2+1 & d_1 = q^4+q^3+q^2+q+1 \\
b_2 = 2q^2+q+1 & c_2 = q^3+3q^2+q+1 & d_2 = q^4+q^2+q+1 \\
b_3 = 2q^2+1 & c_3 = q^3+q^2+1 & f_1 = q^5+q^4+q^3+3q^2+q+1 \\
b_4 = 3q^2+1 & &
\end{array}
$$

[Proof]

We obviously have $|\mathcal{O}_1 \cap W_j| = 1$ for all j. For $j = 1,2,6,7$, we obtain the cardinalities $|\mathcal{O}_i \cap W_j|$ for $1 \le i \le 15$ from Proposition 7.2. For $j = 3,4$, we easily obtain the cardinalities. For $j = 15$, we already calculated the cardinalities in Proposition 11.1. We calculate the rest cardinalities. For $1 \le i,j \le 15$, let $G(i,j) = \{g \in G \mid gx_i \in W_j\}$ and $G(i,j^{\perp}) = \{g \in G \mid gx_i \in W_j^{\perp}\}$ as the proof of Proposition 10.2. Here, for vectors v and w over $\mathbb{F}_{q^2}$, $v /\!/_{\mathbb{F}_q} w$ means that v and w are parallel over $\mathbb{F}_q$, and $v \not\parallel_{\mathbb{F}_q} w$ means that v and w are not parallel over $\mathbb{F}_q$. To calculate cardinalities for some sets, we use the result in Section 7. We refer to $\mathcal{O}_i$, W_j, $W_j^{\perp}$ and M in Section 7 as $\mathcal{O}_i^{2H2}$, W_j^{2H2}, ${W_j^{\perp}}^{2H2}$ and M^{2H2}, respectively.

We consider W_5. We write an element $x \in W_5$ as $x = (\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & a_{33} \end{bmatrix}, \begin{bmatrix} 0 & 0 & b_{13} \\ 0 & 0 & b_{23} \\ \overline{b_{13}} & \overline{b_{23}} & b_{33} \end{bmatrix})$. Let $W_5^0 = (\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & * \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & * \\ 0 & * & * \end{bmatrix})$ and $W_5^1 = \{x \in W_5 \mid b_{13} \neq 0\}$. We have $|\mathcal{O}_i \cap W_5| = |\mathcal{O}_i \cap W_5^0| + |\mathcal{O}_i \cap W_5^1|$. We already counted the cardinalities $|\mathcal{O}_i \cap W_5^0|$ in the proof of Proposition 7.2; $|\mathcal{O}_i^{2H2} \cap W_4^{2H2}| = |\mathcal{O}_i \cap W_5^0|$ for $1 \leq i \leq 3$ and $|\mathcal{O}_i^{2H2} \cap W_4^{2H2}| = |\mathcal{O}_{i+1} \cap W_5^0|$ for $4 \leq i \leq 6$. Thus we count $|\mathcal{O}_i \cap W_5^1|$. If $a_{33} = 0$, we have $x \in \mathcal{O}_3$. If $a_{33} \neq 0$, we have $x \in \mathcal{O}_5$. Thus we obtain $|\mathcal{O}_3 \cap W_5^1| = q^3(q^2-1)$ and $|\mathcal{O}_5 \cap W_5^1| = q^3(q^2-1)(q-1)$.

Next we consider W_8. We write an element $x \in W_8$ as $x = (\begin{bmatrix} 0 & 0 & a_{13} \\ 0 & 0 & a_{23} \\ \overline{a_{13}} & \overline{a_{23}} & a_{33} \end{bmatrix}, \begin{bmatrix} 0 & 0 & b_{13} \\ 0 & 0 & b_{23} \\ \overline{b_{13}} & \overline{b_{23}} & b_{33} \end{bmatrix})$. If $\begin{bmatrix} a_{13} & a_{23} \\ b_{13} & b_{23} \end{bmatrix} = 0$ and $\begin{bmatrix} a_{33} \\ b_{33} \end{bmatrix} \neq 0$, we have $x \in \mathcal{O}_2$. If $\begin{bmatrix} a_{13} & a_{23} \\ b_{13} & b_{23} \end{bmatrix} = 1$ and $\mathrm{rank}(\begin{bmatrix} a_{13} & a_{23} & a_{33} \\ b_{13} & b_{23} & b_{33} \end{bmatrix}) = 1$, we have $x \in \mathcal{O}_3$. If $\begin{bmatrix} a_{13} & a_{23} \\ b_{13} & b_{23} \end{bmatrix} = 1$, $\mathrm{rank}(\begin{bmatrix} a_{13} & a_{23} & a_{33} \\ b_{13} & b_{23} & b_{33} \end{bmatrix}) = 2$ and $\begin{bmatrix} a_{13} \\ a_{23} \end{bmatrix} /\!/_{\mathbb{F}_q} \begin{bmatrix} b_{13} \\ b_{23} \end{bmatrix}$, then we have $x \in \mathcal{O}_5$. If $\begin{bmatrix} a_{13} & a_{23} \\ b_{13} & b_{23} \end{bmatrix} = 1$, $\mathrm{rank}(\begin{bmatrix} a_{13} & a_{23} & a_{33} \\ b_{13} & b_{23} & b_{33} \end{bmatrix}) = 2$ and $\begin{bmatrix} a_{13} \\ a_{23} \end{bmatrix} \not/\!/_{\mathbb{F}_q} \begin{bmatrix} b_{13} \\ b_{23} \end{bmatrix}$, then we have $x \in \mathcal{O}_7$. If $\begin{bmatrix} a_{13} & a_{23} \\ b_{13} & b_{23} \end{bmatrix} = 2$, we have $x \in \mathcal{O}_8$. Thus we obtain $|\mathcal{O}_2 \cap W_8| = q^2 - 1$, $|\mathcal{O}_3 \cap W_8| = q(q^4-1)(q+1)$, $|\mathcal{O}_5 \cap W_8| = (q^4-1)(q+1)(q^2-q)$, $|\mathcal{O}_7 \cap W_8| = q^2(q^4-1)(q^2-q)$ and $|\mathcal{O}_8 \cap W_8| = q^2(q^4-1)(q^4-q^2)$.

Next we consider W_9. We write an element $x \in W_9$ as $x = (\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & a_{33} \end{bmatrix}, \begin{bmatrix} 0 & 0 & b_{13} \\ 0 & b_{22} & b_{23} \\ \overline{b_{13}} & \overline{b_{23}} & b_{33} \end{bmatrix})$. Let $W_9^1 = \{x \in W_9 \mid b_{22} \neq 0\}$. We have $|\mathcal{O}_i \cap W_9| = |\mathcal{O}_i \cap W_5| + |\mathcal{O}_i \cap W_9^1|$. Thus we count $|\mathcal{O}_i \cap W_9^1|$. Let $x \in W_9^1$. If $a_{33} = b_{31} = 0$ and $\mathrm{rank}(\begin{bmatrix} b_{22} & b_{23} \\ \overline{b_{23}} & b_{33} \end{bmatrix}) = 1$, we have $x \in \mathcal{O}_2$. If $a_{33} = b_{31} = 0$ and $\mathrm{rank}(\begin{bmatrix} b_{22} & b_{23} \\ \overline{b_{23}} & b_{33} \end{bmatrix}) = 2$, we have $x \in \mathcal{O}_3$. If $a_{33} = 0$ and $b_{31} \neq 0$, we have $x \in \mathcal{O}_4$. If $a_{33} \neq 0$ and $b_{31} = 0$, we have $x \in \mathcal{O}_6$. If $a_{33}b_{31} \neq 0$, we have $x \in \mathcal{O}_9$. Thus we obtain $|\mathcal{O}_2 \cap W_9^1| = q^2(q-1)$, $|\mathcal{O}_3 \cap W_9^1| = q^2(q-1)^2$, $|\mathcal{O}_4 \cap W_9^1| = q^3(q-1)(q^2-1)$, $|\mathcal{O}_6 \cap W_9^1| = q^3(q-1)^2$ and $|\mathcal{O}_9 \cap W_9^1| = q^3(q^2-1)(q-1)^2$.

Next we consider W_{10}. We write an element $x \in W_{10}$ as $x = (\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & a_{23} \\ 0 & \overline{a_{23}} & a_{33} \end{bmatrix}, \begin{bmatrix} 0 & 0 & b_{13} \\ 0 & b_{22} & b_{23} \\ \overline{b_{13}} & \overline{b_{23}} & b_{33} \end{bmatrix})$. Let $W_{10}^0 = (\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & * \\ 0 & * & * \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 \\ 0 & * & * \\ 0 & * & * \end{bmatrix})$ and $W_{10}^1 = \{x \in W_{10} \mid b_{13} \neq 0\}$. We have $|\mathcal{O}_i \cap W_{10}| = |\mathcal{O}_i \cap W_{10}^0| + |\mathcal{O}_i \cap W_{10}^1|$. To calculate $|\mathcal{O}_i \cap W_{10}^0|$, we use the Fourier transform for $2 \otimes \mathrm{H}_2(\mathbb{F}_{q^2})$. Let $W = (\begin{bmatrix} 0 & * \\ * & * \end{bmatrix}, \begin{bmatrix} * & * \\ * & * \end{bmatrix}) \in 2 \otimes \mathrm{H}_2(\mathbb{F}_{q^2})$. The orthogonal complement of W is $W^\perp = W_2^{2H2}$. We have $|\mathcal{O}_i \cap W_{10}^0| = |c\mathcal{O}_i^{2H2} \cap W|$ for $i = 1, 2, 3$ and $|\mathcal{O}_i \cap W_{10}^0| = |\mathcal{O}_{i-1}^{2H2} \cap W|$ for $i = 5, 6, 7$. By Proposition 2.2, we have

$$\begin{bmatrix} |\mathcal{O}_1^{2H2} \cap W| \\ \vdots \\ |\mathcal{O}_6^{2H2} \cap W| \end{bmatrix} = \frac{|V|}{|W_2^{2H2}|} \begin{bmatrix} |\mathcal{O}_1^{2H2}| & & 0 \\ & \ddots & \\ 0 & & |\mathcal{O}_6^{2H2}| \end{bmatrix} M^{2H2} \begin{bmatrix} |\mathcal{O}_1^{2H2}| & & 0 \\ & \ddots & \\ 0 & & |\mathcal{O}_6^{2H2}| \end{bmatrix}^{-1} \begin{bmatrix} |\mathcal{O}_1^{2H2} \cap W_2^{2H2}| \\ \vdots \\ |\mathcal{O}_6^{2H2} \cap W_2^{2H2}| \end{bmatrix}.$$

The matrix M^{2H2} is explicitly determined in Theorem 7.3. We have $\begin{bmatrix} |\mathcal{O}_1^{2H2} \cap W_2^{2H2}| & \cdots & |\mathcal{O}_6^{2H2} \cap W_2^{2H2}| \end{bmatrix}^T = \begin{bmatrix} 1 & (q-1) & 0 & 0 & 0 & 0 \end{bmatrix}^T$. Thus we obtain the cardinalities $|\mathcal{O}_i \cap W_{10}^0|$; $|\mathcal{O}_2 \cap W_{10}^0| = q^3 - 1$, $|\mathcal{O}_3 \cap W_{10}^0| = 2q^4 - q^3 - q$, $|\mathcal{O}_5 \cap W_{10}^0| = q^6 - q^4 - q^3 + q$, $|\mathcal{O}_6 \cap W_{10}^0| = \frac{1}{2}q^7 - q^6 + q^5 - q^4 + \frac{1}{2}q^3$, $|\mathcal{O}_7 \cap W_{10}^0| = \frac{1}{2}q^3(q^2-1)^2$. Next we calculate $|\mathcal{O}_i \cap W_{10}^1|$. We write $x \in W_{10}^1$ as $x = (\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & a_{23} \\ 0 & \overline{a_{23}} & a_{33} \end{bmatrix}, \begin{bmatrix} 0 & 0 & b_{13} \\ 0 & b_{22} & b_{23} \\ \overline{b_{13}} & \overline{b_{23}} & b_{33} \end{bmatrix})$.

If $b_{22}=a_{23}=a_{33}=0$, we have $x\in\mathcal{O}_3$. If $b_{22}=a_{23}=0$ and $a_{33}\neq 0$, we have $x\in\mathcal{O}_5$. If $b_{22}=0$ and $a_{23}\neq 0$, we have $x\in\mathcal{O}_8$. If $b_{22}\neq 0$ and $a_{23}=a_{33}=0$, we have $x\in\mathcal{O}_4$. If $b_{22}a_{33}\neq 0$ and $a_{23}=0$, we have $x\in\mathcal{O}_9$. If $b_{22}a_{23}\neq 0$, we have $x\in\mathcal{O}_{10}$. Thus we obtain $|\mathcal{O}_3\cap W_{10}^1|=q^3(q^2-1)$, $|\mathcal{O}_5\cap W_{10}^1|=q^3(q-1)(q^2-1)$, $|\mathcal{O}_8\cap W_{10}^1|=q^4(q^2-1)^2$, $|\mathcal{O}_4\cap W_{10}^1|=q^3(q-1)(q^2-1)$, $|\mathcal{O}_9\cap W_{10}^1|=q^3(q-1)^2(q^2-1)$, $|\mathcal{O}_{10}\cap W_{10}^1|=q^4(q-1)(q^2-1)^2$.

Next we consider W_{11}. We write an element $x\in W_{11}$ as $x=(A,B)=(\begin{bmatrix}0&0&0\\0&0&0\\0&0&a_{33}\end{bmatrix},\begin{bmatrix}0&b_{12}&b_{13}\\\overline{b_{12}}&b_{22}&b_{23}\\\overline{b_{13}}&\overline{b_{23}}&b_{33}\end{bmatrix})$. Let $W_{11}^0=(\begin{bmatrix}0&0&0\\0&0&0\\0&0&0\end{bmatrix},\begin{bmatrix}0&*&*\\*&*&*\\*&*&*\end{bmatrix})$, $W_{11}^1=(\begin{bmatrix}0&0&0\\0&0&0\\0&0&0\end{bmatrix},\begin{bmatrix}0&0&*\\0&*&*\\*&*&*\end{bmatrix})$ and $W_{11}^2=\{x\in W_{11}\mid a_{33}b_{12}\neq 0\}$. We have $|\mathcal{O}_i\cap W_{11}|=|\mathcal{O}_i\cap W_{11}^0|+|\mathcal{O}_i\cap W_9|-|\mathcal{O}_i\cap W_{11}^1|+|\mathcal{O}_i\cap W_{11}^2|$. First, we count $|\mathcal{O}_i\cap W_{11}^0|$. For $2\leq i\leq 4$, let $t_{i-1}=\begin{bmatrix}I_{i-1}&O_{i-1,4-i}\\O_{4-i,i-1}&O_{4-i,4-i}\end{bmatrix}$. Let

$$W_{11}^3:=\left\{g\in\mathrm{GL}_3(\mathbb{F}_{q^2})\;\middle|\;gt_{i-1}\overline{g}^T\in\begin{bmatrix}0&*&*\\*&*&*\\*&*&*\end{bmatrix}\subset\mathrm{H}_3(\mathbb{F}_{q^2})\right\}.$$

For $2\leq i\leq 4$, we have

$$\begin{aligned}|\mathcal{O}_i\cap W_{11}^0|=&\ |\{A=[a_{ij}]_{1\leq i,j\leq 3}\in\mathrm{H}_3(\mathbb{F}_{q^2})\mid a_{11}=0,\mathrm{rank}(A)=i-1\}|\\=&\ \frac{|W_{11}^3|}{|\{g\in\mathrm{GL}_3(\mathbb{F}_{q^2})\mid gt_{i-1}\overline{g}^T=t_{i-1}\}|}\\=&\ \frac{(q^6-q^2)(q^6-q^4)\cdot|\{v=\begin{bmatrix}g_{11}&g_{12}&g_{13}\end{bmatrix}\in\mathbb{F}_{q^2}^3\mid v\neq 0,vt_{i-1}\overline{v}^T=0\}|}{q^{2(i-1)(4-i)}\cdot\mathrm{gl}_{4-i}(\mathbb{F}_{q^2})\cdot|\mathrm{U}_{i-1}(\mathbb{F}_{q^2})|}.\end{aligned}$$

For $j\in\mathbb{Z}_{\geq 1}$, let

$$X_j=|\{v=\begin{bmatrix}z_1&\cdots&z_j\end{bmatrix}\in\mathbb{F}_{q^2}^j\mid\mathrm{N}_2(z_1)+\cdots+\mathrm{N}_2(z_j)=0\}|.$$

Then we have

$$|\mathcal{O}_i\cap W_{11}^0|=\frac{(q^6-q^2)(q^6-q^4)(q^{8-2i}\cdot X_{i-1}-1)}{q^{2(i-1)(4-i)}\cdot\mathrm{gl}_{4-i}(\mathbb{F}_{q^2})\cdot|\mathrm{U}_{i-1}(\mathbb{F}_{q^2})|}.$$

Thus we calculate X_j. Let

$$Y_j=|\{v=\begin{bmatrix}z_1&\cdots&z_j\end{bmatrix}\in\mathbb{F}_{q^2}^j\mid\mathrm{N}_2(z_1)+\cdots+\mathrm{N}_2(z_j)\neq 0\}|.$$

X_j and Y_j satisfy the following recurrence relations:

$$\begin{aligned}X_1&=1,\\Y_1&=q^2-1,\\X_{k+1}&=X_k+(q+1)Y_k,\\Y_{k+1}&=(q^2-1)X_k+(q^2-q-1)Y_k.\end{aligned}$$

By solving these equations, we obtain

$$\begin{bmatrix}X_j\\Y_j\end{bmatrix}=\begin{bmatrix}q^{2j-1}-(-q)^{j-1}(q-1)\\(q-1)q^{2j-1}+(-q)^{j-1}(q-1)\end{bmatrix}.$$

Therefore we obtain $|\mathcal{O}_2\cap W_{11}^0|=q^3-q^2+q-1$, $|\mathcal{O}_3\cap W_{11}^0|=q^7+q^4-2q^3+q^2-q$, $|\mathcal{O}_4\cap W_{11}^0|=q^8-q^7-q^4+q^3$. Next we count $|\mathcal{O}_i\cap W_{11}^1|$. We write an element $x\in W_{11}^1$ as $x=(A,B)=(\begin{bmatrix}0&0&0\\0&0&0\\0&0&0\end{bmatrix},\begin{bmatrix}0&0&b_{13}\\0&b_{22}&b_{23}\\\overline{b_{13}}&\overline{b_{23}}&b_{33}\end{bmatrix})$. If $b_{13}=0$ and $\mathrm{rank}(\begin{bmatrix}b_{22}&b_{23}\\\overline{b_{23}}&b_{33}\end{bmatrix})=1$, then $x\in\mathcal{O}_2$. If $b_{13}=0$ and $\mathrm{rank}(\begin{bmatrix}b_{22}&b_{23}\\\overline{b_{23}}&b_{33}\end{bmatrix})=2$, then $x\in\mathcal{O}_3$. If $b_{13}\neq 0$ and $b_{22}=0$, then $x\in\mathcal{O}_3$. If $b_{13}b_{22}\neq 0$, then $x\in\mathcal{O}_4$. Thus we obtain $|\mathcal{O}_2\cap W_{11}^1|=|\mathrm{H}(2,1)|$, $|\mathcal{O}_3\cap W_{11}^1|=|\mathrm{H}(2,2)|+q^3(q^2-1)$, $|\mathcal{O}_4\cap W_{11}^1|=q^3(q-1)(q^2-1)$. Lastly, we easily see that if $x\in W_{11}^2$, then $x\in\mathcal{O}_{11}$. Thus we obtain $|\mathcal{O}_{11}\cap W_{11}^2|=q^6(q-1)(q^2-1)$.

Next we consider W_{12}. We write an element $x \in W_{12}$ as $x = (\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & a_{23} \\ 0 & \overline{a_{23}} & a_{33} \end{bmatrix}, \begin{bmatrix} 0 & b_{12} & b_{13} \\ \overline{b_{12}} & b_{22} & b_{23} \\ \overline{b_{13}} & \overline{b_{23}} & b_{33} \end{bmatrix})$. For $2 \le i \le 4$, we have

$$|\mathcal{O}_i \cap W_{12}| = |\mathcal{O}_i \cap W_{11}^0| + \left| \left\{ x = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & a \\ 0 & \overline{a} & b \end{bmatrix} \in \mathrm{H}_3(\mathbb{F}_{q^2}) \,\middle|\, \mathrm{rank}(x) = i-1 \right\} \right|.$$

Thus we easily see that $|\mathcal{O}_2 \cap W_{12}| = q^3 - q^2 + 2q - 2$, $|\mathcal{O}_3 \cap W_{12}| = q^7 + q^4 - q^3 + q^2 - 2q$, $|\mathcal{O}_4 \cap W_{12}| = q^8 - q^7 - q^4 + q^3$. Next we count $|G(i, 12)|$ for $5 \le i \le 12$. Let $g = (g_1, g_2) = ((g_{ij})_{1 \le i,j \le 2}, (h_{ij})_{1 \le i,j \le 3}) \in G$ and we consider when $g \in G(i, 12)$. $g \cdot (A, B) \in W_{12}$ holds if and only if

$$\begin{cases} \begin{bmatrix} \begin{bmatrix} h_{11} & h_{12} & h_{13} \end{bmatrix} A \begin{bmatrix} \overline{h_{11}} & \overline{h_{12}} & \overline{h_{13}} \end{bmatrix}^T \\ \begin{bmatrix} h_{11} & h_{12} & h_{13} \end{bmatrix} B \begin{bmatrix} \overline{h_{11}} & \overline{h_{12}} & \overline{h_{13}} \end{bmatrix}^T \end{bmatrix} & = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \\ \begin{bmatrix} h_{11} & h_{12} & h_{13} \end{bmatrix} (g_{11}A + g_{12}B) & = \begin{bmatrix} 0 & 0 & 0 \end{bmatrix}, \\ \begin{bmatrix} h_{21} & h_{22} & h_{23} \end{bmatrix} (g_{11}A + g_{12}B) \begin{bmatrix} \overline{h_{21}} & \overline{h_{22}} & \overline{h_{23}} \end{bmatrix}^T & = 0. \end{cases}$$

Let us count $|G(5, 12)|$. Let $g = (g_1, g_2) = ((g_{ij})_{1 \le i,j \le 2}, (h_{ij})_{1 \le i,j \le 3}) \in G(5, 12)$. We have

$$\begin{cases} \begin{bmatrix} \mathrm{Tr}_2(h_{12}\overline{h_{13}}) & \mathrm{N}_2(h_{13}) \end{bmatrix} & = \begin{bmatrix} 0 & 0 \end{bmatrix}, \\ \begin{bmatrix} 0 & g_{11}h_{13} & g_{11}h_{12} + g_{12}h_{13} \end{bmatrix} & = \begin{bmatrix} 0 & 0 & 0 \end{bmatrix}, \\ g_{11}\mathrm{Tr}_2(h_{22}\overline{h_{23}}) + g_{12}\mathrm{N}_2(h_{23}) & = 0. \end{cases}$$

It follows that $h_{13} = g_{11}h_{23}\overline{h_{12}} = g_{11}h_{33}\overline{h_{22}} = 0$, $g_{11}\mathrm{Tr}_2(h_{22}\overline{h_{23}}) + g_{12}\mathrm{N}_2(h_{23}) = 0$. If $g_{11} = 0$, then $h_{23} = 0$. Therefore $|\{((g_{ij})_{1 \le i,j \le 2}, (h_{ij})_{1 \le i,j \le 3}) \in G(5, 12) \mid g_{11} = 0\}| = q(q-1)^2 \cdot (q^2-1)(q^6-q^2)(q^6-q^4)$. If $g_{11} \ne 0$ and $h_{23} = 0$, then $h_{12} = 0$ and $h_{33} \ne 0$. Therefore $|\{((g_{ij})_{1 \le i,j \le 2}, (h_{ij})_{1 \le i,j \le 3}) \in G(5, 12) \mid g_{11} \ne 0, h_{23} = 0\}| = q^2(q-1)^2 \cdot (q^2-1)^3 q^6$. If $g_{11}h_{23} \ne 0$, then $h_{12} = 0$ and $g_{12} = -g_{11}\frac{\mathrm{Tr}_2(h_{22}\overline{h_{23}})}{\mathrm{N}_2(h_{23})}$. Therefore $|\{((g_{ij})_{1 \le i,j \le 2}, (h_{ij})_{1 \le i,j \le 3}) \in G(5, 12) \mid g_{11}h_{23} \ne 0\}| = (q^2-1)^2 q^4 (q^6-q^4) \cdot q(q-1)^2$. Thus we obtain $|G(5, 12)| = q^7(q-1)^5(q+1)^3(2q^2+q+1)$. The counts of the cardinalities $|G(i, 12)|$ for $6 \le i \le 12$ are carried out in the same way, and we omit the detail.

Next we consider W_{13}. We easily see that $|\mathcal{O}_2 \cap W_{13}| = q \cdot |\mathrm{H}(2, 1)| + |\mathrm{H}(3, 1)|$, $|\mathcal{O}_3 \cap W_{13}| = q \cdot |\mathrm{H}(2, 2)| + |\mathrm{H}(3, 2)|$, and $|\mathcal{O}_4 \cap W_{13}| = |\mathrm{H}(3, 3)|$. Next we count $|G(i, 13)|$ for $5 \le i \le 13$. Let $g = (g_1, g_2) = ((g_{ij})_{1 \le i,j \le 2}, (h_{ij})_{1 \le i,j \le 3}) \in G$ and we consider when $g \in G(i, 13)$. $g \cdot (A, B) \in W_{13}$ holds if and only if

$$\begin{bmatrix} h_{11} & h_{12} & h_{13} \end{bmatrix} (g_{11}A + g_{12}B) = \begin{bmatrix} 0 & 0 & 0 \end{bmatrix}.$$

Let us count $|G(5, 13)|$. Let $g = (g_1, g_2) = ((g_{ij})_{1 \le i,j \le 2}, (h_{ij})_{1 \le i,j \le 3}) \in G(5, 13)$. Then we have $\begin{bmatrix} 0 & g_{11}h_{13} & g_{11}h_{12} + g_{12}h_{13} \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \end{bmatrix}$. When $g_{11} = 0$, we have $h_{13} = 0$. Therefore we obtain $|\{((g_{ij})_{1 \le i,j \le 2}, (h_{ij})_{1 \le i,j \le 3}) \in G(5, 13) \mid g_{11} = 0\}| = q(q-1)^2 \cdot (q^4-1)(q^6-q^2)(q^6-q^4)$. When $g_{11} \ne 0$, we have $h_{12} = h_{13} = 0$. Therefore $|\{((g_{ij})_{1 \le i,j \le 2}, (h_{ij})_{1 \le i,j \le 3}) \in G(5, 13) \mid g_{11} \ne 0\}| = q^2(q-1)^2 \cdot (q^2-1)(q^6-q^2)(q^6-q^4)$. Thus we obtain $|G(5, 13)| = q(q-1)^2 \cdot (q^2+1+q)(q^2-1)(q^4-q^2)(q^6-q^4) = q^7(q-1)^5(q+1)^3(q^2+1)(q^2+q+1)$. The counts of the cardinalities $|G(i, 13)|$ for $6 \le i \le 14$ are carried out in the same way, and we omit the detail.

Next we consider W_{14}. We write an element $x \in W_{14}$ as $x = (A, B) = (\begin{bmatrix} 0 & 0 & a_{13} \\ 0 & a_{22} & a_{23} \\ \overline{a_{13}} & \overline{a_{23}} & a_{33} \end{bmatrix}, \begin{bmatrix} 0 & 0 & b_{13} \\ 0 & b_{22} & b_{23} \\ \overline{b_{13}} & \overline{b_{23}} & b_{33} \end{bmatrix})$. Let $W_{14}^0 = \{x \in W_{14} \mid a_{13} = 0, a_{22}b_{13} \ne 0\}$ and $W_{14}^1 = \{x \in W_{14} \mid a_{13} \ne 0\}$. Then we have $|\mathcal{O}_i \cap W_{14}| = |\mathcal{O}_i \cap W_6| + |\mathcal{O}_i \cap W_{10}^1| + |\mathcal{O}_i \cap W_{14}^0| + |\mathcal{O}_i \cap W_{14}^1|$ for $1 \le i \le 15$. Furthermore, let $W_{14}^2 = \{x \in W_{14}^1 \mid b_{13} /\!/_{\mathbb{F}_q} a_{13}\}$ and $W_{14}^3 = \{x \in W_{14}^1 \mid b_{13} \not/\!/_{\mathbb{F}_q} a_{13}\}$. We have the map

$$W_{14}^2 \ni (A, B) \mapsto (B - \frac{b_{13}}{a_{13}}A, A) \in W_{10}^1 \sqcup W_{14}^0 = \{x \in W_{14} \mid a_{13} = 0, b_{13} \ne 0\}.$$

We easily see that this map is surjective. The inverse image of $(A, B) \in W_{10}^1 \sqcup W_{14}^0$ is $\{(B, A + aB) \mid a \in \mathbb{F}_q\}$. Therefore we obtain $|\mathcal{O}_i \cap W_{14}| = |\mathcal{O}_i \cap W_6| + (q+1) \cdot (|\mathcal{O}_i \cap W_{10}^1| + |\mathcal{O}_i \cap W_{14}^0|) + |\mathcal{O}_i \cap W_{14}^3|$. Thus we count $|\mathcal{O}_i \cap W_{14}^0|$ and $|\mathcal{O}_i \cap W_{14}^3|$. First we count $|\mathcal{O}_i \cap W_{14}^0|$. Since $a_{13} \ne 0$, we have $1 \le \mathrm{rank}(x) \le 2$. If $\mathrm{rank}(A) = 1$, we have $x \in \mathcal{O}_{11}$. If $\mathrm{rank}(A) = 2$, we have $x \in \mathcal{O}_{12}$. Thus we obtain $|\mathcal{O}_{11} \cap W_{14}^0| = q^6(q-1)(q^2-1)$ and $|\mathcal{O}_{12} \cap W_{14}^0| = q^6(q-1)^2(q^2-1)$. Next we count $|\mathcal{O}_i \cap W_{14}^3|$. If $a_{22} = b_{22} = 0$ and $\begin{bmatrix} a_{13} & b_{13} \end{bmatrix} /\!/ \begin{bmatrix} a_{23} & b_{23} \end{bmatrix}$, we have $x \in \mathcal{O}_6$. If $a_{22} = b_{22} = 0$ and $\begin{bmatrix} a_{13} & b_{13} \end{bmatrix} \not/\!/ \begin{bmatrix} a_{23} & b_{23} \end{bmatrix}$, we have $x \in \mathcal{O}_8$. If $\begin{bmatrix} a_{22} & b_{22} \end{bmatrix} \neq \begin{bmatrix} 0 & 0 \end{bmatrix}$, we have $x \in \mathcal{O}_{14}$. Thus we have $|\mathcal{O}_6 \cap W_{14}^3| = q^4(q^2-1)(q^2-q)$, $|\mathcal{O}_8 \cap W_{14}^3| = q^2(q^2-1)(q^2-q)(q^4-q^2)$, and $|\mathcal{O}_{14} \cap W_{14}^3| = q^6(q^2-1)(q^2-1)(q^2-q)$.

Next we consider $W_2^{\perp}$. We easily see that $|\mathcal{O}_i \cap W_2^{\perp}| = (q+1) \cdot |\mathcal{O}_i \cap W_{11}^0|$ for $2 \leq i \leq 4$. Next we count $|G(i, 2^{\perp})|$ for $5 \leq i \leq 13$, $i \neq 7$. Let $g = (g_1, g_2) = ((g_{ij})_{1 \leq i,j \leq 2}, (h_{ij})_{1 \leq i,j \leq 3}) \in G$ and we consider when $g \in G(i, 2^{\perp})$. $g \cdot (A, B) \in W_2^{\perp}$ holds if and only if

$$\begin{bmatrix} \begin{bmatrix} h_{11} & h_{12} & h_{13} \end{bmatrix} A \begin{bmatrix} \overline{h_{11}} & \overline{h_{12}} & \overline{h_{13}} \end{bmatrix}^T \\ \begin{bmatrix} h_{11} & h_{12} & h_{13} \end{bmatrix} B \begin{bmatrix} \overline{h_{11}} & \overline{h_{12}} & \overline{h_{13}} \end{bmatrix}^T \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

Let us count $|G(5, 2^{\perp})|$. Let $g = (g_1, g_2) = ((g_{ij})_{1 \leq i,j \leq 2}, (h_{ij})_{1 \leq i,j \leq 3}) \in G(5, 2^{\perp})$. Then we have $\begin{bmatrix} \mathrm{Tr}_2(h_{12}\overline{h_{13}}) & \mathrm{N}_2(h_{13}) \end{bmatrix} = \begin{bmatrix} 0 & 0 \end{bmatrix}$, and therefore $h_{13} = 0$. Thus we obtain $|G(5, 2^{\perp})| = \mathrm{gl}_2 \cdot (q^4-1)(q^6-q^2)(q^6-q^4) = q^7(q-1)^5(q+1)^4(q^2+1)^2$. The counts of the cardinalities $|G(i, 2^{\perp})|$ for $6 \leq i \leq 13$, $i \neq 7$ are carried out in the same way, and we omit the detail. Next we count $|\mathcal{O}_7 \cap W_2^{\perp}|$. We write an element $x \in \mathcal{O}_7 \cap W_2^{\perp}$ as $x = (A, B) = (\begin{bmatrix} 0 & a_{12} & a_{13} \\ \overline{a_{12}} & a_{22} & a_{23} \\ \overline{a_{13}} & \overline{a_{23}} & a_{33} \end{bmatrix}, \begin{bmatrix} 0 & b_{12} & b_{13} \\ \overline{b_{12}} & b_{22} & b_{23} \\ \overline{b_{13}} & \overline{b_{23}} & b_{33} \end{bmatrix})$. Since $\mathrm{r}_1(x) = 2$, we have $\mathrm{rank}(\begin{bmatrix} a_{12} & a_{13} \\ b_{12} & b_{13} \end{bmatrix}) \leq 1$. If $\mathrm{rank}(\begin{bmatrix} a_{12} & a_{13} \\ b_{12} & b_{13} \end{bmatrix}) = 0$, $x \in W_6$. Let

$$W_2^{\perp 0} = \left\{ x \in W_2^{\perp} \;\middle|\; \mathrm{rank}(\begin{bmatrix} a_{12} & a_{13} \\ b_{12} & b_{13} \end{bmatrix}) = 1, \begin{bmatrix} a_{13} \\ b_{13} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \right\}$$

and

$$W_2^{\perp 1} = \left\{ x \in W_2^{\perp} \;\middle|\; \mathrm{rank}(\begin{bmatrix} a_{12} & a_{13} \\ b_{12} & b_{13} \end{bmatrix}) = 1, \begin{bmatrix} a_{13} \\ b_{13} \end{bmatrix} \neq \begin{bmatrix} 0 \\ 0 \end{bmatrix} \right\}.$$

Clearly $|\mathcal{O}_7 \cap W_2^{\perp}| = |\mathcal{O}_7 \cap W_6| + |\mathcal{O}_7 \cap W_2^{\perp 0}| + |\mathcal{O}_7 \cap W_2^{\perp 1}|$. We easily see that $|\mathcal{O}_7 \cap W_2^{\perp 0}| = |\mathcal{O}_7 \cap W_{14}| - |\mathcal{O}_7 \cap W_6|$. Next we calculate $|\mathcal{O}_7 \cap W_2^{\perp 1}|$. Since $\mathrm{rank}(\begin{bmatrix} a_{12} & a_{13} \\ b_{12} & b_{13} \end{bmatrix}) = 1$, we write $\begin{bmatrix} a_{12} & b_{12} \end{bmatrix} = a_x \begin{bmatrix} b_{13} & b_{13} \end{bmatrix}$ for $a \in \mathbb{F}_{q^2}$. We have the map

$$|\mathcal{O}_7 \cap W_2^{\perp 1}| \ni x \mapsto g_0 x \in |\mathcal{O}_7 \cap W_2^{\perp 0}|$$

where $g_0 = (I_2, \begin{pmatrix} 1 & 0 & 0 \\ 0 & -\overline{a_x} & 1 \\ 0 & 1 & 0 \end{pmatrix})$. We easily see that this map is surjective. The inverse image of $x \in \mathcal{O}_7 \cap W_2^{\perp 0}$ is $\left\{ (I_2, \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & \overline{a} \end{pmatrix})x \;\middle|\; a \in \mathbb{F}_{q^2} \right\}$. Thus we obtain $|\mathcal{O}_7 \cap W_2^{\perp 1}| = q^2 \cdot |\mathcal{O}_7 \cap W_2^{\perp 0}|$, and $|\mathcal{O}_7 \cap W_2^{\perp}| = (q^2+1)|\mathcal{O}_7 \cap W_{14}| - q^2|\mathcal{O}_7 \cap W_6|)$. Next we count $|G(14, 2^{\perp})|$. We have $\begin{bmatrix} \mathrm{N}_2(h_{11}) + \mathrm{N}_2(h_{12}) + \mathrm{N}_2(h_{13}) & \mu_0 \mathrm{Tr}_2(h_{12}\overline{h_{13}}) + \mu_1 \mathrm{N}_2(h_{13}) \end{bmatrix} = \begin{bmatrix} 0 & 0 \end{bmatrix}$. Let

$$X_2^{\perp} = \{(a, b) \in \mathbb{F}_{q^2}^2 \mid \mathrm{N}_2(a) + \mathrm{N}_2(b) = \mu_0 \mathrm{Tr}_2(a\bar{b}) + \mu_1 \mathrm{N}_2(b) = 0\}.$$

Then we have

$$|\{x \in G(14, 2^{\perp}) \mid h_{11} = 0\}| = \mathrm{gl}_2 \cdot (q^6-q^2)(q^6-q^4) \cdot (|X_2^{\perp}| - 1),$$
$$|\{x \in G(14, 2^{\perp}) \mid h_{11} \neq 0\}| = \mathrm{gl}_2 \cdot (q^6-q^2)(q^6-q^4) \cdot (q+1)(q^3+q^2-q-|X_2^{\perp}|).$$

Thus we count $|X_2^{\perp}|$. We use the result of $G(7, 2^{\perp})$. We proved $|G(7, 2^{\perp})| = q^3(q-1)^2(q+1)(q^2+1)(2q^2+1)$. Moreover, we have $|G(7, 2^{\perp})| = \mathrm{gl}_2 \cdot (q^6-q^2)(q^6-q^4) \cdot (q^2|X_2^{\perp}|-1)$. Therefore $|X_2^{\perp}| = 2q^2-1$. Hence we obtain

$$|\{x \in G(14, 2^{\perp}) \mid h_{11} = 0\}| = 2\mathrm{gl}_2 \cdot (q^2-1)(q^6-q^2)(q^6-q^4),$$
$$|\{x \in G(14, 2^{\perp}) \mid h_{11} \neq 0\}| = \mathrm{gl}_2 \cdot (q-1)^2(q+1)^2(q^6-q^2)(q^6-q^4),$$

and thus $|G(14, 2^{\perp})| = q^7(q-1)^5(q+1)^4(q^2+1)^2$. Lastly, $|\mathcal{O}_{15} \cap W_2^{\perp}| = q^{16} - \sum_{i=1}^{14} |\mathcal{O}_i \cap W_2^{\perp}|$.

Next we consider $W_5^{\perp}$. For $2 \leq i \leq 4$, we easily see that $|\mathcal{O}_i \cap W_5^{\perp}| = q|\mathrm{H}(2, i-1)| + |\mathcal{O}_i \cap W_{11}^0|$. Next we count $|G(i, 5^{\perp})|$ for $5 \leq i \leq 12$. Let $g = (g_1, g_2) = ((g_{ij})_{1 \leq i,j \leq 2}, (h_{ij})_{1 \leq i,j \leq 3}) \in G$ and we consider when $g \in G(i, 5^{\perp})$. $g \cdot (A, B) \in W_5^{\perp}$ holds if and only if

$$\begin{cases} \begin{bmatrix} \begin{bmatrix} h_{11} & h_{12} & h_{13} \end{bmatrix} A \begin{bmatrix} \overline{h_{11}} & \overline{h_{12}} & \overline{h_{13}} \end{bmatrix}^T \\ \begin{bmatrix} h_{11} & h_{12} & h_{13} \end{bmatrix} B \begin{bmatrix} \overline{h_{11}} & \overline{h_{12}} & \overline{h_{13}} \end{bmatrix}^T \end{bmatrix} & = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \\ \begin{bmatrix} h_{11} & h_{12} & h_{13} \end{bmatrix} (g_{11}A + g_{12}B) & = \begin{bmatrix} 0 & 0 & 0 \end{bmatrix}. \end{cases}$$

Let us count $|G(5, 5^{\perp})|$. Let $g = (g_1, g_2) = ((g_{ij})_{1 \le i,j \le 2}, (h_{ij})_{1 \le i,j \le 3}) \in G(5, 5^{\perp})$. We have

$$\begin{cases} \begin{bmatrix} \mathrm{Tr}_2(h_{12}\overline{h_{13}}) & \mathrm{N}_2(h_{13}) \end{bmatrix} & = \begin{bmatrix} 0 & 0 \end{bmatrix}, \\ \begin{bmatrix} 0 & g_{11}h_{13} & g_{11}h_{12} + g_{12}h_{13} \end{bmatrix} & = \begin{bmatrix} 0 & 0 & 0 \end{bmatrix}. \end{cases}$$

and therefore $h_{13} = 0$ and $g_{11}h_{12} = 0$. When $g_{11} = 0$, we have $\begin{bmatrix} h_{11} & h_{12} \end{bmatrix} \neq \begin{bmatrix} 0 & 0 \end{bmatrix}$. Hence $|\{g \in G(5, 5^{\perp}) \mid g_{11} = 0\}| = q(q-1)^2 \cdot (q^4 - 1)(q^6 - q^2)(q^6 - q^4)$. When $g_{11} \neq 0$, we have $h_{12} = 0$. Hence $|\{g \in G(5, 5^{\perp}) \mid g_{11} = 0\}| = q^2(q-1)^2 \cdot (q^2 - 1)(q^6 - q^2)(q^6 - q^4)$. Thus we obtain $|G(5, 5^{\perp})| = q^7(q-1)^5(q+1)^3(q^2+1)(q^2+q+1)$. The counts of the cardinalities $|G(i, 5^{\perp})|$ for $6 \le i \le 12$ are carried out in the same way, and we omit the detail. □

11.3. Fourier transform.

Theorem 11.3. *The representation matrix M of the Fourier transform on $\mathcal{F}_V^G$ with respect to the basis $e_1, ..., e_{15}$ is given as follows:*

$$q^{-18} \left[\begin{array}{cccccccc} 1 & [101101] & [111111] & [231011] & [212111] & \frac{1}{2}[231111] & \frac{1}{2}[231111] & [242111] \\ 1 & -1 & [110010]d_1 & [130010]c_1 & -[111010] & -\frac{1}{2}[130010]d_2 & \frac{1}{2}[231020] & [242020] \\ 1 & [100000]d_1 & qg_1 & [130000]e_1 & [111000]e_2 & \frac{1}{2}[230000]d_3 & \frac{1}{2}[131000]c_2 & [142000]c_2 \\ 1 & [000100]c_1 & [010100]e_1 & q^3 f_1 & -[111101] & -\frac{1}{2}[130100]b_1 & -\frac{1}{2}[131100] & -[142100] \\ 1 & -1 & qe_2 & -[130001] & qg_2 & -\frac{1}{2}[130101] & \frac{1}{2}[130000]d_4 & [141000]d_4 \\ 1 & -d_2 & [110000]d_3 & -[130000]b_1 & -[111101] & -\frac{1}{2}q^3 e_3 & \frac{1}{2}[231010] & [242010] \\ 1 & [101010] & [0110000]c_2 & -[131000] & [111000]d_4 & \frac{1}{2}[231010] & \frac{1}{2}q^3 e_4 & [141000]d_5 \\ 1 & [101010] & [011000]c_2 & -[131000] & [111000]d_4 & \frac{1}{2}[231010] & \frac{1}{2}[130000]d_5 & -q^4 d_4 \\ 1 & -d_2 & qe_5 & -q^3 c_3 & -qf_2 & q^3 \frac{1}{2} q^3 d_6 & -\frac{1}{2}[130000] & -[141000] \\ 1 & -1 & -[010010] & q^3 & qe_6 & -\frac{1}{2}[130010] & -\frac{1}{2}[130010] & q^4 \\ 1 & -d_7 & qe_7 & -[130010] & [111000]c_3 & -\frac{1}{2}[130000]b_2 & -\frac{1}{2}[131000] & -[142000] \\ 1 & [100001] & qc_4 & -[130000] & qe_8 & -\frac{1}{2}[130100] & -\frac{1}{2}[130001] & [041000] \\ 1 & c_5 & qc_6 & -q^3 a_1 & -[011000]c_7 & \frac{1}{2}q^3 c_8 & -\frac{1}{2}[132000] & [042000] \\ 1 & -1 & -[010010] & q^3 & -[111010] & -\frac{1}{2}[130010] & \frac{1}{2}q^3 c_9 & -[141000] \\ 1 & -[001100] & -[011100] & [031000] & [011100] & \frac{1}{2}[021100] & -\frac{1}{2}[130100] & [040100] \end{array} \right.$$

$$\left. \begin{array}{ccccccc} [332111] & [343111] & [261111] & [362111] & \frac{1}{6}[471111] & \frac{1}{2}[372111] & \frac{1}{3}[473011] \\ -[231010]d_2 & -[242010] & -[160010]d_7 & [361011] & \frac{1}{6}[370010]c_5 & -\frac{1}{2}[271010] & -\frac{1}{3}[373010] \\ [231000]e_5 & -[242010] & [160000]e_7 & [261000]c_4 & \frac{1}{6}[370000]c_6 & -\frac{1}{2}[271010] & -\frac{1}{3}[373000] \\ -[131100]c_3 & [142100] & -[160110] & -[261100] & -\frac{1}{6}[270100]a_1 & \frac{1}{2}[171100] & \frac{1}{3}[273000] \\ -[130000]f_2 & [141000]e_6 & [160000]c_3 & [160000]e_8 & -\frac{1}{6}[270000]c_7 & -\frac{1}{2}[271010] & \frac{1}{3}[272000] \\ [131000]d_6 & -[242010] & -[160000]b_2 & -[261100] & \frac{1}{6}[270000]c_8 & -\frac{1}{2}[271010] & \frac{1}{3}[273000] \\ -[231000] & -[242010] & -[161000] & -[261001] & -\frac{1}{6}[372000] & \frac{1}{2}[171000]c_9 & -\frac{1}{3}[372000] \\ -[231000] & [141000] & -[161000] & [161000] & \frac{1}{6}[271000] & -\frac{1}{2}[271000] & \frac{1}{3}[271000] \\ q^3 f_3 & [142100] & -[160100] & [361000] & -\frac{1}{6}[170000]a_1^2 & \frac{1}{2}[170000] & -\frac{1}{3}[172000] \\ [131100] & q^4 e_9 & -[160000] & -[261000] & \frac{1}{6}[170000]a_1 & \frac{1}{2}[170000] & -\frac{1}{3}[171000] \\ -[231100] & -[242000] & q^6 b_3 & -[161000]a_2 & \frac{1}{2}[270000] & \frac{1}{2}[171000] & 0 \\ [331000] & -[242000] & -q^6 a_2 & q^6 b_4 & -\frac{1}{2}[170000] & \frac{1}{2}[170000] & 0 \\ -[031000]a_1^2 & [042000]a_1 & 3q^6 & -3[061000] & q^7 & 0 & 0 \\ [130000] & [141000] & q^6 & [160000] & 0 & -q^7 & 0 \\ -[031100] & -[041100] & 0 & 0 & 0 & 0 & q^7 \end{array} \right]$$

Here, we put $[a,b,c,d,e,f] = (q-1)^a q^b (q+1)^c (q^2-q+1)^d (q^2+1)^e (q^2+q+1)^f$ and

$a_1 = 2q-1$
$a_2 = q-2$
$b_1 = 2q^2+q+1$
$b_2 = 2q^2-q+1$
$b_3 = q^2-2q+2$
$b_4 = q^2-2$
$c_1 = q^3-q-1$
$c_2 = q^3-2q^2+q-1$
$c_3 = q^3-q^2-1$
$c_4 = q^3-2q^2-1$
$c_5 = 2q^3-1$
$c_6 = 2q^3-3q^2-1$
$c_7 = 3q^3-q^2+q-1$
$c_8 = q^3+3q^2-3q+1$
$c_9 = q^3+q^2-q+1$
$d_1 = q^4+q^3+q^2+q+1$
$d_2 = q^4+1$
$d_3 = 2q^4+q^3+2q^2+q+1$
$d_4 = q^4-q^2-1$
$d_5 = q^4-2q^2-1$
$d_6 = 2q^4-2q^3+2q^2-q+1$
$d_7 = q^4-q^3+1$
$e_1 = q^5+q^3-q^2-q-1$
$e_2 = q^5-q^2-1$
$e_3 = q^5-3q^4+2q^3-2q^2+q-1$
$e_4 = 3q^5-q^4-2q^3+2q^2-q+1$
$e_5 = q^5-q^4+q^3-q^2-1$
$e_6 = q^5-q^4+1$
$e_7 = q^5-q^4+2q^3-2q^2-1$
$e_8 = q^5-2q^4-q^3+1$
$e_9 = q^5-q^4-q^3+q^2-1$
$f_1 = q^6-q^5+q^4-2q^3+q^2+1$
$f_2 = q^6-q^5-1$
$f_3 = q^6-2q^5+q^4-q^2+q-1$
$g_1 = q^7-q^6+2q^5-q^4-q^2-1$
$g_2 = q^7-q^4+1$

Corollary 11.4. *The Fourier transform of Ψ is given as follows:*

$$\widehat{\Psi}(x) = \begin{cases} q^{-1}+q^{-3}-q^{-5}-q^{-7}+q^{-8} & x \in \mathcal{O}_1, \\ q^{-4}-2q^{-5}+2q^{-6}-2q^{-7}+q^{-8} & x \in \mathcal{O}_2, \\ 2q^{-6}-4q^{-7}+2q^{-8} & x \in \mathcal{O}_3, \\ -2q^{-7}+3q^{-8}-q^{-9} & x \in \mathcal{O}_4, \\ q^{-6}-2q^{-7}+q^{-8} & x \in \mathcal{O}_5, \\ -q^{-7}+2q^{-8}-q^{-9} & x \in \mathcal{O}_6, \\ -q^{-7}+q^{-9} & x \in \mathcal{O}_7, \\ 0 & x \in \mathcal{O}_8, \mathcal{O}_{10}, \mathcal{O}_{12}, \\ q^{-8}-q^{-9} & x \in \mathcal{O}_9, \\ -q^{-9}+q^{-10} & x \in \mathcal{O}_{11}, \\ -q^{-11} & x \in \mathcal{O}_{13}, \mathcal{O}_{15}, \\ q^{-11} & x \in \mathcal{O}_{14}. \end{cases}$$

In particular we have the following L_1-norm bound of $\widehat{\Psi}$:

$$\sum_{x \in V} |\widehat{\Psi}(x)| = O(q^7).$$

12. Orbit decomposition of $\mathbb{F}_q^2 \otimes \wedge^2(\mathbb{F}_q^5)$

Let $\wedge^2(\mathbb{F}_q^n)$ be the set of all alternating matrices of order n over $\mathbb{F}_q$. We write $A \in \wedge^2(\mathbb{F}_q^5)$ as

$$A = \begin{bmatrix} 0 & a_{12} & a_{13} & a_{14} & a_{15} \\ -a_{12} & 0 & a_{23} & a_{24} & a_{25} \\ -a_{13} & -a_{23} & 0 & a_{34} & a_{35} \\ -a_{14} & -a_{24} & -a_{34} & 0 & a_{45} \\ -a_{15} & -a_{25} & -a_{35} & -a_{45} & 0 \end{bmatrix} \text{ where } a_{ij} \in \mathbb{F}_q.$$

Let $V' = \mathbb{F}_q^2 \otimes \wedge^2(\mathbb{F}_q^5)$ and $G' = \mathrm{GL}_2 \times \mathrm{GL}_5$. We write $x \in V'$ as $x = (A, B)$ where $A, B \in \wedge^2(\mathbb{F}_q^5)$, and write $g \in G'$ as $g = (g_1, g_2)$ where $g_1 \in \mathrm{GL}_2$ and $g_2 \in \mathrm{GL}_5$. The action of G' on V' is defined by

$$gx = (g_2 A g_2^T, g_2 B g_2^T) g_1^T.$$

Define a bilinear form β of V' as

$$\beta((A_1, B_1), (A_2, B_2)) = \mathrm{Tr}(A_1 A_2^T + B_1 B_2^T).$$

In addition, define an automorphism ι of G' as

$$(g_1, g_2)^\iota = ((g_1^T)^{-1}, (g_2^T)^{-1}).$$

These β and ι satisfy Assumption 2.1.

Let $u_{lmn}(1 \leq l \leq 2, 1 \leq n < m \leq 5)$ be the element of V' that the (n,m)-entry and (m,n)-entry of lth matrix is 1 and -1 respectively and the rest are all 0. For example,

$$u_{112} = (\begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}).$$

The set $\{u_{lmn} \mid 1 \leq l \leq 2, 1 \leq n < m \leq 5\}$ is a $\mathbb{F}_q$-basis of V'.

For $x = (A,B) = \sum_{1 \leq i<j \leq 5} a_{ij}u_{1ij} + \sum_{1 \leq i<j \leq 5} b_{ij}u_{2ij} \in V'$, define

$$r_1(x) := \dim(\langle A, B\rangle_{\mathbb{F}_q}), \text{ i.e., the dimension of the subspace of } \wedge^2(\mathbb{F}_q^5) \text{ generated by } A \text{ and } B,$$

$$r_2(x) := \operatorname{rank}(\begin{bmatrix} 0 & a_{12} & a_{13} & a_{14} & a_{15} & 0 & b_{12} & b_{13} & b_{14} & b_{15} \\ -a_{12} & 0 & a_{23} & a_{24} & a_{25} & -b_{12} & 0 & b_{23} & b_{24} & b_{25} \\ -a_{13} & -a_{23} & 0 & a_{34} & a_{35} & -b_{13} & -b_{23} & 0 & b_{34} & b_{35} \\ -a_{14} & -a_{24} & -a_{34} & 0 & a_{45} & -b_{14} & -b_{24} & -b_{34} & 0 & b_{45} \\ -a_{15} & -a_{25} & -a_{35} & -a_{45} & 0 & -b_{15} & -b_{25} & -b_{35} & -b_{45} & 0 \end{bmatrix}),$$

$$\operatorname{mi}(x) := \min\{\operatorname{rank}(rA+sB) | (r,s) \in \mathbb{F}_q^2 \backslash \{(0,0)\}\},$$

$$\operatorname{ma}(x) := \max\{\operatorname{rank}(rA+sB) | (r,s) \in \mathbb{F}_q^2 \backslash \{(0,0)\}\}.$$

$r_1(x)$, $r_2(x)$, $\operatorname{mi}(x)$ and $\operatorname{ma}(x)$ are invariants of the orbits.

Proposition 12.1. *V' consists of 9 G'-orbits in all.*

Orbit name	Representative	$r_1(x)$	$r_2(x)$	$\operatorname{mi}(x)$	$\operatorname{ma}(x)$	Cardinality
$\mathcal{O}_1$	0	0	0	0	0	1
$\mathcal{O}_2$	u_{112}	1	2	0	2	$[1,0,1,0,1,1]$
$\mathcal{O}_3$	$u_{112}+u_{134}$	1	4	0	4	$[2,2,1,1,0,1]$
$\mathcal{O}_4$	$u_{112}+u_{213}$	2	3	2	2	$[2,1,1,1,1,1]$
$\mathcal{O}_5$	$u_{112}+u_{214}-u_{223}$	2	4	2	4	$[3,2,2,1,1,1]$
$\mathcal{O}_6$	$u_{112}+u_{234}$	2	4	2	4	$\frac{1}{2}[2,5,1,1,1,1]$
$\mathcal{O}_7$	$2u_{112}+\mu_1u_{114}-\mu_1u_{123}+(\mu_1^2-2\mu_0)u_{134}$ $+\mu_1u_{212}+(\mu_1^2-2\mu_0)u_{214}$ $-(\mu_1^2-2\mu_0)u_{223}+(\mu_1^3-2\mu_1\mu_0)u_{234}$	2	4	4	4	$\frac{1}{2}[4,5,1,1,0,1]$
$\mathcal{O}_8$	$u_{112}+u_{215}-u_{234}$	2	5	2	4	$[3,5,2,1,1,1]$
$\mathcal{O}_9$	$u_{112}+u_{134}+u_{215}+u_{223}$	2	5	4	4	$[4,6,2,1,1,1]$

Here, we put $[a,b,c,d,e,f] = (q-1)^a q^b (q+1)^c (q^2+q+1)^d (q^2+1)^e (q^4+q^3+q^2+q+1)^f$ and μ_1, $\mu_0 \in \mathbb{F}_q$ are elements such that $X^2+\mu_1X+\mu_0 \in \mathbb{F}_q[X]$ is irreducible.

[Proof]

We count the cardinalities of the orbits of the 9 elements in the "Representative" column of the table. We refer to these elements as $x_1, ..., x_9$ in order from the top, and let $\mathcal{O}_i$ be the orbit of the element x_i For the calculation for the cases of $r_2(x) \leq 4$, we use the result for $V'' := \mathbb{F}_q^2 \otimes \wedge^2(\mathbb{F}_q^4)$. Let $G'' = \mathrm{GL}_2 \times \mathrm{GL}_4$. The action of G'' on V'' is defined by

$$G' \times V'' \ni ((g_1,g_2),(A,B)) = (g_2Ag_2^T, g_2Bg_2^T)g_1^T \in V''.$$

By the embeddings $V'' \ni (A,B) \mapsto (\begin{bmatrix} A & \\ & 0 \end{bmatrix}, \begin{bmatrix} B & \\ & 0 \end{bmatrix}) \in V'$ and $G'' \ni (g_1,g_2) \mapsto (g_1, \begin{pmatrix} g_2 & \\ & 1 \end{pmatrix}) \in G'$, we regard V'' as the subspace of V' and G'' as the subgroup of G'. We consider the induced injective map $G_1 \setminus V_1 \to G \setminus V$. For $x \in V''$, we have

$$|G'x| = \frac{|G'|\mathrm{gl}_{4-r_2(x)}}{q^{r_2(x)}|G''|\mathrm{gl}_{5-r_2(x)}}|G''x|.$$

Therefore we obtain $|\mathcal{O}_i|$ for $1 \leq i \leq 7$. We calculate the cardinalities for the cases $r_2(x) = 5$. Let $\mathrm{Stab}(x_i)$ be the group of stabilizers of x_i. Since $|\mathcal{O}_i| = |G'|/|\mathrm{Stab}(x_i)|$, it is enough to count $|\mathrm{Stab}(x_i)|$.

For $x = (A, B) \in V'$, let $r_2(x) = 5$. Considering the rank of $[A\ B] \in \mathrm{M}(5, 10)(\mathbb{F}_q)$, we have $r_1(x) = 2$ and $\mathrm{mi}(x) \geq 2$. Let $\mathrm{mi}(x) = 2$. We have

$$\begin{aligned}
x &\sim u_{112} + b_{13}u_{213} + b_{14}u_{214} + b_{15}u_{215} + b_{23}u_{223} + b_{24}u_{224} + b_{25}u_{225} + b_{34}u_{234} + b_{35}u_{235} + b_{45}u_{245} \\
&\quad \text{where } \mathrm{rank}(\begin{bmatrix} b_{13} & b_{14} & b_{15} \\ b_{23} & b_{24} & b_{25} \\ 0 & b_{34} & b_{35} \\ -b_{34} & 0 & b_{45} \\ -b_{35} & -b_{45} & 0 \end{bmatrix}) = 3, \mathrm{rank}(\begin{bmatrix} 0 & b_{34} & b_{35} \\ -b_{34} & 0 & b_{45} \\ -b_{35} & -b_{45} & 0 \end{bmatrix}) = 2 \\
&\sim u_{112} + u_{213} + u_{245} \\
&\sim x_8.
\end{aligned}$$

Let $\mathrm{mi}(x) = 4$. We have

$$\begin{aligned}
x &\sim u_{112} + u_{134} + u_{215} + b_{23}u_{223} + b_{24}u_{224} + b_{34}u_{234} \\
&\quad \text{where } \begin{bmatrix} 0 & b_{23} & b_{24} \\ -b_{23} & 0 & b_{34} \\ -b_{24} & -b_{34} & 0 \end{bmatrix}) \neq 0 \\
&\sim u_{112} + u_{134} + u_{215} + u_{223} \\
&\sim x_9.
\end{aligned}$$

Therefore we have $V' = \bigsqcup_{i=1}^{9} \mathcal{O}_i$. Next we count $|\mathcal{O}_8|$. Let $g = (g_1^{-1}, g_2) = ((g_{ij})_{1\leq i,j\leq 2}^{-1}, (h_{ij})_{1\leq i,j\leq 5}) \in \mathrm{Stab}(x_9) = \{g \in G' | gx_9 = x_9\}$. We have $(g_1, 1) \cdot x = (1, g_2) \cdot x$. By comparing the rank of first entry, we have $g_{12} = 0$, $\begin{vmatrix} h_{11} & h_{12} \\ h_{21} & h_{22} \end{vmatrix} = g_{11}$ and $\begin{bmatrix} h_{31} & h_{41} & h_{51} \\ h_{32} & h_{42} & h_{52} \end{bmatrix} = 0$. Furthermore, by comparing the second entry, we have $h_{21} = h_{23} = h_{24} = h_{53} = h_{54} = 0$, $h_{11}h_{55} = \begin{vmatrix} h_{33} & h_{34} \\ h_{43} & h_{44} \end{vmatrix} = g_{22}$, $\begin{bmatrix} h_{33} & h_{34} \\ h_{43} & h_{44} \end{bmatrix} \begin{bmatrix} h_{14} \\ -h_{13} \end{bmatrix} = -h_{11} \begin{bmatrix} h_{35} \\ h_{45} \end{bmatrix}$ and $h_{11}h_{25} = g_{21}$. Therefore we obtain $|\mathrm{Stab}(x_8)| = q^5 \mathrm{gl}_1^2 \mathrm{gl}_2$, and $|\mathcal{O}_8| = |G'|/|\mathrm{Stab}(x_8)|$.

Lastly, we have $|\mathcal{O}_9| = q^{20} - \sum_{i=1}^{8} |\mathcal{O}_i|$. □

13. $\mathbb{F}_q^2 \otimes \wedge^2(\mathbb{F}_q^6)$

We write $A \in \wedge^2(\mathbb{F}_q^6)$ as

$$A = \begin{bmatrix} 0 & a_{12} & a_{13} & a_{14} & a_{15} & a_{16} \\ -a_{12} & 0 & a_{23} & a_{24} & a_{25} & a_{26} \\ -a_{13} & -a_{23} & 0 & a_{34} & a_{35} & a_{36} \\ -a_{14} & -a_{24} & -a_{34} & 0 & a_{45} & a_{46} \\ -a_{15} & -a_{25} & -a_{35} & -a_{45} & 0 & a_{56} \\ -a_{16} & -a_{26} & -a_{36} & -a_{46} & -a_{56} & 0 \end{bmatrix} \text{ where } a_{ij} \in \mathbb{F}_q.$$

The Pfaffian of A is defined by

$$\begin{aligned}
\mathrm{Pfaff}(A) =& a_{12}a_{34}a_{56} - a_{13}a_{24}a_{56} + a_{14}a_{23}a_{56} - a_{14}a_{25}a_{36} + a_{14}a_{26}a_{35} + a_{15}a_{24}a_{36} - a_{15}a_{26}a_{34} - a_{16}a_{24}a_{35} \\
&+ a_{16}a_{25}a_{34} + a_{13}a_{25}a_{46} - a_{13}a_{26}a_{45} - a_{15}a_{23}a_{46} + a_{16}a_{23}a_{45} - a_{12}a_{35}a_{46}. + a_{12}a_{36}a_{45}.
\end{aligned}$$

Let $V = \mathbb{F}_q^2 \otimes \wedge^2(\mathbb{F}_q^6)$ and $G = G_1 \times G_2 = \mathrm{GL}_2 \times \mathrm{GL}_6$. We write $x \in V$ as $x = (A, B)$ where $A, B \in \wedge^2(\mathbb{F}_q^6)$, and write $g \in G$ as $g = (g_1, g_2)$ where $g_1 \in \mathrm{GL}_2$ and $g_2 \in \mathrm{GL}_6$. The action of G on V is defined by

$$gx = (g_2Ag_2^T, g_2Bg_2^T)g_1^T.$$

Define a bilinear form β of V as

$$\beta((A_1, B_1), (A_2, B_2)) = \mathrm{Tr}(A_1A_2^T + B_1B_2^T).$$

In addition, define an automorphism ι of G as

$$(g_1, g_2)^\iota = ((g_1^T)^{-1}, (g_2^T)^{-1}).$$

These β and ι satisfy Assumption 2.1.

13.1. **Orbit decomposition.** Let $u_{lmn}(1 \le l \le 2, 1 \le n < m \le 6)$ be the element of V that the (n,m)-entry and (m,n)-entry of lth matrix is 1 and -1 respectively and the rest are all 0. For example,

$$u_{112} = (\begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}).$$

The set $\{u_{lmn} \mid 1 \le l \le 2, 1 \le n < m \le 6\}$ is a $\mathbb{F}_q$-basis of V.

For $x = (A, B) = \sum_{1 \le i < j \le 6} a_{ij} u_{1ij} + \sum_{1 \le i < j \le 6} b_{ij} u_{2ij} \in V$, define

$$r_1(x) := \dim(\langle A, B \rangle_{\mathbb{F}_q}), \text{ i.e., the dimension of the subspace of } \wedge^2(\mathbb{F}_q^6) \text{ generated by } A \text{ and } B,$$

$$r_2(x) := \operatorname{rank}(\begin{bmatrix} 0 & a_{12} & a_{13} & a_{14} & a_{15} & a_{16} & 0 & b_{12} & b_{13} & b_{14} & b_{15} & b_{16} \\ -a_{12} & 0 & a_{23} & a_{24} & a_{25} & a_{26} & -b_{12} & 0 & b_{23} & b_{24} & b_{25} & b_{26} \\ -a_{13} & -a_{23} & 0 & a_{34} & a_{35} & a_{36} & -b_{13} & -b_{23} & 0 & b_{34} & b_{35} & b_{36} \\ -a_{14} & -a_{24} & -a_{34} & 0 & a_{45} & a_{46} & -b_{14} & -b_{24} & -b_{34} & 0 & b_{45} & b_{46} \\ -a_{15} & -a_{25} & -a_{35} & -a_{45} & 0 & a_{56} & -b_{15} & -b_{25} & -b_{35} & -b_{45} & 0 & b_{56} \\ -a_{16} & -a_{26} & -a_{36} & -a_{46} & -a_{56} & 0 & -b_{16} & -b_{26} & -b_{36} & -b_{46} & -b_{56} & 0 \end{bmatrix}),$$

$$\operatorname{mi}(x) := \min\{\operatorname{rank}(rA + sB) | (r,s) \in \mathbb{F}_q^2 \backslash \{(0,0)\}\},$$

$$\operatorname{ma}(x) := \max\{\operatorname{rank}(rA + sB) | (r,s) \in \mathbb{F}_q^2 \backslash \{(0,0)\}\}.$$

$r_1(x)$, $r_2(x)$, $\operatorname{mi}(x)$ and $\operatorname{ma}(x)$ are invariants of the orbits. We also define

$$\operatorname{Pfaff}_x(u,v) := \operatorname{Pfaff}(uA + vB) \in \operatorname{Sym}^3(\mathbb{F}_q^2) \text{ where } u, v \text{ are variables,}$$

$$\operatorname{T}(x) := \langle\!\langle \alpha \rangle\!\rangle \text{ if and only if } \operatorname{Pfaff}_x(u,v) \in O_{\langle\!\langle \alpha \rangle\!\rangle} \text{ in } \operatorname{Sym}^3(\mathbb{F}_q^2).$$

For $x \in V$ and $g = (g_1, g_2) \in G$, we have

$$\operatorname{Pfaff}_{gx}(u,v) = \det(g_2)\operatorname{Pfaff}_x((u,v)g_1).$$

Therefore $\operatorname{T}(x)$ is also an invariant of the orbits.

Proposition 13.1. *V consists of 18 G-orbits in all.*

Orbit name	Representative	$r_1(x)$	$r_2(x)$	$\operatorname{T}(x)$	$\operatorname{mi}(x)$	$\operatorname{ma}(x)$	Cardinality
$\mathcal{O}_1$	0	0	0	$\langle\!\langle 0 \rangle\!\rangle$	0	0	1
$\mathcal{O}_2$	u_{212}	1	2	$\langle\!\langle 0 \rangle\!\rangle$	0	2	$[1,0,1,1,0,1,1]$
$\mathcal{O}_3$	$u_{212} + u_{234}$	1	4	$\langle\!\langle 0 \rangle\!\rangle$	0	4	$[2,2,1,2,0,1,1]$
$\mathcal{O}_4$	$u_{212} + u_{234} + u_{256}$	1	6	$\langle\!\langle 1^3 \rangle\!\rangle$	0	6	$[3,6,1,1,0,1,0]$
$\mathcal{O}_5$	$u_{112} + u_{213}$	2	3	$\langle\!\langle 0 \rangle\!\rangle$	2	2	$[2,1,2,1,1,1,1]$
$\mathcal{O}_6$	$u_{112} + u_{214} - u_{223}$	2	4	$\langle\!\langle 0 \rangle\!\rangle$	2	4	$[3,2,2,2,1,1,1]$
$\mathcal{O}_7$	$u_{112} + u_{234}$	2	4	$\langle\!\langle 0 \rangle\!\rangle$	2	4	$\frac{1}{2}[2,5,1,2,1,1,1]$
$\mathcal{O}_8$	$2u_{112} + \mu_1 u_{114} - \mu_1 u_{123} + (\mu_1^2 - 2\mu_0)u_{134}$ $+\mu_1 u_{212} + (\mu_1^2 - 2\mu_0)u_{214}$ $-(\mu_1^2 - 2\mu_0)u_{223} + (\mu_1^3 - 2\mu_1\mu_0)u_{234}$	2	4	$\langle\!\langle 0 \rangle\!\rangle$	4	4	$\frac{1}{2}[4,5,1,2,0,1,1]$
$\mathcal{O}_9$	$u_{112} + u_{215} - u_{234}$	2	5	$\langle\!\langle 0 \rangle\!\rangle$	2	4	$[3,5,3,2,1,1,1]$
$\mathcal{O}_{10}$	$u_{112} + u_{134} + u_{215} + u_{223}$	2	5	$\langle\!\langle 0 \rangle\!\rangle$	4	4	$[4,6,3,2,1,1,1]$
$\mathcal{O}_{11}$	$u_{116} - u_{125} + u_{236} - u_{245}$	2	6	$\langle\!\langle 0 \rangle\!\rangle$	4	4	$[4,8,2,2,1,1,1]$
$\mathcal{O}_{12}$	$u_{112} + u_{216} - u_{225} + u_{234}$	2	6	$\langle\!\langle 1^3 \rangle\!\rangle$	2	6	$[4,6,2,2,1,1,1]$
$\mathcal{O}_{13}$	$u_{114} - u_{123}$ $+u_{216} - u_{225} + u_{234}$	2	6	$\langle\!\langle 1^3 \rangle\!\rangle$	4	6	$[5,8,3,2,1,1,1]$
$\mathcal{O}_{14}$	$u_{112} + u_{234} + u_{256}$	2	6	$\langle\!\langle 1^2 1 \rangle\!\rangle$	2	6	$[3,11,1,2,0,1,1]$
$\mathcal{O}_{15}$	$u_{112} + u_{134} + u_{236} - u_{245}$	2	6	$\langle\!\langle 1^2 1 \rangle\!\rangle$	4	6	$[4,11,2,2,1,1,1]$
$\mathcal{O}_{16}$	$u_{112} + u_{134} + u_{234} + u_{256}$	2	6	$\langle\!\langle 111 \rangle\!\rangle$	4	6	$\frac{1}{6}[4,13,1,2,1,1,1]$
$\mathcal{O}_{17}$	$2u_{112} + \mu_1 u_{114} - \mu_1 u_{123} + (\mu_1^2 - 2\mu_0)u_{134}$ $+\mu_1 u_{212} + (\mu_1^2 - 2\mu_0)u_{214}$ $-(\mu_1^2 - 2\mu_0)u_{223} + (\mu_1^3 - 2\mu_1\mu_0)u_{234}$	2	6	$\langle\!\langle 12 \rangle\!\rangle$	4	6	$\frac{1}{2}[5,13,2,2,0,1,1]$
$\mathcal{O}_{18}$	$u_{112} + u_{134} + u_{156} + \nu_2 u_{212}$ $+u_{216} + u_{223} + \nu_1 u_{225} + \nu_0 u_{245}$	2	6	$\langle\!\langle 3 \rangle\!\rangle$	6	6	$\frac{1}{3}[6,13,3,1,1,1,0]$

Here, we put $[a,b,c,d,e,f,g] = (q-1)^a q^b (q+1)^c (q^2+q+1)^d (q^2+1)^e (q^4+q^3+q^2+q+1)^f (q^2-q+1)^g$ *and* μ_1, μ_0, ν_2, ν_1, $\nu_0 \in \mathbb{F}_q$ *are elements such that* $X^2+\mu_1 X+\mu_0, X^3+\nu_2 X^2+\nu_1 X+\nu_0 \in \mathbb{F}_q[X]$ *are irreducible.*

[Proof]

We count the cardinalities of the orbits of the 18 elements in the "Representative" column of the table. We refer to these elements as $x_1, ..., x_{18}$ in order from the top, and let $\mathcal{O}_i$ be the orbit of the element x_i. For the calculation for the cases of $r_2(x) \leq 5$, we use the result for $(G', V') = (\mathrm{GL}_2 \times \mathrm{GL}_5, \mathbb{F}_q^2 \otimes \wedge^2(\mathbb{F}_q^5))$. By the embeddings $V' \ni (A, B) \mapsto (\begin{bmatrix} A & \\ & 0 \end{bmatrix}, \begin{bmatrix} B & \\ & 0 \end{bmatrix}) \in V$ and $G' \ni (g_1, g_2) \mapsto (g_1, \begin{pmatrix} g_2 & \\ & 1 \end{pmatrix}) \in G$, we regard V' as the subspace of V and G' as the subgroup of G. We consider the induced injective map $G_1 \setminus V_1 \to G \setminus V$. This map is injective. For $x \in V'$, we have

$$|Gx| = \frac{|G|\mathrm{gl}_{5-r_2(x)}}{q^{r_2(x)}|G'|\mathrm{gl}_{6-r_2(x)}} |G'x|.$$

Therefore we obtain $|\mathcal{O}_i|$ for $1 \leq i \leq 10$, $i \neq 4$. We calculate the rest cardinalities. Let $\mathrm{Stab}(x_i)$ be the group of stabilizers of x_i. Since $|\mathcal{O}_i| = |G|/|\mathrm{Stab}(x_i)|$, it is enough to count $|\mathrm{Stab}(x_i)|$. Let $J_2 = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$, and $J_{2n} = \begin{bmatrix} J_2 & & \\ & \ddots & \\ & & J_2 \end{bmatrix} \in \mathrm{M}_{2n}(\mathbb{F}_q)$. Let $\mathrm{Sp}_{2n}(\mathbb{F}_q) = \{g \in \mathrm{GL}_{2n} \mid gJ_{2n}g^T = J_{2n}\}$. We use the following fact for the calculation:

$$|\mathrm{Sp}_{2n}(\mathbb{F}_q)| = q^{n^2} \prod_{i=1}^{n} (q^{2i} - 1).$$

Let $\mathrm{GSp}_{2n}(\mathbb{F}_q) = \{g \in \mathrm{GL}_{2n} \mid \exists h \in \mathrm{GL}_1, gJ_{2n}g^T = hJ_{2n}\}$. We have $\mathrm{Sp}_{2n}(\mathbb{F}_q) \triangleleft \mathrm{GSp}_{2n}(\mathbb{F}_q)$. For $g \in \mathrm{GSp}_{2n}(\mathbb{F}_p)$, let $\lambda_{2n}(g)$ be the element of GL_1 such that $gJ_{2n}g^T = \lambda_{2n}(g)J_{2n}$. We define a map λ_{2n} as follows:

$$\lambda_{2n} : \mathrm{GSp}(\mathbb{F}_q) \ni g \mapsto \lambda_{2n}(g) \in \mathrm{GL}_1.$$

This λ_{2n} is an surjective group homomorphism. In addition, λ_{2n} induces a group isomorphism

$$\lambda'_{2n} : \mathrm{GSp}(\mathbb{F}_q)/\mathrm{Sp}(\mathbb{F}_q) \ni [g] \mapsto \lambda_{2n}(g) \in \mathrm{GL}_1.$$

Therefore we obtain

$$|\mathrm{GSp}_{2n}(\mathbb{F}_q)| = (q-1)q^{n^2} \prod_{i=1}^{n} (q^{2i} - 1).$$

The structure and the order of $\mathrm{Stab}(x_i)$ for $i = 4, 11 \leq i \leq 18$ is summarized as follows:

x_i	$\mathrm{Stab}(x_i) \cong$	$\lvert\mathrm{Stab}(x_i)\rvert$
x_4	$(\mathrm{GL}_1 \times \mathrm{GSp}_6(\mathbb{F}_q)) \ltimes \mathbb{F}_q$	$(q-1)q \cdot \lvert\mathrm{GSp}_6(\mathbb{F}_q)\rvert$
x_{11}	$((\mathrm{GL}_2) \times (\mathrm{GL}_1)^2) \ltimes \mathbb{F}_q^2$	$q^6 \cdot \mathrm{gl}_2^2$
x_{12}	$(\mathrm{GL}_2)^2 \ltimes \mathbb{F}_q^8$	$q^8 \cdot \mathrm{gl}_2^2$
x_{13}	$(\mathrm{GL}_1 \times (\mathrm{GL}_2)^2) \ltimes \mathbb{F}_q^7$	$q^7(q-1) \cdot \mathrm{gl}_2$
x_{14}	$\mathrm{GL}_2 \times \mathrm{GSp}_4(\mathbb{F}_q)$	$\mathrm{gl}_2 \cdot \lvert\mathrm{GSp}_4(\mathbb{F}_q)\rvert$
x_{15}	$(\mathrm{GL}_1)^2 \times (\mathrm{SL}_2)^2 \ltimes \mathbb{F}_q^3$	$q^3(q-1)\mathrm{gl}_2 \cdot \mathrm{sl}_2$
x_{16}	$\mathfrak{S}_3 \ltimes (\{(g_1, g_2, g_3) \in (\mathrm{GL}_2)^3 \mid \det(g_1) = \det(g_2) = \det(g_3)\})$	$6\mathrm{gl}_2 \cdot ({}_2)^2$
x_{17}	$\mathbb{Z}/2\mathbb{Z} \ltimes (\{(g_1, g_2) \in \mathrm{GL}_2 \times \mathrm{GL}_2(\mathbb{F}_{q^2}) \mid \det(g_1) = \det(g_2) \in \mathrm{GL}_1\})$	$2\mathrm{gl}_2 \cdot (q^4-1)(q^4-q^2)/(q^2-1)$
x_{18}	$\mathbb{Z}/3\mathbb{Z} \ltimes (\{g \in \mathrm{GL}_2(\mathbb{F}_{q^3}) \mid \det(g) \in \mathrm{GL}_1\})$	$3(q^6-1)(q^6-q^3)/(q-1)$

First we consider $\mathrm{Stab}(x_4)$. Assume $g = ((g_{ij})_{1 \leq i,j \leq 2}, g_2) \in \mathrm{Stab}(x_4)$. Since the rank of the first entry of x_4 is 0, we have $g_{12} \neq 0$, and $g_2 J_6 g_2^T = g_{11} J_6$. Thus we obtain $\mathrm{Stab}(x_4) \cong (\mathrm{GL}_1 \times \mathrm{GSp}_6(\mathbb{F}_q)) \ltimes \mathbb{F}_q$.

Next we consider $\mathrm{Stab}(x_{11})$. Assume $g = ((g_{ij})_{1\le i,j\le 2}^{-1}, (h_{ij})_{1\le i,j\le 6}) \in \mathrm{Stab}(x_{11})$. We have

$$
\begin{aligned}
&[h_{ij}]_{1\le i\le 4, 5\le j\le 6} = O_{2,4},\\
&[h_{ij}]_{1\le i\le 4, 1\le j\le 4} \cdot \begin{bmatrix} g_{11}I_2 & g_{21}I_2 \\ g_{12}I_2 & g_{22}I_2 \end{bmatrix} = \begin{bmatrix} J_2[h_{ij}]_{5\le i,j\le 6}^T J_2^{-1} & O_2 \\ O_2 & J_2[h_{ij}]_{5\le i,j\le 6}^T J_2^{-1} \end{bmatrix},\\
&[h_{ij}]_{5\le i\le 6, 1\le j\le 2} \cdot J_2 \cdot [h_{ij}]_{5\le i,j\le 6}^T \in \mathrm{Sym}^2(\mathbb{F}_q^2),\\
&[h_{ij}]_{5\le i\le 6, 3\le j\le 4} \cdot J_2 \cdot [h_{ij}]_{5\le i,j\le 6}^T \in \mathrm{Sym}^2(\mathbb{F}_q^2).
\end{aligned}
$$

Thus we obtain $\mathrm{Stab}(x_{11}) \cong (\mathrm{GL}_2)^2 \ltimes \mathbb{F}_q^6$. Next we consider $\mathrm{Stab}(x_{12})$. Assume $g = ((g_{ij})_{1\le i,j\le 2}^{-1}, (h_{ij})_{1\le i,j\le 6}) \in \mathrm{Stab}(x_{12})$. We obtain $g_{12} = 0$, $\begin{bmatrix} h_{31} & h_{32} \\ h_{41} & h_{42} \end{bmatrix} = \begin{bmatrix} h_{51} & h_{52} \\ h_{61} & h_{62} \end{bmatrix} = \begin{bmatrix} h_{53} & h_{54} \\ h_{63} & h_{64} \end{bmatrix} = 0$, $\begin{vmatrix} h_{11} & h_{12} \\ h_{21} & h_{22} \end{vmatrix} = g_{11}$, $\begin{vmatrix} h_{33} & h_{33} \\ h_{43} & h_{44} \end{vmatrix} = g_{22}$, $\begin{bmatrix} h_{55} & h_{56} \\ h_{65} & h_{66} \end{bmatrix} J_2 \begin{bmatrix} h_{11} & h_{21} \\ h_{12} & h_{22} \end{bmatrix} = g_{22}J_2$, $\begin{bmatrix} h_{33} & h_{34} \\ h_{43} & h_{44} \end{bmatrix} J_2 \begin{bmatrix} h_{13} & h_{23} \\ h_{14} & h_{24} \end{bmatrix} + \begin{bmatrix} h_{35} & h_{36} \\ h_{45} & h_{46} \end{bmatrix} J_2 \begin{bmatrix} h_{11} & h_{21} \\ h_{12} & h_{22} \end{bmatrix} = 0$ and $\begin{bmatrix} h_{11} & h_{12} \\ h_{21} & h_{22} \end{bmatrix} J_2 \begin{bmatrix} h_{15} & h_{25} \\ h_{16} & h_{26} \end{bmatrix} + \begin{bmatrix} h_{15} & h_{16} \\ h_{25} & h_{26} \end{bmatrix} J_2 \begin{bmatrix} h_{11} & h_{21} \\ h_{12} & h_{22} \end{bmatrix} + \begin{bmatrix} h_{13} & h_{14} \\ h_{23} & h_{24} \end{bmatrix} J_2 \begin{bmatrix} h_{13} & h_{23} \\ h_{14} & h_{24} \end{bmatrix} = g_{21}J_2$. It follows that $\mathrm{Stab}(x_{12}) \cong (\mathrm{GL}_2)^2 \ltimes \mathbb{F}_q^8$. Next we consider $\mathrm{Stab}(x_{13})$. Assume $g = ((g_{ij})_{1\le i,j\le 2}^{-1}, (h_{ij})_{1\le i,j\le 6}) \in \mathrm{Stab}(x_{13})$. We obtain $g_{12} = 0$, $\begin{vmatrix} h_{33} & h_{34} \\ h_{43} & h_{44} \end{vmatrix} = g_{22}$, $\begin{bmatrix} h_{31} & h_{32} \\ h_{41} & h_{42} \end{bmatrix} = \begin{bmatrix} h_{51} & h_{52} \\ h_{61} & h_{62} \end{bmatrix} = \begin{bmatrix} h_{53} & h_{54} \\ h_{63} & h_{64} \end{bmatrix} = 0$, $\begin{bmatrix} h_{33} & h_{34} \\ h_{43} & h_{44} \end{bmatrix} J_2 \begin{bmatrix} h_{11} & h_{21} \\ h_{12} & h_{22} \end{bmatrix} = g_{11}J_2$, $\begin{bmatrix} h_{55} & h_{55} \\ h_{66} & h_{66} \end{bmatrix} J_2 \begin{bmatrix} h_{11} & h_{21} \\ h_{12} & h_{22} \end{bmatrix} = g_{22}J_2$, $\begin{bmatrix} h_{11} & h_{12} \\ h_{21} & h_{22} \end{bmatrix} J_2 \begin{bmatrix} h_{13} & h_{23} \\ h_{14} & h_{24} \end{bmatrix} - \begin{bmatrix} h_{13} & h_{14} \\ h_{23} & h_{24} \end{bmatrix} J_2 \begin{bmatrix} h_{11} & h_{21} \\ h_{12} & h_{22} \end{bmatrix} = 0$, $\begin{bmatrix} h_{11} & h_{12} \\ h_{21} & h_{22} \end{bmatrix} J_2 \begin{bmatrix} h_{15} & h_{25} \\ h_{16} & h_{26} \end{bmatrix} - \begin{bmatrix} h_{15} & h_{16} \\ h_{16} & h_{26} \end{bmatrix} J_2 \begin{bmatrix} h_{11} & h_{21} \\ h_{12} & h_{22} \end{bmatrix} + \begin{bmatrix} h_{13} & h_{14} \\ h_{23} & h_{24} \end{bmatrix} J_2 \begin{bmatrix} h_{13} & h_{23} \\ h_{14} & h_{24} \end{bmatrix} = 0$ and $\begin{bmatrix} h_{33} & h_{34} \\ h_{43} & h_{44} \end{bmatrix} J_2 \begin{bmatrix} h_{13} & h_{23} \\ h_{14} & h_{24} \end{bmatrix} + \begin{bmatrix} h_{35} & h_{36} \\ h_{45} & h_{46} \end{bmatrix} J_2 \begin{bmatrix} h_{11} & h_{21} \\ h_{12} & h_{22} \end{bmatrix} = g_{21}J_2$. It follows that $\mathrm{Stab}(x_{13}) \cong (\mathrm{GL}_1 \times (\mathrm{GL}_2)^2) \ltimes \mathbb{F}_q^7$. Next we consider $\mathrm{Stab}(x_{14})$. Assume $g = (g_1^{-1}, g_2) \in \mathrm{Stab}(x_{14})$. By comparing the rank of the two entries, we see that g_1 must be diagonal. Now it is easy to see that $\mathrm{Stab}(x_{14}) \cong \mathrm{GL}_2 \times \mathrm{GSp}_4(\mathbb{F}_q)$. Next we consider $\mathrm{Stab}(x_{15})$. Assume $g = ((g_{ij})_{1\le i,j\le 2}^{-1}, (h_{ij})_{1\le i,j\le 6}) \in \mathrm{Stab}(x_{15})$. We have $g_{12} = g_{21} = 0$, $\begin{bmatrix} h_{13} & h_{14} \\ h_{23} & h_{24} \end{bmatrix} = \begin{bmatrix} h_{15} & h_{16} \\ h_{25} & h_{26} \end{bmatrix} = \begin{bmatrix} h_{31} & h_{32} \\ h_{41} & h_{42} \end{bmatrix} = \begin{bmatrix} h_{51} & h_{52} \\ h_{61} & h_{62} \end{bmatrix} = \begin{bmatrix} h_{53} & h_{54} \\ h_{63} & h_{64} \end{bmatrix} = 0$, $\begin{vmatrix} h_{11} & h_{12} \\ h_{21} & h_{22} \end{vmatrix} = \begin{vmatrix} h_{33} & h_{34} \\ h_{43} & h_{44} \end{vmatrix} = g_{11}$, $\begin{bmatrix} h_{33} & h_{34} \\ h_{43} & h_{44} \end{bmatrix} J_2 \begin{bmatrix} h_{55} & h_{65} \\ h_{56} & h_{66} \end{bmatrix} = g_{22}J_2$ and $\begin{bmatrix} h_{35} & h_{36} \\ h_{45} & h_{46} \end{bmatrix} J_2 \begin{bmatrix} h_{33} & h_{43} \\ h_{34} & h_{44} \end{bmatrix} + \begin{bmatrix} h_{33} & h_{34} \\ h_{43} & h_{44} \end{bmatrix} J_2 \begin{bmatrix} h_{35} & h_{45} \\ h_{36} & h_{46} \end{bmatrix} = 0$. It follows that $\mathrm{Stab}(x_{15}) \cong (\mathrm{GL}_1)^2 \times (\mathrm{SL}_2)^2 \ltimes \mathbb{F}_q^3$. For $i = 16, 17, 18$, the structure of $\mathrm{Stab}(x_i)$ is determined by Wright and Yukie [9, Proposition 4.2, 4.7]. (It is assumed that V is defined over an infinite field in [9], but the method to determine the structures of $\mathrm{Stab}(x_{16})$, $\mathrm{Stab}(x_{17})$, and $\mathrm{Stab}(x_{18})$ holds for the $\mathbb{F}_q$.)

Lastly, these cardinalities $|\mathcal{O}_1|, ..., |\mathcal{O}_{18}|$ are all distinct and their sum total is $q^{30} = |V|$. Therefore we obtain $V = \bigsqcup_{i=1}^{18} \mathcal{O}_i$. □

13.2. **The intersection between the orbits and the subspaces.** The subspaces we choose to calculate the Fourier transform are as follows:

$$\begin{aligned}
W_1 =&\{0\},\\
W_2 =&\langle u_{112}, u_{113}, u_{114}, u_{115}, u_{116}\rangle_{\mathbb{F}_q},\\
W_3 =&\langle u_{123}, u_{124}, u_{125}, u_{126}, u_{134}, u_{135}, u_{136}, u_{145}, u_{146}, u_{156}\rangle_{\mathbb{F}_q},\\
W_4 =&\langle u_{112}, u_{113}, u_{114}, u_{115}, u_{116}, u_{123}, u_{124}, u_{125}, u_{126}, u_{134}, u_{135}, u_{136}, u_{145}, u_{146}, u_{156}\rangle_{\mathbb{F}_q},\\
W_5 =&\langle u_{112}, u_{113}, u_{114}, u_{115}, u_{116}, u_{212}, u_{213}, u_{214}, u_{215}, u_{216}\rangle_{\mathbb{F}_q},\\
W_6 =&\langle u_{112}, u_{113}, u_{114}, u_{115}, u_{116}, u_{123}, u_{124}, u_{125}, u_{126}, u_{212}\rangle_{\mathbb{F}_q},\\
W_7 =&\langle u_{112}, u_{113}, u_{114}, u_{115}, u_{116}, u_{212}, u_{223}, u_{224}, u_{225}, u_{226}\rangle_{\mathbb{F}_q},\\
W_8 =&\langle u_{134}, u_{135}, u_{136}, u_{145}, u_{146}, u_{156}, u_{234}, u_{235}, u_{236}, u_{245}, u_{246}, u_{256}\rangle_{\mathbb{F}_q},\\
W_9 =&\langle u_{112}, u_{113}, u_{114}, u_{115}, u_{116}, u_{212}, u_{213}, u_{214}, u_{215}, u_{216}, u_{223}, u_{224}, u_{225}, u_{226}\rangle_{\mathbb{F}_q},\\
W_{10} =&\langle u_{123}, u_{124}, u_{125}, u_{126}, u_{134}, u_{135}, u_{136}, u_{145}, u_{146}, u_{156}, u_{223}, u_{224}, u_{225}, u_{226}, u_{234}, u_{235}, u_{236}, u_{245}, u_{246}, u_{256}\rangle_{\mathbb{F}_q},\\
W_{11} =&\langle u_{112}, u_{113}, u_{114}, u_{115}, u_{116}, u_{123}, u_{124}, u_{125}, u_{126}, u_{212}, u_{213}, u_{214}, u_{215}, u_{216}, u_{223}, u_{224}, u_{225}, u_{226}\rangle_{\mathbb{F}_q},\\
W_{12} =&\langle u_{114}, u_{115}, u_{116}, u_{124}, u_{125}, u_{126}, u_{134}, u_{135}, u_{136}, u_{145}, u_{146}, u_{156}, u_{245}, u_{246}, u_{256}\rangle_{\mathbb{F}_q},\\
W_{13} =&\langle u_{112}, u_{113}, u_{114}, u_{123}, u_{124}, u_{212}, u_{213}, u_{214}, u_{215}, u_{216}, u_{223}, u_{224}, u_{225}, u_{226}, u_{234}\rangle_{\mathbb{F}_q},\\
W_{14} =&\langle u_{112}, u_{113}, u_{114}, u_{115}, u_{116}, u_{212}, u_{213}, u_{214}, u_{215}, u_{216}, u_{223}, u_{224}, u_{225}, u_{226}, u_{234}, u_{235}, u_{236}, u_{245}, u_{246}, u_{256}\rangle_{\mathbb{F}_q},\\
W_{15} =&\langle u_{134}, u_{135}, u_{136}, u_{145}, u_{146}, u_{156}, u_{213}, u_{214}, u_{215}, u_{216}, u_{223}, u_{224}, u_{225}, u_{226}, u_{234}, u_{235}, u_{236}, u_{245}, u_{246}, u_{256}\rangle_{\mathbb{F}_q},\\
W_{16} =&\langle u_{123}, u_{124}, u_{125}, u_{126}, u_{134}, u_{135}, u_{136}, u_{145}, u_{146}, u_{156}, u_{213}, u_{214}, u_{215}, u_{216}, u_{234}, u_{235}, u_{236}, u_{245}, u_{246}, u_{256}\rangle_{\mathbb{F}_q},\\
W_{17} =&\langle u_{123}, u_{124}, u_{125}, u_{126}, u_{134}, u_{135}, u_{136}, u_{145}, u_{146}, u_{156}, u_{212}, u_{213}, u_{214}, u_{215}, u_{216}, u_{223}\\
&, u_{224}, u_{225}, u_{226}, u_{234}, u_{235}, u_{236}, u_{245}, u_{246}, u_{256}\rangle_{\mathbb{F}_q},\\
W_{18} =&V.
\end{aligned}$$

The orthogonal complements of these subspaces are as follows:
$W_1^{\perp} = W_{18}$, $W_2^{\perp} = W_{17}$, $W_3^{\perp} = W_{14}$, $W_4^{\perp} = W_4$, $W_5^{\perp} = W_{10}$, $W_6^{\perp} = W_{15}$, $W_7^{\perp} = W_{16}$, $W_8^{\perp} = W_{11}$, $W_{12}^{\perp} = W_{12}$, $W_{13}^{\perp} = W_{13}$ and

$$W_9^{\perp} = \langle u_{123}, u_{124}, u_{125}, u_{126}, u_{134}, u_{135}, u_{136}, u_{145}, u_{146}, u_{156}, u_{234}, u_{235}, u_{236}, u_{245}, u_{246}, u_{256}\rangle_{\mathbb{F}_q}.$$

Proposition 13.2. *The cardinalities $|\mathcal{O}_i \cap W_j|$ for the orbit $\mathcal{O}_i$ and the subspace W_j are given as follows:*

	W_1	W_2	W_3	W_4	W_5	W_6	W_7
$\mathcal{O}_1$	1	1	1	1	1	1	1
$\mathcal{O}_2$	0	$[1,0,0,0,0,1,0]$	$[1,0,0,0,1,1,0]$	$[1,0,0,1,0,1,1]$	$[1,0,1,0,0,1,0]$	$[1,0,1,0,0,1,0]$	$[1,0,1,0,0,0,0]c_1$
$\mathcal{O}_3$	0	0	$[2,2,0,1,0,1,0]$	$[2,2,0,2,0,1,1]$	0	$[2,2,1,1,1,0,0]$	0
$\mathcal{O}_4$	0	0	0	$[3,6,0,1,0,1,0]$	0	0	0
$\mathcal{O}_5$	0	0	0	0	$[2,1,1,0,1,1,0]$	$[2,1,2,0,1,0,0]$	$[2,1,1,0,1,0,0]a_1$
$\mathcal{O}_6$	0	0	0	0	0	$[3,2,1,1,1,0,0]$	0
$\mathcal{O}_7$	0	0	0	0	0	0	$[2,3,1,1,1,0,0]$
$\mathcal{O}_8$	0	0	0	0	0	0	0
$\mathcal{O}_9$	0	0	0	0	0	0	0
$\mathcal{O}_{10}$	0	0	0	0	0	0	0
$\mathcal{O}_{11}$	0	0	0	0	0	0	0
$\mathcal{O}_{12}$	0	0	0	0	0	0	0
$\mathcal{O}_{13}$	0	0	0	0	0	0	0
$\mathcal{O}_{14}$	0	0	0	0	0	0	0
$\mathcal{O}_{16}$	0	0	0	0	0	0	0
$\mathcal{O}_{17}$	0	0	0	0	0	0	0
$\mathcal{O}_{18}$	0	0	0	0	0	0	0

W_8	W_9	W_{10}	W_{11}	W_{12}	W_{13}
1	1	1	1	1	1
$[1,0,1,1,1,0,0]$	$[1,0,1,0,0,0,0]d_1$	$[1,1,0,0,1,1,0]$	$[1,0,1,1,0,0,0]c_2$	$[1,0,0,1,0,1,0]$	$[1,0,0,0,0,0,0]e_1$
$[2,2,1,1,0,0,0]$	$[2,2,1,1,1,0,0]$	$[2,2,1,1,0,1,0]$	$[2,2,2,1,1,0,0]$	$[2,2,0,2,0,0,0]c_2$	$[2,2,0,0,0,0,0]e_2$
0	0	0	0	$[3,6,1,1,0,0,0]$	$[3,6,1,0,0,0,0]$
$[2,1,2,1,1,0,0]$	$[2,1,1,0,1,0,0]d_2$	$[2,1,1,1,1,1,0]$	$[2,1,2,1,2,0,0]$	$[2,1,2,1,1,0,0]$	$[2,1,2,0,0,0,0]d_3$
$[3,2,2,1,1,0,0]$	$[3,2,2,1,1,0,0]$	$[3,2,2,1,1,1,0]$	$[3,2,2,2,1,0,0]$	$[3,2,1,2,1,0,0]$	$[3,2,1,0,0,0,0]e_3$
$\frac{1}{2}[2,5,1,1,1,0,0]$	$[2,5,1,1,1,0,0]$	$\frac{1}{2}[2,5,1,1,1,1,0]$	$\frac{1}{2}[2,5,3,1,1,0,0]$	$[2,5,0,2,0,0,0]$	$\frac{1}{2}[2,5,0,0,0,0,0]d_4$
$\frac{1}{2}[4,5,1,1,0,0,0]$	0	$\frac{1}{2}[4,5,1,1,0,1,0]$	$\frac{1}{2}[4,5,1,1,1,0,0]$	0	$\frac{1}{2}[4,5,2,0,0,0,0]$
0	$[3,5,2,1,1,0,0]$	$[3,5,2,1,1,1,0]$	$[3,5,4,1,1,0,0]$	$[3,5,2,2,0,0,0]$	$[3,5,2,0,0,0,0]c_3$
0	0	$[4,6,2,1,1,1,0]$	$[4,6,3,1,1,0,0]$	0	$[4,6,4,0,0,0,0]$
0	0	0	$[4,8,2,1,1,0,0]$	0	$[4,8,2,0,0,0,0]$
0	0	0	0	$[4,6,1,2,0,0,0]$	$[4,6,1,0,0,0,0]c_4$
0	0	0	0	0	$[5,8,2,0,0,0,0]$
0	0	0	0	0	0
0	0	0	0	0	0
0	0	0	0	0	0
0	0	0	0	0	0
0	0	0	0	0	0

W_{14}	W_{15}	W_{16}	W_{17}	W_{18}	$W_9^{\perp}$
1	1	1	1	1	1
$[1,0,0,0,0,1,0]d_5$	$[1,0,1,0,1,1,0]$	$[1,0,1,0,1,0,0]c_5$	$[1,0,0,0,0,2,0]$	$[1,0,1,1,0,1,1]$	$[1,0,2,0,2,0,0]$
$[2,2,0,2,0,1,1]$	$[2,2,1,1,0,0,1]d_2$	$[2,2,1,1,0,0,0]c_1$	$[2,2,0,0,1,1,0]d_5$	$[2,2,1,2,0,1,1]$	$[2,2,1,1,0,0,0]c_6$
$[3,6,0,1,0,1,0]$	$[3,6,1,1,1,0,0]$	0	$[3,6,0,1,0,1,0]$	$[3,6,1,1,0,1,0]$	0
$[2,1,2,0,1,1,0]$	$[2,1,2,1,2,0,0]$	$[2,1,1,0,1,0,0]e_4$	$[2,1,2,0,2,1,0]$	$[2,1,2,1,1,1,1]$	$[2,1,1,2,1,0,0]$
$[3,2,1,1,1,1,0]$	$[3,2,1,1,1,0,0]d_2$	$[3,2,2,1,1,0,0]a_2$	$[3,2,1,2,1,1,0]$	$[3,2,2,2,1,1,1]$	$[3,2,3,1,1,0,0]$
$[2,5,0,1,1,1,0]$	$\frac{1}{2}[2,5,1,1,1,0,0]b_1$	$\frac{1}{2}[2,5,1,1,1,0,0]b_2$	$\frac{1}{2}[2,5,0,1,1,1,0]b_4$	$\frac{1}{2}[2,5,1,2,1,1,1]$	$\frac{1}{2}[2,5,1,1,1,0,0]a_2$
0	$\frac{1}{2}[4,5,1,1,0,0,0]$	$\frac{1}{2}[4,5,1,1,0,0,0]$	$\frac{1}{2}[4,5,1,1,0,1,0]$	$\frac{1}{2}[4,5,1,2,0,1,1]$	$\frac{1}{2}[4,5,1,1,0,0,0]$
$[3,5,2,1,1,1,0]$	$[3,5,3,2,1,0,0]$	$[3,5,2,1,1,0,0]b_3$	$[3,5,2,2,1,1,0]$	$[3,5,3,2,1,1,1]$	$[3,5,3,1,1,0,0]$
0	$[4,6,3,1,1,0,0]$	$[4,6,2,1,1,0,0]a_1$	$[4,6,3,1,1,1,0]$	$[4,6,3,2,1,1,1]$	$[4,6,2,1,1,0,0]$
0	$[4,8,2,1,1,0,0]$	$[4,7,3,1,1,0,0]$	$[4,8,2,1,1,1,0]$	$[4,8,2,2,1,1,1]$	0
$[4,6,1,1,1,1,0]$	$[4,6,1,1,1,0,0]c_4$	0	$[4,6,1,1,2,1,0]$	$[4,6,2,2,1,1,1]$	0
0	$[5,8,2,1,1,0,0]$	0	$[5,8,2,1,1,1,0]$	$[5,8,3,2,1,1,1]$	0
$[3,11,0,1,0,1,0]$	$[3,11,1,1,1,0,0]$	$2[3,9,1,1,1,0,0]$	$[3,11,0,1,0,1,0]b_5$	$[3,11,1,2,0,1,1]$	0
0	$[4,11,1,1,1,0,0]$	$2[4,9,2,1,1,0,0]$	$2[4,11,1,1,1,1,0]$	$[4,11,2,2,1,1,1]$	0
0	0	$[4,11,1,1,1,0,0]$	$\frac{1}{2}[4,13,0,1,1,1,0]$	$\frac{1}{6}[4,13,1,2,1,1,1]$	0
0	0	0	$\frac{1}{2}[5,13,1,1,0,1,0]$	$\frac{1}{2}[5,13,2,2,0,1,1]$	0
0	0	0	0	$\frac{1}{3}[6,13,3,1,1,1,0]$	0

Here, we put $[a,b,c,d,e,f,g] = (q-1)^a q^b (q+1)^c (q^2+q+1)^d (q^2+1)^e (q^4+q^3+q^2+q+1)^f (q^2-q+1)^g$ *and*

$$\begin{aligned}
&a_1 = q+2, && c_1 = 2q^3+2q+1, && d_1 = 2q^4+q^3+2q^2+q+1,\\
&a_2 = 2q+1, && c_2 = q^3+q^2+1, && d_2 = q^4+q^3+2q^2+2q+1,\\
&b_1 = 2q^2+2q+1, && c_3 = 2q^3+5q^2+3q+1, && d_3 = q^4+2q^3+3q^2+q+1,\\
&b_2 = 2q^2+4q+1, && c_4 = q^3+2q^2+q+1, && d_4 = 3q^4+8q^3+10q^2+4q+1,\\
&b_3 = 2q^2+3q+2, && c_5 = 2q^3+q^2+q+1, && d_5 = q^4+q^2+q+1,\\
&b_4 = 2q^2+q+1, && c_6 = q^3+q+1, && e_1 = q^5+4q^4+4q^3+3q^2+2q+1,\\
&b_5 = q^2+2, && && e_2 = 2q^5+4q^4+4q^3+5q^2+3q+1,\\
& && && e_3 = q^5+5q^4+6q^3+6q^2+3q+1,\\
& && && e_4 = q^5+2q^4+3q^3+4q^2+2q+1.
\end{aligned}$$

[Proof]

We obviously have $|\mathcal{O}_1 \cap W_j| = 1$ for all j. For $j = 1, 8, 10$, we obtain the cardinalities $|\mathcal{O}_i \cap W_j|$ form Proposition 8.2 and Proposition 12.1. For $j = 2, 3, 4$, we easily obtain the cardinalities. For $j = 18$, we already calculated the cardinalities in Proposition 13.1. We calculate the rest cardinalities. For $1 \le i, j \le 18$, let $G(i,j) = \{g \in G \mid gx_i \in W_j\}$ and $G(i, j^{\perp}) = \{g \in G \mid gx_i \in W_j^{\perp}\}$ as the proof of Proposition 10.2. Let $\mathrm{Alt}(n, 2m) = \{x \in \wedge^2(\mathbb{F}_q^n) \mid \mathrm{rank}(x) = 2m\}$. Then we have

$$|\mathrm{Alt}(n,2m)| = \frac{|\mathrm{GL}_n|}{q^{2m(n-2m)}|\mathrm{Sp}_{2m}(\mathbb{F}_q)||\mathrm{GL}_{n-2m}|}.$$

To calculate cardinalities for some sets, we use the result in Sections 6 and 8. We refer to $\mathcal{O}_i$, W_j, $W_j^{\perp}$ and M in Section 6 as $\mathcal{O}_i^{224}$, W_j^{224}, $W_j^{\perp\,224}$ and M^{224}, respectively. We refer to $\mathcal{O}_i$, W_j, $W_j^{\perp}$ and M in Section 8 as $\mathcal{O}_i^{2A4}$, W_j^{2A4}, $W_j^{\perp\,2A4}$ and M^{2A4}, respectively.

First we consider W_5. Assume $x = (A, B) \in W_5$. If $x \neq 0$ and $A /\!/ B$, $x \in \mathcal{O}_2$. If $A /\!/ B$, $x \in \mathcal{O}_5$. Thus we obtain $|\mathcal{O}_2 \cap W_5| = (q + 1)(q^5 - 1)$ and $|\mathcal{O}_5 \cap W_5| = (q^5 - 1)(q^5 - q)$.

Next we consider W_6. Let $x = (A, B) = a_{12}u_{112} + b_{12}u_{212} + \sum_{1\leq i\leq 2, i<j\leq 6} b_{ij}u_{2ij} \in W_6$. If $a_{12} = 0$ and $\mathrm{rank}(B) = 2$, we have $x \in \mathcal{O}_2$. If $a_{12} = 0$ and $\mathrm{rank}(B) = 4$, we have $x \in \mathcal{O}_3$. If $a_{12} \neq 0$ and $\mathrm{rank}([b_{ij}]_{1\leq i\leq 2,3\leq j\leq 6}) = 0$, we have $x \in \mathcal{O}_2$. If $a_{12} \neq 0$ and $\mathrm{rank}([b_{ij}]_{1\leq i\leq 2,3\leq j\leq 6}) = 1$, we have $x \in \mathcal{O}_5$. If $a_{12} \neq 0$ and $\mathrm{rank}([b_{ij}]_{1\leq i\leq 2,3\leq j\leq 6}) = 2$, we have $x \in \mathcal{O}_6$. Thus we obtain $|\mathcal{O}_2 \cap W_6| = q^4(q-1) + |(2, 4), 1| + q(q-1)$, $|\mathcal{O}_3 \cap W_6| = q^4(q-1)(q^4-1) + |(2, 4), 2|$, $|\mathcal{O}_5 \cap W_6| = q(q-1)|(2, 4), 1|$, and $|\mathcal{O}_6 \cap W_6| = q(q-1)|(2, 4), 2|$.

Next we consider W_7. Let $x = (A, B) = \sum_{2\leq j\leq 6} a_{1j}u_{11j} + b_{12}u_{212} + \sum_{3\leq j\leq 6} b_{2j}u_{22j} \in W_7$, and

$$r_7(x) := (\mathrm{rank}([a_{1j}]_{3\leq j\leq 6}), \mathrm{rank}([b_{2j}]_{3\leq j\leq 6})).$$

$r_7(x)$ and some additional conditions determine the orbits to which x belongs:

$r_7(x)$	additional condition	x is in
$(0,0)$	$[a_{12}, b_{12}] \neq 0$	$\mathcal{O}_2$
(1,0)	$b_{12} = 0$	$\mathcal{O}_2$
	$b_{12} \neq 0$	$\mathcal{O}_5$
(0,1)	$a_{12} = 0$	$\mathcal{O}_2$
	$a_{12} \neq 0$	$\mathcal{O}_5$
(1,1)	$[a_{1j}]_{3\leq j\leq 6} /\!/ [b_{2j}]_{3\leq j\leq 6}$	$\mathcal{O}_5$
	$[a_{1j}]_{3\leq j\leq 6} \not/\!/ [b_{2j}]_{3\leq j\leq 6}$	$\mathcal{O}_7$

Thus we obtain the cardinalities $|\mathcal{O}_i \cap W_7|$.

Next we consider W_9. Let $W_9^0 = W_9 \setminus W_5$, and $x = (A, B) = \sum_{2\leq j\leq 6} a_{1j}u_{11j} + \sum_{2\leq j\leq 6} b_{1j}u_{21j} + \sum_{3\leq j\leq 6} b_{2j}u_{22j} \in W_9^0$. Then we have $|\mathcal{O}_i \cap W_9| = |\mathcal{O}_i \cap W_9^0| + |\mathcal{O}_i \cap W_5|$. We count $|\mathcal{O}_i \cap W_9^0|$. Some conditions determine the orbits to which x belongs:

$[a_{1j}]_{3\leq j\leq 6}$	additional condition	x is in
$= O_{1,4}$	$a_{12} = 0$, $[b_{1j}]_{3\leq j\leq 6} /\!/ [b_{2j}]_{3\leq j\leq 6}$	$\mathcal{O}_2$
	$a_{12} = 0$, $[b_{1j}]_{3\leq j\leq 6} \not/\!/ [b_{2j}]_{3\leq j\leq 6}$	$\mathcal{O}_3$
	$a_{12} \neq 0$, $[b_{1j}]_{3\leq j\leq 6} /\!/ [b_{2j}]_{3\leq j\leq 6}$	$\mathcal{O}_5$
	$a_{12} \neq 0$, $[b_{1j}]_{3\leq j\leq 6} \not/\!/ [b_{2j}]_{3\leq j\leq 6}$	$\mathcal{O}_6$
$\neq O_{1,4}$, $/\!/ [b_{2j}]_{3\leq j\leq 6}$	$[b_{1j}]_{3\leq j\leq 6} /\!/ [b_{2j}]_{3\leq j\leq 6}$	$\mathcal{O}_5$
	$[b_{1j}]_{3\leq j\leq 6} \not/\!/ [b_{2j}]_{3\leq j\leq 6}$	$\mathcal{O}_6$
$\neq O_{1,4}$, $\not/\!/ [b_{2j}]_{3\leq j\leq 6}$	$[b_{1j}]_{3\leq j\leq 6} \in \langle [a_{1j}]_{3\leq j\leq 6}, [b_{2j}]_{3\leq j\leq 6}\rangle_{\mathbb{F}_q}$	$\mathcal{O}_7$
	$[b_{1j}]_{3\leq j\leq 6} \notin \langle [a_{1j}]_{3\leq j\leq 6}, [b_{2j}]_{3\leq j\leq 6}\rangle_{\mathbb{F}_q}$	$\mathcal{O}_9$

Thus we obtain the cardinalities $|\mathcal{O}_i \cap W_9^0|$.

Next we consider W_{11}. Let $x = (A, B) = \sum_{1\leq i\leq 2, i<j\leq 6} a_{ij}u_{1ij} + \sum_{1\leq i\leq 2, i<j\leq 6} b_{ij}u_{2ij} \in W_{11}$. We define the map

$$F_{11} : W_{11} \ni x \mapsto (\begin{bmatrix} b_{13} & a_{13} \\ b_{23} & a_{23} \end{bmatrix}, \begin{bmatrix} b_{14} & a_{14} \\ b_{24} & a_{24} \end{bmatrix}, \begin{bmatrix} b_{15} & a_{15} \\ b_{25} & a_{25} \end{bmatrix}, \begin{bmatrix} b_{16} & a_{16} \\ b_{26} & a_{26} \end{bmatrix}) \in \mathbb{F}_q^2 \otimes \mathbb{F}_q^2 \otimes \mathbb{F}_q^4.$$

$F_{11}(x)$ and some additional conditions determine the orbits to which x belongs:

$F_{11}(x)$ is in	additional condition	x is in
$\mathcal{O}_3^{224}$	-	$\mathcal{O}_5$
$\mathcal{O}_4^{224}$	-	$\mathcal{O}_5$
$\mathcal{O}_6^{224}$	-	$\mathcal{O}_6$
$\mathcal{O}_7^{224}$	-	$\mathcal{O}_7$
$\mathcal{O}_8^{224}$	-	$\mathcal{O}_8$
$\mathcal{O}_9^{224}$	-	$\mathcal{O}_9$
$\mathcal{O}_{10}^{224}$	-	$\mathcal{O}_{10}$
$\mathcal{O}_{11}^{224}$	-	$\mathcal{O}_{11}$
$\mathcal{O}_2^{224}$	$\begin{bmatrix} a_{13} & a_{14} & a_{15} & a_{16} \\ b_{13} & b_{14} & b_{15} & b_{16} \end{bmatrix} = O_{2,4}$, $\operatorname{rank}(\begin{bmatrix} a_{12} & a_{23} & a_{24} & a_{25} & a_{26} \\ b_{12} & b_{23} & b_{24} & b_{25} & b_{26} \end{bmatrix}) = 1$	$\mathcal{O}_2$
	$\begin{bmatrix} a_{13} & a_{14} & a_{15} & a_{16} \\ b_{13} & b_{14} & b_{15} & b_{16} \end{bmatrix} = O_{2,4}$, $\operatorname{rank}(\begin{bmatrix} a_{12} & a_{23} & a_{24} & a_{25} & a_{26} \\ b_{12} & b_{23} & b_{24} & b_{25} & b_{26} \end{bmatrix}) = 2$	$\mathcal{O}_5$
	$\begin{bmatrix} a_{13} & a_{14} & a_{15} & a_{16} \\ b_{13} & b_{14} & b_{15} & b_{16} \end{bmatrix} \neq O_{2,4}$, $\operatorname{rank}(\begin{bmatrix} a_{12} & a_{13} & a_{14} & a_{15} & a_{16} \\ b_{12} & b_{13} & b_{14} & b_{15} & b_{16} \end{bmatrix}) = 1$	$\mathcal{O}_2$
	$\begin{bmatrix} a_{13} & a_{14} & a_{15} & a_{16} \\ b_{13} & b_{14} & b_{15} & b_{16} \end{bmatrix} \neq O_{2,4}$, $\operatorname{rank}(\begin{bmatrix} a_{12} & a_{13} & a_{14} & a_{15} & a_{16} \\ b_{12} & b_{13} & b_{14} & b_{15} & b_{16} \end{bmatrix}) = 2$	$\mathcal{O}_5$
$\mathcal{O}_5^{224}$	$\operatorname{rank}(\begin{bmatrix} a_{12} & a_{13} & a_{14} & a_{15} & a_{16} \\ b_{12} & b_{13} & b_{14} & b_{15} & b_{16} \end{bmatrix}) = 1$	$\mathcal{O}_3$
	$\operatorname{rank}(\begin{bmatrix} a_{12} & a_{13} & a_{14} & a_{15} & a_{16} \\ b_{12} & b_{13} & b_{14} & b_{15} & b_{16} \end{bmatrix}) = 2$	$\mathcal{O}_6$

Thus we obtain the cardinalities $|\mathcal{O}_i \cap W_{11}|$.

Next we consider W_{12}. We write an element $x \in W_{12}$ as $x = (A, B) = (\begin{bmatrix} O_{3,3} & C \\ -C^T & D \end{bmatrix}, \begin{bmatrix} O_{3,3} & O_{3,3} \\ O_{3,3} & E \end{bmatrix})$, where $C \in \mathrm{M}_3(\mathbb{F}_q)$ and $D, E \in \wedge^2(\mathbb{F}_q^3)$. For this x, let $F = \begin{bmatrix} O_{3,3} & C \\ -C^T & E \end{bmatrix}$, and let $r_{12}(x) = (\operatorname{rank}(F), \operatorname{rank}(A))$. If $E = 0$ and $\operatorname{rank}(A) = 2$, we have $x \in \mathcal{O}_2$. If $E = 0$ and $\operatorname{rank}(A) = 4$, we have $x \in \mathcal{O}_3$. If $E = 0$ and $\operatorname{rank}(A) = 6$, we have $x \in \mathcal{O}_4$. If $E \neq 0$, $\operatorname{rank}(C) = 0$ and $D /\!/ E$, then we have $x \in \mathcal{O}_2$. If $E \neq 0$, $\operatorname{rank}(C) = 0$ and $D \not\!/\!/ E$, then we have $x \in \mathcal{O}_5$. If $E \neq 0$, $\operatorname{rank}(C) = 1$ and $r_{12}(x) = (2, 2)$, then we have $x \in \mathcal{O}_5$. If $E \neq 0$, $\operatorname{rank}(C) = 1$ and $r_{12}(x) = (2, 4)$, then we have $x \in \mathcal{O}_6$. If $E \neq 0$, $\operatorname{rank}(C) = 1$ and $\operatorname{rank}(F) = 4$, then we have $x \in \mathcal{O}_7$. Let $G' = \left\{ \begin{pmatrix} G_{11} & O_{3,3} \\ G_{21} & G_{22} \end{pmatrix} \in \mathrm{GL}_6 \right\}$ and $V' = \left\{ \begin{bmatrix} O_{3,3} & C \\ -C^T & E \end{bmatrix} \in \wedge^2(\mathbb{F}_q^6) \right\}$. G' acts on V' by $G' \times V' \ni (g, x) \mapsto gxg^T \in V'$. When $\operatorname{rank}(C) = 2$, there is an element $g \in G$ such that $gF = \begin{bmatrix} O_{3,3} & C' \\ -C'^T & E' \end{bmatrix}$ where $C' = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$. E' is uniquely determined. Let $E' = \begin{bmatrix} 0 & e_1 & e_2 \\ -e_1 & 0 & e_3 \\ -e_2 & -e_3 & 0 \end{bmatrix}$. If $E \neq 0$, $\operatorname{rank}(C) = 2$ and $e_1 = e_2 = 0$, then we have $x \in \mathcal{O}_6$. If $E \neq 0$, $\operatorname{rank}(C) = 2$ and $(e_1, e_2) \neq 0$, then we have $x \in \mathcal{O}_9$. If $E \neq 0$ and $\operatorname{rank}(C) = 3$, then we have $x \in \mathcal{O}_{12}$. Thus we obtain $|\mathcal{O}_2 \cap W_{12}| = q(q^3 - 1) + (q^3 - 1) + q^2|3, 1|$, $|\mathcal{O}_3 \cap W_{12}| = (q^3 - q^2)|3, 1| + q^3|3, 2|$, $|\mathcal{O}_4 \cap W_{12}| = q^3 \cdot \mathrm{GL}_3$, $|\mathcal{O}_5 \cap W_{12}| = (q^3 - 1)(q^3 - q) + q^2(q^2 - 1)|3, 1|$, $|\mathcal{O}_6 \cap W_{12}| = (q^2 - 1)(q^3 - q^2)|3, 1| + q^3(q - 1)|3, 2|$, $|\mathcal{O}_7 \cap W_{12}| = q^3(q^3 - q^2)|3, 1|$, $|\mathcal{O}_9 \cap W_{12}| = q^4(q^2 - 1)|3, 2|$, and $|\mathcal{O}_{12} \cap W_{12}| = q^3(q^3 - 1)\mathrm{GL}_3$.

Next we consider W_{13}. Let

$$W_{13}^0 = \langle u_{112}, u_{113}, u_{114}, u_{123}, u_{124}, u_{212}, u_{213}, u_{214}, u_{215}, u_{216}, u_{223}, u_{224}, u_{225}, u_{226} \rangle_{\mathbb{F}_q}.$$

and $W_{13}^1 = W_{13} \setminus W_{13}^0$. Then we have $|\mathcal{O}_i \cap W_{13}| = |\mathcal{O}_i \cap W_{13}^0| + |\mathcal{O}_i \cap W_{13}^1|$. First we count $|\mathcal{O}_i \cap W_{13}^0|$. Let $x = (A, B) = \sum_{1 \le i \le 2, i < j \le 4} a_{ij} u_{1ij} + \sum_{1 \le i \le 2, i < j \le 6} b_{ij} u_{2ij} \in W_{13}^0$. We define the map

$$F_{13} : W_{13}^0 \ni x \mapsto (\begin{bmatrix} b_{13} & a_{13} \\ b_{23} & a_{23} \end{bmatrix}, \begin{bmatrix} b_{14} & a_{14} \\ b_{24} & a_{24} \end{bmatrix}, \begin{bmatrix} b_{15} & 0 \\ b_{25} & 0 \end{bmatrix}, \begin{bmatrix} b_{16} & 0 \\ b_{26} & 0 \end{bmatrix}) \in \mathbb{F}_q^2 \otimes \mathbb{F}_q^2 \otimes \mathbb{F}_q^4.$$

$F_{13}(x)$ and some additional conditions determine the orbits to which x belongs:

$F_{13}(x)$ is in	additional condition	x is in
$\mathcal{O}_3^{224}$	-	$\mathcal{O}_5$
$\mathcal{O}_4^{224}$	-	$\mathcal{O}_5$
$\mathcal{O}_6^{224}$	-	$\mathcal{O}_6$
$\mathcal{O}_7^{224}$	-	$\mathcal{O}_7$
$\mathcal{O}_8^{224}$	-	$\mathcal{O}_8$
$\mathcal{O}_9^{224}$	-	$\mathcal{O}_9$
$\mathcal{O}_{10}^{224}$	-	$\mathcal{O}_{10}$
$\mathcal{O}_{11}^{224}$	-	$\mathcal{O}_{11}$
$\mathcal{O}_2^{224}$	$\begin{bmatrix} a_{13} & a_{14} & 0 & 0 \\ b_{13} & b_{14} & b_{15} & b_{16} \end{bmatrix} = O_{2,4}$, $\operatorname{rank}(\begin{bmatrix} a_{12} & a_{23} & a_{24} & 0 & 0 \\ b_{12} & b_{23} & b_{24} & b_{25} & b_{26} \end{bmatrix}) = 1$	$\mathcal{O}_2$
	$\begin{bmatrix} a_{13} & a_{14} & 0 & 0 \\ b_{13} & b_{14} & b_{15} & b_{16} \end{bmatrix} = O_{2,4}$, $\operatorname{rank}(\begin{bmatrix} a_{12} & a_{23} & a_{24} & 0 & 0 \\ b_{12} & b_{23} & b_{24} & b_{25} & b_{26} \end{bmatrix}) = 2$	$\mathcal{O}_5$
	$\begin{bmatrix} a_{13} & a_{14} & 0 & 0 \\ b_{13} & b_{14} & b_{15} & b_{16} \end{bmatrix} \neq O_{2,4}$, $\operatorname{rank}(\begin{bmatrix} a_{12} & a_{13} & a_{14} & 0 & 0 \\ b_{12} & b_{13} & b_{14} & b_{15} & b_{16} \end{bmatrix}) = 1$	$\mathcal{O}_2$
	$\begin{bmatrix} a_{13} & a_{14} & 0 & 0 \\ b_{13} & b_{14} & b_{15} & b_{16} \end{bmatrix} \neq O_{2,4}$, $\operatorname{rank}(\begin{bmatrix} a_{12} & a_{13} & a_{14} & 0 & 0 \\ b_{12} & b_{13} & b_{14} & b_{15} & b_{16} \end{bmatrix}) = 2$	$\mathcal{O}_5$
$\mathcal{O}_5^{224}$	$\operatorname{rank}(\begin{bmatrix} a_{12} & a_{13} & a_{14} & 0 & 0 \\ b_{12} & b_{13} & b_{14} & b_{15} & b_{16} \end{bmatrix}) = 1$	$\mathcal{O}_3$
	$\operatorname{rank}(\begin{bmatrix} a_{12} & a_{13} & a_{14} & 0 & 0 \\ b_{12} & b_{13} & b_{14} & b_{15} & b_{16} \end{bmatrix}) = 2$	$\mathcal{O}_6$

Let

$$W_{13}^2 := (\begin{bmatrix} * & * \\ * & * \end{bmatrix}, \begin{bmatrix} * & * \\ * & * \end{bmatrix}, \begin{bmatrix} * & 0 \\ * & 0 \end{bmatrix}, \begin{bmatrix} * & 0 \\ * & 0 \end{bmatrix}) \subset \mathbb{F}_q^2 \otimes \mathbb{F}_q^2 \otimes \mathbb{F}_q^4.$$

Since $\operatorname{Im}(F_{13}) = W_{13}^2$, we have $|\mathcal{O}_2 \cap W_{13}^0| = q|\mathcal{O}_2^{224} \cap W_{13}^2|$, $|\mathcal{O}_3 \cap W_{13}^0| = q|\mathcal{O}_5^{224} \cap W_{13}^2|$, $|\mathcal{O}_5 \cap W_{13}^0| = (q^2 - q)|\mathcal{O}_2^{224} \cap W_{13}^2| + |\mathcal{O}_3^{224} \cap W_{13}^2| + |\mathcal{O}_4^{224} \cap W_{13}^2|$, $|\mathcal{O}_6 \cap W_{13}^0| = (q^2 - q)|\mathcal{O}_5^{224} \cap W_{13}^2| + |\mathcal{O}_6^{224} \cap W_{13}^2|$ and $|\mathcal{O}_i \cap W_{13}^0| = q|\mathcal{O}_i^{224} \cap W_{13}^2|$ for $7 \le i \le 11$. To calculate $|\mathcal{O}_i^{224} \cap W_{13}^2|$, we use the Fourier transform for $2 \otimes 2 \otimes 4$. We see ${W_{13}^2}^\perp = (\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & * \\ 0 & * \end{bmatrix}, \begin{bmatrix} 0 & * \\ 0 & * \end{bmatrix})$. By Proposition 2.2, we have

$$\begin{bmatrix} |\mathcal{O}_1^{224} \cap W_{13}^2| \\ \vdots \\ |\mathcal{O}_{11}^{224} \cap W_{13}^2| \end{bmatrix} = \frac{|V|}{|{W_{13}^2}^\perp|} \begin{bmatrix} |\mathcal{O}_1^{224}| & & 0 \\ & \ddots & \\ 0 & & |\mathcal{O}_{11}^{224}| \end{bmatrix} M^{224} \begin{bmatrix} |\mathcal{O}_1^{224}| & & 0 \\ & \ddots & \\ 0 & & |\mathcal{O}_{11}^{224}| \end{bmatrix}^{-1} \begin{bmatrix} |\mathcal{O}_1^{224} \cap {W_{13}^2}^\perp| \\ \vdots \\ |\mathcal{O}_{11}^{224} \cap {W_{13}^2}^\perp| \end{bmatrix}.$$

The matrix M^{224} is explicitly determined in Theorem 6.3. Since $\begin{bmatrix}|\mathcal{O}_1^{224}\cap {W_{13}^2}^\perp| & \cdots & |\mathcal{O}_{11}^{224}\cap {W_{13}^2}^\perp|\end{bmatrix}^T = \begin{bmatrix}1 & |2,1| & 0 & 0 & \mathrm{gl}_2 & 0 & \cdots & 0\end{bmatrix}^T$, we have

$$[|cO_i^{224}\cap W_{13}^2|] = \begin{bmatrix} 1 \\ (q-1)(q+1)^2(q^2+q+1) \\ q(q^2-1)^2 \\ q(q^2-1)^2(q^2+q+1) \\ q(q-1)^2(q+1)(q^4+q^3+2q^2+2q+1) \\ q(q-1)^3(q+1)^4 \\ \frac{1}{2}q^3(q-1)^2(q+1)^3(2q+1) \\ \frac{1}{2}q^3(q-1)^4(q+1) \\ q^3(q-1)^3(q+1)^5 \\ q^4(q-1)^4(q+1)^3 \\ q^6(q-1)^4(q+1)^2 \end{bmatrix}.$$

Thus we obtain the cardinalities $|\mathcal{O}_i\cap W_{13}^0|$.

Next we count $|\mathcal{O}_i\cap W_{13}^1|$. Let $x=(A,B)=\sum_{1\le i\le 2, i<j\le 4} a_{ij}u_{1ij}+\sum_{1\le i\le 2, i<j\le 6} b_{ij}u_{2ij}+b_{34}u_{234}\in W_{13}^1$. Let $W_{13}^3 = \left\{x\in W_{13}^1 \;\middle|\; \det(\begin{bmatrix} b_{15} & b_{16} \\ b_{25} & b_{26}\end{bmatrix})=0\right\}$, $W_{13}^4 = \left\{x\in W_{13}^1 \;\middle|\; \det(\begin{bmatrix} b_{15} & b_{16} \\ b_{25} & b_{26}\end{bmatrix})=1\right\}$, and $W_{13}^5 = \left\{x\in W_{13}^1 \;\middle|\; \det(\begin{bmatrix} b_{15} & b_{16} \\ b_{25} & b_{26}\end{bmatrix})=2\right\}$. Then we have $|\mathcal{O}_i\cap W_{13}^1| = |\mathcal{O}_i\cap W_{13}^3|+|\mathcal{O}_i\cap W_{13}^4|+|\mathcal{O}_i\cap W_{13}^5|$. First we count $|\mathcal{O}_i\cap W_{13}^3|$. Let $W_{13}^6 = \left\{x\in W_{13} \;\middle|\; \det(\begin{bmatrix} b_{15} & b_{16} \\ b_{25} & b_{26}\end{bmatrix})=0\right\}$, and $W_{13}^7 = \left\{x\in W_{13}^0 \;\middle|\; \det(\begin{bmatrix} b_{15} & b_{16} \\ b_{25} & b_{26}\end{bmatrix})=0\right\}$. Then we have $|\mathcal{O}_i\cap W_{13}^3| = |\mathcal{O}_i\cap W_{13}^6|-|\mathcal{O}_i\cap W_{13}^7|$. We define the map

$$F'_{13}: W_{13}^6\ni x=\sum_{1\le i\le 2, i<j\le 4} a_{ij}u_{1ij}+\sum_{1\le i<j\le 4} b_{ij}u_{2ij}\mapsto \begin{bmatrix} a_{12} & a_{13} & a_{14} & a_{23} & a_{24} & 0 \\ b_{12} & b_{13} & b_{14} & b_{23} & b_{24} & b_{34}\end{bmatrix}\in \mathbb{F}_q^2\otimes\wedge^2(\mathbb{F}_q^4).$$

Then we have

$$\begin{aligned} {F'_{13}}^{-1}(\mathcal{O}_i^{2A4}) &\subset \mathcal{O}_i (1\le i\le 3),\\ {F'_{13}}^{-1}(\mathcal{O}_i^{2A4}) &\subset \mathcal{O}_{i+1} (4\le i\le 7).\end{aligned}$$

We count $|\mathcal{O}_1^{2A4}\cap F'_{13}(W_{13}^6)|-|\mathcal{O}_1^{2A4}\cap F'_{13}(W_{13}^7)|$. Since $F'_{13}(W_{13}^6) = \begin{bmatrix} 0&0&0&0&0&* \\ 0&0&0&0&0&0\end{bmatrix}^\perp$ and $F'_{13}(W_{13}^7) = \begin{bmatrix} 0&0&0&0&0&* \\ 0&0&0&0&0&*\end{bmatrix}^\perp$. By Proposition 2.2, we have

$$\begin{bmatrix} |\mathcal{O}_1^{2A4}\cap F'_{13}(W_{13}^6)|-|\mathcal{O}_1^{2A4}\cap F'_{13}(W_{13}^7)| \\ \vdots \\ |\mathcal{O}_7^{2A4}\cap F'_{13}(W_{13}^6)|-|\mathcal{O}_7^{2A4}\cap F'_{13}(W_{13}^7)|\end{bmatrix} = \begin{bmatrix} |\mathcal{O}_1^{2A4}| & & 0 \\ & \ddots & \\ 0 & & |\mathcal{O}_7^{2A4}|\end{bmatrix} M^{2A4} \begin{bmatrix} |\mathcal{O}_1^{2A4}| & & 0 \\ & \ddots & \\ 0 & & |\mathcal{O}_7^{2A4}|\end{bmatrix}^{-1}$$

$$\times \begin{bmatrix} \frac{|V|}{F'_{13}(W_{13}^6)^\perp}|\mathcal{O}_1^{2A4}\cap F'_{13}(W_{13}^6)^\perp| - \frac{|V|}{F'_{13}(W_{13}^7)^\perp}|\mathcal{O}_1^{2A4}\cap F'_{13}(W_{13}^7)^\perp| \\ \vdots \\ \frac{|V|}{F'_{13}(W_{13}^6)^\perp}|\mathcal{O}_7^{2A4}\cap F'_{13}(W_{13}^6)^\perp| - \frac{|V|}{F'_{13}(W_{13}^7)^\perp}|\mathcal{O}_7^{2A4}\cap F'_{13}(W_{13}^7)^\perp| \end{bmatrix}.$$

The matrix M^{2A4} is explicitly determined in Theorem 8.3. Since

$$\begin{bmatrix}|\mathcal{O}_1^{2A4}\cap F'_{13}(W_{13}^6)^\perp| & \cdots & |\mathcal{O}_7^{2A4}\cap F'_{13}(W_{13}^6)^\perp|\end{bmatrix}^T = \begin{bmatrix}1 & q-1 & 0 & 0 & 0 & 0 & 0\end{bmatrix}^T$$

and

$$\begin{bmatrix}|\mathcal{O}_1^{2A4}\cap F'_{13}(W_{13}^7)^\perp| & \cdots & |\mathcal{O}_7^{2A4}\cap F'_{13}(W_{13}^7)^\perp|\end{bmatrix}^T = \begin{bmatrix}1 & q^2-1 & 0 & 0 & 0 & 0 & 0\end{bmatrix}^T,$$

we obtain

$$\begin{bmatrix} |\mathcal{O}_1^{2A4}\cap F'_{13}(W_{13}^6)|-|\mathcal{O}_1^{2A4}\cap F'_{13}(W_{13}^7)| \\ \vdots \\ |\mathcal{O}_7^{2A4}\cap F'_{13}(W_{13}^6)|-|\mathcal{O}_7^{2A4}\cap F'_{13}(W_{13}^7)| \end{bmatrix}$$

$$=|V|\begin{bmatrix} |\mathcal{O}_1^{2A4}| & & 0 \\ & \ddots & \\ 0 & & |\mathcal{O}_7^{2A4}| \end{bmatrix} M^{2A4} \begin{bmatrix} |\mathcal{O}_1^{2A4}| & & 0 \\ & \ddots & \\ 0 & & |\mathcal{O}_{11}^{2A4}| \end{bmatrix}^{-1} \begin{bmatrix} q^{11}-q^{10} \\ -q^{11}+q^{10} \\ 0 \\ \vdots \\ 0 \end{bmatrix}$$

$$=\begin{bmatrix} 0 \\ q^4(q-1) \\ q^4(q-1)^2 \\ q^4(q-1)^2(q+1)^2 \\ q^4(q-1)^3(q+1)(q^2+q+1) \\ \frac{1}{2}q^6(q-1)^2(q^3+q^2+q-1) \\ \frac{1}{2}q^6(q-1)^4(q+1) \end{bmatrix}.$$

Next we count $|\mathcal{O}_i\cap W_{13}^4|$. We consider the subgroup of G

$$G_{13} := \left\{ (I_2, \begin{pmatrix} H_{11} & H_{12} & H_{13} \\ O_{2,2} & H_{22} & H_{23} \\ O_{2,2} & O_{2,2} & H_{33} \end{pmatrix}) \in G \;\middle|\; H_{ij}\in M_2(\mathbb{F}_q) \right\}.$$

G_{13} acts on W^4. Let

$$\begin{aligned} y_1 &= u_{215}+u_{234}\in \mathcal{O}_3\cap W_{13}^4, \\ y_2 &= u_{113}+u_{114}+u_{215}+u_{234}\in \mathcal{O}_{10}\cap W_{13}^4, \\ y_3 &= u_{113}+u_{215}+u_{234}\in \mathcal{O}_6\cap W_{13}^4, \\ y_4 &= u_{112}+u_{215}+u_{234}\in \mathcal{O}_9\cap W_{13}^4, \\ y_5 &= u_{112}+u_{113}+u_{215}+u_{234}\in \mathcal{O}_9\cap W_{13}^4, \\ y_6 &= u_{123}+u_{215}+u_{234}\in \mathcal{O}_9\cap W_{13}^4. \end{aligned}$$

We easily see that $G_{13}y_1=\{(A,B)\in W_{13}^4 \mid A=0\}$ and $G_{13}y_2=\{(A,B)\in W_{13}^4 \mid \det(A)=4\}$. Thus we obtain $|G_{13}y_1|=q^5(q-1)|2,1|$ and $G_{13}y_2=q^5(q-1)|2,1|\cdot q\mathrm{gl}_2$. Next we count $|G_{13}y_i|$ for $i=3,4,5$. The structures of the stabilizer subgroups for y_3,y_4,y_5,y_6 are easy to determine. We have

$$\begin{aligned} \{g\in G_{13}\mid gy_3=y_3\} &\cong (\mathrm{GL}_1)^3\ltimes \mathbb{F}_q^{10}, \\ \{g\in G_{13}\mid gy_4=y_4\} &\cong ((\mathrm{GL}_1)^2\times \mathrm{SL}_2)\ltimes \mathbb{F}_q^{9}, \\ \{g\in G_{13}\mid gy_5=y_5\} &\cong (\mathrm{GL}_1)^2\ltimes \mathbb{F}_q^{10}, \\ \{g\in G_{13}\mid gy_6=y_6\} &\cong (\mathrm{GL}_1)^3\ltimes \mathbb{F}_q^{8}. \end{aligned}$$

Therefore we have $|G_{13}y_3|=q^5(q-1)^3(q+1)^3$, $|G_{13}y_4|=q^5(q-1)^3(q+1)^2$, $|G_{13}y_5|=q^5(q-1)^4(q+1)^3$, and $|G_{13}y_6|=q^7(q-1)^3(q+1)^3$. Since the cardinalities $|G_{13}y_i|$ are all distinct and $\sum_{i=1}^6|G_{13}y_i|=(q-1)q^{10}|2,1|=|W_{13}^4|$, we have $\bigcup_{i=1}^6 Gy_i=W_{13}^4$, and therefore $|\mathcal{O}_3\cap W_{13}^4|=|G_{13}y_1|$, $|\mathcal{O}_3\cap W_{13}^{10}|=|G_{13}y_2|$, $|\mathcal{O}_3\cap W_{13}^6|=|G_{13}y_3|$, and $|\mathcal{O}_3\cap W_{13}^9|=\sum_{i=4}^6|G_{13}y_i|$. Next we count $|\mathcal{O}_i\cap W_{13}^5|$. Let $x=(A,B)\in W_{13}^5$. If $\det(A)=0$, we have $x\in\mathcal{O}_4$. If $\det(A)=1$, we have $x\in\mathcal{O}_{12}$. If $\det(A)=2$, we have $x\in\mathcal{O}_{13}$. Thus we obtain $|\mathcal{O}_4\cap W_{13}^5|=q^5(q-1)\mathrm{gl}_2$, $|\mathcal{O}_4\cap W_{13}^5|=q^5(q-1)\mathrm{gl}_2\cdot((q-1)+q|2,1|)$, $|\mathcal{O}_4\cap W_{13}^5|=q^5(q-1)\mathrm{gl}_2\cdot q\mathrm{gl}_2$.

Next we consider W_{14}. Let $x=(A,B)=\sum_{2\le j\le 6}a_{1j}u_{11j}+\sum_{1\le i<j\le 6}b_{ij}u_{2ij}\in W_{14}$. Let $y_1^{14}=x_2$, $y_2^{14}=u_{223}\in\mathcal{O}_2$, $y_3^{14}=x_3$, $y_4^{14}=x_4$, $y_5^{14}=x_5$, $y_6^{14}=u_{112}+0u_{223}\in\mathcal{O}_5$. Define the subgroup G_{14} of G:

$$G_{14}=\{(g_{ij})_{1\le i,j\le 2},(h_{ij})_{1\le i,j\le 6}\in G\mid g_{12}=0, h_{i1}=0 (2\le i\le 6)\}.$$

G_{14} acts on W_{14} by

$$G_{14}\times W_{14}\ni ((g_1,g_2),(A,B))\mapsto (g_2Ag_2^T, g_2Bg_2^T)g_1^T\in W_{14}.$$

We easily see that $|G_{14}y_1^{14}| = q(q^5-1)$, $|G_{14}y_2^{14}| = |\mathrm{GL}_6|/q^8|\mathrm{GL}_4||\mathrm{Sp}_2(\mathbb{F}_q)|$, $|G_{14}y_3^{14}| = |\mathrm{GL}_6|/q^8|\mathrm{GL}_2||\mathrm{Sp}_4(\mathbb{F}_q)|$, $|G_{14}y_4^{14}| = |\mathrm{GL}_6|/|\mathrm{Sp}_6(\mathbb{F}_q)|$. Next we count $|G_{14}y_5^{14}|$, $|G_{14}y_6^{14}|$. We have

$$\{g \in G_{14} \mid gy_i^{14} = y_5^{14}\} \cong ((\mathrm{GL}_1)^3 \times \mathrm{GL}_3) \ltimes \mathbb{F}_q^{12},$$
$$\{g \in G_{14} \mid gy_i^{14} = y_6^{14}\} \cong ((\mathrm{GL}_1)^3 \times \mathrm{GL}_3) \ltimes \mathbb{F}_q^{11}.$$

Thus we obtain $|G_{14}y_5^{14}| = |G_{14}|/q^{15}(q-1)^6(q+1)(q^2+q+1)$ and $|G_{14}y_6^{14}| = |G_{14}|/q^{15}(q-1)^6(q+1)(q^2+q+1)$. Next we count $|G(i,14)|$ for $i = 6, 9, 12, 14$. Write an element of G as $g := ((g_{ij})_{1\le i,j\le 2}, (h_{ij})_{1\le i,j\le 6})$. Assume $gx_i \in W_{14}$. Then we have $g_{12} = 0$ and $\mathrm{rank}(\begin{bmatrix} h_{11} & h_{21} & h_{31} & h_{41} & h_{51} \\ h_{12} & h_{22} & h_{32} & h_{42} & h_{52}\end{bmatrix}) = 1$. Thus we obtain $|G(i,14)| = q(q-1)^2 \cdot (q^5-1)(q+1)(q^2-q)(q^6-q^2)(q^6-q^3)(q^6-q^4)(q^6-q^5)$ for $i = 6, 9, 12, 14$. Next we count $|G(7,14)|$. Let

$$G(7,14)^0 = \{((g_{ij})_{1\le i,j\le 2}, (h_{ij})_{1\le i,j\le 6}) \in G(7,14) \mid g_{12} = 0\},$$
$$G(7,14)^1 = \{((g_{ij})_{1\le i,j\le 2}, (h_{ij})_{1\le i,j\le 6}) \in G(7,14) \mid g_{12} \ne 0\}.$$

If $g = ((g_{ij})_{1\le i,j\le 2}, (h_{ij})_{1\le i,j\le 6}) \in G(7,14)^0$, then we have $\mathrm{rank}(\begin{bmatrix} h_{11} & h_{21} & h_{31} & h_{41} & h_{51} \\ h_{12} & h_{22} & h_{32} & h_{42} & h_{52}\end{bmatrix}) = 1$. Moreover, we easily see that

$$G(7,14)^1 = (-I_2, \begin{pmatrix} 0 & 0 & 1 & \\ 0 & 1 & 0 & \\ 1 & 0 & 0 & \\ & & & I_3 \end{pmatrix}) \cdot G(7,14)^0.$$

Thus we obtain $|G(i,14)| = 2q(q-1)^2 \cdot (q^5-1)(q+1)(q^2-q)(q^6-q^2)(q^6-q^3)(q^6-q^4)(q^6-q^5)$.

Next we consider W_{15}. For $2 \le i < j \le 6$, let v_{ij} be the element of $\wedge^2(\mathbb{F}_q^6)$ whose (i,j)-entry and (j,i)-entry is 1 and -1 respectively and the rest are all zero. Let $W_{15}^0 = \{\sum_{1\le i<j\le 6} a_{ij}v_{ij} \in \wedge^2(\mathbb{F}_q^6) \mid a_{ij} \in \mathbb{F}_q, a_{12} = 0\}$. We easily see that $|\mathcal{O}_i \cap W_{15}| = q|\mathrm{Alt}(4, 2i-2)| + |\{x \in W_{15}^0 \mid \mathrm{rank}(x) = 2i-2\}|$ for $i = 2, 3, 4$. We calculate the cardinalities of $X_i^{15} := \{x \in W_{15}^0 \mid \mathrm{rank}(x) = 2i\}$ for $i = 1, 2, 3$. To count $|X_i^{15}|$, we calculate the Fourier transform for $(\mathrm{GL}_6, \wedge^2(\mathbb{F}_q^6))$. GL_6 acts on $\wedge^2(\mathbb{F}_q^6)$ by $(g, x) \mapsto gxg^T$. The orbits of this action are characterized by the rank of matrices. Let

$$\mathcal{O}_1^{A6} = \{x \in \wedge^2(\mathbb{F}_q^6) \mid \mathrm{rank}(x) = 0\},$$
$$\mathcal{O}_2^{A6} = \{x \in \wedge^2(\mathbb{F}_q^6) \mid \mathrm{rank}(x) = 1\},$$
$$\mathcal{O}_3^{A6} = \{x \in \wedge^2(\mathbb{F}_q^6) \mid \mathrm{rank}(x) = 2\},$$
$$\mathcal{O}_4^{A6} = \{x \in \wedge^2(\mathbb{F}_q^6) \mid \mathrm{rank}(x) = 3\}.$$

We choose the subspace of $\wedge^2(\mathbb{F}_q^6)$ as follows:

$$W_1^{A6} := \{0\}$$
$$W_2^{A6} := \{\sum_{1\le i\le 6, \max(i,4)<j\le 6} a_{ij}v_{ij} \mid a_{ij}\mathbb{F}_q\},$$
$$W_3^{A6} := W_2^{A6\perp} = \{\sum_{1\le i<j\le 4} a_{ij}v_{ij} \mid a_{ij}\mathbb{F}_q\}$$
$$W_4^{A6} := \wedge^2(\mathbb{F}_q^6)$$

By counting $|\mathcal{O}_i^{A6} \cap W_j^{A6}|$, we obtain the Fourier transform for $\wedge^2(\mathbb{F}_q^6)$:

$$\begin{bmatrix} \widehat{e_1^{A6}} \\ \widehat{e_2^{A6}} \\ \widehat{e_3^{A6}} \\ \widehat{e_4^{A6}} \end{bmatrix} = q^{-15} \begin{bmatrix} 1 & [1,0,0,1,0,1,1] & [2,2,0,2,0,1,1] & [3,6,0,1,0,1,0] \\ 1 & g_1 & [1,2,0,1,0,0,0]e_1 & -[2,6,0,1,0,0,0] \\ 1 & e_1 & -[0,2,0,0,0,0,0]e_2 & [1,6,0,0,0,0,0] \\ 1 & -[0,0,0,1,0,0,1] & [0,2,0,1,0,0,1] & -[0,6,0,0,0,0,0] \end{bmatrix} \begin{bmatrix} e_1^{A6} \\ e_2^{A6} \\ e_3^{A6} \\ e_4^{A6} \end{bmatrix}.$$

Here, $e_i^{A6} : \wedge^2(\mathbb{F}_q^6) \to \{0,1\}$ is the indicator function of $\mathcal{O}_i^{A6}$, $e_1 = q^5-q^4-q^2-1$, $e_2 = q^5-q^4+q^3-q^2-1$, and $g_1 = q^7+q^5-q^4-q^2-1$. Moreover, we easily see that $[|W_{15}^{0\,\perp} \cap W_j^{A6}|] = [1, q-1, 0, 0]^T$. Therefore we obtain

$$\begin{bmatrix}|X_0^{15}|\\|X_1^{15}|\\|X_2^{15}|\\|X_3^{15}|\end{bmatrix}=\begin{bmatrix}|W_{15}^0\cap W_1^{A6}|\\|W_{15}^0\cap W_2^{A6}|\\|W_{15}^0\cap W_3^{A6}|\\|W_{15}^0\cap W_4^{A6}|\end{bmatrix}=\begin{bmatrix}1\\(q-1)(q^2+1)(q^5+2q^4+q^3+q^2+q+1)\\q^2(q-1)^2(q^2+q+1)(q^5+q^6+2q^5+3q^4+2q^3+2q^2+q+1)\\q^6(q-1)^3(q+1)(q^2+q+1)\end{bmatrix}.$$

Next we count $|G(i,15)|$ for $5\le i\le 15$. Let $g=(g_1,g_2)=((g_{ij})_{1\le i,j\le 2},(h_{ij})_{1\le i,j\le 6})\in G$ and we consider when $g\in G(i,15)$. $g\cdot(A,B)\in W_{15}$ holds if and only if

$$\begin{cases}[h_{ij}]_{1\le i\le 2,1\le j\le 6}\cdot A\cdot[h_{ij}]^T_{1\le i\le 2,1\le j\le 6} &= O_{2,2},\\ [h_{ij}]_{1\le i\le 2,1\le j\le 6}\cdot B\cdot[h_{ij}]^T_{1\le i\le 2,1\le j\le 6} &= O_{2,2},\\ [h_{ij}]_{1\le i\le 2,1\le j\le 6}\cdot(g_{11}A+g_{12}B) &= O_{2,6}.\end{cases}$$

Let us count $|G(5,15)|$. Let $g=(g_1,g_2)=((g_{ij})_{1\le i,j\le 2},(h_{ij})_{1\le i,j\le 6})\in G(5,15)$. We have $\begin{vmatrix}h_{11}&h_{12}\\h_{21}&h_{22}\end{vmatrix}=\begin{vmatrix}h_{11}&h_{13}\\h_{21}&h_{23}\end{vmatrix}=0$ and $\begin{bmatrix}-g_{11}h_{12}-g_{12}h_{13}&g_{11}h_{11}&g_{12}h_{11}\\-g_{11}h_{22}-g_{12}h_{23}&g_{11}h_{21}&g_{12}h_{21}\end{bmatrix}=O_{2,3}$. It follows that $h_{11}=h_{12}=0$ and $\begin{bmatrix}g_{11}&g_{12}\end{bmatrix}\begin{bmatrix}h_{12}&h_{13}\\h_{22}&h_{23}\end{bmatrix}=O_{1,2}$. Since $\begin{bmatrix}g_{11}&g_{12}\end{bmatrix}\neq O_{1,2}$, we have $\mathrm{rank}(\begin{bmatrix}h_{12}&h_{13}\\h_{22}&h_{23}\end{bmatrix})\le 1$. When $\mathrm{rank}(\begin{bmatrix}h_{12}&h_{13}\\h_{22}&h_{23}\end{bmatrix})=0$, then we have $\mathrm{rank}(\begin{bmatrix}h_{14}&h_{15}&h_{26}\\h_{24}&h_{25}&h_{36}\end{bmatrix})=2$. When $\mathrm{rank}(\begin{bmatrix}h_{12}&h_{13}\\h_{22}&h_{23}\end{bmatrix})=1$ and $\mathrm{rank}(\begin{bmatrix}h_{14}&h_{15}&h_{26}\\h_{24}&h_{25}&h_{36}\end{bmatrix})=1$, we have $\mathrm{rank}(\begin{bmatrix}h_{12}&h_{13}&h_{14}&h_{15}&h_{26}\\h_{22}&h_{23}&h_{24}&h_{25}&h_{36}\end{bmatrix})=2$. When $\mathrm{rank}(\begin{bmatrix}h_{12}&h_{13}\\h_{22}&h_{23}\end{bmatrix})=1$ and $\mathrm{rank}(\begin{bmatrix}h_{14}&h_{15}&h_{26}\\h_{24}&h_{25}&h_{36}\end{bmatrix})=2$, we have no other condition. Thus we obtain $|G(5,15)|=\mathrm{gl}_2\cdot((q^3-1)(q^3-q)+(q^2-1)((q^2-q)(q^2+q+1)+(q^3-1)(q^3-q)))(q^6-q^2)(q^6-q^3)(q^6-q^4)(q^6-q^5)=q^16(q-1)^8(q+1)^4(q^2+q+1)^2(q^2+1)^2$. The counts of the cardinalities $|G(i,15)|$ for $6\le i\le 15$ are carried out in the same way, and we omit the detail.

Next we consider W_{16}. First we count $|\mathcal{O}_2\cap W_{16}|$ and $|\mathcal{O}_3\cap W_{16}|$. Let

$$W_{16}^0=\{x=\sum_{1\le i<j\le 6}a_{ij}u_{1ij}+b_{ij}u_{2ij}\in W_{16}\mid a_{2j}\neq 0,b_{1j}=0(3\le j\le 6)\},$$

$$W_{16}^1=\{x=\sum_{1\le i<j\le 6}a_{ij}u_{1ij}+b_{ij}u_{2ij}\in W_{16}\mid a_{2j}=0,b_{1j}\neq 0(3\le j\le 6)\}.$$

For $i=2,3$, We have $|\mathcal{O}_i\cap W_{16}|=|\mathcal{O}_i\cap W_{16}^0|+|\mathcal{O}_i\cap W_{16}^1|+|\mathcal{O}_i\cap W_8|$. Since $|\mathcal{O}_i\cap W_{16}^0|=|\mathcal{O}_i\cap W_{16}^1|$, we obtain $|\mathcal{O}_i\cap W_{16}|=2|\mathcal{O}_i\cap W_{16}^0|+|\mathcal{O}_i\cap W_8|$ for $i=2,3$. Moreover, we easily see that $|\mathcal{O}_i\cap W_{16}^0|=|\mathrm{Alt}(5,2i-2)|-|\mathrm{Alt}(4,2i-2)|$ for $i=2,3$. Therefore $|\mathcal{O}_i\cap W_{16}|=2|\mathrm{Alt}(5,2i-2)|-2|\mathrm{Alt}(4,2i-2)|+|\mathcal{O}_i\cap W_8|$ for $i=2,3$. Next we count $|G(i,16)|$ for $5\le i\le 16$. Let $g=(g_1,g_2)=((g_{ij})_{1\le i,j\le 2},(h_{ij})_{1\le i,j\le 6})\in G$ and we consider when $g\in G(i,16)$. $g\cdot(A,B)\in W_{16}$ holds if and only if

$$\begin{cases}[h_{1j}]_{1\le j\le 6}\cdot(g_{11}A+g_{12}B) &= O_{1,6},\\ [h_{2j}]_{1\le j\le 6}\cdot(g_{12}A+g_{22}B) &= O_{1,6}.\end{cases}$$

Let us count $|G(5,16)|$. Let $g=(g_1,g_2)=((g_{ij})_{1\le i,j\le 2},(h_{ij})_{1\le i,j\le 6})\in G(5,16)$. We have $h_{11}=h_{21}=0$ and $g_{11}h_{12}+g_{12}h_{13}=g_{21}h_{22}+g_{22}h_{23}=0$. When $\mathrm{rank}(\begin{bmatrix}h_{12}&h_{13}\\h_{22}&h_{23}\end{bmatrix})=0$, we have $\mathrm{rank}(\begin{bmatrix}h_{14}&h_{15}&h_{16}\\h_{24}&h_{25}&h_{26}\end{bmatrix})=2$. When $\mathrm{rank}(\begin{bmatrix}h_{12}&h_{13}\\h_{22}&h_{23}\end{bmatrix})=1$, then we have $h_{12}=h_{13}=0$ or $h_{22}=h_{23}=0$. It follows that $\max\left\{\mathrm{rank}(\begin{bmatrix}h_{1i}&h_{1j}\\h_{2i}&h_{2j}\end{bmatrix})\;\middle|\;\begin{matrix}i=2,3,\\j=4,5,6\end{matrix}\right\}=2$. When $\mathrm{rank}(\begin{bmatrix}h_{12}&h_{13}\\h_{22}&h_{23}\end{bmatrix})=2$, we have $g_{11}h_{12}+g_{12}h_{13}=g_{21}h_{22}+g_{22}h_{23}=0$, $\begin{bmatrix}h_{12}&h_{13}\end{bmatrix}\neq 0$, and $\begin{bmatrix}h_{22}&h_{23}\end{bmatrix}\neq 0$, . Thus we obtain $|G(5,16)|=\mathrm{gl}_2\cdot((q^3-1)(q^3-q)+2(q-1)(q^2-q)(q^6-q^3)+q^6\mathrm{gl}_2)(q^6-q^2)(q^6-q^3)(q^6-q^4)(q^6-q^5)$. The counts of the cardinalities $|G(i,16)|$ for $6\le i\le 16$ are carried out in the same way, and we omit the detail.

Next we consider W_{17}. We easily see that $|\mathcal{O}_i\cap W_{17}|=\mathrm{Alt}(6,2i-2)+q\cdot\mathrm{Alt}(5,2i-2)$ for $i=2,3,4$. Next we count $|G(i,17)|$ for $5\le i\le 17$. Let $g=(g_1,g_2)=((g_{ij})_{1\le i,j\le 2},(h_{ij})_{1\le i,j\le 6})\in G$ and we consider when $g\in G(i,17)$. $g\cdot(A,B)\in W_{17}$ holds if and only if

$$[h_{1j}]_{1\le j\le 6}\cdot(g_{11}A+g_{12}B)=O_{1,6}.$$

Let us count $|G(5,17)|$. Let $g=(g_1,g_2)=((g_{ij})_{1\le i,j\le 2},(h_{ij})_{1\le i,j\le 6})\in G(5,17)$. We have $h_{11}=0$ and $g_{11}h_{12}+g_{12}h_{13}=0$. Thus we obtain $|G(5,17)|=\mathrm{gl}_2\cdot(q^3(q-1)+(q^3-1))(q^6-q)(q^6-q^2)(q^6-q^3)(q^6-q^4)(q^6-q^5)$. The counts of the cardinalities $|G(i,17)|$ for $6\le i\le 17$ are carried out in the same way, and we omit the detail.

Next we consider $W_9^{\perp}$. We easily see that $|\mathcal{O}_i\cap W_9^{\perp}|=\mathrm{Alt}(5,2i-2)+q\cdot\mathrm{Alt}(4,2i-2)$ for $i=2,3,4$. Next we count $|G(i,9^{\perp})|$ for $5\le i\le 9$. Let $g=(g_1,g_2)=((g_{ij})_{1\le i,j\le 2},(h_{ij})_{1\le i,j\le 6})\in G$ and we consider when $g\in G(i,9^{\perp})$. $g\cdot(A,B)\in W_9^{\perp}$ holds if and only if

$$\begin{cases} [h_{1j}]_{1\le j\le 6}\cdot(g_{11}A+g_{12}B) &= O_{1,6},\\ [h_{ij}]_{1\le i\le 2,1\le j\le 6}\cdot(g_{21}A+g_{22}B) &= O_{2,6}.\end{cases}$$

Let us count $|G(5,9^{\perp})|$. Let $g=(g_1,g_2)=((g_{ij})_{1\le i,j\le 2},(h_{ij})_{1\le i,j\le 6})\in G(5,9^{\perp})$. We have $h_{11}=h_{12}=h_{13}=g_{21}=0$ and $g_{11}h_{22}-g_{12}h_{23}=0$. Thus we obtain $|G(5,9^{\perp})|=\mathrm{gl}_2\cdot(q^3(q-1)(q^3-1)+|3,2|)(q^6-q^2)(q^6-q^3)(q^6-q^4)(q^6-q^5)$. The counts of the cardinalities $|G(i,9^{\perp})|$ for $6\le i\le 9$ are carried out in the same way, and we omit the detail. □

13.3. Fourier transform.

Theorem 13.3. *The representation matrix M of the Fourier transform on $\mathcal{F}_V^G$ with respect to the basis $e_1, \dots, e_{18}$ is given as follows:*

$$
q^{-18}\left[\begin{array}{ccccccc}
1 & [1,0,1,1,0,1,1] & [2,2,1,2,0,1,1] & [3,6,1,1,0,1,0] & [2,1,2,1,1,1,1] & [3,2,2,2,1,1,1] & \frac{1}{2}[2,5,1,2,1,1,1] \\
1 & [0,0,0,0,0,0,0]i_1 & [1,2,0,1,0,0,0]i_2 & [2,6,0,1,0,0,0]e_1 & [1,1,2,0,1,0,0]g_1 & [2,2,1,2,1,0,0]g_2 & \frac{1}{2}[1,5,0,1,1,0,0]g_2 \\
1 & [0,0,0,0,0,0,0]i_2 & [0,2,0,0,0,0,0]l_1 & [1,6,0,0,0,0,0]h_1 & [1,1,2,0,1,0,0]e_3 & [1,2,1,0,1,0,0]h_2 & \frac{1}{2}[1,5,0,0,1,0,0]g_5 \\
1 & [0,0,0,1,0,0,1]e_1 & [0,2,0,1,0,0,1]h_1 & [0,6,0,0,0,0,0]i_3 & -[1,1,2,1,1,0,1] & -[1,2,1,1,1,0,1]c_1 & -\frac{1}{2}[1,5,0,1,1,0,1]b_1 \\
1 & [0,0,1,0,0,0,0]g_1 & [1,2,1,1,0,0,0]e_3 & -[2,6,1,1,0,0,0] & [0,1,0,0,0,0,0]k_1 & [1,2,1,1,0,0,0]i_4 & \frac{1}{2}[1,5,1,2,0,0,0]e_8 \\
1 & [0,0,0,1,0,0,0]e_2 & [0,2,0,0,0,0,0]h_2 & -[1,6,0,0,0,0,0]c_1 & [0,1,1,0,0,0,0]i_4 & [0,2,0,0,0,0,0]l_2 & \frac{1}{2}[1,5,0,0,0,0,0]g_7 \\
1 & [0,0,0,0,0,0,0]G_2 & [1,2,0,0,0,0,0]g_5 & -[2,6,0,0,0,0,0]b_1 & [1,1,2,1,0,0,0]e_{10} & [2,2,1,0,0,0,0]g_7 & \frac{1}{2}[0,5,0,0,0,0,0]i_7 \\
1 & [0,0,1,0,0,0,0]e_3 & -[0,2,1,0,0,0,0]e_6 & [1,6,1,0,0,0,0] & [0,1,1,0,1,0,0]g_6 & -[0,2,1,0,1,0,0]g_8 & -\frac{1}{2}[1,5,1,1,2,0,0] \\
1 & [0,0,0,1,0,0,0]e_2 & [0,2,0,0,0,0,0]h_2 & -[1,6,0,0,0,0,0]c_1 & [0,1,0,0,0,0,0]i_5 & [1,2,1,0,0,0,0]h_4 & \frac{1}{2}[0,5,0,0,0,0,0]h_7 \\
1 & [0,0,1,0,0,0,0]e_3 & -[0,2,1,0,0,0,0]e_6 & [1,6,1,0,0,0,0] & [0,1,0,0,0,0,0]i_6 & -[0,2,1,0,0,0,0]g_9 & -\frac{1}{2}[0,5,1,0,0,0,0]e_{11} \\
1 & [0,0,1,0,0,0,0]e_3 & -[0,2,1,0,0,0,0]e_6 & [1,6,1,0,0,0,0] & -[0,1,1,0,0,0,0]e_8 & [1,2,1,0,0,0,0]g_{10} & \frac{1}{2}[0,5,1,0,0,0,0]f_3 \\
1 & [0,0,0,0,0,0,0]g_3 & [0,2,0,1,0,0,0]f_1 & -[0,6,0,0,0,0,0]d_3 & -[0,1,1,0,0,0,0]f_2 & [0,2,0,0,0,0,0]j_1 & -\frac{1}{2}[0,5,0,0,0,0,0]g_{11} \\
1 & -[0,0,0,0,0,1,0] & [0,2,0,0,0,0,0]d_1 & -[0,6,0,0,0,0,0] & [0,1,1,0,1,0,0] & [0,2,0,0,0,0,0]h_5 & -\frac{1}{2}[0,5,0,0,1,0,0]c_2 \\
1 & [0,0,0,0,0,0,0]g_4 & [0,2,0,0,0,0,0]h_3 & -[2,6,0,1,0,0,0] & -[1,1,2,0,2,0,0] & -[1,2,1,0,1,0,0]e_9 & -\frac{1}{2}[0,5,0,0,1,0,0]c_3 \\
1 & [0,0,0,0,0,0,0]e_4 & -[0,2,0,0,0,0,0]e_7 & [1,6,0,0,0,0,0] & -[0,1,1,0,0,0,0]d_4 & [0,2,0,0,0,0,0]h_6 & -\frac{1}{2}[0,5,0,0,0,0,0]e_{12} \\
1 & [0,0,0,0,0,0,0]e_5 & -[1,2,0,0,0,0,0]d_2 & [0,6,0,0,0,0,0]a_1 & -[1,1,2,0,0,0,0]b_2 & -[2,2,1,0,0,0,0]d_5 & \frac{1}{2}[0,5,0,0,0,0,0]e_{13} \\
1 & -[0,0,0,0,0,1,0] & [0,2,0,0,0,0,0]d_1 & -[0,6,0,0,0,0,0] & [0,1,1,0,1,0,0] & -[0,2,1,0,1,0,0]d_3 & -\frac{1}{2}[0,5,0,0,1,0,0]c_2 \\
1 & -[0,0,1,1,0,0,1] & [0,2,1,1,0,0,1] & -[0,6,1,0,0,0,0] & [0,1,1,1,0,0,1] & -[0,2,1,1,0,0,1] & \frac{1}{2}[0,5,1,1,0,0,1]
\end{array}\right.
$$

$$
\begin{array}{cccccc}
\frac{1}{2}[4,5,1,2,0,1,1] & [3,5,3,2,1,1,1] & [4,6,3,2,1,1,1] & [4,8,2,2,1,1,1] & [4,6,2,2,1,1,1] & [5,8,3,2,1,1,1] \\
\frac{1}{2}[3,5,1,1,0,0,0]e_3 & [2,5,2,2,1,0,0]e_2 & [3,6,3,1,1,0,0]e_3 & [3,8,2,1,1,0,0]e_3 & [3,6,1,1,1,0,0]g_3 & -[4,8,2,1,1,1,0] \\
-\frac{1}{2}[2,5,1,0,0,0,0]e_6 & [1,5,2,0,1,0,0]h_2 & -[2,6,3,0,1,0,0]e_6 & -[2,8,2,0,1,0,0]e_6 & [2,6,1,1,1,0,0]f_1 & [3,8,2,0,1,0,0]d_1 \\
\frac{1}{2}[2,5,1,1,0,0,1] & -[1,5,2,1,1,0,1]c_1 & [2,6,3,1,1,0,1] & [2,8,2,1,1,0,1] & -[1,6,1,1,1,0,1]d_3 & -[2,8,2,1,1,0,1] \\
\frac{1}{2}[2,5,0,1,0,0,0]g_6 & [1,5,1,1,0,0,0]i_5 & [2,6,1,1,0,0,0]i_6 & -[2,8,1,1,0,0,0]e_8 & -[2,6,1,1,0,0,0]f_2 & [3,8,2,1,1,0,0] \\
-\frac{1}{2}[1,5,0,0,0,0,0]g_8 & [1,5,2,0,0,0,0]h_4 & -[1,6,2,0,0,0,0]g_9 & [2,8,1,0,0,0,0]g_{10} & [1,6,0,0,0,0,0]j_1 & [2,8,1,0,0,0,0]h_5 \\
-\frac{1}{2}[3,5,1,1,1,0,0] & [1,5,2,0,0,0,0]h_7 & -[2,6,3,0,0,0,0]e_{11} & [2,8,2,0,0,0,0]f_3 & -[2,6,1,0,0,0,0]g_{11} & -[3,8,2,0,1,0,0]c_2 \\
\frac{1}{2}[0,5,0,0,0,0,0]i_8 & [1,5,2,1,1,0,0] & -[1,6,2,0,1,0,0]d_3 & [2,8,1,0,1,0,0]e_{14} & -[1,6,1,0,1,0,0] & -[2,8,2,0,1,0,0]d_6 \\
\frac{1}{2}[2,5,0,1,0,0,0] & [0,5,0,0,0,0,0]j_2 & -[1,6,1,0,0,0,0]g_{12} & -[1,8,1,0,0,0,0]f_4 & [1,6,0,0,0,0,0]f_5 & -[2,8,1,1,0,0,0] \\
-\frac{1}{2}[1,5,0,0,0,0,0]d_3 & -[0,5,1,0,0,0,0]g_{12} & [0,6,0,0,0,0,0]g_{13} & -[1,8,0,1,0,0,0] & -[1,6,1,0,1,0,0] & [1,8,1,0,0,0,0] \\
\frac{1}{2}[2,5,0,0,0,0,0]e_{14} & -[0,5,2,0,0,0,0]f_4 & -[1,6,1,1,0,0,0] & [0,8,0,0,0,0,0]e_{16} & -[1,6,1,0,1,0,0] & [1,8,1,0,0,0,0] \\
-\frac{1}{2}[1,5,0,0,0,0,0] & [0,5,1,0,0,0,0]f_5 & -[1,6,2,0,1,0,0] & -[1,8,1,0,1,0,0] & [0,6,0,0,0,0,0]i_9 & -[1,8,2,0,0,0,1]c_5 \\
-\frac{1}{2}[1,5,0,0,0,0,0]d_6 & -[0,5,1,1,0,0,0] & [0,6,1,0,0,0,0] & [0,8,0,0,0,0,0] & -[0,6,1,0,0,0,1]c_5 & [0,8,0,0,0,0,0]g_{14} \\
\frac{1}{2}[2,5,1,0,0,0,0] & -[1,5,2,0,1,0,0]c_2 & [2,6,3,0,1,0,0] & [2,8,2,0,1,0,0] & -[2,6,1,0,1,0,0]c_5 & [3,8,2,0,1,0,0] \\
-\frac{1}{2}[1,5,0,0,0,0,0]d_7 & [0,5,1,0,0,0,0]d_8 & -[1,6,2,0,0,0,0] & -[1,8,1,0,0,0,0] & -[1,6,1,0,0,0,0]d_9 & -[2,8,2,0,0,0,1] \\
-\frac{1}{2}[3,5,2,0,0,0,0] & -[1,5,2,0,0,0,0]c_4 & [2,6,3,0,0,0,0] & [2,8,2,0,0,0,0] & [1,6,1,0,0,0,0]c_6 & [2,8,2,0,0,0,0]a_1 \\
\frac{1}{2}[0,5,0,0,0,0,0]e_{15} & [0,5,1,0,1,0,0]c_1 & -[1,6,2,0,1,0,0] & -[1,8,1,0,1,0,0] & [0,6,0,0,1,0,0] & [1,8,1,0,1,0,0] \\
-\frac{1}{2}[1,5,0,1,0,0,1] & -[0,5,2,1,0,0,1] & [0,6,1,1,0,0,1] & [0,8,0,1,0,0,1] & [0,6,1,1,0,0,1] & -[0,8,1,1,0,0,1]
\end{array}
$$

$$
\left.\begin{array}{ccccc}
[3,11,1,2,0,1,1] & [4,11,2,2,1,1,1] & \frac{1}{6}[4,13,1,2,1,1,1] & \frac{1}{2}[5,13,2,2,0,1,1] & \frac{1}{3}[6,13,3,1,1,1,0] \\
[2,11,0,1,0,0,0]g_4 & [3,11,1,1,1,0,0]e_4 & \frac{1}{6}[3,13,0,1,1,0,0]e_5 & -\frac{1}{2}[4,13,1,1,0,1,0] & -\frac{1}{3}[5,13,3,1,1,0,0] \\
[1,11,0,0,0,0,0]h_3 & -[2,11,1,0,1,0,0]e_7 & -\frac{1}{6}[3,13,0,0,1,0,0] & \frac{1}{2}[3,13,1,0,0,0,0]d_1 & \frac{1}{3}[4,13,3,0,1,0,0] \\
-[2,11,0,2,0,0,1] & [2,11,1,1,1,0,1] & \frac{1}{6}[1,13,0,1,1,0,1]a_1 & -\frac{1}{2}[2,13,1,1,0,0,1] & -\frac{1}{3}[3,13,3,0,1,0,0] \\
-[2,11,1,1,1,0,0] & -[2,11,1,1,0,0,0]d_4 & -\frac{1}{6}[3,13,1,1,0,0,0]b_2 & \frac{1}{2}[3,13,1,1,0,0,0] & \frac{1}{3}[4,13,2,1,0,0,0] \\
-[1,11,0,0,0,0,0]e_9 & [1,11,0,0,0,0,0]h_6 & -\frac{1}{6}[3,13,0,0,0,0,0]d_5 & -\frac{1}{2}[2,13,1,0,0,0,0]d_3 & -\frac{1}{3}[3,13,2,0,0,0,0] \\
-[1,11,0,0,0,0,0]c_3 & -[2,11,1,0,0,0,0]e_{12} & \frac{1}{6}[2,13,0,0,0,0,0]e_{13} & -\frac{1}{2}[3,13,1,0,0,0,0]c_2 & \frac{1}{3}[4,13,3,0,0,0,0] \\
[1,11,1,0,0,0,0] & -[1,11,1,0,1,0,0]d_7 & -\frac{1}{6}[3,13,2,0,1,0,0] & \frac{1}{2}[1,13,1,0,0,0,0]e_{15} & -\frac{1}{3}[3,13,2,0,1,0,0] \\
-[1,11,0,0,0,0,0]c_2 & [1,11,0,0,0,0,0]d_8 & -\frac{1}{6}[2,13,0,0,0,0,0] & \frac{1}{2}[2,13,0,0,0,0,0]c_1 & -\frac{1}{3}[3,13,2,0,0,0,0] \\
[1,11,1,0,0,0,0] & -[1,11,1,0,0,0,0] & \frac{1}{6}[2,13,1,0,0,0,0] & -\frac{1}{2}[2,13,1,0,0,0,0] & \frac{1}{3}[2,13,1,0,0,0,0] \\
[1,11,1,0,0,0,0] & -[1,11,1,0,0,0,0] & \frac{1}{6}[2,13,1,0,0,0,0] & -\frac{1}{2}[2,13,1,0,0,0,0] & \frac{1}{3}[2,13,1,0,0,0,0] \\
-[1,11,0,0,0,0,0]c_5 & -[1,11,1,0,0,0,0]d_9 & \frac{1}{6}[1,13,0,0,0,0,0]c_6 & \frac{1}{2}[1,13,0,0,0,0,0] & \frac{1}{3}[2,13,2,0,0,0,0] \\
[1,11,0,0,0,0,0] & -[1,11,1,0,0,0,1] & \frac{1}{6}[1,13,0,0,0,0,0]a_1 & \frac{1}{2}[1,13,0,0,0,0,0] & -\frac{1}{3}[1,13,1,0,0,0,0] \\
[0,11,0,0,0,0,0]c_7 & [1,11,1,0,1,0,0]a_2 & -\frac{1}{2}[1,13,0,0,1,0,0] & -\frac{1}{2}[2,13,1,0,0,0,0] & 0 \\
[0,11,0,0,0,0,0]a_2 & [0,11,0,0,0,0,0]c_8 & -\frac{1}{2}[1,13,0,0,0,0,0] & \frac{1}{2}[1,13,0,0,0,0,0] & 0 \\
-3[0,11,0,0,0,0,0] & -3[1,11,1,0,0,0,0] & [0,13,0,0,0,0,0] & 0 & 0 \\
-[0,11,0,0,0,0,0] & [0,11,0,0,1,0,0] & 0 & -[0,13,0,0,0,0,0] & 0 \\
0 & 0 & 0 & 0 & [0,13,0,0,0,0,0]
\end{array}\right]
$$

Here, we put $[a,b,c,d,e,f] = (q-1)^a q^b (q+1)^c (q^2+q+1)^d (q^2+1)^e (q^4+q^3+q^2+q+1)^f (q^2+q+1)^g$ and

$a_1 = 2q-1,$
$a_2 = q-2,$
$b_1 = 2q^2+2q+1,$
$b_2 = 2q^2+1,$
$c_1 = q^3-q-1,$
$c_2 = q^3-q^2-q-1,$
$c_3 = 2q^3-2q^2-q-1,$
$c_4 = q^3-2q^2-q-1,$
$c_5 = q^3-q^2-1,$
$c_6 = 4q^3-2q^2+2q-1,$
$c_7 = q^3-q^2+q-2,$
$c_8 = q^3-q^2-q+2d_1 = q^4+q^2+q+1,$
$d_2 = 2q^4+q^3+3q^2+2q+1,$
$d_3 = q^4-q^3+1,$
$d_4 = q^4-q^2-1,$
$d_5 = 3q^4+3q^3+4q^2+2q+1,$
$d_6 = q^4+1,$
$d_7 = q^4-q^2+1,$
$d_8 = q^4+q^3-q^2-q-1,$
$d_9 = q^4-2q^3+2q^2-q+1,$
$e_1 = q^5-q-1,$
$e_2 = q^5-q^3-1,$
$e_3 = q^5-q^4-q^2-1,$
$e_4 = q^5-q^4-q^3-q^2-q-1,$
$e_5 = 2q^5-q^4-q^3-q^2-q-1,$
$e_6 = q^5-q^4+q^3-q^2-1,$
$e_7 = q^5-q^4+q^3-q^2-q-1,$
$e_8 = q^5-q^2-1,$
$e_9 = q^5+q^3-q^2-q-1,$
$e_{10} = q^5-q^4+q^3-2q^2+q-1,$
$e_{11} = 2q^5-q^4+q^3-2q^2-1,$
$e_{12} = q^5-q^4+q^3-3q^2-q-1,$
$e_{13} = q^5+q^4-2q^3+4q^2+q+1,$
$e_{14} = q^5+q^4+q^2+q+1,$
$e_{15} = q^5+q^4-q+1,$
$e_{16} = q^5-q^3+1,$
$f_1 = q^6-2q^5+q^4+1,$
$f_2 = q^6-q^2-1,$
$f_3 = q^6-2q^5+q^4-q^3+2q^2+1,$
$f_4 = q^6-q^5-q^3+q^2+1,$
$f_5 = q^6+q^5-q^2-q-1,$
$g_1 = q^7+q^5-q^4-q^2-1,$
$g_2 = 2q^7+q^6-q^4-q^3-q^2-q-1,$
$g_3 = q^7-q^4-q^3-q^2-q-1,$
$g_4 = q^7+q^5-q^4-q^3-q^2-q-1,$
$g_5 = 2q^7-q^5-3q^4-3q^3-3q^2-2q-1,$
$g_6 = q^7-q^5+1,$
$g_7 = q^7-2q^6-4q^5-6q^4-5q^3-4q^2-2q-1,$
$g_8 = q^7-q^3+1,$
$g_9 = q^7-q^5-q^3+q^2+1,$
$g_{10} = q^7+q^4+q^3+2q^2+q+1,$
$g_{11} = 2q^7+2q^5-2q^4-2q^2-q-1,$
$g_{12} = q^7-q^6-q^5-q^4+q^2+q+1,$
$g_{13} = q^7-q^4-q^3+q+1,$
$g_{14} = q^7-q^6+q^4-q^3-1,$
$h_1 = q^8-q^5-q^3+q+1,$
$h_2 = q^8-q^7-q^6-q^5+q^2+q+1,$
$h_3 = q^8-q^7-2q^5+q^4-q^3+q^2+q+1,$
$h_4 = q^8-q^7-q^6-q^5+q^4+q^3+2q^2+q+1,$
$h_5 = q^8-q^7-q^2-q-1,$
$h_6 = q^8-2q^7+q^5+q^4+q^3-q^2-q-1,$
$h_7 = 2q^8-2q^7-2q^6-2q^5+q^4+q^3+2q^2+q+1,$
$i_1 = q^9+q^8+q^7+q^6-q^4-q^3-q^2-q-1,$
$i_2 = q^9+q^7+q^6-q^4-q^3-q^2-q-1,$
$i_3 = q^9-q^8-q^6+q^5-q^4+q^3-1,$
$i_4 = q^9-q^6-q^5+q^2+1,$
$i_5 = q^9-q^7-2q^6-q^5+q^3+q^2+q+1,$
$i_6 = q^9-q^6-q^5+q^3+q^2+q+1,$
$i_7 = 2q^9+3q^8-5q^7-2q^6-3q^5+2q^4+q^3+2q^2+q+1,$
$i_8 = 2q^9-q^8+q^7-q^5+2q^4-q^3-q+1,$
$i_9 = q^9-2q^8+q^7-q^6+q^4-q^3+q^2+1,$
$j_1 = q^{10}-q^9+q^6+q^5-q^2-q-1,$
$j_2 = q^{10}-2q^9-3q^8+q^7+4q^6+3q^5+q^4-q^3-2q^2-2q-1,$
$k_1 = q^{11}+q^{10}+q^9-q^8-2q^7-2q^6-q^5+q^3+q^2+q+1,$
$l_1 = q^{12}+q^{10}-q^9+q^8-2q^7-q^6-q^5+q^2+q+1,$
$l_2 = q^{12}-2q^9-2q^8+2q^6+2q^5+q^4-q^2-q-1.$

Corollary 13.4. *The Fourier transform of* Ψ *is given as follows:*

$$\widehat{\Psi}(x) = \begin{cases} q^{-1}+q^{-2}-q^{-4}-q^{-7}+q^{-9}+q^{-10}-q^{-11} & x \in \mathcal{O}_1, \\ q^{-6}-2q^{-7}+q^{-8}-q^{-9}+2q^{-10}-q^{-11} & x \in \mathcal{O}_2, \\ -q^{-9}+3q^{-10}-3q^{-11}+q^{-12} & x \in \mathcal{O}_3, \\ q^{-10}-2q^{-11}+q^{-12}-q^{-13}+q^{-14} & x \in \mathcal{O}_4, \\ 0 & x \in \mathcal{O}_5, \mathcal{O}_9, \mathcal{O}_{10}, \mathcal{O}_{11}, \mathcal{O}_{13}, \mathcal{O}_{15}, \\ q^{-10}-2q^{-11}+q^{-12} & x \in \mathcal{O}_6, \\ -q^{-11}+2q^{-12}-q^{-13} & x \in \mathcal{O}_7, \\ -q^{-11}+q^{-13} & x \in \mathcal{O}_8, \\ -q^{-13}+q^{-14} & x \in \mathcal{O}_{12}, \\ q^{-14}-q^{-15} & x \in \mathcal{O}_{14}, \\ -1/q^{17} & x \in \mathcal{O}_{16}, \mathcal{O}_{18} \\ 1/q^{17} & x \in \mathcal{O}_{17}. \end{cases}$$

In particular we have the following L_1*-norm bound of* $\widehat{\Psi}$*:*

$$\sum_{x \in V} |\widehat{\Psi}(x)| = O(q^{13}).$$

14. Concluding Remarks

Here, we state some notices of the paper and what are observed from the results of the calculations.

14.1. **Verification for the calculation.** For a general linear representation (G, V) over $\mathbb{F}_q$, the matrix M stated after Proposition 2.2 satisfies the following properties:

Lemma 14.1. [5, Lemma 7] *1. Let $S = \mathrm{diag}(|\mathcal{O}_i|)$. Then SM is symmetric.*
2. Suppose that x and $-x$ lie in the same G-orbit for each $x \in V$. Then $M^2 = |V|^{-1} I_r$.

These properties are not needed to calculate the explicit formula, but it is effective to verify our calculations for the explicit formulas. We confirmed that for the prehomogeneous vector spaces in this paper, the matrices M satisfy this lemma.

14.2. **Eigenvalue of M.** Let $\dim V$ be the dimension over $\mathbb{F}_q$ of V. By *2* of Lemma 14.1, the possible eigenvalues of M are either $q^{-\frac{\dim V}{2}}$ or $-q^{-\frac{\dim V}{2}}$. Let m_+ and m_- be the multiplicity of the eigenvalues $q^{-\frac{\dim V}{2}}$ and $-q^{-\frac{\dim V}{2}}$, respectively. We easily see that

$$m_+ + m_- = r$$

and

$$m_+ - m_- = q^{\frac{\dim V}{2}} \mathrm{Tr}(M).$$

Therefore we have m_+ and m_- for each prehomogeneous vector space V in this paper.

Corollary 14.2. *The multiplicities m_+ and m_- for each V are given as follows:*

V	m_+	m_-
$2 \otimes 2 \otimes 2$	6	2
$2 \otimes 2 \otimes 3$	7	3
$2 \otimes 2 \otimes 4$	8	3
$2 \otimes \mathrm{H}_2(\mathbb{F}_{q^2})$	4	2
$2 \otimes \wedge^2(4)$	5	2
binary tri-Hermitian forms over $\mathbb{F}_{q^3}$	3	2
$2 \otimes 3 \otimes 3$	13	8
$2 \otimes \mathrm{H}_3(\mathbb{F}_{q^2})$	9	6
$2 \otimes \wedge^2(6)$	11	7

It may be interesting if a way to calculate the m_+ and m_- systematically is found.

14.3. **Similarity of $2\otimes2\otimes2$, $2\otimes\mathrm{H}_2(\mathbb{F}_{q^2})$ and the space of binary tri-Hermitian forms.** $2\otimes\mathrm{H}_2(\mathbb{F}_{q^2})$ and the space of binary tri-Hermitian forms over $\mathbb{F}_{q^3}$ are the non-split $\mathbb{F}_q$-forms of $2 \otimes 2 \otimes 2$. In all of the three cases, we have

$$|\{x \in V \mid \mathrm{Disc}(\det_x(u, v)) \neq 0\}| = q^3(q-1)^2(q+1)(q^2+1)$$

and

$$\widehat{\Psi}(x) = \begin{cases} q^{-1} + q^{-4} - q^{-5} & x = 0, \\ q^{-4} - q^{-5} & x \neq 0, \mathrm{Disc}(\det_x(u, v)) = 0, \\ -q^{-5} & \mathrm{Disc}(\det_x(u, v)) \neq 0. \end{cases}$$

By these coincidences, we find that the L_1-norms of $\widehat{\Psi}$ for these three spaces also coincide:

$$\sum_{x \in V} |\widehat{\Psi}(x)| = 2q^3 - 2q^2 + 1 - 2q^{-1} + 2q^{-2}.$$

The reason for these coincidences are not yet found.

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