Effect of gravitational waves on magnons

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Asuka Ito

Contents

Abstract

In this thesis, we propose a novel method for detecting gravitational waves around GHz range with magnons. The magnons as corrective spin excitations have been studied extensively in the field of the cavity quantum electrodynamics both in theory and experiment. We investigate the possibility to use magnons for detecting gravitational waves. It is shown that gravitational waves can excite magnons. Therefore, gravitational waves can be probed by measuring resonance fluorescence of magnons. Moreover, in the process of deriving the interactions between gravitational waves and magnons, we reveal all possible gravitational effects on a non-relativistic fermion with a mass m in Fermi normal coordinates up to order of $1/m$. Finally, we give experimental upper limits on the amplitude of continuous gravitational waves around GHz range by utilizing the experimental results of resonance fluorescence of magnons. In terms of the spectral density of gravitational waves, the upper limits at 95 % C.L. are given by 7.5×10^{-19} [$\text{Hz}^{-1/2}$] at 14 GHz and 8.7×10^{-18} [$\text{Hz}^{-1/2}$] at 8.2 GHz, respectively.

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Asuka Ito

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Chapter 1

Introduction

In 2015, the gravitational wave interferometer detector LIGO [1] opened up full-blown multi-messenger astronomy and cosmology, where electromagnetic waves, gravitational waves, neutrinos, and cosmic rays are utilized to explore the universe. In future, as the history of electromagnetic wave astronomy tells us, multi-frequency gravitational wave observations will be required to boost the multi-messenger astronomy and cosmology.

The purpose of this thesis is to present a novel idea for extending the frequency frontier in gravitational wave observations and to report the first limit on GHz gravitational waves. As we will see below, there are experimental and theoretical motivations to probe GHz gravitational waves.

First, it is useful to review the current status of gravitational wave observations [2]. It should be stressed that there exists the lowest measurable frequency. Indeed, the lowest frequency we can measure is around 10^{-18} Hz below which the wave length of gravitational waves exceeds the current Hubble horizon. Measuring the temperature anisotropy and the B-mode polarization of the cosmic microwave background [3, 4], we can probe gravitational waves with frequencies between 10^{-18} Hz and 10^{-16} Hz. Astrometry of extragalactic radio sources is sensitive to gravitational waves with frequencies between 10^{-16} Hz and 10^{-9} Hz [5, 6]. The pulsar timing arrays, like EPTA [7, 8] and NANOGrav [9], observe the gravitational waves in the frequency band from 10^{-9} Hz to 10^{-7} Hz. Doppler tracking of a space craft, which uses a measurement similar to the pulsar timing arrays, can search for gravitational waves in the frequency band from 10^{-7} Hz to 10^{-3} Hz [10]. The space interferometers LISA [11] and DECIGO [12] can cover the range between 10^{-3} Hz and 10 Hz. The interferometer detectors LIGO [13], Virgo [14], and KAGRA [15] with km size arm lengths can search for gravitational waves with frequencies from 10 Hz to 1 kHz. In this frequency band, resonant bar experiments [16] are complementary to the interferometers [17]. Furthermore, interferometers can be used to measure gravitational waves with the frequencies between 1 kHz and 100 MHz. In fact, recently, the limit on gravitational waves at MHz was reported [18]. To our best knowledge, the measurement of 100 MHz gravitational waves with a 0.75m arm length interferometer [19] is the highest frequency gravitational wave experiment to date. Thus, the frequency range higher than 100 MHz is remaining to be explored. Given this experimental situation, experiments for GHz gravitational waves are desired to extend the frequency frontier.

Theoretically, GHz gravitational waves are interesting from various points of view. As is well known, inflation can produce primordial gravitational waves. Among the features of primordial gravitational waves, the most clear signature is the break of the spectrum, determined by the energy scale of inflation, which locates at around GHz. Moreover, corresponding to the end of inflation or just after inflation, there may be a high frequency peak of gravitational waves [20, 21]. Remarkably, there is a chance to observe non-classical nature of primordial gravitational waves with frequency between MHz and GHz [22]. On the other hand, there are many astrophysical sources producing high frequency gravitational waves [23]. Among them, primordial black holes may be the most interesting one because they give rise to a hint of information loss problem. Exotic signals from extra dimensions may exist in the GHz band [24, 25]. Hence, GHz gravitational waves could be a window to the extra dimensions [26]. Therefore, it is worth investigating GHz gravitational waves to understand the astrophysical process, the early universe, and quantum gravity.

In this thesis, we propose a novel method for detecting GHz gravitational waves with a magnon detector, based on our paper [27]. The thesis is organized as follows. In the chapter 2, we review a formulation of gravitational waves as perturbations of a spacetime metric. Observables of gravitational waves are explained while introducing the energy of gravitational waves. In the chapter 3, we introduce a proper reference frame, which is a coordinate used in real experiments. Also, we explain how to treat the effect of Earth's gravity on the proper reference frame. It will turn out Earth's gravity is negligible in our discussion. In the chapter 4, we study the Dirac equation in curved spacetime in order to investigate effects of gravitational waves on a fermion. We will see, by taking the non-relativistic limit, that gravitational waves can cause spin resonance of the fermion. Furthermore, all possible gravitational interactions with a non-relativistic fermion (mass m) in Fermi normal coordinates up to order of $1/m$ are found. In the chapter 5, we first explain what a magnon is. Moreover, it is shown that gravitational waves excite magnons in a ferromagnetic insulator in the presence of external magnetic fields. In the chapter 6, using experimental results of measurements of resonance fluorescence of magnons, we give upper limits on the spectral density of gravitational waves, 7.5×10^{-19} [$\mathrm{Hz}^{-1/2}$] at 14 GHz and 8.7×10^{-18} [$\mathrm{Hz}^{-1/2}$] at 8.2 GHz, respectively. Finally, discussion and future prospects are given in the section 6.3.

Chapter 2

Gravitational waves

In this chapter, we derive the wave equation for metric perturbations by expanding the Einstein equation around the Minkowski spacetime up to linear order. It will turn out that the propagating degrees of freedom are nothing but the gravitational waves. Furthermore, we will define the energy and several characteristic parameters of gravitational waves in the following sections.

The discussion in the section 2.1 is based on [28] and the definition of the variables in the section 2.3 follows [16].

2.1 The wave equation of gravitational waves

Let us consider the Einstein equation at a vacuum. It is given by

$$R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} = 0 \ , \tag{2.1}$$

where $g_{\mu\nu}$ is a metric, $R_{\mu\nu}$ is the Ricci tensor and R is the Ricci scalar. As a solution of Eq. (2.1), our universe is described by the Minkowski spacetime, $g_{\mu\nu} = \eta_{\mu\nu}$, when there are no matters. Now we assume that there are small perturbations on the Minkowski spacetime and consider a small perturbation of the metric as

$$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu} \ . \tag{2.2}$$

Then the inverse of the metric is given by

$$g^{\mu\nu} = \eta^{\mu\nu} - h^{\mu\nu} \ , \tag{2.3}$$

because $g^{\mu\alpha}g_{\alpha\nu} \simeq \delta^\mu_\nu$ at linear order. We now study the solution of the Einstein equation (2.1) at linear order under the metric ansatz (2.2). From Eq. (2.2), one can calculate the Christoffel symbol up to linear order as

$$\begin{aligned} \Gamma^\alpha_{\beta\gamma} &= \frac{1}{2} g^{\alpha\delta}(g_{\beta\delta,\gamma} + g_{\gamma\delta,\beta} - g_{\beta\gamma,\delta}) \\ &\simeq \frac{1}{2}\eta^{\alpha\delta}(h_{\beta\delta,\gamma} + h_{\gamma\delta,\beta} - h_{\beta\gamma,\delta}) \\ &= \frac{1}{2}(h^\alpha{}_{\beta,\gamma} + h^\alpha{}_{\gamma,\beta} - h_{\beta\gamma}{}^{,\alpha}) \ . \end{aligned} \tag{2.4}$$

Here we can raise and lower the index of $h_{\mu\nu}$ by $\eta_{\mu\nu}$. Thus, $h_{\mu\nu}$ can be treated as a tensor on the flat spacetime. The Riemann tensor is given by

$$\begin{aligned} R^\alpha_{\mu\beta\nu} &\simeq \Gamma^\alpha_{\mu\nu,\beta} - \Gamma^\alpha_{\mu\beta,\nu} \\ &= \frac{1}{2}(h^\alpha{}_{\mu,\nu} + h^\alpha{}_{\nu,\mu} - h_{\mu\nu}{}^{,\alpha})_{,\beta} - \frac{1}{2}(h^\alpha{}_{\mu,\beta} + h^\alpha{}_{\beta,\mu} - h_{\mu\beta}{}^{,\alpha})_{,\nu} \\ &= \frac{1}{2}(h^\alpha{}_{\nu,\mu\beta} - h_{\mu\nu}{}^{,\alpha}{}_{,\beta} - h^\alpha{}_{\beta,\mu\nu} + h_{\mu\beta}{}^{,\alpha}{}_{,\nu}) \ . \end{aligned} \tag{2.5}$$

The Ricci tensor is

$$\begin{aligned} R_{\mu\nu} &= \frac{1}{2}(h^\alpha{}_{\nu,\mu\alpha} - h_{\mu\nu}{}^{,\alpha}{}_{,\alpha} - h^\alpha{}_{\alpha,\mu\nu} + h_{\mu\alpha}{}^{,\alpha}{}_{,\nu}) \\ &= \frac{1}{2}(h^\alpha{}_{\nu,\mu\alpha} - \Box h_{\mu\nu} - h_{,\mu\nu} + h_{\mu\alpha}{}^{,\alpha}{}_{,\nu}) \ , \end{aligned} \tag{2.6}$$

where $\Box = \partial^\alpha\partial_\alpha$, $h = h^\alpha_\alpha$. Also the Ricci scalar is

$$R = h^{\alpha\beta}{}_{,\alpha\beta} - \Box h \ . \tag{2.7}$$

Finally we obtain the Einstein tensor as

$$\begin{aligned} G_{\mu\nu} &= R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} \\ &= \frac{1}{2}\left[h^\alpha{}_{\nu,\mu\alpha} - \Box h_{\mu\nu} - h_{,\mu\nu} + h_{\mu\alpha}{}^{,\alpha}{}_{,\nu} + (-h^{\alpha\beta}{}_{,\alpha\beta} + \Box h)\eta_{\mu\nu}\right] \ . \end{aligned} \tag{2.8}$$

Therefore, the Einstein equation around vacua at linear order is given by

$$h^{\alpha}{}_{\nu,\mu\alpha} - \Box h_{\mu\nu} - h_{,\mu\nu} + h_{\mu\alpha}{}^{,\alpha}{}_{,\nu} + (-h^{\alpha\beta}{}_{,\alpha\beta} + \Box h)\eta_{\mu\nu} = 0 \ . \tag{2.9}$$

Before solving the equation, we have to consider the gauge freedom, indeed, there remain unphysical degree of freedoms in $h_{\mu\nu}$. Why does the gauge freedom exist and how do we deal with it mathematically?

Recall that we separated the metric into the background one $\eta_{\mu\nu}$ and the perturbation $h_{\mu\nu}$. However there exists arbitrariness how to map a point in the background spacetime to a point in the perturbed spacetime and it gives rise to the gauge freedom. Choosing a particular mapping is called a gauge fixing and transformation among other gauges is called a gauge transformation. It is important to deal with the gauge freedom appropriately, otherwise one may misunderstand a gauge artifact as a physical observable.

There are two ways to solve the gauge problem, one is to use gauge invariant variables and the other is to fix the gauge. The former is that we find gauge invariant variables which made by linear combination of initial variables and solve the equations about them. The latter is that we fix the gauge freedom completely and solve the equations about the remaining physical variables. When we fix the gauge, we have to choose an appropriate gauge fixing to simplify the equations. It is usual that we choose a coordinate where the physics looks like simple. In the following, we will learn the gauge fixing method, to do so, we first explain how to define the gauge transformation.

Let us consider an infinitesimal transformation of the coordinate:

$$x'^{\mu} = x^{\mu} - \xi^{\mu}(x) \ . \tag{2.10}$$

Then the metric is transformed as

$$\begin{aligned} g'_{\mu\nu}(x') &= \frac{\partial x^{\alpha}}{\partial x'^{\mu}}\frac{\partial x^{\beta}}{\partial x'^{\nu}} g_{\alpha\beta}(x) \\ &\simeq (\delta^{\alpha}{}_{\mu} + \xi^{\alpha}{}_{,\mu}(x))(\delta^{\beta}{}_{\nu} + \xi^{\beta}{}_{,\nu}(x)) g_{\alpha\beta}(x) \\ &\simeq g_{\mu\nu}(x) + g_{\mu\beta}\xi^{\beta}{}_{,\nu}(x) + g_{\alpha\nu}\xi^{\alpha}{}_{,\mu}(x) \ . \end{aligned} \tag{2.11}$$

Moreover we pull back the metric from the coordinate x' to x,

$$\begin{aligned} g'_{\mu\nu}(x') &= g'_{\mu\nu}(x-\xi) \\ &\simeq g'_{\mu\nu}(x) - \frac{\partial g'_{\mu\nu}(x)}{\partial x^\alpha}\xi^\alpha \\ &\simeq g'_{\mu\nu}(x) - g_{\mu\nu,\alpha}(x)\xi^\alpha \ . \end{aligned} \tag{2.12}$$

From Eqs. (2.11) and (2.12), we obtain

$$g'_{\mu\nu}(x) - g_{\mu\nu}(x) = g_{\mu\beta}(x)\xi^\beta{}_{,\nu} + g_{\alpha\nu}(x)\xi^\alpha{}_{,\mu} + g_{\mu\nu,\alpha}(x)\xi^\alpha \ . \tag{2.13}$$

This is the gauge transformation. It is the variation of the functional form of the metric by an infinitesimal transformation of the coordinate and the pull back. Note that this procedure is called the Lie dragging or the Lie derivative. After the gauge transformation, the perturbative metric becomes

$$\begin{aligned} h'_{\mu\nu}(x) &= g'_{\mu\nu}(x) - \eta_{\mu\nu} \\ &= g_{\mu\beta}\xi^\beta{}_{,\nu}(x) + g_{\alpha\nu}\xi^\alpha{}_{,\mu}(x) + g_{\mu\nu,\alpha}(x)\xi^\alpha + g_{\mu\nu}(x) - \eta_{\mu\nu} \\ &= g_{\mu\beta}\xi^\beta{}_{,\nu}(x) + g_{\alpha\nu}\xi^\alpha{}_{,\mu}(x) + g_{\mu\nu,\alpha}(x)\xi^\alpha + h_{\mu\nu}(x) \\ &= \xi_{\mu;\nu} + \xi_{\nu;\mu} + h_{\mu\nu}(x) \ . \end{aligned} \tag{2.14}$$

Corresponding to the arbitrariness of ξ, there are freedoms to define the perturbative metric $h_{\mu\nu}$ as deviation from the background metric $\eta_{\mu\nu}$. These freedoms are nothing but the gauge freedoms.[1]

Now we have learned what is the gauge freedom and how to describe it mathematically. Let us return to Eq. (2.9) and solve it with considering the gauge freedom. First, it is useful to define a new variable,

$$\tilde{h}_{\mu\nu} = h_{\mu\nu} - \frac{1}{2}\eta_{\mu\nu}h \ , \tag{2.15}$$

whose trace has inverse sign of the original one, i.e., $\tilde{h} = -h$. Using this variable, we can rewrite Eq. (2.9) as

$$\tilde{h}^\alpha{}_{\nu,\mu\alpha} - \Box\tilde{h}_{\mu\nu} + \tilde{h}_{\mu\alpha}{}^{,\alpha}{}_{,\nu} - \tilde{h}^{\alpha\beta}{}_{,\alpha\beta}\eta_{\mu\nu} = 0 \ . \tag{2.16}$$

[1]Notice that the linearized Riemann tensor (2.5) on flat spacetime background is invariant under the gauge transformation (2.14). This fact will be used in the chapter 4.

On the other hand, the gauge transformation of $\tilde{h}_{\mu\nu}$ is, from Eqs. (2.14) and (2.15),

$$\tilde{h}'_{\mu\nu} = \tilde{h}_{\mu\nu} + \xi_{\mu;\nu} + \xi_{\nu;\mu} - \eta_{\mu\nu}\xi^{\alpha}_{,\alpha} \ . \tag{2.17}$$

We now fix the gauge by taking the Lorentz gauge defined by

$$\tilde{h}'^{\alpha}_{\mu,\alpha} = 0 \ . \tag{2.18}$$

Then from Eq. (2.17), ξ must satisfies

$$\Box \xi^{\mu} = -\tilde{h}^{\mu,\alpha}_{\alpha} \ . \tag{2.19}$$

This is a non homogeneous wave equation and thus analytic solutions exist. However, adding a homogeneous solution to the solution is also a solution of Eq. (2.19). It implies that there still remains uncertainty of the gauge fixing, which is called the residual gauge. More precisely, the residual gauge is specified by the solution of

$$\Box \xi^{\mu} = 0 \ . \tag{2.20}$$

We will come back the problem of the residual gauge soon after. In the Lorentz gauge, the Einstein equation (2.16) is reduced as

$$\Box \tilde{h}_{\mu\nu} = 0 \ . \tag{2.21}$$

The homogeneous wave equation has plane wave solutions:

$$\tilde{h}^{\mu\nu} = A^{\mu\nu} e^{ik_{\alpha}x^{\alpha}} \ , \tag{2.22}$$

where $A_{\mu\nu} = A_{\nu\mu}$ and $k_{\alpha}k^{\alpha} = 0$. Moreover, from the gauge condition (2.18),

$$A^{\mu\alpha}k_{\alpha} = 0 \ . \tag{2.23}$$

This shows that the wave is transverse wave.

On the other hand, a residual gauge which satisfies the equation (2.20) is

$$\xi^{\mu} = B^{\mu} e^{ik_{\alpha}x^{\alpha}} \ . \tag{2.24}$$

We can transform $A^{\mu\nu}$ with the residual gauge as follows:

$$A'^{\mu\nu} = A^{\mu\nu} + i(B^{\mu}k^{\nu} + B^{\nu}k^{\mu} - \eta^{\mu\nu}B^{\alpha}k_{\alpha}) \ . \tag{2.25}$$

Then using the four components of B_{μ}, one can constrain $A_{\mu\nu}$ to be

$$A'^{0\mu} = 0 \ . \tag{2.26}$$

Note that now

$$A'^{\alpha}_{\ \alpha} = 0 \ , \tag{2.27}$$

is automatically satisfied[2] . The conditions (2.18), (2.26) and (2.27) are called the transverse traceless gauge. In this gauge, $h_{\mu\nu} = \tilde{h}_{\mu\nu}$, and therefore $h_{\mu\nu}$ in the transverse traceless gauge satisfies

$$h^{0\mu} = h^{ij}_{\ ,j} = h = 0 \ . \tag{2.28}$$

Above conditions reduce the freedoms of $h_{\mu\nu}$ by 8, so that the remaining physical freedoms are 2. These propagating physical degree of freedoms represent gravitational waves and 2 degree of freedoms are corresponding to the polarizations of the gravitational waves. For example, if we consider a gravitational wave propagating along z-direction, the components of the gravitational wave become

$$h_{\mu\nu} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & h_{+} & h_{\times} & 0 \\ 0 & h_{\times} & -h_{+} & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \tag{2.29}$$

It shows that space is distorted by the gravitational wave when it goes through and two kind of distortion occur corresponding to polarization modes h_{+} and $h_{\times}$.

2.2 Energy of gravitational waves

We saw that gravitational waves are propagating on spacetime as a solution of the perturbative Einstein equation. Gravitational waves carry energy and momentum as well as

[2]Instead, one can consider giving constraints $A'^{0i} = A'^{\alpha}_{\ \alpha} = 0$. Then from the transverse condition (2.23), $A'^{00}|k| - A'^{0i}k_i = 0$, and $A'^{0i} = 0$ give rise to $A'^{00} = 0$

ordinary waves like electromagnetic waves. We want to define the energy momentum tensor of gravitational waves, however, it is not so clear how to achieve it. Considering the energy of gravitational waves, the Einstein equation tells us that spacetime is bended:

$$G^{(\mathrm{GW})}_{\mu\nu} = 8\pi G\, T^{(\mathrm{GW})}_{\mu\nu} \ , \tag{2.30}$$

where $G^{(\mathrm{GW})}_{\mu\nu}$ is the Einstein tensor sourced by the energy momentum tensor of gravitational waves $T^{(\mathrm{GW})}_{\mu\nu}$.

Let us examine what $T^{(\mathrm{GW})}_{\mu\nu}$ means by evaluating each order of magnitudes of variables. First, a metric can be separated into a background one and a perturbative one as

$$g_{\mu\nu} = g^{(0)}_{\mu\nu} + h_{\mu\nu} \ . \tag{2.31}$$

Here order of each variables have been set as $g^{(0)}_{\mu\nu} \sim \mathcal{O}(1)$ and $h_{\mu\nu} \sim \mathcal{O}(\epsilon)$, respectively. ϵ ($\ll 1$) represents a certain small numerical value.

We now assume that $\frac{\lambda}{L} \ll 1$ where L is the radius of curvature of the background spacetime and λ is the wave length of gravitational waves. Then order of derivatives of the metrics are

$$\begin{cases} g^{(0)}_{\mu\nu,\alpha} \sim \mathcal{O}\left(\frac{1}{L}\right) \ , \\ g^{(0)}_{\mu\nu,\alpha\beta} \sim \mathcal{O}\left(\frac{1}{L^2}\right) \ , \\ h_{\mu\nu,\alpha} \sim \mathcal{O}\left(\frac{\epsilon}{\lambda}\right) \ , \\ h_{\mu\nu,\alpha\beta} \sim \mathcal{O}\left(\frac{\epsilon}{\lambda^2}\right) \ . \end{cases} \tag{2.32}$$

Expanding the Einstein equation at a vacuum in series of ϵ, we get

$$G_{\mu\nu}(g^{(0)}_{\mu\nu} + h_{\mu\nu}) \ = \ G^{(0)}_{\mu\nu} + G^{(1)}_{\mu\nu} + G^{(2)}_{\mu\nu} + \cdots = 0 \ . \tag{2.33}$$

Order of each terms are evaluated as follows:

$$\begin{cases} G^{(0)}_{\mu\nu} \sim \mathcal{O}\left(\frac{1}{L^2}\right) \ , \\ G^{(1)}_{\mu\nu} \sim \mathcal{O}\left(\frac{\epsilon}{\lambda^2}\right), \mathcal{O}\left(\frac{\epsilon}{L^2}\right), \mathcal{O}\left(\frac{\epsilon}{\lambda L}\right) \ , \\ G^{(2)}_{\mu\nu} \sim \mathcal{O}\left(\frac{\epsilon^2}{\lambda^2}\right), \mathcal{O}\left(\frac{\epsilon^2}{L^2}\right), \mathcal{O}\left(\frac{\epsilon^2}{\lambda L}\right) \ . \end{cases} \tag{2.34}$$

We now average the Einstein tensors over a scale l which satisfies $\lambda \ll l \ll L$ and then the background spacetime can be regarded as flat spacetime locally, namely, one can take

$g^{(0)}_{\mu\nu} = g^{(0)}_{\mu\nu,\alpha} = 0$. Therefore, the relations (2.34) are reduced to

$$\begin{cases} < G^{(0)} >_{\mu\nu} \sim \ \mathcal{O}(\frac{1}{L^2}) \ , \\ < G^{(1)} >_{\mu\nu} \sim \ \mathcal{O}(\frac{\epsilon}{\lambda^2}), \mathcal{O}(\frac{\epsilon}{L^2}) \ , \\ < G^{(2)} >_{\mu\nu} \sim \ \mathcal{O}(\frac{\epsilon^2}{\lambda^2}), \mathcal{O}(\frac{\epsilon^2}{L^2}) \ . \end{cases} \tag{2.35}$$

On the other hand, since gravitational waves are oscillating,

$$< G^{(1)}_{\mu\nu} >= 0 \ , \tag{2.36}$$

should hold. Finally, up to the second order of ϵ, Eq. (2.33) yields

$$\mathcal{O}(\frac{1}{L^2}) = \mathcal{O}(\frac{\epsilon^2}{\lambda^2}) \ \text{ or } \ \mathcal{O}(\frac{\epsilon^2}{L^2}) \ . \tag{2.37}$$

Since $\epsilon \sim 1$ is forbidden by the assumption $\epsilon \ll 1$, we conclude that

$$\lambda \sim \epsilon L \ . \tag{2.38}$$

In this case, from the relations (2.35), we observe that

$$\begin{cases} < G^{(0)}_{\mu\nu} >, < G^{(2)}_{\mu\nu} > \sim \ \mathcal{O}(\frac{1}{L^2}) \ , \\ < G^{(1)}_{\mu\nu} > \sim \ \mathcal{O}(\frac{1}{\epsilon L^2}) \ . \end{cases} \tag{2.39}$$

Now at each order of powers of ϵ, the Einstein equation (2.30) is given by following two equations:

$$\begin{cases} < G^{(1)}_{\mu\nu} > = \ 0 \ , \\ < G^{(0)}_{\mu\nu} > = \ - < G^{(2)}_{\mu\nu} > \ . \end{cases} \tag{2.40}$$

The first equation is for the wave equation of gravitational waves we derived in the previous section. On the other hand, comparing the second equation with Eq. (2.30), we find that the energy momentum tensor of the gravitational wave can be defined by

$$< T^{(\mathrm{GW})}_{\mu\nu} > \equiv \ -\frac{1}{8\pi G} < G^{(2)}_{\mu\nu} > \ . \tag{2.41}$$

It should be noted the fact that $T^{(\mathrm{GW})}_{\mu\nu}$ should variate for the scale of L and $G^{(2)}_{\mu\nu}$ variates for the scale of λ seems to be incompatible. However, they are reconciled by taking average over the intermediate scale l.

Now let us calculate Eq. (2.41) explicitly. The Einstein tensor can be written as

$$G_{\mu\nu} = (\delta^\alpha_\mu \delta^\beta{}_\nu - \frac{1}{2} g_{\mu\nu} g^{\alpha\beta}) R_{\alpha\beta} \ . \tag{2.42}$$

Then, remembering that $g^{\mu\nu} = g^{(0)\mu\nu} - h^{\mu\nu} + h^{\mu\alpha} h_\alpha{}^\nu$, the part of the second order perturbation of the Einstein tensor is

$$\begin{aligned} G^{(2)}_{\mu\nu} &= (\delta^\alpha_\mu \delta^\beta{}_\nu - \frac{1}{2} g^{(0)}_{\mu\nu} g^{(0)\alpha\beta}) R^{(2)}_{\alpha\beta} - \frac{1}{2}(h_{\mu\nu} g^{(0)\alpha\beta} - g^{(0)}_{\mu\nu} h^{\alpha\beta}) R^{(1)}_{\alpha\beta} \\ &\quad + \frac{1}{2}(h_{\mu\nu} h^{\alpha\beta} - g^{(0)}_{\mu\nu} h^{\alpha\gamma} h_\gamma{}^\beta) R^{(0)}_{\alpha\beta} \\ &\simeq (\delta^\alpha_\mu \delta^\beta{}_\nu - \frac{1}{2} g^{(0)}_{\mu\nu} g^{(0)\alpha\beta}) R^{(2)}_{\alpha\beta} \ . \end{aligned}$$

In the second equality, we have extracted the part of order of $\mathcal{O}(\frac{\epsilon^2}{\lambda^2})$ by regarding the background spacetime as flat one. Also, $R^{(2)}_{\mu\nu}$ is given by the Christoffel symbol as

$$R^{(2)}_{\mu\nu} = \Gamma^{(2)\alpha}_{\mu\nu,\alpha} - \Gamma^{(2)\alpha}_{\mu\alpha,\nu} + \Gamma^{(1)\alpha}_{\beta\alpha} \Gamma^{(1)\beta}_{\mu\nu} - \Gamma^{(1)\alpha}_{\beta\nu} \Gamma^{(1)\beta}_{\mu\alpha} \ . \tag{2.43}$$

When we take an average, $< \Gamma^{(2)\alpha}_{\mu\nu,\beta} >$ becomes order of $\mathcal{O}(\frac{\epsilon^2}{\lambda L})$ because it is evaluated by a surface integral and thus the term is negligible. Moreover, taking the transverse traceless gauge, $h^{\mu\nu}_{\mathrm{TT},\nu} = h_{\mathrm{TT}} = 0$, $\Gamma^{(1)\alpha}_{\beta\alpha}$ is zero. Therefore,

$$\begin{aligned} < R^{(2)}_{\mu\nu} >_{\mathrm{TT}} &= - < \Gamma^{(1)\alpha}_{\beta\nu} \Gamma^{(1)\beta}_{\mu\alpha} >_{\mathrm{TT}} \\ &= -\frac{1}{4} < (h^\alpha{}_{\nu,\beta} + h^\alpha{}_{\beta,\nu} - h_{\beta\nu}{}^{;\alpha})(h^\beta{}_{\alpha,\mu} + h^\beta{}_{\mu,\alpha} - h_{\mu\alpha}{}^{;\beta}) >_{\mathrm{TT}} \\ &= -\frac{1}{4} < -2 h_{\beta\nu}^{;\alpha} h^\beta_{\mu,\alpha} + h^\alpha_{\beta,\nu} h^\beta_{\alpha,\mu} >_{\mathrm{TT}} \\ &= -\frac{1}{4} < h^\alpha_{\beta,\nu} h^\beta_{\alpha,\mu} >_{\mathrm{TT}} \ . \end{aligned} \tag{2.44}$$

The Ricci scalar is

$$\begin{aligned} \eta^{\mu\nu} < R^{(2)}_{\mu\nu} >_{\mathrm{TT}} &= -\frac{1}{4} \eta^{\mu\nu} < h^\alpha_{\beta,\nu} h^\beta_{\alpha,\mu} >_{\mathrm{TT}} \\ &\simeq 0 \ , \end{aligned}$$

where we used the integration by parts and the equation of gravitational waves. We now have

$$
\begin{aligned}
< G^{(2)}_{\mu\nu} >_{\rm TT} &= (\delta^\alpha_\mu \delta^\beta_{\ \nu} - \frac{1}{2} g^{(0)}_{\mu\nu} g^{(0)\alpha\beta}) < R^{(2)}_{\alpha\beta} >_{\rm TT} \\
&= < R^{(2)}_{\mu\nu} >_{\rm TT} \\
&= -\frac{1}{4} < h^{\alpha\beta}_{\ \ ,\mu} h_{\alpha\beta,\nu} >_{\rm TT} \ . \qquad (2.45)
\end{aligned}
$$

From Eqs. (2.41) and (2.45), we obtain

$$T^{\rm (GW)}_{\mu\nu} = \frac{1}{32\pi G} < h^{\alpha\beta}_{\ \ ,\mu} h_{\alpha\beta,\nu} >_{\rm TT} \ . \qquad (2.46)$$

In a covariant way, it can be written

$$T^{\rm (GW)}_{\mu\nu} = \frac{1}{32\pi G} < h^{\alpha\beta}_{\ \ ;\mu} h_{\alpha\beta;\nu} >_{\rm TT} \ . \qquad (2.47)$$

In particular, the energy density of gravitational waves is

$$T^{\rm (GW)}_{00} = \frac{1}{32\pi G} < \dot{h}^{\alpha\beta} \dot{h}_{\alpha\beta} >_{\rm TT} \ . \qquad (2.48)$$

We will move on to the Fourier space and define several parameters to characterize gravitational waves in the next section.

2.3 Observables of gravitational waves

We introduce three parameters characterizing gravitational waves. Although they are not independent and are related with each other, we use them properly depending on the situation.

2.3.1 Spectral density $S_h(f)$

Let us consider gravitational waves, $h_{ij}(t, \vec{x})$, at a time t and a position $\vec{x}$. In the Minkowski spacetime, it can be expanded with plane waves as

$$h_{ij}(t, \vec{x}) = \sum_A \int_{-\infty}^{\infty} df \int d\hat{\Omega}\ e^{2\pi i \left(ft - |f| \hat{\Omega}\cdot\vec{x}\right)} \tilde{h}_A(f, \hat{\Omega}) e^A_{ij}(\hat{\Omega}) \ , \qquad (2.49)$$

where $\hat{\Omega}$ denotes the direction of propagation of a gravitational wave, A labels two polarizations and $e_{ij}^{A}(\hat{\Omega})$ is a polarization tensor which satisfies

$$e_{ij}^{(A)}e_{ij}^{(A')} = \delta_{AA'} \ . \tag{2.50}$$

Note that $\tilde{h}_A^*(f,\hat{\Omega}) = \tilde{h}_A(-f,-\hat{\Omega})$ because $h_{ij}(t,\vec{x})$ is a real valued function.

We now consider an ensemble average of the two point function of the Fourier coefficient of gravitational waves:

$$< \tilde{h}_A^*(f,\hat{\Omega})\tilde{h}_{A'}(f',\hat{\Omega}') >= \frac{1}{2}\delta(f-f')\frac{1}{4\pi}\delta(\hat{\Omega}-\hat{\Omega}')\delta_{AA'}2S_h(f) \ , \tag{2.51}$$

where we defined a variable $S_h(f)$ called the spectral density[3]. Notice that it has a dimension Hz^{-1}. We mention that we assumed homogeneity and isotropy of the background spacetime[4] and no polarizations of gravitational waves in Eq. (2.51). Furthermore, the factor 1/2 comes from the fact that the actual integration range of f is $0 \sim \infty$, the factor 1/4 is a normalization for the angular integral and the factor 2 in front of S_h is just a convention. From Eqs. (2.49)

[3]It is usually called the power spectrum apart from the difference of the coefficient, in particular, in statistics.

[4]Considerer an ensemble average of a two point function $< h(\vec{x}_1)h(\vec{x}_2) >$ and assume that it only depends on the distance of the two points, namely $\xi(|\vec{x}_1-\vec{x}_2|) =< h(\vec{x}_1)h(\vec{x}_2) >$. It's Fourier coefficient is

$$< \tilde{h}(f_1,\hat{\Omega}_1)\tilde{h}(f_2,\hat{\Omega}_2) > \ = \ \iint d^3x_1 d^3x_2 e^{2\pi i f_1(\hat{\Omega}_1\cdot\vec{x}_1)}e^{2\pi i f_2(\hat{\Omega}_2\cdot\vec{x}_2)}\xi(|\vec{x}_1-\vec{x}_2|) \ .$$

Using a new variable $\vec{x}=\vec{x}_1-\vec{x}_2$ instead of $\vec{x}_2$ and carrying out the integral with respect to $\vec{x}_1$, we obtain

$$< \tilde{h}(f_1,\hat{\Omega}_1)\tilde{h}(f_2,\hat{\Omega}_2) >= \delta(f_1\hat{\Omega}_1+f_2\hat{\Omega}_2)\int d^3x e^{-2\pi i f_2\hat{\Omega}_2\cdot\vec{x}}\xi(|\vec{x}|) \ .$$

The delta function is come from homogeneity of the background spacetime and therefore represents the momentum conservation. Furthermore, doing the angular integration,

$$< \tilde{h}(f_1,\hat{\Omega}_1)\tilde{h}(f_2,\hat{\Omega}_2) >= \delta(f_1\hat{\Omega}_1+f_2\hat{\Omega}_2)\times -\frac{2}{f_2}\int dxx\sin(2\pi f_2 x)\xi(|\vec{x}|) \ .$$

We see that the integrand does not depend on $\hat{\Omega}_2$, so that $< \tilde{h}(f_1,\hat{\Omega}_1)\tilde{h}(f_2,\hat{\Omega}_2) >$ is free from $\hat{\Omega}_2$. This is a consequence that the back ground spacetime is isotropic.

and (2.51), we have

$$\begin{aligned} < h^*_{ij}(t,\vec{x})h^{ij}(t,\vec{x}) > &= \sum_{A,A'}\iint_{-\infty}^{\infty} df df' \iint d\hat{\Omega}d\hat{\Omega}'\ e^{2\pi i(f-f')t}e^{-2\pi i(|f|\hat{\Omega}-|f'|\hat{\Omega}')\cdot\vec{x}} \\ &\quad \times e^{(A)}_{ij}(\hat{\Omega})e^{(A')ij}(\hat{\Omega}')\frac{1}{2}\delta(f-f')\frac{1}{4\pi}\delta(\hat{\Omega}-\hat{\Omega}')\delta_{AA'}S_h(f) \\ &= 2\int_{-\infty}^{\infty} df S_h(f) \\ &= 4\int_{f=0}^{f=\infty} d(\log f) f S_h(f)\ . \end{aligned} \tag{2.52}$$

2.3.2 Characteristic amplitudes $h_c(f)$

We define the characteristic amplitudes $h_c(f)$ as follows:

$$< h_{\mu\nu}(t,\vec{x})h^{\mu\nu}(t,\vec{x}) >= 2\int_{f=0}^{f=\infty} d(\log f) h_c^2(f)\ . \tag{2.53}$$

Notice that $h_c(f)$ is a dimensionless parameter. The factor 2 is for the number of polarizations. Comparing Eqs. (2.52) and (2.53), we find a relation

$$h_c^2(f) = 2fS_h(f)\ . \tag{2.54}$$

2.3.3 The energy density parameter

Finally, we define the energy density parameter $\Omega_{GW}(f)$ as

$$\Omega_{GW}(f) = \frac{1}{\rho_c}\frac{d\rho_{GW}}{d\log(f)}\ . \tag{2.55}$$

It is the energy density of gravitational waves divided by the critical density $\rho_c = \frac{3H_0^2}{8\pi G} = \frac{3c}{8\pi\hbar G}h_0^2 \times (100\text{km/s Mpc})^2$, which is the current energy density of the Universe. Note that $\Omega_{GW}(f)$ is a dimensionless parameter. Practically, $h_0^2\Omega_{GW}$ is used rather than Ω_{GW} because h_0 contains observational uncertainty. On the other hand, using Eq. (2.52) in Eq. (2.48), we have

$$\begin{aligned} \rho_{GW} &= \frac{1}{32\pi G} < \dot{h}_{\mu\nu}(t,\vec{x})\dot{h}^{\mu\nu}(t,\vec{x}) > \\ &= \frac{(2\pi)^2}{32\pi G}\, 4\int_{f=0}^{f=\infty} d(\log f) f^3 S_h(f)\ . \end{aligned} \tag{2.56}$$

Hence,

$$\frac{d\rho_{GW}}{d(\log(f))} = \frac{\pi}{2G} f^3 S_h(f) \tag{2.57}$$

$$= \frac{\pi}{4G} f^2 h_c^2(f) \ . \tag{2.58}$$

Therefore from Eq. (2.55), one obtain

$$\Omega_{GW}(f) = \frac{4\pi^2}{3H_0^2} f^3 S_h(f) \tag{2.59}$$

$$= \frac{2\pi^2}{3H_0^2} f^2 h_c^2(f) \ . \tag{2.60}$$

Of these observables characterizing gravitational waves, the most useful one is used according to each situation. However they are related with each other through Eqs. (2.59) and (2.60) and thus we can always convert from one to the others. An observation of gravitational waves mean the measurement of these variables.

Chapter 3

A proper reference frame

As we saw in the section 2.1, gravitational waves as the perturbations of the metric are propagating on the (flat) spacetime. Then, we chose a coordinate system where the metric satisfies the transverse traceless condition (2.28). However, if one wants to observe effects of gravitational waves with a certain detector, a coordinate system which is fixed with the detector, we call it the proper reference frame, should be used to examine the effects. Otherwise, we may get wrong conclusion since gravity is closely related to the coordinate system due to the equivalence principle. In the section 3.1, we introduce Fermi normal coordinates which origin is moving along a geodesic of a particle. We also investigate the effect of Earth's gravity on Fermi normal coordinates in the section 3.2.

I referred [29, 30, 31] at some parts in this chapter and they would also be helpful for readers.

3.1 Fermi normal coordinates

One can construct locally inertial coordinates along a geodesic of a particle, it is called Fermi normal coordinates [32]. An observer on the earth is freely falling assuming that the earth's gravity, which will be examind in the next section, is negligible, so that a Fermi normal coordinate corresponds to a frame which is used in a real experiment. In this section, we briefly review how to construct Fermi normal coordinates [32].

We consider a timelike geodesic γ_τ parametrized by a proper time τ and specify the points on the geodesic as $P(\tau)$. Moreover, we consider a spacelike geodesic γ_s orthogonal to γ_τ, which is parametrized by a proper distance s[5] and is crossing a point $P(\tau)$ on γ_τ when $s = 0$. The situation is illustrated in Fig. 3.1.

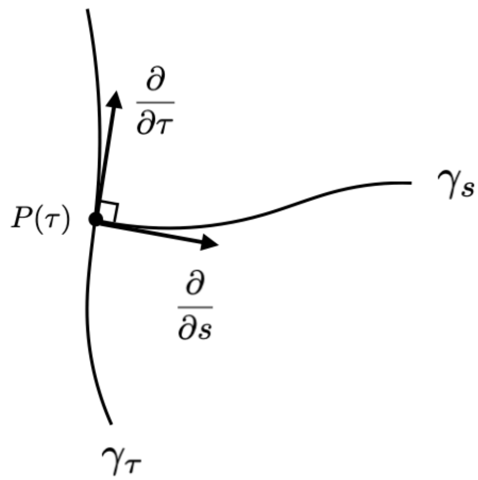

Figure 3.1: A timelike geodesic γ_τ parametrized by a proper time τ and a spacelike geodesic γ_s parametrized by a proper distance s, which is orthogonal to γ_τ, are illustrated.

Then, Fermi normal coordinates which is locally inertial frames along γ_τ are defined as follows:

$$x^0 = \tau, \quad x^i = \alpha^i s \ , \tag{3.1}$$

where the bases of Fermi normal coordinates, $\frac{\partial}{\partial x^\mu}$, are defined to be parallelly transformed along with γ_τ and α^i are components of the tangent vector $\frac{\partial}{\partial s}$ in Fermi normal coordinates, actually,

$$\frac{\partial}{\partial s} = \alpha^i \frac{\partial}{\partial x^i} \ . \tag{3.2}$$

Also, the bases, $\frac{\partial}{\partial x^\mu}$, are taken to be orthonormal by utilizing the arbitrariness of rescaling α^i. Thus in Fermi normal coordinates, a metric is given by $\eta_{\mu\nu}$ on γ_τ.[6]

[5] Although one can use affine parameters instead of s, it does not change the following discussion.

[6] Note that orthonormality is hold at every point on γ_τ if it is satisfied at one point on γ_τ, because a parallel transformation keeps orthonormality.

Let us show that Fermi normal coordinates (3.1) indeed are locally inertial frames, namely, the Christoffel symbols are zero on γ_τ. First, because the bases of Fermi normal coordinates are parallelly transformed along γ_τ, we have

$$\begin{aligned} 0 &= \left(\frac{\partial}{\partial x^\nu}\right)^\mu_{;\alpha}\left(\frac{\partial}{\partial \tau}\right)^\alpha \\ &= (\delta^\mu_\nu)_{;\alpha}\,\delta^\alpha_0 \\ &= \Gamma^\mu_{\nu 0}(x^0=\tau, x^i=0) \\ &= \Gamma^\mu_{\nu 0}|_{\gamma_\tau}\ , \end{aligned} \tag{3.3}$$

where we used the fact that vector components of the bases of Fermi normal coordinates are $\left(\frac{\partial}{\partial x^\nu}\right)^\mu = \delta^\mu_\nu$. On the other hand, on the spacelike geodesic γ_s, the geodesic equation

$$\frac{d^2x^\mu}{ds^2} + \Gamma^\mu_{\alpha\beta}\frac{dx^\alpha}{ds}\frac{dx^\beta}{ds} = 0\ , \tag{3.4}$$

is satisfied. Using (3.1) in Eq. (3.4), we obtain

$$\Gamma^\mu_{ij}(x^0=\tau, x^i=\alpha^i s)\,\alpha^i\alpha^j = 0\ . \tag{3.5}$$

In particular on γ_τ, namely at $s=0$, we conclude that

$$\Gamma^\mu_{ij}(x^0=\tau, x^i=0) = \Gamma^\mu_{ij}|_{\gamma_\tau} = 0\ . \tag{3.6}$$

Therefore, from Eqs. (3.3) and (3.6), we see that the christoffel symbols on the timelike geodesic γ_τ are all zero and thus Fermi normal coordinates are locally inertial frames along γ_τ.

Now our question is what the form of a metric in Fermi normal coordinates is. Locally inertial coordinates mean that considering an expansion of a metric in powers of the coordinates x^μ, a nonzero derivative term of the metric first appears at quadratic order. The quadratic term is the leading one, namely higher derivative terms are negligible, in a situation that a curvature scale is much larger than that of a system we treat, which is specified by the coordinates x^μ. The situation agrees with what we will consider in following chapters, so that we can ignore the higher derivative terms of the metric than quadratic. Therefore, in order to find the form of a metric in Fermi normal coordinates, it is enough to get the second derivative term of the metric for our purpose.

Because the second derivative of a metric is related to the first derivative of christoffel symbols, we investigate the latter one to reveal the former one. Since the christoffel symbols are all zero along the geodesic γ_τ, it implies

$$\Gamma^\mu_{\nu\lambda,0}|_{\gamma_\tau} = 0 \ . \tag{3.7}$$

Then, by the definition of a Riemann tensor, we find

$$\Gamma^\mu_{\nu 0,\lambda}|_{\gamma_\tau} = R^\mu_{\nu\lambda 0}|_{\gamma_\tau} \ . \tag{3.8}$$

To go further, we use the geodesic deviation equation (A.5):

$$\frac{d^2\xi^\mu}{d\lambda^2} + 2\frac{d\xi^\alpha}{d\lambda}\Gamma^\mu_{\alpha\beta}u^\beta + \left(R^\mu_{\alpha\gamma\beta} + \Gamma^\mu_{\alpha\gamma,\beta} + \Gamma^\mu_{\beta\delta}\Gamma^\delta_{\alpha\gamma} - \Gamma^\mu_{\gamma\delta}\Gamma^\delta_{\alpha\beta}\right) u^\alpha u^\beta \xi^\gamma = 0 \ , \tag{3.9}$$

where λ takes τ or s in general. We notice that a point on γ_s is specified by the parameters (τ, s, α^i). Then, as to the spacelike geodesic γ_s, one can consider two deviation vectors; one is $\left(\frac{\partial}{\partial\tau}\right)_{s,\alpha^i}$ and the other is $\left(\frac{\partial}{\partial\alpha^i}\right)_{\tau,s}$. $\left(\frac{\partial}{\partial\tau}\right)_{s,\alpha^i}$ represents a deviation between two spacelike geodesics which stem from different points on γ_τ and $\left(\frac{\partial}{\partial\alpha^i}\right)_{\tau,s}$ represents a deviation between two spacelike geodesics which stem from a same point $P(\tau)$ on γ_τ. Substituting $\xi^\mu = \left(\frac{\partial}{\partial\tau}\right)^\mu_{s,\alpha^i} = \delta^\mu_0$ into Eq. (3.9) yields

$$\left(\Gamma^\mu_{i0,j}|_{\gamma_\tau} - R^\mu_{ij0}|_{\gamma_\tau}\right)\alpha^j\alpha^k = 0 \ , \tag{3.10}$$

but we have already known that the inside of the parenthesis is 0 because of Eq. (3.8). On the other hand, substituting $\xi^\mu = \left(\frac{\partial}{\partial\alpha^i}\right)_{\tau,s} = s\delta^\mu_i$ into Eq. (3.9), we obtain

$$2\Gamma^\mu_{ij}\alpha^j + sR^\mu_{jik}|_{\gamma_\tau}\alpha^j\alpha^k + s\Gamma^\mu_{ij,k}|_{\gamma_\tau}\alpha^j\alpha^k + \mathcal{O}\left(s^2\right) = 0 \ . \tag{3.11}$$

The first term in Eq. (3.11) can be expanded in powers of s as

$$\begin{aligned} 2\Gamma^\mu_{ij}\alpha^j &= 2\Gamma^\mu_{ij}|_{\gamma_\tau}\alpha^j + 2s\left(\frac{\partial}{\partial s}\Gamma^\mu_{ij}\right)_{\text{at }\gamma_\tau}\alpha^j \\ &= 2s\Gamma^\mu_{ij,k}|_{\gamma_\tau}\alpha^j\alpha^k \ . \end{aligned} \tag{3.12}$$

From Eqs. (3.11) and (3.12), at order of s, we find that an equality

$$\left(\Gamma^{\mu}_{ij,k}|_{\gamma_\tau} + \frac{1}{3}R^{\mu}_{jik}|_{\gamma_\tau}\right)\alpha^j\alpha^k = 0\ , \tag{3.13}$$

holds. It implies that the symmetric part about indeies of j and k in the parenthesis should be zero, i.e.,

$$\Gamma^{\mu}_{ij,k}|_{\gamma_\tau} + \Gamma^{\mu}_{ik,j}|_{\gamma_\tau} = -\frac{1}{3}\left(R^{\mu}_{jik}|_{\gamma_\tau} + R^{\mu}_{kij}|_{\gamma_\tau}\right)\ . \tag{3.14}$$

After little algebras, this can be solved with respect to the derivative of the Christoffel symbol as

$$\Gamma^{\mu}_{ij,k}|_{\gamma_\tau} = -\frac{1}{3}\left(R^{\mu}_{ijk}|_{\gamma_\tau} + R^{\mu}_{jik}|_{\gamma_\tau}\right)\ . \tag{3.15}$$

Finally, we express the second derivative of the metric by the first derivative of the Christoffel symbols and then relations between the second derivative of the metric and the Riemann tensor are obtained. From the definition of the Christoffel symbol, we have

$$g_{\mu\nu,\lambda} = g_{\mu\alpha}\Gamma^{\alpha}_{\nu\lambda} + g_{\nu\alpha}\Gamma^{\alpha}_{\mu\lambda}\ . \tag{3.16}$$

Differentiating it with respect to x^σ leads

$$g_{\mu\nu,\lambda\sigma}|_{\gamma_\tau} = \eta_{\mu\alpha}\Gamma^{\alpha}_{\nu\lambda,\sigma}|_{\gamma_\tau} + \eta_{\nu\alpha}\Gamma^{\alpha}_{\mu\lambda,\sigma}|_{\gamma_\tau}\ . \tag{3.17}$$

Using Eqs. (3.7), (3.8) and (3.15) in Eq. (3.17), one can deduce following equations:

$$\begin{aligned} g_{\mu\nu,0\lambda} &= 0\ , \\ g_{00,ij} &= -2R_{0i0j}|_{\gamma_\tau}\ , \\ g_{0i,jk} &= -\frac{2}{3}\left(R_{0jik}|_{\gamma_\tau} + R_{0kij}|_{\gamma_\tau}\right)\ , \\ g_{ij,kl} &= -\frac{1}{3}\left(R_{ikjl}|_{\gamma_\tau} + R_{iljk}|_{\gamma_\tau}\right)\ . \end{aligned} \tag{3.18}$$

Therefore, in fermi normal coordinates, up to quadratic order of the coordinates, a metric is given by

$$g_{00} = -1 - R_{0i0j}|_{\gamma_\tau}x^ix^j\ , \tag{3.19}$$

$$g_{0i} = -\frac{2}{3}R_{0jik}|_{\gamma_\tau}x^jx^k\ , \tag{3.20}$$

$$g_{ij} = \delta_{ij} - \frac{1}{3}R_{ikjl}|_{\gamma_\tau}x^kx^l\ . \tag{3.21}$$

We note that the Riemann tensor is evaluated on the timelike geodesic γ_τ, so that it only depends on x^0. It should be mentioned that the Riemann tensor in Eqs. (3.19)-(3.21) is constructed on Fermi normal coordinates. Thus, we generally have to transform a Riemann tensor which is evaluated on a metric we consider to a Riemann tensor constructed on Fermi normal coordinates. However, the two Riemann tensors coincide with each other in special cases. For example, a linearized Riemann tensor on the flat spacetime background is invariant under a gauge transformation as mentioned in the footnote 1. It implies that the Riemann tensor constructed in Fermi normal coordinates is the same as that in the transverse traceless gauge. Therefore, we can use (2.5) in Eqs. (3.19)-(3.21) when we consider gravitational waves on the flat spacetime background. It simplifies discussions in following chapters a little bit.

3.2 Earth's gravity

In the previous section, we constructed locally inertial coordinates along a geodesic for a freely falling observer, namely Fermi normal coordinates. However, an observer is not freely falling if he is bounded on Earth because of Earth's gravity. Thus, he is accelerating by receiving a force from the ground; first, he accelerates against the gravity of Earth $g = 9.8\,\mathrm{m/s^2}$. Second, he is rotationally accelerating because of Earth's rotation. We will evaluate these gravitational effects of Earth [30, 33]. It will turn out that these effects are negligible in discussion we will develop in following chapters and so skipping this section and proceeding to the next chapter does not cause any problem. Nevertheless, it is worth studying how the effects of Earth's gravity appears and why they are negligible for our purpose to detect gravitational waves with magnons.

The set up to consider Earth's gravity is almost the same as the case of construction of Fermi normal coordinates; we first consider a timelike geodesic γ_τ parametrized by a proper time τ and second construct a spacelike geodesic γ_s parametrized by a proper distance s, which crosses γ_τ at $s = 0$. The situation is illustrated by Fig. 3.1. A difference compared with the construction of Fermi normal coordinates appears in the way of transformation of the orthonormal bases $\boldsymbol{e}_\mu$ which cover small region around a point on γ_τ. Although the bases

$\boldsymbol{e}_\mu$ are parallelly transformed along γ_τ, i.e. $\frac{d}{d\tau}\boldsymbol{e}_\mu = 0$, in the construction of Fermi normal coordinates[7], now e_μ are transformed as following, due to Earth's gravity [30]:

$$\begin{aligned} \frac{D}{d\tau}\boldsymbol{e}_\mu &= -\boldsymbol{\Omega}\cdot\boldsymbol{e}_\alpha \\ &= \Omega^{\mu\nu}\left(\boldsymbol{e}_\mu \wedge \boldsymbol{e}_\nu\right)\cdot\boldsymbol{e}_\alpha \\ &= -\boldsymbol{e}_\alpha \Omega^{\alpha}{}_{\mu} \ , \end{aligned} \tag{3.22}$$

where $\Omega^{\mu\nu}$ is an infinitesimal Lorentz transformation defined by

$$\begin{aligned} \Omega^{\mu\nu} &= (a^\mu u^\nu - a^\nu u^\mu) + u_\alpha \omega_\beta \epsilon^{\alpha\beta\mu\nu} \\ &= {}_{\rm (F)}\Omega^{\mu\nu} + {}_{\rm (R)}\,\Omega^{\mu\nu} \ . \end{aligned} \tag{3.23}$$

Also,

$$u^\mu = \frac{dx^\mu}{d\tau} \ , \tag{3.24}$$

is a four velocity,

$$a^\mu = \frac{du^\mu}{d\tau} \ , \tag{3.25}$$

is a four acceleration and ω_μ represents an angular velocity of rotation of spatial bases $\boldsymbol{e}_i$. Note that orthonormality of the bases are hold under the evolution (3.22) as a consequence of anti symmetricity of $\Omega^{\mu\nu}$.

One finds that ${}_{\rm (R)}\Omega^{\mu\nu}$ represents just a three dimensional rotation in terms of four dimensional covariant form by considering a rest frame, i.e. $u^\mu = (1,0,0,0)$, because

$$\begin{aligned} -\boldsymbol{e}_{\alpha}\,{}_{\rm (R)}\Omega^{\alpha}_{\mu} &= -\boldsymbol{e}_\alpha u_\gamma \omega_\beta \epsilon^{\gamma\beta\alpha}{}_{\mu} \\ &= \omega_i \boldsymbol{e}_j \epsilon^{0ij}{}_{\mu} \\ &= \left(\boldsymbol{\omega}\times\boldsymbol{e}_j\right)_{\mu=k} \ , \end{aligned} \tag{3.26}$$

where we identified the label of the bases $\boldsymbol{e}_\mu$ as the component of them due to orthonormality to obtain the last equality and $\mu = k$ denotes that μ takes a spatial index. In order to take into account rotationally acceleration due to Earth's gravity, $\boldsymbol{\omega}$ would correspond to the angular velocity of the Earth's rotation.

[7] At this time, we do not limit the discussion to the coordinate bases given by Eq. (3.1).

The transformation ${}_{(\mathrm{F})}\Omega^{\mu\nu}$ is called the Fermi-Walker transport. Let us reveal what the the Fermi-Walker transport is. Begin by considering an observer who is accelerating with magnitude of the gravity of Earth, $a^\mu a_\mu = g^2$, along x^1-coordinate in an inertial frame[8]. Then, because an acceleration vector defined by (3.25) is orthogonal to the four velocity, we have

$$a^\mu u_\mu = -a^0 u^0 + a^i u^i = 0 \ . \tag{3.27}$$

Using it and an explicit relation

$$a^\mu a_\mu = -a^0 a^0 + a^1 a^1 = g^2 \ , \tag{3.28}$$

one can obtain following equations

$$\begin{cases} a^0 = \frac{u^0}{d\tau} = g u^1 \ , \\ a^1 = \frac{u^1}{d\tau} = g u^0 \ . \end{cases} \tag{3.29}$$

A solution of Eqs. (3.29) is given by

$$\begin{cases} t = g^{-1} \sinh (g\tau) \ , \\ x^1 = g^{-1} \cosh (g\tau) \ . \end{cases} \tag{3.30}$$

This represents a hyperbola world line, indeed, $x^2 - t^2 = g^{-2}$ and the hyperbola line is a set of Lorentz transformation (Lorentz boost in this case), apart from a freedom of scaling, from the inertial coordinate (t, x^1) to another one. Moreover, since τ dependence appears in Eqs. (3.30), one can construct the rest frame for the accelerating observer at instant τ by doing a Lorentz boost transformation depending on τ. Such a Lorentz boost, which is a four dimensional rotation of a plane spanned by u^μ and a^μ, would be expressed by ${}_{(\mathrm{F})}\Omega^{\mu\nu}$. Indeed, if one consider a rest frame of an observer who is accelerating along x^1-direction, we have

$${}_{(\mathrm{F})}\Omega^{0x^1} = -g \ , \tag{3.31}$$

[8]Considering a rest frame of the observer, a Newtonian equation like, $\frac{d^2x^i}{(dx^0)^2} - g = 0$ holds, where we used the fact that the 0-component of a^μ is zero because a^μ is orthogonal to u^μ and $u^\mu = \delta^\mu_0$ in the rest frame. Therefore, the relation of the relativistically invariant quantity, $a^\mu a_\mu = g$, is satisfied as expected in Newtonian gravity.

and other components of ${}_{(\mathrm{F})}\Omega^{\mu\nu}$ are all zero. Then, the infinitesimal Lorentz transformation conducted by ${}_{(\mathrm{F})}\Omega^{\mu\nu}$ for the four vector $x^\mu = (\tau, 0, 0, 0)$ is

$$\begin{aligned} d\left(x^{0'} - x^0\right) &= x^\mu \,{}_{(\mathrm{F})}\Omega_{\mu 0'} d\tau \\ &= 0 \ . \end{aligned} \tag{3.32}$$

Thus,

$$\frac{dx^{0'}}{d\tau} = 0 \ . \tag{3.33}$$

This is consistent with the first equation in (3.30) when $g\tau \ll 1$. Furthermore,

$$\begin{aligned} d\left(x^{1'} - x^1\right) &= x^\mu \,{}_{(\mathrm{F})}\Omega_{\mu 1'} d\tau \\ &= \tau g d\tau \ . \end{aligned} \tag{3.34}$$

Then,

$$\frac{dx^{1'}}{d\tau} = g\tau \ , \tag{3.35}$$

is obtained. This is consistent with the second equation in (3.30) when $g\tau \ll 1$. Therefore, we find that ${}_{(\mathrm{F})}\Omega^{\mu\nu}$ correctly represents an infinitesimal Lorentz transformation which connects a rest frame to an accelerating frame relative to the rest frame. Now, we can understand the meaning of the Fermi-Walker transport in Eq. (3.22). At one point on γ_τ, one can construct a rest frame of an accelerating observer, but after certain duration the frame is not a rest frame for the observer anymore. In order to keep a frame as a rest frame at any τ, the base of the frame should be developed by the Fermi-Walker transport. Then we obtain a coordinate system moving with an accelerating observer.

From now on, we use coordinate bases specified by Eq. (3.1):

$$x^0 = \tau, \quad x^i = \alpha^i s \ , \tag{3.36}$$

and get an explicit expression of a metric in the proper detector coordinate which is moving with an accelerating observer due to Earth's gravity. The procedure is similar to the case of Fermi normal coordinates in the previous section, that is, we evaluate the Christoffel symbols and its first derivatives and next, we relate them to the first and second derivatives of the metric and then, an expression of the metric expanded up to second order of spacetime coordinates is found.

From Eq. (3.22), we obtain a relation:

$$\Gamma^{\alpha}_{\mu 0} = \Omega^{\alpha}{}_{\mu} \ . \tag{3.37}$$

Using $u^\mu = (1, 0, 0, 0)$ and $a^\mu = (0, a^i)$ in the definition of $\Omega^{\mu\nu}$, (3.23), we have

$$\Omega^0{}_i = a_i, \quad \Omega^i{}_j = -\epsilon^{0ijk}\omega_k \ . \tag{3.38}$$

Thus together with Eqs. (3.37) and (3.38), we obtain

$$\Gamma^0_{00} = 0 \ , \quad \Gamma^0_{i0}|_{\gamma_\tau} = \Gamma^i_{00}|_{\gamma_\tau} = a^i \ , \quad \Gamma^i_{j0}|_{\gamma_\tau} = -\omega_k \epsilon^{0ijk} \ . \tag{3.39}$$

We see that the proper reference frame is not a locally inertial frame anymore. Furthermore, considering a spacelike geodesic equation along γ_s,

$$\frac{d^2 x^\mu}{ds^2} + \Gamma^\mu_{\alpha\beta} \frac{dx^\alpha}{ds} \frac{dx^\beta}{ds} = 0 \ , \tag{3.40}$$

we can deduce

$$\Gamma^\mu_{ij}(x^0 = \tau, x^i = \alpha^i s)\, \alpha^i \alpha^j = 0 \ . \tag{3.41}$$

Especially, at $s = 0$, we conclude that

$$\Gamma^\mu_{ij}|_{\gamma_\tau} = 0 \ . \tag{3.42}$$

From Eqs. (3.39), (3.42) and the relation between a metric and a christoffel symbol

$$g_{\mu\nu,\lambda} = g_{\mu\alpha}\Gamma^\alpha_{\nu\lambda} + g_{\nu\alpha}\Gamma^\alpha_{\mu\lambda} \ , \tag{3.43}$$

one can observe that

$$\begin{aligned} g_{\mu\nu,0} &= 0 \ , \\ g_{00,i} &= -2a^i \ , \\ g_{0i,j} &= -\omega_k \epsilon^{0ijk} \ , \\ g_{ij,k} &= 0 \ , \end{aligned} \tag{3.44}$$

along the timelike geodesic γ_τ.

Next, we evaluate the second derivatives of the metric. Differentiating Eqs. (3.39) and (3.42) with respect to τ, we get

$$\begin{aligned} \Gamma^0_{00,0}|_{\gamma_\tau} &= \Gamma^\mu_{ij,0}|_{\gamma_\tau} = 0 \ , \\ \Gamma^0_{i0,0}|_{\gamma_\tau} &= \Gamma^i_{00,0}|_{\gamma_\tau} = \dot{a}^i \ , \\ \Gamma^i_{j0,0}|_{\gamma_\tau} &= -\dot{\omega}_k \epsilon^{0ijk} \ , \end{aligned} \tag{3.45}$$

where a dot represents a derivative with respect to τ. On the geodesic γ_τ, by the definition of a Riemann tensor, we find

$$\Gamma^\mu_{\nu 0,\lambda} = R^\mu_{\nu\lambda 0} + \Gamma^\mu_{\nu\lambda,0} - \Gamma^\mu_{\lambda\alpha}\Gamma^\alpha_{\nu 0} + \Gamma^\mu_{0\alpha}\Gamma^\alpha_{\nu\lambda} \ . \tag{3.46}$$

Substituting Eqs. (3.45) into Eq. (3.46) yields

$$\begin{aligned} \Gamma^0_{00,i}|_{\gamma_\tau} &= \dot{a}^i + a^j\omega^k\epsilon^{0ijk} \ , \\ \Gamma^0_{i0,j}|_{\gamma_\tau} &= R^0_{ij0}|_{\gamma_\tau} - a^i a^j \ , \\ \Gamma^i_{00,j}|_{\gamma_\tau} &= R^i_{0j0}|_{\gamma_\tau} - \dot{\omega}_k\epsilon^{0ijk} + a^i a^j + \omega^i\omega^j - \delta_{ij}\omega^k\omega_k \ , \\ \Gamma^i_{j0,k}|_{\gamma_\tau} &= R^i_{jk0}|_{\gamma_\tau} + a^j\omega_l\epsilon^{0ikl} \ . \end{aligned} \tag{3.47}$$

In order to obtain an expression of $\Gamma^\mu_{ij,k}|_{\gamma_\tau}$, one can utilize a geodesic deviation equation on γ_s and the procedure is completely same as that in construction of Fermi normal coordinates. Thus the result is given by Eq. (3.15):

$$\Gamma^\mu_{ij,k}|_{\gamma_\tau} = -\frac{1}{3}\left(R^\mu_{ijk}|_{\gamma_\tau} + R^\mu_{jik}|_{\gamma_\tau}\right) \ . \tag{3.48}$$

Finally, we express the second derivative of the metric by the Christoffel symbols and their first derivatives, and then relations between the second derivatives of the metric and the Riemann tensor are obtained. From the definition of the Christoffel symbol, we have

$$g_{\mu\nu,\lambda} = g_{\mu\alpha}\Gamma^\alpha_{\nu\lambda} + g_{\nu\alpha}\Gamma^\alpha_{\mu\lambda} \ . \tag{3.49}$$

Differentiating it with respect to x^σ leads

$$g_{\mu\nu,\lambda\sigma}|_{\gamma_\tau} = \eta_{\mu\alpha}\Gamma^\alpha_{\nu\lambda,\sigma}|_{\gamma_\tau} + \eta_{\nu\alpha}\Gamma^\alpha_{\mu\lambda,\sigma}|_{\gamma_\tau} + g_{\mu\alpha,\sigma}|_{\gamma_\tau}\Gamma^\alpha_{\nu\lambda}|_{\gamma_\tau} + g_{\nu\alpha,\sigma}|_{\gamma_\tau}\Gamma^\alpha_{\mu\lambda}|_{\gamma_\tau} \ . \tag{3.50}$$

Using Eqs. (3.47) and (3.48) in Eq. (3.50), one can deduce following equations:

$$\begin{aligned}
g_{\mu\nu,00} &= 0 \ , \\
g_{00,0i} &= -2\dot{a}^i \ , \\
g_{00,ij} &= -2R^0_{ij0} - 2a^i a^j - 2\omega^i\omega^j + 2\delta_{ij}\omega^k\omega_k \ , \\
g_{0i,0j} &= \dot{\omega}_k \epsilon^{0ijk} \ , \\
g_{0i,jk} &= -\frac{2}{3}\left(R_{0jik}|_{\gamma_\tau} + R_{0kij}|_{\gamma_\tau}\right) \ , \\
g_{ij,0k} &= 0 \ , \\
g_{ij,kl} &= -\frac{1}{3}\left(R_{ikjl}|_{\gamma_\tau} + R_{iljk}|_{\gamma_\tau}\right) \ . \qquad (3.51)
\end{aligned}$$

Therefore, in a proper reference coordinate, up to quadratic order of the coordinates, a metric is given by

$$g_{00} = -1 - 2a_i x^i - \left(a^i x^i\right)^2 - \left(\omega^i x^i\right)^2 + \omega^i\omega^i x^j x^j - R_{0i0j}|_{\gamma_\tau} x^i x^j \ , \qquad (3.52)$$

$$g_{0i} = -\omega_k \epsilon^{0ijk} x^j - \frac{2}{3} R_{0jik}|_{\gamma_\tau} x^j x^k \ , \qquad (3.53)$$

$$g_{ij} = \delta_{ij} - \frac{1}{3} R_{ikjl}|_{\gamma_\tau} x^k x^l \ . \qquad (3.54)$$

We see that the effects of Earth's gravity enter even at linear both for a^i and ω^i. Even if so, the effects are quite small, for examples, setting the scale of experimental apparatus to be $x^i \sim 1\,\mathrm{m}$ and using the values $a^i \sim 9.8\,\mathrm{m/s^2}$, $\omega^i \sim 2.0 \times 10^{-7}\,\mathrm{rad/s}$, we can estimate $a^i x^i \sim 1.1 \times 10^{-16}$ and $\omega^i x^i \sim 6.7 \times 10^{-16}$. In general, these small corrections are negligible in experiments because; first they are small, second their effects are static and so usually not distinguishable from other signals we want to see. In fact, Earth's gravity is negligible in magnon experiments because we utilize a phenomenon of resonance between gravitational waves and magnons to detect gravitational waves and then the effects of Earth's gravity does not concern it. Therefore, we will neglect the acceleration due to Earth's gravity, i.e. a^i and ω^i and use the Fermi normal coordinates for a freely falling observer in following chapters.

Chapter 4

Gravitational effects on fermions

In this chapter, we study gravitational effects on fermions, especially in the non-relativistic regime. To do so, we first consider the Dirac equation in curved spacetime with a Fermi normal coordinate by reviewing the discussion of [34] in the section 4.1. Next, in the section 4.2, we will take the non-relativistic limit of the Dirac equation and reveal all possible gravitational interactions with a non-relativistic fermion (mass m) in Fermi normal coordinates up to order of $1/m$. We then find the effect of gravitational waves on non-relativistic fermions, in particular an interaction between the spin and gravitational waves.

4.1 Dirac fields in curved spacetime

The Dirac equation in curved spacetime with a metric $g_{\mu\nu}$ is given by (See [35] for details)

$$i\gamma^{\hat{\alpha}} e^{\mu}_{\hat{\alpha}} \left(\partial_\mu - \Gamma_\mu - ieA_\mu\right)\psi = m\psi \ , \tag{4.1}$$

where $\gamma^{\hat{\alpha}}$, e, A_μ are the gamma matrices, the elementary charge, and a vector potential, respectively. A tetrad $e^{\mu}_{\hat{\alpha}}$ satisfies

$$e^{\hat{\alpha}}_{\mu} e^{\hat{\beta}}_{\nu} \eta_{\hat{\alpha}\hat{\beta}} = g_{\mu\nu} \ . \tag{4.2}$$

Note that $\hat{\alpha}$ is used for the locally inertial frame. The spin connection is defined by

$$\Gamma_\mu = \frac{i}{2} e^{\hat{\alpha}}_{\nu} \sigma_{\hat{\alpha}\hat{\beta}} \left(\partial_\mu e^{\nu\hat{\beta}} + \Gamma^{\nu}_{\lambda\mu} e^{\lambda\hat{\beta}}\right) , \tag{4.3}$$

where $\sigma_{\hat{\alpha}\hat{\beta}} = \frac{i}{4}[\gamma_{\hat{\alpha}}, \gamma_{\hat{\beta}}]$ is a generator of the Lorentz group and $\Gamma^{\mu}_{\nu\lambda}$ is the Christoffel symbol.

We now consider a proper reference frame, and thus it is a Fermi normal coordinate approximately, to evaluate the Dirac equation (4.1) because the coordinate is the one used in real experiments. In the section 3.1, we derived an explicit expression of the metric in Fermi normal coordinates as

$$g_{00} = -1 - R_{0i0j}x^i x^j \ , \tag{4.4}$$
$$g_{0i} = -\frac{2}{3}R_{0jik}x^j x^k \ , \tag{4.5}$$
$$g_{ij} = \delta_{ij} - \frac{1}{3}R_{ikjl}x^k x^l \ , \tag{4.6}$$

where the Riemann tensor is evaluated on $x^i = 0$ and thus it only depends on time x^0. Moreover, inverse of the metric is approximately given by

$$g^{00} = -1 + R^{0i0j}x_i x_j \ , \tag{4.7}$$
$$g^{0i} = +\frac{2}{3}R^{0jik}x_j x_k \ , \tag{4.8}$$
$$g^{ij} = \delta_{ij} + \frac{1}{3}R^{ikjl}x_k x_l \ . \tag{4.9}$$

From the metric (4.4)-(4.9), by a standard calculation, one can obtain the Christoffel symbols:

$$\begin{cases} \Gamma^0_{00} = 0 \ , \quad \Gamma^0_{0i} = R_{0i0j}x^j \ , \quad \Gamma^0_{ij} = \frac{1}{3}\left(R_{0ijk} + R_{0jik}\right)x^k \ , \\ \Gamma^i_{00} = R_{0i0j}x^j \ , \quad \Gamma^i_{0j} = R_{0kji}x^k \ , \quad \Gamma^i_{jk} = \frac{1}{3}\left(R_{kijl} + R_{jikl}\right)x^l \ . \end{cases} \tag{4.10}$$

The tetrad is constructed to satisfy (4.2) as

$$e^{\hat{\alpha}}_0 = \delta^{\hat{\alpha}}_0 - \frac{1}{2}\delta^{\hat{\alpha}}_{\alpha}R^{\alpha}{}_{k0l}x^k x^l \ , \tag{4.11}$$
$$e^{\hat{\alpha}}_i = \delta^{\hat{\alpha}}_i - \frac{1}{6}\delta^{\hat{\alpha}}_{\alpha}R^{\alpha}{}_{kil}x^k x^l \ . \tag{4.12}$$
$$e^0_{\hat{\alpha}} = \delta^{\hat{0}}_{\hat{\alpha}} + \frac{1}{2}\delta^{\hat{0}}_{\hat{\alpha}}R^0{}_{k0l} - \frac{1}{6}\eta_{\hat{\alpha}j}R^j{}_{k0l}x^k x^l \ , \tag{4.13}$$
$$e^i_{\hat{\alpha}} = \delta^i_{\hat{\alpha}} - \frac{1}{2}\delta^{\hat{0}}_{\hat{\alpha}}R^0{}_k{}^i{}_l x^k x^l + \frac{1}{6}\eta_{\hat{\alpha}j}R^i{}_k{}^j{}_l x^k x^l \ . \tag{4.14}$$

Substituting Eqs. (4.10)-(4.14) into Eq. (4.3) results in

$$\Gamma_0 = \frac{1}{2}\gamma^{\hat{0}}\gamma^{\hat{i}}R_{0i0j}x^j + \frac{1}{4}\gamma^{\hat{i}}\gamma^{\hat{j}}R_{ij0k}x^k \ , \tag{4.15}$$
$$\Gamma_i = \frac{1}{4}\gamma^{\hat{0}}\gamma^{\hat{j}}R_{0jik}x^k + \frac{1}{8}\gamma^{\hat{j}}\gamma^{\hat{k}}R_{jkil}x^l \ . \tag{4.16}$$

Here we have rewritten $\delta^{\mu}_{\hat{\alpha}}\gamma^{\hat{\alpha}}$ as $\gamma^{\hat{\mu}}$ and we will do so here after.

On the other hand, the Dirac equation (4.1) can be rewritten as

$$\begin{aligned} i\gamma^0\partial_0\psi &= \left[i\gamma^0\left(\Gamma_0 + ieA_0\right) + i\gamma^j\left(\partial_j + \Gamma_j + ieA_j\right) + m\right]\psi \\ &= \gamma^0 H\psi \ , \end{aligned} \tag{4.17}$$

where we defined the non-relativistic Hamiltonian H and $\gamma^{\mu} = e^{\mu}_{\hat{\alpha}}\gamma^{\hat{\alpha}}$ is a gamma matrix in curved spacetime , which satisfies

$$\{\gamma^{\mu}, \gamma^{\nu}\} = -2g^{\mu\nu} \ . \tag{4.18}$$

Let us express the Hamiltonian in terms of the gamma matrix of the locally inertial frame instead of that of curved spacetime. First, because of $\gamma^0\gamma^0 = -g^{00}$,

$$H = -(g^{00})^{-1}\left[ig^{00}\left(\Gamma_0 + ieA_0\right) + i\gamma^0\gamma^j\left(\partial_j - \Gamma_j - ieA_j\right) + \gamma^0 m\right] \ . \tag{4.19}$$

Using Eqs. (4.13) and (4.14), we calculate

$$\begin{aligned} \gamma^0\gamma^j &= \left(e^0_{\hat{\alpha}}\gamma^{\hat{\alpha}}\right)\left(e^j_{\hat{\beta}}\gamma^{\hat{\beta}}\right) \\ &= \left(e^0_{\hat{0}}\gamma^{\hat{0}} + e^0_{\hat{a}}\gamma^{\hat{a}}\right)\left(e^j_{\hat{0}}\gamma^{\hat{0}} + e^j_{\hat{b}}\gamma^{\hat{b}}\right) \\ &= \gamma^{\hat{0}}\gamma^{\hat{j}} - \frac{1}{2}\gamma^{\hat{0}}\gamma^{\hat{0}}R^{0\ \ j}_{\ k\ \ l}x^k x^l + \frac{1}{6}\gamma^{\hat{0}}\gamma^{\hat{b}}R^{j}_{\ k\hat{b}l}x^k x^l \\ &\quad + \frac{1}{2}\gamma^{\hat{0}}\gamma^{\hat{j}}R^{0}_{\ k0l}x^k x^l - \frac{1}{6}\gamma^{\hat{a}}\gamma^{\hat{j}}R_{\hat{a}k0l}x^k x^l \ . \end{aligned} \tag{4.20}$$

Together with Eq. (4.7), we have

$$\begin{aligned} (g^{00})^{-1}\gamma^0\gamma^j &\simeq -\gamma^{\hat{0}}\gamma^{\hat{j}} - \frac{1}{2}R_{0kjl}x^k x^l - \frac{1}{6}\gamma^{\hat{0}}\gamma^{\hat{a}}R_{jk\hat{a}l}x^k x^l \\ &\quad - \frac{1}{2}\gamma^{\hat{0}}\gamma^{\hat{j}}R_{0k0l}x^k x^l + \frac{1}{6}\gamma^{\hat{a}}\gamma^{\hat{j}}R_{\hat{a}k0l}x^k x^l \ . \end{aligned} \tag{4.21}$$

Similarly, one can obtain

$$(g^{00})^{-1}\gamma^0 \simeq -\gamma^{\hat{0}} - \frac{1}{2}\gamma^{\hat{0}}R_{0k0l}x^k x^l + \frac{1}{6}\gamma^{\hat{a}}R_{\hat{a}k0l}x^k x^l \ . \tag{4.22}$$

Therefore, from Eqs. (4.19), (4.21) and (4.22), the Hamiltonian expressed in the locally inertial coordinate becomes

$$\begin{aligned} H &= i\Gamma_0 + i\gamma^{\hat{0}}\gamma^{\hat{j}}\Gamma_j - eA_0 \\ &\quad + \left[\gamma^{\hat{0}}\gamma^{\hat{j}} + \frac{1}{2}R_{0kjl}x^k x^l + \frac{1}{6}\gamma^{\hat{0}}\gamma^{\hat{a}}R_{jk\hat{a}l}x^k x^l \right. \\ &\qquad \left. + \frac{1}{2}\gamma^{\hat{0}}\gamma^{\hat{j}}R_{0k0l}x^k x^l - \frac{1}{6}\gamma^{\hat{a}}\gamma^{\hat{j}}R_{\hat{a}k0l}x^k x^l\right](-i\partial_j - eA_j) \\ &\quad + \left[\gamma^{\hat{0}} + \frac{1}{2}\gamma^{\hat{0}}R_{0k0l}x^k x^l - \frac{1}{6}\gamma^{\hat{a}}R_{\hat{a}k0l}x^k x^l\right] m \ . \end{aligned} \tag{4.23}$$

Furthermore, substituting Eqs. (4.15) and (4.16) into the above Hamiltonian and arranging terms, we have

$$\begin{aligned} H &= \frac{i}{2}\gamma^{\hat{0}}\gamma^{\hat{i}}R_{0i0j}x^j + \frac{i}{4}\gamma^{\hat{i}}\gamma^{\hat{j}}R_{0ikj}x^k + \frac{i}{8}\gamma^{\hat{0}}\gamma^{\hat{i}}\gamma^{\hat{j}}\gamma^{\hat{k}}R_{jkil}x^l - eA_0 \\ &\quad + \left[\gamma^{\hat{0}}\gamma^{\hat{i}}\left(\delta^{\hat{j}}_{\hat{i}} + \theta^{\hat{j}}_{\hat{i}}\right) + \frac{1}{2}R_{0kjl}x^k x^l - \frac{1}{6}\gamma^{\hat{i}}\gamma^{\hat{j}}R_{ik0l}x^k x^l\right](-i\partial_j - eA_j) \\ &\quad + \gamma^{\hat{0}}\left[1 + \frac{1}{2}R_{0k0l}x^k x^l - \frac{1}{6}\gamma^{\hat{0}}\gamma^{\hat{i}}R_{ik0l}x^k x^l\right] m \ , \end{aligned} \tag{4.24}$$

where we have defined

$$\theta^{\hat{j}}_{\hat{i}} = \frac{1}{2}\delta^{\hat{j}}_{\hat{i}}R_{0k0l}x^k x^l + \frac{1}{6}R_{jkil}x^k x^l \ . \tag{4.25}$$

This is the 4×4 matrix including both of a fermi particle and a anti-fermi particle. The situation we are interested in is that there exist non-relativistic fermi particles. To take the non-relativistic limit of the fermi particle in the Hamiltonian (4.24), we have to separate the particle and the anti-particle while expanding the Hamiltonian in powers of $1/m$. We will explicitly see how to perform it in the next section.

4.2 Non-relativistic limit of the Dirac equation

In the previous section, we derived the (non-relativistic) Hamiltonian of a Dirac field in general curved spacetime with a Fermi normal coordinate. Assuming that a fermi particle has a velocity well below the speed of light, which is the situation we will consider in the section 5.2, we take the non-relativistic limit of the Hamiltonian. The procedure in the flat spacetime is known as the Foldy Wouthuysen transformation [36, 37]. We generalize it to the case of curved spacetime.

We first separate the Hamiltonian (4.24) into the even part, the odd part and the terms multiplied by m as

$$\begin{aligned} H &= \frac{i}{2}\alpha^i R_{0i0j}x^j - \frac{i}{8}\alpha^i\alpha^j\alpha^k R_{jkil}x^l + \alpha^i\left(\delta_i^j + \theta_i^j\right)\Pi_j \\ &\quad -eA_0 - \frac{i}{4}\alpha^i\alpha^j R_{0ikj}x^k + \left[\frac{1}{2}R_{0kjl}x^k x^l + \frac{1}{6}\alpha^i\alpha^j R_{ik0l}x^k x^l\right]\Pi_j \\ &\quad + \left[\beta\left(1 + \frac{1}{2}R_{0k0l}x^k x^l\right) - \frac{1}{6}\beta\alpha^i R_{ik0l}x^k x^l\right] m \\ &= \mathcal{O} + \mathcal{E} + \left[\beta\left(1 + \frac{1}{2}R_{0k0l}x^k x^l\right) - \frac{1}{6}\beta\alpha^i R_{ik0l}x^k x^l\right] m \ , \end{aligned} \tag{4.26}$$

where we have defined $\beta = \gamma^{\hat{0}}$, $\alpha^i = \gamma^{\hat{0}}\gamma^{\hat{i}}$ and $\Pi_j = -i\partial_j - eA_j$ for brevity. The even, $\mathcal{E}$, means that the matrix has only block diagonal elements and the odd, $\mathcal{O}$, means that the matrix has only block off-diagonal elements. Any product of two even (odd) matrices is even and a product of even (odd) and odd (even) matrices becomes odd. To take the non-relativistic limit of a fermi particle, we have to diagonalize the Hamiltonian (4.26) and expand the upper block diagonal part in powers of $1/m$. It is known that this can be done in flat spacetime by repeating unitary transformations order by order in powers of $1/m$ [36, 37]. Let us generalize the method to the case of arbitrary curved spacetime in a Fermi normal coordinate.

We now consider a unitary transformation,

$$\psi' = e^{iS}\psi \ , \tag{4.27}$$

where S is a time-dependent Hermitian 4×4 matrix. Observing that

$$\begin{aligned} i\frac{\partial\psi'}{\partial t} &= i\frac{\partial}{\partial t}\left(e^{iS}\psi\right) \\ &= e^{iS}\left(i\frac{\partial\psi}{\partial t}\right) + i\left(\frac{\partial}{\partial t}e^{iS}\right)\psi \\ &= \left[e^{iS}He^{-iS} + i\left(\frac{\partial}{\partial t}e^{iS}\right)e^{-iS}\right]\psi' \ , \end{aligned} \tag{4.28}$$

we find that the Hamiltonian after the unitary transformation is given by

$$H' = e^{iS}He^{-iS} + i\left(\frac{\partial}{\partial t}e^{iS}\right)e^{-iS} \ . \tag{4.29}$$

We now assume that S is proportional to $1/m$ and consider expanding the transformed Hamiltonian (4.29) in powers of S up to order of $1/m$. Using Eqs. (B.4) and (B.7) in Eq. (4.29), we obtain

$$\begin{aligned} H' &= H + i\big[S, H\big] - \frac{1}{2}\big[S, \big[S, H\big]\big] - \frac{i}{6}\big[S, \big[S, \big[S, H\big]\big]\big] + \cdots \\ &\quad - \dot{S} - \frac{i}{2}\big[S, \dot{S}\big] + \cdots \ . \end{aligned} \tag{4.30}$$

First, let us eliminate the off-diagonal part of the Hamiltonian (4.26) at order of m by a unitary transformation. Then we will drop the higher order terms with respect to the Riemann tensor, which only depends on time, and derivatives of the Riemann tensor with respect to the time by assuming that they are small enough[9]. To cancel the last term in the square bracket of (4.26), we take

$$S = -\frac{i}{2m}\beta \left(-\frac{1}{6}\beta\alpha^i R_{ik0l} x^k x^l m \right) \ . \tag{4.31}$$

We then obtain

$$\begin{aligned} i\big[S, H\big] &\simeq \frac{1}{6}\beta\alpha^i R_{ik0l} x^k x^l m - \frac{1}{12}\big[\alpha^i, \alpha^j\big] R_{ik0l} x^k x^l \Pi_j \\ &\quad + \frac{i}{6}\alpha^i\alpha^j R_{0ikj} x^k + \frac{i}{12}\alpha^i\alpha^j R_{0jik} x^k \ . \end{aligned} \tag{4.32}$$

Therefore, from Eqs. (4.30) and (4.32), we have the transformed Hamiltonian as

$$\begin{aligned} H' &\simeq H + i\big[S, H\big] \\ &\simeq \frac{i}{2}\alpha^i R_{0i0j} x^j - \frac{i}{8}\alpha^i\alpha^j\alpha^k R_{jkil} x^l + \alpha^i \left(\delta_i^j + \theta_i^j\right) \Pi_j \\ &\quad - eA_0 - \frac{i}{6} R_{0iki} x^k + \frac{2}{3} R_{0kil} x^k x^l \Pi_i \\ &\quad + \beta \left(1 + \frac{1}{2} R_{0k0l} x^k x^l \right) m \\ &= \mathcal{O} + \mathcal{E}' + \beta \left(1 + \frac{1}{2} R_{0k0l} x^k x^l \right) m \ , \end{aligned} \tag{4.33}$$

where we have used the relation $\left\{\alpha^i, \alpha^j\right\} = 2\delta^{ij}$. One can see that only even terms remain at order of m as expected.

[9]Then, the Hermiticity of the non-relativistic Hamiltonian is guaranteed [38].

Next, we focus on order of m^0 and eliminate the odd terms by a unitary transformation. In order to do so, we choose the Hermitian operator to be

$$S' = -\frac{i}{2m}\beta\left(\mathcal{O} - \frac{1}{2}\alpha^j R_{0k0l}x^k x^l \Pi_j + \frac{i}{2}\alpha^j R_{0k0j}x^k\right) . \tag{4.34}$$

One then calculate

$$\begin{aligned} i\left[S', H'\right] \simeq\; & -\mathcal{O} + \frac{1}{m}\beta\mathcal{O}^2 + \frac{1}{2m}\beta\left[\mathcal{O}, \mathcal{E}'\right] \\ & -\frac{1}{2m}\beta\alpha^i\alpha^j R_{0k0l}x^k x^l \Pi_i\Pi_j + \frac{i}{m}\beta R_{0k0i}x^k\Pi_i \\ & +\frac{i}{4m}\beta\left[\alpha^i, \alpha^j\right] R_{0k0i}x^k\Pi_j + \frac{1}{4m}\beta R_{0i0i} \\ & -\frac{i}{4m}\beta\alpha^j R_{0k0l}x^k x^l \left(\partial_j eA_0\right) , \end{aligned} \tag{4.35}$$

Furthermore, up to order of $1/m$,

$$\begin{aligned} -\frac{1}{2}\left[S', \left[S', H'\right]\right] \simeq\; & -\frac{1}{2}\left[S', i\mathcal{O}\right] \\ \simeq\; & -\frac{1}{2m}\beta\mathcal{O}^2 + \frac{1}{4m}\beta\alpha^i\alpha^j R_{0k0l}x^k x^l \Pi_i\Pi_j - \frac{i}{2m}\beta R_{0k0i}x^k\Pi_i \\ & -\frac{i}{8m}\beta\left[\alpha^i, \alpha^j\right] R_{0k0i}x^k\Pi_j - \frac{1}{8m}\beta R_{0i0i} , \end{aligned} \tag{4.36}$$

and

$$-\dot{S}' \simeq \frac{i}{2m}\beta\dot{\mathcal{O}} + \frac{i}{4m}\beta\alpha^j R_{0k0l}x^k x^l e\dot{A}_j . \tag{4.37}$$

Therefore, the Hamiltonian after the unitary transformation is given by

$$\begin{aligned} H'' \simeq\; & H' + i\left[S', H'\right] - \frac{1}{2}\left[S', \left[S', H'\right]\right] - \dot{S}' \\ \simeq\; & -\frac{i}{4m}\beta\alpha^j R_{0k0l}x^k x^l eE_j + \frac{1}{2m}\beta\left(\left[\mathcal{O}, \mathcal{E}'\right] + i\dot{\mathcal{O}}\right) \\ & +\mathcal{E}' + \frac{1}{2m}\beta\mathcal{O}^2 - \frac{1}{4m}\beta\alpha^i\alpha^j R_{0k0l}x^k x^l\Pi_i\Pi_j + \frac{i}{2m}\beta R_{0k0i}x^k\Pi_i + \frac{i}{8m}\beta\left[\alpha^i, \alpha^j\right] R_{0k0i}x^k\Pi_j + \frac{1}{8m}\beta R_{0i0i} \\ & +\beta\left(1 + \frac{1}{2}R_{0k0l}x^k x^l\right) m \\ =\; & \mathcal{O}' + \mathcal{E}'' + \beta\left(1 + \frac{1}{2}R_{0k0l}x^k x^l\right) m , \end{aligned} \tag{4.38}$$

where $E_j \equiv \partial_j A_0 - \dot{A}_j$ is an electric field. We see that $\mathcal{O}'$ has only terms of order of $1/m$, so that odd terms at order of m^0 have been removed precisely.

Finally, we will eliminate the odd term $\mathcal{O}'$ and then the Hamiltonian will consist of only even terms up to order of $1/m$, which we want to get. To this end, we now choose the Hermitian operator of a unitary transformation as

$$S'' = -\frac{i}{2m}\beta\left(\mathcal{O}' - \frac{i}{4m}\beta\alpha^i eE_i R_{0k0l}x^k x^l\right) . \tag{4.39}$$

Then up to order of $1/m$,

$$i\big[S'', H''\big] \simeq -\mathcal{O}' . \tag{4.40}$$

Therefore, we have the transformed Hamiltonian as

$$\begin{aligned} H''' &\simeq H'' + i\big[S'', H''\big] \\ &\simeq \mathcal{E}'' + \beta\left(1 + \frac{1}{2}R_{0k0l}x^k x^l\right) m , \end{aligned} \tag{4.41}$$

where $\mathcal{E}''$ is

$$\begin{aligned} \mathcal{E}'' \;=\; & -eA_0 - \frac{i}{6}R_{0iki}x^k + \frac{2}{3}R_{0kil}x^k x^l \Pi_i + \frac{1}{2m}\beta\mathcal{O}^2 - \frac{1}{4m}\beta\alpha^i\alpha^j R_{0k0l}x^k x^l \Pi_i\Pi_j \\ & + \frac{i}{2m}\beta R_{0k0i}x^k\Pi_i + \frac{i}{8m}\beta\big[\alpha^i, \alpha^j\big] R_{0k0i}x^k\Pi_j + \frac{1}{8m}\beta R_{0i0i} . \end{aligned} \tag{4.42}$$

Moreover, the fourth term in the first line of Eq. (4.42) can be evaluated as

$$\begin{aligned} \frac{1}{2m}\beta\mathcal{O}^2 \;\simeq\; & \frac{i}{8m}\beta\big[\alpha^i, \alpha^j\big]\epsilon_{klm}eB^m\left(\delta_{ki}\delta_{lj} + 2\delta_{ki}\theta_{lj}\right) \\ & - \frac{i}{8m}\beta\big[\alpha^i, \alpha^j\big]\left(\frac{1}{2}R_{lmji} + 2\delta^l_j R_{0i0m}\right) x^m\Pi_l \\ & + \frac{1}{2m}\beta\Pi_i^2 + \frac{i}{2m}\beta R_{0i0j}x^j\Pi_i + \frac{1}{4m}\beta R_{0i0i} \\ & + \frac{1}{16m}\beta\alpha^i\alpha^j\alpha^k\alpha^l R_{ijkl} - \frac{i}{16m}\beta\big\{\alpha^i, \alpha^j\alpha^k\alpha^l\big\}R_{kljm}x^m\Pi_i , \end{aligned} \tag{4.43}$$

where $B^i \equiv \frac{1}{2}\epsilon^{ijk}(\partial_j A_k - \partial_k A_j)$ is a magnetic field. Using Eqs. (4.42), (4.43) and a relation, $\big[\alpha^i, \alpha^j\big] = 2i\epsilon_{ijk}\sigma^k$, in the transformed Hamiltonian (4.41), we finally arrive at the

Hamiltonian for a non-relativistic fermion up to order of $1/m$ as

$$\begin{aligned} H''' &= \left(1+\frac{1}{2}R_{0k0l}x^k x^l\right)m - eA_0 - \frac{i}{6}R_{0iki}x^k + \frac{2}{3}R_{0kil}x^k x^l \Pi_i + \frac{1}{2m}\left(1-\frac{1}{2}R_{0k0l}x^k x^l\right)\Pi_i^2 \\ &\quad -\frac{e}{2m}\sigma^i B^j \left[\delta_{ij}\left(1+\frac{1}{2}R_{0k0l}x^k x^l + \frac{1}{6}R_{akal}x^k x^l\right) - \frac{1}{6}\delta_{ia}\delta_{jb}R_{akbl}x^k x^l\right] \\ &\quad +\frac{1}{8m}\epsilon_{ijk}\sigma^k \left(R_{ijlm} + 2\delta_{jm}R_{0i0l}\right) x^l \Pi_m \\ &\quad +\frac{1}{m}\left(\frac{3}{8}R_{0i0i} - \frac{1}{2}R_{ijij}\right) + \frac{i}{m}\left(R_{0j0k} - \frac{1}{4}R_{ijik}\right) x^k \Pi_j \ . \end{aligned} \tag{4.44}$$

The first term is the rest mass and its modification from gravity at a point x^i. The third term represents gravitational redshift, namely energy shift due to gravity. The first term in the last line gives same effect at order of $1/m$. Considering an equation of motion of a particle, we find that the fourth and the fifth terms are gravitational effects on motion of a particle. However we notice that the former contains the time derivative of the curvature, which has been assumed to be tiny, in the equation of motion. Therefore, the second term in the last line is also tiny one. The second line represents interactions between gravity and a spin in the presence of an external magnetic field. This is what causes the spin resonance and/or the excitation of magnons as we will see in following chapters. The third line is a spin-orbit coupling mediated by gravity.

Let us focus on gravitational wave as gravitational effects. In the section 2.1, we derived the Riemann tensor for a general perturbed metric at linear order:

$$R^{\alpha}_{\mu\beta\nu} = \frac{1}{2}(h^{\alpha}{}_{\nu,\mu\beta} - h_{\mu\nu}{}^{,\alpha}{}_{\beta} - h^{\alpha}{}_{\beta,\mu\nu} + h_{\mu\beta}{}^{,\alpha}{}_{,\nu}) \ . \tag{4.45}$$

Taking the linear perturbation $h_{\mu\nu}$ as gravitational waves, i.e., $h_{0\mu} = h_{ii} = h_{0i,i} = 0$, one can obtain

$$\begin{aligned} R_{0i0j} &= -\frac{1}{2}\ddot{h}_{ij} \ , \\ R_{0ijk} &= \frac{1}{2}\left(\dot{h}_{ij,k} - \dot{h}_{ik,j}\right) \ , \\ R_{ijkl} &= \frac{1}{2}\left(h_{il,jk} + h_{jk,il} - h_{jl,ik} - h_{ik,jl}\right) \ . \end{aligned} \tag{4.46}$$

Substituting (4.46) into (4.44) results in

$$
\begin{aligned}
H''' \; = \; & \left(1-\frac{1}{4}h_{ij}x^ix^j\right)m - eA_0 + \frac{1}{3}\left(h_{ki,l}-h_{kl,i}\right)x^kx^l\Pi_i + \frac{1}{2m}\left(1+\frac{1}{4}\ddot{h}_{ij}x^ix^j\right)\Pi_k^2 \\
& -\frac{e}{2m}\sigma^iB^j\left[\delta_{ij}\left(1-\frac{1}{4}\ddot{h}_{kl}x^kx^l\right)-\frac{1}{12}\delta_{ia}\delta_{jb}\left(h_{al,kb}+h_{kb,al}-h_{kl,ab}-h_{ab,kl}\right)x^kx^l\right] \\
& +\frac{1}{8m}\epsilon_{ijk}\sigma^k\left(\frac{1}{2}\left(h_{im,jl}+h_{jl,im}-h_{jm,il}-h_{il,jm}\right)-\delta_{jm}\ddot{h}_{il}\right)x^l\Pi_m \\
& -\frac{i}{2m}\ddot{h}_{jk}x^k\Pi_j \; , \qquad (4.47)
\end{aligned}
$$

where we have used the equation of motion of gravitational waves, i.e., $\Box h_{ij}=0$.

In the next chapter, we will explain magnons, which are corrective excitation of spins. After that, in the section 5.2, we will see that gravitational waves excite magnons through the interaction at the second line in Eq. (4.47).

Chapter 5

Magnons

In this chapter, we first find what a magnons is in the section 5.1. Moreover, in the section 5.2, we study the interaction between magnons and gravitational waves. It will turn out that gravitational waves excite magnons from the ground state.

5.1 Magnons as corrective spin excitations

A magnon is a quantum of spin waves, which are corrective spin excitations. To see what magnons are more precisely, let us consider a specific situation where a spherical ferromagnetic sample which has N electronic spins is put in an external magnetic field. Such a system is well described by the Heisenberg model [39]:

$$\mathscr{H} = -2\mu_B B_z \sum_i \hat{S}^z_{(i)} - \sum_{i,j} J_{ij} \hat{\boldsymbol{S}}_{(i)} \cdot \hat{\boldsymbol{S}}_{(j)} \ , \tag{5.1}$$

where the Bohr magneton $\mu_B = e/2m_e$ is defined by the elementary electric charge e and the mass of electrons m_e. We applied an external magnetic field along the z-direction, B_z, without loss of generality because of isotropy. i specifies each site of spins. The first term is the conventional Pauli term, which turns the spin direction to be along the external magnetic field. The second term represents the exchange interactions between spins with the strength J_{ij} and it is taken to be positive.

The spin system (5.1) can be rewritten by using the Holstein-Primakoff transformation [40]:

$$\begin{cases} \hat{S}^z_{(i)} = S - \hat{C}^\dagger_i \hat{C}_i \ , \\ \hat{S}^+_{(i)} = \sqrt{2S - \hat{C}^\dagger_i \hat{C}_i}\ \hat{C}_i \ , \\ \hat{S}^-_{(i)} = \hat{C}^\dagger_i \sqrt{2S - \hat{C}^\dagger_i \hat{C}_i} \ , \end{cases} \tag{5.2}$$

where S denotes the amplitude of the spins, bosonic operators $\hat{C}_i$ and $\hat{C}^\dagger_i$ satisfy commutation relations $[\hat{C}_i, \hat{C}^\dagger_j] = \delta_{ij}$ and $S^\pm_{(j)} = S^x_{(j)} \pm i S^y_{(j)}$ are the ladder operators. It is easy to check that the SU(2) algebra, $[S^i, S^j] = i\epsilon_{ijk} S^k$ $(i, j, k = x, y, z)$, is satisfied even after the transformation (5.2). We note that $\hat{C}^\dagger_i \hat{C}_i$ represents the particle numbers of the boson, namely the magnon, created by the creation operator $\hat{C}^\dagger_i$.

We first examine the first term in Eq. (5.1). Substituting the Holstein-Primakoff transformation (5.2) into it, we obtain

$$-2\mu_B B_z \sum_i \hat{S}^z_{(i)} = 2\mu_B B_z \sum_i \hat{C}^\dagger_i \hat{C}_i \ , \tag{5.3}$$

where we have dropped the constant term in the Hamiltonian since it is not important for our purpose. Furthermore, provided that contributions from the surface of the sample are negligible, one can expand the bosonic operators by plane waves as

$$\hat{C}_i = \sum_{\boldsymbol{k}} \frac{e^{-i\boldsymbol{k}\cdot\boldsymbol{r}_i}}{\sqrt{N}} \hat{c}_{\boldsymbol{k}} \ , \tag{5.4}$$

where $\boldsymbol{k}$ denotes the discrete wave numbers and $\boldsymbol{r}_i$ is the position vector of the i spin from the center of the ferromagnetic sample. Substituting Eq. (5.4) into Eq. (5.3) yields

$$\begin{aligned} 2\mu_B B_z \sum_i \hat{C}^\dagger_i \hat{C}_i &= 2\mu_B B_z \sum_i \sum_{\boldsymbol{k},\boldsymbol{k}'} \frac{e^{i(\boldsymbol{k}-\boldsymbol{k}')\boldsymbol{r}_i}}{N} \hat{c}^\dagger_{\boldsymbol{k}} \hat{c}_{\boldsymbol{k}'} \\ &= 2\mu_B B_z \sum_{\boldsymbol{k}} \hat{c}^\dagger_{\boldsymbol{k}} \hat{c}_{\boldsymbol{k}} \ . \end{aligned} \tag{5.5}$$

To get the second line, we used the relation $\sum_i e^{i(\boldsymbol{k}-\boldsymbol{k}')\boldsymbol{r}_i} = N\delta(\boldsymbol{k}-\boldsymbol{k}')$. Eq. (5.5) is a finite set of harmonic oscillators, and thus it represents so-called spin waves. Especially, a quantum of the spin waves created by $\hat{c}^\dagger_k$ is called a magnon.

Next, we consider the second term in Eq. (5.1) and from now on, we assume that the particle numbers of magnons are always much less than unity, i.e., $\hat{C}_i^\dagger \hat{C}_i \ll 1$. In fact, this is the situation we will consider in following sections. Then, after the Holstein-Primakoff transformation (5.2), the second term of Eq. (5.1) becomes

$$\begin{aligned} -\sum_{i,j} J_{ij} \hat{\boldsymbol{S}}_{(i)} \cdot \hat{\boldsymbol{S}}_{(j)} &= -\sum_{i,j} J_{ij} \left[\frac{1}{4}(S^+_{(i)} + S^-_{(i)})(S^+_{(j)} + S^-_{(j)}) - \frac{1}{4}(S^+_{(i)} - S^-_{(i)})(S^+_{(j)} - S^-_{(j)}) + S^z_{(i)} S^z_{(j)} \right] , \\ &\simeq -\sum_{i,j} J_{ij} \left[S \left(\hat{C}_i \hat{C}_j^\dagger + \hat{C}_i^\dagger \hat{C}_j - \hat{C}_i^\dagger \hat{C}_i - \hat{C}_j^\dagger \hat{C}_j \right) + S^2 \right] . \end{aligned} \tag{5.6}$$

We now proceed to the Fourier space, first, we calculate

$$\begin{aligned} \sum_{i,j} J_{ij} \hat{C}_i^\dagger \hat{C}_j &= \sum_{i,j} J(\boldsymbol{r}_i - \boldsymbol{r}_j) \sum_{\boldsymbol{k}} \frac{e^{i\boldsymbol{k}\cdot\boldsymbol{r}_i}}{\sqrt{N}} \hat{c}^\dagger_{\boldsymbol{k}} \sum_{\boldsymbol{k}'} \frac{e^{-i\boldsymbol{k}'\cdot\boldsymbol{r}_j}}{\sqrt{N}} \hat{c}_{\boldsymbol{k}'} \\ &= \sum_{i,l} J(\boldsymbol{r}_l) \sum_{\boldsymbol{k},\boldsymbol{k}'} \frac{1}{N} e^{i(\boldsymbol{k}-\boldsymbol{k}')\cdot\boldsymbol{r}_i} e^{i\boldsymbol{k}'\cdot\boldsymbol{r}_l} \ \hat{c}^\dagger_{\boldsymbol{k}} \hat{c}_{\boldsymbol{k}'} \\ &= \sum_{l} J(\boldsymbol{r}_l) \sum_{\boldsymbol{k},\boldsymbol{k}'} \delta(\boldsymbol{k}-\boldsymbol{k}') e^{i\boldsymbol{k}'\cdot\boldsymbol{r}_l} \ \hat{c}^\dagger_{\boldsymbol{k}} \hat{c}_{\boldsymbol{k}'} \\ &= \sum_{l} J(\boldsymbol{r}_l) \sum_{\boldsymbol{k}} e^{i\boldsymbol{k}\cdot\boldsymbol{r}_l} \ \hat{c}^\dagger_{\boldsymbol{k}} \hat{c}_{\boldsymbol{k}} \\ &= \sum_{l} \sum_{\boldsymbol{k}''} \frac{e^{-i\boldsymbol{k}''\cdot\boldsymbol{r}_l}}{\sqrt{N}} \tilde{J}(\boldsymbol{k}'') \sum_{\boldsymbol{k}} e^{i\boldsymbol{k}\cdot\boldsymbol{r}_l} \ \hat{c}^\dagger_{\boldsymbol{k}} \hat{c}_{\boldsymbol{k}} \\ &= \sqrt{N} \sum_{\boldsymbol{k}} \tilde{J}(\boldsymbol{k}) \ \hat{c}^\dagger_{\boldsymbol{k}} \hat{c}_{\boldsymbol{k}} . \end{aligned} \tag{5.7}$$

In the second line, we defined $\boldsymbol{r}_l = \boldsymbol{r}_i - \boldsymbol{r}_j$ and $\tilde{J}(\boldsymbol{k})$ defined at the fifth line is the Fourier coefficient of the coupling strength between spins J_{ij}. We can also calculate

$$\begin{aligned} \sum_{i,j} J_{ij} \hat{C}_i \hat{C}_j^\dagger &= \sum_{i,j} J_{ij} \left(\hat{C}_j^\dagger \hat{C}_i + \delta_{ij} \right) \\ &= \sum_{i,j} J_{ij} \hat{C}_j^\dagger \hat{C}_i + \sum_i J_{ii} \\ &= \sqrt{N} \sum_{\boldsymbol{k}} \tilde{J}^*(\boldsymbol{k}) \ \hat{c}^\dagger_{\boldsymbol{k}} \hat{c}_{\boldsymbol{k}} + \sqrt{N} \sum_{\boldsymbol{k}} \tilde{J}(\boldsymbol{k}) , \end{aligned} \tag{5.8}$$

where we have used the relation $\tilde{J}(-\boldsymbol{k}) = \tilde{J}^*(\boldsymbol{k})$ stem from the fact J_{ij} is real[10]. Furthermore,

[10]Moreover, requireing $J_{ij} = J_{ji}$ of being probable, $\tilde{J}(\boldsymbol{k})$ is a real function.

repeating similar calculations, we get following relations:

$$\begin{aligned}\sum_{i,j} J_{ij} &= N\sqrt{N}\tilde{J}(\mathbf{0}) \ , \\ \sum_{i,j} J_{ij}\hat{C}_i^\dagger \hat{C}_i &= \sqrt{N}\sum_{\boldsymbol{k}} \tilde{J}(\mathbf{0})\ \hat{c}_{\boldsymbol{k}}^\dagger \hat{c}_{\boldsymbol{k}} \ , \\ \sum_{i,j} J_{ij}\hat{C}_j^\dagger \hat{C}_j &= \sqrt{N}\sum_{\boldsymbol{k}} \tilde{J}^*(\mathbf{0})\ \hat{c}_{\boldsymbol{k}}^\dagger \hat{c}_{\boldsymbol{k}} \ . \end{aligned} \tag{5.9}$$

Using Eqs. (5.7)-(5.9) in Eq. (5.6), we obtain

$$-\sum_{i,j} J_{ij}\hat{\boldsymbol{S}}_{(i)}\cdot\hat{\boldsymbol{S}}_{(j)} = -2S\sqrt{N}\sum_{\boldsymbol{k}} \mathrm{Re}\left(\tilde{J}(\boldsymbol{k})-\tilde{J}(\mathbf{0})\right)\ \hat{c}_{\boldsymbol{k}}^\dagger \hat{c}_{\boldsymbol{k}} - S\sum_{\boldsymbol{k}}\sqrt{N}\tilde{J}(\boldsymbol{k}) - S^2N\sqrt{N}\tilde{J}(\mathbf{0}) \ . \tag{5.10}$$

Finally, combing Eqs. (5.5) and (5.10), the Hamiltonian (5.1) can be rewritten in terms of magnons instead of spins as

$$\mathscr{H} = \sum_{\boldsymbol{k}}\left[2\mu_B B_z + 2\sqrt{N}S\ \mathrm{Re}\left(\tilde{J}(\mathbf{0})-\tilde{J}(\boldsymbol{k})\right)\right]\hat{c}_{\boldsymbol{k}}^\dagger \hat{c}_{\boldsymbol{k}} \ , \tag{5.11}$$

where we have omitted the parts which give a shift of a constant in the Hamiltonian. We see that the magnons has a dispersion relation

$$\omega_k = 2\mu_B B_z + 2\sqrt{N}S\ \mathrm{Re}\left(\tilde{J}(\mathbf{0})-\tilde{J}(\boldsymbol{k})\right) \ , \tag{5.12}$$

where ω_k represents the angular frequency of a k magnon mode, namely, the energy of a magnon particle with the momentum k. One can see that in particular the angular frequency of the uniform mode, $k = 0$, is given by the Larmor frequency $2\mu_B B_z$, which consists of the Bohr magneton μ_B, defined by the mass and charge of the particle, and the external magnetic field B_z. The angular frequencies of other modes except the uniform one further consist of the coupling strength between spins J_{ij}.

Several points should be mentioned about the dispersion relation of magnons (5.12); First, $\tilde{J}(0) = \frac{1}{\sqrt{N}}\sum_{\boldsymbol{r}_i} J(\boldsymbol{r}_i) > 0$ holds because $J(\boldsymbol{r}_i) > 0$. Second, admitting $J_{ij} = J_{ji}$ (see also the footnote 10),

$$\tilde{J}(\boldsymbol{k}) = \sum_i \frac{1}{\sqrt{N}}\cos(\boldsymbol{k}\cdot\boldsymbol{r}_i)J(\boldsymbol{r}_i) \ , \tag{5.13}$$

and thus in the long wave length limit, $\boldsymbol{k}\cdot\boldsymbol{r}_i \ll 1$, we have

$$\omega_k \simeq 2\mu_B B_z + S\sum_i (\boldsymbol{k}\cdot\boldsymbol{r}_i)^2\, J(\boldsymbol{r}_i)\ . \tag{5.14}$$

Finally, if one assume nearest neighbor interactions as J_{ij}, an approximated expression of (5.14) is

$$\omega_k \simeq 2\mu_B B_z + SJa^2k^2\ , \tag{5.15}$$

where J is the coupling strength of the nearest neighbor interaction between spins separated by a lattice constant a. We see that ω_k depends on k quadratically. This is a characteristic feature of magnons, for example, in contrast, it is linear dependence in the case of phonons.

As Eq. (5.12) shows, the magnon picture instead of spins is useful because we can solve the system analytically when the magnon occupancy is much less than unity. In the next section, we will include the effect of gravitational waves on magnons and reveal that magnons are excited by gravitational waves.

5.2 Graviton-magnon resonance

In the section 4.2, we revealed gravitational effects on a non-relativistic Dirac fermion in Fermi normal coordinates. As you can see in Eq. (4.47), if one consider a freely falling point particle and set a Fermi normal coordinate whose origin traces the particle, the particle does not feel perturbative gravity, h_{ij}, entirely. This is because of the equivalence principle. However, gravitational effects are canceled, of course, only at one point and thus an object with finite dimension feels gravity. In the case of magnons, we prepare, for example, a ferromagnetic sample in an external magnetic sample and then the sample feels gravity since it has finite size. It implies magnons can be excited by gravitational effects, especially by gravitational waves. To examine the effect of gravitational waves on magnons, it is appropriate to set a Fermi normal coordinate whose origin is placed at the center of the ferromagnetic sample. Then, we can apply the discussion of the section 4.2.

As in the previous section, we consider a ferromagnetic sample in an external magnetic field. Such system is described by the Heisenberg model (5.1):

$$H_{\rm spin} = -2\mu_B B_z \sum_i \hat{S}^z_{(i)} - \sum_{i,j} J_{ij} \hat{\boldsymbol{S}}_{(i)} \cdot \hat{\boldsymbol{S}}_{(j)} \ , \tag{5.16}$$

where the magnetic field direct the z-direction. In addition, we take into account the effect of gravitational waves on the system. From Eq. (4.47), the interaction Hamiltonian between gravitational waves and a spin in the ferromagnetic sample is

$$H_{\rm GW} = -\mu_B B_a \hat{S}^b_{(i)} Q_{ab} \ , \tag{5.17}$$

where we have defined

$$Q_{ij} = \frac{1}{4}\delta_{ij}\ddot{h}_{kl}x^k x^l - \frac{1}{12}\delta_{ia}\delta_{jb}\left(h_{al,kb} + h_{kb,al} - h_{kl,ab} - h_{ab,kl}\right) x^k x^l \ . \tag{5.18}$$

It represents the effect of gravitational waves in a Fermi normal coordinate. Indeed, at the origin, $x^i = 0$, we see that $Q_{ij} = 0$. From Eqs. (5.16) and (5.17), the total Hamiltonian of the system is

$$\begin{aligned} H_{\rm tot} &= H_{\rm spin} + H_{\rm GW} \\ &= -\mu_B \left(2\delta_{za} + Q_{za}\right) B_z \sum_i \hat{S}^a_{(i)} - \sum_{i,j} J_{ij} \hat{\boldsymbol{S}}_{(i)} \cdot \hat{\boldsymbol{S}}_{(j)} \ . \end{aligned} \tag{5.19}$$

We now rewrite the spin system by magnons with the Holstein-Primakoff transformation (5.2) and then we only focus on the homogeneous mode of magnons, so that the second term in the total Hamiltonian (5.19) is irrelevant (see Eq. (5.11)). Furthermore, because Q_{zz} does not contribute the resonance of spins, namely excitation of magnons, we will drop it. Thus we have

$$H_{\rm tot} = \mu_B B_z \sum_i \left[2\hat{C}^\dagger_i \hat{C}_i + \frac{\hat{C}_i + \hat{C}^\dagger_i}{2} Q_{zx} + \frac{\hat{C}_i - \hat{C}^\dagger_i}{2i} Q_{zy} \right] \ . \tag{5.20}$$

Now let us consider a planar gravitational wave propagating in the z-x plane, namely, the wave number vector of the gravitational waves $\boldsymbol{k}$ has a direction $\hat{k} = (\sin\theta, 0, \cos\theta)$. Moreover, we assume that the wave length of the gravitational wave is much longer than the dimension of the sample. This is the case of cavity experiments which we will utilize in the next chapter. We can expand the metric perturbations in terms of linear polarization tensors satisfying $e^{(\sigma)}_{ij} e^{(\sigma')}_{ij} = \delta_{\sigma\sigma'}$ as

$$h_{ij}(t) = h^{(+)}(t) e^{(+)}_{ij} + h^{(\times)}(t) e^{(\times)}_{ij} \ , \tag{5.21}$$

where we used the fact that the amplitude is approximately uniform over the sample. More explicitly, we took the representation

$$\begin{cases} h^{(+)}(t) = \dfrac{h^{(+)}}{2} \left(e^{-iw_h t} + e^{iw_h t} \right) \ , & (5.22) \\ h^{(\times)}(t) = \dfrac{h^{(\times)}}{2} \left(e^{-i(w_h t+\alpha)} + e^{i(w_h t+\alpha)} \right) \ , & (5.23) \end{cases}$$

where ω_h is an angular frequency of the gravitational wave and α represents a difference of the phases of polarizations. Note that the polarization tensors can be explicitly constructed as

$$e^{(+)}_{ij} = \frac{1}{\sqrt{2}} \begin{pmatrix} \cos\theta^2 & 0 & -\cos\theta\sin\theta \\ 0 & -1 & 0 \\ -\cos\theta\sin\theta & 0 & \sin\theta^2 \end{pmatrix} , \tag{5.24}$$

$$e^{(\times)}_{ij} = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & \cos\theta & 0 \\ \cos\theta & 0 & -\sin\theta \\ 0 & -\sin\theta & 0 \end{pmatrix} . \tag{5.25}$$

In above Eqs. (5.24) and (5.25), we defined + mode as a deformation in the y-direction.

Then substituting Eq. (5.21)-(5.25) into the total Hamiltonian (5.20), moving on to the Fourier space and using the rotating wave approximation, one can deduce

$$H_{tot} \simeq 2\mu_B B_z \hat{c}^\dagger \hat{c} + g_{eff} \left(\hat{c}^\dagger e^{-i\omega_h t} + \hat{c} e^{i\omega_h t} \right) , \tag{5.26}$$

where $\hat{c} = \hat{c}_{k=0}$ and

$$g_{eff} = \frac{\sqrt{2}\pi^2}{60} \left(\frac{l}{\lambda} \right)^2 \mu_B B_z \sin\theta \sqrt{N} \left[\cos^2\theta \, (h^{(+)})^2 + (h^{(\times)})^2 + 2\cos\theta \sin\alpha \, h^{(+)} h^{(\times)} \right]^{1/2} \ , \tag{5.27}$$

is an effective coupling constant between the gravitational wave and the magnons. The parameters l and $\lambda = 2\pi/\omega_h$ are the radius of the (spherical) ferromagnetic sample and the wavelength of the gravitational wave. We note that the sum over the spin sites i was evaluated as

$$\sum_i xx = \sum_i yy = \sum_i zz \simeq \frac{1}{L^3} \iiint_0^l r^2 \sin\zeta \left(r\cos\zeta\right)^2 dr d\zeta d\phi = \frac{4\pi}{15} \frac{l^5}{L^3} \ , \tag{5.28}$$

where L is a lattice constant, which is related to the number of spins as $N = \left(\frac{4\pi}{3} l^3\right)/L^3$. From Eq. (5.27), we see that the effective coupling constant has gotten a factor $\sqrt{N}$. Moreover, in order to obtain a coordinate-independent expression of g_{eff}, it is useful to use the Stokes parameters:

$$g_{eff} = \frac{\sqrt{2}\pi^2}{60} \left(\frac{l}{\lambda}\right)^2 \mu_B B_z \sin\theta \sqrt{N} \left[\frac{1+\cos^2\theta}{2} I - \frac{\sin^2\theta}{2} Q + \cos\theta\, V\right]^{1/2} \ , \tag{5.29}$$

where the Stokes parameters are defined by

$$\begin{cases} I = (h^{(+)})^2 + (h^{(\times)})^2 \ , \\ Q = (h^{(+)})^2 - (h^{(\times)})^2 \ , \\ U = 2\cos\alpha\, h^{(+)} h^{(\times)} \ , \\ V = 2\sin\alpha\, h^{(+)} h^{(\times)} \ . \end{cases} \tag{5.30}$$

They satisfy $I^2 = U^2 + Q^2 + V^2$. We see that the effective coupling constant depends on the polarizations. Note that the stokes parameters Q and U transform as

$$\begin{pmatrix} Q' \\ U' \end{pmatrix} = \begin{pmatrix} \cos 4\Psi & \sin 4\Psi \\ -\sin 4\Psi & \cos 4\Psi \end{pmatrix} \begin{pmatrix} Q \\ U \end{pmatrix} \tag{5.31}$$

where Ψ is the rotation angle around $\boldsymbol{k}$.

The second term in Eq. (5.26) shows that planar gravitational waves induce the resonant spin precessions if the angular frequency of the gravitational waves is near the Lamor frequency, $2\mu_B B_z$. It is worth noting that the situation is similar to the resonant bar experiments [16] where planar gravitational waves excite phonons in a bar detector.

In the next section, utilizing the graviton-magnon resonance, we will search for planar gravitational waves and give upper limits on GHz gravitational waves.

Chapter 6

Limits on GHz gravitational waves with magnons

In the previous section, we showed that planar gravitational waves can induce resonant spin precession of electrons, namely excitation of magnons. It is our observation that the same resonance is caused by coherent oscillation of the axion dark matter [41]. Recently, measurements of resonance fluorescence of magnons induced by the axion dark matter was conducted and upper bounds on an axion-electron coupling constant have been obtained [42, 43]. The point is that we can utilize these experimental results to give upper bounds on the amplitude of GHz gravitational waves. We will review how the axion-magnon resonance occurs [41] and draw a parallel between the axion dark matter and gravitational waves in the section 6.1. Next, in the section 6.2, we will give upper limits on planar gravitational waves in GHz range with the experimental results [27].

6.1 Axion-magnon resonance

The axion emerges as a Nambu-Goldstone boson of the broken Peccei-Quinn symmetry [44, 45, 46]. An axion field $a(x)$ can interact with the electron as

$$\mathscr{L}_{\rm int} = -ig_{aee}a(x)\bar{\psi}(x)\gamma_5\psi(x) \ , \tag{6.1}$$

where $\psi(x)$ denotes the electron and g_{aee}[11] is a dimensionless coupling constant. Especially, the interaction (6.1) is realized at tree level in the DFSZ model [47]. Taking the non-relativistic limit of the interaction term as is done in the section 4.2, one can get the Hamiltonian concerned with the spin of the electron as[12]

$$\mathscr{H}_{\rm int} \simeq -\frac{g_{aee}\hbar}{2m_e}\hat{\boldsymbol{\sigma}}\cdot\boldsymbol{\nabla}a = -2\mu_{\rm B}\hat{\boldsymbol{S}}\cdot\left(\frac{g_{aee}}{e}\boldsymbol{\nabla}a\right). \tag{6.2}$$

Here m_e is the mass of the electron, e is the elementary electric charge, $\mu_{\rm B} = e\hbar/2m_e$ is the Bohr magneton and the spin of the electron $\hat{\boldsymbol{S}}$ is related to the Pauli matrices $\hat{\boldsymbol{\sigma}}$ as $\hat{\boldsymbol{S}} = \hat{\boldsymbol{\sigma}}/2$. Note that we do not consider loop corrections of the Landé g-factor from the value 2.

As an analogue of the Pauli term, the term in the parenthesis of Eq. (6.2) can be regarded as an effective magnetic field defined by

$$\boldsymbol{B}_a(x) = \frac{g_{aee}}{e}\boldsymbol{\nabla}a(x)\ . \tag{6.3}$$

If the dark matter is the axion, such effective magnetic fields are ubiquitous around us. Also, properties of the effective magnetic field (6.3) reflect features of the axion dark matter.

The axion dark matter can be regarded as a classical (pseudo) scalar field oscillating at the bottom of the potential of the axion field [48, 49]. As a solution of the classical equation of motion, the axion dark matter is oscillating in time determined by the mass of the axion, so that the effective magnetic filed is oscillating with the frequency:

$$f_a = 0.24\left(\frac{m_a}{1.0\ \mu{\rm eV}}\right)\ {\rm GHz}\ . \tag{6.4}$$

We assume that the axion dark matter forms coherently oscillating solitonic objects which are the stable solution of the Schrödinger Poisson equation [48, 49]. The radius, namely the Jeans length, of such axion clumps can be estimated by applying the virial theorem to the object and assuming that the Jeans length is roughly equal to the de Broglie length. It leads

$$r_{\rm ob} \sim 6.8\times 10^{11}\left(\frac{1.0\mu{\rm eV}}{m_a}\right)^{1/2}\left(\frac{0.45\ {\rm GeV/cm^3}}{\rho_{\rm ob}}\right)^{1/4}[{\rm m}], \tag{6.5}$$

[11] We can rewrite the interaction (6.1) by using the background Dirac equation as $\tilde{g}_{aee}(\partial_\mu a)\bar{\psi}\gamma^\mu\gamma_5\psi(x)$, where $\tilde{g}_{aee} = \frac{g_{aee}}{2m_e}$. It clearly shows the shift symmetry of the axion field.

[12] In this case, it is more useful to use the exact Foldy Wouthuysen transformation [38] than the conventional one [36, 37].

where $\rho_{\rm ob}$ is the energy density of the object. Here, $\rho_{\rm ob}$ is assumed to be the local dark matter density, although it could be higher by several orders. Since the objects are moving with the virial velocity in the Galaxy v, the coherence time is estimated as

$$t_{\rm ob} \sim \frac{r_{\rm ob}}{v} = 2.3 \times 10^6 \times \left(\frac{1.0\ \mu eV}{m_a}\right)^{1/2} \left(\frac{0.45\ {\rm GeV/cm^3}}{\rho_{\rm ob}}\right)^{1/4} \left(\frac{300\ {\rm km/s}}{v}\right) [{\rm s}]. \quad (6.6)$$

We note that the coherence time in which the effective magnetic field keeps coherence is much longer than the observation time in magnon experiments we will consider in the next section. In the coherently oscillating object, we expect that the distribution of the axion field is almost homogeneous due to quantum effects. Even if so, the spatial gradient of the axion field is not zero since the object is moving with velocity v [50] relative to the laboratory frame. Then, we have a relation $\partial_i a \simeq m_a v a$ because the time for the moving object depends on the coordinates of our frame. Therefore, the effective magnetic field can be written as

$$\boldsymbol{B}_a(t) = \frac{\boldsymbol{B}_a}{2}\left(e^{-im_a t} + e^{im_a t}\right). \quad (6.7)$$

Let us consider the effect of the axion dark matter on magnons. We now consider a ferromagnetic sample which has N electronic spins in the axion dark matter background. Such a system is well described by the Heisenberg model [39]:

$$\mathscr{H}_{\rm tot} = -2\mu_{\rm B}\sum_i \hat{\boldsymbol{S}}_i \cdot (\boldsymbol{B}_0 + \boldsymbol{B}_a(t)) - \sum_{i,j} J_{ij}\hat{\boldsymbol{S}}_i \cdot \hat{\boldsymbol{S}}_j , \quad (6.8)$$

where $\boldsymbol{B}_0$ is an external magnetic field and i specifies each site of spins. The second term represents the exchange interactions between spins with the strength J_{ij}. We apply an external magnetic field along the z-direction. Without loss of generality, we can consider the direction of the effective magnetic field to lie in the z-x plane. Moreover, using the fact $B_a \ll B_0$, we have

$$\boldsymbol{B}_0 = (0,0,B_0) , \quad \boldsymbol{B}_a \simeq (B_a \sin\theta, 0, 0) . \quad (6.9)$$

Here, θ is an angle between the z-axis and the effective magnetic filed $\boldsymbol{B}_a$. We further move on to the magnon picture with the Holstein-Primakoff transformation (5.2). As is done in the section 5.2, substituting Eqs. (5.2), (6.7) and (6.9) into Eq. (6.8), in the Fourier space, we have

$$\mathscr{H}_{\rm tot} = 2\mu_{\rm B} B_z \hat{c}^\dagger \hat{c} + \tilde{g}_{eff} \left(\hat{c}^\dagger e^{-i m_a t} + \hat{c} e^{i m_a t} \right) \ , \tag{6.10}$$

where $\hat{c} = \hat{c}_{k=0}$ and we used the rotating wave approximation. Also, the effective coupling constant between the axion and the magnon is defined by

$$\tilde{g}_{eff} = 2\mu_{\rm B} \frac{B_a \sin\theta}{4} \sqrt{N} \ . \tag{6.11}$$

Comparing Eq. (5.26) and Eq. (6.10), we find that the effect of the gravitational wave has a same form as that of the axion dark matter. In fact, axion dark matter searches with measurements of resonance fluorescence of magnons was operated and upper bounds on the axion-electron coupling constant (6.1) have been obtained [42, 43]. Reading off upper bonds on $\tilde{g}_{eff}$ from the results enables us to give constraints on the g_{eff} as we will see soon. Moreover, we will see that one can constrain the amplitude of gravitational waves in GHz range.

6.2 Measurement of resonance fluorescence of magnons

Recently, measurements of resonance fluorescence of magnons induced by the axion dark matter was conducted and upper bounds on an axion-electron coupling constant have been obtained [42, 43]. We can utilize these experimental results to give upper bounds on the amplitude of GHz gravitational waves.

As we saw in the previous section, the interaction hamiltonian which describe excitation of magnons in the axion dark matter background is

$$H_{\rm axion} = \tilde{g}_{eff} \left(\hat{c}^\dagger e^{-i m_a t} + \hat{c} e^{i m_a t} \right) \ , \tag{6.12}$$

where $\tilde{g}_{eff}$ is an effective coupling constant between the axion and the magnons. Notice that the axion dark matter, which can be regarded as a classical field, oscillates with a frequency determined by the axion mass m_a (6.4). One can see that the form of (6.12) is same as the interaction term in Eq. (5.26). Through Eqs. (6.3) and (6.11), $\tilde{g}_{eff}$ is related to the axion-electron coupling constant g_{aee}. More explicitly, g_{aee} can be converted to $\tilde{g}_{eff}$ by using parameters, such as the energy density of the axion dark matter, the external magnetic fields, the numbers of spins in the ferromagnetic samples, etc, which are explicitly given in [42, 43]. Therefore constraints on $\tilde{g}_{eff}$ (95% C.L.) can be read from the constraints on the axion-electron coupling constant given in [42] and [43], respectively, as follows:

$$\tilde{g}_{eff} < \begin{cases} 7.0 \times 10^{-12}\ \text{eV}\ , \\ 6.2 \times 10^{-11}\ \text{eV}\ . \end{cases} \tag{6.13}$$

It is easy to convert the above constraints to those on the amplitude of gravitational waves appearing in the effective coupling constant (5.29). Indeed, we can read off the external magnetic field B_z and the number of electrons N as $(B_z, N) = (0.5\,\text{T},\ 5.6 \times 10^{19})$ from [42] and $(B_z, N) = (0.3\,\text{T},\ 9.2 \times 10^{19})$ from [43], respectively. The external magnetic field B_z determines the frequency of gravitational waves we can detect. Therefore, using Eqs. (5.29), (6.13) and above parameters, one can put upper limits on gravitational waves at frequencies determined by B_z. Since [42] and [43] focused on the direction of Cygnus and set the external magnetic fields to be perpendicular to it, we probe continuous gravitational waves coming from Cygnus with $\theta = \frac{\pi}{2}$ (More precisely, $\sin\theta = 0.9$ in [43]). We also assume no linear and circular polarizations, i.e., $Q' = U' = V = 0$. Consequently, experimental data [42] and [43] let us put upper bounds on the characteristic amplitude of gravitational waves defined by $h_c = h^{(+)} = h^{(\times)}$ as

$$h_c \sim \begin{cases} 1.3 \times 10^{-13} \quad \text{at 14 GHz}\ , \\ 1.1 \times 10^{-12} \quad \text{at 8.2 GHz}\ , \end{cases} \tag{6.14}$$

at 95 % C.L., respectively. In terms of the spectral density defined by $S_h = h_c^2/2f$ and the energy density parameter defined by $\Omega_{GW} = 2\pi^2 f^2 h_c^2/3H_0^2$ (H_0 is the Hubble parameter), the upper limits at 95 % C.L. are

$$\sqrt{S_h} \sim \begin{cases} 7.5 \times 10^{-19}\ [\text{Hz}^{-1/2}] \quad \text{at 14 GHz}\ , \\ 8.7 \times 10^{-18}\ [\text{Hz}^{-1/2}] \quad \text{at 8.2 GHz}\ , \end{cases} \tag{6.15}$$

and

$$h_0^2\Omega_{GW} \sim \begin{cases} 2.1\times 10^{29} & \text{at } 14\text{ GHz} \,, \\ 5.5\times 10^{30} & \text{at } 8.2\text{ GHz} \,. \end{cases} \tag{6.16}$$

We depicted the limits on the spectral density with several other gravitational wave experiments in Fig. 6.1.

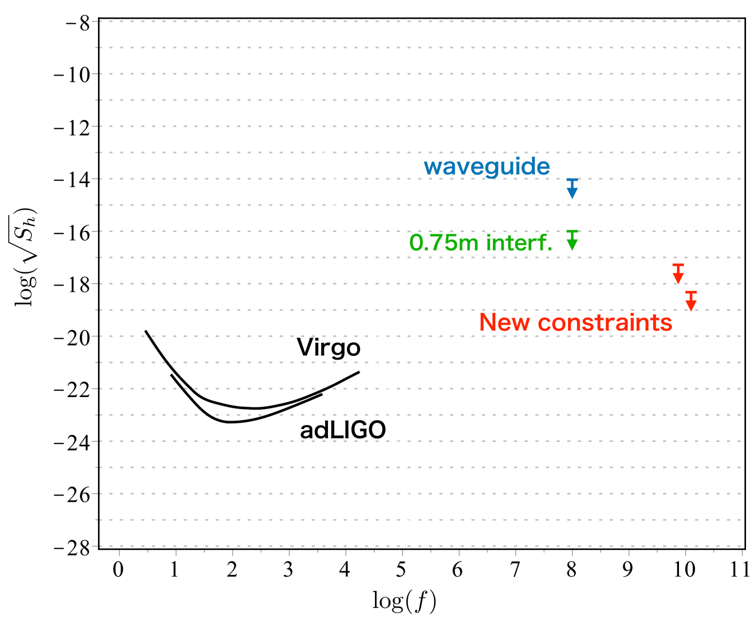

Figure 6.1: Several experimental sensitivities and constraints on high frequency gravitational waves are depicted. LIGO and Virgo have the sensitivity around 10^2 Hz [13, 14]. The blue color represents an upper limit on stochastic gravitational waves by waveguide experiment using an interaction between electromagnetic fields and gravitational waves [51]. The green one is the upper limit on stochastic gravitational waves, obtained by the 0.75 m interferometer [19]. Our new constraints on continuous gravitational waves are plotted with a red color, which also represent the sensitivity of the graviton-magnon detector for stochastic gravitational waves.

6.3 Discussion and future prospects

Interestingly, there are several theoretical models predicting high frequency gravitational waves which are with in the scope of our method [2]. Although we focussed on continuous gravitational waves to put experimental upper bounds, the graviton-magnon resonance is also useful for probing stochastic gravitational waves with almost the same sensitivity illustrated in Fig. 6.1. Moreover, we can probe burst gravitational waves of any wave form if the duration time is smaller than the relaxation time of a system. The situation is the same as resonant bar detectors [29]. For instance, in the measurements [42, 43], the relaxation time is about 0.1 μs, which is determined by the line width of the ferromagnetic sample and the cavity. If a duration of burst gravitational waves is smaller than 0.1 μs, we can detect it.

Taking a look at Eq. (6.16), we see that further improvement of the sensitivity is required to observe, for instance, stochastic gravitational waves. In order to improve the sensitivity, there are several potentials to pursue. Noises in a system of a measurement decide the actual sensitivity of the magnon detector and they are characterized by the line width of the ferromagnetic sample and the cavity [52, 53]. Therefore, improving the line width by purifying the sample and/or reducing the noises in the system leads to improvement of the sensitivity. Recall the effective coupling constant between a gravitational wave and magnons (5.29):

$$g_{eff} = \frac{\sqrt{2}\pi^2}{60}\left(\frac{l}{\lambda}\right)^2 \mu_B B_z \sin\theta\sqrt{N}\left[\frac{1+\cos^2\theta}{2}\,I - \frac{\sin^2\theta}{2}\,Q + \cos\theta\,V\right]^{1/2} .$$

Although it seems that getting the external magnetic field strong leads to larger coupling constant, one then have to remind that the detectable frequency of gravitational waves is also changed because it is determined by the Larmor frequency $2\mu_B B_z$. The most simple way to make the coupling constant large is to increase the number of spins in the ferromagnetic sample N. It is doable by finding a new ferromagnetic material or using a bigger sample. The former does not seem easy, but the latter would be possible. (Then the factor $\left(\frac{l}{\lambda}\right)^2$ in g_{eff} is also improved.) However, in general, a sufficiently bigger sample has a larger line width because inhomogeneity of cavity modes appleid on the sample becomes obvious and then the quality of the detector drops as a whole [54]. Then it seems that using several samples at the same time may be one of a best way to increase N. Actually, it has been done in [42]. Furthermore, there would be room for improvement on the method for analyzing data. For example, a matched filtering is useful for increasing the signal to noise ratio a few decades hopefully. As another way to improve sensitivity, quantum nondemolition measurement may be promising [55, 56, 57]. It would enable us to overcome the quantum limit of measurements. In particular, although we assumed that a gravitational wave was approximately monochromatic, there might be cases the approximation is not valid. In such cases, quantum nondemolition measurement would be useful.

Appendix. A

The geodesic deviation equation

In the section 3.1, we used the geodesic deviation equation to obtain the expression of second derivative of the metric in Fermi normal coordinates. A derivation of the geodesic deviation equation is given in this appendix. The discussion is based on [58] and [59].

Let us consider two geodesics γ_1 and γ_2 parameterized by an affine parameter λ and two geodesics are continuously connected by a parameter s. Then we define a tangent vector along γ_1:

$$u^\mu = \left(\frac{\partial x^\mu}{\partial \lambda}\right)_s \quad , \tag{A.1}$$

and a deviation vector evaluated on γ_1:

$$\xi^\mu = \left(\frac{\partial x^\mu}{\partial s}\right)_\lambda \quad . \tag{A.2}$$

ξ^μ which took to be orthogonal to u^μ, i.e., $\xi^\mu u_\mu = 0$, represents the deviation of the two neighbouring geodesics and our goal is to derive the evolution equation for ξ^μ with respect to λ.

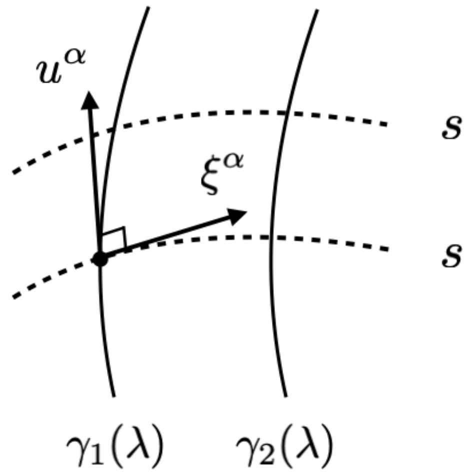

Figure A.1: Two neighbouring geodesics and the deviation vector.

We note that the orthogonality between ξ^μ and u^μ is hold at every point on the geodesic γ_1 if initially they were orthogonal at a some point on γ_1, indeed,

$$\begin{aligned} \frac{d(\xi^\mu u_\mu)}{d\lambda} &= (\xi^\mu u_\mu)_{;\beta} u^\beta \\ &= u^\mu{}_{;\beta} \xi^\beta u_\mu \\ &= \frac{1}{2}(u^\mu u_\mu)_{;\beta}\xi^\beta \\ &= 0 \ , \end{aligned} \tag{A.3}$$

where we used the fact $u_{\mu;\beta}u^\beta$ and $u^\mu{}_{;\beta}\xi^\beta = \xi^\mu{}_{;\beta}u^\beta$ which is a consequence of interchangeability of order of derivatives with respect to λ and s to obtain the second equality. Therefore, the ξ^μ takes a role of a geodesic deviation vector for any value of λ on γ_1.

We now consider the evolution of ξ^μ along the geodesic γ_1:

$$\begin{aligned} \frac{D^2\xi^\mu}{d\lambda^2} &= \left(\xi^\mu{}_{;\nu}u^\nu\right)_{;\lambda} u^\lambda \\ &= \left(u^\mu{}_{;\nu}\xi^\nu\right)_{;\lambda} u^\lambda \\ &= \left(-R^\mu_{\sigma\nu\lambda}u^\sigma + u^\mu{}_{;\lambda\nu}\right)\xi^\nu u^\lambda + u^\mu{}_{;\nu}\xi^\nu{}_{;\lambda}u^\lambda \\ &= -R^\mu_{\sigma\nu\lambda}u^\sigma u^\lambda \xi^\nu + \left(u^\mu{}_{;\lambda}u^\lambda\right)_{;\nu}\xi^\nu - u^\mu{}_{;\lambda}u^\lambda{}_{;\nu}\xi^\nu + u^\mu{}_{;\nu}u^\nu{}_{;\lambda}\xi^\lambda \\ &= -R^\mu_{\alpha\gamma\beta}u^\alpha u^\beta \xi^\gamma \ , \end{aligned} \tag{A.4}$$

where the left-hand side means a covariant derivative with respect to λ and we have used the definition of the Riemann tensor $\left[\nabla_\mu, \nabla_\nu\right] V^\alpha = R^\alpha{}_{\beta\mu\nu}$ at the third equality. We see that the geodesic deviation equation (A.4) is determined by the Riemann tensor, so that the right-hand side is zero in the flat spacetime. Actually, in the flat spacetime, the deviation of two freely falling objects should be a constant (if they were parallel initially) or proportional to λ and thus the second derivative of the deviation with respect to λ is zero. In contrast, no neighbouring geodesics always run parallel with each other even if they were parallel initially in general curved spacetime.

Finally, instead of the covariant form (A.4), we give a convenient expression of the geodesic deviation equation for the discussion in the section 3.1:

$$\frac{d^2\xi^\mu}{d\lambda^2} + 2\frac{d\xi^\alpha}{d\lambda}\Gamma^\mu_{\alpha\beta}u^\beta + \left(R^\mu_{\alpha\gamma\beta} + \Gamma^\mu_{\alpha\gamma,\beta} + \Gamma^\mu_{\beta\delta}\Gamma^\delta_{\alpha\gamma} - \Gamma^\mu_{\gamma\delta}\Gamma^\delta_{\alpha\beta}\right) u^\alpha u^\beta \xi^\gamma = 0 \ . \tag{A.5}$$

Appendix. B

Expansion in powers of S

We consider expanding $e^{iS}He^{-iS}$ and $\left(\frac{\partial}{\partial t}e^{iS}\right)e^{-iS}$ in powers of S. To do so, we introduce a fake parameter λ as

$$f(\lambda, S) \equiv e^{i\lambda S}He^{-i\lambda S} \ , \tag{B.1}$$

and set $\lambda = 1$ finally. Expanding it with respect to λ, we obtain

$$f(\lambda, S) = \sum_{n=0}^{\infty} \frac{\lambda^n}{n!} \left(\frac{\partial^n f(\lambda, S)}{\partial \lambda^n} \right)_{\lambda=0} \ . \tag{B.2}$$

We find that

$$\begin{aligned} \frac{\partial f(\lambda, S)}{\partial \lambda} &= e^{i\lambda S} i \big[S, H\big] e^{-i\lambda S} \ , \\ \frac{\partial^2 f(\lambda, S)}{\partial \lambda^2} &= e^{i\lambda S} i^2 \big[S, \big[S, H\big]\big] e^{-i\lambda S} \ , \\ &\ \vdots \\ \frac{\partial^n f(\lambda, S)}{\partial \lambda^n} &= e^{i\lambda S} i^n \big[S, \big[S, \ldots, \big[S, H\big]\big] \cdots \big]\big] e^{-i\lambda S} \ . \end{aligned} \tag{B.3}$$

Therefore, one can deduce

$$\begin{aligned} e^{iS}He^{-iS} &= f(1, S) \\ &= \sum_{n=0}^{\infty} \frac{i^n}{n!} \big[S, \big[S, \ldots, \big[S, H\big]\big] \cdots \big]\big] \ . \end{aligned} \tag{B.4}$$

Eq. (B.4) is called the Campbell-Baker-Hausdorff formula.

Next, let us consider the expansion of $\left(\frac{\partial}{\partial t}e^{iS}\right)e^{-iS}$ in powers of S. Again we introduce a fake parameter λ and expand it with respect to λ:

$$\begin{aligned} g(\lambda, S) &= \left(\frac{\partial}{\partial t}e^{i\lambda S}\right)e^{-i\lambda S} \\ &= \sum_{n=0}^{\infty}\frac{\lambda^n}{n!}\left(\frac{\partial^n g(\lambda, S)}{\partial \lambda^n}\right)_{\lambda=0} . \end{aligned} \tag{B.5}$$

We see that

$$\begin{aligned} \frac{\partial g(\lambda, S)}{\partial \lambda} &= e^{i\lambda S}\, i\dot{S}\, e^{-i\lambda S} , \\ \frac{\partial^2 g(\lambda, S)}{\partial \lambda^2} &= e^{i\lambda S} i^2 \big[S, \dot{S}\big] e^{-i\lambda S} , \\ &\vdots \\ \frac{\partial^{n+1} f(\lambda, S)}{\partial \lambda^{n+1}} &= e^{i\lambda S} i^{n+1} \big[S, \big[S, \ldots, \big[S, \dot{S}\big]\big] \cdots \big]\big] e^{-i\lambda S} . \end{aligned} \tag{B.6}$$

Note that the right-hand side of the last equation has n pieces of S. Hence, we finally arrive at

$$\begin{aligned} \left(\frac{\partial}{\partial t}e^{iS}\right)e^{-iS} &= g(1, S) \\ &= \sum_{n=0}^{\infty}\frac{i^{n+1}}{(n+1)!}\big[S, \big[S, \ldots, \big[S, \dot{S}\big]\big] \cdots \big]\big] . \end{aligned} \tag{B.7}$$

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