



Voros coefficients for hypergeometric differential equations and Eynard-Orantin's topological recursion

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博 士 論 文

Voros coefficients for hypergeometric differential
equations and Eynard-Orantin's topological recursion

(超幾何微分方程式の Voros 係数と Eynard-Orantin の
位相的漸化式)

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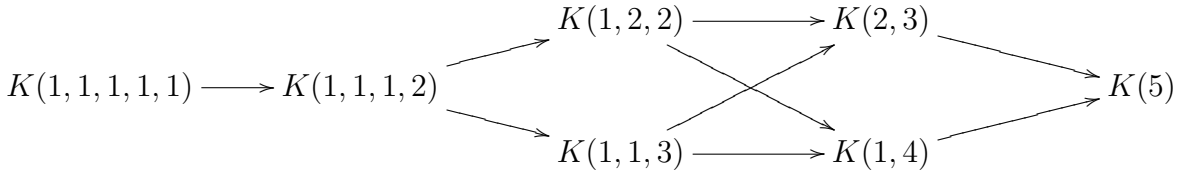
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1 Introduction

In this paper we consider a confluent family of Gauss hypergeometric differential equations and that of hypergeometric differential equations associated with 2-dimensional Garnier systems.

The N -dimensional Garnier system is a Hamiltonian system with N variables obtained through monodromy preserving deformations of second order differential equations on $\mathbb{P}^1$ with $N + 3$ regular singular points. In the case of $N = 1$, the system reduces to the sixth Painlevé equation P_{VI} . It is known that the Gauss hypergeometric function gives a particular solution of P_{VI} and in this sense the Gauss hypergeometric equation is associated with P_{VI} and its underlying monodromy preserving deformations of linear equations. Note that other members of the family of the Gauss hypergeometric equations are obtained from the Gauss equation and P_{VI} by the so-called confluence process. In the same manner a confluent family of hypergeometric differential systems with two independent variables is obtained from the 2-dimensional Garnier system according to the following diagram of degeneration;



Here $K(1, 1, 1, 1, 1)$ designates the 2-dimensional Garnier system and, in general, the symbol $(\#) = (r_1, \dots, r_m)$ means that its underlying monodromy preserving deformation is concerned with a linear equation with m singular points of Poincaré ranks $r_1 - 1, \dots, r_m - 1$. In what follows a hypergeometric differential system with two independent variables is called a hypergeometric system of type $(\#)$ when it is associated with a confluent 2-dimensional Garnier system $K(\#)$. In this article we consider a third order differential equation obtained from the hypergeometric system of type $(1, 1, 3)$, $(2, 3)$ and $(1, 4)$ by fixing the second variable $x_2 = t$. The first one is

$$(1.1) \quad \left\{ 2t^2 \hbar^3 \frac{d^3}{dx^3} + (2x - 3t^2) \hbar^2 \frac{d^2}{dx^2} - (2x - t^2 - 2\hat{\lambda}_{\infty,0} - 2\hat{\lambda}_{\infty,1} - 4\hbar) \hbar \frac{d}{dx} - 2\hat{\lambda}_{\infty,0} - 2\hbar \right\} \psi = 0.$$

We call this equation the hypergeometric equation of type $(1, 1, 3)$. The second one is

$$(1.2) \quad \left\{ 4\hbar^3 \frac{d^3}{dx^3} - 2x \hbar^2 \frac{d^2}{dx^2} + 2(\hat{\lambda}_{\infty} - \hbar) \hbar \frac{d}{dx} - t \right\} \psi = 0.$$

We call this equation the hypergeometric equation of type $(2, 3)$. The third one is

$$(1.3) \quad \left\{ 3\hbar^3 \frac{d^3}{dx^3} + 2t \hbar^2 \frac{d^2}{dx^2} + x \hbar \frac{d}{dx} - \hat{\lambda}_{\infty} \right\} \psi = 0.$$

We call this equation the hypergeometric equation of type $(1, 4)$. The purpose of this article is to analyze a confluent family of Gauss hypergeometric differential equations and the hypergeometric equations of type $(1, 1, 3)$, $(2, 3)$ and $(1, 4)$ from two different points of view, that is, the exact WKB analysis and the topological recursion.

When we study a linear ordinary differential equation by using the exact WKB analysis, the so-called Voros coefficients play an important role. The Voros coefficient is defined as a contour integral of the logarithmic derivative of WKB solutions. Its importance in the global study of solutions of differential equations has been already recognized by the work of Voros ([V1]). As a matter of fact, the Voros coefficient is a key ingredient for describing the Stokes phenomena and the monodromy group of a Schrödinger equation. Moreover, the explicit form of Voros coefficients enables us to analyze the parametric Stokes phenomena. The explicit form of Voros coefficients is now known for the Weber equation, the Whittaker equation, the Kummer equation, the Gauss hypergeometric equation, and the hypergeometric equation of type (1,4) ([SS, T1, KoT, ATT1, AT, AIT, IKo]). The Voros coefficient also plays an essentially important role in discussing the relationship between the exact WKB analysis and cluster algebras ([IN]).

On the other hand, the topological recursion introduced by B. Eynard and N. Orantin ([EO1]) is a generalization of the loop equations that the correlation functions of the matrix model satisfy. For a Riemann surface Σ and meromorphic functions x and y on Σ , it produces an infinite tower of meromorphic differentials $W_{g,n}(z_1, \dots, z_n)$ on Σ . A triplet (Σ, x, y) is called a spectral curve and $W_{g,n}(z_1, \dots, z_n)$ is called a correlation function. Moreover, for a spectral curve, we can define free energies (also called symplectic invariants) F_g . The topological recursion attracts interests both of mathematicians and of physicists since the quantities produced by this method are expected to encode information of various geometric or enumerative invariants. The topological recursion is also closely related to integrable systems. For more details see, e.g., the survey paper [EO3].

The theory of quantum curves, developed by many works including [GS, DM, BE2], is a bridge between the topological recursion (or matrix models) and the WKB analysis. It claims that, under a certain admissibility assumption ([BE2]), a generating function

$$(1.4) \quad \psi(z, \hbar) = \exp \left(\sum_{\substack{g \geq 0 \\ n \geq 1}} \frac{\hbar^{2g-2+n}}{n!} \int^z \cdots \int^z \left(W_{g,n}(z_1, \dots, z_n) - \delta_{g,0} \delta_{n,2} \frac{dx(z_1)dx(z_2)}{(x(z_1) - x(z_2))^2} \right) \right)$$

of Eynard-Orantin's correlation functions $\{W_{g,n}(z_1, \dots, z_n)\}_{g \geq 0, n \geq 1}$ defined by the topological recursion ([EO1]) gives a WKB solution of a linear differential (or difference) equation whose classical limit coincides with the initial spectral curve for the topological recursion. Interestingly enough, the theory of quantum curves also suggests that the expansion coefficients of WKB solutions may contain information about various enumerative invariants which the topological recursion computes (see [E, EO2, EO3, Z] etc).

In previous works for quantum curves, the roles of the free energy and Voros coefficients or their relationships are not discussed even though they are crucially important in the topological recursion and the exact WKB analysis. For example, the following natural question is not answered yet:

What quantity corresponds to the Voros coefficient in the topological recursion ?

In this paper we answer this question for all members of the confluent family of Gauss hypergeometric differential equations and for hypergeometric differential equations of type (1, 1, 3),

(2, 3) and (1, 4). That is, we show that the Voros coefficients are expressed as the difference values of the free energy of the spectral curve corresponding to the classical limit of these hypergeometric equations.

Let us formulate our result in a more specific manner for the Gauss curve

$$(1.5) \quad y^2 - \frac{\lambda_\infty^2 x^2 - (\lambda_\infty^2 + \lambda_0^2 - \lambda_1^2)x + \lambda_0^2}{x^2(1-x)^2} = 0$$

and the associated quantum curve (quantum Gauss curve) given in (3.113). We also set the parameters $\nu_{j\pm} = 1/6$ ($j \in \{0, 1, \infty\}$) in the quantum Gauss curve (3.113) for simplicity. Since the quantum Gauss curve has three singular points at $x = 0, 1, \infty$ on $\mathbb{P}^1$, three Voros coefficients $V^{(j)}(\underline{\lambda}; \hbar)$ ($\underline{\lambda} = (\lambda_0, \lambda_1, \lambda_\infty)$) can be defined depending on the choice of singular points $j \in \{0, 1, \infty\}$ (see §4.1.1 for the precise definition). Having this in mind, we find the following relation between the Voros coefficient and the free energy for the Gauss curve:

Theorem 1.1 (Theorem 4.1 (i)). *The Voros coefficient $V^{(j)}(\underline{\lambda}; \hbar)$ for the singular point $j \in \{0, 1, \infty\}$ of the quantum Gauss curve (3.113) and the free energy $F(\underline{\lambda})$ of the Gauss curve (1.5) are related as follows:*

$$(1.6) \quad V^{(0)}(\underline{\lambda}; \hbar) = F\left(\lambda_0 + \frac{\hbar}{2}, \lambda_1, \lambda_\infty; \hbar\right) - F\left(\lambda_0 - \frac{\hbar}{2}, \lambda_1, \lambda_\infty; \hbar\right) - \frac{1}{\hbar} \frac{\partial F_0}{\partial \lambda_0},$$

$$(1.7) \quad V^{(1)}(\underline{\lambda}; \hbar) = F\left(\lambda_0, \lambda_1 + \frac{\hbar}{2}, \lambda_\infty; \hbar\right) - F\left(\lambda_0, \lambda_1 - \frac{\hbar}{2}, \lambda_\infty; \hbar\right) - \frac{1}{\hbar} \frac{\partial F_0}{\partial \lambda_1},$$

$$(1.8) \quad V^{(\infty)}(\underline{\lambda}; \hbar) = F\left(\lambda_0, \lambda_1, \lambda_\infty + \frac{\hbar}{2}; \hbar\right) - F\left(\lambda_0, \lambda_1, \lambda_\infty - \frac{\hbar}{2}; \hbar\right) - \frac{1}{\hbar} \frac{\partial F_0}{\partial \lambda_\infty}.$$

Note that a similar result holds for all examples in §3.2.5 and §3.3.2. For the proof we employ case-by-case study without using the confluence diagram; this is because coalescence of ramification points occurs and poles of $W_{0,1}(z)$ appear in the confluence process and hence we need more careful treatment to use the confluence diagram.

In the case of second order differential equations we also study quantizations without assuming the “admissibility condition” of [BE2] to obtain the above result. This is done in Theorem 3.5. As the admissibility condition is always assumed in the work of [BE2] to study quantizations, this is an extension of [BE2]. We need this extension because the spectral curve (1.5) for the Gauss hypergeometric differential equation does not satisfy the admissibility condition.

It should be noted that, although there exist several Voros coefficients for one differential equation when it has several singular points (the Gauss hypergeometric equation is a typical example), they are all described in terms of just one quantity, that is, the generating function of the free energies. These results suggest that the free energy is a more fundamental object that controls the Voros coefficients.

The above formula has several applications. Firstly, by combining the above relation and the contiguity relations for hypergeometric equations (cf. §C), we can derive three-term difference equations which the free energy satisfies (Theorem 4.1 (ii) and Theorem 4.9 (ii)). By solving them, we obtain explicit forms of the free energies and those of the Voros coefficients as well

(Theorem 4.1 (iii), (iv) and Theorem 4.9 (iii), (iv)). It is very intriguing that all the free energies discussed in this paper are expressed in terms of the Bernoulli numbers. Consequently, the Voros coefficients of the quantum curves are written in terms of the Bernoulli polynomials. We note that the idea to use the contiguity relations to obtain the explicit form of Voros coefficients is due to [T1].

Finally we note that the biggest difference between the case of second order differential equations and third order differential equations is the existence of the parameter t which comes from the second variable. To obtain explicit forms of the free energies and those of the Voros coefficients in the case of third order equations, we have thus to determine the dependence on the parameter t . This is done in Theorem 4.12 in §4.2.2.

The paper is organized as follows: In §2 we recall some fundamental facts about the exact WKB analysis and Eynard-Orantin's topological recursion. In §3 we study quantization of the spectral curve. We consider second order differential equations in §3.2 and third order differential equations in §3.3. One of our main results is Theorem 3.5 in §3.2, which gives quantizations without assuming the admissibility condition in the case when the spectral curve is described by $P(x, y) = 0$ with a polynomial $P(x, y)$ of degree 2 in y . In §4 we state our main theorem for the confluent family of Gauss hypergeometric differential equations (Theorem 4.1 in §4.1) and for hypergeometric differential equations of type (1,1,3), (2,3) and (1,4) (Theorem 4.9 in §4.2).

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2 Voros coefficients and topological recursion

In this section we briefly recall some basic facts about the exact WKB analysis, and Eynard-Orantin's theory. See [KT] and [EO1] (or [EO3]) respectively for the details of them.

2.1 WKB type formal solution

2.1.1 In the case of second order differential equations

We first recall some basic facts about the exact WKB analysis for second order ordinary differential equations with a small parameter $\hbar \neq 0$ of the form

$$(2.1) \quad \left\{ \hbar^2 \frac{d^2}{dx^2} + q(x, \hbar) \hbar \frac{d}{dx} + r(x, \hbar) \right\} \psi = 0,$$

where $x \in \mathbb{C}$ and

$$(2.2) \quad q(x, \hbar) = q_0(x) + \hbar q_1(x), \quad r(x, \hbar) = r_0(x) + \hbar r_1(x) + \hbar^2 r_2(x)$$

with rational functions $q_j(x)$ and $r_j(x)$ ($j = 0, 1, 2$). We consider (2.1) as a differential equation on the Riemann sphere $\mathbb{P}^1$ with regular or irregular singular points. A WKB type solution of (2.1) is a (formal) solution of (2.1) of the form

$$(2.3) \quad \psi(x, \hbar) = \exp \left[\frac{1}{\hbar} f_{-1}(x) + f_0(x) + \hbar f_1(x) + \cdots \right].$$

A typical way of constructing WKB type solutions is the following: The logarithmic derivative $S(x, \hbar)$ of solutions of (2.1) satisfies the Riccati equation

$$(2.4) \quad \hbar^2 \left(\frac{d}{dx} S(x, \hbar) + S(x, \hbar)^2 \right) + \hbar q(x, \hbar) S(x, \hbar) + r(x, \hbar) = 0.$$

Eq. (2.4) admits a solution of the form

$$(2.5) \quad S(x, \hbar) := \hbar^{-1} S_{-1}(x) + S_0(x) + \hbar S_1(x) + \cdots = \sum_{m=-1}^{\infty} \hbar^m S_m(x).$$

In fact, by substituting (2.5) into (2.4), and equating like powers of both sides with respect to $\hbar$, we obtain

$$(2.6) \quad S_{-1}^2 + q_0(x) S_{-1} + r_0(x) = 0,$$

$$(2.7) \quad (2S_{-1} + q_0(x)) S_0 + q_1(x) S_{-1} + r_1(x) + \frac{dS_{-1}}{dx} = 0,$$

$$(2.8) \quad (2S_{-1} + q_0(x)) S_1 + S_0^2 + q_1(x) S_0 + r_2(x) + \frac{dS_0}{dx} = 0,$$

and

$$(2.9) \quad (2S_{-1} + q_0(x)) S_{m+1} + \sum_{j=0}^m S_{m-j} S_j + q_1(x) S_m + \frac{dS_m}{dx} = 0 \quad (m \geq 1).$$

Eq. (2.6) has two solutions, and once we fix one of them, we can determine S_m for $m \geq 0$ uniquely and recursively by (2.7) – (2.9). Thus we obtain two WKB type solutions of the form

$$(2.10) \quad \psi(x, \hbar) := \exp \left(\int^x S(x, \hbar) dx \right).$$

Here the integral of $S(x, \hbar)$ is defined as the term-wise integral.

Let $\text{Disc}(x)$ be the discriminant of (2.6), i.e.,

$$(2.11) \quad \text{Disc}(x) := \{q_0(x)\}^2 - 4r_0(x) = \{y_+(x) - y_-(x)\}^2.$$

Here $y_{\pm}(x)$ are two solutions of $y^2 + q_0(x)y + r_0(x) = 0$. A point $a \in \mathbb{C}$ is called a turning point of (2.1) if it satisfies $\text{Disc}(a) = 0$. If it is a simple zero, we say it is a simple turning point.

A simple pole $a \in \mathbb{C}$ of $\text{Disc}(x)$ is called a simple-pole type turning point (cf. [Ko2]). We also call $\infty \in \mathbb{P}^1$ is a turning point (resp., simple-pole type turning point) if $X = 0$ is a zero (resp., simple-pole) of $\text{Disc}(1/X)/X^4$ (i.e. if $X = 0$ is a turning point (resp., simple-pole type turning point) of the differential equation obtained by the coordinate change $X = 1/x$ from (2.1)).

Turning points and Stokes curves, which is defined by

$$(2.12) \quad \text{Im} \int_a^x \sqrt{\text{Disc}(x)} dx = 0,$$

where a is a turning point or a simple-pole type turning point, play a central role in the exact WKB analysis. They are used to describe the region where Borel summability of WKB type solutions holds, and to give connection formulas among the Borel sum of WKB type solutions (see [KT, Section 2]). Although we do not discuss analytic properties of WKB type solutions in this paper, turning points and Stokes curves are still useful to visualize a path of integration for Voros coefficients.

The so-called SL-form of a second order linear differential equation, i.e., a second order equation with no first order terms, is more convenient when we apply the exact WKB analysis to it (the word ‘‘SL’’ comes from the fact that the monodromy matrices of the equation of SL-form belong to $\text{SL}(2, \mathbb{C})$). A gauge transformation

$$(2.13) \quad \varphi := \exp \left(\frac{1}{2} \int^x q(x, \hbar) dx \right) \psi$$

of the unknown function ψ makes (2.1) into the SL-form:

$$(2.14) \quad \left\{ \hbar^2 \frac{d^2}{dx^2} - Q(x, \hbar) \right\} \varphi = 0,$$

where

$$(2.15) \quad \begin{aligned} Q(x, \hbar) &:= \frac{1}{4} q(x, \hbar)^2 - r(x, \hbar) + \frac{1}{2} \hbar \frac{\partial}{\partial x} q(x, \hbar) \\ &= \frac{1}{4} q_0(x)^2 - r_0(x) + \frac{\hbar}{2} \left(q_0(x) q_1(x) - 2r_1(x) + \frac{dq_0}{dx} \right) + \frac{\hbar^2}{4} \left(q_1(x)^2 + 2 \frac{dq_1}{dx} - 4r_2(x) \right). \end{aligned}$$

Note that the leading term of the potential $Q(x, \hbar)$ is $\text{Disc}(x)/4$.

2.1.2 In the case of third order differential equations

In this paper we also discuss third order ordinary differential equations with a small parameter $\hbar \neq 0$.

$$(2.16) \quad \left\{ \hbar^3 \frac{d^3}{dx^3} + p(x) \hbar^2 \frac{d^2}{dx^2} + q(x, \hbar) \hbar \frac{d}{dx} + r(x, \hbar) \right\} \psi = 0,$$

where we assume that the functions $p(x)$, $q(x, \hbar) = q_0(x) + \hbar q_1(x)$ and $r(x, \hbar) = r_0(x) + \hbar r_1(x)$ are polynomials in $\hbar$ with coefficients being rational functions in x . We consider (2.16) as a

differential equation on the Riemann sphere $\mathbb{P}^1$ with regular or irregular singular points. For (2.16) we can construct a formal solution, called a WKB solution, of the form

$$(2.17) \quad \psi(x, \hbar) = \exp \left[\int^x S(x, \hbar) dx \right].$$

The logarithmic derivative $S(x, \hbar)$ of WKB solutions of (2.16) satisfies the equation

$$(2.18) \quad \hbar^3 \left(\frac{d^2}{dx^2} S(x, \hbar) + 3S(x, \hbar) \frac{d}{dx} S(x, \hbar) + S(x, \hbar)^3 \right) + p(x) \hbar^2 \left(\frac{d}{dx} S(x, \hbar) + S(x, \hbar)^2 \right) + \hbar q(x, \hbar) S(x, \hbar) + r(x, \hbar) = 0.$$

Eq. (2.18) is a counterpart of the Riccati equation in the second-order case and admits a solution of the form

$$(2.19) \quad S(x, \hbar) := \hbar^{-1} S_{-1}(x) + S_0(x) + \hbar S_1(x) + \cdots = \sum_{m=-1}^{\infty} \hbar^m S_m(x).$$

In fact, by substituting (2.19) into (2.18), and equating like powers of both sides with respect to $\hbar$, we obtain

$$(2.20) \quad S_{-1}^3 + p(x) S_{-1}^2 + q_0(x) S_{-1} + r_0(x) = 0,$$

$$(2.21) \quad (3S_{-1}^2 + 2p(x) S_{-1} + q_0(x)) S_0 + 3S_{-1} \frac{dS_{-1}}{dx} + p(x) \frac{dS_{-1}}{dx} + q_1(x) S_{-1} + r_1(x) = 0,$$

and

$$(2.22) \quad (3S_{-1}^2 + 2p(x) S_{-1} + q_0(x)) S_{m+1} + \sum_{\substack{i+j+k=m-1 \\ i,j,k \geq 0}} S_i S_j S_k + 3 \sum_{j=0}^{m-1} S_{m-j-1} S_j + 3S_m \frac{dS_{-1}}{dx} + 3S_{-1} \frac{dS_m}{dx} + \frac{d^2 S_{m-1}}{dx^2} + p(x) \sum_{j=0}^m S_{m-j} S_j + p(x) \frac{dS_m}{dx} + q_1(x) S_m = 0 \quad (m \geq 0).$$

Eq. (2.20) has three solutions, and once we fix one of them, we can determine S_m for $m \geq 0$ uniquely and recursively by (2.21) and (2.22).

Remark 2.1. In general, WKB solutions don't converge. However, they are Borel summable under suitable conditions.

2.2 Voros coefficients

A Voros coefficient is defined as a properly regularized integral of $S(x, \hbar)$ along a path connecting singular points of (2.1) or (2.16). We assume that $S_m(x)$ with $m \geq 1$ is integrable at any singular point of (2.1) and (2.16). In this situation we can define Voros coefficients by

$$(2.23) \quad V_{\gamma_{b_1, b_2}}(\hbar) := \int_{\gamma_{b_1, b_2}} (S(x, \hbar) - \hbar^{-1} S_{-1}(x) - S_0(x)) dx = \sum_{m=1}^{\infty} \hbar^m \int_{\gamma_{b_1, b_2}} S_m(x) dx,$$

where γ_{b_1, b_2} is a path from a singular point b_1 to a singular point b_2 . Note that Voros coefficients only depend on the class $[\gamma_{b_1, b_2}]$ of paths in the relative homology group

$$(2.24) \quad H_1(\mathbb{P}^1 \setminus \{\text{Turning points}\}, \{\text{Singular points}\}; \mathbb{Z}).$$

Such an integration contour (or a relative homology class) can be understood as a lift of path on x -plane onto the Riemann surface of $S_{-1}(x)$ (i.e., two sheeted covering of x -plane in the case of (2.1), and three sheeted covering of x -plane in the case of (2.16)) after drawing branch cuts and distinguishing the first, second and third sheets of the Riemann surface.

Remark 2.2 ([KT]). By applying the WKB analysis, the monodromy group of Schrödinger equations (2.1) can be computed explicitly. This explicit computation verifies that the monodromy group is described in terms of the characteristic exponents at regular singular points and the contour integrals $\oint_{\alpha} S(x, \hbar) dx$ ($\alpha \in (2.24)$). In particular these contour integrals provide the global data necessary for the description of the monodromy group. These contour integrals can be expressed as the sum of Voros coefficients. Therefore, the Voros coefficient is an important object in the theory of differential equations in a complex domain.

Remark 2.3. The Voros coefficient also enables us to analyze the parametric Stokes phenomenon. The parametric Stokes phenomenon is a Stokes phenomenon which occurs when the parameter like λ_j in the Gauss hypergeometric differential equation varies. The formula describing the parametric Stokes phenomenon plays an important role also in establishing the relationship between the exact WKB analysis and the cluster algebra ([IN]). In particular, the Borel sum of $\exp(V_{\alpha})$ ($\alpha \in (2.24)$) expresses the cluster variable.

2.3 Topological recursion

A starting point of Eynard-Orantin's theory ([EO1]) is a spectral curve. Because we will not discuss the general case in this paper, we restrict ourselves to the case when a spectral curve is of genus 0 (see [EO1] for the general definition; see also Remark 2.11 (ii) below).

Definition 2.4. A spectral curve (of genus 0) is a pair $(x(z), y(z))$ of non-constant rational functions on $\mathbb{P}^1$, such that their exterior differentials dx and dy never vanish simultaneously.

Let R be the set of ramification points of $x(z)$, i.e., R consists of zeros of $dx(z)$ of any order and poles of $x(z)$ whose orders are greater than or equal to two (here we consider x as a branched covering map from $\mathbb{P}^1$ to itself). We further assume that

- (A1) A function field $\mathbb{C}(x(z), y(z))$ coincides with $\mathbb{C}(z)$.
- (A2) If r is a ramification point which is a pole of $x(z)$, and if $Y(z) = -x(z)^2 y(z)$ is holomorphic near r , then $dY(r) \neq 0$.
- (A3) All of the ramification points of $x(z)$ are simple, i.e., the ramification index of each ramification point is two.

(A4) We assume branch points are all distinct, where a branch point is defined as the image of a ramification point by $x(z)$.

Because of the assumption (A1), we can find an irreducible polynomial $P(x, y) \in \mathbb{C}[x, y]$ for which $P(x(z), y(z)) = 0$ holds for any z . Hence $(x(z), y(z))$ becomes a regular meromorphic parametrization in the sense of [SWP, §4] of the plane curve $\mathcal{C} := \{(x, y) \in \mathbb{C} \mid P(x, y) = 0\}$. We also call this curve $\mathcal{C}$ a spectral curve if there is no fear of confusions.

In the assumption (A2), note that the transformation $(x, y) \mapsto (X, Y) := (1/x, -x^2y)$ satisfies $ydx = YdX$ (hence it is symplectic), and that r is a ramification point of $X(z)$. This assumption ensures that Eynard-Orantin's correlation function $W_{g,n}(z_1, z_2, \dots, z_n)$ becomes symmetric in their variables. (See Theorem 2.12 below.) In this assumption we also allow the case when $y(z)$ has a pole at a ramification point.

The assumption (A3) is equivalent to saying that any zero of $dx(z)$ is simple, and that the order of any pole of $x(z)$ is less than or equal to two. From this assumption, for each $r \in R$, there exists a neighborhood U of r such that $x^{-1}(x(z)) \cap U$ consists of only two points for $z \in U \setminus \{r\}$. Let $x^{-1}(x(z)) \cap U = \{z, \bar{z}\}$. Then, we define a map $\bar{\cdot} : U \setminus \{r\} \rightarrow \mathbb{P}^1$, called a conjugate map, by $z \mapsto \bar{z}$. This conjugate map is characterized by the following three conditions:

$$(2.25) \quad \text{(a) } \bar{z} \neq z \text{ if } z \neq r, \quad \text{(b) } x(\bar{z}) = x(z), \quad \text{(c) } \lim_{z \rightarrow r} \bar{z} = r.$$

The conjugate map can be extended to U as a holomorphic map: if $r \in \mathbb{C}$ is a zero of $dx(z)$, and $x(z) = x_0 + x_2(z - r)^2 + x_3(z - r)^3 + \dots$ with a nonzero x_2 , then

$$(2.26) \quad \bar{z} = r - (z - r) - \frac{x_3}{x_2}(z - r)^2 + \dots$$

near r . If $r \in \mathbb{C}$ is a pole of $x(z)$, and $x(z) = (z - r)^{-2}\{x_0 + x_1(z - r) + \dots\}$ with a nonzero x_0 , then

$$(2.27) \quad \bar{z} = r - (z - r) + \frac{x_1}{x_0}(z - r)^2 + \dots.$$

The assumption (A4) will be imposed for variational formulas of [EO1, §5]; see also §2.5.

We need to introduce some notation to define the topological recursion.

Definition 2.5. We introduce

$$(2.28) \quad \tau(z) = \pi^{-1}(\pi(z)), \quad \tau'(z) = \tau(z) \setminus \{z\}.$$

$\tau(z)$ is an analytic map that takes a point $p \in \mathbb{P}^1$ to the set of preimages (with multiplicity) of the inverse image of its projection with respect to the branched covering $\pi : \mathbb{P}^1 \rightarrow \mathbb{P}^1$.

Definition 2.6. For two finite sets A, B , let $A \subseteq_k B$ if $A \subseteq B$ and $|A| = k$.

Definition 2.7. Let $\mathcal{S}(\mathbf{t})$ be the set of set partitions of an ensemble $\mathbf{t} = \{t_1, \dots, t_k\}$.

Then, we define the recursive structure:

Definition 2.8 ([BE2, Definition 3.4]). Let $\{W_{g,n}\}$ be an arbitrary collection of symmetric multidifferential on $(\mathbb{P}^1)^n$ with $g \geq 0$ and $n \geq 1$. Let $k \geq 1$, $\mathbf{t} = \{t_1, \dots, t_k\}$ and $\mathbf{z} = \{z_1, \dots, z_n\}$. Then, we define

$$(2.29) \quad \mathcal{R}^{(k)} W_{g,n+1}(\mathbf{t}; \mathbf{z}) := \sum_{\mu \in \mathcal{S}(\mathbf{t})} \sum_{\sqcup_{i=1}^{l(\mu)} J_i = \mathbf{z}} \sum'_{\sum_{i=1}^{l(\mu)} g_i = g + l(\mu) - k} \left\{ \prod_{i=1}^{l(\mu)} W_{g_i, |\mu_i| + |J_i|}(\mu_i, J_i) \right\}.$$

The first summation in (2.29) is over set partitions of $\mathbf{t}$, $l(\mu)$ is the number of subsets in the set partition μ . The third summation in (2.29) is over all $l(\mu)$ -tuple of non-negative integers $(g_1, \dots, g_{l(\mu)})$ such that $\sum_{i=1}^{l(\mu)} g_i = g + l(\mu) - k$. $\sqcup$ denotes the disjoint union, and the prime ' on the summation symbol in (2.29) means that we exclude terms for $(g_i, |\mu_i| + |J_i|) = (0, 1)$ ($i = 1, \dots, l(\mu)$) (so that $W_{0,1}$ does not appear) in the sum. We also define

$$(2.30) \quad \mathcal{R}^{(0)} W_{g,n+1}(\mathbf{z}) := \delta_{g,0} \delta_{n,0},$$

where $\delta_{i,j}$ is the Kronecker delta symbol.

We now define the topological recursion.

Definition 2.9 ([BE2, Definition 3.6]). Eynard-Orantin's correlation function $W_{g,n}(z_1, \dots, z_n)$ for $g \geq 0$ and $n \geq 1$ is defined as a meromorphic multidifferential on $(\mathbb{P}^1)^n$ using the recurrence relation

$$(2.31) \quad W_{g,n+1}(z_0, z_1, \dots, z_n) := \sum_{r \in R} \operatorname{Res}_{z=r} \left\{ \sum_{k=1}^{r-1} \sum_{\beta(z) \subseteq_k \tau'(z)} (-1)^{k+1} \frac{\int_{\zeta=\alpha}^{\zeta=z} B(\zeta, z_0)}{E^{(k)}(z; \beta(z))} \mathcal{R}^{(k+1)} W_{g,n+1}(z, \beta(z); z_1, \dots, z_n) \right\}$$

for $2g + n \geq 2$ with initial conditions

$$(2.32) \quad W_{0,1}(z_0) := y(z_0) dx(z_0), \quad W_{0,2}(z_0, z_1) = B(z_0, z_1) := \frac{dz_0 dz_1}{(z_0 - z_1)^2}.$$

Here we set $W_{g,n} \equiv 0$ for a negative g and

$$(2.33) \quad E^{(k)}(z; t_1, \dots, t_k) := \prod_{i=1}^k (W_{0,1}(z) - W_{0,1}(t_i)).$$

The second and third summations in (2.31) together mean that we are summing over all subsets of $\tau'(z)$. α is an arbitrary base point on $\mathbb{P}^1$, but it can be checked (see [BE1]) that the definition is actually independent of the choice of base point α . We have also used the multi-index notation: for $I = \{i_1, \dots, i_m\} \subset \{1, 2, \dots, n\}$ with $i_1 < i_2 < \dots < i_m$, $z_I := (z_{i_1}, \dots, z_{i_m})$.

Note that this recursion was called ‘‘global topological recursion’’ in [BE1]. It was shown in [BE1, §3.4] that it is indeed equivalent to the following usual local formulation of the topological recursion when the ramification points are all simple.

Definition 2.10 ([EO1, Definition 4.2]). Eynard-Orantin's correlation function $W_{g,n}(z_1, \dots, z_n)$ for $g \geq 0$ and $n \geq 1$ is defined as a meromorphic multidifferential on $(\mathbb{P}^1)^n$ using the recurrence relation (called Eynard-Orantin's topological recursion)

$$(2.34) \quad W_{g,n+1}(z_0, z_1, \dots, z_n) := \sum_{r \in R} \operatorname{Res}_{z=r} K_r(z_0, z) \left[W_{g-1,n+2}(z, \bar{z}, z_1, \dots, z_n) + \sum'_{\substack{g_1+g_2=g \\ I_1 \sqcup I_2 = \{1,2,\dots,n\}}} W_{g_1,|I_1|+1}(z, z_{I_1}) W_{g_2,|I_2|+1}(\bar{z}, z_{I_2}) \right]$$

for $2g + n \geq 2$ with initial conditions

$$(2.35) \quad W_{0,1}(z_0) := y(z_0)dx(z_0), \quad W_{0,2}(z_0, z_1) = B(z_0, z_1) := \frac{dz_0 dz_1}{(z_0 - z_1)^2}.$$

Here we set $W_{g,n} \equiv 0$ for a negative g ,

$$(2.36) \quad K_r(z_0, z) := \frac{1}{2(y(z) - y(\bar{z}))dx(z)} \int_{\zeta=\bar{z}}^{\zeta=z} B(z_0, \zeta)$$

is a recursion kernel defined near a ramification point $r \in R$, $\sqcup$ denotes the disjoint union, and the prime ' on the summation symbol in (2.34) means that we exclude terms for $(g_1, I_1) = (0, \emptyset)$ and $(g_2, I_2) = (0, \emptyset)$ (so that $W_{0,1}$ does not appear) in the sum. We have also used the multi-index notation: for $I = \{i_1, \dots, i_m\} \subset \{1, 2, \dots, n\}$ with $i_1 < i_2 < \dots < i_m$, $z_I := (z_{i_1}, \dots, z_{i_m})$.

Some remarks are necessary here:

- Remark 2.11.* (i) Eq. (2.31) and (2.34) becomes a recurrence relation with respect to $2g + n$.
- (ii) In general setting formulated by [EO1], a spectral curve is defined as a triplet (Σ, x, y) of compact Riemann surface Σ together with its Torelli marking, and meromorphic functions x, y on Σ . When Σ has genus ≥ 1 , $B(z_0, z_1)$ is replaced by the fundamental bilinear differential (also called the Bergman kernel in the literature) of the second kind with respect to the Torelli marking.
- (iii) Eynard and Orantin defined R as a set of zeros of $dx(z)$ in [EO1], and they call an element of R a branch point. As far as we know, it is [BE2] which pointed out that poles of $x(z)$ of order two or more also play the same role as zeros of $dx(z)$ in the topological recursion. In this paper we use this modified version. See also [BHLMR, BE1], where higher multiple zero of $dx(z)$ is studied.

Theorem 2.12. *Under the assumptions (A1) – (A3), we obtain*

- (i) $W_{g,n}(z_1, \dots, z_n)$ is a symmetric meromorphic multidifferential.
- (ii) Only singular points of $W_{g,n}(z_1, \dots, z_n)$ for $2g + n > 2$ (i.e., $(g, n) \neq (0, 1), (0, 2)$) with respect to each variable are ramification points. They are poles with no residue.

If we further assume that the degree of $x(z)$ is two, then

(iii) The following relation holds for $2g + n \geq 2$:

$$(2.37) \quad W_{g,n}(z_1, \dots, z_n) + W_{g,n}(\bar{z}_1, \dots, z_n) = \delta_{g,0} \delta_{n,2} \frac{dx(z_1)dx(z_2)}{(x(z_1) - x(z_2))^2}.$$

See [EO1, Theorem 4.6], [EO1, Theorem 4.2] and [EO1, Theorem 4.4] respectively for the proof. (Although the original paper [EO1] does not include higher order poles of $x(z)$ to the set of ramification points, the proof can be given in the same way.)

It turns out that, ramification points given as higher order poles of $x(z)$ sometimes do not contribute to the topological recursion; namely, the correlation functions $W_{g,n}(z_1, \dots, z_n)$ may be holomorphic at such ramification points for each variable z_i ($i = 1, \dots, n$) except for $(g, n) = (0, 1)$, and the residue at those points in (2.34) may become zero. This happens for examples which will be considered in this paper. Therefore we introduce the following notion.

Definition 2.13. A ramification point r is said to be ineffective if the correlation functions $W_{g,n}(z_1, \dots, z_n)$ for $(g, n) \neq (0, 1)$ are holomorphic at $z_i = r$ for each $i = 1, \dots, n$. A ramification point which is not ineffective is called effective. The set of effective ramification points is denoted by $R^* (\subset R)$.

The following properties of ineffective ramification points are important in this paper.

Proposition 2.14. Under the assumptions (A1) – (A3), we obtain the following:

- (i) Let r be a ramification point. Then, r is an ineffective ramification point if and only if $(y(z) - y(\bar{z}))dx(z)$ has a pole at r .
- (ii) If $r \in R$ is an ineffective ramification point, then the residue at r in (2.34) is zero.

The proof will be given in §B (cf. Propositions B.1 and B.4). The property (i) above implies that an ineffective ramification point often appears as a double pole of $x(z)$.

These properties (Theorem 2.12 and Proposition 2.14) of the correlation functions are fundamental, and we often use them without any reference.

Remark 2.15 (cf. [BE1], [BE2, Remark 3.10]). Correlation functions also satisfy (2.34), where $K_r(z_0, z)$ is now replaced by

$$(2.38) \quad K_{D,r}(z_0, z) := \frac{1}{(y(z) - y(\bar{z}))dx(z)} \int_{\zeta \in D(z)} B(z_0, \zeta)$$

with any divisor $D(z) = [z] - \sum_{j=1}^m \nu_j [\beta_j]$ with $\beta_j \in \mathbb{P}^1 \setminus R^*$ and $\nu_j \in \mathbb{C}$ satisfying $\sum_{j=1}^m \nu_j = 1$. (See §A for the definition of the integral with a divisor.)

2.4 Free energy through the topological recursion

The g -th free energy F_g ($g \geq 0$) is a complex number defined for the spectral curve, and one of the most important objects in Eynard-Orantin's theory. It is also called a symplectic invariant since it is “almost” invariant under symplectic transformations of spectral curves (see [EO4] for the details).

Definition 2.16 ([EO1, Definition 4.3]). The g -th free energy F_g is defined by

$$(2.39) \quad F_g := \frac{1}{2-2g} \sum_{r \in R} \operatorname{Res}_{z=r} [\Phi(z) W_{g,1}(z)] \quad (g \geq 2),$$

where $\Phi(z)$ is a primitive of $y(z)dx(z)$. The free energies F_0 and F_1 for $g = 0$ and 1 are also defined, but in a different manner (see [EO1, §4.2.2 and §4.2.3] for the definition).

Note that the right-hand side of (2.39) does not depend on the choice of the primitive because $W_{g,1}$ has no residue at each ramification point (see Theorem 2.12 (ii)).

In applications (and in our article), the generating series

$$(2.40) \quad F := \sum_{g=0}^{\infty} \hbar^{2g-2} F_g$$

of F_g 's is crucially important. We also call the generating series (2.40) the free energy of the spectral curve.

2.5 Variational formulas for the correlation functions

In this article we will consider a family of spectral curves parametrized by complex parameters. For our purpose, we briefly recall the variational formulas obtained by [EO1, §5] which describe the differentiation of the correlation functions $W_{g,n}$ and the free energies F_g with respect to the parameters.

Suppose that we have given a family $(x_\varepsilon(z), y_\varepsilon(z))$ of spectral curves parametrized by a complex parameter ε which lies on a certain domain $U \subset \mathbb{C}$ such that

- $x_\varepsilon(z), y_\varepsilon(z)$ depend holomorphically on $\varepsilon \in U$.
- $x_\varepsilon(z), y_\varepsilon(z)$ satisfy the assumptions (A1) – (A4) for any $\varepsilon \in U$.
- The cardinality of the set R_ε of ramification points of $x_\varepsilon(z)$ is constant on $\varepsilon \in U$ (i.e. ramification points of $x_\varepsilon(z)$ are distinct for any $\varepsilon \in U$).

Then, the correlation functions $W_{g,n}(z_1, \dots, z_n; \varepsilon)$ and the g -th free energy $F_g(\varepsilon)$ defined from the spectral curve $(x_\varepsilon(z), y_\varepsilon(z))$ are holomorphic in $\varepsilon \in U$ as long as $z_i \notin R_\varepsilon$ for any $i = 1, \dots, n$.

In order to formulate a variational formula for correlation functions, we need to introduce the notion of “differentiation with fixed x ”. For a meromorphic differential $\omega(z; \varepsilon)$ on $\mathbb{P}^1$, which depends on ε holomorphically, define

$$(2.41) \quad \delta_\varepsilon \omega(z; \varepsilon) := \left(\frac{\partial}{\partial \varepsilon} \omega(z_\varepsilon(x); \varepsilon) \right) \Big|_{x=x_\varepsilon(z)} \quad (z \notin R_\varepsilon),$$

where $z_\varepsilon(x)$ is (any branch of) the inverse function of $x = x_\varepsilon(z)$ which is defined away from branchpoints (i.e. points in $x_\varepsilon(R_\varepsilon)$). In [EO1] the notation $\delta_\Omega \omega(z; \varepsilon)|_{x(z)}$ is used for $\delta_\varepsilon \omega(z; \varepsilon)$ defined above. Such differentiation δ_ε can be generalized to multidifferentials in an obvious way. Then, under these assumptions, the variational formula is formulated as follows.

Theorem 2.17 ([EO1, Theorem 5.1]). *In addition to the above conditions, for any $\varepsilon \in U$, we further assume that*

- *If $r_\varepsilon \in R_\varepsilon$ is a zero of $dx_\varepsilon(z)$, then the functions $\partial x_\varepsilon / \partial \varepsilon$ and $\partial y_\varepsilon / \partial \varepsilon$ are holomorphic (as functions of z) at r_ε , and $dy_\varepsilon(z)$ does not vanish (as a differential of z) at r_ε .*
- *If $r_\varepsilon \in R_\varepsilon$ is a pole of $x_\varepsilon(z)$ with an order greater than or equal to two, then*

$$\frac{\Omega_\varepsilon(z) B(z_1, z) B(z_2, z)}{dy_\varepsilon(z) dx_\varepsilon(z)}$$

is holomorphic (as a differential in z) at $r(\varepsilon)$, where

$$(2.42) \quad \Omega_\varepsilon(z) := \frac{\partial y_\varepsilon}{\partial \varepsilon}(z) dx_\varepsilon(z) - \frac{\partial x_\varepsilon}{\partial \varepsilon}(z) dy_\varepsilon(z).$$

- *There exist a path γ in $\mathbb{P}^1$ passing through no ramification point and a function $\Lambda_\varepsilon(z)$ holomorphic in a neighborhood of γ for which the following holds.*

$$(2.43) \quad \Omega_\varepsilon(z) = \int_{\zeta \in \gamma} \Lambda_\varepsilon(\zeta) B(z, \zeta).$$

Then, $W_{g,n}(z_1, \dots, z_n; \varepsilon)$ and $F_g(\varepsilon)$ defined from the spectral curve $(x_\varepsilon(z), y_\varepsilon(z))$ satisfy the following relations:

(i) *For $2g + n \geq 2$,*

$$(2.44) \quad \delta_\varepsilon W_{g,n}(z_1, \dots, z_n; \varepsilon) = \int_{\zeta \in \gamma} \Lambda_\varepsilon(\zeta) W_{g,n+1}(z_1, \dots, z_n, \zeta; \varepsilon)$$

holds on $\varepsilon \in U$ as long as each of $z_1, \dots, z_n$ satisfies $z_i \notin R_\varepsilon$.

(ii) *For $g \geq 1$,*

$$(2.45) \quad \frac{\partial F_g}{\partial \varepsilon}(\varepsilon) = \int_{\gamma} \Lambda_\varepsilon(z) W_{g,1}(z; \varepsilon)$$

holds on $\varepsilon \in U$.

See [EO1, §5.1] (based on the Rauch's variation formula; see [KK1] for example) for the proof. We note that, since we modify the definition of the topological recursion by adding higher order poles of $x(z)$ as ramification point, we also need to require the second condition in the above claim.

In examples discussed in §3.2.5, we will use the variational formula in a situation that $\Lambda_\varepsilon(z) = \Lambda$ is a constant function (which is also independent of the parameter ε). In that case, applying the above formulas iteratively, we have

$$(2.46) \quad \frac{\partial^n F_g}{\partial \varepsilon^n}(\varepsilon) = \Lambda^n \int_{\zeta_1 \in \gamma} \dots \int_{\zeta_n \in \gamma} W_{g,n}(\zeta_1, \dots, \zeta_n; \varepsilon).$$

3 Quantization of spectral curves

3.1 The WKB type formal series from the topological recursion

With correlation functions $W_{g,n}(z_1, \dots, z_n)$ of a spectral curve $(x(z), y(z))$ we associate a formal series defined by

$$(3.1) \quad \varphi(z; \underline{\nu}, \hbar) := \exp \left[\frac{1}{\hbar} \int_{\zeta_1 \in D(z; \underline{\nu})} W_{0,1}(\zeta_1) + \frac{1}{2!} \int_{\zeta_1 \in D(z; \underline{\nu})} \int_{\zeta_2 \in D(z; \underline{\nu})} \left(W_{0,2}(\zeta_1, \zeta_2) - \frac{dx(\zeta_1) dx(\zeta_2)}{(x(\zeta_1) - x(\zeta_2))^2} \right) \right. \\ \left. + \sum_{m=1}^{\infty} \hbar^m \left\{ \sum_{\substack{2g+n-2=m \\ g \geq 0, n \geq 1}} \frac{1}{n!} \int_{\zeta_1 \in D(z; \underline{\nu})} \cdots \int_{\zeta_n \in D(z; \underline{\nu})} W_{g,n}(\zeta_1, \dots, \zeta_n) \right\} \right],$$

where

$$(3.2) \quad D(z; \underline{\nu}) = [z] - \sum_{\beta \in B} \nu_{\beta} [\beta]$$

is a divisor on $\mathbb{P}^1$ with a finite set $B \subset \mathbb{P}^1 \setminus R^*$ and $\underline{\nu} = (\nu_{\beta})_{\beta \in B}$ which is a tuple of complex numbers satisfying $\sum_{\beta \in B} \nu_{\beta} = 1$ (see §A, for the integral with a divisor). Precisely speaking, the above integrals of $W_{0,1}$ and $W_{0,2}$ are not well-defined when $\beta \in B$ is a singular point of these differentials. In the situation, we define these integrals through a certain regularization technique (e.g., method used in Remark 4.4). We do not consider the regularization in detail because only their z -derivatives are important in the following discussion (in particular, in the proof of Theorem 3.5).

Note that the quantization by using the divisor with the parameter $\underline{\nu}$ was first introduced by [BE2]. This $\varphi(z; \underline{\nu}, \hbar)$ has the same form as WKB type solutions (2.17), and it is known that $\psi(x; \underline{\nu}, \hbar) = \varphi(z(x); \underline{\nu}, \hbar)$ for some class of spectral curves with some specified divisors $D(z, \underline{\nu})$ satisfies a linear ordinary differential equation, where $z(x)$ is an inverse function of $x(z)$. Such a linear ordinary differential equation is called a quantum curve of a spectral curve in question.

A typical example of a quantization is the Airy curve

$$(3.3) \quad x(z) = z^2, \quad y(z) = z \quad (z \in \mathbb{P}^1)$$

(so that $P(x(z), y(z)) = 0$ with $P(x, y) = y^2 - x$). In this case we choose $D = [z] - [\infty]$, and the corresponding quantum curve is the Airy equation

$$(3.4) \quad \left\{ \hbar^2 \frac{d^2}{dx^2} - x \right\} \psi = 0$$

([GS, Z], see also [EO1], [EO3]). Another example is

$$(3.5) \quad x(z) = z^2, \quad y(z) = 1/z \quad (z \in \mathbb{P}^1)$$

called Bessel curve in [DN]. In this case $P(x(z), y(z)) = 0$ with $P(x, y) = xy^2 - 1$, and the corresponding quantum curve is obtained in [DN]:

$$(3.6) \quad \left\{ \hbar^2 \frac{d^2}{dx^2} - \frac{1}{x} \right\} \psi = 0.$$

A systematic study of a quantization is done in [BE2], and explicit forms of quantum curves are given for spectral curves satisfying the admissibility assumption ([BE2, Definition 2.7]). The spectral curve (1.5) of the Gauss hypergeometric equation in the SL-form, however, does not satisfy the admissibility condition. In the next subsection we study quantizations without the admissibility condition, while we restrict ourselves to the case when the degree of a polynomial $P(x, y)$ with respect to y is two.

3.2 In the case of second order differential equations

In this subsection we study quantizations without the admissibility condition, while we restrict ourselves to the case when the degree of a polynomial $P(x, y)$ with respect to y is two.

To be more precise, we assume the following conditions in addition to (A1) – (A4):

(AQ1) The rational functions $(x(z), y(z))$ satisfy $P(x(z), y(z)) = 0$ with an irreducible polynomial $P(x, y) = p_0(x)y^2 + p_1(x)y + p_2(x) \in \mathbb{C}[x, y]$, where $p_0(x)$ is a nonzero polynomial.

(AQ2) The differential $(y(z) - y(\bar{z}))dx(z)$ does not vanish on $\mathbb{P}^1 \setminus R$.

By the assumption (AQ1), the conjugate map is now a rational map defined globally. Hence the assumption (AQ2) makes sense. We also note that $y(z)$ and $y(\bar{z})$ are two solutions of $P(x(z), y) = 0$. These solutions are distinct when $z \notin R$.

Remark 3.1. The assumption (AQ2) ensures that two solutions $y(z)$ and $y(\bar{z})$ of $P(x(z), y) = 0$ merge to each other only at ramification points. An example of a spectral curve which does not satisfy (AQ2) is

$$(3.7) \quad P(x, y) = y^2 - x^3 - x^2, \quad x(z) = -1 + z^2, \quad y(z) = z(z^2 - 1).$$

Because $R = \{0, \infty\}$ and $\bar{z} = -z$, we have $y(1) = y(\bar{1}) = 0$, whereas $z = 1$ is not a ramification point of $x(z)$. Thus (AQ2) is violated. At $(x(z), y(z))|_{z=1} = (0, 0)$, a curve defined by $\{(x, y) \mid P(x, y) = 0\}$ has a normal crossing. Hence the point $x(1) = 0$ should become a double turning point of its quantum curve (if it exists). As illustrated by this example, (AQ2) excludes multiple turning points. A quantum curve with a double turning point is discussed in [IS], where the infinitely many $\hbar$ -correction terms appear in the quantum curve. It was also shown in [IS] that the correction terms are related to asymptotic expansion of Painlevé transcendents.

3.2.1 An index of the defining polynomials

To state assumptions for the divisor $D(z; \underline{\nu})$ in (3.1), we introduce an index at $x_0 \in \mathbb{P}^1$ by

$$(3.8) \quad \rho(x_0; P) := \begin{cases} \text{ord}_{x_0} Q_0(x) & (x_0 \neq \infty), \\ \text{ord}_0 Q_0^{(\infty)}(x) & (x_0 = \infty) \end{cases}$$

for a polynomial $P(x, y) = p_0(x)y^2 + p_1(x)y + p_2(x) \in \mathbb{C}[x, y]$, where

$$(3.9) \quad Q_0(x) := \frac{1}{4} \left(\frac{p_1(x)}{p_0(x)} \right)^2 - \frac{p_2(x)}{p_0(x)} \quad \text{and} \quad Q_0^{(\infty)}(x) := \frac{1}{x^4} Q_0(1/x).$$

Here $\text{ord}_{x_0} g(x)$ for a rational function $g(x)$ is defined as an integer p for which

$$(3.10) \quad g(x) = (x - x_0)^p (c_0 + c_1(x - x_0) + \cdots), \quad c_0 \neq 0$$

holds as the Laurent expansion of $g(x)$ at x_0 .

Remark 3.2. (i) This index $\rho(x_0; P)$ is related to the type of singularities of the quantum curve. We have expected that the leading term with respect to $\hbar$ of the quantum curve is given by the second order differential operator $\mathcal{L} := (p_0(x))^{-1} P(x, \hbar(d/dx))$, where we use the normal ordering with respect to x and $\hbar(d/dx)$. Then $Q_0(x)$ is the leading term with respect to $\hbar$ of the potential of the SL-form of $\mathcal{L}$ (cf. (3.152)). It is well-known that

- (a) $x_0 \in \mathbb{P}^1$ is a regular singular point of the SL-form of $\mathcal{L}$ iff $\rho(x_0; P) = -1, -2$.
 $(x_0$ with $\rho(x_0; P) = -1$ is a simple-pole type turning point in the WKB analysis, we have mentioned in §2.1.)

- (b) $x_0 \in \mathbb{P}^1$ is an irregular singular point of the SL-form of $\mathcal{L}$ iff $\rho(x_0; P) \leq -3$.

- (ii) Our index $\rho(x_0; P)$ remains invariant under a symplectic transformation of the form $(x, y) \mapsto (X, Y) = (x, y + g(x))$ for some function $g(x)$, or, a gauge transformation $\mathcal{L} \mapsto e^{-\int g(x)dx} \circ \mathcal{L} \circ e^{\int g(x)dx}$.

Proposition 3.3 ([IKoT, Proposition 3.3]). *We assume (A1) – (A4), (AQ1) and (AQ2). Then, for $\alpha \in \mathbb{P}^1$,*

$$(3.11) \quad \text{ord}_\alpha \left[(y(z) - y(\bar{z})) \frac{dx}{dz}(z) \right] = \begin{cases} \frac{1}{2} \rho(x(\alpha); P) & (\alpha \notin R), \\ \rho(x(\alpha); P) + 1 & (\alpha \in R). \end{cases}$$

Proof. Because $y(z)$ and $y(\bar{z})$ are two solutions of $P(x(z), y) = 0$, we have

$$(3.12) \quad \{y(z) - y(\bar{z})\}^2 = \left\{ \left(\frac{p_1(x)}{p_0(x)} \right)^2 - 4 \frac{p_2(x)}{p_0(x)} \right\} \Big|_{x=x(z)} = 4Q_0(x(z)).$$

Hence if α is a regular point of $x(z)$, we have

$$(3.13) \quad \text{ord}_\alpha [y(z) - y(\bar{z})] = \frac{1}{2} \text{ord}_{x(\alpha)} Q_0(x) = \frac{1}{2} \rho(x(\alpha); P).$$

If α is a pole of $x(z)$, we utilize a transformation

$$(3.14) \quad X(z) = \frac{1}{x(z)}, \quad Y(z) = -x(z)^2 y(z).$$

By this transformation, we have

$$(3.15) \quad (y(z) - y(\bar{z}))dx(z) = (Y(z) - Y(\bar{z}))dX(z).$$

Furthermore $Y(z)$ and $Y(\bar{z})$ are two solutions of $P^{(\infty)}(X, Y) = 0$, where

$$(3.16) \quad P^{(\infty)}(X, Y) = P(1/X, -X^2 Y) = X^4 p_0(1/X)Y^2 - X^2 p_1(1/X)Y + p_2(1/X).$$

Hence

$$(3.17) \quad \{Y(z) - Y(\bar{z})\}^2 = \frac{1}{X^4} \left\{ \left(\frac{p_1(1/X)}{p_0(1/X)} \right)^2 - 4 \frac{p_2(1/X)}{p_0(1/X)} \right\} \Big|_{X=X(z)} = 4Q_0^{(\infty)}(X(z)).$$

Thus

$$(3.18) \quad \text{ord}_\alpha [Y(z) - Y(\bar{z})] = \frac{1}{2} \text{ord}_0 Q_0^{(\infty)}(x) = \frac{1}{2} \rho(x(\alpha); P).$$

Now we divide the proof into four cases.

(a) $\alpha \notin R$ and α is a regular point of $x(z)$: In this case $\text{ord}_\alpha [dx/dz] = 0$ follows, and hence

$$(3.19) \quad \text{ord}_\alpha \left[(y(z) - y(\bar{z})) \frac{dx}{dz}(z) \right] = \text{ord}_\alpha [y(z) - y(\bar{z})] \stackrel{(3.13)}{=} \frac{1}{2} \rho(x(\alpha); P).$$

(b) $\alpha \notin R$ and α is a simple pole of $x(z)$: Since α is a regular point of $X(z)$ with $X(\alpha) = 0$ and $X'(\alpha) \neq 0$, we obtain

$$(3.20) \quad \begin{aligned} \text{ord}_\alpha \left[(y(z) - y(\bar{z})) \frac{dx}{dz}(z) \right] &\stackrel{(3.15)}{=} \text{ord}_\alpha \left[(Y(z) - Y(\bar{z})) \frac{dX}{dz}(z) \right] \\ &= \text{ord}_\alpha [Y(z) - Y(\bar{z})] \\ &\stackrel{(3.18)}{=} \frac{1}{2} \rho(x(\alpha); P). \end{aligned}$$

(c) $\alpha \in R$ and α is a simple zero of $dx(z)$: In this case

$$(3.21) \quad \text{ord}_\alpha [x(z) - x(\alpha)] = 2 \quad \text{and} \quad \text{ord}_\alpha \left[\frac{dx}{dz}(z) \right] = 1$$

hold. Hence

$$\begin{aligned} \text{ord}_\alpha \left[(y(z) - y(\bar{z})) \frac{dx}{dz}(z) \right] &= \frac{1}{2} \rho(x(\alpha); P) \times (\text{ord}_\alpha [x(z) - x(\alpha)]) + 1 \\ &= \rho(x(\alpha); P) + 1. \end{aligned}$$

(d) $\alpha \in R$ and α is a double pole of $x(z)$: Since

$$(3.22) \quad \text{ord}_\alpha [X(z) - X(\alpha)] = 2 \quad \text{and} \quad \text{ord}_\alpha \left[\frac{dX}{dz}(z) \right] = 1,$$

we have

$$(3.23) \quad \begin{aligned} \text{ord}_\alpha \left[(y(z) - y(\bar{z})) \frac{dx}{dz}(z) \right] &= \text{ord}_\alpha \left[(Y(z) - Y(\bar{z})) \frac{dX}{dz}(z) \right] \\ &= \frac{1}{2} \rho(x(\alpha); P) \times (\text{ord}_\alpha [X(z) - X(\alpha)]) + 1 \\ &= \rho(x(\alpha); P) + 1. \end{aligned}$$

□

Remark 3.4. It follows from Proposition 3.3 that, if $r \in R$ satisfies $\rho(x(r); P) \leq -2$, then r is an ineffective ramification point (cf. Proposition 2.14).

3.2.2 Quantum curves

In order to give our theorem in a clear form, we introduce

$$(3.24) \quad \text{Sing}(P) := \{b \in \mathbb{P}^1 \mid \rho(b; P) \leq -2\}, \quad \text{Sing}_2(P) := \{b \in \mathbb{P}^1 \mid \rho(b; P) = -2\}$$

for a polynomial $P(x, y) = p_0(x)y^2 + p_1(x)y + p_2(x) \in \mathbb{C}[x, y]$. We also use the following symbols:

$$(3.25) \quad \Delta(z) := y(z) - y(\bar{z}),$$

$$(3.26) \quad C_\beta := \text{Res}_{z=\beta} \Delta(z) dx(z).$$

Theorem 3.5 ([IKoT, Theorem 3.5]). *We assume (A1) – (A4), (AQ1) and (AQ2). Let*

$$(3.27) \quad D(z; \underline{\nu}) := [z] - \sum_{\beta \in B} \nu_\beta [\beta]$$

be a divisor on $\mathbb{P}^1$, where $B := x^{-1}(\text{Sing}(P))$, and $\underline{\nu} = (\nu_\beta)_{\beta \in B}$ is a tuple of complex numbers satisfying

$$(3.28) \quad \sum_{\beta \in B} \nu_\beta = 1.$$

Then,

$$(3.29) \quad \psi(x; \underline{\nu}, \hbar) := \varphi(z(x); \underline{\nu}, \hbar),$$

where $z(x)$ denotes an inverse function of $x(z)$ and $\varphi(z, \hbar)$ is defined by (3.1) with the integration divisor (3.27), is a WKB type formal solution of $\hat{P}\psi = 0$, where $\hat{P}$ is defined by

$$(3.30) \quad \hat{P} := \hbar^2 \frac{d^2}{dx^2} + q(x, \hbar) \hbar \frac{d}{dx} + r(x, \hbar)$$

with

$$(3.31) \quad q(x, \hbar) = q_0(x) + \hbar q_1(x), \quad r(x, \hbar) = r_0(x) + \hbar r_1(x) + \hbar^2 r_2(x)$$

whose leading terms are respectively given by

$$(3.32) \quad q_0(x) = \frac{p_1(x)}{p_0(x)}, \quad r_0(x) = \frac{p_2(x)}{p_0(x)}$$

and their lower order terms are determined by

$$(3.33) \quad x'(z)q_1(x(z)) = -\frac{\Delta'(z)}{\Delta(z)} + \frac{2}{z - \bar{z}} - \sum_{\beta \in B} \frac{\nu_\beta + \nu_{\bar{\beta}}}{z - \beta},$$

$$(3.34) \quad x'(z)r_1(x(z)) = \frac{1}{2}x'(z)\frac{\partial q_0}{\partial x}\Big|_{x=x(z)} + \frac{1}{2}x'(z)q_0(x(z))q_1(x(z)) + \frac{1}{2}\Delta(z)\sum_{\beta \in B} \frac{\nu_\beta - \nu_{\bar{\beta}}}{z - \beta},$$

$$(3.35) \quad x'(z)r_2(x(z)) = \Delta(z)\sum_{\beta \in B_1} \frac{\nu_\beta \nu_{\bar{\beta}}}{C_\beta} \frac{1}{z - \beta}.$$

Here we set $B_1 := x^{-1}(\text{Sing}_2(P))$.

We call the equation $\hat{P}\psi = 0$ given by Theorem 3.5 a quantum curve of the spectral curve.

Remark 3.6. (i) The set B may contain the infinity. In Theorem 3.5, we use the convention that if $\beta \in B$ is the infinity, $1/(z - \beta)$ in q_1 , r_1 and r_2 is zero. In what follows, we use this convention.

(ii) It follows from Proposition 3.3 that

$$(3.36) \quad B = \text{the set of poles of } \Delta(z)dx(z),$$

$$(3.37) \quad B_1 = \text{the set of simple poles of } \Delta(z)dx(z).$$

Hence, $C_\beta \neq 0$ for $\beta \in B_1$.

(iii) The set B may contain ramification points; this happens in examples treated in this paper. It follows, however, from the definition of $\text{Sing}(P)$ and Remark 3.4 that ramification points contained in B are ineffective. Hence, $W_{g,n}$'s are holomorphic there, and hence, the formal series (3.1) is well-defined. Note also that, since $\text{Sing}(P)$ and $\text{Sing}_2(P)$ are defined in terms of order of $Q_0(x)$ or $Q_\infty(x)$, they are closed by the conjugate map; that is, if $\beta \in B$ (resp. $\beta \in B_1$), then $\bar{\beta} \in B$ (resp. $\bar{\beta} \in B_1$).

(iv) It is not obvious that $q_1(x)$, $r_1(x)$ and $r_2(x)$ defined by (3.33) – (3.35) are rational functions in x . But for the examples which will be considered in §3.2.5, $q_1(x)$, $r_1(x)$ and $r_2(x)$ become rational functions in x .

(v) Because the degree of $x(z)$ is two by our assumption (AQ1), we have two inverse functions of $x(z)$. By Theorem 3.5, we can construct two WKB type formal solutions from these two inverse functions, and they form a basis of the space of formal solutions of $\hat{P}\psi = 0$.

3.2.3 Preparations for the proof

Our proof of Theorem 3.5 is based on “diagonal specialization” which allows us to relate the topological recursion (2.34) to the recursive relations (2.6)–(2.9) satisfied by the coefficients of WKB solution ([DM]; see also [BE2, IS]). In this subsection we prepare several statements for the proof.

In the remaining part of this section we denote $D(z; \underline{\nu})$ by $D(z)$ for simplicity. We also assume (A1) – (A4), (AQ1) and (AQ2) as we have assumed in Theorem 3.5.

Let us define

$$(3.38) \quad \hat{T}(z, \hbar) = \frac{1}{\hbar} \hat{T}_{-1}(z) + \hat{T}_0(z) + \hbar \hat{T}_1(z) + \cdots := \frac{d}{dz} \log \varphi(z; \underline{\nu}, \hbar),$$

that is,

$$(3.39) \quad \begin{aligned} \hat{T}(z, \hbar) dz = & \frac{1}{\hbar} W_{0,1}(z) + \int_{\zeta \in D(z)} \left(W_{0,2}(z, \zeta) - \frac{dx(z) dx(\zeta)}{(x(z) - x(\zeta))^2} \right) \\ & + \sum_{m=1}^{\infty} \hbar^m \left\{ \sum_{\substack{2g+n-2=m \\ g \geq 0, n \geq 1}} \frac{1}{(n-1)!} \int_{\zeta_1 \in D(z)} \cdots \int_{\zeta_{n-1} \in D(z)} W_{g,n}(z, \zeta_1, \dots, \zeta_{n-1}) \right\}. \end{aligned}$$

Then, in order to prove Theorem 3.5, it is enough to show that $\hat{T}(z, \hbar)$ satisfies the Riccati equation associated with $\hat{P}$ after a change of variable $x = x(z)$. Comparing the coefficients, we have

$$(3.40) \quad \hat{T}_m(z) dz = \begin{cases} W_{0,1}(z) & (m = -1), \\ \int_{\zeta \in D(z)} \left\{ W_{0,2}(z, \zeta) - \frac{dx(z) dx(\zeta)}{(x(z) - x(\zeta))^2} \right\} & (m = 0), \\ \sum_{\substack{2g+n-2=m \\ g \geq 0, n \geq 1}} \frac{1}{(n-1)!} G_{g,n}(z, \dots, z) & (m \geq 1), \end{cases}$$

where

$$(3.41) \quad G_{g,n}(z_0, z_1, \dots, z_{n-1}) := \begin{cases} W_{g,1}(z_0) & (n = 1), \\ \int_{\zeta_1 \in D(z_1)} \cdots \int_{\zeta_{n-1} \in D(z_{n-1})} W_{g,n}(z_0, \zeta_1, \dots, \zeta_{n-1}) & (n \geq 2) \end{cases}$$

for $2g+n \geq 2$. Note that $G_{g,n}(z_0, z_1, \dots, z_{n-1})$ is a meromorphic differential (1-form) in the first variable z_0 , and is a meromorphic functions (0-form) in the remaining variables $z_1, \dots, z_{n-1}$. We will derive a recurrence relation satisfied by $\hat{T}_m$ by using several properties (especially the topological recursion (2.34)) of $W_{g,n}$.

For the later convenience, we set $G_{g,n} = W_{g,n} = 0$ for either $g < 0$ or $n \leq 0$.

Proposition 3.7 ([IKoT, Proposition 3.7]).

(i) $G_{g,n}(z_0, z_1, \dots, z_{n-1})$ for $2g + n \geq 3$ is holomorphic in each variable on $\mathbb{P}^1 \setminus (R^* \cup B)$. All singular points of them are poles.

(ii) For $(g, n) = (0, 2)$,

$$(3.42) \quad G_{0,2}(z_0, z_1) = \left(\frac{1}{z_0 - z_1} - \sum_{\beta \in B} \frac{\nu_\beta}{z_0 - \beta} \right) dz_0.$$

Proof. Because only singularities of $W_{g,n}$ for $2g + n \geq 3$ are poles whose residues vanish, we obtain (i). By a straightforward computation using $W_{0,2}(z_0, z_1) = B(z_0, z_1) = dz_0 dz_1 / (z_0 - z_1)^2$, we can prove (ii). $\square$

Proposition 3.8 ([IKoT, Proposition 3.8]).

$$(i) \quad G_{0,2}(\overline{z_0}, z_1) = \left(\frac{1}{z_0 - \overline{z_1}} - \sum_{\beta \in B} \frac{\nu_\beta}{z_0 - \overline{\beta}} \right) dz_0.$$

(ii) For $2g + n \geq 3$,

$$(3.43) \quad G_{g,n}(\overline{z_0}, z_1, \dots, z_{n-1}) = -G_{g,n}(z_0, z_1, \dots, z_{n-1}).$$

Proof. By Theorem 2.12 (iii), we have

$$(3.44) \quad W_{0,2}(z_0, z_1) + W_{0,2}(\overline{z_0}, z_1) = \frac{dx(z_0)dx(z_1)}{(x(z_0) - x(z_1))^2}.$$

Because $W_{0,2}$ is symmetric in its variables, we also have

$$(3.45) \quad W_{0,2}(z_0, z_1) + W_{0,2}(z_0, \overline{z_1}) = \frac{dx(z_0)dx(z_1)}{(x(z_0) - x(z_1))^2}.$$

From these two relations, we have

$$(3.46) \quad W_{0,2}(z_0, \overline{z_1}) = W_{0,2}(\overline{z_0}, z_1).$$

Thus

$$(3.47) \quad \begin{aligned} G_{0,2}(\overline{z_0}, z_1) &= \int_{\zeta_1 \in D(z_1)} W_{0,2}(\overline{z_0}, \zeta_1) = \int_{\zeta_1 \in D(z_1)} W_{0,2}(z_0, \overline{\zeta_1}) \\ &= \sum_{\beta \in B} \nu_\beta \left(\int_{\beta}^{z_1} \frac{d\overline{\zeta_1}}{(z_0 - \overline{\zeta_1})^2} \right) dz_0 = \sum_{\beta \in B} \nu_\beta \left(\int_{\overline{\beta}}^{\overline{z_1}} \frac{d\xi_1}{(z_0 - \xi_1)^2} \right) dz_0 \\ &= \sum_{\beta \in B} \nu_\beta \left(\frac{1}{z_0 - \overline{z_1}} - \frac{1}{z_0 - \overline{\beta}} \right) dz_0 = \left(\frac{1}{z_0 - \overline{z_1}} - \sum_{\beta \in B} \frac{\nu_\beta}{z_0 - \overline{\beta}} \right) dz_0. \end{aligned}$$

The relation (ii) is a direct consequence of Theorem 2.12 (iii). $\square$

We now state two propositions which give recurrence relations of $\{G_{g,n}\}$. These relations play key roles in the proof of Theorem 3.5.

Proposition 3.9 ([IKoT, Proposition 3.9]). *We have*

$$(3.48) \quad G_{1,1}(z_0) = \frac{B(z_0, \bar{z}_0)}{\Delta(z_0)dx(z_0)}$$

and

$$(3.49) \quad \begin{aligned} G_{0,3}(z_0, z_1, z_2) &= G_{0,2}(z_0, z_1) \left\{ \frac{G_{0,2}(\bar{z}_0, z_2)}{\Delta(z_0)dx(z_0)} - \frac{G_{0,2}(\bar{z}_1, z_2)}{\Delta(z_1)dx(z_1)} \right\} + \frac{G_{0,2}(z_0, \bar{z}_1)G_{0,2}(\bar{z}_1, z_2)}{\Delta(z_1)dx(z_1)} \\ &\quad + G_{0,2}(z_0, z_2) \left\{ \frac{G_{0,2}(\bar{z}_0, z_1)}{\Delta(z_0)dx(z_0)} - \frac{G_{0,2}(\bar{z}_2, z_1)}{\Delta(z_2)dx(z_2)} \right\} + \frac{G_{0,2}(z_0, \bar{z}_2)G_{0,2}(\bar{z}_2, z_1)}{\Delta(z_2)dx(z_2)} \\ &\quad - \sum_{\beta \in B_1} \frac{\nu_\beta \nu_{\bar{\beta}}}{C_\beta} \{G_{0,2}(z_0, \beta) - G_{0,2}(z_0, \bar{\beta})\}. \end{aligned}$$

Proposition 3.10 ([IKoT, Proposition 3.10]). *For $2g + n \geq 3$ we have*

$$(3.50) \quad \begin{aligned} G_{g,n+1}(z_0, z_I) &= \frac{1}{\Delta(z_0)dx(z_0)} \int_{\zeta_1 \in D(z_1)} \cdots \int_{\zeta_n \in D(z_n)} W_{g-1,n+2}(z_0, \bar{z}_0, \zeta_I) \\ &\quad + \sum_{j=1}^n G_{0,2}(z_0, z_j) \left\{ \frac{G_{g,n}(\bar{z}_0, z_{I \setminus \{j\}})}{\Delta(z_0)dx(z_0)} - \frac{G_{g,n}(\bar{z}_j, z_{I \setminus \{j\}})}{\Delta(z_j)dx(z_j)} \right\} \\ &\quad + \sum_{j=1}^n \left\{ \frac{G_{0,2}(\bar{z}_0, z_j)G_{g,n}(z_0, z_{I \setminus \{j\}})}{\Delta(z_0)dx(z_0)} + \frac{G_{0,2}(z_0, \bar{z}_j)G_{g,n}(\bar{z}_j, z_{I \setminus \{j\}})}{\Delta(z_j)dx(z_j)} \right\} \\ &\quad + \frac{1}{\Delta(z_0)dx(z_0)} \sum_{\substack{g_1+g_2=g, \\ I_1 \sqcup I_2=I, \\ 2g_1+|I_1| \geq 2, \\ 2g_2+|I_2| \geq 2}} G_{g_1,|I_1|+1}(z_0, z_{I_1}) G_{g_2,|I_2|+1}(\bar{z}_0, z_{I_2}), \end{aligned}$$

where $I = \{1, 2, \dots, n\}$.

Proof of Proposition 3.9. It follows from the topological recursion (2.34) for $(g, n) = (1, 0)$ that

$$(3.51) \quad G_{1,1}(z_0) = \sum_{r \in R^*} \text{Res}_{\xi=r} \left[\frac{G_{0,2}(z_0, \xi)}{\Delta(\xi)dx(\xi)} W_{0,2}(\xi, \bar{\xi}) \right]$$

holds (cf. Proposition 2.14 (ii)). Here we remind the readers that the set R^* consists of the effective ramification points in the sense of Definition 2.13. Because, under the assumption (AQ2), the singular points of the integrand are contained in $R^* \cup \{z_0\}$, the residue theorem gives

$$(3.52) \quad G_{1,1}(z_0) = -\text{Res}_{\xi=z_0} \left[\frac{G_{0,2}(z_0, \xi)}{\Delta(\xi)dx(\xi)} W_{0,2}(\xi, \bar{\xi}) \right] = \frac{W_{0,2}(z_0, \bar{z}_0)}{\Delta(z_0)dx(z_0)}.$$

Here we have used (3.42) to compute the residue. Similarly, it follows from the topological recursion (2.34) for $(g, n) = (0, 2)$ that

$$(3.53) \quad G_{0,3}(z_0, z_1, z_2) = \sum_{r \in R^*} \operatorname{Res}_{\xi=r} [g_{0,3}(z; z_0, z_1, z_2)],$$

where

$$(3.54) \quad \begin{aligned} g_{0,3}(\xi; z_0, z_1, z_2) &= K_{D(\xi), r}(z_0, \xi) \{G_{0,2}(\xi, z_1)G_{0,2}(\bar{\xi}, z_2) + G_{0,2}(\xi, z_2)G_{0,2}(\bar{\xi}, z_1)\} \\ &= \frac{G_{0,2}(z_0, \xi)}{\Delta(\xi)dx(\xi)} \{G_{0,2}(\xi, z_1)G_{0,2}(\bar{\xi}, z_2) + G_{0,2}(\xi, z_2)G_{0,2}(\bar{\xi}, z_1)\}. \end{aligned}$$

Since $g_{0,3}(\xi; z_0, z_1, z_2)$ is holomorphic in ξ except for points in $R^* \cup B \cup \{z_0, z_1, \bar{z}_1, z_2, \bar{z}_2\}$, the residue theorem gives

$$(3.55) \quad G_{0,3}(z_0, z_1, z_2) = - \sum_{r \in \{z_0, z_1, \bar{z}_1, z_2, \bar{z}_2\}} \operatorname{Res}_{\xi=r} [g_{0,3}(\xi; z_0, z_1, z_2)] - \sum_{\beta \in B} \operatorname{Res}_{\xi=\beta} [g_{0,3}(\xi; z_0, z_1, z_2)].$$

By Proposition 3.8, we find that $g_{0,3}(\xi; z_0, z_1, z_2)$ has simple poles at $z_0, z_1, \bar{z}_1, z_2, \bar{z}_2$, and its residues are given respectively by

$$\begin{aligned} &\operatorname{Res}_{\xi=z_0} [g_{0,3}(\xi; z_0, z_1, z_2)] \\ &= -\frac{1}{\Delta(z_0)dx(z_0)} \{G_{0,2}(z_0, z_1)G_{0,2}(\bar{z}_0, z_2) + G_{0,2}(\bar{z}_0, z_1)G_{0,2}(z_0, z_2)\}, \\ &\operatorname{Res}_{\xi=z_1} [g_{0,3}(\xi; z_0, z_1, z_2)] + \operatorname{Res}_{\xi=\bar{z}_1} [g_{0,3}(\xi; z_0, z_1, z_2)] \\ &= \frac{G_{0,2}(z_0, z_1)G_{0,2}(\bar{z}_1, z_2)}{\Delta(z_1)dx(z_1)} - \frac{G_{0,2}(z_0, \bar{z}_1)G_{0,2}(\bar{z}_1, z_2)}{\Delta(z_1)dx(z_1)}, \\ &\operatorname{Res}_{\xi=z_2} [g_{0,3}(\xi; z_0, z_1, z_2)] + \operatorname{Res}_{\xi=\bar{z}_2} [g_{0,3}(\xi; z_0, z_1, z_2)] \\ &= \frac{G_{0,2}(z_0, z_2)G_{0,2}(\bar{z}_2, z_1)}{\Delta(z_2)dx(z_2)} - \frac{G_{0,2}(z_0, \bar{z}_2)G_{0,2}(\bar{z}_2, z_1)}{\Delta(z_2)dx(z_2)}. \end{aligned}$$

Next we compute the residue at $\beta \in B$. By Proposition 3.7 (ii) and Proposition 3.8 (i), we have the Laurent expansion as

$$G_{0,2}(\xi, z_1)G_{0,2}(\bar{\xi}, z_2) = \frac{\nu_\beta \nu_{\bar{\beta}}}{(\xi - \beta)^2} (1 + O(\xi - \beta))(d\xi)^2$$

near $\xi = \beta$ (note that $\bar{\beta} \in B$ since B is closed by the conjugate map; see Remark 3.6 (iii)). Hence, if $\Delta(\xi)dx(\xi)$ has a double or higher order pole at $\xi = \beta$, then

$$(3.56) \quad \frac{G_{0,2}(z, \xi)G_{0,2}(\xi, z_1)G_{0,2}(\bar{\xi}, z_2)}{\Delta(\xi)dx(\xi)}$$

is holomorphic at $\xi = \beta$ and its residue is zero. If $\Delta(\xi)dx(\xi)$ has a simple pole at β , i.e., if $\beta \in B_1$ (cf. Remark 3.6 (ii)), then

$$(3.56) = \frac{\nu_\beta \nu_{\bar{\beta}}}{C_\beta} G_{0,2}(z, \beta) \left(\frac{1}{\xi - \beta} + O(1) \right) d\xi$$

when $\xi \rightarrow \beta$. Hence we obtain

$$(3.57) \quad \operatorname{Res}_{\xi=\beta} \left[\frac{G_{0,2}(z, \xi) G_{0,2}(\xi, z_1) G_{0,2}(\bar{\xi}, z_2)}{\Delta(\xi) dx(\xi)} \right] = \begin{cases} 0 & (\beta \notin B_1), \\ \frac{\nu_\beta \nu_{\bar{\beta}}}{C_\beta} G_{0,2}(z, \beta) & (\beta \in B_1). \end{cases}$$

Then, the desired equality (3.49) follows from

$$(3.58) \quad \sum_{\beta \in B_1} \frac{\nu_\beta \nu_{\bar{\beta}}}{C_\beta} G_{0,2}(z, \beta) = \sum_{\beta \in B_1} \frac{\nu_\beta \nu_{\bar{\beta}}}{C_{\bar{\beta}}} G_{0,2}(z, \bar{\beta})$$

(cf. Remark 3.6 (iii)) and

$$(3.59) \quad C_{\bar{\beta}} = \operatorname{Res}_{z=\bar{\beta}} \Delta(z) dx(z) = \operatorname{Res}_{z=\beta} \Delta(\bar{z}) dx(\bar{z}) = -\operatorname{Res}_{z=\beta} \Delta(z) dx(z) = -C_\beta \quad (\beta \in B).$$

This completes the proof. $\square$

Proof of Proposition 3.10. The topological recursion gives

$$(3.60) \quad G_{g,n+1}(z_0, z_I) = \sum_{r \in R^*} \operatorname{Res}_{\xi=r} [f_1(\xi; z_0, z_I) + f_2(\xi; z_0, z_I) + f_3(\xi; z_0, z_I)].$$

Here

$$(3.61) \quad f_1(\xi; z_0, z_I) = \frac{G_{0,2}(z_0, \xi)}{\Delta(\xi) dx(\xi)} \int_{\zeta_1 \in D(z_1)} \cdots \int_{\zeta_n \in D(z_n)} W_{g-1, n+2}(\xi, \bar{\xi}, \zeta_I),$$

$$(3.62) \quad f_2(\xi; z_0, z_I) = \frac{G_{0,2}(z_0, \xi)}{\Delta(\xi) dx(\xi)} \sum_{j=1}^n \{ G_{0,2}(\xi, z_j) G_{g,n}(\bar{\xi}, z_{I \setminus \{j\}}) + G_{0,2}(\bar{\xi}, z_j) G_{g,n}(\xi, z_{I \setminus \{j\}}) \},$$

$$(3.63) \quad f_3(\xi; z_0, z_I) = \frac{G_{0,2}(z_0, \xi)}{\Delta(\xi) dx(\xi)} \sum_{\substack{g_1+g_2=g, \\ I_1 \sqcup I_2=I}}'' G_{g_1, |I_1|+1}(\xi, z_{I_1}) G_{g_2, |I_2|+1}(\bar{\xi}, z_{I_2}),$$

where $\sum''$ in $f_3(\xi; z_0, z_I)$ means that we take the sum for

$$(3.64) \quad 2g_1 + |I_1| \geq 2 \quad \text{and} \quad 2g_2 + |I_2| \geq 2.$$

Because $f_1(\xi; z_0, z_I)$ and $f_3(\xi; z_0, z_I)$ are holomorphic except for $R^* \cup \{z_0\}$, the residue theorem gives

$$(3.65) \quad \begin{aligned} & \sum_{r \in R^*} \operatorname{Res}_{\xi=r} [f_1(\xi; z_0, z_I) + f_3(\xi; z_0, z_I)] \\ &= \frac{1}{\Delta(z_0) dx(z_0)} \int_{\zeta_1 \in D(z_1)} \cdots \int_{\zeta_n \in D(z_n)} W_{g-1, n+2}(z_0, \bar{z}_0, \zeta_I) \\ & \quad + \frac{1}{\Delta(z_0) dx(z_0)} \sum_{\substack{g_1+g_2=g, \\ I_1 \sqcup I_2=I}}'' G_{g_1, |I_1|+1}(z_0, z_{I_1}) G_{g_2, |I_2|+1}(\bar{z}_0, z_{I_2}). \end{aligned}$$

Although $G_{0,2}(\xi, z_j)$ has a simple pole at $\xi = \beta \in B$, $f_2(\xi; z_0, z_I)$ is holomorphic at $\xi = \beta$ because $\Delta(\xi)dx(\xi)$ has a pole at β (cf. Remark 3.6 (ii)). It is also holomorphic near $\xi = \bar{\beta}$ with $\beta \in B$. Hence the singular points of $f_2(\xi; z_0, z_I)$ are contained in $R^* \cup \{z_0\} \cup \{z_j, \bar{z}_j\}_{j=1}^n$. Then, by the residue theorem we obtain

$$(3.66) \quad \sum_{r \in R^*} \operatorname{Res}_{\xi=r} f_2(\xi; z_0, z_I) \\ = \frac{1}{\Delta(z_0)dx(z_0)} \sum_{j=1}^n \{G_{0,2}(z_0, z_j)G_{g,n}(\bar{z}_0, z_{I \setminus \{j\}}) + G_{0,2}(\bar{z}_0, z_j)G_{g,n}(z_0, z_{I \setminus \{j\}})\} \\ - \sum_{j=1}^n \frac{G_{0,2}(z_0, z_j)}{\Delta(z_j)dx(z_j)} G_{g,n}(\bar{z}_j, z_{I \setminus \{j\}}) + \sum_{j=1}^n \frac{G_{0,2}(z_0, \bar{z}_j)}{\Delta(z_j)dx(z_j)} G_{g,n}(\bar{z}_j, z_{I \setminus \{j\}}).$$

Here the first term comes from the residue at z_0 , the second term from the residue at z_j , and the third term from the residue at $\bar{z}_j$. In computing the residue at $\bar{z}_j$, we have used the relation $\Delta(\bar{z}_j)dx(\bar{z}_j) = -\Delta(z_j)dx(z_j)$. $\square$

Let us consider the diagonal specialization of $G_{g,n}$. For the purpose, we define

$$(3.67) \quad H_{g,n}(z) := \frac{1}{(n-1)!} G_{g,n}(z_0, z_1, \dots, z_{n-1}) \Big|_{z_0=z_1=\dots=z_{n-1}=z}.$$

This is a meromorphic differential in z .

Proposition 3.11 ([IKoT, Proposition 3.11]).

$$(3.68) \quad H_{1,1}(z) = \frac{W_{0,2}(z, \bar{z})}{\Delta(z)dx(z)},$$

$$(3.69) \quad H_{0,3}(z) = \frac{\partial}{\partial z_0} \left(\frac{G_{0,2}(\bar{z}_0, z)}{\Delta(z_0)x'(z_0)} \right) \Big|_{z_0=z} + \frac{G_{0,2}(z, \bar{z})G_{0,2}(\bar{z}, z)}{\Delta(z)dx(z)} \\ - \sum_{\beta \in B_1} \frac{\nu_\beta \nu_{\bar{\beta}}}{2C_\beta} (G_{0,2}(z, \beta) - G_{0,2}(z, \bar{\beta})),$$

$$(3.70) \quad H_{g,n+1}(z) = -\frac{1}{\Delta(z)x'(z)} \left[\frac{\partial}{\partial z} H_{g-1,n+2}(z) - \frac{\partial}{\partial z_0} \left(\frac{G_{g-1,n+2}(z_0, z, \dots, z)}{(n+1)!} \right) \Big|_{z_0=z} \right] \\ - \frac{d}{dz} \left(\frac{1}{\Delta(z)x'(z)} \right) H_{g,n}(z) - \frac{1}{\Delta(z)x'(z)} \frac{\partial}{\partial z_0} \left(\frac{G_{g,n}(z_0, z, \dots, z)}{(n-1)!} \right) \Big|_{z_0=z} \\ + \frac{G_{0,2}(\bar{z}, z) - G_{0,2}(z, \bar{z})}{\Delta(z)dx(z)} H_{g,n}(z) \\ - \frac{1}{\Delta(z)dx(z)} \sum_{\substack{g_1+g_2=g, \\ n_1+n_2=n, \\ 2g_1+n_1 \geq 2, \\ 2g_2+n_2 \geq 2}} H_{g_1,n_1+1}(z) H_{g_2,n_2+1}(z) \quad (2g+n \geq 3).$$

Here and in what follows, we use the following convention: For a meromorphic differential $f(z)dz$, its z -derivative means

$$\frac{\partial}{\partial z} (f(z)dz) := \frac{\partial f}{\partial z}(z)dz.$$

Proof. Because $G_{0,2}(z_j, z_k)$ is singular along $z_j = z_k$, we need a careful treatment for the terms which contain it. For example, the first term in the right-hand side of (3.49) becomes

$$\begin{aligned}
(3.71) \quad & \lim_{z_0, z_1, z_2 \rightarrow z} \left[G_{0,2}(z_0, z_1) \left(\frac{G_{0,2}(\bar{z}_0, z_2)}{\Delta(z_0)dx(z_0)} - \frac{G_{0,2}(\bar{z}_1, z_2)}{\Delta(z_1)dx(z_1)} \right) \right] \\
&= \lim_{z_0, z_1, z_2 \rightarrow z} \left[\left(\frac{dz_0}{z_0 - z_1} + (\text{holomorphic along } z_0 = z_1) \right) \times \left(\frac{G_{0,2}(\bar{z}_0, z_2)}{\Delta(z_0)dx(z_0)} - \frac{G_{0,2}(\bar{z}_1, z_2)}{\Delta(z_1)dx(z_1)} \right) \right] \\
&= \frac{\partial}{\partial z_0} \left(\frac{G_{0,2}(\bar{z}_0, z)}{\Delta(z_0)x'(z_0)} \right) \Big|_{z_0=z}
\end{aligned}$$

after the diagonal specialization. In the same manner, we can compute the third term in the right-hand side of (3.49), and obtain the same result. This proves (3.69). We use the same computation to obtain (3.70) together with Theorem 2.12 (iii) and the relation

$$\begin{aligned}
(3.72) \quad & \frac{1}{dx(z)} \int_{\zeta_1 \in D(z)} \cdots \int_{\zeta_n \in D(z)} \frac{W_{g-1, n+2}(z, \bar{z}, \zeta_I)}{n!} \\
&= -\frac{1}{dx(z)} \int_{\zeta_1 \in D(z)} \cdots \int_{\zeta_n \in D(z)} \frac{W_{g-1, n+2}(z, z, \zeta_I)}{n!} \\
&= \frac{1}{x'(z)} \left\{ -\frac{\partial}{\partial z} H_{g-1, n+2}(z) + \frac{\partial}{\partial z_0} \left(\frac{G_{g-1, n+2}(z_0, z, \dots, z)}{(n+1)!} \right) \Big|_{z_0=z} \right\}.
\end{aligned}$$

□

Proposition 3.12 ([IKoT, Proposition 3.12]).

$$(3.73) \quad \hat{T}_0(z)dz = -G_{0,2}(\bar{z}, z),$$

$$\begin{aligned}
(3.74) \quad \hat{T}_1(z) = & -\frac{1}{\Delta(z)x'(z)} \frac{d\hat{T}_0}{dz}(z) - \frac{\partial}{\partial z} \left(\frac{1}{\Delta(z)x'(z)} \right) \hat{T}_0(z) \\
& - \frac{G_{0,2}(z, \bar{z}) + G_{0,2}(\bar{z}, z)}{\Delta(z)dx(z)} \hat{T}_0(z) - \frac{1}{\Delta(z)x'(z)} \hat{T}_0(z)^2 \\
& - \sum_{\beta \in B_1} \frac{\nu_\beta \nu_{\bar{\beta}}}{2C_\beta} \frac{G_{0,2}(z, \beta) - G_{0,2}(z, \bar{\beta})}{dz},
\end{aligned}$$

$$\begin{aligned}
(3.75) \quad \hat{T}_{m+1}(z) = & -\frac{1}{\Delta(z)x'(z)} \frac{d\hat{T}_m}{dz}(z) - \frac{\partial}{\partial z} \left(\frac{1}{\Delta(z)x'(z)} \right) \hat{T}_m(z) \\
& - \frac{G_{0,2}(\bar{z}, z) + G_{0,2}(z, \bar{z})}{\Delta(z)dx(z)} \hat{T}_m(z) - \frac{1}{\Delta(z)x'(z)} \sum_{j=0}^m \hat{T}_{m-j}(z) \hat{T}_j(z) \quad (m \geq 1).
\end{aligned}$$

Proof. It follows from the definition (3.40) of $\hat{T}_m(z)$ and Theorem 2.12 (iii) that

$$\hat{T}_0(z)dz = \int_{\zeta \in D(z)} \left(W_{0,2}(z, \zeta) - \frac{dx(z)dx(\zeta)}{(x(z) - x(\zeta))^2} \right) = - \int_{\zeta \in D(z)} W_{0,2}(\bar{z}, \zeta) = -G_{0,2}(\bar{z}, z).$$

Thus we obtain (3.73).

We may write $\hat{T}_m$ for $m \geq 1$ as

$$(3.76) \quad \hat{T}_m(z)dz = \sum_{\substack{2g+n-2=m, \\ g \geq 0, n \geq 1}} H_{g,n}(z).$$

Then, by using Proposition 3.11, we find

$$(3.77) \quad \begin{aligned} \hat{T}_1(z)dz &= H_{0,3}(z) + H_{1,1}(z) \\ &= \frac{\partial}{\partial z_0} \left(\frac{G_{0,2}(\bar{z}_0, z)}{\Delta(z_0)x'(z_0)} \right) \Big|_{z_0=z} + \frac{G_{0,2}(z, \bar{z})G_{0,2}(\bar{z}, z)}{\Delta(z)dx(z)} \\ &\quad - \sum_{\beta \in B_1} \frac{\nu_\beta \nu_{\bar{\beta}}}{2C_\beta} (G_{0,2}(z, \beta) - G_{0,2}(z, \bar{\beta})) + \frac{W_{0,2}(z, \bar{z})}{\Delta(z)dx(z)}. \end{aligned}$$

Because

$$W_{0,2}(z, \bar{z}) = W_{0,2}(\bar{z}, z) = \frac{\partial}{\partial z_1} (G_{0,2}(\bar{z}, z_1)) \Big|_{z_1=z} dz$$

holds, we conclude that

$$(3.78) \quad \begin{aligned} \hat{T}_1(z)dz &= \frac{\partial}{\partial z} \left(\frac{1}{\Delta(z)x'(z)} \right) G_{0,2}(\bar{z}, z) + \frac{1}{\Delta(z)x'(z)} \frac{\partial}{\partial z} (G_{0,2}(\bar{z}, z)) \\ &\quad + \frac{G_{0,2}(z, \bar{z})G_{0,2}(\bar{z}, z)}{\Delta(z)dx(z)} - \sum_{\beta \in B_1} \frac{\nu_\beta \nu_{\bar{\beta}}}{2C_\beta} (G_{0,2}(z, \beta) - G_{0,2}(z, \bar{\beta})) \\ &= \frac{\partial}{\partial z} \left(\frac{1}{\Delta(z)x'(z)} \right) G_{0,2}(\bar{z}, z) + \frac{1}{\Delta(z)x'(z)} \frac{\partial}{\partial z} (G_{0,2}(\bar{z}, z)) \\ &\quad + \frac{G_{0,2}(z, \bar{z}) + G_{0,2}(\bar{z}, z)}{\Delta(z)dx(z)} G_{0,2}(\bar{z}, z) - \frac{G_{0,2}(\bar{z}, z)^2}{\Delta(z)dx(z)} \\ &\quad - \sum_{\beta \in B_1} \frac{\nu_\beta \nu_{\bar{\beta}}}{2C_\beta} (G_{0,2}(z, \beta) - G_{0,2}(z, \bar{\beta})). \end{aligned}$$

Then, the equality (3.74) follows from this equality together with (3.73).

Similarly, for $m \geq 1$, (3.40) gives

$$\begin{aligned}
(3.79) \quad \hat{T}_{m+1}(z)dz &= \sum_{\substack{2g+n-2=m+1, \\ g \geq 0, n \geq 1}} H_{g,n}(z) = \sum_{\substack{2g+n-2=m, \\ g \geq 0, n \geq 0}} H_{g,n+1}(z) \\
&= -\frac{1}{\Delta(z)x'(z)} \frac{\partial}{\partial z_0} \left(\sum_{\substack{2g+n-2=m, \\ g \geq 0, n \geq 0}} H_{g-1,n+2}(z_0) \right. \\
&\quad \left. - \sum_{\substack{2g+n-2=m, \\ g \geq 0, n \geq 0}} \frac{G_{g-1,n+2}(z_0, z, \dots, z)}{(n+1)!} + \sum_{\substack{2g+n-2=m, \\ g \geq 0, n \geq 0}} \frac{G_{g,n}(z_0, z, \dots, z)}{(n-1)!} \right) \Big|_{z_0=z} \\
&\quad - \frac{d}{dz} \left(\frac{1}{\Delta(z)x'(z)} \right) \hat{T}_m(z)dz + \frac{G_{0,2}(\bar{z}, z) - G_{0,2}(z, \bar{z})}{\Delta(z)dx(z)} \hat{T}_m(z)dz \\
&\quad - \frac{1}{\Delta(z)dx(z)} \sum_{\substack{2g+n-2=m, \\ g \geq 0, n \geq 0}} \sum_{\substack{g_1+g_2=g, \\ n_1+n_2=n, \\ 2g_1+n_1 \geq 2, \\ 2g_2+n_2 \geq 2}} H_{g_1,n_1+1}(z) H_{g_2,n_2+1}(z).
\end{aligned}$$

From this expression we obtain (3.75), because we can compute as

$$\begin{aligned}
(3.80) \quad &\sum_{\substack{2g+n-2=m, \\ g \geq 0, n \geq 0}} H_{g-1,n+2}(z_0) + \sum_{\substack{2g+n-2=m, \\ g \geq 0, n \geq 0}} \left(\frac{G_{g,n}(z_0, z, \dots, z)}{(n-1)!} - \frac{G_{g-1,n+2}(z_0, z, \dots, z)}{(n+1)!} \right) \\
&= \begin{cases} \sum_{\substack{2g+n-2=m, \\ g \geq 0, n \geq 2}} H_{g,n}(z_0) & \text{if } m \text{ is even} \\ \sum_{\substack{2g+n-2=m, \\ g \geq 0, n \geq 2}} H_{g,n}(z_0) + G_{\frac{m+1}{2},1}(z_0) & \text{if } m \text{ is odd} \end{cases} \\
&= \hat{T}_m(z_0)dz_0,
\end{aligned}$$

and

$$\begin{aligned}
(3.81) \quad &\frac{1}{dx(z)} \sum_{\substack{2g+n-2=m, \\ g \geq 0, n \geq 0}} \sum_{\substack{g_1+g_2=g, \\ n_1+n_2=n, \\ 2g_1+n_1 \geq 2, \\ 2g_2+n_2 \geq 2}} H_{g_1,n_1+1}(z) H_{g_2,n_2+1}(z) = \frac{1}{x'(z)} \sum_{\substack{m_1+m_2=m, \\ m_1, m_2 \geq 1}} \hat{T}_{m_1}(z) \hat{T}_{m_2}(z) dz \\
&= \frac{1}{x'(z)} \left(\sum_{j=0}^m \hat{T}_{m-j}(z) \hat{T}_j(z) - 2\hat{T}_0(z) \hat{T}_m(z) \right) dz \\
&= \frac{1}{x'(z)} \sum_{j=0}^m \hat{T}_{m-j}(z) \hat{T}_j(z) dz + \frac{2G_{0,2}(\bar{z}, z)}{x'(z)} \hat{T}_m(z),
\end{aligned}$$

where we set $m_j = 2g_j + (n_j + 1) - 2$ for $j = 1, 2$ to rewrite summation. $\square$

3.2.4 Proof of Theorem 3.5

Now we give a proof of Theorem 3.5. We will compare the recursive relations (3.73)–(3.75) satisfied by the functions $\{\hat{T}_m\}_{m \geq -1}$ to those (2.7)–(2.9) satisfied by the expansion coefficients of the WKB solutions (which will be denoted by $\{T_m\}_{m \geq -1}$ below) after the coordinate change $x = x(z)$.

Transformation of the Riccati equation (2.4) to z -variable. Let $\psi(x, \hbar)$ be the WKB solution of $\hat{P}\psi = 0$, where $\hat{P}$ is given by the formula (3.157) with the coefficients specified by (3.31)–(3.35). Let $S(x, \hbar)$ is a logarithmic derivative of $\psi(x, \hbar)$, and $T(z, \hbar)$ is that of $\psi(x(z), \hbar)$; namely,

$$S(x(z), \hbar) = \left. \frac{d}{dx} \log \psi(x, \hbar) \right|_{x=x(z)} = \frac{1}{x'(z)} \frac{d}{dz} \log \psi(x(z), \hbar) = \frac{1}{x'(z)} T(z, \hbar).$$

Since $S(x, \hbar)$ satisfies the Riccati equation (2.4) associated with (3.157), $T(z, \hbar)$ satisfies

$$(3.82) \quad \hbar^2 \left\{ \frac{d}{dz} T + T^2 \right\} + \hbar \left\{ U(z, \hbar) - \hbar \frac{x''(z)}{x'(z)} \right\} T + V(z, \hbar) = 0,$$

where $'$ designates derivative with respect to z , and we set

$$(3.83) \quad U(z, \hbar) = U_0(z) + \hbar U_1(z) := x'(z) q(x(z), \hbar),$$

$$(3.84) \quad V(z, \hbar) = V_0(z) + \hbar V_1(z) + \hbar^2 V_2(z) := (x'(z))^2 r(x(z), \hbar).$$

Hence

$$(3.85) \quad T(z, \hbar) = \frac{1}{\hbar} T_{-1}(z) + T_0(z) + \hbar T_1(z) + \cdots$$

satisfies (3.82) if and only if the following recurrence relations hold:

$$(3.86) \quad T_{-1}^2 + U_0(z) T_{-1} + V_0(z) = 0,$$

$$(3.87) \quad \{2T_{-1}(z) + U_0(z)\} T_{m+1} + \left\{ U_1(z) - \frac{x''(z)}{x'(z)} \right\} T_m \\ + \sum_{j=0}^m T_{m-j} T_j + \frac{d}{dz} T_m + V_{m+2}(z) = 0 \quad (m \geq -1),$$

where we set $V_m(z) = 0$ for $m \geq 3$. These equations determine the coefficients $\{T_m\}_{m \geq -1}$ of $T(z)$ uniquely.

Our task for the proof of Theorem 3.5 is to show that

$$(3.88) \quad T_m(z) = \hat{T}_m(z) \quad (m \geq -1).$$

Equivalently, we must prove that $\{\hat{T}_m\}_{m \geq 0}$ satisfies the recurrence relations (3.86) and (3.87). The rest of this subsection is devoted to a proof of (3.88).

Proof of $T_{-1}(z) = \hat{T}_{-1}(z)$. Since

$$(3.89) \quad U_0(z) = x'(z)q_0(x(z)), \quad V_0(z) = x'(z)^2 r_0(x(z))$$

and $(x(z), y(z))$ satisfies $y(z)^2 + q_0(x(z))y(z) + r_0(x(z)) = 0$, we find

$$(3.90) \quad \hat{T}_{-1}(z) = \frac{W_{0,1}(z)}{dz} = y(z)x'(z)$$

satisfies (3.86). Thus we have verified (3.88) for $m = -1$.

Proof of $T_0(z) = \hat{T}_0(z)$. Since $y(z)$ and $y(\bar{z})$ are two solutions of $P(x(z), y) = 0$,

$$(3.91) \quad y(z) + y(\bar{z}) = -q_0(x(z)).$$

Therefore,

$$(3.92) \quad 2T_{-1}(z) + U_0(z) = x'(z)\{2y(z) + q_0(x(z))\} = x'(z)\{y(z) - y(\bar{z})\} = x'(z)\Delta(z)$$

holds. Then, (3.87) for $m = -1$ implies that the desired relation $T_0(z) = \hat{T}_0(z)$ is equivalent to

$$(3.93) \quad x'(z)\Delta(z)\hat{T}_0 + \left\{U_1(z) - \frac{x''(z)}{x'(z)}\right\}\hat{T}_{-1} + \frac{d}{dz}\hat{T}_{-1} + V_1(z) = 0.$$

In order to verify (3.93), we first prove the following.

Lemma 3.13 ([IKoT, Lemma 3.13]). *The function $U_1(z)$ is expressed as*

$$(3.94) \quad U_1(z) = -\frac{\Delta'(z)}{\Delta(z)} + \frac{G_{0,2}(\bar{z}, z) + G_{0,2}(z, \bar{z})}{dz}.$$

Proof of Lemma 3.13. It follows from Proposition 3.7 (ii) and Proposition 3.8 (i) that

$$(3.95) \quad \frac{G_{0,2}(\bar{z}, z) + G_{0,2}(z, \bar{z})}{dz} = \frac{2}{z - \bar{z}} - \sum_{\beta \in B} \nu_\beta \left(\frac{1}{z - \beta} + \frac{1}{z - \bar{\beta}} \right) = \frac{2}{z - \bar{z}} - \sum_{\beta \in B} \frac{\nu_\beta + \nu_{\bar{\beta}}}{z - \beta}.$$

Here we have used the fact that B is closed under the conjugate map (cf. Remark 3.6 (iii)) in the last equality. This proves the relation (3.94). $\square$

Let us prove

$$(3.96) \quad V_1(z) = -x'(z)\Delta(z)\hat{T}_0 - \left\{U_1(z) - \frac{x''(z)}{x'(z)}\right\}\hat{T}_{-1} - \frac{d}{dz}\hat{T}_{-1},$$

which is equivalent to (3.93). By using Proposition 3.12 and Lemma 3.13, the right-hand side becomes

$$(3.97) \quad \begin{aligned} & -x'(z)\Delta(z)\hat{T}_0 - \left\{U_1(z) - \frac{x''(z)}{x'(z)}\right\}\hat{T}_{-1} - \frac{d}{dz}\hat{T}_{-1} \\ &= x'(z)\Delta(z)\frac{G_{0,2}(\bar{z}, z)}{dz} - \left\{-\frac{\Delta'(z)}{\Delta(z)} + \frac{G_{0,2}(\bar{z}, z) + G_{0,2}(z, \bar{z})}{dz} - \frac{x''(z)}{x'(z)}\right\}x'(z)y(z) \\ & \quad - \frac{d}{dz}(x'(z)y(z)) \\ &= x'(z)\Delta(z)\frac{G_{0,2}(\bar{z}, z)}{dz} + \frac{\Delta'(z)}{\Delta(z)}x'(z)y(z) \\ & \quad - \frac{G_{0,2}(\bar{z}, z) + G_{0,2}(z, \bar{z})}{dz}x'(z)y(z) - x'(z)y'(z). \end{aligned}$$

Substituting

$$y(z) = \frac{y(z) - y(\bar{z})}{2} + \frac{y(z) + y(\bar{z})}{2} = \frac{1}{2}\Delta(z) - \frac{1}{2}q_0(x(z))$$

into the last expression, we find

$$(3.98) \quad \begin{aligned} & \text{(RHS of (3.96))} \\ &= \frac{1}{2}x'(z)\Delta(z)\frac{G_{0,2}(\bar{z}, z) - G_{0,2}(z, \bar{z})}{dz} - \frac{1}{2}x'(z)\frac{\Delta'(z)}{\Delta(z)}q_0(x(z)) \\ & \quad + \frac{1}{2}x'(z)^2\frac{dq_0}{dx}\Big|_{x=x(z)} + \frac{1}{2}x'(z)q_0(x(z))\frac{G_{0,2}(\bar{z}, z) + G_{0,2}(z, \bar{z})}{dz}. \end{aligned}$$

Then, thanks to the equalities (3.94) and

$$(3.99) \quad \frac{G_{0,2}(\bar{z}, z) - G_{0,2}(z, \bar{z})}{dz} = -\sum_{\beta \in B} \frac{\nu_\beta}{z - \bar{\beta}} + \sum_{\beta \in B} \frac{\nu_\beta}{z - \beta} = \sum_{\beta \in B} \frac{\nu_\beta - \nu_{\bar{\beta}}}{z - \beta},$$

we conclude that the right-hand side of (3.96) is equal to $V_1(z)$. Thus we have verified (3.93), and hence, (3.88) holds for $m = 0$.

Proof of $T_1(z) = \hat{T}_1(z)$. It follows from Proposition 3.12 and Lemma 3.13 that $\hat{T}_1(z)$ satisfies

$$(3.100) \quad \begin{aligned} & x'(z)\Delta(z)\hat{T}_1(z) + \left\{ U_1(z) - \frac{x''(z)}{x'(z)} \right\} \hat{T}_0(z) + \hat{T}_0(z)^2 + \frac{d\hat{T}_0}{dz}(z) \\ & \quad + x'(z)\Delta(z) \sum_{\beta \in B_1} \frac{\nu_\beta \nu_{\bar{\beta}}}{2C_\beta} \frac{G_{0,2}(z, \beta) - G_{0,2}(z, \bar{\beta})}{dz} = 0. \end{aligned}$$

In view of (3.87) for $m = 0$, the desired relation $T_1(z) = \hat{T}_1(z)$ is equivalent to the following claim:

Lemma 3.14 ([IKoT, Lemma 3.14]). *The function $V_2(z)$ is expressed as*

$$(3.101) \quad V_2(z) = x'(z)\Delta(z) \sum_{\beta \in B_1} \frac{\nu_\beta \nu_{\bar{\beta}}}{2C_\beta} \frac{G_{0,2}(z, \beta) - G_{0,2}(z, \bar{\beta})}{dz}.$$

Proof of Lemma 3.14. It follows from Proposition 3.7 (ii) and the relation (3.59) that

$$(3.102) \quad \begin{aligned} \sum_{\beta \in B_1} \frac{\nu_\beta \nu_{\bar{\beta}}}{2C_\beta} \frac{G_{0,2}(z, \beta) - G_{0,2}(z, \bar{\beta})}{dz} &= \sum_{\beta \in B_1} \frac{\nu_\beta \nu_{\bar{\beta}}}{2C_\beta} \frac{1}{z - \beta} - \sum_{\beta \in B_1} \frac{\nu_\beta \nu_{\bar{\beta}}}{2C_\beta} \frac{1}{z - \bar{\beta}} \\ &= \sum_{\beta \in B_1} \frac{\nu_\beta \nu_{\bar{\beta}}}{2C_\beta} \frac{1}{z - \beta} + \sum_{\beta \in B_1} \frac{\nu_\beta \nu_{\bar{\beta}}}{2C_{\bar{\beta}}} \frac{1}{z - \bar{\beta}} \\ &= \sum_{\beta \in B_1} \frac{\nu_\beta \nu_{\bar{\beta}}}{C_\beta} \frac{1}{z - \beta}. \end{aligned}$$

In the last equality, we have also used the fact that the set B_1 is closed under the conjugate map (cf. Remark 3.6 (iii)). This proves (3.101). $\square$

Thus we have verified that (3.88) holds for $m = 1$.

Proof of $T_m(z) = \hat{T}_m(z)$ for $m \geq 2$. Proposition 3.12 and Lemma 3.13 show that

$$x'(z)\Delta(z)\hat{T}_{m+1}(z) + \left\{ U_1(z) - \frac{x''(z)}{x'(z)} \right\} \hat{T}_0(z) + \sum_{j=0}^m \hat{T}_{m-j}(z)\hat{T}_j(z) + \frac{d\hat{T}_m}{dz} = 0 \quad (m \geq 1)$$

holds. This is the same as the equation (3.87) for $m \geq 1$ satisfied by $T_{m+1}(z)$. Thus we have proved the desired relation (3.88) for all $m \geq 2$.

This completes the proof of Theorem 3.5.

3.2.5 Some examples

Here we show the list of quantum curves associated with the spectral curves arising from the confluent family of the Gauss hypergeometric equations. We can observe that these equations satisfied by classical special functions are recovered as the quantum curve. An application of the result will be presented in next section.

3.2.5.1 Quantum Gauss curve ([IKoT2, §2.3.1])

Let us consider the Gauss curve defined by

$$(3.103) \quad P(x, y) = x^2(1-x)^2y^2 - \{\lambda_\infty^2x^2 - (\lambda_\infty^2 + \lambda_0^2 - \lambda_1^2)x + \lambda_0^2\} = 0.$$

Note that the curve does not satisfy the admissibility condition of [BE2, Definition 2.7]; that is, the Newton polygon of (3.103) contains an interior point. Our spectral curve is obtained as a rational parameterization of (3.103) given by

$$(3.104) \quad \begin{cases} x = x(z) = \frac{\sqrt{\Lambda}}{4\lambda_\infty^2} \left(z + \frac{1}{z} \right) + \frac{\lambda_\infty^2 + \lambda_0^2 - \lambda_1^2}{2\lambda_\infty^2} = \frac{\sqrt{\Lambda}}{4\lambda_\infty^2} \frac{(z - \beta_{0+})(z - \beta_{0-})}{z}, \\ y = y(z) = \frac{4\lambda_\infty^3 z(z^2 - 1)}{\sqrt{\Lambda}(z - \beta_{0+})(z - \beta_{0-})(z - \beta_{1+})(z - \beta_{1-})}, \end{cases}$$

where

$$(3.105) \quad \Lambda = (\lambda_0 + \lambda_1 + \lambda_\infty)(\lambda_0 + \lambda_1 - \lambda_\infty)(\lambda_0 - \lambda_1 + \lambda_\infty)(\lambda_0 - \lambda_1 - \lambda_\infty),$$

$$(3.106) \quad \beta_{0\pm} = -\frac{(\lambda_0 \pm \lambda_\infty)^2 - \lambda_1^2}{\sqrt{\Lambda}}, \quad \beta_{1\pm} = \frac{(\lambda_1 \pm \lambda_\infty)^2 - \lambda_0^2}{\sqrt{\Lambda}}.$$

We impose the following generic conditions for parameters $\underline{\lambda} = (\lambda_0, \lambda_1, \lambda_\infty)$ so that our assumptions are satisfied:

$$(3.107) \quad \lambda_0, \lambda_1, \lambda_\infty \neq 0 \quad \text{and} \quad \Lambda \neq 0.$$

Note that $R = \{1, -1\}$ ($= R^*$) and the conjugate map is given by $\bar{z} = 1/z$.

We also introduce the notation $\beta_{\infty+} := 0$ and $\beta_{\infty-} := \infty$. Then, for each $j \in \{0, 1, \infty\}$, we can verify that $\beta_{j\pm}$ are two preimages of j by the map $x(z)$; that is, $x(\beta_{j\pm}) = j$ holds. Note also that the parameters are chosen so that

$$(3.108) \quad \pm\lambda_j = \operatorname{Res}_{z=\beta_{j\pm}} y(z)dx(z)$$

holds for each $j \in \{0, 1, \infty\}$. (We will use similar notations in the subsequent subsections so that the relation (3.108) holds.)

First few terms of the correlation functions and free energies are computed as

$$\begin{aligned} W_{0,3}(z_1, z_2, z_3) &= \left\{ \frac{(\beta_{0+} + \beta_{0-} + 2)(\beta_{1+} + \beta_{1-} + 2)}{(z_1 + 1)^2(z_2 + 1)^2(z_3 + 1)^2} \right. \\ &\quad \left. - \frac{(\beta_{0+} + \beta_{0-} - 2)(\beta_{1+} + \beta_{1-} - 2)}{(z_1 - 1)^2(z_2 - 1)^2(z_3 - 1)^2} \right\} \frac{dz_1 dz_2 dz_3}{4\lambda_\infty}, \\ W_{1,1}(z) &= -\frac{z(z - \beta_{0+})(z - \beta_{0-})(z - \beta_{1+})(z - \beta_{1-})}{2\lambda_\infty(z^2 - 1)^4} dz, \\ F_0 &= \frac{(\lambda_0 + \lambda_1 + \lambda_\infty)^2}{2} \log(\lambda_0 + \lambda_1 + \lambda_\infty) + \frac{(\lambda_0 - \lambda_1 + \lambda_\infty)^2}{2} \log(\lambda_0 - \lambda_1 + \lambda_\infty) \\ &\quad + \frac{(\lambda_0 + \lambda_1 - \lambda_\infty)^2}{2} \log(\lambda_0 + \lambda_1 - \lambda_\infty) + \frac{(\lambda_0 - \lambda_1 - \lambda_\infty)^2}{2} \log(\lambda_0 - \lambda_1 - \lambda_\infty) \\ &\quad - 2\lambda_0^2 \log(2\lambda_0) - 2\lambda_1^2 \log(2\lambda_1) - 2\lambda_\infty^2 \log(2\lambda_\infty), \\ F_1 &= -\frac{1}{12} \log\left(\frac{\Lambda}{\lambda_0 \lambda_1 \lambda_\infty}\right), \\ F_2 &= \frac{1}{960\lambda_0^2 \lambda_1^2 \lambda_\infty^2 \Lambda^2} \{(\lambda_0^2 + \lambda_1^2)\lambda_\infty^{10} - (4\lambda_0^4 + 23\lambda_0^2 \lambda_1^2 + 4\lambda_1^4)\lambda_\infty^8 \\ &\quad + 2(\lambda_0^2 + \lambda_1^2)(3\lambda_0^4 + 8\lambda_0^2 \lambda_1^2 + 3\lambda_1^4)\lambda_\infty^6 - 2(2\lambda_0^8 - 11\lambda_0^6 \lambda_1^2 + 74\lambda_0^4 \lambda_1^4 \\ &\quad - 11\lambda_0^2 \lambda_1^6 + 2\lambda_1^8)\lambda_\infty^4 + (\lambda_0^2 - \lambda_1^2)^2(\lambda_0^2 + \lambda_1^2)(\lambda_0^4 - 22\lambda_0^2 \lambda_1^2 + \lambda_1^4)\lambda_\infty^2 \\ &\quad + \lambda_0^2 \lambda_1^2 (\lambda_0^2 - \lambda_1^2)^4\}. \end{aligned}$$

Since

$$(3.109) \quad Q_0(x) = \frac{\lambda_\infty^2 x^2 - (\lambda_\infty^2 + \lambda_0^2 - \lambda_1^2)x + \lambda_0^2}{x^2(1-x)^2}, \quad Q_\infty(x) = \frac{\lambda_0^2 x^2 - (\lambda_\infty^2 + \lambda_0^2 - \lambda_1^2)x + \lambda_\infty^2}{x^2(1-x)^2},$$

we have

$$(3.110) \quad \operatorname{Sing}(P) = \operatorname{Sing}_2(P) = \{0, 1, \infty\}, \quad B = B_1 = \{\beta_{0+}, \beta_{0-}, \beta_{1+}, \beta_{1-}, \beta_{\infty+}, \beta_{\infty-}\}.$$

Therefore we choose

$$(3.111) \quad D(z; \underline{\nu}) = [z] - \nu_{0+}[\beta_{0+}] - \nu_{0-}[\beta_{0-}] - \nu_{1+}[\beta_{1+}] - \nu_{1-}[\beta_{1-}] - \nu_{\infty+}[\beta_{\infty+}] - \nu_{\infty-}[\beta_{\infty-}]$$

as the divisor for the quantization, where the parameters $\nu_{0\pm}$, $\nu_{1\pm}$ and $\nu_{\infty\pm}$ satisfy

$$(3.112) \quad \nu_{0+} + \nu_{0-} + \nu_{1+} + \nu_{1-} + \nu_{\infty+} + \nu_{\infty-} = 1.$$

We may easily verify that $C_{\beta_{j\pm}} = \pm 2\lambda_j$ for $j \in \{0, 1, \infty\}$. Then, Theorem 3.5 gives the quantum curve of the Gauss curve (quantum Gauss curve):

$$(3.113) \quad \left(\hbar^2 \frac{d^2}{dx^2} + q(x, \hbar) \hbar \frac{d}{dx} + r(x, \hbar) \right) \psi = 0,$$

where $q(x, \hbar) = q_0(x) + \hbar q_1(x)$, $r(x, \hbar) = r_0(x) + \hbar r_1(x) + r_2(x)$ and

$$\begin{aligned} q_0(x) &= 0, \quad q_1(x) = \frac{1 - \nu_{0+} - \nu_{0-}}{x} + \frac{1 - \nu_{1+} - \nu_{1-}}{x-1}, \\ r_0(x) &= -\frac{\lambda_\infty^2 x^2 - (\lambda_\infty^2 + \lambda_0^2 - \lambda_1^2)x + \lambda_0^2}{x^2(1-x)^2}, \\ r_1(x) &= -\frac{(\nu_{0+} - \nu_{0-})\lambda_0}{x^2(x-1)} + \frac{(\nu_{1+} - \nu_{1-})\lambda_1}{x(x-1)^2} + \frac{(\nu_{\infty+} - \nu_{\infty-})\lambda_\infty}{x(x-1)}, \\ r_2(x) &= -\frac{\nu_{0+}\nu_{0-}}{x^2(x-1)} + \frac{\nu_{1+}\nu_{1-}}{x(x-1)^2} + \frac{\nu_{\infty+}\nu_{\infty-}}{x(x-1)}. \end{aligned}$$

The equation (3.113) has three regular singular points at 0, 1 and ∞ , and hence, it must be equivalent to the Gauss hypergeometric equation. Indeed, (3.113) is related to the standard form of the Gauss hypergeometric equation

$$(3.114) \quad \left\{ x(1-x) \frac{d^2}{dx^2} + (\gamma - (\alpha + \beta + 1)x) \frac{d}{dx} - \alpha\beta \right\} \phi = 0$$

by

$$(3.115) \quad \alpha = \frac{\hat{\lambda}_0 + \hat{\lambda}_1 + \hat{\lambda}_\infty}{\hbar} + \frac{1}{2}, \quad \beta = \frac{\hat{\lambda}_0 + \hat{\lambda}_1 - \hat{\lambda}_\infty}{\hbar} + \frac{1}{2}, \quad \gamma = \frac{2\hat{\lambda}_0}{\hbar} + 1$$

and the gauge transform

$$(3.116) \quad \psi = x^{\frac{\hat{\lambda}_0}{\hbar} + \frac{\nu_{0+} + \nu_{0-}}{2}} (x-1)^{\frac{\hat{\lambda}_1}{\hbar} + \frac{\nu_{1+} + \nu_{1-}}{2}} \phi.$$

Here we used the notation

$$(3.117) \quad \hat{\lambda}_j = \lambda_j - \frac{\hbar \nu_j}{2}, \quad \nu_j = \nu_{j+} - \nu_{j-} \quad (j \in \{0, 1, \infty\}).$$

We will investigate this equation in more detail in §4.

Remark 3.15. The Gauss curve has a symmetry. That is, the symplectic transformations

$$(3.118) \quad (x, y) \mapsto (x_j, y_j) := \begin{cases} (1-x, -y) & \text{for } j = 1, \\ (1/x, -x^2 y) & \text{for } j = \infty, \end{cases}$$

yield the permutation

$$(3.119) \quad (\lambda_0, \lambda_1, \lambda_\infty) \mapsto \begin{cases} (\lambda_1, \lambda_0, \lambda_\infty) & \text{for } j = 1, \\ (\lambda_\infty, \lambda_1, \lambda_0) & \text{for } j = \infty \end{cases}$$

of parameters in the Gauss curve (3.103). Since the symplectic transform preserves the free energy (cf. [EO1, §7]), this proves that the free energy is invariant under any permutation of the parameters $\lambda_0, \lambda_1, \lambda_\infty$.

Moreover, this symmetry of the Gauss curve is lifted to that of quantum Gauss curve (3.113). Namely, the parameters in the quantum Gauss curve are permuted as

$$(3.120) \quad (\lambda_0, \lambda_1, \lambda_\infty, \nu_{0\pm}, \nu_{1\pm}, \nu_{\infty\pm}) \mapsto \begin{cases} (\lambda_1, \lambda_0, \lambda_\infty, \nu_{1\pm}, \nu_{0\pm}, \nu_{\infty\pm}) & \text{for } j = 1, \\ (\lambda_\infty, \lambda_1, \lambda_0, \nu_{\infty\pm}, \nu_{1\pm}, \nu_{0\pm}) & \text{for } j = \infty \end{cases}$$

under the coordinate transform $x \mapsto x_j$ ($j \in \{1, \infty\}$) given above. This will induce a certain symmetry property of the Voros coefficients (see Theorem 4.1 (iv)).

3.2.5.2 Quantum degenerate Gauss curve ([IKoT2, §2.3.2])

In this subsection we consider a spectral curve defined by

$$(3.121) \quad P(x, y) = x(x-1)^2 y^2 - \lambda_\infty^2 x - \lambda_1^2 + \lambda_\infty^2 = 0,$$

which has a parametrization

$$(3.122) \quad \begin{cases} x(z) &= \frac{\lambda_1^2 - \lambda_\infty^2}{z^2 - \lambda_\infty^2}, \\ y(z) &= -\frac{z(z^2 - \lambda_\infty^2)}{z^2 - \lambda_1^2}. \end{cases}$$

Here we impose

$$(3.123) \quad \lambda_1, \lambda_\infty \neq 0, \quad \text{and} \quad \lambda_\infty \pm \lambda_1 \neq 0$$

for $\underline{\lambda} = (\lambda_1, \lambda_\infty)$ so that our assumptions are satisfied. For this curve, we have $R = \{0, \infty\}$ ($= R^*$) and $\bar{z} = -z$. First few correlation functions and free energies are given explicitly as follows:

$$\begin{aligned} W_{0,3}(z_1, z_2, z_3) &= -\frac{\lambda_\infty^2 \lambda_1^2}{2(\lambda_1^2 - \lambda_\infty^2)} \frac{dz_1 dz_2 dz_3}{z_1^2 z_2^2 z_3^2}, \\ W_{1,1}(z_1) &= \frac{1}{16(\lambda_\infty^2 - \lambda_1^2)} \frac{(z_1^2 - \lambda_\infty^2)(z_1^2 - \lambda_1^2)}{z_1^4} dz_1, \end{aligned}$$

$$\begin{aligned} F_0 &= \lambda_\infty^2 \log \left(\frac{\lambda_\infty^2 - \lambda_1^2}{4\lambda_\infty^2} \right) + 2\lambda_\infty \lambda_1 \log \left(\frac{\lambda_\infty + \lambda_1}{\lambda_\infty - \lambda_1} \right) + \lambda_1^2 \log \left(\frac{\lambda_\infty^2 - \lambda_1^2}{4\lambda_1^2} \right), \\ F_1 &= -\frac{1}{12} \log \left(\frac{(\lambda_\infty^2 - \lambda_1^2)^2}{\lambda_\infty \lambda_1} \right), \quad F_2 = \frac{\lambda_\infty^6 - 17\lambda_\infty^4 \lambda_1^2 - 17\lambda_\infty^2 \lambda_1^4 + \lambda_1^6}{960\lambda_\infty^2 \lambda_1^2 (\lambda_\infty^2 - \lambda_1^2)^2}. \end{aligned}$$

Theorem 3.5 implies that, for the choice

$$(3.124) \quad D(z; \underline{\nu}) = [z] - \nu_{1+}[\beta_{1+}] - \nu_{1-}[\beta_{1-}] - \nu_{\infty+}[\beta_{\infty+}] - \nu_{\infty-}[\beta_{\infty-}]$$

(where $\beta_{1\pm} = \pm\lambda_1$, $\beta_{\infty\pm} = \mp\lambda_\infty$, and the parameters satisfy $\nu_{1+} + \nu_{1-} + \nu_{\infty+} + \nu_{\infty-} = 1$) of the divisor, we obtain the following quantum curve:

$$(3.125) \quad \left(\hbar^2 \frac{d^2}{dx^2} + q(x, \hbar) \hbar \frac{d}{dx} + r(x, \hbar) \right) \psi = 0$$

with

$$\begin{aligned} q(x, \hbar) &= \hbar \left(\frac{1}{x} + \frac{1 - \nu_{1+} - \nu_{1-}}{x - 1} \right), \\ r(x, \hbar) &= -\frac{\lambda_\infty^2 x + \lambda_1^2 - \lambda_\infty^2}{x(x-1)^2} + \hbar \left(\frac{(\nu_{1+} - \nu_{1-})\lambda_1}{x(x-1)^2} + \frac{(\nu_{\infty+} - \nu_{\infty-})\lambda_\infty}{x(x-1)} \right) \\ &\quad + \hbar^2 \left(\frac{\nu_{1+}\nu_{1-}}{x(x-1)^2} + \frac{\nu_{\infty+}\nu_{\infty-}}{x(x-1)} \right). \end{aligned}$$

This equation is a special case (i.e. $\lambda_0 = 0$, $\nu_{0\pm} = 0$) of the quantum Gauss curve (3.113), and hence, we call (3.125) the degenerate Gauss equation. We distinguish these equations because the geometry of the spectral curves (or WKB-theoretic structure) are different.

3.2.5.3 Quantum Kummer curve ([IKoT2, §2.3.3])

Let us consider the Kummer curve defined by

$$(3.126) \quad P(x, y) = 4x^2y^2 - (x^2 + 4\lambda_\infty x + 4\lambda_0^2) = 0 \quad (\lambda_0 \neq 0, \lambda_0 \pm \lambda_\infty \neq 0).$$

A rational parameterization of this curve is

$$(3.127) \quad \begin{cases} x = x(z) = \sqrt{\lambda_\infty^2 - \lambda_0^2} \left(z + \frac{1}{z} \right) - 2\lambda_\infty = \frac{\sqrt{\lambda_\infty^2 - \lambda_0^2} (z - \beta_{0+})(z - \beta_{0-})}{z}, \\ y = y(z) = \frac{z^2 - 1}{2(z - \beta_{0+})(z - \beta_{0-})} \end{cases}$$

with $\beta_{0\pm} = (\lambda_\infty \pm \lambda_0)/\sqrt{\lambda_\infty^2 - \lambda_0^2}$. We also set $\beta_{\infty+} = 0$ and $\beta_{\infty-} = \infty$ for the consistency of our presentation. The set of ramification points is given by $R = \{1, -1\}$ ($= R^*$), and $\bar{z} = 1/z$.

Let $W_{g,n}$ and F_g be the correlation functions and the free energies computed from (3.127) respectively. Few of them are computed as

$$\begin{aligned} W_{0,3}(z_1, z_2, z_3) &= -\frac{dz_1 dz_2 dz_3}{2\sqrt{\lambda_\infty^2 - \lambda_0^2}} \left\{ \frac{\beta_{0+} + \beta_{0-} + 2}{(z_1 + 1)^2(z_2 + 1)^2(z_3 + 1)^2} - \frac{\beta_{0+} + \beta_{0-} - 2}{(z_1 - 1)^2(z_2 - 1)^2(z_3 - 1)^2} \right\}, \\ W_{1,1}(z) &= -\frac{z^2(z - \beta_{0+})(z - \beta_{0-})}{\sqrt{\lambda_\infty^2 - \lambda_0^2}(z^2 - 1)^4} dz, \end{aligned}$$

$$\begin{aligned}
F_0 &= \frac{(\lambda_\infty + \lambda_0)^2}{2} \log(\lambda_\infty + \lambda_0) + \frac{(\lambda_\infty - \lambda_0)^2}{2} \log(\lambda_\infty - \lambda_0) \\
&\quad - 2\lambda_0^2 \log(2\lambda_0) - \frac{3}{2}(\lambda_\infty^2 - \lambda_0^2), \\
F_1 &= -\frac{1}{12} \log\left(\frac{\lambda_\infty^2 - \lambda_0^2}{\lambda_0}\right), \quad F_2 = \frac{\lambda_\infty^4 - 10\lambda_\infty^2\lambda_0^2 - 7\lambda_0^4}{960\lambda_0^2(\lambda_\infty^2 - \lambda_0^2)^2}.
\end{aligned}$$

Let us describe the quantization of (3.126). For the divisor

$$(3.128) \quad D(z; \underline{\nu}) = [z] - \nu_{\infty+}[\beta_{\infty+}] - \nu_{\infty-}[\beta_{\infty-}] - \nu_{0+}[\beta_{0+}] - \nu_{0-}[\beta_{0-}]$$

(where the parameters satisfy $\nu_{\infty+} + \nu_{\infty-} + \nu_{0+} + \nu_{0-} = 1$), Theorem 3.5 shows that the formal series $\psi(x; \underline{\nu}, \hbar) = \varphi(z(x); \underline{\nu}, \hbar)$ (cf. (3.1)) is a WKB solution of the following equation (quantum Kummer curve)

$$(3.129) \quad \left(\hbar^2 \frac{d^2}{dx^2} + q(x, \hbar) \hbar \frac{d}{dx} + r(x, \hbar) \right) \psi = 0$$

with

$$\begin{aligned}
q(x, \hbar) &= \hbar \frac{\nu_{\infty+} + \nu_{\infty-}}{x}, \\
r(x, \hbar) &= -\frac{x^2 + 4\lambda_\infty x + \lambda_0^2}{4x^2} + \hbar \left(\frac{\nu_{\infty+} - \nu_{\infty-}}{2x} + \frac{(\nu_{0+} - \nu_{0-})\lambda_0}{x^2} \right) + \hbar^2 \frac{\nu_{0+}\nu_{0-}}{x^2}.
\end{aligned}$$

This equation is equivalent to the Kummer differential equation (i.e., the confluent hypergeometric equation) via a certain gauge transformation. Note also that the quantum curve (3.129) is closely related to the quantum differential equation associated with the degree 1 hypersurface in $\mathbb{CP}^1$. A realization of the equation as a quantum curve has already been done in [FIMS, §4.1].

3.2.5.4 Quantum Legendre curve ([IKoT2, §2.3.4])

This subsection is devoted to study the Legendre curve defined by

$$(3.130) \quad P(x, y) := (x^2 - 1)y^2 - \lambda_\infty^2 = 0 \quad (\lambda_\infty \neq 0).$$

A rational parametrization of this curve is

$$(3.131) \quad \begin{cases} x(z) &= \frac{1}{2} \left(z + \frac{1}{z} \right), \\ y(z) &= \frac{2\lambda_\infty z}{z^2 - 1}. \end{cases}$$

For this curve, we have $R = \{1, -1\}$ ($= R^*$), and the conjugate map $\bar{z} = 1/z$. Examples of correlation functions and free energies are:

$$\begin{aligned}
W_{1,1}(z) &= -\frac{z}{2\lambda_\infty(z^2 - 1)^2} dz, \quad W_{0,3}(z_1, z_2, z_3) = 0, \\
W_{1,2}(z_1, z_2) &= \frac{z_1^2 z_2^2 + z_1^2 + z_2^2 + 4z_1 z_2 + 1}{4\lambda_\infty^2(z_1^2 - 1)^2(z_2^2 - 1)^2} dz_1 dz_2, \quad W_{2,1}(z) = -\frac{z^5 + 7z^3 + z}{8\lambda_\infty^3(z^2 - 1)^4} dz,
\end{aligned}$$

$$F_0 = -\lambda_\infty^2 \log 2, \quad F_1 = -\frac{1}{4} \log \lambda_\infty, \quad F_2 = -\frac{1}{64\lambda_\infty^2}, \quad F_3 = \frac{1}{256\lambda_\infty^4}.$$

Set $\beta_{\infty+} = 0, \beta_{\infty-} = \infty$ and choose a divisor

$$(3.132) \quad D(z; \underline{\nu}) = [z] - \nu_{\infty+}[\beta_{\infty+}] - \nu_{\infty-}[\beta_{\infty-}] \quad (\nu_{\infty+} + \nu_{\infty-} = 1).$$

Then, the associated quantum curve by Theorem 3.5 is given by

$$(3.133) \quad \left\{ \hbar^2 \frac{d^2}{dx^2} + \hbar^2 \frac{2x}{x^2-1} \frac{d}{dx} - \frac{\lambda_\infty^2}{x^2-1} + \hbar \frac{\lambda_\infty(\nu_{\infty+} - \nu_{\infty-})}{x^2-1} + \hbar^2 \frac{\nu_{\infty+}\nu_{\infty-}}{x^2-1} \right\} \psi = 0.$$

This equation (3.133) has two simple-pole type turning points at $x = \pm 1$ which correspond to the ramification points $z = \pm 1$ of the spectral curve.

3.2.5.5 Quantum Bessel curve ([IKoT2, §2.3.5])

Let us consider the Bessel curve defined by

$$(3.134) \quad P(x, y) = y^2 - \frac{x + 4\lambda_0^2}{4x^2} = 0 \quad (\lambda_0 \neq 0).$$

A rational parameterization of this curve is given by

$$(3.135) \quad \begin{cases} x = x(z) = 4\lambda_0^2(z^2 - 1), \\ y = y(z) = \frac{z}{4\lambda_0(z^2 - 1)}. \end{cases}$$

For this curve, we have $R = \{0, \infty\}$. Note that $\infty \in R$ is ineffective, and hence $R^* = \{0\}$. The conjugate map is given by $\bar{z} = -z$. Few of $W_{g,n}$ and F_g are computed as

$$\begin{aligned} W_{0,3}(z_1, z_2, z_3) &= \frac{dz_1 dz_2 dz_3}{2\lambda_0 z_1^2 z_2^2 z_3^2}, \quad W_{1,1}(z) = -\frac{z^2 - 1}{16\lambda_0 z^4} dz, \\ W_{1,2}(z_1, z_2) &= \frac{z_1^4 z_2^4 - 6z_1^4 z_2^2 - 6z_1^2 z_2^4 + 5z_1^4 + 3z_1^2 z_2^2 + 5z_2^4}{32\lambda_0^2 z_1^6 z_2^6} dz_1 dz_2, \\ W_{2,1}(z) &= -\frac{9z^6 - 107z^4 + 203z^2 - 105}{1024\lambda_0^3 z^{10}} dz, \end{aligned}$$

$$F_0 = 3\lambda_0^2 + \lambda_0^2 \log \left(-\frac{1}{16\lambda_0^2} \right), \quad F_1 = -\frac{1}{24} \log \left(-\frac{1}{16\lambda_0^2} \right), \quad F_2 = \frac{1}{960\lambda_0^2}, \quad F_3 = -\frac{1}{16128\lambda_0^4}.$$

Let us take a divisor

$$(3.136) \quad D(z; \underline{\nu}) = [z] - \nu_{0+}[\beta_{0+}] - \nu_{0-}[\beta_{0-}] - \nu_\infty[\infty] \quad (\nu_{0+} + \nu_{0-} + \nu_\infty = 1),$$

where $\beta_{0\pm} = \pm 1$. Then, the quantization of the Bessel curve via Theorem 3.5 is given by

$$(3.137) \quad \left\{ \hbar^2 \frac{d^2}{dx^2} + \hbar^2 \frac{1 - \nu_{0+} - \nu_{0-}}{x} \frac{d}{dx} - \frac{x + 4\lambda_0^2}{4x^2} + \hbar \frac{\lambda_0(\nu_{0+} - \nu_{0-})}{x^2} + \hbar^2 \frac{\nu_{0+}\nu_{0-}}{x^2} \right\} \psi = 0.$$

This equation has a regular singular point at $x = 0$ and an irregular singular point at $x = \infty$. We can verify that the equation (3.137) is equivalent to the Bessel equation via a certain change of variables and a gauge transform. Note also that the special case $\nu_{0+} = \nu_{0-} = 0$ of the equation has already constructed as a quantum curve in [FIMS, §4.1]; this example appears as the quantum differential equation for the equivariant $\mathbb{CP}^1$.

3.2.5.6 Quantum Whittaker curve ([IKoT2, §2.3.6])

In this subsection, we study the Whittaker curve defined by

$$(3.138) \quad P(x, y) := xy^2 - \frac{1}{4}x + \lambda_\infty = 0 \quad (\lambda_\infty \neq 0).$$

We parametrize this curve by

$$(3.139) \quad \begin{cases} x(z) &= -\frac{4\lambda_\infty}{z^2 - 1} = 2\lambda_\infty \left(\frac{1}{z+1} - \frac{1}{z-1} \right), \\ y(z) &= \frac{1}{2}z. \end{cases}$$

We find that $R = \{0, \infty\}$ ($= R^*$) and $\bar{z} = -z$. First few correlation functions and free energies are given as follows.

$$\begin{aligned} W_{1,1}(z) &= -\frac{(z^2 - 1)^2}{32\lambda_\infty z^4} dz, \quad W_{0,3}(z_1, z_2, z_3) = -\frac{dz_1 dz_2 dz_3}{4\lambda_\infty z_1^2 z_2^2 z_3^2}, \\ W_{1,2}(z_1, z_2) &= \frac{5z_1^4 + 3z_1^2 z_2^2 + 5z_2^4 - 12z_1^2 z_2^4 - 12z_1^4 z_2^2 + 10z_1^4 z_2^4 + z_1^6 z_2^6}{128\lambda_\infty^2 z_1^6 z_2^6} dz_1 dz_2, \\ W_{2,1}(z) &= \frac{-9z^{12} + 22z^{10} - 103z^8 + 372z^6 - 583z^4 + 406z^2 - 105}{8192\lambda_\infty^3 z^{10}} dz, \end{aligned}$$

$$F_0 = -\frac{3}{2}\lambda_\infty^2 + \lambda_\infty^2 \log(4\lambda_\infty), \quad F_1 = -\frac{1}{6} \log \lambda_\infty, \quad F_2 = -\frac{1}{120\lambda_\infty^2}.$$

To quantize the Whittaker curve, we use a divisor

$$(3.140) \quad D(z; \underline{\nu}) := [z] - \nu_{\infty+}[\beta_{\infty+}] - \nu_{\infty-}[\beta_{\infty-}] \quad (\nu_{\infty+} + \nu_{\infty-} = 1),$$

where $\beta_{\infty\pm} = \pm 1$. The quantum curve is given by (cf. Theorem 3.5)

$$(3.141) \quad \left\{ \hbar^2 \frac{d^2}{dx^2} + \hbar^2 \frac{1}{x} \frac{d}{dx} - \frac{x - 4\lambda_\infty}{4x} - \hbar \frac{\nu_{\infty+} - \nu_{\infty-}}{2x} \right\} \psi = 0.$$

3.2.5.7 Quantum Weber curve ([IKoT2, §2.3.7])

Computations presented in this subsection have been done in [Ta]. We discuss the Weber curve

$$(3.142) \quad P(x, y) := y^2 - \frac{x^2}{4} + \lambda_\infty = 0 \quad (\lambda_\infty \neq 0)$$

with a parametrization

$$(3.143) \quad \begin{cases} x = x(z) = \sqrt{\lambda_\infty} \left(z + \frac{1}{z} \right), \\ y = y(z) = \frac{\sqrt{\lambda_\infty}}{2} \left(z - \frac{1}{z} \right). \end{cases}$$

The ramification points are given by $R = \{+1, -1\}$ ($= R^*$), and the conjugation map becomes $\bar{z} = 1/z$. Correlation functions and free energies for lower g, n are given as follows.

$$W_{0,3}(z_1, z_2, z_3) = \frac{1}{2\lambda_\infty} \left\{ \frac{1}{(z_1+1)^2(z_2+1)^2(z_3+1)^2} - \frac{1}{(z_1-1)^2(z_2-1)^2(z_3-1)^2} \right\} dz_1 dz_2 dz_3,$$

$$W_{1,1}(z) = -\frac{z^3}{\lambda_\infty(z^2-1)^4} dz, \quad W_{2,1}(z) = -\frac{21(z^{11} + 3z^9 + z^7)}{\lambda_\infty^3(z^2-1)^{10}} dz,$$

$$F_0 = -\frac{3}{4}\lambda_\infty^2 + \frac{1}{2}\lambda_\infty^2 \log \lambda_\infty, \quad F_1 = -\frac{1}{12} \log \lambda_\infty, \quad F_2 = -\frac{1}{240\lambda_\infty^2}.$$

We set $\beta_{\infty+} = \infty$ and $\beta_{\infty-} = 0$, and choose

$$(3.144) \quad D(z; \underline{\nu}) := [z] - \nu_{\infty+}[\beta_{\infty+}] - \nu_{\infty-}[\beta_{\infty-}] \quad (\nu_{\infty+} + \nu_{\infty-} = 1)$$

as the divisor for the quantization. The quantum curve for (3.142) constructed by Theorem 3.5 becomes

$$(3.145) \quad \left\{ \hbar^2 \frac{d^2}{dx^2} - \left(\frac{x^2}{4} - \lambda_\infty \right) - \hbar \frac{\nu_{\infty+} - \nu_{\infty-}}{2} \right\} \psi = 0.$$

3.2.5.8 Quantum degenerate Bessel curve ([IKoT2, §2.3.8])

We discuss the degenerate Bessel curve defined by

$$(3.146) \quad P(x, y) := y^2 - \frac{1}{x} = 0$$

with a parametrization

$$(3.147) \quad \begin{cases} x = x(z) = z^2, \\ y = y(z) = \frac{1}{z}. \end{cases}$$

The ramification points are given by $R = \{0, \infty\}$, but ∞ is ineffective; i.e., $R^* = \{0\}$. The conjugation map is $\bar{z} = -z$. We choose $D(z; \underline{\nu}) := [z] - [\infty]$ as the divisor for the quantization. The quantum curve constructed by Theorem 3.5 becomes

$$(3.148) \quad \left\{ \hbar^2 \frac{d^2}{dx^2} + \hbar^2 \frac{1}{x} \frac{d}{dx} - \frac{1}{x} \right\} \psi = 0.$$

The quantum curve result for this spectral curve was established by [DN, Theorem 8], where the curve (3.146) is called the Bessel curve. Also, this example appears as the quantum differential equation for the (non-equivariant) $\mathbb{CP}^1$, as discussed in [FIMS].

3.2.5.9 Quantum Airy curve ([IKoT2, §2.3.9])

As the last example, let us consider the Airy curve defined by

$$(3.149) \quad P(x, y) := y^2 - x = 0$$

with a parametrization

$$(3.150) \quad \begin{cases} x = x(z) = z^2, \\ y = y(z) = z. \end{cases}$$

For this spectral curve, we have $R = \{0, \infty\}$ and $\bar{z} = -z$. We choose the divisor $D(z) = [z] - [\infty]$ and the associated quantum curve of the Airy curve is given in [Z] by

$$(3.151) \quad \left\{ \hbar^2 \frac{d^2}{dx^2} - x \right\} \psi = 0.$$

Remark 3.16. It is more convenient for the exact WKB analysis if the second order linear differential equation in question is represented in the so-called SL-form, i.e., the second order linear differential equation with no first order term. Any second order linear differential equation of the form (3.113) can be reduced to an SL-form

$$(3.152) \quad \left\{ \hbar^2 \frac{d^2}{dx^2} - Q(x, \hbar) \right\} \varphi = 0, \quad Q(x, \hbar) := \frac{1}{4} q(x, \hbar)^2 - r(x, \hbar) + \frac{\hbar}{2} \frac{\partial}{\partial x} q(x, \hbar)$$

by the gauge transformation $\varphi = \exp\left(\frac{1}{2\hbar} \int^x q(x, \hbar) dx\right) \psi$ of the unknown function. Table 3.1 shows the SL-form of quantum curves discussed in this subsection.

3.3 In the case of third order differential equations

3.3.1 Quantum curves

In this subsection we treat the quantization which was introduced by [BE2]. We consider the defining equation of the spectral curve

$$(3.153) \quad P(x, y) = p_0(x)y^3 + p_1(x)y^2 + p_2(x)y + p_3(x) = 0$$

and we introduce some notation.

Definition 3.17. Let us rewrite (3.153) as

$$(3.154) \quad P(x, y) = \sum_{i,j \in A} \alpha_{i,j} x^i y^j = 0.$$

Then the Newton polygon Δ of is the convex hull of the set A .

Definition 3.18. For $m = 2, 3$, we define the following meromorphic function on $\mathbb{P}^1$:

$$(3.155) \quad P_m(x, y) = \sum_{k=1}^{m-1} p_{m-1-k}(x) y^k = 0.$$

	$Q(x, \underline{\lambda}, \underline{\nu}; \hbar)$
Gauss (§3.2.5.1)	$\frac{\hat{\lambda}_\infty^2 x^2 - (\hat{\lambda}_\infty^2 + \hat{\lambda}_0^2 - \hat{\lambda}_1^2)x + \hat{\lambda}_0^2}{x^2(x-1)^2} - \hbar^2 \frac{x^2 - x + 1}{4x^2(x-1)^2}$
Degenerate Gauss (§3.2.5.2)	$\frac{\hat{\lambda}_\infty^2 x + \hat{\lambda}_1^2 - \hat{\lambda}_\infty^2}{x(x-1)^2} - \hbar^2 \frac{x^2 - x + 1}{4x^2(x-1)^2}$
Kummer (§3.2.5.3)	$\frac{x^2 + 4\hat{\lambda}_\infty x + 4\hat{\lambda}_0^2}{4x^2} - \frac{\hbar^2}{4x^2}$
Legendre (§3.2.5.4)	$\frac{\hat{\lambda}_\infty^2}{x^2 - 1} - \hbar^2 \frac{x^2 + 3}{4(x^2 - 1)^2}$
Bessel (§3.2.5.5)	$\frac{x + \hat{\lambda}_0^2}{4x^2} - \frac{\hbar^2}{4x^2}$
Whittaker (§3.2.5.6)	$\frac{x - 4\hat{\lambda}_\infty}{4x} - \frac{\hbar^2}{4x^2}$
Weber (§3.2.5.7)	$\frac{1}{4}x^2 - \hat{\lambda}_\infty$
Degenerate Bessel (§3.2.5.8)	$\frac{1}{x} - \frac{\hbar^2}{4x^2}$
Airy (§3.2.5.9)	x

Table 3.1: SL-forms $\{(\hbar d/dx)^2 - Q(x, \underline{\lambda}, \underline{\nu}; \hbar)\}\varphi = 0$ of quantum curves of spectral curves. In this table $\hat{\lambda}_j = \lambda_j - \hbar\nu_j/2$ and $\nu_j = \nu_{j+} - \nu_{j-}$ ($j \in \{0, 1, \infty\}$). For all cases, the leading term of the potential is given by $Q_{\text{cl}}(x)$.

Definition 3.19 ([BE2, Definition 2.7]). We say that a spectral curve is admissible if:

1. Its Newton polygon Δ has no interior point;
2. If the origin $(x, y) = (0, 0) \in \mathbb{C}^2$ is on the curve $\{P(x, y) = 0 \subset \mathbb{C}^2\}$, then the curve is smooth at this point.

We assume that our spectral curve $(x(z), y(z))$ is admissible. Then the following theorem holds according to [BE2].

Theorem 3.20 ([BE2, Lemma 5.14]). *Let β_i ($1 \leq i \leq n$) be simple poles of $x(z)$ and*

$$(3.156) \quad D(z; \underline{\nu}) = [z] - \sum_{i=1}^n \nu_i [\beta_i]$$

be a divisor on $\mathbb{P}^1$, where ν_i ($1 \leq i \leq n$) are complex numbers satisfying $\sum_{i=1}^n \nu_i = 1$. Let $W_{g,n}(z_1, \dots, z_n)$ be the correlation functions of a spectral curve $(x(z), y(z))$ defined from (3.153). Then, (3.1) with the integration divisor (3.156) is a WKB type formal solution of

$$(3.157) \quad \left[D_1 D_2 \frac{p_0(x)}{x^{[\alpha_3]}} D_3 + D_1 \frac{p_1(x)}{x^{[\alpha_2]}} D_2 + \frac{p_2(x)}{x^{[\alpha_1]}} D_1 + \frac{p_3(x)}{x^{[\alpha_0]}} - \hbar C_1 D_1 \frac{x^{[\alpha_2]}}{x^{[\alpha_1]}} - \hbar C_2 \frac{x^{[\alpha_1]}}{x^{[\alpha_0]}} \right] \psi = 0,$$

where

$$\begin{aligned} \alpha_m &= \inf\{a \mid (a, m) \in \Delta\} \quad (m = 0, 1, 2, 3), \\ D_i &= \hbar \frac{x^{[\alpha_i]}}{x^{[\alpha_{i-1}]}} \frac{d}{dx} \quad (i = 1, 2, 3), \\ C_k &= \sum_{i=1}^n \nu_i \left(\lim_{z \rightarrow \beta_i} \frac{P_{k+1}(x(z), y(z))}{x(z)^{[\alpha_{3-k}] + 1}} \right) \quad (k = 1, 2). \end{aligned}$$

3.3.2 Some examples

Here we show some quantum curves associated with the spectral curves arising from the confluent family of the hypergeometric equations with two variables. We can observe that these equations satisfied by classical special functions are recovered as the quantum curve. An application of the result will be presented in next section.

3.3.2.1 Quantum (1,1,3) curve Let us consider the (1,1,3) curve defined by

$$(3.158) \quad P(x, y) = 2t^2 y^3 + (2x - 3t^2) y^2 - (2x - t^2 - 2\lambda_{\infty,0} - 2\lambda_{\infty,1}) y - 2\lambda_{\infty,0} = 0,$$

where parameters $t, \lambda_{\infty,0}, \lambda_{\infty,1} \neq 0$. We parametrize (3.158) as

$$(3.159) \quad \begin{cases} x = x(z) = \frac{-2t^2 z^3 + 3t^2 z^2 - (t^2 + 2\lambda_{\infty,0} + 2\lambda_{\infty,1})z + 2\lambda_{\infty,0}}{2z(z-1)}, \\ y = y(z) = z \end{cases}$$

and regard it as a spectral curve.

We also introduce the notation $\beta_{\infty,1} := 0$, $\beta_{\infty,2} := 1$ and $\beta_{\infty,3} := \infty$. Then, we can verify that $\beta_{\infty,j}$ are three preimages of ∞ by the map $x(z)$; that is, $x(\beta_{\infty,j}) = \infty$ holds. Note also that the parameters are chosen so that

$$(3.160) \quad \lambda_{\infty, \beta_{\infty,j}} = \operatorname{Res}_{z=\beta_{\infty,j}} y(z) dx(z)$$

holds for each $j \in \{1, 2\}$. (We will use similar notations in the subsequent subsections so that the relation (3.160) holds.)

First few terms of the correlation functions and free energies are computed as

$$\begin{aligned}
W_{0,3}(z_1, z_2, z_3) = 2 \Bigg[& \left\{ z_1(z_1 - 1) \left\{ t^2 z_1^6 - 3t^2 z_1^5 + 3t^2 z_1^4 - ((\lambda_{\infty,0} + \lambda_{\infty,1})z_2 + t^2 - \lambda_{\infty,1})z_1^3 \right. \right. \\
& + 3\lambda_{\infty,0}z_2z_1^2 - 3\lambda_{\infty,0}z_1z_2 + \lambda_{\infty,0}z_2 \left. \right\} z_3 - z_1^2 \{ 2t^2 z_1^5 - t^2(z_2 + 6)z_1^4 \\
& + (3t^2 z_2 + 6t^2 - \lambda_{\infty,0} - \lambda_{\infty,1})z_1^3 - (3t^2 z_2 - 2t^2 + 3\lambda_{\infty,0} + 2\lambda_{\infty,1})z_1^2 \\
& + ((t^2 - \lambda_{\infty,1})z_2 - 3\lambda_{\infty,0})z_1 + \lambda_{\infty,0} \left. \right\} / \{ (z_3 - z_1)^3 (z_3 - z_2)^3 \\
& \times (t^2 z_1^4 - 2t^2 z_1^3 + (t^2 - \lambda_{\infty,0} - \lambda_{\infty,1})z_1^2 + 2\lambda_{\infty,0}z_1 - \lambda_{\infty,0})^2 \} \\
& + \left\{ z_2(z_2 - 1) \left\{ t^2 z_2^6 - 3t^2 z_2^5 + 3t^2 z_2^4 - ((\lambda_{\infty,0} + \lambda_{\infty,1})z_3 + t^2 - \lambda_{\infty,1})z_2^3 \right. \right. \\
& + 3\lambda_{\infty,0}z_3z_2^2 - 3\lambda_{\infty,0}z_2z_3 + \lambda_{\infty,0}z_3 \left. \right\} z_1 - z_2^2 \{ 2t^2 z_2^5 - t^2(z_3 + 6)z_2^4 \\
& + (3t^2 z_3 + 6t^2 - \lambda_{\infty,0} - \lambda_{\infty,1})z_2^3 - (3t^2 z_3 - 2t^2 + 3\lambda_{\infty,0} + 2\lambda_{\infty,1})z_2^2 \\
& + ((t^2 - \lambda_{\infty,1})z_3 - 3\lambda_{\infty,0})z_2 + \lambda_{\infty,0} \left. \right\} / \{ (z_1 - z_2)^3 (z_1 - z_3)^3 \\
& \times (t^2 z_2^4 - 2t^2 z_2^3 + (t^2 - \lambda_{\infty,0} - \lambda_{\infty,1})z_2^2 + 2\lambda_{\infty,0}z_2 - \lambda_{\infty,0})^2 \} \\
& + \left\{ z_3(z_3 - 1) \left\{ t^2 z_3^6 - 3t^2 z_3^5 + 3t^2 z_3^4 - ((\lambda_{\infty,0} + \lambda_{\infty,1})z_1 + t^2 - \lambda_{\infty,1})z_3^3 \right. \right. \\
& + 3\lambda_{\infty,0}z_1z_3^2 - 3\lambda_{\infty,0}z_1z_3 + \lambda_{\infty,0}z_1 \left. \right\} z_2 - z_3^2 \{ 2t^2 z_3^5 - t^2(z_1 + 6)z_3^4 \\
& + (3t^2 z_1 + 6t^2 - \lambda_{\infty,0} - \lambda_{\infty,1})z_3^3 - (3t^2 z_1 - 2t^2 + 3\lambda_{\infty,0} + 2\lambda_{\infty,1})z_3^2 \\
& + ((t^2 - \lambda_{\infty,1})z_1 - 3\lambda_{\infty,0})z_3 + \lambda_{\infty,0} \left. \right\} / \{ (z_2 - z_3)^3 (z_2 - z_1)^3 \\
& \times (t^2 z_3^4 - 2t^2 z_3^3 + (t^2 - \lambda_{\infty,0} - \lambda_{\infty,1})z_3^2 + 2\lambda_{\infty,0}z_3 - \lambda_{\infty,0})^2 \} \Bigg] dz_1 dz_2 dz_3, \\
W_{1,1}(z) = & \left\{ z^2(z - 1)^2 \{ (\lambda_{\infty,0} + \lambda_{\infty,1})t^4 z^7 - (6\lambda_{\infty,0} + \lambda_{\infty,1})t^4 z^6 + 15\lambda_{\infty,0}t^4 z^5 \right. \\
& - 20\lambda_{\infty,0}t^4 z^4 + \lambda_{\infty,0}(15t^2 - 4\lambda_{\infty,1})t^2 z^3 - 6\lambda_{\infty,0}(t^2 - \lambda_{\infty,1})t^2 z^2 \\
& + \lambda_{\infty,0}(t^4 - 2\lambda_{\infty,1}t^2 + \lambda_{\infty,1}(\lambda_{\infty,0} + \lambda_{\infty,1}))z - \lambda_{\infty,0}^2 \lambda_{\infty,1} \left. \right\} dz \\
& / \left\{ t^2 z^4 - 2t^2 z^3 + (t^2 - \lambda_{\infty,0} - \lambda_{\infty,1})z^2 + 2\lambda_{\infty,0}z - \lambda_{\infty,0} \right\},
\end{aligned}$$

$$\begin{aligned}
F_0 = & -\frac{1}{8} \left\{ 6\lambda_{\infty,0}^2 + 4\lambda_{\infty,0}\lambda_{\infty,1} + 6\lambda_{\infty,1}^2 - (\lambda_{\infty,0} + \lambda_{\infty,1})t^2 - 4\lambda_{\infty,0}\lambda_{\infty,1} \log(-t^4) \right. \\
& \left. - 4\lambda_{\infty,0}^2 \log(\lambda_{\infty,0}t^2) - 4\lambda_{\infty,1}^2 \log(\lambda_{\infty,1}t^2) \right\}, \\
F_1 = & -\frac{1}{12} \log(\lambda_{\infty,0}\lambda_{\infty,1}).
\end{aligned}$$

We choose

$$\begin{aligned}
(3.161) \quad D(z; \nu) = & [z] - \nu_{\infty,0}[0] - \nu_{\infty,1}[1] - (1 - \nu_{\infty,0} - \nu_{\infty,1})[\infty] \\
= & \nu_{\infty,0}([z] - [0]) + \nu_{\infty,1}([z] - [1]) + (1 - \nu_{\infty,0} - \nu_{\infty,1})([z] - [\infty])
\end{aligned}$$

as the divisor for the quantization. Then, Theorem 3.20 gives the quantum curve of the (1,1,3) curve (quantum (1,1,3) curve):

$$(3.162) \quad \left[2t^2 \hbar^3 \frac{d^3}{dx^3} + (2x - 3t^2) \hbar^2 \frac{d^2}{dx^2} - (2x - t^2 - 2\hat{\lambda}_{\infty,0} - 2\hat{\lambda}_{\infty,1} - 4\hbar) \hbar \frac{d}{dx} - 2\hat{\lambda}_{\infty,0} - 2\hbar \right] \psi = 0.$$

Here we used the notation

$$(3.163) \quad \hat{\lambda}_{\infty,j} = \lambda_{\infty,j} - \nu_{\infty,j} \hbar \quad (j \in \{0, 1\}).$$

Remark 3.21. The logarithmic derivative of the WKB solution $S(x, \hbar) = \sum_{m=-1}^{\infty} \hbar^m S_m(x)$ satisfies the higher order analogue of Riccati equation (2.18). Then, we must have

$$(3.164) \quad S_1(x(z))dx(z) = \frac{1}{3!} \frac{d}{dz} \int_{\zeta_1 \in D(z; \mathcal{L})} \cdots \int_{\zeta_3 \in D(z; \mathcal{L})} W_{0,3}(\zeta_1, \zeta_2, \zeta_3) + W_{1,1}(z),$$

which can be verified by the explicit formulas of $W_{0,3}$ and $W_{1,1}$ in the previous page.

3.3.2.2 Quantum (2,3) curve Let us consider the (2,3) curve defined by

$$(3.165) \quad P(x, y) = 4y^3 - 2xy^2 + 2\lambda_{\infty}y - t = 0,$$

where parameters $t, \lambda_{\infty} (= \lambda_{\infty,0}) \neq 0$. A rational parameterization of this curve is

$$(3.166) \quad \begin{cases} x = x(z) = \frac{4z^3 + 2\lambda_{\infty}z - t}{2z^2} = 2z + \frac{\lambda_{\infty}}{z} - \frac{t}{2z^2}, \\ y = y(z) = z. \end{cases}$$

We also introduce the notation $\beta_{\infty,1} (= \beta_{\infty,2}) := 0$ and $\beta_{\infty,3} := \infty$. Then, we can verify that $\beta_{\infty,j}$ are three preimages of ∞ by the map $x(z)$; that is, $x(\beta_{\infty,j}) = \infty$ holds. Note also that the parameter is chosen so that

$$(3.167) \quad \lambda_{\infty, \beta_{\infty,1}} = \operatorname{Res}_{z=\beta_{\infty,1}} y(z)dx(z).$$

First few terms of the correlation functions and free energies are computed as

$$\begin{aligned} W_{0,3}(z_1, z_2, z_3) = & \left\{ -\frac{z_1^2(8z_1^5 - 4(z_2 + z_3)z_1^4 - 2\lambda_{\infty}z_1^3 + tz_1^2 + (2\lambda_{\infty}z_2z_3 + tz_2 + tz_3)z_1 - 3tz_2z_3)}{(z_1 - z_2)^3(z_1 - z_3)^3(2z_1^3 - \lambda_{\infty}z_1 + t)^2} \right. \\ & - \frac{z_2^2(8z_2^5 - 4(z_3 + z_1)z_2^4 - 2\lambda_{\infty}z_2^3 + tz_2^2 + (2\lambda_{\infty}z_3z_1 + tz_3 + tz_1)z_2 - 3tz_3z_1)}{(z_2 - z_3)^3(z_2 - z_1)^3(2z_2^3 - \lambda_{\infty}z_2 + t)^2} \\ & \left. - \frac{z_3^2(8z_3^5 - 4(z_1 + z_2)z_3^4 - 2\lambda_{\infty}z_3^3 + tz_3^2 + (2\lambda_{\infty}z_1z_2 + tz_1 + tz_2)z_3 - 3tz_1z_2)}{(z_3 - z_1)^3(z_3 - z_2)^3(2z_3^3 - \lambda_{\infty}z_3 + t)^2} \right\} \\ & \times dz_1 dz_2 dz_3, \\ W_{1,1}(z) = & -\frac{(4z^3 - t)(8\lambda_{\infty}z^4 - 20tz^3 + 2\lambda_{\infty}tz - t^2)}{8(2z^3 - \lambda_{\infty}z + t)^4} dz, \end{aligned}$$

$$F_0 = -\frac{\lambda_{\infty}^2}{4} \log(-2t), \quad F_1 = -\frac{1}{8} \log t.$$

Remark 3.22. It seems $W_{0,3}$ has singularities at $z_1 = z_2 = z_3$, but we can verify that $W_{0,3}$ is holomorphic there.

We choose

$$(3.168) \quad \begin{aligned} D(z; \nu) &= [z] - (1 - \nu_\infty)[0] - \nu_\infty[\infty] \\ &= (1 - \nu_\infty)([z] - [0]) + \nu_\infty([z] - [\infty]) \end{aligned}$$

as the divisor for the quantization, where $\nu_\infty (= \nu_{\infty,0})$ is a parameter.

Remark 3.23. $z = 0$ is a double pole of $x(z)$, i.e., $0 \in R$, but we can verify that $0 \notin R^*$. Therefore, we can choose $\beta = 0$ as a base point.

Then, Theorem 3.20 gives the quantum curve of the (2,3) curve (quantum (2,3) curve):

$$(3.169) \quad \left\{ 4\hbar^3 \frac{d^3}{dx^3} - 2x\hbar^2 \frac{d^2}{dx^2} + 2(\hat{\lambda}_\infty - \hbar)\hbar \frac{d}{dx} - t \right\} \psi = 0,$$

where

$$(3.170) \quad \hat{\lambda}_\infty = \lambda_\infty - \nu_\infty \hbar.$$

3.3.2.3 Quantum (1,4) curve Let us consider the (1,4) curve defined by

$$(3.171) \quad P(x, y) = 3y^3 + 2ty^2 + xy - \lambda_\infty = 0,$$

where parameters $t, \lambda_\infty (= \lambda_{\infty,0}) \neq 0$. A rational parameterization of this curve is

$$(3.172) \quad \begin{cases} x = x(z) = \frac{-3z^3 - 2tz^2 + \lambda_\infty}{z} = -3z^2 - 2tz + \frac{\lambda_\infty}{z}, \\ y = y(z) = z. \end{cases}$$

We also introduce the notation $\beta_{\infty,1} := 0$ and $\beta_{\infty,2} (= \beta_{\infty,3}) := \infty$. Then, we can verify that $\beta_{\infty,j}$ are three preimages of ∞ by the map $x(z)$; that is, $x(\beta_{\infty,j}) = \infty$ holds. Note also that the parameter is chosen so that

$$(3.173) \quad \lambda_{\infty, \beta_{\infty,1}} = \text{Res}_{z=\beta_{\infty,1}} y(z)dx(z).$$

First few terms of the correlation functions and free energies are computed as

$$\begin{aligned} W_{0,3}(z_1, z_2, z_3) &= \left\{ \frac{2z_1(15z_1^5 - (9z_2 + 9z_3 - 4t)z_1^4 - (2tz_2 + 2tz_3 - 3z_2z_3)z_1^3 + \lambda_\infty z_1^2 - \lambda_\infty z_2z_3)}{(z_1 - z_2)^3(z_1 - z_3)^3(6z_1^3 + 2tz_1^2 + \lambda_\infty)^2} \right. \\ &\quad + \frac{2z_2(15z_2^5 - (9z_3 + 9z_1 - 4t)z_2^4 - (2tz_3 + 2tz_1 - 3z_3z_1)z_2^3 + \lambda_\infty z_2^2 - \lambda_\infty z_3z_1)}{(z_2 - z_3)^3(z_2 - z_1)^3(6z_2^3 + 2tz_2^2 + \lambda_\infty)^2} \\ &\quad \left. + \frac{2z_3(15z_3^5 - (9z_1 + 9z_2 - 4t)z_3^4 - (2tz_1 + 2tz_2 - 3z_1z_2)z_3^3 + \lambda_\infty z_3^2 - \lambda_\infty z_1z_2)}{(z_3 - z_1)^3(z_3 - z_2)^3(6z_3^3 + 2tz_3^2 + \lambda_\infty)^2} \right\} \\ &\quad \times dz_1 dz_2 dz_3, \\ W_{1,1}(z) &= \frac{z^2(27z^6 - 99z^3 - 36\lambda_\infty tz^2 - 4\lambda_\infty t^2z + 3\lambda_\infty^2)}{(6z^3 + 2tz^2 + \lambda_\infty)^4} dz, \end{aligned}$$

$$F_0 = -\frac{t^6}{972} + \frac{2\lambda_\infty t^3}{27} - \frac{3\lambda_\infty^2}{4} + \frac{\lambda_\infty^2}{4} \log(-3\lambda_\infty^2), \quad F_1 = -\frac{1}{12} \log \lambda_\infty.$$

Remark 3.24. It seems $W_{0,3}$ has singularities at $z_1 = z_2 = z_3$, but we can verify that $W_{0,3}$ is holomorphic there.

We choose

$$(3.174) \quad \begin{aligned} D(z; \nu) &= [z] - (1 - \nu_\infty)[0] - \nu_\infty[\infty] \\ &= (1 - \nu_\infty)([z] - [0]) + \nu_\infty([z] - [\infty]) \end{aligned}$$

as the divisor for the quantization, where $\nu_\infty (= \nu_{\infty,0})$ is a parameter.

Remark 3.25. $z = \infty$ is a double pole of $x(z)$, i.e., $\infty \in R$, but we can verify that $\infty \notin R^*$. Therefore, we can choose $\beta = \infty$ as a base point.

Then, Theorem 3.20 gives the quantum curve of the (1,4) curve (quantum (1,4) curve):

$$(3.175) \quad \left\{ 3\hbar^3 \frac{d^3}{dx^3} + 2t\hbar^2 \frac{d^2}{dx^2} + x\hbar \frac{d}{dx} - \hat{\lambda}_\infty \right\} \psi = 0.$$

Here we used the notation

$$(3.176) \quad \hat{\lambda}_\infty = \lambda_\infty - \nu_\infty \hbar.$$

Note that the special case $t = 0$ of the equation has already constructed as a quantum curve in [BE2, §6.2.2].

4 Free energies and Voros coefficients for hypergeometric quantum curves

4.1 In the case of second order differential equations

4.1.1 Main Theorem

In this subsection we formulate our main results for the quantum curves discussed in §3.2.5. Our theorem allows us to express the Voros coefficients of the quantum curves by the free energy with a certain parameter shift (cf. Theorem 4.1 (i)). As a corollary, we can find three term relations satisfied by the free energy (cf. Theorem 4.1 (ii)), together with explicit formulas for the coefficients of the free energy and the Voros coefficients (as well as Theorem 4.1 (iii) and (iv)). We will only give the proof of our results for the (quantum) Gauss curve considered in §3.2.5.1 since the other examples can be treated in a similar manner.

Here we formulate our main results more precisely.

Let (C) be one of the spectral curves constructed in §3.2.5; that is, Gauss (§3.2.5.1), degenerate Gauss (§3.2.5.2), Kummer (§3.2.5.3), Legendre (§3.2.5.4), Bessel (§3.2.5.5), Whittaker (§3.2.5.6), or Weber (§3.2.5.7) curves. Denote by (E) the associated quantum curve by Theorem 3.5. We will not consider the degenerate Bessel (§3.2.5.8) and Airy (§3.2.5.9) in this section because they have only trivial Voros coefficients (i.e., no non-trivial relative homology class on Σ), and accordingly, the free energy F_g vanishes for $g \geq 2$. Recall that the spectral curve (C) is (a parametrization of) the classical limit Σ of the quantum curve (E) .

(E)	Gauss	Degenerate Gauss	Kummer	Legendre	Bessel	Whittaker	Weber
S	$\{0, 1, \infty\}$	$\{1, \infty\}$	$\{0, \infty\}$	$\{\infty\}$	$\{0\}$	$\{\infty\}$	$\{\infty\}$

Table 4.1: The list of singular points (which are not branch points of the spectral curve) of quantum curves. Note that, some of regular singular point (e.g., $x = 0$ for the quantum degenerate Gauss curve) are eliminated from the above list. This is because those points are “turnings point of simple-pole type” in the sense of [Ko2], which play similar roles as turning points in the WKB analysis. Hence, we will not associate the Voros coefficient for those points.

Let S be the set of singular points of (E) , which are not branch points of the spectral curve (C) . The set S is tabulated in Table 4.1. For each $j \in S$, there exist two preimages β_{j+} and β_{j-} of j by the map $x(z)$ which appeared in the definition of the corresponding spectral curves (note that this notation is consistent with those given in §3.2.5.1 – §3.2.5.9). Let γ_j be a path on the Riemann surface Σ defined as the image by $x(z)$ of a path from β_{j-} to β_{j+} on z -plane. (On x -plane, γ_j starts from the point j and come back to j but with a different sheet of the Riemann surface Σ .) Then, we define the Voros coefficient $V^{(j)}$ of the quantum curve (E) for the path γ_j by

$$(4.1) \quad V^{(j)}(\underline{\lambda}, \underline{\nu}; \hbar) = \int_{\gamma_j} \left(S(x, \hbar) - \hbar^{-1} S_{-1}(x) - S_0(x) \right) dx \quad (j \in S).$$

Here $S(x, \hbar)$ is the logarithmic derivative of the WKB solution of the quantum curve (E) . Actually, $V^{(j)}$ depends only on the end and initial points of γ_j since $S_m(x)dx$ for $m \geq 1$ (or the correlation functions $W_{g,n}$ for $2g - 2 + n \geq 1$) has no residue at branch points. We also note that, since the SL-form of the quantum curves in Table 3.1 depends only on the difference ν_j of the parameter $\nu_{j\pm}$, so do the Voros coefficients. Thus we have used the symbol $\underline{\lambda} = (\lambda_j)_{j \in S}$, $\underline{\nu} = (\nu_j)_{j \in S}$ for the tuple of parameters contained in the quantum curve (E) in the expression (4.1).

On the other hand, the topological recursion associates the free energy

$$(4.2) \quad F(\underline{\lambda}; \hbar) = \sum_{g=0}^{\infty} \hbar^{2g-2} F_g(\underline{\lambda})$$

for the spectral curve (C) as we have reviewed in §2.4. Our main result gives a formula which relates the free energy of (C) and the Voros coefficients of (E) . We derive three-terms relations satisfied by the free energy, and moreover, obtain explicit expressions of the free energies and Voros coefficients as a corollary of these relations. The precise statement is formulated as follows.

Theorem 4.1 ([IKoT2, Theorem 3.1]). (i) *The Voros coefficient of the quantum curve (E) and the free energy of the spectral curve (C) are related as follows.*

$$(4.3) \quad V^{(j)}(\underline{\lambda}, \underline{\nu}; \hbar) = F\left(\hat{\underline{\lambda}} + \frac{\hbar}{2}\delta_j; \hbar\right) - F\left(\hat{\underline{\lambda}} - \frac{\hbar}{2}\delta_j; \hbar\right) - \frac{1}{\hbar} \frac{\partial F_0}{\partial \lambda_j} + \frac{1}{2} \sum_{k \in S} \nu_k \frac{\partial^2 F_0}{\partial \lambda_j \partial \lambda_k} \quad (j \in S).$$

Here the symbol $\hat{\lambda} + \alpha\delta_j$ means the parameter shift of $\hat{\lambda}_j$ by α :

$$(4.4) \quad \hat{\lambda} + \alpha\delta_j := (\hat{\lambda}_k + \delta_{kj}\alpha)_{k \in S}.$$

(Recall $\hat{\lambda}_j = \lambda_j - \hbar\nu_j/2$ with $\nu_j = \nu_{j+} - \nu_{j-}$ ($j = 0, 1, \infty$) as we have introduced in (3.117).)

(ii) The free energy for the spectral curve (C) satisfies the following three-term difference equation.

$$(4.5) \quad F(\hat{\lambda} + \hbar\delta_j; \hbar) - 2F(\hat{\lambda}; \hbar) + F(\hat{\lambda} - \hbar\delta_j; \hbar) = \log \Lambda(\hat{\lambda}) + R_j(\hat{\lambda}; \hbar) \quad (j \in S).$$

Here $\Lambda(\hat{\lambda})$ and $R_j(\hat{\lambda}; \hbar)$ in the right hand-side of (4.5) are given as follows:

- For Gauss curve (§3.2.5.1):

$$\Lambda(\hat{\lambda}) = (\lambda_0 + \lambda_1 + \lambda_\infty)(\lambda_0 + \lambda_1 - \lambda_\infty)(\lambda_0 - \lambda_1 + \lambda_\infty)(\lambda_0 - \lambda_1 - \lambda_\infty),$$

$$R_j(\hat{\lambda}; \hbar) = -2\log(2\lambda_j) - \log(2\lambda_j + \hbar) - \log(2\lambda_j - \hbar) \quad (j \in \{0, 1, \infty\}).$$

- For degenerate Gauss curve (§3.2.5.2):

$$\Lambda(\hat{\lambda}) = (\lambda_1 + \lambda_\infty)^2(\lambda_1 - \lambda_\infty)^2,$$

$$R_j(\hat{\lambda}; \hbar) = -2\log(2\lambda_j) - \log(2\lambda_j + \hbar) - \log(2\lambda_j - \hbar) \quad (j \in \{1, \infty\}).$$

- For Kummer curve (§3.2.5.3):

$$\Lambda(\hat{\lambda}) = (\lambda_\infty + \lambda_0)(\lambda_\infty - \lambda_0),$$

$$R_j(\hat{\lambda}; \hbar) = \begin{cases} -2\log(2\lambda_0) - \log(2\lambda_0 + \hbar) - \log(2\lambda_0 - \hbar) & (j = 0), \\ 0 & (j = \infty). \end{cases}$$

- For Legendre curve (§3.2.5.4):

$$\Lambda(\lambda_\infty) = 4\lambda_\infty^4, \quad R_\infty(\lambda_\infty; \hbar) = -2\log(2\lambda_\infty) - \log(2\lambda_\infty + \hbar) - \log(2\lambda_\infty - \hbar).$$

- For Bessel curve (§3.2.5.5):

$$\Lambda(\lambda_0) = \frac{1}{16}, \quad R_0(\lambda_0; \hbar) = -2\log(2\lambda_0) - \log(2\lambda_0 + \hbar) - \log(2\lambda_0 - \hbar).$$

- For Whittaker curve (§3.2.5.6):

$$\Lambda(\lambda_\infty) = \lambda_\infty^2, \quad R_\infty(\lambda_\infty; \hbar) = 0.$$

- For Weber curve (§3.2.5.7):

$$\Lambda(\lambda_\infty) = \lambda_\infty, \quad R_\infty(\lambda_\infty; \hbar) = 0.$$

(iii) For $g \geq 2$, the g -th free energy of the spectral curve (C) has the following expression.

- For Gauss curve (§3.2.5.1):

$$F_g(\underline{\lambda}) = \frac{B_{2g}}{2g(2g-2)} \left\{ \frac{1}{(\lambda_0 + \lambda_1 + \lambda_\infty)^{2g-2}} + \frac{1}{(\lambda_0 - \lambda_1 + \lambda_\infty)^{2g-2}} + \frac{1}{(\lambda_0 + \lambda_1 - \lambda_\infty)^{2g-2}} \right. \\ \left. + \frac{1}{(\lambda_0 - \lambda_1 - \lambda_\infty)^{2g-2}} - \frac{1}{(2\lambda_0)^{2g-2}} - \frac{1}{(2\lambda_1)^{2g-2}} - \frac{1}{(2\lambda_\infty)^{2g-2}} \right\}.$$

(F_0 and F_1 for Gauss curve are given in §3.2.5.1.)

- For degenerate Gauss curve (§3.2.5.2):

$$F_g(\underline{\lambda}) = \frac{B_{2g}}{2g(2g-2)} \left\{ \frac{2}{(\lambda_1 + \lambda_\infty)^{2g-2}} + \frac{2}{(\lambda_1 - \lambda_\infty)^{2g-2}} - \frac{1}{(2\lambda_1)^{2g-2}} - \frac{1}{(2\lambda_\infty)^{2g-2}} \right\}.$$

(F_0 and F_1 for degenerate Gauss curve are given in §3.2.5.2.)

- For Kummer curve (§3.2.5.3):

$$F_g(\underline{\lambda}) = \frac{B_{2g}}{2g(2g-2)} \left\{ \frac{1}{(\lambda_0 + \lambda_\infty)^{2g-2}} + \frac{1}{(\lambda_0 - \lambda_\infty)^{2g-2}} - \frac{1}{(2\lambda_0)^{2g-2}} \right\}.$$

(F_0 and F_1 for Kummer curve are given in §3.2.5.3.)

- For Legendre curve (§3.2.5.4):

$$F_g(\lambda_\infty) = \frac{B_{2g}}{2g(2g-2)} \left\{ \frac{4}{\lambda_\infty^{2g-2}} - \frac{1}{(2\lambda_\infty)^{2g-2}} \right\}.$$

(F_0 and F_1 for Legendre curve are given in §3.2.5.4.)

- For Bessel curve (§3.2.5.5):

$$F_g(\lambda_0) = -\frac{B_{2g}}{2g(2g-2)} \frac{1}{(2\lambda_0)^{2g-2}}.$$

(F_0 and F_1 for Bessel curve are given in §3.2.5.5. The expression was obtained in [IM].)

- For Whittaker curve (§3.2.5.6):

$$F_g(\lambda_\infty) = \frac{B_{2g}}{2g(2g-2)} \frac{2}{\lambda_\infty^{2g-2}}.$$

(F_0 and F_1 for Whittaker curve are given in §3.2.5.6.)

- For Weber curve (§3.2.5.7):

$$F_g(\lambda_\infty) = \frac{B_{2g}}{2g(2g-2)} \frac{1}{\lambda_\infty^{2g-2}}.$$

(F_0 and F_1 for Weber curve are given in §3.2.5.7. The expression was obtained in [HZ]; see also [IM].)

In these expressions, B_m is the m -th Bernoulli number given by (D.1) in §D.

(iv) The Voros coefficients (4.1) for the quantum curve (E) are explicitly given by

$$(4.6) \quad V^{(j)}(\underline{\lambda}, \underline{\nu}; \hbar) = \sum_{m=1}^{\infty} \hbar^m V_m^{(j)}(\underline{\lambda}, \underline{\nu}) \quad (j \in S),$$

where the coefficients $V_m^{(j)}(\underline{\lambda}, \underline{\nu})$ for $m \geq 1$ are given as follows.

- For quantum Gauss curve (§3.2.5.1):

$$\begin{aligned} V_m^{(0)}(\underline{\lambda}, \underline{\nu}) &= V_m^{(0)}(\lambda_0, \lambda_1, \lambda_\infty, \nu_0, \nu_1, \nu_\infty) \\ &= \frac{1}{m(m+1)} \left\{ \frac{B_{m+1}((\nu_0 + \nu_1 + \nu_\infty + 1)/2)}{(\lambda_0 + \lambda_1 + \lambda_\infty)^m} + \frac{B_{m+1}((\nu_0 - \nu_1 + \nu_\infty + 1)/2)}{(\lambda_0 - \lambda_1 + \lambda_\infty)^m} \right. \\ &\quad \left. + \frac{B_{m+1}((\nu_0 + \nu_1 - \nu_\infty + 1)/2)}{(\lambda_0 + \lambda_1 - \lambda_\infty)^m} + \frac{B_{m+1}((\nu_0 - \nu_1 - \nu_\infty + 1)/2)}{(\lambda_0 - \lambda_1 - \lambda_\infty)^m} - \frac{B_{m+1}(\nu_0) + B_{m+1}(\nu_0 + 1)}{(2\lambda_0)^m} \right\} \end{aligned}$$

gives the formula for $j = 0$. The other Voros coefficients (for $j = 1$ and ∞) are obtained from the above formula by permuting the parameters $\underline{\lambda}$ and $\underline{\nu}$:

$$V_m^{(1)}(\underline{\lambda}, \underline{\nu}) = V_m^{(0)}(\lambda_1, \lambda_0, \lambda_\infty, \nu_1, \nu_0, \nu_\infty), \quad V_m^{(\infty)}(\underline{\lambda}, \underline{\nu}) = V_m^{(0)}(\lambda_\infty, \lambda_1, \lambda_0, \nu_\infty, \nu_1, \nu_0).$$

(These expressions were also obtained in [ATT2].)

- For quantum degenerate Gauss curve (§3.2.5.2):

$$\begin{aligned} V_m^{(1)}(\underline{\lambda}, \underline{\nu}) &= V_m^{(1)}(\lambda_1, \lambda_\infty, \nu_1, \nu_\infty) \\ &= \frac{1}{m(m+1)} \left\{ \frac{2B_{m+1}((\nu_1 + \nu_\infty + 1)/2)}{(\lambda_1 + \lambda_\infty)^m} + \frac{2B_{m+1}((\nu_1 - \nu_\infty + 1)/2)}{(\lambda_1 - \lambda_\infty)^m} \right. \\ &\quad \left. - \frac{B_{m+1}(\nu_1) + B_{m+1}(\nu_1 + 1)}{(2\lambda_1)^m} \right\} \end{aligned}$$

gives the formula for $j = 1$. The other Voros coefficient for ∞ is obtained from the above formula by permuting the parameters:

$$V_m^{(\infty)}(\lambda_1, \lambda_\infty, \nu_1, \nu_\infty) = V_m^{(1)}(\lambda_\infty, \lambda_1, \nu_\infty, \nu_1).$$

- For quantum Kummer curve (§3.2.5.3):

$$V_m^{(0)}(\underline{\lambda}, \underline{\nu}) = \frac{1}{m(m+1)} \left\{ \frac{B_{m+1}((\nu_0 + \nu_\infty + 1)/2)}{(\lambda_0 + \lambda_\infty)^m} + \frac{B_{m+1}((\nu_0 - \nu_\infty + 1)/2)}{(\lambda_0 - \lambda_\infty)^m} - \frac{B_{m+1}(\nu_0) + B_{m+1}(\nu_0 + 1)}{(2\lambda_0)^m} \right\},$$

$$V_m^{(\infty)}(\underline{\lambda}, \underline{\nu}) = \frac{1}{m(m+1)} \left\{ \frac{B_{m+1}((\nu_0 + \nu_\infty + 1)/2)}{(\lambda_0 + \lambda_\infty)^m} - \frac{B_{m+1}((\nu_0 - \nu_\infty + 1)/2)}{(\lambda_0 - \lambda_\infty)^m} \right\}.$$

(These expressions were also obtained in [ATT1].)

- For quantum Legendre curve (§3.2.5.4):

$$V_m^{(\infty)}(\lambda_\infty, \nu_\infty) = \frac{1}{m(m+1)} \left\{ \frac{4B_{m+1}((\nu_\infty + 1)/2)}{\lambda_\infty^m} - \frac{B_{m+1}(\nu_\infty) + B_{m+1}(\nu_\infty + 1)}{(2\lambda_\infty)^m} \right\}.$$

(This expression was also obtained in [Ko4].)

- For quantum Bessel curve (§3.2.5.5):

$$V_m^{(0)}(\lambda_0, \nu_0) = -\frac{B_{m+1}(\nu_0) + B_{m+1}(\nu_0 + 1)}{m(m+1)(2\lambda_0)^m}.$$

(The special case $\nu_0 = 0$ of the expression was obtained in [AIT].)

- For quantum Whittaker curve (§3.2.5.6):

$$V_m^{(\infty)}(\lambda_\infty, \nu_\infty) = \frac{2B_{m+1}((\nu_\infty + 1)/2)}{m(m+1)\lambda_\infty^m}.$$

(This expression was obtained in [KoT].)

- For quantum Weber curve (§3.2.5.7):

$$V_m^{(\infty)}(\lambda_\infty, \nu_\infty) = \frac{B_{m+1}((\nu_\infty + 1)/2)}{m(m+1)\lambda_\infty^m}.$$

(This expression was obtained in [Ta]; see [T1] for the special case $\nu_\infty = 0$.)

In these expressions, $B_m(t)$ is the m -th Bernoulli polynomial given by (D.3) in §D.

We will prove these statements in the rest of this section.

4.1.2 Proof of main theorem for quantum Gauss curve

Here we give a proof of Theorem 4.1 for the case of (quantum) Gauss curve. Therefore, in the following proof, we focus on the Gauss curve (3.104) and the associated quantum Gauss curve (3.113) and use the notations in §3.2.5.1; namely, the function $x(z)$ is given in (3.104), and the divisor $D(z; \underline{\nu})$ is given in (3.111), and the set S of singular points becomes $\{0, 1, \infty\}$, etc. (We omit the proof for other examples considered in §3.2.5 because they can be treated in a similar manner.)

4.1.2.1 Proof of Theorem 4.1 (i) For the proof, we need

Lemma 4.2 ([IKoT2, Lemma 3.2]). *The following relations hold for $2g - 2 + n \geq 1$:*

$$(4.7) \quad \frac{\partial^n}{\partial \lambda_j^n} F_g = \int_{\zeta_1=\beta_{j-}}^{\zeta_1=\beta_{j+}} \cdots \int_{\zeta_n=\beta_{j-}}^{\zeta_n=\beta_{j+}} W_{g,n}(\zeta_1, \dots, \zeta_n) \quad (j \in \{0, 1, \infty\}).$$

Proof. Because

$$(4.8) \quad \frac{\partial y(z)}{\partial \lambda_j} \cdot dx(z) - \frac{\partial x(z)}{\partial \lambda_j} \cdot dy(z) = \int_{\zeta=\beta_{j-}}^{\zeta=\beta_{j+}} B(z, \zeta)$$

holds for each $j \in \{0, 1, \infty\}$, the variation formula [EO1, Theorem 5.1] implies (4.7). $\square$

Note that the parameters λ_j and points $\beta_{j\pm}$ in the other examples in §3.2.5 are chosen so that the variation formula (4.7) holds. Therefore, the following proof is applicable to the other examples.

Lemma 4.3 ([IKoT2, Lemma 3.3]). *Let $\omega(z)$ be a meromorphic differential whose poles are only on the set of ramification points of $x(z)$ in (3.104). Suppose also that $\omega(z)$ has no residue at those poles, and is anti-invariant under the conjugation map; that is, $\omega(\bar{z}) = -\omega(z)$. Then, we have*

$$(4.9) \quad \int_{\beta_{j\epsilon}}^{\beta_{k\eta}} \omega(z) = - \int_{\beta_{j(-\epsilon)}}^{\beta_{k(-\eta)}} \omega(z) = \frac{1}{2} \int_{\beta_{j\epsilon}}^{\beta_{j(-\epsilon)}} \omega(z) + \frac{1}{2} \int_{\beta_{k(-\eta)}}^{\beta_{k\eta}} \omega(z)$$

for any $j, k \in \{0, 1, \infty\}$ and $\epsilon, \eta \in \{+, -\}$.

This is proved by a straightforward computation and relations $\overline{\beta_{j\pm}} = \beta_{j\mp}$ ($j \in \{0, 1, \infty\}$).

Now we derive the equality (4.3) with the aid of Lemmas 4.2 and 4.3 and the results in §3.2.5.1 on the quantization of Gauss curve. Let us focus on the Voros coefficient $V^{(0)}(\underline{\lambda}, \underline{\nu}; \hbar)$ for the singular point $j = 0$. It follows from the definition of the path γ_0 and Theorem 3.5 that the Voros coefficient associated with the singular point 0 is expressed in terms of $W_{g,n}$ as follows.

(4.10)

$$\begin{aligned} V^{(0)}(\underline{\lambda}, \underline{\nu}; \hbar) &= \int_{\beta_{0-}}^{\beta_{0+}} \left(S(x(z), \hbar) - \hbar^{-1} S_{-1}(x(z)) - S_0(x(z)) \right) \frac{dx}{dz} dz \\ &= \sum_{m=1}^{\infty} \hbar^m \int_{\beta_{0-}}^{\beta_{0+}} \left\{ \sum_{\substack{2g-2+n=m \\ g \geq 0, n \geq 1}} \frac{1}{n!} \frac{d}{dz} \int_{\zeta_1 \in D(z; \underline{\nu})} \cdots \int_{\zeta_n \in D(z; \underline{\nu})} W_{g,n}(\zeta_1, \dots, \zeta_n) \right\} dz \\ &= \sum_{m=1}^{\infty} \hbar^m \sum_{\substack{2g-2+n=m \\ g \geq 0, n \geq 1}} \frac{1}{n!} \left(\int_{\zeta_1 \in D(\beta_{0+}; \underline{\nu})} \cdots \int_{\zeta_n \in D(\beta_{0+}; \underline{\nu})} \right. \\ &\quad \left. - \int_{\zeta_1 \in D(\beta_{0-}; \underline{\nu})} \cdots \int_{\zeta_n \in D(\beta_{0-}; \underline{\nu})} \right) W_{g,n}(\zeta_1, \dots, \zeta_n). \end{aligned}$$

For simplicity, let us introduce the notation

$$(4.11) \quad \left\{ \int_{\gamma} \right\}^n W_{g,n}(\zeta_1, \dots, \zeta_n) = \int_{\zeta_1 \in \gamma} \cdots \int_{\zeta_n \in \gamma} W_{g,n}(\zeta_1, \dots, \zeta_n).$$

Since

$$(4.12) \quad D(\beta_{0\pm}; \underline{\nu}) = \nu_{0\mp}([\beta_{0\pm}] - [\beta_{0\mp}]) + \sum_{\substack{j \in \{1, \infty\} \\ \epsilon \in \{+, -\}}} \nu_{j\epsilon}([\beta_{0\pm}] - [\beta_{j\epsilon}])$$

(here we have used (3.112)), Lemmas 4.2 and 4.3 imply

$$(4.13) \quad \begin{aligned} & \left\{ \int_{D(\beta_{0\pm}; \underline{\nu})} \right\}^n W_{g,n}(\zeta_1, \dots, \zeta_n) \\ &= \left\{ \left(\nu_{0\mp} + \sum_{\substack{j \in \{1, \infty\} \\ \epsilon \in \{+, -\}}} \frac{\nu_{j\epsilon}}{2} \right) \int_{\beta_{0\mp}}^{\beta_{0\pm}} + \sum_{\substack{j \in \{1, \infty\} \\ \epsilon \in \{+, -\}}} \frac{\nu_{j\epsilon}}{2} \int_{\beta_{j\epsilon}}^{\beta_{j(-\epsilon)}} \right\}^n W_{g,n}(\zeta_1, \dots, \zeta_n) \\ &= \left\{ \frac{-\nu_0 \pm 1}{2} \int_{\beta_{0-}}^{\beta_{0+}} - \sum_{j \in \{1, \infty\}} \frac{\nu_j}{2} \int_{\beta_{j-}}^{\beta_{j+}} \right\}^n W_{g,n}(\zeta_1, \dots, \zeta_n) \\ &= \left\{ \frac{-\nu_0 \pm 1}{2} \frac{\partial}{\partial \lambda_0} - \sum_{j \in \{1, \infty\}} \frac{\nu_j}{2} \frac{\partial}{\partial \lambda_j} \right\}^n F_g(\underline{\lambda}). \end{aligned}$$

Here we used (3.112) again, and set $\nu_j = \nu_{j+} - \nu_{j-}$ ($j \in \{0, 1, \infty\}$) as we have introduced in (3.117). Further computation shows

$$\begin{aligned} V^{(0)}(\underline{\lambda}, \underline{\nu}; \hbar) &= \sum_{m=1}^{\infty} \hbar^m \sum_{\substack{2g-2+n=m \\ g \geq 0, n \geq 1}} \frac{1}{n!} \\ &\times \left[\left\{ \frac{-\nu_0 + 1}{2} \frac{\partial}{\partial \lambda_0} - \sum_{j \in \{1, \infty\}} \frac{\nu_j}{2} \frac{\partial}{\partial \lambda_j} \right\}^n - \left\{ \frac{-\nu_0 - 1}{2} \frac{\partial}{\partial \lambda_0} - \sum_{j \in \{1, \infty\}} \frac{\nu_j}{2} \frac{\partial}{\partial \lambda_j} \right\}^n \right] F_g(\underline{\lambda}) \\ &= \sum_{\substack{g \geq 0, n_1, n_2, n_3 \geq 0 \\ 2g-2+n_1+n_2+n_3 \geq 1}} \frac{1}{n_1! n_2! n_3!} \\ &\times \left[\left\{ \left(-\frac{\hbar \nu_0}{2} + \frac{\hbar}{2} \right) \frac{\partial}{\partial \lambda_0} \right\}^{n_1} \left\{ -\frac{\hbar \nu_1}{2} \frac{\partial}{\partial \lambda_1} \right\}^{n_2} \left\{ -\frac{\hbar \nu_{\infty}}{2} \frac{\partial}{\partial \lambda_{\infty}} \right\}^{n_3} \hbar^{2g-2} F_g(\underline{\lambda}) \right. \\ &\quad \left. - \left\{ \left(-\frac{\hbar \nu_0}{2} - \frac{\hbar}{2} \right) \frac{\partial}{\partial \lambda_0} \right\}^{n_1} \left\{ -\frac{\hbar \nu_1}{2} \frac{\partial}{\partial \lambda_1} \right\}^{n_2} \left\{ -\frac{\hbar \nu_{\infty}}{2} \frac{\partial}{\partial \lambda_{\infty}} \right\}^{n_3} \hbar^{2g-2} F_g(\underline{\lambda}) \right] \\ &= F \left(\lambda_0 - \frac{\hbar \nu_0}{2} + \frac{\hbar}{2}, \lambda_1 - \frac{\hbar \nu_1}{2}, \lambda_{\infty} - \frac{\hbar \nu_{\infty}}{2}; \hbar \right) \\ &\quad - F \left(\lambda_0 - \frac{\hbar \nu_0}{2} - \frac{\hbar}{2}, \lambda_1 - \frac{\hbar \nu_1}{2}, \lambda_{\infty} - \frac{\hbar \nu_{\infty}}{2}; \hbar \right) \\ &\quad - \frac{1}{\hbar} \frac{\partial F_0}{\partial \lambda_0} + \frac{1}{2} \left(\nu_0 \frac{\partial}{\partial \lambda_0} + \nu_1 \frac{\partial}{\partial \lambda_1} + \nu_{\infty} \frac{\partial}{\partial \lambda_{\infty}} \right) \frac{\partial F_0}{\partial \lambda_0}. \end{aligned}$$

Thus we have proved (4.3) for $j = 0$. The other equalities for $j = 1$ and $j = \infty$ can be derived similarly.

Remark 4.4. In the definition (4.1) of the Voros coefficient, we subtracted the first two terms $\hbar^{-1}S_{-1}$ and S_0 because these terms are singular at end points of the path γ_j . However, a regularization procedure of divergent integral (see [V2] for example) allows us to define the regularized Voros coefficient:

$$(4.14) \quad V_{\text{reg}}^{(j)}(\underline{\lambda}, \underline{\nu}; \hbar) = \hbar^{-1}V_{-1}^{(j)}(\underline{\lambda}, \underline{\nu}) + V_0^{(j)}(\underline{\lambda}, \underline{\nu}) + V^{(j)}(\underline{\lambda}, \underline{\nu}; \hbar) \quad (j \in \{0, 1, \infty\}),$$

where $V_{-1}^{(j)}(\underline{\lambda}, \underline{\nu})$ and $V_0^{(j)}(\underline{\lambda}, \underline{\nu})$ are obtained by solving

$$(4.15) \quad \frac{\partial^2}{\partial \lambda_j^2} V_{-1}^{(j)} = \int_{\gamma_j} \frac{\partial^2}{\partial \lambda_j^2} S_{-1}(x) dx, \quad \frac{\partial}{\partial \lambda_j} V_0^{(j)} = \int_{\gamma_j} \frac{\partial}{\partial \lambda_j} S_0(x) dx \quad (j \in \{0, 1, \infty\}).$$

Actually, we can verify that $\partial_{\lambda_j}^2 S_{-1}(x)$ and $\partial_{\lambda_j} S_0(x)$ are holomorphic at $x = j$ although S_{-1} and S_0 are singular there. Hence, the equations in (4.59) make sense and we can find $V_{-1}^{(j)}$ and $V_0^{(j)}$. For example, in the case of quantum Gauss curve, the equations (4.59) for $j = 0$ are solved explicitly by

$$(4.16) \quad \begin{aligned} V_{-1}^{(0)} &= (\lambda_0 + \lambda_1 + \lambda_\infty) \log(\lambda_0 + \lambda_1 + \lambda_\infty) + (\lambda_0 - \lambda_1 + \lambda_\infty) \log(\lambda_0 - \lambda_1 + \lambda_\infty) \\ &\quad + (\lambda_0 + \lambda_1 - \lambda_\infty) \log(\lambda_0 + \lambda_1 - \lambda_\infty) + (\lambda_0 - \lambda_1 - \lambda_\infty) \log(\lambda_0 - \lambda_1 - \lambda_\infty) \\ &\quad - 4\lambda_0 \log(2\lambda_0), \end{aligned}$$

$$(4.17) \quad \begin{aligned} V_0^{(0)} &= -\frac{\nu_0}{2} \left(\log(\lambda_0 + \lambda_1 + \lambda_\infty) + \log(\lambda_0 - \lambda_1 + \lambda_\infty) + \log(\lambda_0 + \lambda_1 - \lambda_\infty) + \log(\lambda_0 - \lambda_1 - \lambda_\infty) \right. \\ &\quad \left. - 4 \log(2\lambda_0) \right) \\ &\quad - \frac{\nu_1}{2} \left(\log(\lambda_0 + \lambda_1 + \lambda_\infty) - \log(\lambda_0 - \lambda_1 + \lambda_\infty) + \log(\lambda_0 + \lambda_1 - \lambda_\infty) - \log(\lambda_0 - \lambda_1 - \lambda_\infty) \right) \\ &\quad - \frac{\nu_\infty}{2} \left(\log(\lambda_0 + \lambda_1 + \lambda_\infty) + \log(\lambda_0 - \lambda_1 + \lambda_\infty) - \log(\lambda_0 + \lambda_1 - \lambda_\infty) - \log(\lambda_0 - \lambda_1 - \lambda_\infty) \right). \end{aligned}$$

Actually, we can verify that the regularized integrals are realized by the correction terms

$$(4.18) \quad V_{-1}^{(j)} = \frac{\partial F_0}{\partial \lambda_j}, \quad V_0^{(j)} = -\frac{1}{2} \left(\nu_0 \frac{\partial}{\partial \lambda_0} + \nu_1 \frac{\partial}{\partial \lambda_1} + \nu_\infty \frac{\partial}{\partial \lambda_\infty} \right) \frac{\partial F_0}{\partial \lambda_j} \quad (j \in \{0, 1, \infty\})$$

in the right hand-side of the relation (4.3). Thus we can conclude that the regularized Voros coefficient satisfies

$$(4.19) \quad V_{\text{reg}}^{(j)}(\underline{\lambda}, \underline{\nu}; \hbar) = F\left(\hat{\underline{\lambda}} + \frac{\hbar}{2}\delta_j; \hbar\right) - F\left(\hat{\underline{\lambda}} - \frac{\hbar}{2}\delta_j; \hbar\right) \quad (j \in \{0, 1, \infty\}).$$

4.1.2.2 Proof of Theorem 4.1 (ii) Using the contiguity relations for the Gauss hypergeometric equation (see §C), we can find difference equations (with respect to the parameter ν_j) satisfied by the Voros coefficients.

Lemma 4.5 ([IKoT2, Lemma 3.5]). *The Voros coefficient $V^{(0)}$ (for the singular point 0) of the quantum Gauss curve (3.113) satisfies the following relations.*

$$(4.20) \quad V^{(0)}(\underline{\lambda}, \nu_0, \nu_1, \nu_\infty; \hbar) - V^{(0)}(\underline{\lambda}, \nu_0 - 1, \nu_1 - 1, \nu_\infty; \hbar) \\ = -\log \left[\frac{\lambda_0^2}{\hat{\lambda}_0(\hat{\lambda}_0 + \hbar/2)} \frac{(\hat{\lambda}_0 + \hat{\lambda}_1 + \hat{\lambda}_\infty + \hbar/2)}{(\lambda_0 + \lambda_1 + \lambda_\infty)} \frac{(\hat{\lambda}_0 + \hat{\lambda}_1 - \hat{\lambda}_\infty + \hbar/2)}{(\lambda_0 + \lambda_1 - \lambda_\infty)} \right],$$

$$(4.21) \quad V^{(0)}(\underline{\lambda}, \nu_0, \nu_1, \nu_\infty; \hbar) - V^{(0)}(\underline{\lambda}, \nu_0, \nu_1 - 1, \nu_\infty + 1; \hbar) \\ = -\log \left[\frac{(\hat{\lambda}_0 + \hat{\lambda}_1 - \hat{\lambda}_\infty + \hbar/2)}{(\lambda_0 + \lambda_1 - \lambda_\infty)} \frac{(\lambda_0 - \lambda_1 + \lambda_\infty)}{(\hat{\lambda}_0 - \hat{\lambda}_1 + \hat{\lambda}_\infty - \hbar/2)} \right],$$

$$(4.22) \quad V^{(0)}(\underline{\lambda}, \nu_0, \nu_1, \nu_\infty; \hbar) - V^{(0)}(\underline{\lambda}, \nu_0 - 1, \nu_1, \nu_\infty + 1; \hbar) \\ = -\log \left[\frac{\lambda_0^2}{\hat{\lambda}_0(\hat{\lambda}_0 + \hbar/2)} \frac{(\hat{\lambda}_0 + \hat{\lambda}_1 - \hat{\lambda}_\infty + \hbar/2)}{(\lambda_0 + \lambda_1 - \lambda_\infty)} \frac{(\hat{\lambda}_0 - \hat{\lambda}_1 - \hat{\lambda}_\infty + \hbar/2)}{(\lambda_0 - \lambda_1 - \lambda_\infty)} \right].$$

We will prove Lemma 4.5 in §C.2. Here we derive (4.5) for $j = 0$ by using this result.

In the proof $V^{(0)}(\underline{\lambda}, \underline{\nu}; \hbar)$ is abbreviated as $V^{(0)}(\nu_0, \nu_1, \nu_\infty)$ since we are only interested in dependence on $\underline{\nu}$. Substituting $(\nu_0, \nu_1, \nu_\infty) = (1, 0, 0)$ into (4.20), $(\nu_0, \nu_1, \nu_\infty) = (0, 0, -1)$ into (4.21), and $(\nu_0, \nu_1, \nu_\infty) = (0, 0, -1)$ into (4.22), we have

$$(4.23) \quad V^{(0)}(1, 0, 0) - V^{(0)}(0, -1, 0) = -\log \left[\frac{2\lambda_0}{2\lambda_0 - \hbar} \right],$$

$$(4.24) \quad V^{(0)}(0, 0, -1) - V^{(0)}(0, -1, 0) = 0,$$

$$(4.25) \quad V^{(0)}(0, 0, -1) - V^{(0)}(-1, 0, 0) = -\log \left[\frac{2\lambda_0}{2\lambda_0 + \hbar} \right].$$

Combining these equalities, we have

$$(4.26) \quad V^{(0)}(1, 0, 0) - V^{(0)}(-1, 0, 0) = -\log \left[\frac{4\lambda_0^2}{(2\lambda_0 + \hbar)(2\lambda_0 - \hbar)} \right].$$

On the other hand, (4.3) for $j = 0$ with $(\nu_0, \nu_1, \nu_\infty) = (1, 0, 0)$ and $(\nu_0, \nu_1, \nu_\infty) = (-1, 0, 0)$ gives

$$(4.27) \quad V^{(0)}(1, 0, 0) = F(\lambda_0, \lambda_1, \lambda_\infty; \hbar) - F(\lambda_0 - \hbar, \lambda_1, \lambda_\infty; \hbar) - \frac{1}{\hbar} \frac{\partial F_0}{\partial \lambda_0} + \frac{1}{2} \frac{\partial^2 F_0}{\partial \lambda_0^2},$$

$$(4.28) \quad V^{(0)}(-1, 0, 0) = F(\lambda_0 + \hbar, \lambda_1, \lambda_\infty; \hbar) - F(\lambda_0, \lambda_1, \lambda_\infty; \hbar) - \frac{1}{\hbar} \frac{\partial F_0}{\partial \lambda_0} - \frac{1}{2} \frac{\partial^2 F_0}{\partial \lambda_0^2}.$$

Then, the desired equality (4.5) for $j = 0$ follows immediately from (4.26), (4.27), (4.28) and the explicit expression of F_0 given in §3.2.5.1.

As we have seen in Remark 3.15, the free energy is symmetric with respect to the parameters $(\lambda_0, \lambda_1, \lambda_\infty)$. Therefore, (4.5) for $j = 0$ implies the other equalities for $j = 1$ and ∞ . Thus we have proved Theorem 4.1 (ii).

4.1.2.3 Proof of Theorem 4.1 (iii) Let X_j ($j \in S$) be the shift operator defined by

$$(4.29) \quad X_j = e^{-\hbar \partial_{\lambda_j}} \left(e^{\hbar \partial_{\lambda_j}} - 1 \right)^2 \quad : \quad F(\lambda_j) \mapsto F(\lambda_j + \hbar) - 2F(\lambda_j) + F(\lambda_j - \hbar),$$

where $\partial_{\lambda_j} = \partial / \partial \lambda_j$. Using this operator, the three term relation (4.5) is written as

$$(4.30) \quad X_j F(\underline{\lambda}; \hbar) = \log \Lambda(\underline{\lambda}) - 2 \log(2\lambda_j) - \log(2\lambda_j + \hbar) - \log(2\lambda_j - \hbar).$$

Our goal is to solve the difference equation (or invert the operator X_j) and obtain the explicit expression of the coefficients of F . For this purpose, we use

Lemma 4.6 ([IKoT2, Lemma 3.6]). (i) *Let $G(\underline{\lambda}; \hbar) = \sum_{g=0}^{\infty} \hbar^{2g-2} G_g(\underline{\lambda})$ be the formal series with the coefficients*

$$(4.31) \quad \begin{aligned} G_0(\underline{\lambda}) &= \frac{(\lambda_0 + \lambda_1 + \lambda_{\infty})^2}{2} \log(\lambda_0 + \lambda_1 + \lambda_{\infty}) + \frac{(\lambda_0 - \lambda_1 + \lambda_{\infty})^2}{2} \log(\lambda_0 - \lambda_1 + \lambda_{\infty}) \\ &\quad + \frac{(\lambda_0 + \lambda_1 - \lambda_{\infty})^2}{2} \log(\lambda_0 + \lambda_1 - \lambda_{\infty}) + \frac{(\lambda_0 - \lambda_1 - \lambda_{\infty})^2}{2} \log(\lambda_0 - \lambda_1 - \lambda_{\infty}) \\ &\quad - 3(\lambda_0^2 + \lambda_1^2 + \lambda_{\infty}^2), \end{aligned}$$

$$(4.32) \quad G_1(\underline{\lambda}) = -\frac{1}{12} \log \Lambda,$$

$$(4.33) \quad G_g(\underline{\lambda}) = \frac{B_{2g}}{2g(2g-2)} \left\{ \frac{1}{(\lambda_0 + \lambda_1 + \lambda_{\infty})^{2g-2}} + \frac{1}{(\lambda_0 - \lambda_1 + \lambda_{\infty})^{2g-2}} \right. \\ \left. + \frac{1}{(\lambda_0 + \lambda_1 - \lambda_{\infty})^{2g-2}} + \frac{1}{(\lambda_0 - \lambda_1 - \lambda_{\infty})^{2g-2}} \right\} \quad (g \geq 2).$$

Then, $G(\underline{\lambda}; \hbar)$ satisfies the following difference equation:

$$(4.34) \quad X_j G(\underline{\lambda}; \hbar) = \log \Lambda \quad (j \in \{0, 1, \infty\}).$$

(ii) *Let $H(\underline{\lambda}; \hbar) = \sum_{g=0}^{\infty} \hbar^{2g-2} H_g(\underline{\lambda})$ be the formal series with the coefficients*

$$(4.35) \quad H_0(\underline{\lambda}) = -2\lambda_0^2 \log(2\lambda_0) - 2\lambda_1^2 \log(2\lambda_1) - 2\lambda_{\infty}^2 \log(2\lambda_{\infty}) + 3(\lambda_0^2 + \lambda_1^2 + \lambda_{\infty}^2),$$

$$(4.36) \quad H_1(\underline{\lambda}) = \frac{1}{12} \log(\lambda_0 \lambda_1 \lambda_{\infty}),$$

$$(4.37) \quad H_g(\underline{\lambda}) = -\frac{B_{2g}}{2g(2g-2)} \left\{ \frac{1}{(2\lambda_0)^{2g-2}} + \frac{1}{(2\lambda_1)^{2g-2}} + \frac{1}{(2\lambda_{\infty})^{2g-2}} \right\} \quad (g \geq 2).$$

Then, $H(\underline{\lambda}; \hbar)$ satisfies the following difference equation:

$$(4.38) \quad X_j H(\underline{\lambda}; \hbar) = -\log(2\lambda_j + \hbar) - 2 \log(2\lambda_j) - \log(2\lambda_j - \hbar) \quad (j \in \{0, 1, \infty\}).$$

Proof. Firstly, we note that the terms G_0, H_0 are chosen so that the relations

$$(4.39) \quad \partial_{\lambda_j}^2 G_0(\underline{\lambda}) = \log \Lambda, \quad \partial_{\lambda_j}^2 H_0(\underline{\lambda}) = -4 \log(2\lambda_j)$$

hold for each $j \in \{0, 1, \infty\}$. Therefore, the claim (i) (resp., (ii)) immediately follows from Proposition D.3 (i) (resp., Proposition D.3 (ii)) which will be proved in §D.2. $\square$

Note that the formal series F and $G + H$ have the same first two terms:

$$(4.40) \quad F_0 = G_0 + H_0, \quad F_1 = G_1 + H_1.$$

Lemma 4.6 shows that they satisfy the same difference equations; that is,

$$(4.41) \quad X_j(F - (G + H)) = 0 \quad (j \in \{0, 1, \infty\}).$$

Then, Proposition D.6 implies that

$$(4.42) \quad F - (G + H) = \sum_{g=2}^{\infty} \hbar^{2g-2} \left(C_g + \sum_{j \in S} C_g^{(j)} \lambda_j + \sum_{\substack{j, k \in S \\ j \neq k}} C_g^{(jk)} \lambda_j \lambda_k + C_g^{(01\infty)} \lambda_0 \lambda_1 \lambda_{\infty} \right)$$

holds. Here $C_g, C_g^{(j)}, C_g^{(jk)}$ and $C_g^{(01\infty)}$ are some complex numbers which are independent of $\lambda_0, \lambda_1, \lambda_{\infty}$. To show that these constants vanish, we use the homogeneity of F_g 's.

Lemma 4.7 ([IKoT2, Lemma 3.7]). *The following relations hold for any nonzero complex number t .*

$$(4.43) \quad F_g(t\lambda_0, t\lambda_1, t\lambda_{\infty}) = t^{2-2g} F_g(\lambda_0, \lambda_1, \lambda_{\infty}) \quad (g \geq 2),$$

$$(4.44) \quad G_g(t\lambda_0, t\lambda_1, t\lambda_{\infty}) = t^{2-2g} G_g(\lambda_0, \lambda_1, \lambda_{\infty}) \quad (g \geq 2),$$

$$(4.45) \quad H_g(t\lambda_0, t\lambda_1, t\lambda_{\infty}) = t^{2-2g} H_g(\lambda_0, \lambda_1, \lambda_{\infty}) \quad (g \geq 2).$$

The first equality (4.43) is a consequence of [EO1, Theorem 5.3]. Therefore, we can conclude that the right hand-side of (4.42) must vanish due to the homogeneity. Consequently, we obtain the equality $F = G + H$ which proves the desired equality for the free energy of the Gauss curve. This completes the proof of Theorem 4.1 (iii).

Remark 4.8. The other results in Theorem 4.1 (iii) are easily obtained from the difference equation (4.5) by a similar method presented here. In particular, the formal series in Proposition D.3 (i) appears when we compute the action of inverse operator of X_j on (each factor of) $\log \Lambda(\underline{\lambda})$ appearing in (4.5). Similarly, the other formal series in Proposition D.3 (ii) is the inverse of $R_j(\underline{\lambda}; \hbar)$. Although the solution of the difference equation is not unique, we can fix the ambiguity with the aid of homogeneity, as we have done here.

4.1.2.4 Proof of Theorem 4.1 (iv) Here we derive the explicit formula of the coefficient of the Voros coefficient $V^{(0)}(\underline{\lambda}, \underline{\nu}; \hbar)$ of the quantum Gauss curve by using the relation (4.3) between the Voros coefficient and the free energy. Firstly, applying the difference operator $e^{\hbar\partial_{\lambda_0}} - 1$ to the both sides of the relation (4.3) for $j = 0$, we have

$$(4.46) \quad (e^{\hbar\partial_{\lambda_0}} - 1) V^{(0)}(\underline{\lambda}, \underline{\nu}; \hbar) = e^{\hbar\partial_{\lambda_0}/2} X_0 F(\hat{\lambda}_0, \hat{\lambda}_1, \hat{\lambda}_\infty; \hbar) \\ + (e^{\hbar\partial_{\lambda_0}} - 1) \left\{ -\frac{1}{\hbar} \frac{\partial F_0}{\partial \lambda_0} + \frac{1}{2} \left(\nu_0 \frac{\partial^2 F_0}{\partial \lambda_0^2} + \nu_1 \frac{\partial^2 F_0}{\partial \lambda_0 \lambda_1} + \nu_\infty \frac{\partial^2 F_0}{\partial \lambda_0 \lambda_\infty} \right) \right\},$$

where X_0 is the operator given in (4.29) for $j = 0$. Using (4.30), we have

$$(4.47) \quad e^{\hbar\partial_{\lambda_0}/2} X_0 F(\hat{\lambda}_0, \hat{\lambda}_1, \hat{\lambda}_\infty; \hbar) \\ = e^{(1-\nu_0-\nu_1-\nu_\infty)\hbar\partial_{\lambda_0}/2} \log(\lambda_0 + \lambda_1 + \lambda_\infty) + e^{(1-\nu_0+\nu_1-\nu_\infty)\hbar\partial_{\lambda_0}/2} \log(\lambda_0 - \lambda_1 + \lambda_\infty) \\ + e^{(1-\nu_0-\nu_1+\nu_\infty)\hbar\partial_{\lambda_0}/2} \log(\lambda_0 + \lambda_1 - \lambda_\infty) + e^{(1-\nu_0+\nu_1+\nu_\infty)\hbar\partial_{\lambda_0}/2} \log(\lambda_0 - \lambda_1 - \lambda_\infty) \\ - e^{(1-\nu_0)\hbar\partial_{\lambda_0}/2} \{ \log(2\lambda_0 + \hbar) + 2\log(2\lambda_0) + \log(2\lambda_0 - \hbar) \}.$$

To obtain the expression of the Voros coefficient, let us compute the action of the inverse operator of $(e^{\hbar\partial_{\lambda_0}} - 1)$ to each terms appearing in the right hand-side of (4.47). For the purpose, Proposition D.4 is quite useful. For example, applying Proposition D.4 (i) for $t = \nu_0 + \nu_1 + \nu_\infty$, $\lambda = \lambda_0$, $\mu = \lambda_1 + \lambda_\infty$, we can verify that the formal series

$$(4.48) \quad G(\underline{\lambda}, \underline{\nu}; \hbar) = \hbar^{-1} \left((\lambda_0 + \lambda_1 + \lambda_\infty) \log(\lambda_0 + \lambda_1 + \lambda_\infty) - \lambda_0 \right) - \frac{\nu_0 + \nu_1 + \nu_\infty}{2} \log(\lambda_0 + \lambda_1 + \lambda_\infty) \\ + \sum_{m=1}^{\infty} \frac{\hbar^m}{m(m+1)} \frac{B_{m+1}((\nu_0 + \nu_1 + \nu_\infty + 1)/2)}{(\lambda_0 + \lambda_1 + \lambda_\infty)^m}$$

is a particular solution of the difference equation

$$(4.49) \quad (e^{\hbar\partial_{\lambda_0}} - 1) G(\underline{\lambda}, \underline{\nu}; \hbar) = e^{(1-\nu_0-\nu_1-\nu_\infty)\hbar\partial_{\lambda_0}/2} \log(\lambda_0 + \lambda_1 + \lambda_\infty).$$

Namely, (4.48) coincides with the result of action of $(e^{\hbar\partial_{\lambda_0}} - 1)^{-1}$ on the first term in the right hand side of (4.47) modulo constant terms. Thanks to Proposition D.4, we can compute the action of $(e^{\hbar\partial_{\lambda_0}} - 1)^{-1}$ on the other terms in (4.47) similarly. Although (4.48) contains terms

which are proportional to $\hbar^{-1}$ and $\hbar^0$, those terms cancel with the other terms

$$\begin{aligned}
\frac{\partial F_0}{\partial \lambda_0} &= (\lambda_0 + \lambda_1 + \lambda_\infty) \log(\lambda_0 + \lambda_1 + \lambda_\infty) + (\lambda_0 - \lambda_1 + \lambda_\infty) \log(\lambda_0 - \lambda_1 + \lambda_\infty) \\
&\quad + (\lambda_0 + \lambda_1 - \lambda_\infty) \log(\lambda_0 + \lambda_1 - \lambda_\infty) + (\lambda_0 - \lambda_1 - \lambda_\infty) \log(\lambda_0 - \lambda_1 - \lambda_\infty) \\
&\quad - 4\lambda_0 \log(2\lambda_0), \\
\frac{\partial^2 F_0}{\partial \lambda_0^2} &= \log(\lambda_0 + \lambda_1 + \lambda_\infty) + \log(\lambda_0 - \lambda_1 + \lambda_\infty) + \log(\lambda_0 + \lambda_1 - \lambda_\infty) + \log(\lambda_0 - \lambda_1 - \lambda_\infty) \\
&\quad - 4 \log(2\lambda_0), \\
\frac{\partial^2 F_0}{\partial \lambda_0 \partial \lambda_1} &= \log(\lambda_0 + \lambda_1 + \lambda_\infty) - \log(\lambda_0 - \lambda_1 + \lambda_\infty) + \log(\lambda_0 + \lambda_1 - \lambda_\infty) - \log(\lambda_0 - \lambda_1 - \lambda_\infty), \\
\frac{\partial^2 F_0}{\partial \lambda_0 \partial \lambda_\infty} &= \log(\lambda_0 + \lambda_1 + \lambda_\infty) + \log(\lambda_0 - \lambda_1 + \lambda_\infty) - \log(\lambda_0 + \lambda_1 - \lambda_\infty) - \log(\lambda_0 - \lambda_1 - \lambda_\infty)
\end{aligned}$$

in the right hand side of (4.46). In the end, we can verify that the formal series

$$\begin{aligned}
(4.50) \quad V^{(0)}(\underline{\lambda}, \underline{\nu}; \hbar) &= \sum_{m=1}^{\infty} \frac{\hbar^m}{m(m+1)} \left\{ \frac{B_{m+1}((\nu_0 + \nu_1 + \nu_\infty + 1)/2)}{(\lambda_0 + \lambda_1 + \lambda_\infty)^m} + \frac{B_{m+1}((\nu_0 - \nu_1 + \nu_\infty + 1)/2)}{(\lambda_0 - \lambda_1 + \lambda_\infty)^m} \right. \\
&\quad \left. + \frac{B_{m+1}((\nu_0 + \nu_1 - \nu_\infty + 1)/2)}{(\lambda_0 + \lambda_1 - \lambda_\infty)^m} + \frac{B_{m+1}((\nu_0 - \nu_1 - \nu_\infty + 1)/2)}{(\lambda_0 - \lambda_1 - \lambda_\infty)^m} - \frac{B_{m+1}(\nu_0) + B_{m+1}(\nu_0 + 1)}{(2\lambda_0)^m} \right\}
\end{aligned}$$

satisfies the difference equation (4.46). Since we derive the formula (4.50) by solving the difference equation, the above expression only holds modulo formal series of $\hbar$ whose coefficients are independent of λ_j a priori (cf. Proposition D.5). However, by a similar argument used in the previous subsection, we can conclude that (4.50) is indeed an explicit expression of the Voros coefficient for the quantum Gauss curve. This completes the proof of Theorem 4.1 (iv).

4.2 In the case of third order differential equations

4.2.1 Main Theorem

In this subsection we formulate our main results for the quantum curves discussed in §3.3.2.

Let (C) be one of the spectral curves constructed in §3.3.2; that is, the (1,1,3) (§3.3.2.1), the (2,3) (§3.3.2.2) and the (1,4) (§3.3.2.3) curve. Denote by (E) the associated quantum curve by Theorem 3.20. Recall that the spectral curve (C) is (a parametrization of) the classical limit Σ of the quantum curve (E) .

Let S be the set of singular points of (E) , which are not branch points of the spectral curve (C) . The set S is tabulated in Table 4.2. For each $j \in S$, there exist three preimages $\beta_{j,1}$, $\beta_{j,2}$ and $\beta_{j,3}$ of j by the map $x(z)$ which appeared in the definition of the corresponding spectral curves (note that this notation is consistent with those given in §3.3.2.1 – §3.3.2.3). We also introduce $T = \{\beta_{\infty,1}, \beta_{\infty,2}, \beta_{\infty,3}\} \setminus \{\infty\}$. The set T is tabulated in Table 4.2. Let $\gamma_{\infty,j}$ ($j \in T$) be a path on the Riemann surface Σ defined as the image by $x(z)$ of a path from ∞ to $\beta_{\infty,j}$ on

(E)	$(1,1,3)$	$(2,3)$	$(1,4)$
S	$\{\infty\}$	$\{\infty\}$	$\{\infty\}$
T	$\{0, 1\}$	$\{0\}$	$\{0\}$

Table 4.2: The list of singular points (which are not branch points of the spectral curve) of quantum curves.

z -plane. (On x -plane, $\gamma_{\infty,j}$ starts from the point ∞ and come back to ∞ but with a different sheet of the Riemann surface Σ .) Note that $\lambda_{\infty,j}$ and $\gamma_{\infty,j}$ satisfy the variation formula

$$(4.51) \quad \frac{\partial^n F_g}{\partial \lambda_{\infty,j}^n} = \int_{\zeta_1 \in \gamma_{\infty,j}} \cdots \int_{\zeta_n \in \gamma_{\infty,j}} W_{g,n}(\zeta_1, \dots, \zeta_n) \quad (j \in T)$$

([EO1, §5.3]). Then, we define the Voros coefficient $V^{(\infty,j)}$ of the quantum curve (E) for the path $\gamma_{\infty,j}$ by

$$(4.52) \quad V^{(\infty,j)}(\underline{\lambda}, \underline{\nu}; \hbar) = \int_{\gamma_{\infty,j}} \left(S(x, \hbar) - \hbar^{-1} S_{-1}(x) - S_0(x) \right) dx \quad (j \in T).$$

Here $S(x, \hbar)$ is the logarithmic derivative of the WKB solution of the quantum curve (E) . Actually, $V^{(\infty,j)}$ depends only on the end and initial points of $\gamma_{\infty,j}$ since $S_m(x)dx$ for $m \geq 1$ (or the correlation functions $W_{g,n}$ for $2g - 2 + n \geq 1$) has no residue at branch points. We also note that, since the quantum curves discussed in §3.3.2 depends only on the parameters $\nu_{j,1}$, $\nu_{j,2}$, and $\nu_{j,3}$, so do the Voros coefficients. Thus we have used the symbol $\underline{\lambda} = (\lambda_{\infty,j})_{j \in T}$, $\underline{\nu} = (\nu_{\infty,j})_{j \in T}$ for the tuple of parameters contained in the quantum curve (E) in the expression (4.52).

On the other hand, the topological recursion associates the free energy

$$(4.53) \quad F(\underline{\lambda}, t; \hbar) = \sum_{g=0}^{\infty} \hbar^{2g-2} F_g(\underline{\lambda}, t)$$

for the spectral curve (C) as we have reviewed in §2.4. Our main result gives a formula which relates the free energy of (C) and the Voros coefficients of (E) . We derive three-terms relations satisfied by the free energy, and moreover, obtain explicit expressions of the free energies and Voros coefficients as a corollary of these relations. The precise statement is formulated as follows.

Theorem 4.9. (i) *The Voros coefficient of the quantum curve (E) and the free energy of the spectral curve (C) are related as follows.*

$$(4.54) \quad V^{(\infty,j)}(\underline{\lambda}, t, \underline{\nu}; \hbar) = F\left(\hat{\underline{\lambda}} + \hbar \delta_j, t; \hbar\right) - F\left(\hat{\underline{\lambda}}, t; \hbar\right) - \frac{1}{\hbar} \frac{\partial F_0}{\partial \lambda_{\infty,j}} + \sum_{k \in T} \left(\nu_{\infty,k} - \frac{\delta_{j,k}}{2} \right) \frac{\partial^2 F_0}{\partial \lambda_{\infty,j} \partial \lambda_{\infty,k}} \quad (j \in T).$$

Here the symbol $\underline{\lambda} + \alpha\delta_j$ means the parameter shift of $\hat{\lambda}_{\infty,j}$ by α :

$$(4.55) \quad \underline{\lambda} + \alpha\delta_j := (\hat{\lambda}_{\infty,k} + \delta_{kj}\alpha)_{k \in T}.$$

(Recall $\hat{\lambda}_{\infty,j} = \lambda_{\infty,j} - \nu_{\infty,j}\hbar$ ($j \in T$) as we have introduced in (3.163).)

- (ii) The free energy for the spectral curve (C) satisfies the following three-term difference equation.

$$(4.56) \quad F(\underline{\lambda} + \hbar\delta_j, t; \hbar) - 2F(\underline{\lambda}, t; \hbar) + F(\underline{\lambda} - \hbar\delta_j, t; \hbar) = R_j(\underline{\lambda}, t; \hbar) \quad (j \in T).$$

Here $R_j(\underline{\lambda}, t; \hbar)$ in the right hand-side of (4.56) are given as follows:

- For the $(1,1,3)$ curve (§3.3.2.1):

$$R_j(\underline{\lambda}, t; \hbar) = \log(\lambda_{\infty,j}) + 2\log(t) \quad (j \in \{0, 1\}).$$

- For the $(2,3)$ curve (§3.3.2.2):

$$R_0(\underline{\lambda}, t; \hbar) = -\frac{1}{2}\log(-2t).$$

- For the $(1,4)$ curve (§3.3.2.3):

$$R_0(\underline{\lambda}, t; \hbar) = \frac{1}{2}\log(-3\lambda_{\infty}^2).$$

- (iii) For $g \geq 2$, the g -th free energy of the spectral curve (C) has the following expression.

- For the $(1,1,3)$ curve (§3.3.2.1):

$$F_g(\underline{\lambda}, t) = \frac{B_{2g}}{2g(2g-2)} \left\{ \frac{1}{\lambda_{\infty,0}^{2g-2}} + \frac{1}{\lambda_{\infty,1}^{2g-2}} \right\}.$$

(F_0 and F_1 for the $(1,1,3)$ curve are given in §3.3.2.1.)

- For the $(2,3)$ curve (§3.3.2.2):

$$F_g(\underline{\lambda}, t) = 0.$$

(F_0 and F_1 for the $(2,3)$ curve are given in §3.3.2.2.)

- For the $(1,4)$ curve (§3.3.2.3):

$$F_g(\underline{\lambda}, t) = \frac{B_{2g}}{2g(2g-2)} \frac{1}{\lambda_{\infty}^{2g-2}}.$$

(F_0 and F_1 for the $(1,4)$ curve are given in §3.3.2.3.)

In these expressions, B_m is the m -th Bernoulli number given by (D.1) in §D.

(iv) The Voros coefficients (4.52) for the quantum curve (E) are explicitly given by

$$(4.57) \quad V^{(\infty,j)}(\underline{\lambda}, t, \underline{\nu}; \hbar) = \sum_{m=1}^{\infty} \hbar^m V_m^{(\infty,j)}(\underline{\lambda}, t, \underline{\nu}) \quad (j \in T),$$

where the coefficients $V_m^{(\infty,j)}(\underline{\lambda}, t, \underline{\nu})$ for $m \geq 1$ are given as follows.

- For quantum (1,1,3) curve (§3.3.2.1):

$$V_m^{(\infty,0)}(\underline{\lambda}, t, \underline{\nu}) = \frac{1}{m(m+1)} \frac{B_{m+1}(\nu_{\infty,0})}{\lambda_{\infty,0}^m},$$

$$V_m^{(\infty,1)}(\underline{\lambda}, t, \underline{\nu}) = \frac{1}{m(m+1)} \frac{B_{m+1}(\nu_{\infty,1})}{\lambda_{\infty,1}^m}.$$

- For quantum (2,3) curve (§3.3.2.2):

$$V_m^{(\infty,0)}(\underline{\lambda}, t, \underline{\nu}) = 0.$$

- For quantum (1,4) curve (§3.3.2.3):

$$V_m^{(\infty,0)}(\underline{\lambda}, t, \underline{\nu}) = \frac{1}{m(m+1)} \frac{B_{m+1}(\nu_{\infty})}{\lambda_{\infty}^m}.$$

(This expression was obtained in [IKo] when $\nu_{\infty} = 1$.)

In these expressions, $B_m(t)$ is the m -th Bernoulli polynomial given by (D.3) in §D.

4.2.2 Proof of main theorem for quantum (1,1,3) curve

Here we give a proof of Theorem 4.9 for the case of (quantum) (1,1,3) curve. Therefore, in the following proof, we focus on the (1,1,3) curve (3.159) and the associated quantum (1,1,3) curve (3.162) and use the notations in §3.3.2.1; namely, the function $x(z)$ is given in (3.159), and the divisor $D(z; \underline{\nu})$ is given in (3.161), and the set S of singular points becomes $\{\infty\}$, and the set T becomes $\{0, 1\}$, etc. (We omit the proof for other examples considered in §3.3.2 because they can be treated in a similar manner.)

4.2.2.1 Proof of Theorem 4.9 (i) Theorem 4.9 is proved in a similar manner to Theorem 4.1 (i). Then, we only note that the key equation is the variation formula (4.51) and omit the proof of Theorem 4.9 (i).

Remark 4.10. In the definition (4.52) of the Voros coefficient, we subtracted the first two terms $\hbar^{-1}S_{-1}$ and S_0 because these terms are singular at end points of the path γ . However, a regularization procedure of divergent integral (see [V2] for example) allows us to define the regularized Voros coefficient:

$$(4.58) \quad V_{\text{reg}}^{(\infty,j)}(\underline{\lambda}, t, \underline{\nu}; \hbar) := \hbar^{-1}V_{-1}^{(\infty,j)}(\underline{\lambda}, t, \underline{\nu}) + V_0^{(\infty,j)}(\underline{\lambda}, t, \underline{\nu}) + V^{(\infty,j)}(\underline{\lambda}, t, \underline{\nu}; \hbar),$$

where $V_{-1}^{(\infty,j)}(\underline{\lambda}, t, \underline{\nu})$ and $V_0^{(\infty,j)}(\underline{\lambda}, t, \underline{\nu})$ are obtained by solving

$$(4.59) \quad \frac{\partial^2}{\partial \lambda_{\infty,j}^2} V_{-1}^{(\infty,j)} = \int_{\gamma_{\infty,j}} \frac{\partial^2}{\partial \lambda_{\infty,j}^2} S_{-1}(x) dx, \quad \frac{\partial}{\partial \lambda_{\infty,j}} V_0^{(\infty,j)} = \int_{\gamma_{\infty,j}} \frac{\partial}{\partial \lambda_{\infty,j}} S_0(x) dx.$$

Actually, we can verify that $\partial_{\lambda_{\infty,j}}^2 S_{-1}(x)dx$ and $\partial_{\lambda_{\infty,j}} S_0(x)dx$ are holomorphic at $x = \infty$ although S_{-1} and S_0 are singular there. Hence, the equations in (4.59) make sense and we can find $V_{-1}^{(\infty,j)}$ and $V_0^{(\infty,j)}$. For example, in the case of the (1,1,3) quantum curve, we obtain

$$(4.60) \quad \frac{\partial^2}{\partial \lambda_{\infty,j}^2} V_{-1}^{(\infty,j)} = \frac{1}{\lambda_{\infty,j}}, \quad \frac{\partial}{\partial \lambda_{\infty,j}} V_0^{(\infty,j)} = - \left(\nu_{\infty,j} - \frac{1}{2} \right) \frac{1}{\lambda_{\infty,j}}.$$

Actually, we can verify that the regularized integrals are realized by the correction terms

$$(4.61) \quad V_{-1}^{(\infty,j)} = \frac{\partial F_0}{\partial \lambda_{\infty,j}}, \quad V_0^{(\infty,j)} = - \left(\nu_{\infty,j} - \frac{1}{2} \right) \frac{\partial^2 F_0}{\partial \lambda_{\infty,j}^2}$$

in the right hand-side of the relation (4.54). Thus we can conclude that the regularized Voros coefficient satisfies

$$(4.62) \quad V_{\text{reg}}^{(\infty,j)}(\underline{\lambda}, t, \underline{\nu}; \hbar) = F(\hat{\underline{\lambda}} + \hbar \delta_j, t; \hbar) - F(\hat{\underline{\lambda}}, t; \hbar).$$

4.2.2.2 Proof of Theorem 4.9 (ii) Using the contiguity relations for the (1,1,3) equation, we can find difference equations (with respect to the parameter $\nu_{\infty,j}$) satisfied by the Voros coefficients.

Lemma 4.11. *The Voros coefficient $V^{(\infty,j)}$ of the quantum (1,1,3) curve (3.162) satisfies the following relation.*

$$(4.63) \quad V^{(\infty,0)}(\underline{\lambda}, t, \nu_{\infty,0} - 1, \nu_{\infty,1}, \hbar) - V^{(\infty,0)}(\underline{\lambda}, t, \nu_{\infty,0}, \nu_{\infty,1}, \hbar) = \log \left(1 - \frac{(\nu_{\infty,0} - 1)\hbar}{\lambda_{\infty,0}} \right).$$

We will prove Lemma 4.11 in §C.3. Here we derive (4.56) by using this result.

In the proof $V^{(\infty,0)}(\underline{\lambda}, t, \underline{\nu}; \hbar)$ is abbreviated as $V^{(\infty,0)}(\nu_{\infty,0}, \nu_{\infty,1})$ since we are only interested in dependence on $\underline{\nu}$. Substituting $(\nu_{\infty,0}, \nu_{\infty,1}) = (1, 0)$ into (4.63), we have

$$(4.64) \quad V^{(\infty,0)}(1, 0) = V^{(\infty,0)}(0, 0).$$

On the other hand, (4.54) with $(\nu_{\infty,0}, \nu_{\infty,1}) = (1, 0)$ and $(\nu_{\infty,0}, \nu_{\infty,1}) = (0, 0)$ gives

$$(4.65) \quad V^{(\infty,0)}(1, 0) = F(\lambda_{\infty,0}, \lambda_{\infty,1}, t; \hbar) - F(\lambda_{\infty,0} - \hbar, \lambda_{\infty,1}, t; \hbar) - \frac{\partial F_0}{\partial \lambda_{\infty,0}} \hbar^{-1} + \frac{1}{2} \frac{\partial^2 F_0}{\partial \lambda_{\infty,0}^2},$$

$$(4.66) \quad V^{(\infty,0)}(0, 0) = F(\lambda_{\infty,0} + \hbar, \lambda_{\infty,1}, t; \hbar) - F(\lambda_{\infty,0}, \lambda_{\infty,1}, t; \hbar) - \frac{\partial F_0}{\partial \lambda_{\infty,0}} \hbar^{-1} - \frac{1}{2} \frac{\partial^2 F_0}{\partial \lambda_{\infty,0}^2}.$$

Then, the desired equality (4.56) follows immediately from (4.64), (4.65), (4.66) and the explicit expression of F_0 given in §3.3.2.1. Thus we have proved Theorem 4.9 (ii).

4.2.2.3 Proof of Theorem 4.9 (iii) Because we can determine the dependence of $F_g(\underline{\lambda}, t)$ ($g \geq 2$) with respect to parameter $\underline{\lambda}$ in a similar manner to the proof of Theorem 4.1 (iii), we only determine the dependence with respect to parameter t . Note that this point is the biggest difference between the case of second order differential equations and third order differential equations. Then, we prove the following theorem.

Theorem 4.12.

$$(4.67) \quad \frac{\partial F_g}{\partial t} = 0 \quad (g \geq 2).$$

To prove this theorem we use the following two lemmas.

Lemma 4.13. *For the free energy for the $(1,1,3)$ curve*

$$(4.68) \quad \frac{\partial F_g}{\partial t} = -\operatorname{Res}_{z=\infty} t(z^2 - z)W_{g,1}(z) \quad (g \geq 1)$$

holds.

Lemma 4.14. *For the $(1,1,3)$ equation and the $(1,1,3)$ curve the following relations hold:*

$$(4.69) \quad \operatorname{Res}_{z=\infty} t(z^2 - z) \sum_{m=-1}^{\infty} \hbar^m S_m(x(z)) dx(z) = C_{-1}(\underline{\lambda}, t, \underline{\nu}) \hbar^{-1} + C_0(\underline{\lambda}, t, \underline{\nu}),$$

$$(4.70) \quad \operatorname{Res}_{z=\infty} t(z^2 - z) \sum_{\substack{g \geq 0, n \geq 2 \\ (g,n) \neq (0,2)}} \frac{\hbar^{2g-2+n}}{(n-1)!} \int_{\zeta_2=\infty}^{\zeta_2=z} \cdots \int_{\zeta_n=\infty}^{\zeta_n=z} W_{g,n}(z, \zeta_2, \dots, \zeta_n) = \sum_{g \geq 1} \hbar^{2g} C_{g,2}(\underline{\lambda}, t, \underline{\nu}),$$

where C_{-1} , C_0 and $C_{g,2}$ ($g \geq 1$) are constant with respect to $\hbar$.

Proof of Lemma 4.13. Because

$$(4.71) \quad \Omega(z) = \frac{\partial y(z)}{\partial t} \cdot dx(z) - \frac{\partial x(z)}{\partial t} \cdot dy(z) = 2z dz = -\frac{1}{2\pi i} \int_{\zeta \in \gamma} \zeta^2 B(z, \zeta)$$

holds, Theorem 2.17 gives (4.68). □

Proof of Lemma 4.14. By using Lemma C.3, we can verify (4.69) directly. Because $W_{g,n}(z_1, \dots, z_n)$ are holomorphic at $z_i = \infty$ ($1 \leq i \leq n$) for $2g - 2 + n \geq 1$, we find that

$$W_{g,n}(z, z_2, \dots, z_n) \sim \frac{dz_i}{z_i^2} (C_{g,n}^{(i)} + O(1/z_i)) \quad (z_i \rightarrow \infty).$$

Then, since lower order terms $O(1/z_i)$ vanish in the limit $z_i \rightarrow \infty$ ($1 \leq i \leq n$), we obtain

$$\int_{\zeta_2=\infty}^{\zeta_2=z_2} W_{g,n}(z, \zeta_2, \dots, \zeta_n) = \int_{\zeta_2=\infty}^{\zeta_2=z_2} \frac{C_{g,n} d\zeta_2 \cdots d\zeta_n}{z^2 \zeta_2^2 \cdots \zeta_n^2} dz = -\frac{C_{g,n} d\zeta_3 \cdots d\zeta_n}{z^2 z_2 \zeta_3^2 \cdots \zeta_n^2} dz.$$

Therefore,

$$\int_{\zeta_2=\infty}^{\zeta_2=z_2} \cdots \int_{\zeta_n=\infty}^{\zeta_n=z_n} W_{g,n}(z, \zeta_2, \dots, \zeta_n) \Big|_{z_2=\dots=z_n=z} \sim \left(\frac{(-1)^{n+1} C_{g,n}}{z^{n+1}} + \dots \right) dz$$

holds. Multiplying the both sides of the equation by $t(z^2 - z)$ and calculating residues, we get (4.70). □

Proof of Theorem 4.12. By taking $(\nu_{\infty,0}, \nu_{\infty,1}) = (0, 0)$ in Theorem 3.20 we obtain

$$(4.72) \quad \log \psi|_{x=x(z)} = \sum_{m=-1}^{\infty} \hbar^m \int^{x(z)} S_m dx \\ = \sum_{m=-1}^{\infty} \hbar^m \left\{ \sum_{\substack{2g+n-2=m \\ g \geq 0, n \geq 1}} \frac{1}{n!} \int_{\infty}^z \cdots \int_{\infty}^z \left(W_{g,n}(z_1, \dots, z_n) - \delta_{g,0} \delta_{n,2} \frac{dx(z_1) dx(z_2)}{(x(z_1) - x(z_2))^2} \right) \right\}.$$

It follows from this equation that

$$(4.73) \quad \sum_{g \geq 0} \hbar^{2g-1} \left(-\operatorname{Res}_{z=\infty} t(z^2 - z) W_{g,1}(z) \right) \\ = -\operatorname{Res}_{z=\infty} t(z^2 - z) \sum_{m=-1}^{\infty} \hbar^m S_m(x(z)) dx(z) + \operatorname{Res}_{z=\infty} t(z^2 - z) \int_{\infty}^z \left(W_{0,2}(z, z_2) - \frac{dx(z) dx(z_2)}{(x(z) - x(z_2))^2} \right) \\ + \operatorname{Res}_{z=\infty} t(z^2 - z) \sum_{\substack{g \geq 0, n \geq 2 \\ (g,n) \neq (0,2)}} \frac{\hbar^{2g-2+n}}{(n-1)!} \int_{\infty}^z \cdots \int_{\infty}^z W_{g,n}(z, z_2, \dots, z_n) \\ = -C_{-1}(\underline{\lambda}, t, \underline{\nu}) \hbar^{-1} - C_0(\underline{\lambda}, t, \underline{\nu}) + \operatorname{Res}_{z=\infty} t(z^2 - z) \int_{\infty}^z \left(W_{0,2}(z, z_2) - \frac{dx(z) dx(z_2)}{(x(z) - x(z_2))^2} \right) \\ + \sum_{g \geq 1} \hbar^{2g} C_{g,2}(\underline{\lambda}, t, \underline{\nu}).$$

Here we have used Lemma 4.14 in the last equality. Because the left hand side of this equation is written by

$$\sum_{g \geq 0} \hbar^{2g-1} \left(-\operatorname{Res}_{z=\infty} t(z^2 - z) W_{g,1}(z) \right) = -\hbar^{-1} \operatorname{Res}_{z=\infty} t(z^2 - z) W_{0,1}(z) + \sum_{g \geq 1} \hbar^{2g-1} \frac{\partial F_g}{\partial t},$$

we compare the odd terms with respect to $\hbar$ of both sides. Then, we find that there is no odd term whose order with respect to $\hbar$ is greater than or equal to one in the right hand side. It means that (4.67) holds. □

4.2.2.4 Proof of Theorem 4.9 (iv) We omit the proof of Theorem 4.9 (iv) because we can obtain the Voros coefficients directly by using the relation (4.54) and the explicit form of the free energy.

A Meromorphic multidifferentials

The correlation function $W_{g,n}(z_1, z_2, \dots, z_n)$ is a meromorphic multidifferential, i.e., a meromorphic section of the line bundle $\pi_1^*(T^*\mathbb{P}^1) \otimes \pi_2^*(T^*\mathbb{P}^1) \otimes \dots \otimes \pi_n^*(T^*\mathbb{P}^1)$ on $(\mathbb{P}^1)^n$, where $\pi_j : (\mathbb{P}^1)^n \rightarrow \mathbb{P}^1$ denotes the j -th projection ([DN]). Thus a multidifferential is a meromorphic differential in $\mathbb{P}^1$ for each variable. If all of the residue with respect to each variable vanish, then we call it a multidifferential of the second kind. We summarize here some notations on multidifferential which we use in this paper.

Local coordinate representation. In a local coordinate, we express a meromorphic multidifferential Ω on $(\mathbb{P}^1)^n$ as

$$(A.1) \quad \Omega = \Omega(z_1, z_2, \dots, z_n) = f(z_1, z_2, \dots, z_n) dz_1 dz_2 \dots dz_n,$$

where f is a meromorphic function (we omit the tensor product $\otimes$ in its expression). If Ω is symmetric under the permutation of variables, i.e.,

$$(A.2) \quad \Omega(z_1, \dots, z_j, \dots, z_k, \dots, z_n) = \Omega(z_1, \dots, z_k, \dots, z_j, \dots, z_n)$$

for any $j, k \in \{1, 2, \dots, n\}$, Ω is said to be a symmetric multidifferential.

Integration. Its integral with respect to j -th variable is denoted by

$$(A.3) \quad \int_{z_j=a}^{z_j=b} \Omega(z_1, z_2, \dots, z_n) := \left(\int_a^b f(z_1, z_2, \dots, z_n) dz_j \right) dz_1 \dots dz_{j-1} dz_{j+1} \dots dz_n$$

or

$$(A.4) \quad \int_{z_j \in \gamma} \Omega(z_1, z_2, \dots, z_n) := \left(\int_{\gamma} f(z_1, z_2, \dots, z_n) dz_j \right) dz_1 \dots dz_{j-1} dz_{j+1} \dots dz_n$$

for an integration path γ in $\mathbb{P}^1$. If Ω is symmetric under the permutation of the variables, we write a multiple integral with a same integration contour γ like

$$(A.5) \quad \underbrace{\int_{\gamma} \dots \int_{\gamma}}_{n\text{-th}} \Omega(z_1, \dots, z_n) := \int_{\gamma} dz_1 \dots \int_{\gamma} dz_n f(z_1, \dots, z_n).$$

Further we sometimes drop off the word “ n -th” if it is clear from the context.

Integration with a divisor. Let ω be a meromorphic differential on some domain U in $\mathbb{P}^1$ of the second kind. Following [BE2], for a divisor $D(z; \underline{\nu}) = [z] - \sum_{k=1}^m \nu_k [p_k]$, where $z, p_1, p_2, \dots, p_m \in \mathbb{P}^1 \setminus \{\text{poles of } \omega\}$, and $\underline{\nu} = (\nu_k)_{k=1}^m$ being a tuple of complex numbers satisfying $\sum_{k=1}^m \nu_k = 1$, we define the integral of ω with $D(z; \underline{\nu})$ by

$$(A.6) \quad \int_{D(z; \underline{\nu})} \omega := \sum_{k=1}^m \nu_k \int_{p_k}^z \omega.$$

Because ω is of the second kind, this integral does not depend on the choice of paths in U from p_j to z ($j = 1, 2, \dots, m$). For a multidifferential $\Omega = \Omega(z_1, z_2, \dots, z_n)$ of the second kind, we define

$$(A.7) \quad \int_{z_j \in D(z; \mathcal{V})} \Omega(z_1, \dots, z_j, \dots, z_n) := \sum_{k=1}^m \nu_k \int_{z_j=p_k}^{z_j=z} \Omega(z_1, \dots, z_j, \dots, z_n).$$

As in (A.5), we write the multiple integral as

$$(A.8) \quad \underbrace{\int_{D(z; \mathcal{V})} \cdots \int_{D(z; \mathcal{V})}}_{n\text{-th}} \Omega(z_1, \dots, z_n) := \int_{z_1 \in D(z; \mathcal{V})} dz_1 \cdots \int_{z_n \in D(z; \mathcal{V})} dz_n f(z_1, \dots, z_n)$$

for a symmetric meromorphic multidifferential of the second kind.

Pull-back. Finally, for a holomorphic map ϕ from some domain in $\mathbb{P}^1$ to $\mathbb{P}^1$, we write the pullback of Ω by ϕ with respect to the j -th variable as

$$\Omega(z_1, \dots, \phi(z_j), \dots, z_n) := f(z_1, \dots, \phi(z_j), \dots, z_n) dz_1 \cdots d\phi(z_j) \cdots dz_n.$$

We frequently use this expression mainly when ϕ is conjugate map defined near a ramification point.

B Ineffectiveness of ramification points

Following [BE2], we define R as a set of ramification points of $x(z)$, not as a set of zeros of $dx(z)$ as in [EO1]. However, this modification does not cause difference when the ramification points are ineffective (in the sense of Definition 2.13). Here we give a criterion for the ineffectiveness of ramification points (cf. Proposition 2.14).

Proposition B.1. *For a ramification point r , the followings are equivalent:*

- (a) *the correlation function $W_{g,n}(z_1, \dots, z_n)$ with $(g, n) \neq (1, 0)$ is holomorphic at $z_i = r$ for each $i = 1, \dots, n$ (i.e., r is an ineffective ramification point).*
- (b) *The differential $(y(z) - y(\bar{z}))dx(z)$ has a pole at r .*

Proof. First let us give a remark on the pole order of correlation functions at a ramification point satisfying the above condition.

Lemma B.2. *If $(y(z) - y(\bar{z}))dx(z)$ has a pole at a ramification point r , then the pole order of $(y(z) - y(\bar{z}))dx(z)$ at $z = r$ is greater than or equal to two.*

Proof of Lemma B.2. It is enough to prove that $(y(z) - y(\bar{z}))dx(z)$ never has a simple pole at ramification point. In other words, it suffices to prove that, there is no $r \in R$ satisfying $\rho(x(r); P) = -2$ (cf. Proposition 3.3).

Suppose for contradiction that a point $r \in R$ satisfies $\rho(x(r); P) = -2$.

We also assume that $x(r) \neq \infty$ for simplicity. (The case $x(r) = \infty$ can be treated by a similar way.) Then, the function $Q_0(x)$ defined by (3.9) has an expression

$$Q_0(x) = \frac{c_0}{(x - x(r))^2} (1 + f(x)),$$

where c_0 is a nonzero constant, and $f(x)$ is a rational function of x which vanishes at $x = x(r)$. Taking a square root with an appropriate branch, we obtain

$$y(z) - y(\bar{z}) = \frac{2\sqrt{c_0}}{x(z) - x(r)} (1 + f(x(z)))^{1/2}.$$

Since the left hand side is anti-invariant under the involution $z \mapsto \bar{z}$, the above equality and the relation $x(z) = x(\bar{z})$ imply

$$\frac{1}{x(z) - x(r)} = -\frac{1}{x(z) - x(r)}$$

which leads a contradiction. This proves that no $r \in R$ satisfying $\rho(x(r); P) = -2$. $\square$

Remark B.3. By a similar argument presented in the proof of Lemma B.2, we can also show that, there is no $r \in R$ satisfying $\rho(x(r); P) = -2m$ for some $m \geq 1$.

Now let us prove Proposition B.1.

Let us assume (b), that is, $(y(z) - y(\bar{z}))dx(z)$ has a pole at r , and look at the behavior of $W_{0,3}(z_0, z_1, z_2)$ and $W_{1,1}(z_0)$ when z_0 approaches to r . By deforming the residue contour around r , we can decompose the contribution of residue at $z = r$ to $W_{0,3}(z_0, z_1, z_2)$ as

$$\begin{aligned} \text{(B.1)} \quad & \text{Res}_{z=r} K_r(z_0, z) (B(z, z_1)B(\bar{z}, z_2) + B(\bar{z}, z_1)B(z, z_2)) \\ &= \frac{1}{2\pi i} \left(\oint_{z \in C_{r, z_0, \bar{z}_0}} - \oint_{z \in C_{z_0}} - \oint_{z \in C_{\bar{z}_0}} \right) K_r(z_0, z) (B(z, z_1)B(\bar{z}, z_2) + B(\bar{z}, z_1)B(z, z_2)). \end{aligned}$$

Here $C_{r, z_0, \bar{z}_0}$ is a contour satisfying

$$\{\text{the domain bounded by } C_{r, z_0, \bar{z}_0}\} \cap R \cap \{z_0, \bar{z}_0, \dots, z_n, \bar{z}_n\} = \{r, z_0, \bar{z}_0\},$$

and the other contours C_{z_0} and $C_{\bar{z}_0}$ are defined similarly. Then, the first integral in the right hand-side of (B.1) is holomorphic at $z_0 = r$, while the other two integrals are evaluated as

$$\begin{aligned} \text{(B.2)} \quad & -\frac{1}{2\pi i} \left(\oint_{z \in C_{z_0}} + \oint_{z \in C_{\bar{z}_0}} \right) K_r(z_0, z) (B(z, z_1)B(\bar{z}, z_2) + B(\bar{z}, z_1)B(z, z_2)) \\ &= \frac{1}{(y(z_0) - y(\bar{z}_0))dx(z_0)} (B(z_0, z_1)B(\bar{z}_0, z_2) + B(\bar{z}_0, z_1)B(z_0, z_2)). \end{aligned}$$

Therefore $W_{0,3}(z_0, z_1, z_2)$ is holomorphic at $z_0 = r$ (and hence, it is holomorphic at $z_i = r$ for $i = 1, 2$ as well) under the assumption on the pole property of $(y(z) - y(\bar{z}))dx(z)$ at r .

On the other hand, the behavior of $W_{1,1}(z_0)$ when z_0 approaches to r is described in a similar manner:

$$(B.3) \quad \text{Res}_{z=r} K_r(z_0, z)B(z, \bar{z}) = \frac{1}{2\pi i} \oint_{z \in C_{r, z_0, \bar{z}_0}} K_r(z_0, z)B(z, \bar{z}) + \frac{1}{(y(z_0) - y(\bar{z}_0))dx(z_0)} B(z_0, \bar{z}_0).$$

Then we can conclude that $W_{1,1}(z_0)$ is holomorphic at $z_0 = r$ because $(y(z_0) - y(\bar{z}_0))dx(z_0)$ has a double or higher order pole at r (cf. Lemma B.2) while $B(z_0, \bar{z}_0)$ has a double pole there.

Using the induction on $2g - 2 + n$, we can conclude that the correlation functions $W_{g,n}$ are holomorphic at r with respect to each variable z_i , as follows. For general (g, n) , the contribution of the residue at $z = r$ to $W_{g,n+1}(z_0, z_1, \dots, z_n)$ is given by the following form

$$(B.4) \quad \text{Res}_{z=r} K_r(z_0, z)F_{g,n}(z, \bar{z}, z_1, \dots, z_n) \\ = \frac{1}{2\pi i} \left(\oint_{z \in C_{r, z_0, \bar{z}_0}} - \oint_{z \in C_{z_0}} - \oint_{z \in C_{\bar{z}_0}} \right) K_r(z_0, z)F_{g,n}(z, \bar{z}, z_1, \dots, z_n).$$

Although we omit an explicit expression of $F_{g,n}(z, \bar{z}, z_1, \dots, z_n)$ (which can be read off from (2.34)), we know it has at most double pole at $z = r$ under the induction hypothesis. Then, by the similar argument for $W_{1,1}(z_0)$ presented above, we can verify that the right hand-side of (B.4) is holomorphic at $z_0 = r$. Thus we have verified that r is ineffective.

Conversely, let us assume (a). Then, it follows from the definition of ineffectiveness that the correlation functions $W_{0,3}(z_0, z_1, z_2)$ must be holomorphic at $z_0 = r$. Then, in view of (B.2), we can conclude that the differential $(y(z_0) - y(\bar{z}_0))dx(z_0)$ must have a pole at $z_0 = r$ (otherwise the right-hand side of (B.2) never becomes holomorphic at $z_0 = r$).

Thus we have proved the equivalence between the conditions (a) and (b). This also completes the proof of Proposition 2.14 (i). $\square$

The remaining task for a proof of Proposition 2.14 is to show

Proposition B.4. *If r is an ineffective ramification point, then the residue at r in (2.34) becomes zero.*

Proof. Since $\int_{\zeta=\bar{z}}^{\zeta=z} B(z_0, \zeta)$ is holomorphic and vanishes at $z = r$, we can verify that $K_r(z_0, z)$ has a double (or more higher order) zero at $z = r$ if r is ineffective (cf. Proposition B.1). Therefore, $K_r(z_0, z)F_{g,n}(z, \bar{z}, z_1, \dots, z_n)$ in (B.4) is holomorphic and has no residue at $z = r$. This completes the proof. $\square$

C Contiguity relations of the Voros coefficient

Here we give a proof of Lemma 4.5 by using the contiguity relations of the Gauss hypergeometric equation. A recent result of Oshima (see [O] and references therein) shows that his framework is efficient in a study of linear differential equations of rational coefficients theoretically and in practice. As is discussed in [IKo], it would give a computation method of Voros coefficients in a unified and algorithmic manner. Although it is enough to use the traditional contiguity relations because we only discuss second order hypergeometric equations in this paper, here we give them along Oshima's theory for future works.

C.1 Contiguity relations of the Gauss hypergeometric equation

In our study we need contiguity relations of the following operator defined through the middle convolution.

$$(C.1) \quad \begin{aligned} P_{\text{Gauss}}(\kappa_1, \kappa_2, \mu) &:= \text{RAd}(\partial^{-\mu/\hbar}) \text{RAd}(x^{\kappa_1/\hbar} (1-x)^{\kappa_2/\hbar}) \partial_x \\ &= x(1-x)(\hbar \partial_x)^2 + (c - (a+b+\hbar)x)(\hbar \partial_x) - ab. \\ &\left(\begin{array}{l} a = -(\mu + \kappa_1 + \kappa_2), \quad b = -\mu + \hbar, \quad c = -\mu - \kappa_1 + \hbar \\ \iff \quad \kappa_1 = b - c, \quad \kappa_2 = -a + c - \hbar, \quad \mu = -b + \hbar \end{array} \right) \end{aligned}$$

Here RAd denotes the reduced adjoint operator ([O, §2.2]). We specify the parameters

$$(C.2) \quad a = \hat{\lambda}_0 + \hat{\lambda}_1 + \hat{\lambda}_\infty + \frac{\hbar}{2}, \quad b = \hat{\lambda}_0 + \hat{\lambda}_1 - \hat{\lambda}_\infty + \frac{\hbar}{2}, \quad c = 2\hat{\lambda}_0 + \hbar$$

so that the equation $P_{\text{Gauss}}\psi_{\text{Gauss}} = 0$ is equivalent to the quantum Gauss curve (3.113) by a gauge transform (cf. (3.116)). These relations are equivalent to

$$(C.3) \quad \hat{\lambda}_0 = -\frac{\mu}{2} - \frac{\kappa_1}{2}, \quad \hat{\lambda}_1 = -\frac{\mu}{2} - \frac{\kappa_2}{2}, \quad \hat{\lambda}_\infty = -\frac{\kappa_1}{2} - \frac{\kappa_2}{2} - \frac{\hbar}{2}$$

in terms of the parameters related to the middle convolution.

We will use the following contiguity relations of the operator P_{Gauss} (cf. [O, §5 and §4]).

Theorem C.1. *The differential operator (C.1) satisfies the following contiguity relations.*

$$\begin{aligned} \partial_x \circ P_{\text{Gauss}}(\kappa_1, \kappa_2, \mu) &= P_{\text{Gauss}}(\kappa_1, \kappa_2, \mu - \hbar) \circ \partial_x, \\ (x + (\hbar - \mu)(\hbar \partial_x)^{-1}) \circ P_{\text{Gauss}}(\kappa_1, \kappa_2, \mu) &= P_{\text{Gauss}}(\kappa_1 + \hbar, \kappa_2, \mu) \circ (x - \mu(\hbar \partial_x)^{-1}), \\ (x - 1 + (\hbar - \mu)(\hbar \partial_x)^{-1}) \circ P_{\text{Gauss}}(\kappa_1, \kappa_2, \mu) &= P_{\text{Gauss}}(\kappa_1, \kappa_2 + \hbar, \mu) \circ (x - 1 - \mu(\hbar \partial_x)^{-1}). \end{aligned}$$

In particular, the parameter shift $(\kappa_1, \kappa_2, \mu) \mapsto (\kappa_1, \kappa_2, \mu - \hbar)$ in the first line can be translated into the shift $(\nu_0, \nu_1, \nu_\infty) \mapsto (\nu_0 - 1, \nu_1 - 1, \nu_\infty)$ of the parameters ν_j 's.

C.2 Proof of Lemma 4.5

Now we prove Lemma 4.5 with the aid of Theorem C.1. The following proof is based on the same ideas of [T1] and [IKo]. We will only give a proof of (4.20) in Lemma 4.5 since the other equalities can be derived similarly (see Remark C.2).

C.2.1 Preliminary for the proof

Let $\psi_{\text{Gauss}}(x, \hbar)$ be the WKB solution to the equation

$$(C.4) \quad P_{\text{Gauss}}\psi_{\text{Gauss}} = 0,$$

and $S_{\text{Gauss}} = \sum_{m \geq -1} \hbar^m S_{\text{Gauss},m}(x)$ be its logarithmic derivative. In view of (3.116), the logarithmic derivative S of the WKB solution of the quantum Gauss curve (3.113) (which is considered in §4.1.2) is related to S_{Gauss} as follows:

$$(C.5) \quad S_{\text{Gauss}}(x, \hbar) = S(x, \hbar) - \left(\frac{\hat{\lambda}_0}{\hbar} + \frac{\nu_{0+} + \nu_{0-}}{2} \right) \frac{1}{x} - \left(\frac{\hat{\lambda}_1}{\hbar} + \frac{\nu_{1+} + \nu_{1-}}{2} \right) \frac{1}{x-1}.$$

The relation (C.5) implies that the formal series S and S_{Gauss} differ only by first two terms (i.e., terms proportional to $\hbar^{-1}$ and $\hbar^0$). Therefore, the Voros coefficients

$$(C.6) \quad V_{\text{Gauss}}^{(j)}(\underline{\lambda}, \underline{\nu}; \hbar) := \int_{\gamma_j} (S_{\text{Gauss}}(x, \hbar) - \hbar^{-1} S_{\text{Gauss},-1}(x) - S_{\text{Gauss},0}(x)) dx$$

of the equation (C.4) defined for the same path γ_j ($j \in \{0, 1, \infty\}$) given in §4.1.1 coincide with those for the quantum Gauss curve; that is, $V_{\text{Gauss}}^{(j)} = V^{(j)}$ holds for each $j \in \{0, 1, \infty\}$. Therefore, our task is to show that the Voros coefficient $V_{\text{Gauss}}^{(0)}$ satisfies (4.20).

We will use the symbols x_1 and x_2 for points on γ_0 which eventually tend to the end point and initial point of γ_0 . We will also use the following obvious expression:

$$(C.7) \quad (V^{(0)}(\underline{\lambda}, \underline{\nu}; \hbar) =) V_{\text{Gauss}}^{(0)}(\underline{\lambda}, \underline{\nu}; \hbar) = \lim_{\substack{x_1 \rightarrow 0 \\ x_2 \rightarrow 0^\dagger}} V_{\text{Gauss}}^{x_1, x_2}(\underline{\lambda}, \underline{\nu}; \hbar)$$

with

$$(C.8) \quad V_{\text{Gauss}}^{x_1, x_2}(\underline{\lambda}, \underline{\nu}; \hbar) := \int_{x_2}^{x_1} (S_{\text{Gauss}}(x, \hbar) - \hbar^{-1} S_{\text{Gauss},-1}(x) - S_{\text{Gauss},0}(x)) dx.$$

Here the integration in (C.8) is taken along a part of γ_0 , and 0 and $0^\dagger$ are the end point and initial point of γ_0 , respectively. (These points are preimages of $x = 0$ by the projection $\Sigma \rightarrow \mathbb{P}^1$, or in other words, 0 corresponds to β_{0+} on z -plane while $0^\dagger$ corresponds to β_{0-} .)

In the following proof we need to analyze the behavior of $S_{\text{Gauss},m}(x)$ when x approaches to 0 and $0^\dagger$. A straightforward computation shows

$$(C.9) \quad S_{\text{Gauss},-1}(x) = \begin{cases} O(x^0) & \text{when } x \rightarrow 0, \\ -\frac{2\lambda_0}{x} + O(x^0) & \text{when } x \rightarrow 0^\dagger, \end{cases}$$

$$(C.10) \quad S_{\text{Gauss},0}(x) = \begin{cases} O(x^0) & \text{when } x \rightarrow 0, \\ \frac{\nu_0}{x} + O(x^0) & \text{when } x \rightarrow 0^\dagger, \end{cases}$$

$$(C.11) \quad S_{\text{Gauss},m}(x) = O(x^0) \quad \text{when } x \rightarrow 0 \text{ and } x \rightarrow 0^\dagger.$$

The last property (C.11) is a consequence of the fact that the correlation function $W_{g,n}(z_1, \dots, z_n)$ is holomorphic at $z_i = \beta_{0\pm}$ (which are not ramification points). Thus S_{Gauss} behaves as

$$(C.12) \quad S_{\text{Gauss}}(x, \hbar) = \begin{cases} A(\hbar) + O(x^1) & \text{when } x \rightarrow 0, \\ -\frac{2\hat{\lambda}_0}{\hbar x} + O(x^0) & \text{when } x \rightarrow 0^\dagger \end{cases}$$

near the origin. Here $A(\hbar)$, which is independent of x , can be determined by looking the behavior of coefficients of (C.4) near the origin; the explicit form is given by

$$(C.13) \quad A(\hbar) = \frac{(2\hat{\lambda}_0 + 2\hat{\lambda}_1 + 2\hat{\lambda}_\infty + \hbar)(2\hat{\lambda}_0 + 2\hat{\lambda}_1 - 2\hat{\lambda}_\infty + \hbar)}{4\hbar(2\hat{\lambda}_0 + \hbar)}.$$

(Precisely speaking, since $S_{\text{Gauss}}(x, \hbar)$ is a formal series of $\hbar$, so does $A(\hbar)$. Therefore, $A(\hbar)$ should be understood as the Taylor expansion of the right hand-side of the formula (C.13).)

C.2.2 Derivation of contiguity relation for the Voros coefficients

Now we use Theorem C.1 to derive the contiguity relations satisfied by the Voros coefficient. It follows from the first relation in Theorem C.1 that

$$(C.14) \quad \partial_x \psi_{\text{Gauss}}(x, \nu_0, \nu_1) = \text{const} \times \psi_{\text{Gauss}}(x, \nu_0 - 1, \nu_1 - 1)$$

holds. (In this proof we omit the dependence on other variables since we are mainly interested in dependences on ν_0 and ν_1 .) Then, the logarithmic derivative S_{Gauss} of ψ_{Gauss} satisfies

$$(C.15) \quad S_{\text{Gauss}}(x, \nu_0, \nu_1) - S_{\text{Gauss}}(x, \nu_0 - 1, \nu_1 - 1) = -\frac{d}{dx} \log S_{\text{Gauss}}(x, \nu_0, \nu_1).$$

Integrating the both sides from x_2 to x_1 , we have

$$(C.16) \quad V_{\text{Gauss}}^{x_1, x_2}(\nu_0, \nu_1) - V_{\text{Gauss}}^{x_1, x_2}(\nu_0 - 1, \nu_1 - 1) = -\log \left(\frac{S_{\text{Gauss}}(x_1, \nu_0, \nu_1)}{S_{\text{Gauss}}(x_2, \nu_0, \nu_1)} \right) \\ - \sum_{m \in \{-1, 0\}} \hbar^m \left(\int_{x_2}^{x_1} S_{\text{Gauss}, m}(x, \nu_0, \nu_1) dx - \int_{x_2}^{x_1} S_{\text{Gauss}, m}(x, \nu_0 - 1, \nu_1 - 1) dx \right).$$

In order to derive a difference equation satisfied by $V_{\text{Gauss}}^{(0)}$, let us look at the behavior of the right hand-side of (C.16) when $x_1 \rightarrow 0$ and $x_2 \rightarrow 0^\dagger$. Since $S_{\text{Gauss}, -1}(x)$ is independent of ν_i 's, we have $S_{\text{Gauss}, -1}(x, \nu_0, \nu_1) - S_{\text{Gauss}, -1}(x, \nu_0 - 1, \nu_1 - 1) = 0$. Moreover, a straightforward computation shows

$$(C.17)$$

$$\begin{aligned} & \int^x S_{\text{Gauss}, 0}(x, \nu_0, \nu_1) dx - \int^x S_{\text{Gauss}, 0}(x, \nu_0 - 1, \nu_1 - 1) dx \\ &= -\frac{1}{2} \log \left(\frac{2\lambda_0 \sqrt{\lambda_\infty^2 x^2 - (\lambda_0^2 - \lambda_1^2 + \lambda_\infty^2)x + \lambda_0^2} - (\lambda_0^2 - \lambda_1^2 + \lambda_\infty^2)x + 2\lambda_0^2}{x^2} \right) \\ & \quad + \frac{1}{2} \log \left(-2\lambda_1 \sqrt{\lambda_\infty^2 x^2 - (\lambda_0^2 - \lambda_1^2 + \lambda_\infty^2)x + \lambda_0^2} + (\lambda_0^2 - \lambda_1^2 - \lambda_\infty^2)x - (\lambda_0^2 + \lambda_1^2 - \lambda_\infty^2) \right) \\ &= \begin{cases} \frac{1}{2} \log \left(-\frac{4\lambda_0^2}{(\lambda_0 + \lambda_1 - \lambda_\infty)(\lambda_0 + \lambda_1 + \lambda_\infty)} \right) + O(x^1) & \text{when } x \rightarrow 0, \\ \frac{1}{2} \log \left(-\frac{(\lambda_0 + \lambda_1 - \lambda_\infty)(\lambda_0 + \lambda_1 + \lambda_\infty)}{4\lambda_0^2} x^2 \right) + O(x^1) & \text{when } x \rightarrow 0^\dagger. \end{cases} \end{aligned}$$

Therefore, the right hand-side of (C.16) coincides with

$$\begin{aligned}
(C.18) \quad & -\log A(\hbar) - \frac{1}{2} \log \left(-\frac{4\lambda_0^2}{(\lambda_0 + \lambda_1 - \lambda_\infty)(\lambda_0 + \lambda_1 + \lambda_\infty)} \right) \\
& + \log \left(-\frac{2\hat{\lambda}_0}{\hbar x_2} \right) + \frac{1}{2} \log \left(-\frac{(\lambda_0 + \lambda_1 + \lambda_\infty)(\lambda_0 + \lambda_1 - \lambda_\infty)}{4\lambda_0^2} x_2^2 \right) \\
& = -\log \left[\frac{\lambda_0^2}{\hat{\lambda}_0(\hat{\lambda}_0 + \hbar/2)} \frac{(\hat{\lambda}_0 + \hat{\lambda}_1 + \hat{\lambda}_\infty + \hbar/2)}{(\lambda_0 + \lambda_1 + \lambda_\infty)} \frac{(\hat{\lambda}_0 + \hat{\lambda}_1 - \hat{\lambda}_\infty + \hbar/2)}{(\lambda_0 + \lambda_1 - \lambda_\infty)} \right]
\end{aligned}$$

modulo terms which vanish in the limit $x_1 \rightarrow 0$ and $x_2 \rightarrow 0^\dagger$. Therefore, taking this limit in the both sides of (C.16), we obtain the desired relation (4.20). This completes the proof.

Remark C.2. Combining the equalities in Theorem C.1, we can derive

$$(C.19) \quad (x\hbar\partial_x - \mu + \hbar) \psi_{\text{Gauss}}(\nu_0, \nu_1, \nu_\infty) = \cos nt \times \psi_{\text{Gauss}}(\nu_0, \nu_1 - 1, \nu_\infty + 1),$$

$$(C.20) \quad ((x-1)\hbar\partial_x - \mu + \hbar) \psi_{\text{Gauss}}(\nu_0, \nu_1, \nu_\infty) = \cos nt \times \psi_{\text{Gauss}}(\nu_0 - 1, \nu_1, \nu_\infty + 1)$$

as counterparts of (C.15). Then we can perform a similar calculation as above to derive the other equalities (4.21) and (4.22) in Lemma 4.5. We omit the details.

C.3 Proof of Lemma 4.11

Here we only state some lemmas which we use to prove Lemma 4.11 and omit the details because we can prove Lemma 4.11 in a similar manner to §C.2. In this subsection we omit the dependence on other variables since we are mainly interested in dependences on $\nu_{\infty,0}$ and $\nu_{\infty,1}$.

Lemma C.3. *Let $\infty^{(j)}$ ($j = 1, 2, 3$) are preimages of $x = \infty$ by the projection $\Sigma \rightarrow \mathbb{P}^1$, or in other words, for each $j \in \{1, 2, 3\}$, $\infty^{(j)}$ corresponds to $\beta_{\infty,j}$ on z -plane. Then, a straightforward computation shows*

$$(C.21) \quad S_{(1,1,3),-1}(x) = \begin{cases} -\frac{\lambda_{\infty,0}}{x} + O(x^{-2}) & \text{when } x \rightarrow \infty^{(1)}, \\ 1 - \frac{\lambda_{\infty,1}}{x} + O(x^{-2}) & \text{when } x \rightarrow \infty^{(2)}, \\ -\frac{x}{t^2} + \frac{1}{2} + O(x^{-1}) & \text{when } x \rightarrow \infty^{(3)}, \end{cases}$$

$$(C.22) \quad S_{(1,1,3),0}(x) = \begin{cases} \frac{\nu_{\infty,0} - 1}{x} + O(x^{-2}) & \text{when } x \rightarrow \infty^{(1)}, \\ \frac{\nu_{\infty,1} - 1}{x} + O(x^{-2}) & \text{when } x \rightarrow \infty^{(2)}, \\ -\frac{\nu_{\infty,0} + \nu_{\infty,1}}{x} + O(x^{-2}) & \text{when } x \rightarrow \infty^{(3)}, \end{cases}$$

$$(C.23) \quad S_{(1,1,3),m}(x) = O(x^{-2}) \quad \text{when } x \rightarrow \infty^{(j)} \quad (j = 1, 2, 3) \quad (m \geq 1).$$

Lemma C.4. *For the (1,1,3) equation*

$$(C.24) \quad \partial_x \psi_{(1,1,3)}(\nu_{\infty,0}, \nu_{\infty,1}) = \cos nt \times \psi_{(1,1,3)}(\nu_{\infty,0} - 1, \nu_{\infty,1}),$$

$$(C.25) \quad (-\hbar \partial_x + 1) \psi_{(1,1,3)}(\nu_{\infty,0}, \nu_{\infty,1}) = \cos nt \times \psi_{(1,1,3)}(\nu_{\infty,0}, \nu_{\infty,1} - 1)$$

hold.

From (C.24) the logarithmic derivative $S_{(1,1,3)}$ of $\psi_{(1,1,3)}$ satisfies

$$(C.26) \quad S_{(1,1,3)}(x, \nu_{\infty,0}, \nu_{\infty,1}) - S_{(1,1,3)}(x, \nu_{\infty,0} - 1, \nu_{\infty,1}) = -\frac{d}{dx} \log S_{(1,1,3)}(x, \nu_{\infty,0}, \nu_{\infty,1}).$$

Integrating the both sides from x_2 to x_1 , we have

$$(C.27) \quad V_{(1,1,3)}^{x_1, x_2}(\nu_{\infty,0}, \nu_{\infty,1}) - V_{(1,1,3)}^{x_1, x_2}(\nu_{\infty,0} - 1, \nu_{\infty,1}) = -\log \left(\frac{S_{(1,1,3)}(x_1, \nu_{\infty,0}, \nu_{\infty,1})}{S_{(1,1,3)}(x_2, \nu_{\infty,0}, \nu_{\infty,1})} \right) \\ - \sum_{m \in \{-1, 0\}} \hbar^m \left(\int_{x_2}^{x_1} S_{(1,1,3),m}(x, \nu_{\infty,0}, \nu_{\infty,1}) dx - \int_{x_2}^{x_1} S_{(1,1,3),m}(x, \nu_{\infty,0} - 1, \nu_{\infty,1}) dx \right).$$

By using Lemma C.3 we get

$$(C.28) \quad \int^x S_{(1,1,3),0}(x, \nu_{\infty,0}, \nu_{\infty,1}) dx - \int^x S_{(1,1,3),0}(x, \nu_{\infty,0} - 1, \nu_{\infty,1}) dx \\ = \begin{cases} -\log \left(-\frac{\lambda_{\infty,0}}{x} \right) + O(x^{-1}) & \text{when } x \rightarrow \infty^{(1)}, \\ -\log \left(-\frac{x}{t^2} \right) + O(x^{-1}) & \text{when } x \rightarrow \infty^{(3)}. \end{cases}$$

Therefore, the right hand-side of (C.27) coincides with

$$(C.29) \quad -\log \left\{ \frac{x_1^2 x_2}{t^2} \right\} + \log \{x_1 (\lambda_{\infty,0} - (\nu_{\infty,0} - 1)\hbar)\} + \log \left\{ -\frac{x_1}{t^2} \right\} - \log \left\{ -\frac{\lambda_{\infty,0}}{x_2} \right\} \\ = \log \left\{ 1 - \frac{(\nu_{\infty,0} - 1)\hbar}{\lambda_{\infty,0}} \right\}$$

modulo terms which vanish in the limit $x_1 \rightarrow \infty^{(3)}$ and $x_2 \rightarrow \infty^{(1)}$. Therefore, taking this limit in the both sides of (C.27), we obtain the desired relation (4.63).

D Bernoulli numbers and difference equations

Here we summarize several useful formulas of Bernoulli numbers which are applied to solve difference equations satisfied by free energies and Voros symbols in §4.

D.1 Definitions of Bernoulli numbers and Bernoulli polynomials

The Bernoulli number $\{B_n\}_{n \geq 0}$ can be defined through the generating function as

$$(D.1) \quad \frac{w}{e^w - 1} = \sum_{n=0}^{\infty} B_n \frac{w^n}{n!}.$$

From this definition we find

$$(D.2) \quad B_0 = 1, \quad B_1 = -\frac{1}{2}, \quad B_2 = \frac{1}{6}, \quad B_4 = -\frac{1}{30}, \quad B_{2k+1} = 0 \text{ for } k \geq 1.$$

The Bernoulli polynomials, which we denote by $\{B_n(t)\}_{n \geq 0}$, can also be defined through the generating function:

$$(D.3) \quad \frac{we^{tw}}{e^w - 1} = \sum_{n=0}^{\infty} B_n(t) \frac{w^n}{n!}.$$

Other useful expressions are also known, e.g.,

$$(D.4) \quad B_n(t) = \sum_{k=0}^n \binom{n}{k} B_{n-k} t^k.$$

Important relations for the Bernoulli polynomials are

$$(D.5) \quad B_n(1-t) = (-1)^n B_n(t),$$

$$(D.6) \quad (-1)^n B_n(-t) = B_n(t) + nt^{n-1},$$

$$(D.7) \quad B_n(mx) = m^{n-1} \sum_{k=0}^{m-1} B_n\left(x + \frac{k}{m}\right) \quad (m \geq 1).$$

We will also use formulas about special values of the Bernoulli polynomials:

$$(D.8) \quad B_n(0) = B_n,$$

$$(D.9) \quad B_n\left(\frac{1}{2}\right) = \left(\frac{1}{2^{n-1}} - 1\right) B_n.$$

We use these Bernoulli numbers and Bernoulli polynomials to give particular solutions of some difference equations. Its basic idea is due to [AT].

Proposition D.1.

$$(i) \quad \frac{1}{e^w - 1} = \frac{1}{w} + \sum_{n=0}^{\infty} \frac{B_{n+1}}{n+1} \frac{w^n}{n!}.$$

$$(ii) \quad \frac{e^w}{(e^w - 1)^2} = \frac{1}{w^2} - \sum_{n=0}^{\infty} \frac{B_{n+2}}{n+2} \frac{w^n}{n!}.$$

Proof. The claim (i) follows from (D.1). By differentiating (i) with respect to w , we get (ii). $\square$

Proposition D.2.

$$(i) \quad \frac{e^{tw}}{e^w - 1} = \frac{1}{w} + \sum_{n=0}^{\infty} \frac{B_{n+1}(t)}{n+1} \frac{w^n}{n!}.$$

$$(ii) \quad \frac{e^{(1+t)w}}{(e^w - 1)^2} = \frac{1}{w^2} + \frac{t}{w} + \sum_{n=0}^{\infty} \left\{ t \frac{B_{n+1}(t)}{n+1} - \frac{B_{n+2}(t)}{n+2} \right\} \frac{w^n}{n!}.$$

Proof. The first statement is a direct consequence of (D.3). After differentiating (i) with respect to w , we have

$$(D.10) \quad \frac{te^{tw}}{e^w - 1} - \frac{e^{(t+1)w}}{(e^w - 1)^2} = -\frac{1}{w^2} + \sum_{n=0}^{\infty} \frac{B_{n+2}(t)}{n+2} \frac{w^n}{n!}.$$

Then we substitute (i) into the first term of the right-hand side to obtain (ii). $\square$

D.2 Application to difference equations

Let X be the difference operator defined by

$$(D.11) \quad X = e^{-\hbar\partial_\lambda} (e^{\hbar\partial_\lambda} - 1)^2 \quad : \quad F(\lambda) \mapsto F(\lambda + \hbar) - 2F(\lambda) + F(\lambda - \hbar).$$

In §4.1.2.3 we need to solve difference equations of the form $XF(\lambda) = R(\lambda)$ to obtain the explicit formula for free energies. For the purpose, we use

Proposition D.3. *Particular solutions of a difference equation*

$$(D.12) \quad XF(\lambda) = F(\lambda + \hbar) - 2F(\lambda) + F(\lambda - \hbar) = R(\lambda)$$

for

(i) $R(\lambda) = \log(\lambda + \mu)$ (where μ is independent of λ), and

(ii) $R(\lambda) = 2\log(2\lambda) + \log(2\lambda + \hbar) + \log(2\lambda - \hbar)$,

are respectively given by

$$(i) \quad F(\lambda) = (\hbar\partial_\lambda)^{-2} \log(\lambda + \mu) - \frac{1}{12} \log(\lambda + \mu) + \sum_{n=2}^{\infty} \frac{B_{2n}}{2n(2n-2)} \left(\frac{\hbar}{\lambda + \mu} \right)^{2n-2}.$$

$$(ii) \quad F(\lambda) = 4(\hbar\partial_\lambda)^{-2} \log(2\lambda) - \frac{1}{12} \log \lambda + \sum_{n=2}^{\infty} \frac{B_{2n}}{2n(2n-2)} \left(\frac{\hbar}{2\lambda} \right)^{2n-2}.$$

Here $\partial_\lambda^{-1} f(\lambda)$ means an indefinite integral of $f(\lambda)$.

Proof. (i) It follows from

$$(D.13) \quad e^{-w}(e^w - 1)^2 \left\{ \frac{1}{w^2} - \sum_{n=0}^{\infty} \frac{B_{n+2}}{n+2} \frac{w^n}{n!} \right\} = 1$$

(cf. Proposition D.1 (ii)) that

$$(D.14) \quad e^{-\hbar\partial_\lambda}(e^{\hbar\partial_\lambda} - 1)^2 \left\{ (\hbar\partial_\lambda)^{-2} - \sum_{n=0}^{\infty} \frac{B_{n+2}}{n+2} \frac{(\hbar\partial_\lambda)^n}{n!} \right\} = \text{id}.$$

Hence we find that

$$(D.15) \quad \left\{ (\hbar\partial_\lambda)^{-2} - \sum_{n=0}^{\infty} \frac{B_{n+2}}{n+2} \frac{(\hbar\partial_\lambda)^n}{n!} \right\} \log(\lambda + \mu)$$

gives a solution of (D.12) for $R(\lambda) = \log(\lambda + \mu)$. Thus we have proved (i). (Note that the Bernoulli number B_n vanishes for odd $n \geq 3$ as we see in (D.2).)

(ii) Let us first solve auxiliary difference equations

$$(D.16) \quad XG(\lambda) = \log(2\lambda),$$

$$(D.17) \quad XH_{\pm}(\lambda) = \log(2\lambda \pm \hbar).$$

By a similar method used in the proof of (i), we can verify that

$$(D.18) \quad G(\lambda) = (\hbar\partial_\lambda)^{-2} \log(2\lambda) - \frac{1}{12} \log(2\lambda) + \sum_{n=2}^{\infty} \frac{B_{2n}}{2n(2n-2)} \left(\frac{\hbar}{\lambda} \right)^{2n-2}$$

satisfies (D.16). On the other hand, (D.17) is equivalent to

$$(D.19) \quad e^{-(1 \pm \frac{1}{2})\hbar\partial_\lambda} (e^{\hbar\partial_\lambda} - 1)^2 H_{\pm}(\lambda) = \log(2\lambda).$$

Then, Proposition D.2 (ii) with $t = \pm 1/2$ implies that the formal series

$$(D.20) \quad H_{\pm}(\lambda) = (\hbar\partial_\lambda)^{-2} \log(2\lambda) \pm \frac{1}{2} (\hbar\partial_\lambda)^{-1} \log(2\lambda) + \frac{1}{24} \log(2\lambda) - \sum_{n=1}^{\infty} \left\{ \pm \frac{B_{n+1}(\pm 1/2)}{2n(n+1)} - \frac{B_{n+2}(\pm 1/2)}{n(n+2)} \right\} \left(-\frac{\hbar}{\lambda} \right)^n$$

is a particular solution of (D.17). Therefore, the formal series $F(\lambda) = 2G(\lambda) + H_+(\lambda) + H_-(\lambda)$ gives a particular solution of the original problem $XF(\lambda) = 2\log(2\lambda) + \log(2\lambda + \hbar) + \log(2\lambda - \hbar)$. Then, the claim (ii) follows from the identities (D.6) and (D.9) of Bernoulli polynomials.

□

The following result is useful to obtain an explicit expression of Voros coefficients.

Proposition D.4. *Particular solutions of a difference equation*

$$(D.21) \quad (e^{\hbar\partial_\lambda} - 1) F(\lambda) = e^{(1-t)\hbar\partial_\lambda/2} R(\lambda)$$

for

(i) $R = \log(\lambda + \mu)$ (where μ is independent of λ), and

(ii) $R = 2\log(2\lambda) + \log(2\lambda + \hbar) + \log(2\lambda - \hbar)$

are respectively given by

$$\begin{aligned} \text{(i)} \quad F(\lambda) &= (\hbar \partial_\lambda)^{-1} \log(\lambda + \mu) - \frac{t}{2} \log(\lambda + \mu) + \sum_{m=1}^{\infty} \frac{\hbar^m}{m(m+1)} \frac{B_{m+1}((t+1)/2)}{(\lambda + \mu)^m}, \\ \text{(ii)} \quad F(\lambda) &= 4(\hbar \partial_\lambda)^{-1} \log(2\lambda) - 2t \log(2\lambda) + \sum_{m=1}^{\infty} \frac{\hbar^m}{m(m+1)} \frac{B_{m+1}(t) + B_{m+1}(t+1)}{(2\lambda)^m}. \end{aligned}$$

Proof. The claim (i) follows directly from Proposition D.2 (i) and (D.5). The second claim (ii) can be verified by a similar argument used in the proof of Proposition D.3 (ii). $\square$

In the last, we study solutions of homogeneous difference equations to discuss a uniqueness of solutions.

Proposition D.5. *A formal power series solution $F(\lambda) = \sum_{n \geq 0} F_n(\lambda) \hbar^n$ of a difference equation*

$$(D.22) \quad F(\lambda + \hbar) - F(\lambda) = 0$$

should satisfy $\partial_\lambda F_n = 0$ ($n \geq 0$). Here we regard $\hbar$ as a small parameter and $F_n(\lambda + \hbar)$ for $n \geq 0$ is defined by the Taylor expansion:

$$(D.23) \quad F_n(\lambda + \hbar) = \sum_{k=0}^{\infty} \frac{\hbar^k}{k!} \frac{d^k}{d\lambda^k} F_n(\lambda).$$

Proof. From (D.23), we have

$$(D.24) \quad F(\lambda + \hbar) - F(\lambda) = \sum_{n=0}^{\infty} \sum_{k=0}^{n-1} \frac{\hbar^n}{(n-k)!} \frac{d^{n-k}}{d\lambda^{n-k}} F_k.$$

Hence $F(\lambda + \hbar) - F(\lambda) = 0$ implies

$$\frac{d}{d\lambda} F_0 = 0, \quad \frac{d}{d\lambda} F_1 + \frac{1}{2!} \frac{d^2}{d\lambda^2} F_0 = 0, \quad \frac{d}{d\lambda} F_2 + \frac{1}{2!} \frac{d^2}{d\lambda^2} F_1 + \frac{1}{3!} \frac{d^3}{d\lambda^3} F_0 = 0, \dots$$

and so on. Hence, by induction, we can show $\partial_\lambda F_n = 0$ for $n \geq 0$. $\square$

Proposition D.6. *A formal power series solution $F(\lambda) = \sum_{n \geq 0} F_n(\lambda) \hbar^n$ of a difference equation*

$$(D.25) \quad F(\lambda + \hbar) - 2F(\lambda) + F(\lambda - \hbar) = 0$$

should satisfy $\partial_\lambda^2 F_n = 0$ ($n \geq 0$).

This can be proved in the same way as Proposition D.5. We omit the details.

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