



Stability analysis of various tensor fields on black holes

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Doctoral Dissertation

Stability analysis of various tensor fields
on black holes

(ブラックホール上の様々なテンソル場の安定性解析)

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Abstract

The problem of solving field equations on black holes has been a long-standing challenge in physics and mathematics. A simple example is gravitational waves which were predicted in [1]. It was detected in 2015 [2]^{*1}. But gravitational waves are not all the fields of interest on black holes. In principle, there can be other fields on the black hole. Moreover, from the point of a unified theory, the dimension is not limited to four dimensions. Therefore, it is important to consider various fields, establish a method for analyzing them, and derive their properties. However, the behavior of differential forms on a black hole has not been studied. Differential forms are a wider class of geometric quantities that include scalar fields and vector fields. In this thesis, we focus on the dynamics of the differential form on the black hole. Then, we will do the following. (i) We explain an appropriate decomposition method for a general form field under the spherical symmetry. (ii) Using the decomposition method, we derive the master equations of the arbitrary rank form field in any dimension. (iii) We study the master equation of the form field by the S -deformation method and show that the form field on the black hole is stable. (iv) By applying the WKB method, we calculate the quasinormal mode of the form field on the black hole. We also reveal the dimension dependence and the duality of the form field. Thus, we clarify the general properties of form fields on spherically symmetric black holes.

^{*1} Of course, the detection and analysis of the binary pulsar by [3, 4] gave the evidence of the existence of gravitational waves.

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Part I

Fundamentals

In this part, we note some key tools to study our final result. Firstly, we give a short review of the interest in gravitational waves. In 1915, Einstein established general relativity. In the next year of Einstein's work, Schwarzschild found the solution of general relativity [5]. This solution was named a black hole. In 2015, we have firstly detected the shape of the gravitational waves [2].

Chapter 1

Introduction

Black holes are important objects from various perspectives. Classically, general relativity, describing strong gravity, predicts the existence of black holes. At the same time, general relativity also predicts the existence of gravitational waves [1], like electromagnetic waves in Maxwell's theory. A black hole is a massive celestial object, and its motion causes spacetime distortion, propagating to its surroundings. Recent remarkable technological developments in the experimental field have made it possible to directly observe the waveform of the gravitational waves, which give us the knowledge of the physics of black holes in general relativity [2].

On the other hand, the quantization of gravity is a longstanding problem in physics. Despite the tremendous efforts of the predecessors, the full quantization of gravity theory has not been achieved. However, Hawking predicted an interesting phenomenon, that is, the Hawking radiation [6]. This is a prediction of the fundamental quantum effect associated with black holes that may lead to its mystery. When thinking of the spacetime of a black hole, certain particles, for example, a scalar field, a vector field, and a second-order symmetric tensor field^{*1} might be generated by quantum effects. Additionally, string theories, which unifies gravity and all other physics, assumes the existence of higher dimensions. Indeed, higher-dimensional black holes may be created at accelerators such as the large hadron collider [7]. Given these facts, it is more natural to consider the interaction of the black hole with the differential form fields $\mathbf{A}_p$ which include the scalar and vector fields. Also exploring these whole quantum effects may reveal the nature of quantum gravity theory. The purpose of this research

^{*1} Here the quantum effect of perturbational gravity is assumed. In this sense, investigating the linearized Einstein equation for the perturbation metric is equivalent to considering the motion of a second-order symmetric tensor field $h_{\mu\nu}$ on a black hole as the background spacetime.

is to understand the dynamics of the various tensor fields on such a black hole.

What is needed to achieve this goal is the derivation of the equation of motion of the field on the black hole. Here the structure of this equation of motion determines the difficulty of the problem.

The research on the interaction between a black hole and matter fields originated from the study of the gravitational wave equation on a Schwarzschild black hole by Regge and Wheeler [8]. When considering scalar fields, the Klein-Gordon equation is simple and separable. However, the case of a second-order symmetric tensor $h_{\mu\nu}$ is not so simple. Here, the key is the symmetry of the black hole in the background spacetime. Regge and Wheeler focused on the spherical symmetry and examined the linearized Einstein equation. After choosing an appropriate variable (called a master variable), the equation of motion in the radial direction was found. They were able to derive the equation of motion only for the vector component of the perturbation metric $h_{\mu\nu}$, but Zerilli [9], derived the equation of motion for the remaining scalar component, and all the components of equations of motion for the gravitational waves on a Schwarzschild black hole in four dimensions were obtained.

The independent equations of motion corresponding to the two degrees of freedom derived in this way are second-order linear ordinary differential equations that are written in the form of the Schrödinger equation. These equations are called the master equations. Although we have been focusing on the case of the gravitational perturbation, the equations of motion for the scalar and vector on a spherical black hole can be expressed by the master equations of the same form. The quantity corresponding to the potential of the Schrödinger equation reflects all physical information such as the type of field (such as a scalar, vector, or perturbation tensor) on the black hole and the mass and charge of the black hole in the background spacetime. The equation of motion provides physical information such as the stability of the field concerning time evolution, for example, see [10–12], and the quasinormal modes which reflects the geometric quantity of the background black hole and the matter field [13–18].

These equations of motion were derived in a four-dimensional spherically symmetric black hole. As mentioned earlier, higher-dimensional black holes are also important objects when considering quantum gravity theory represented by string theory. Some solutions predicted by Einstein’s general relativity (for example, Schwarzschild and

Kerr solutions) are naturally extended to higher dimensions [19, 20], and the stability of spherical solutions is discussed by [21–26]. Another candidate of the generally covariant theory having at most the second-order derivative is Lovelock theory [27]. A spherically symmetric solution is also known in this theory by [28–30], and the stability of gravitational perturbation is discussed by [31–34] and its quasinormal modes are calculated by [35]. However, as mentioned above, in quantum gravity theory assuming high dimensions, the research on differential form fields $\mathbf{A}_p$ on black holes is needed, which is still lacking to our best knowledge. Along this line, we focus on the spherical black holes in this thesis.

In this thesis, we will do the following:

1. We explain an appropriate decomposition method for a general form field on the spherical black hole.
2. Using the decomposition method, we derive the master equations of the arbitrary rank form field in any dimension.
3. We study the master equation of the form field by the S -deformation method and show that the form field on the black hole is stable.
4. By applying the WKB method, we calculate the quasinormal modes of the form field on the black hole.

Thus, we clarify the general properties of form fields on spherically symmetric black holes. These are the main results of this research.

The organization of this thesis consists of two parts.

■ **In Part I** We describe the tools used in Part II.

In Chap. 2 We review the perturbation theory of black holes. We derive the master equation in the cases of the scalar Φ , vector A_μ , perturbed metric $h_{\mu\nu}$ on black hole spacetime.

In Chap. 3 We introduce the S -deformation method by [21]. This is useful to study the stability of the master variable. This method gives the general property of the field on a black hole.

In Chap. 4 We mention the quasinormal modes. In general, they are damping oscillations with imaginary discrete frequencies. Their frequencies are determined by the mass of black hole, dimensions, and type of fields and does not depend on the

initial condition of the field on a black hole.

■ In Part II We give the main result of this thesis.

In Chap. 5 We review the properties of the p -form field. In particular, we count the physical degrees of freedom of the p -form field and study its gauge fixing. Then, we represent the p -form field by coexact form fields on the sphere. We also discuss spherical harmonics.

In Chap. 6 We obtain the master equations for a p -form field in spherical black holes in arbitrary dimensions. The master variables are only the 0-form, coexact p -form and the coexact $(p - 1)$ -form only on the sphere. We check their degrees of freedom match the physical degrees of freedom of the p -form field. We also find useful duality relations for the effective potentials.

In Chap. 7 We explicitly verify the stability of the p -form field using the S -deformation technique, and it is shown that the massless general form fields on a spherical black hole are stable in arbitrary dimension.

In Chap. 8 We investigate the quasinormal modes of the p -form fields using the WKB approximation.

In Chap. 9 We summarize this thesis and conclude with an outlook.

Chapter 2

Basics of classical field theory in higher dimensions

2.1 General relativity

In this section, we introduce the action for the general relativity in 4 dimensions and higher dimensions. The general relativity is the theory of spacetime. The spacetime specify by the metric,

$$ds^2 = g_{\mu\nu} dx^\mu dx^\nu \quad (2.1)$$

is called an infinitesimal world displacement. By the existence of matter, the metric is “distorted” .

The action of general relativity is given by

$$S_{\text{GR}} = \kappa \int d^D x \sqrt{-g} (R - 2\Lambda). \quad (2.2)$$

Here, Λ is a cosmological constant. The action for gravity and matter is given by

$$S = S_{\text{GR}} + S_{\text{matter}}. \quad (2.3)$$

For example, when we consider a scalar field, the action is given by

$$S_{\text{matter}} = S_{\text{scalar}} = \int d^D x \sqrt{-g} \left[-\frac{1}{2} \nabla_\mu \Phi \nabla^\mu \Phi - V(\Phi) \right]. \quad (2.4)$$

In the case of electromagnetic fields, the action is given by

$$S_{\text{matter}} = S_{\text{EM}} = \int d^D x \sqrt{-g} \left[-\frac{1}{4} F_{\mu\nu} F^{\mu\nu} - A_\mu J^\mu \right]. \quad (2.5)$$

Here, the field strength $F_{\mu\nu}$ is defined by

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu, \quad (2.6)$$

and J^μ is a current.

Taking the variation with respect to the metric $g_{\mu\nu}$, we can derive the equations of motion.

$$\begin{aligned} 0 = \delta S_{\text{EH}} &= \kappa \int d^D x \delta \{ \sqrt{-g} (R - 2\Lambda) \} \\ &= \kappa \int d^D x \sqrt{-g} \left(R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} + \Lambda g_{\mu\nu} \right) \delta g^{\mu\nu} + \kappa \int d^D x \sqrt{-g} g^{\mu\nu} \delta R_{\mu\nu} \end{aligned} \quad (2.7)$$

The second term can be arranged as follows,

$$\begin{aligned} \int d^D x \sqrt{-g} g^{\mu\nu} \delta R_{\mu\nu} &= \int d^D x \sqrt{-g} g^{\mu\nu} (\nabla_\alpha \delta \Gamma_{\mu\nu}^\alpha - \nabla_\mu \delta \Gamma_{\nu\alpha}^\alpha) \\ &= \int d^D x \sqrt{-g} \{ \nabla_\alpha (g^{\mu\nu} \delta \Gamma_{\mu\nu}^\alpha) - \nabla_\mu (g^{\mu\nu} \delta \Gamma_{\nu\alpha}^\alpha) \} \\ &= \int d^D x \sqrt{-g} \nabla_\alpha (g^{\mu\nu} \delta \Gamma_{\mu\nu}^\alpha - g^{\alpha\nu} \delta \Gamma_{\nu\mu}^\mu). \end{aligned} \quad (2.8)$$

This term is the surface integral. When we treat the entropy of the system, this term becomes important. Here, we neglect this term. In this situation the variation respect to the metric becomes

$$\begin{aligned} \delta S_{\text{EH}} &= \kappa \int d^D x \sqrt{-g} \left(R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} + \Lambda g_{\mu\nu} \right) \delta g^{\mu\nu} \\ &= \kappa \int d^D x \sqrt{-g} (G_{\mu\nu} + \Lambda g_{\mu\nu}) \delta g^{\mu\nu}, \end{aligned} \quad (2.9)$$

where $G_{\mu\nu}$ is the Einstein tensor is defined by

$$G_{\mu\nu} \equiv R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu}. \quad (2.10)$$

The divergence of this tensor is nothing but the Bianchi identity, that is,

$$\nabla_\mu G^{\mu\nu} = 0. \quad (2.11)$$

On the other hand, the action of matter becomes

$$\delta S_{\text{matter}} = -\frac{1}{2} \int d^D x \sqrt{-g} T_{\mu\nu} \delta g^{\mu\nu}. \quad (2.12)$$

Here, $T_{\mu\nu}$ is the energy momentum tensor, is defined by

$$T_{\mu\nu} = -2 \frac{1}{\sqrt{-g}} \frac{\delta S_{\text{matter}}}{\delta g^{\mu\nu}}. \quad (2.13)$$

This tensor satisfies $\nabla^\mu T_{\mu\nu} = 0$. Then, the equation of motion is derived as

$$\begin{aligned} 0 &= \delta S \\ &= \delta S_{\text{EH}} + \delta S_{\text{matter}} \\ &= \kappa \int d^D x \sqrt{-g} (G_{\mu\nu} + \Lambda g_{\mu\nu}) \delta g^{\mu\nu} - \frac{1}{2} \int d^D x \sqrt{-g} T_{\mu\nu} \delta g^{\mu\nu} \\ &= \int d^D x \sqrt{-g} \kappa \left(G_{\mu\nu} + \Lambda g_{\mu\nu} - \frac{1}{2\kappa} T_{\mu\nu} \right) \delta g^{\mu\nu}. \end{aligned} \quad (2.14)$$

Thus we get the Einstein equation

$$G_{\mu\nu} + \Lambda g_{\mu\nu} = \frac{1}{2\kappa} T_{\mu\nu}. \quad (2.15)$$

This equation is often written down in an alternative form. If we take the trace of the Einstein tensor, the left hand side becomes

$$\begin{aligned} g^{\mu\nu} G_{\mu\nu} &= g^{\mu\nu} \left(R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} + \Lambda g_{\mu\nu} \right) \\ &= R - \frac{D}{2} R + D\Lambda \\ &= \frac{2-D}{2} R + D\Lambda. \end{aligned} \quad (2.16)$$

The trace of the energy momentum tensor T is defined

$$T \equiv g^{\mu\nu} T_{\mu\nu}. \quad (2.17)$$

Then, Eq. (2.16) becomes

$$\begin{aligned} g^{\mu\nu} (G_{\mu\nu} + \Lambda g_{\mu\nu}) &= \frac{1}{2\kappa} g^{\mu\nu} T_{\mu\nu} \\ \Leftrightarrow \frac{2-D}{2} R + D\Lambda &= \frac{1}{2\kappa} T \\ \Leftrightarrow R &= \frac{2}{2-D} \left(\frac{1}{2\kappa} T - D\Lambda \right). \end{aligned} \quad (2.18)$$

Finally, the Einstein equation becomes

$$R_{\mu\nu} + \frac{2}{2-D} \Lambda g_{\mu\nu} = \frac{1}{2\kappa} \left(T_{\mu\nu} + \frac{1}{2-D} T g_{\mu\nu} \right). \quad (2.19)$$

In four dimensions the constant κ is defined by

$$\kappa = \frac{1}{16\pi G}, \quad (2.20)$$

where G is the Newton constant. This constant is determinant by the consistency with the Newton gravity.

2.2 Spherical black hole solution

In this section, we consider a solution of the Einstein equation founded by Karl Schwarzschild in 1916 [5]. This solution is derived by assuming the spherical symmetry. This solution have the following form,

$$ds^2 = -f(r) dt^2 + \frac{1}{f(r)} dr^2 + r^2 d\Omega^2. \quad (2.21)$$

Where, f is the function of r , which is given by

$$f = 1 - \frac{2GM}{r} \quad (2.22)$$

and we defined

$$d\Omega^2 = d\theta^2 + \sin^2 \theta d\phi^2.$$

The metric is time-independent, that is, the solution is static. If $M \rightarrow 0$, this metric becomes the flat spacetime in spherical coordinate.

It is not difficult to generalize the Schwarzschild black hole to higher dimensions $D > 4$, and the solutions are called the Schwarzschild-Tangherlini black hole [19] expressed by the metric

$$ds^2 = -f(r) dt^2 + \frac{1}{f(r)} dr^2 + r^2 q_{AB} dx^A dx^B, \quad (2.23)$$

where $f(r)$ is given by

$$f(r) = 1 - \frac{\mathcal{M}}{r^{n-1}}, \quad (2.24)$$

and n is defined as $n \equiv D - 2$. Here, q_{AB} is the metric of the sphere S^n . The spherical coordinate x^A is expressed by

$$x^A = (\theta^1, \dots, \theta^n). \quad (2.25)$$

The constant $\mathcal{M}$ is related to the ADM mass of the black hole in higher dimension. If $\mathcal{M} \rightarrow 0$, this metric becomes the flat spacetime in the spherical coordinate in higher dimensions. The derivation of this metric can be found in appendix. C.

2.3 Field equation on black hole

In this section, we derive equations of motion for various fields to investigate the behavior of fields on a black hole. However, if the field equations have a mathematically complex structure, they cannot be solved and are difficult to analyze. However, by focusing on the symmetry of black holes in the background spacetime, the equation of motion can be simplified. This thesis deals with spherically symmetric black holes. Due to this spherical symmetry, the angular direction and radial direction can be separated by using the spherical harmonics. In order to execute procedure, here we will derive the equation of motion through the following three steps. First, we choose the appropriate variables for the field. Second, the equation of motion is separated using the spherical harmonics. Third, we select the coordinate transformation of radial coordinates and take a variable called the master variable. Through these steps, a second-order linear ordinary differential equation in the form of the Schrödinger equation, called the master equation, can be derived. The field equations can be made into differential equations in the form known in quantum mechanics, which makes subsequent analysis easier. This section deals with a scalar field, vector field, and a gravitational field. In Part. II, we treat p -form field which is the totally antisymmetric tensor field with a rank p . In this thesis, we assume the time dependence of fields is $e^{-i\omega t}$.

2.3.1 Scalar field in 4 dimensions

A scalar field is organized by the Klein-Gordon equation

$$\square\Phi = 0, \tag{2.26}$$

where the operator $\square$ is defined by

$$\square\Phi \equiv \nabla_\mu \nabla^\mu \Phi \tag{2.27}$$

$$= \frac{1}{\sqrt{-g}} \partial_\mu (\sqrt{-g} g^{\mu\nu} \partial_\nu \Phi). \tag{2.28}$$

For a scalar field, the first step can be skipped. The next step is the expansion by the spherical harmonics $Y_{\ell m}(\theta, \phi)$. Then, Φ becomes

$$\Phi = \sum_{l=0}^{\infty} \sum_{m=-\ell}^{\ell} e^{-i\omega t} R_{\ell}(r) Y_{\ell m}(\theta, \phi) \quad (2.29)$$

$$= \sum_{l=0}^{\infty} \Phi_{\ell}. \quad (2.30)$$

Here, we assume the time dependence of Φ have $e^{-i\omega t}$ and define Φ_{ℓ} as

$$\Phi_{\ell} \equiv \sum_{m=-\ell}^{\ell} e^{-i\omega t} R_{\ell}(r) Y_{\ell m}(\theta, \phi). \quad (2.31)$$

So, the Klein-Gordon equation for Φ_{ℓ} becomes

$$0 = \frac{1}{\sqrt{-g}} \partial_{\mu} (\sqrt{-g} g^{\mu\nu} \partial_{\nu} \Phi_{\ell}) \quad (2.32)$$

$$= -\frac{1}{f} \partial_t^2 \Phi_{\ell} + \frac{1}{r^2} \partial_r (r^2 f \partial_r \Phi_{\ell}) + \frac{1}{r^2} \hat{\Delta} \Phi_{\ell} \quad (2.33)$$

$$= \frac{1}{f} \omega^2 \Phi_{\ell} + \frac{1}{r^n} \partial_r (r^n f \partial_r \Phi_{\ell}) - \frac{1}{r^2} \gamma \Phi_{\ell}. \quad (2.34)$$

Here, $\hat{\Delta}$ means the Laplace operator with the spherical coordinate, and the spherical harmonics satisfies

$$\hat{\Delta} Y_{\ell m} = -\gamma Y_{\ell m} = -\ell(\ell+1) Y_{\ell m}. \quad (2.35)$$

Using this property of the spherical harmonics, we can get the simple equation of motion for a scalar field.

The third step is the coordinate transformation of the radial direction r and the choice of the master variable. We change the radial coordinate r to x which is defined by

$$\frac{dx}{dr} = \frac{1}{f}, \quad (2.36)$$

or

$$x = \int^x dx' = \int^r \frac{1}{f} dr' = \int^r \frac{r}{r - \mathcal{M}} dr' = r + \mathcal{M} \ln(\mathcal{M} - r). \quad (2.37)$$

This radial coordinate is called tortoise coordinate. The range of this coordinate x is

$$x : -\infty \rightarrow \infty.$$

Here, $x = -\infty$ corresponds to the horizon at $r = \mathcal{M}$, and $x = \infty$ corresponds to the radial infinity at $r = \infty$.

For a scalar field, we can choose the master variable Ψ defined by

$$\Psi \equiv r\Phi_\ell. \quad (2.38)$$

So, the Klein-Gordon equation becomes

$$\begin{aligned} 0 &= \frac{1}{\sqrt{-g}} \partial_\mu (\sqrt{-g} g^{\mu\nu} \partial_\nu \Phi_\ell) \\ &= \frac{1}{rf} \left(\omega^2 \Psi + \frac{d^2 \Psi}{dx^2} - f \left(\frac{\gamma + rf'}{r^2} \right) \Psi \right) \\ \Rightarrow \quad \omega^2 \Psi &= -\frac{d^2 \Psi}{dx^2} + f \left(\frac{\gamma + rf'}{r^2} \right) \Psi. \end{aligned} \quad (2.39)$$

and this equation is the master equation for a scalar field. Here, $'$ means the derivative with respect to r . This is the simplest example of the master equations.

2.3.2 Vector field in 4 dimensions

In this section, we treat the case of a massless vector field A_μ which obeys the Maxwell equation. There are the two differences from the case of a scalar field. The one is that A_μ has four functional degrees of freedom in four dimensions, and the other is that A_μ has a gauge degree of freedom. So the first step for the derivation of the master equations becomes important.

■ **Gauge fixing of vector fields** Firstly, in four-dimensional spacetime, a vector field can have four independent degrees of freedom and can be written as

$$A_\mu = \begin{pmatrix} A_t \\ A_r \\ A_A \end{pmatrix}. \quad (2.40)$$

Under the spherical coordinate transformation, A_t and A_r are transformed as scalar fields. So, we rewrite them $\mathcal{A}$ and $\bar{\mathcal{A}}$ respectively. On the other hand, for this transformation A_A behaves a vector field under the spherically coordinate transformation. Additionally, A_A can be decomposed as

$$A_A \equiv V_{:A} + v_A, \quad (2.41)$$

where V is a scalar function and a vector v_A satisfies

$$v_A{}^{;A} = 0. \quad (2.42)$$

Here, $:$ means a covariant derivative with the metric q_{AB} . So, v_A has one functional degree of freedom. Thus, we can express A_μ as

$$A_\mu = \begin{pmatrix} \mathcal{A} \\ \bar{\mathcal{A}} \\ V_{;A} + v_A \end{pmatrix}. \quad (2.43)$$

From the action of the vector field, we see the Maxwell equation is invariant under the gauge transformation with parameter Λ ,^{*1},

$$A_\mu \rightarrow \underline{A}_\mu = A_\mu + \partial_\mu \Lambda.$$

Such an arbitrariness of the field A_μ is called a gauge degree of freedom. With this freedom, each component of A_μ is transformed as follows,

$$\begin{aligned} \underline{\mathcal{A}} &= \mathcal{A} + \dot{\Lambda}, \\ \underline{\bar{\mathcal{A}}} &= \bar{\mathcal{A}} + \Lambda', \\ \underline{V}_{;A} + \underline{v}_A &= V_{;A} + v_A + \Lambda_{;A} \text{ and} \\ &= (V + \Lambda)_{;A} + v_A. \end{aligned}$$

Here, the dot $\dot{\cdot}$ means the derivative with respect to the time t . So, we can choose Λ so that

$$\underline{V} = 0. \quad (2.44)$$

This is called the choice of the gauge, and the constraint in Eq. (2.44) is called the gauge condition. In this gauge, there is no arbitrary gauge transformation that maintains this gauge condition in Eq. (2.44). This is called complete gauge fixing. After all, we succeeded in reducing the components of a vector field. It is represented by the two scalars $\mathcal{A}$ and $\bar{\mathcal{A}}$ and the vector v_A which have one degree of freedom, that is, A_μ becomes

$$A_\mu = \begin{pmatrix} \mathcal{A} \\ \bar{\mathcal{A}} \\ v_A \end{pmatrix}. \quad (2.45)$$

This is different from the case of a scalar field.

^{*1} This Λ is not related to cosmological constant. Here, this is an arbitrary function for x^μ .

The second step is the use of the spherical harmonics. We can expand the two scalars $\mathcal{A}$ and $\bar{\mathcal{A}}$ by the spherical harmonics. While, the vector field v_A can be expanded by the vector spherical harmonics $\mathcal{Y}_{\ell m}^A(\theta, \phi)$ as

$$v^A(t, r, \theta, \phi) = \sum_{\ell, m} v_{\ell m}(t, r) \mathcal{Y}_{\ell m}^A(\theta, \phi). \quad (2.46)$$

These harmonics $\mathcal{Y}_{\ell m}^A(\theta, \phi)$ satisfy

$$(\mathcal{Y}_{\ell m}^A)_{:A} = 0, \quad (2.47)$$

and

$$(\mathcal{Y}_{\ell m}^A)_{:B}{}^{:B} = \hat{\Delta} \mathcal{Y}_{\ell m}^A = -\gamma_{\mathcal{V}} \mathcal{Y}_{\ell m}^A. \quad (2.48)$$

The operator $\hat{\Delta}$ means the Laplacian with respect to the spherical coordinate. In the four dimensions, we can construct of the vector harmonics from the scalar harmonics $Y_{\ell m}$ as

$$\mathcal{Y}_{\ell m}^A \equiv \epsilon^{AB} (Y_{\ell m})_{:B}. \quad (2.49)$$

These harmonics satisfy the divergence condition in Eq. (2.42), and the spectrum can be derived as

$$\begin{aligned} \hat{\Delta} (\mathcal{Y}_{\ell m}^A) &\equiv (\mathcal{Y}_{\ell m}^A)_{:B}{}^{:B} \\ &= -(\ell(\ell+1) - 1) \mathcal{Y}_{\ell m}^A \end{aligned} \quad (2.50)$$

by using $(Y_{\ell m})_{:B}{}^{:B} = \hat{\Delta} Y_{\ell m} = -\ell(\ell+1) Y_{\ell m}$. So the spectrum of the vector harmonics is given by

$$\gamma_{\mathcal{V}} = \ell(\ell+1) - 1. \quad (2.51)$$

Here, we assumed the time dependence of $v_{\ell m}$ has $e^{-i\omega t}$.

■ **Equation of motion of a vector field on spherical black hole** Recalling the Maxwell theory in flat spacetime, a massless vector field had two physical degrees of freedom. The master variables should correspond to the physical degrees of freedom. So this system should be written by the two master variables. The conclusion is that one appears in the scalar quantities $\mathcal{A}$ and $\bar{\mathcal{A}}$ and the other appears in the vector degrees of freedom v_A . This is the third step for the derivation of the master equation.

The Maxwell equation reads

$$0 = \frac{1}{\sqrt{-g}} \partial_\nu (\sqrt{-g} F^{\mu\nu}). \quad (2.52)$$

If we write down each component, they are

$$\begin{aligned} 0 &= \frac{1}{\sqrt{-g}} \partial_\nu (\sqrt{-g} F^{t\nu}) \\ &= -\frac{1}{r^2} \partial_r \left(r^2 (\dot{\mathcal{A}} - \mathcal{A}') \right) + \frac{1}{f} \frac{1}{r^2} \hat{\Delta} \mathcal{A}, \end{aligned} \quad (2.53)$$

$$\begin{aligned} 0 &= \frac{1}{\sqrt{-g}} \partial_\nu (\sqrt{-g} F^{r\nu}) \\ &= \partial_t (\dot{\mathcal{A}} - \mathcal{A}') - \frac{f}{r^2} \hat{\Delta} \bar{\mathcal{A}} \end{aligned} \quad (2.54)$$

and

$$\begin{aligned} 0 &= \frac{1}{\sqrt{-g}} \partial_\nu (\sqrt{-g} F^{A\nu}) \\ &= \frac{1}{r^2} q^{AB} \left(-\frac{1}{f} \dot{\mathcal{A}} + \partial_r (f \bar{\mathcal{A}}) \right)_{;B} + \frac{1}{r^2 f} q^{AB} \left(\ddot{v}_B - f \partial_r (f v'_B) + \frac{1}{r^2} f (1 - \hat{\Delta}) v_B \right). \end{aligned} \quad (2.55)$$

Master equation of the vector component First, the equation of motion for the vector component v_B becomes

$$\ddot{v}_B - f \partial_r (f v'_B) + \frac{1}{r^2} f (1 - \hat{\Delta}) v_B.$$

After using the vector spherical harmonics, the equation of motion in the radial direction becomes

$$\ddot{v} - f \partial_r (f \partial_r v) + \frac{1}{r^2} f (1 + \gamma_{\mathcal{V}}) v = 0.$$

Here, the ℓ and m indicating the eigenvalue of the spherical harmonic are omitted for simplicity. At this time, v itself is actually the master variable Ψ . Taking the tortoise coordinates as in the case of the scalar field in Eq. (2.36), we obtain

$$\omega^2 \Psi = -\frac{d^2 \Psi}{dx^2} + V_{\mathcal{V}} \Psi,$$

where the effective potential $V_{\mathcal{V}}$ is defined by

$$V_{\mathcal{V}} \equiv f \frac{1 + \gamma_{\mathcal{V}}}{r^2} = f \frac{\gamma}{r^2}.$$

Master equation of the scalar component Next, let's consider the scalar component of the vector field A_μ . At this time, there are three simultaneous differential equations. In Eq. (2.54), $\bar{\mathcal{A}}$ has the second time derivative with respect to the time t . Substituting Eq. (2.55) into Eq. (2.54) yields the time evolution equation for

$$\begin{aligned} 0 &= \partial_t \left(\dot{\bar{\mathcal{A}}} - \mathcal{A}' \right) - \frac{f}{r^2} \hat{\Delta} \bar{\mathcal{A}} \\ &= \ddot{\bar{\mathcal{A}}} - \partial_r (f \partial_r (f \bar{\mathcal{A}}_{\mathcal{S}})) + \frac{f}{r^2} \gamma \bar{\mathcal{A}}. \end{aligned} \quad (2.56)$$

So, this should be the master variable. In fact, choosing the master variable as

$$\Psi = f \bar{\mathcal{A}} \quad (2.57)$$

and using tortoise coordinates, we get the equation of motion for the scalar component becomes

$$\begin{aligned} 0 &= \ddot{\bar{\mathcal{A}}} - \partial_r (f \partial_r (f \bar{\mathcal{A}})) - \frac{f}{r^2} \hat{\Delta} \bar{\mathcal{A}} \\ &= -\frac{1}{f} \left(\omega^2 \Psi + \frac{d^2 \Psi}{dx^2} - V_{\mathcal{S}} \Psi \right) \\ \Rightarrow \omega^2 \Psi &= -\frac{d^2 \Psi}{dx^2} + V_{\mathcal{S}} \Psi. \end{aligned} \quad (2.58)$$

Here, we defined the effective potential $V_{\mathcal{S}}$ as

$$V_{\mathcal{S}} = f \frac{\gamma}{r^2}. \quad (2.59)$$

Finally, the scalar $\mathcal{A}$ is expressed by the master variable Ψ as

$$\mathcal{A} = f \int \Psi' dt. \quad (2.60)$$

From the two results for a vector field A_μ , the effective potentials of them are the same each other. This is an example of the self-duality in even dimensions for differential form fields. This property will be summarized in Part II, Sec. 6.3 with various examples.

2.3.3 Gravitational perturbation

2.3.3.1 Perturbed equation of the metric on the spherical black hole in 4-dimension

In this section, we note the master equations of the perturbed metric in the Schwarzschild spacetime. The metric $g_{\mu\nu}$ has the 10 components in four dimensions,

and they are decomposed into scalar, vector, and tensor components for the spherical metric q_{AB} . In four dimensions, the tensor component does not have the physical degree of freedom in this formulation. So, they are decoupled into the equations for scalars and vectors. The master equation for the vector components is called the Regge-Wheeler equation, and the master equation for the scalar components is called the Zerilli equation. Here, we follow Takahashi's work [34].

■**SV decomposition** Firstly, the background metric is the Schwarzschild metric which has the spherical symmetry, so the perturbed metric should be decomposed using the symmetry. Let's assume the metric can be approximately expanded as

$$g_{\mu\nu} = \bar{g}_{\mu\nu} + h_{\mu\nu}, \quad (2.61)$$

where $\bar{g}_{\mu\nu}$ is the Schwarzschild solution, and $h_{\mu\nu}$ is the perturbed metric. Here, if we use the coordinate transformation $\frac{\partial x'^\alpha}{\partial x^\mu}$ for a background spacetime $\bar{g}_{\mu\nu}$, then the transformation of the whole metric $g_{\mu\nu}$ becomes the coordinate transformation of the perturbed metric $h_{\mu\nu}$ as

$$\begin{aligned} g_{\mu\nu}(x) &= \frac{\partial x'^\alpha}{\partial x^\mu} \frac{\partial x'^\beta}{\partial x^\nu} g'_{\alpha\beta}(x') \\ \bar{g}_{\mu\nu}(x) + h_{\mu\nu}(x) &= \frac{\partial x'^\alpha}{\partial x^\mu} \frac{\partial x'^\beta}{\partial x^\nu} \bar{g}'_{\alpha\beta}(x') + \frac{\partial x'^\alpha}{\partial x^\mu} \frac{\partial x'^\beta}{\partial x^\nu} h'_{\alpha\beta}(x') \\ \bar{g}_{\mu\nu}(x) + h_{\mu\nu}(x) &= \bar{g}_{\mu\nu}(x) + \frac{\partial x'^\alpha}{\partial x^\mu} \frac{\partial x'^\beta}{\partial x^\nu} h'_{\alpha\beta}(x') \\ h_{\mu\nu}(x) &= \frac{\partial x'^\alpha}{\partial x^\mu} \frac{\partial x'^\beta}{\partial x^\nu} h'_{\alpha\beta}(x'). \end{aligned}$$

So, the perturbed metric $h_{\mu\nu}$ behaves the symmetric tensor field on the background spacetime in the linear order of $h_{\mu\nu}$.

The first step is the selection of field variables. Using the spherical symmetry of the Schwarzschild solution, the perturbed metric can be expressed by

$$h_{\mu\nu} = \begin{pmatrix} S^{(1)} & S^{(2)} & V_A \\ & S^{(3)} & W_A \\ & & T_{AB} \end{pmatrix}. \quad (2.62)$$

Here, $S^{(i)}$ is the scalar and V_A, W_A is the vector. The vector can be decomposed into the two vectors v_A and w_A and the two scalars V and W as follows,

$$V_A = v_A + V_{;A} \quad \text{and} \quad W_A = w_A + W_{;A} \quad (2.63)$$

as in the case of a vector field A_μ . The colon $:$ is the covariant derivative with respect to the spherical metric q_{AB} . The vectors v_A and w_A satisfy

$$v_A{}^{;A} = 0 \quad \text{and} \quad w_A{}^{;A} = 0. \quad (2.64)$$

And the tensor T_{AB} is expressed by

$$T_{AB} = S_{AB} + 2t_{(A:B)} + T_{:AB} + \mathcal{T}q_{AB}. \quad (2.65)$$

Here, S_{AB} is a tensor, and this tensor satisfies

$$q^{AB}S_{AB} = 0 \quad \text{and} \quad S_{AB}{}^{;A} = 0. \quad (2.66)$$

In this case, S_{AB} has the 2×2 symmetric matrix, so three independent components exist. But above constraints kill these degrees of freedom. So, S_{AB} vanishes trivially. Then, the tensor T_{AB} can be expressed by a vector t_A and two scalars T and $\mathcal{T}$ as follows:

$$T_{AB} = 2t_{(A:B)} + T_{:AB} + \mathcal{T}q_{AB}. \quad (2.67)$$

■**Shift vector on spherical symmetry** Einstein's theory has the invariance under the general coordinate transformation. This property is converted into the arbitrary choice of the gauge degrees of freedom. Here, we'd like to use the gauge fixing method as in the case of a vector field A_μ .

Firstly, using the infinitesimal transformation ξ_μ , the transformed perturbed metric $\underline{h}_{\mu\nu}$ is related to the original perturbed metric $h_{\mu\nu}$,

$$\underline{h}_{\mu\nu} = h_{\mu\nu} + 2\bar{\nabla}_{(\mu}\xi_{\nu)}. \quad (2.68)$$

In this case, the infinitesimal vector ξ_μ is written by

$$\xi_\mu = \begin{pmatrix} \Xi_t \\ \Xi_r \\ \Xi_A + \Xi_{:A} \end{pmatrix}. \quad (2.69)$$

Here, Ξ_A satisfies

$$\Xi_A{}^{;A} = 0. \quad (2.70)$$

■ **Gauge transformations on spherical symmetry** Under the gauge transformation, the components of the perturbed metric $h_{\mu\nu}$ are transformed as

$$\underline{S}^{(1)} = S^{(1)} + 2\dot{\Xi}_t - f'f\Xi_r, \quad (2.71)$$

$$\underline{S}^{(2)} = S^{(2)} + \dot{\Xi}_r + \Xi'_t - \frac{f'}{f}\Xi_t, \quad (2.72)$$

$$\underline{S}^{(3)} = S^{(3)} + 2\Xi'_r + \frac{f'}{f}\Xi_r, \quad (2.73)$$

$$\begin{aligned} \underline{V}_A &= V_A + \dot{\xi}_A + \Xi_{t:A} \\ &\Leftrightarrow \underline{v}_A + \underline{V}_{:A} = \left(v_A + \dot{\Xi}_A\right) + \left(V + \dot{\Xi} + \xi_t\right)_{:A}, \end{aligned} \quad (2.74)$$

$$\begin{aligned} \underline{W}_A &= W_A + \xi'_A + \Xi_{r:A} - \frac{2}{r}\xi_A \\ &\Leftrightarrow \underline{w}_A + \underline{W}_{:A} = \left(w_A + \Xi'_A - \frac{2}{r}\Xi_A\right) + \left(W + \Xi_r + \Xi' - \frac{2}{r}\Xi\right)_{:A}, \end{aligned} \quad (2.75)$$

$$\begin{aligned} \text{and } \underline{T}_{AB} &= T_{AB} + 2\xi_{(A:B)} + 2rf\Xi_r q_{AB} \\ &\Leftrightarrow 2\underline{t}_{(A:B)} + \underline{T}_{:AB} + \underline{T}q_{AB} = 2\left(t_{(A} + \Xi_{(A)}\right)_{:B)} + (T + \Xi)_{:AB} + (\mathcal{T} + 2rf\Xi_r)q_{AB}. \end{aligned} \quad (2.76)$$

From the above transformation, the scalar components are transformed as

$$\underline{S}^{(1)} = S^{(1)} + 2\dot{\Xi}_t - f'f\Xi_r, \quad (2.77)$$

$$\underline{S}^{(2)} = S^{(2)} + \dot{\Xi}_r + \Xi'_t - \frac{f'}{f}\Xi_t, \quad (2.78)$$

$$\underline{S}^{(3)} = S^{(3)} + 2\Xi'_r + \frac{f'}{f}\Xi_r, \quad (2.79)$$

$$\underline{V} = V + \dot{\Xi} + \Xi_t, \quad (2.80)$$

$$\underline{W} = W + \Xi_r + \Xi' - \frac{2}{r}\Xi, \quad (2.81)$$

$$\underline{\mathcal{T}} = \mathcal{T} + 2rf\Xi_r, \quad (2.82)$$

$$\text{and } \underline{T} = T + \Xi. \quad (2.83)$$

The vector components are transformed as

$$\underline{v}_A = v_A + \dot{\Xi}_A, \quad (2.84)$$

$$\underline{w}_A = w_A + \Xi'_A - \frac{2}{r}\Xi_A, \quad \text{and} \quad (2.85)$$

$$\underline{t}_A = t_A + \Xi_A. \quad (2.86)$$

■ **Gauge fixing** Next, we'd like to choose the simple expression of the perturbed metric. In the next paragraph, we explain the gauge fixing method for the vector components and the scalar components.

Regge-Wheeler gauge The Regge-Wheeler gauge is the gauge fixing of the vector components of the perturbed metric [8]. From the relation (2.86), we can choose $t_A = 0$ by the use of Ξ_A . Then, this gauge is completely fixed. This gauge is called the Regge-Wheeler gauge. In this gauge, the perturbed metric becomes

$$h_{\mu\nu}^{\text{RW}} = \begin{pmatrix} 0 & 0 & v_A \\ & 0 & w_A \\ & & \text{O} \end{pmatrix}. \quad (2.87)$$

The vectors v_A and w_A can be expanded by the vector spherical harmonics $\mathcal{Y}_{\ell m}^A$ in Eq. (2.46).

Zerilli gauge The Zerilli gauge is about the gauge fixing of the scalar components of the perturbed metric [8, 9]. The Regge and Wheeler found this gauge, but they couldn't find the master variable in this gauge. After the work by Regge and Wheeler, Zerilli derived the master equation in this gauge, so this gauge is called Zerilli gauge. The gauge transformation for the scalar components are

$$\underline{S}^{(1)} = S^{(1)} + 2\dot{\Xi}_t - f'f\Xi_r, \quad (2.88)$$

$$\underline{S}^{(2)} = S^{(2)} + \dot{\Xi}_r + \Xi'_t - \frac{f'}{f}\Xi_t, \quad (2.89)$$

$$\underline{S}^{(3)} = S^{(3)} + 2\Xi'_r + \frac{f'}{f}\Xi_r, \quad (2.90)$$

$$\underline{V} = V + \dot{\Xi} + \Xi_t, \quad (2.91)$$

$$\underline{W} = W + \Xi_r + \Xi' - \frac{2}{r}\Xi \quad (2.92)$$

$$\underline{\mathcal{T}} = \mathcal{T} + 2rf\Xi_r, \quad (2.93)$$

$$\text{and } \underline{\mathcal{T}} = \mathcal{T} + \Xi. \quad (2.94)$$

First, we can eliminate $\mathcal{T}$ by the choice of Ξ . Next, V and W can be deleted by using the remaining Ξ_t and Ξ_r . From these procedures, the perturbed metric is completely gauge-fixed and given by

$$h_{\mu\nu}^Z = \begin{pmatrix} S^{(1)} & S^{(2)} & \mathbf{0} \\ \text{sym.} & S^{(3)} & \mathbf{0} \\ \text{sym.} & \text{sym.} & \mathcal{T}_{qAB} \end{pmatrix}. \quad (2.95)$$

So there are four scalars in the perturbed metric. This gauge is called the Zerilli gauge. These scalars are expanded by the spherical harmonics $Y_{\ell m}$. This is the end of the first step.

■ **Equation of motion of the perturbed metric on a spherical black hole** In this paragraph, we'd like to derive the equation of motion of the perturbed metric. The second step and third step for the derivation of the master equations are contained.

In Regge-Wheeler gauge In this gauge, the components of the Ricci tensor become

$$R_{tt} = R_{rr} = R_{tr} = 0, \quad (2.96)$$

$$R_{tA} = \frac{1}{2r^2} \left((1 + \gamma_{\mathcal{V}} - 2rf') v_A - r^2 f \left(v_A'' - \dot{w}_A' - \frac{2}{r} \dot{w}_A \right) \right), \quad (2.97)$$

$$R_{rA} = \frac{1}{2r^2} \left((\gamma_{\mathcal{V}} - 1) w_A - \frac{r^2}{f} \left(\dot{v}_A' - \frac{2}{r} \dot{v}_A - \ddot{w}_A \right) \right) \quad \text{and} \quad (2.98)$$

$$R_{AB} = -\frac{1}{f} \dot{v}_{(A:B)} + (fw_{(A:B)})'. \quad (2.99)$$

So, from the Einstein equation in vacuum

$$R_{\mu\nu} = 0, \quad (2.100)$$

we obtain three equations

$$0 = (1 + \gamma_{\mathcal{V}} - 2rf') v_A - r^2 f \left(v_A'' - \dot{w}_A' - \frac{2}{r} \dot{w}_A \right), \quad (2.101)$$

$$0 = (\gamma_{\mathcal{V}} - 1) w_A - \frac{r^2}{f} \left(\dot{v}_A' - \frac{2}{r} \dot{v}_A - \ddot{w}_A \right) \quad \text{and} \quad (2.102)$$

$$0 = -\frac{1}{f} \dot{v}_{(A:B)} + (fw_{(A:B)})'. \quad (2.103)$$

From these expressions, we can get the information on the time evolution of this system. The second equation contains the second order time derivative of w_A and the first order time derivative of v_A , and the third equation has the first order time derivative of v_A . Using the third equation, we can eliminate the v_A from the second equation. Finally, we can get the equation of motion for w_A . After solving the equation of w_A , then we can determine the time evolution of v_A . The perturbed

quantities v_A and w_A can be expanded by the spherical harmonics

$$v^A(t, r, \theta, \phi) = \sum_{\ell, m} v_{\ell m}(t, r) \mathcal{Y}_{\ell m}^A(\theta, \phi) \quad \text{and}$$

$$w^A(t, r, \theta, \phi) = \sum_{\ell, m} w_{\ell m}(t, r) \mathcal{Y}_{\ell m}^A(\theta, \phi).$$

From now on, for simplicity, we omit the index ℓ and m . By using this decomposition, we get the equation of w as follows

$$0 = (\gamma_{\mathcal{V}} - 1)w - \frac{r^2}{f} \left(\frac{d}{dr} \left(f \frac{d}{dr} (fw) \right) - \frac{2}{r} f \frac{d}{dr} (fw) - \ddot{w} \right) \quad (2.104)$$

$$\Leftrightarrow 0 = \frac{\gamma_{\mathcal{V}} - 1}{r^2} fw - \left(\frac{d}{dr} \left(f \frac{d}{dr} (fw) \right) - \frac{2}{r} f \frac{d}{dr} (fw) - \ddot{w} \right). \quad (2.105)$$

We define the master variable Ψ as

$$\Psi \equiv f \frac{w}{r}. \quad (2.106)$$

Then, the radial equation becomes

$$0 = \left(\frac{\gamma_{\mathcal{V}} - 1}{r} + 2 \frac{f}{r} - f' \right) \Psi - r \frac{d}{dr} \left(f \frac{d\Psi}{dr} \right) + \frac{r}{f} \ddot{\Psi}, \quad (2.107)$$

If we use the tortoise coordinate

$$\frac{dx}{dr} = \frac{1}{f},$$

the final expression is given by

$$\omega^2 \Psi = - \frac{d^2 \Psi}{dx^2} + V_{\text{RW}} \Psi. \quad (2.108)$$

Here, the potential V_{RW} is given by

$$\begin{aligned} V_{\text{RW}} &\equiv \frac{f}{r^2} (\gamma_{\mathcal{V}} - 1 + 2f - r \partial_r f) \\ &= \frac{f}{r^2} \left(\ell(\ell + 1) - 3 \cdot \frac{2M}{r} \right). \end{aligned} \quad (2.109)$$

This is the one of the master equation for gravitational waves. This equation is often called the Regge-Wheeler equation.

In Zerilli gauge The case of the vector perturbation is not so complicated, but in the Zerilli gauge the situation becomes more difficult. To study the equation of motion for this perturbation, we redefine the perturbed components of metric as

$$h_{\mu\nu} = \begin{pmatrix} S^{(1)} & S^{(2)} & 0 \\ \text{sym.} & S^{(3)} & 0 \\ \text{sym.} & \text{sym.} & \mathcal{T}q_{AB} \end{pmatrix} \supset \begin{pmatrix} f\bar{H}Y & H_1Y & 0 \\ \text{sym.} & f^{-1}HY & 0 \\ \text{sym.} & \text{sym.} & r^2KYq_{AB} \end{pmatrix}. \quad (2.110)$$

Here, $Y \equiv Y_{\ell m}$. This was proposed by Regge and Wheeler [8], but they couldn't find the master variable for this perturbation. The components of the perturbed Ricci tensor become

$$\begin{aligned} R_{tt} = & f^2 \left(\frac{1}{f} \dot{H}'_1 + \frac{1}{r} \left(2 + \frac{rf'}{2f} \right) \frac{1}{f} \dot{H}_1 \right) Y + \frac{1}{2} f^2 \left(-\frac{2}{f^2} \ddot{K} + \frac{f'}{f} K' \right) Y \\ & - \frac{1}{4} \frac{f^2}{r^2} \left(2r^2 \bar{H}'' + \left(4 + 3\frac{rf'}{f} \right) r\bar{H}' + \frac{(2r^2 f'' + 4rf' - 2\gamma)}{f} \bar{H} \right) Y \\ & - \frac{1}{2} f^2 \left(\frac{1}{f^2} \ddot{H} + \frac{1}{2} \frac{f'}{f} H' + \left(\frac{f''}{f} + \frac{2}{r} \frac{f'}{f} \right) H \right) Y, \end{aligned} \quad (2.111)$$

$$R_{tr} = -\frac{1}{2r^2} (r^2 f'' + 2rf' - \gamma) H_1 Y + \frac{1}{r} \dot{H} Y - \left(\partial_r + \frac{1}{r} \left(1 - \frac{rf'}{2f} \right) \right) \dot{K} Y \quad (2.112)$$

$$\begin{aligned} R_{rr} = & \frac{1}{2} Y \left(\left(\bar{H}'' + \frac{3}{2} \frac{f'}{f} \bar{H}' \right) + \left(\frac{1}{f^2} \ddot{H} + \left(\frac{2}{r} + \frac{f'}{2f} \right) H' + \frac{\gamma}{r^2 f} H \right) \right. \\ & \left. - \frac{2}{f} \left(\dot{H}'_1 + \frac{f'}{2f} \dot{H}_1 \right) - 2 \left(K'' + \left(\frac{2}{r} + \frac{f'}{2f} \right) K' \right) \right) \end{aligned} \quad (2.113)$$

$$R_{tA} = \frac{1}{2} Y_{;A} \left((fH_1)' - \dot{H} - \dot{K} \right) \quad (2.114)$$

$$R_{rA} = \frac{1}{2} Y_{;A} \left(-\frac{1}{f} \dot{H}_1 + \bar{H}' - \left(\frac{1}{r} - \frac{f'}{2f} \right) \bar{H} + \left(\frac{1}{r} + \frac{f'}{2f} \right) H - K' \right) \quad (2.115)$$

$$\begin{aligned} R_{AB} = & \frac{1}{2} q_{AB} Y \left(-2r\dot{H}_1 + rf\bar{H}' + rfH' + 2f \left(1 + \frac{rf'}{f} \right) H + \frac{r^2}{f} \ddot{K} \right. \\ & \left. - r^2 f K'' - rf \left(4 + \frac{rf'}{f} \right) K' + \left(\gamma - 2f \left(1 + \frac{rf'}{f} \right) \right) K \right) \\ & + \frac{1}{2} Y_{;AB} (\bar{H} - H) \end{aligned} \quad (2.116)$$

Then, the set of equations of the linearized Einstein equation in vacuum is expressed by the set of the following equations,

$$\begin{aligned}
0 = & f^2 \left(\frac{1}{f} \dot{H}_1' + \frac{1}{r} \left(2 + \frac{rf'}{2f} \right) \frac{1}{f} \dot{H}_1 \right) + \frac{1}{2} f^2 \left(-\frac{2}{f^2} \ddot{K} + \frac{f'}{f} K' \right) \\
& - \frac{1}{4} \frac{f^2}{r^2} \left(2r^2 \bar{H}'' + \left(4 + 3 \frac{rf'}{f} \right) r \bar{H}' + \frac{(2r^2 f'' + 4rf' - 2\gamma)}{f} \bar{H} \right) \\
& - \frac{1}{2} f^2 \left(\frac{1}{f^2} \ddot{H} + \frac{1}{2} \frac{f'}{f} H' + \left(\frac{f''}{f} + \frac{2}{r} \frac{f'}{f} \right) H \right), \tag{2.117}
\end{aligned}$$

$$0 = -\frac{1}{2r^2} (r^2 f'' + 2rf' - \gamma) H_1 + \frac{1}{r} \dot{H} - \left(\partial_r + \frac{1}{r} \left(1 - \frac{rf'}{2f} \right) \right) \dot{K}, \tag{2.118}$$

$$\begin{aligned}
0 = & \left(\bar{H}'' + \frac{3}{2} \frac{f'}{f} \bar{H}' \right) + \left(\frac{1}{f^2} \ddot{H} + \left(\frac{2}{r} + \frac{f'}{2f} \right) H' + \frac{\gamma}{r^2 f} H \right) \\
& - \frac{2}{f} \left(\dot{H}_1' + \frac{f'}{2f} \dot{H}_1 \right) - 2 \left(K'' + \left(\frac{2}{r} + \frac{f'}{2f} \right) K' \right), \tag{2.119}
\end{aligned}$$

$$0 = -\dot{H} + f \left(H_1' + \frac{f'}{f} H_1 \right) - \dot{K}, \tag{2.120}$$

$$0 = -\frac{1}{f} \dot{H}_1 + \bar{H}' - \left(\frac{1}{r} - \frac{f'}{2f} \right) \bar{H} + \left(\frac{1}{r} + \frac{f'}{2f} \right) H - K', \tag{2.121}$$

$$\begin{aligned}
0 = & -\frac{2}{f} \dot{H}_1 + \bar{H}' + \left(H' + \frac{2}{r} \left(1 + \frac{rf'}{f} \right) H \right) \\
& + \left(\frac{r}{f^2} \ddot{K} - rK'' - \left(4 + \frac{rf'}{f} \right) K' + \left(\frac{\gamma}{rf} - \frac{2}{r} \left(1 + \frac{rf'}{f} \right) \right) K \right) \quad \text{and} \tag{2.122}
\end{aligned}$$

$$0 = \bar{H} - H. \tag{2.123}$$

Zerilli found the master variable Ψ in this set of equations. Firstly, H_1 can be expressed as

$$H_1 = \frac{r}{f} (\dot{\psi} + \dot{K}). \tag{2.124}$$

Note that ψ is related to the master variable Ψ . By using the last equation

$$\bar{H} = H, \tag{2.125}$$

the equations reduce to the following

$$0 = 2 \left(\frac{r}{f} \dot{H}_1 + \left(2 + \frac{rf'}{2f} \right) \frac{r}{f} \dot{H}_1 \right) + \left(-\frac{2r^2}{f^2} \ddot{K} + \frac{rf'}{f} r K' \right) - \left(\frac{r^2}{f^2} \ddot{H} + r^2 H'' + 2 \left(1 + \frac{rf'}{f} \right) r H' + \frac{2r^2 f'' + 4rf' - \gamma}{f} H \right), \quad (2.126)$$

$$0 = -\frac{1}{2r^2} (r^2 f'' + 2rf' - \gamma) H_1 + \frac{1}{r} \dot{H} - \left(\partial_r + \frac{1}{r} \left(1 - \frac{rf'}{2f} \right) \right) \dot{K}, \quad (2.127)$$

$$0 = -2 \frac{r}{f} \left(r \partial_r + \frac{rf'}{2f} \right) \dot{H}_1 + \frac{r^2}{f^2} \ddot{H} + r^2 H'' + 2 \left(1 + \frac{rf'}{f} \right) r H' + \frac{\gamma}{f} H - 2 \left(r^2 K'' + \left(2 + r \frac{f'}{2f} \right) r K' \right), \quad (2.128)$$

$$0 = \left(r \partial_r + \frac{rf'}{f} \right) H_1 - \frac{r}{f} \dot{H} - \frac{r}{f} \dot{K}, \quad (2.129)$$

$$0 = \frac{r}{f} \dot{H}_1 - \left(r \partial_r + \frac{rf'}{f} \right) H + r K', \quad (2.130)$$

$$0 = -2 \frac{r}{f} \dot{H}_1 + 2 \left(r \partial_r + 1 + \frac{rf'}{f} \right) H \quad \text{and} \quad + \frac{r^2}{f^2} \ddot{K} - r^2 K'' - 2 \left(2 + \frac{rf'}{2f} \right) r K' + \left(\frac{\gamma}{f} - 2 \left(1 + \frac{rf'}{f} \right) \right) K. \quad (2.131)$$

From Eq. (2.127), H is given by K and ψ as

$$\begin{aligned} \dot{H} &= \left(r \partial_r + 1 - \frac{rf'}{2f} \right) \dot{K} + \frac{1}{2r} (r^2 f'' + 2rf' - \gamma) H_1 \\ &= \left(r \partial_r + 1 - \frac{rf'}{2f} \right) \dot{K} + \frac{1}{2r} (r^2 f'' + 2rf' - \gamma) \left(\frac{r}{f} (\dot{\psi} + \dot{K}) \right) \\ &= r \dot{K}' - \frac{1}{2f} (\gamma - 2f - r^2 f'' - rf') \dot{K} + \frac{1}{2f} (r^2 f'' + 2rf' - \gamma) \dot{\psi} \\ \Rightarrow H &= r K' - \frac{1}{2f} (\gamma - 2f - r^2 f'' - rf') K + \frac{1}{2f} (r^2 f'' + 2rf' - \gamma) \psi. \end{aligned} \quad (2.132)$$

Then, $H, \bar{H}, H_1$ can be expressed by K and ψ ,

$$\bar{H} = H = r K' - \frac{1}{2f} A K + \frac{1}{2f} B \psi \quad \text{and} \quad (2.133)$$

$$H_1 = \frac{r}{f} (\dot{\psi} + \dot{K}).$$

Here, A, B are defined by

$$A \equiv \gamma + 1 - 3f \quad (2.134)$$

$$\text{and } B \equiv -\gamma. \quad (2.135)$$

We need to express K by ψ . From Eq. (2.129), we can deduce the following

$$\begin{aligned}
0 &= \left(r\partial_r + \frac{rf'}{f} \right) H_1 - \frac{r}{f} \dot{H} - \frac{r}{f} \dot{K} \\
&= \left(r\partial_r + \frac{rf'}{f} \right) \left(\frac{r}{f} (\dot{\psi} + \dot{K}) \right) - \frac{r}{f} \left(rK' - \frac{1}{2f} AK + \frac{1}{2f} B\psi \right) - \frac{r}{f} \dot{K} \\
&= \frac{r}{2f^2} \left(2fr\dot{\psi}' + (2f - B)\dot{\psi} \right) + \frac{r}{2f^2} A\dot{K} \\
\Rightarrow A\dot{K} &= - \left(2fr\dot{\psi}' + (2f - B)\dot{\psi} \right) \\
\dot{K} &= - \frac{2fr\dot{\psi}' + (2f - B)\dot{\psi}}{A} \\
\Rightarrow K &= - \frac{2fr\psi' + (2f - B)\psi}{A}.
\end{aligned} \tag{2.136}$$

$$\Rightarrow K = - \frac{2fr\psi' + (2f - B)\psi}{A}. \tag{2.137}$$

So, we showed that all the variables $H, \bar{H}, H_1$ and K can be expressed by ψ . Then we can deduce the time evolution of ψ . From Eq.(2.130) and Eq.(2.131), we can get the equation of motion for ψ as

$$\begin{aligned}
0 &= \frac{r}{f} \dot{H}_1 - \left(r\partial_r + \frac{rf'}{f} \right) H + rK' \\
&\quad - 2\frac{r}{f} \dot{H}_1 + 2 \left(r\partial_r + 1 + \frac{rf'}{f} \right) H + \\
&\quad + \frac{r^2}{f^2} \ddot{K} - r^2 K'' - 2 \left(2 + \frac{rf'}{2f} \right) rK' + \left(\frac{\gamma}{f} - 2 \left(1 + \frac{rf'}{f} \right) \right) K \\
\Rightarrow 0 &= - \frac{r}{f} \dot{H}_1 + \left(r\partial_r + \frac{rf'}{f} \right) H + 2H + rK' + \frac{r^2}{f^2} \ddot{K} \\
&\quad - r^2 K'' - 2 \left(2 + \frac{rf'}{2f} \right) rK' + \left(\frac{\gamma}{f} - 2 \left(1 + \frac{rf'}{f} \right) \right) K \\
0 &= - \frac{r^2}{f^2} \ddot{\psi} + \frac{r}{2f} (B\psi - AK)' + \frac{1}{f} (B\psi - AK) + \frac{1}{f} (\gamma - 2(f + rf')) K
\end{aligned} \tag{2.138}$$

$$\Rightarrow 0 = - \frac{r}{f} \dot{H}_1 + \left(r\partial_r + \frac{rf'}{f} \right) H + 2H + rK' + \frac{r^2}{f^2} \ddot{K} - r^2 K'' - 2 \left(2 + \frac{rf'}{2f} \right) rK' + \left(\frac{\gamma}{f} - 2 \left(1 + \frac{rf'}{f} \right) \right) K \tag{2.139}$$

$$0 = - \frac{r^2}{f^2} \ddot{\psi} + \frac{r}{2f} (B\psi - AK)' + \frac{1}{f} (B\psi - AK) + \frac{1}{f} (\gamma - 2(f + rf')) K \tag{2.140}$$

Here, from the relation between K and ψ ,

$$B\psi - AK = 2f(r\psi' + \psi), \tag{2.141}$$

Thus, the equation of ψ becomes

$$\begin{aligned}
0 &= -\frac{r^2}{f^2}\ddot{\psi} + \frac{r}{f}(f(r\psi' + \psi))' + 2(r\psi' + \psi) - \frac{1}{f}(\gamma - 2(f + rf'))\frac{2fr\psi' + (2f - B)\psi}{A} \\
&= -\frac{r^2}{f^2}\ddot{\psi} + r^2\psi'' + \left(4 + \frac{rf'}{f} - \frac{2(\gamma - 2(f + rf'))}{A}\right)r\psi' \\
&\quad + \left(2 + \frac{rf'}{f} - \frac{1}{f}(\gamma - 2(f + rf'))\frac{(2f - B)}{A}\right)\psi \\
\Rightarrow 0 &= -\ddot{\psi} + f^2\psi'' + \frac{f}{rA}((3f + 1)A - 2f(\gamma - 2))\psi' + \frac{f}{r^2A}\bar{q}\psi.
\end{aligned} \tag{2.142}$$

Here, we used the Einstein equations

$$r^2 f'' = 2f - 2 \tag{2.143}$$

$$\text{and } rf' = 1 - f. \tag{2.144}$$

Here, we defined a new constant $\bar{q}$,

$$\bar{q} \equiv A(f + 1) - (\gamma - 2)(2f + \gamma). \tag{2.145}$$

Using the master variable Ψ defined by

$$\psi = \frac{A}{r}\Psi, \tag{2.146}$$

we get the equation of ψ as

$$0 = -\ddot{\Psi} + f^2\Psi'' + \frac{f}{r}(1 - f)\Psi' - V_Z\Psi. \tag{2.147}$$

Here, we defined the effective potential V_Z

$$\begin{aligned}
V_Z &\equiv -\frac{f}{r^2}\left(\frac{1}{A^2}((3f + 1)A - 2f(\gamma - 2))(\gamma - 2 - 2A) - \frac{2f}{A}(2\gamma - 4 - 3A) + \frac{\bar{q}}{A}\right) \\
&= -\frac{f}{r^2}\frac{(9f^3 + (\gamma + 1)(9f^2 - 3f(\gamma + 1) + (\gamma^2 - 2\gamma + 3)))}{(3f - \gamma - 1)^2}.
\end{aligned} \tag{2.148}$$

If we change the coordinate from r to x , then the equation of Ψ becomes

$$\begin{aligned}
0 &= -\ddot{\Psi} + f^2\Psi'' + \frac{f}{r}(1 - f)\Psi' + V_Z\Psi \\
&= -\ddot{\Psi} + f^2\frac{d^2\Psi}{dr^2} + \frac{f}{r}(1 - f)\partial_r\Psi + V_Z\Psi \\
&= -\ddot{\Psi} + f^2\frac{d}{dr}\left(\frac{1}{f}\frac{d\Psi}{dx}\right) + \frac{f}{r}(1 - f)\frac{1}{f}\partial_x\Psi + V_Z\Psi \\
\omega^2\Psi &= -\frac{d^2\Psi}{dx^2} + V_Z\Psi.
\end{aligned} \tag{2.149}$$

This is the Zerilli equation.

■ Master equations for the metric perturbation on a Schwarzschild black hole We have derived the master equation for each perturbation,

$$\omega^2 \ddot{\Psi} = -\frac{d^2 \Psi}{dx^2} + V_{\text{eff}} \Psi. \quad (2.150)$$

Here, the effective potential is given by

$$V_{\text{eff}} = \begin{cases} V_{\text{RW}} = \frac{f}{r^2} \left(\ell(\ell+1) + (-3) \cdot \frac{2M}{r} \right) \\ V_{\text{Z}} = -\frac{f}{r^2} \frac{(9f^3 + (\gamma+1)(9f^2 - 3f(\gamma+1) + (\gamma^2 - 2\gamma + 3)))}{(3f - \gamma - 1)^2} \end{cases} \quad (2.151)$$

These potentials only depend on the background spacetime and a field configuration^{*2}, and this potential determine the fate of fluctuations of gravitational waves around a Schwarzschild black hole, for example, the stability of black holes.

In the massless limit $M \rightarrow 0$, these potentials becomes equivalent to

$$V_{\text{RW}}, V_{\text{Z}} \rightarrow \frac{\gamma}{r^2}. \quad (2.152)$$

In higher dimensions, the analysis can be extended straightforwardly as will be seen in the next section.

2.3.3.2 Perturbed equation of the metric on the spherical black hole in D-dimension

In this section, we derive the master equation of the perturbed metric in the Schwarzschild-Tangherlini metric. This metric is a natural extension of the Schwarzschild solution to higher dimensions. The difference of the case in 4 dimensions is the existence of the tensor perturbation S_{AB} , that is, three types of the perturbations of the metric. The decomposition is the same in the case of 4-dimension. The perturbed metric can be written as Eq. (2.62), but the dimension n for the spherical coordinate x^A becomes

$$n = D - 2.$$

^{*2} In the effective potential of V_{RW} , the parameter -3 is given by the gravitational perturbation. In the case of the Maxwell equation, this value becomes 0, and in the case of the Klein-Gordon equation, this value becomes 1.

Since the back ground spacetime has the spherical symmetry, we can also decompose them into scalar (S), vector (V) and tensor (T) components like as Eqs.(2.63) and (2.65). The expression of scalar and vector components are same in 4-dimension, but S_{AB} does not vanish in higher dimension $D > 4$. More over, S_{AB} is gauge invariant under the gauge transformation ξ_μ in Eq. (2.68). So, each metric perturbation can be expressed by

$$h_{\mu\nu}^{(S)} = \begin{pmatrix} S^{(1)} & S^{(2)} & \mathbf{0} \\ \text{sym.} & S^{(3)} & \mathbf{0} \\ \text{sym.} & \text{sym.} & \mathcal{T}q_{AB} \end{pmatrix}, \quad (2.153)$$

$$h_{\mu\nu}^{(V)} = \begin{pmatrix} 0 & 0 & v_A \\ \text{sym.} & 0 & w_A \\ \text{sym.} & \text{sym.} & 0 \end{pmatrix}, \quad (2.154)$$

$$\text{and } h_{\mu\nu}^{(T)} = \begin{pmatrix} 0 & 0 & \mathbf{0} \\ \text{sym.} & 0 & \mathbf{0} \\ \text{sym.} & \text{sym.} & S_{AB} \end{pmatrix}. \quad (2.155)$$

The stability of the higher dimensional Schwarzschild black hole is firstly proved by Ishibashi and Kodama [21, 22, 24] using the gauge invariant method. Takahashi also showed their stabilities using the gauge fixing method [34]. The master equations for each perturbation can be found in appendix D. The master equation is written by

$$\omega^2 \Psi = -\frac{d^2 \Psi}{dx^2} + V_{\text{eff}} \Psi.$$

Here, we list the effective potential of the master equation for each perturbed component,

$$V_{\text{eff}} : \begin{cases} V_{\mathcal{T}}^{(n)} = \frac{f}{r^2} \left(\gamma_{\mathcal{T}}^{(n)} + 2n + \frac{1}{4} (n^2 - 10n + 8) f + \frac{1}{2} (n - 4) r f' \right) & \text{tensor,} \\ V_{\mathcal{V}}^{(n)} = \frac{f}{r^2} \left(\frac{1}{4} n (n + 2) f - \frac{1}{2} (n + 4) r f' + (n - 1) (1 - 2f) + \gamma_{\mathcal{V}}^{(n)} \right) & \text{vector, and} \\ V_{\mathcal{S}}^{(n)} = -\frac{f}{rN} (r f N'' + A_3 N' + A_4 N) & \text{scalar.} \end{cases} \quad (2.156)$$

The parameter $\gamma_{\mathcal{T}}^{(n)}$, $\gamma_{\mathcal{V}}^{(n)}$ and $\gamma_{\mathcal{S}}^{(n)}$ are given by [22]

$$\gamma_{\mathcal{T}}^{(n)} = \ell (\ell + n - 1) - 2, \quad (2.157)$$

$$\gamma_{\mathcal{V}}^{(n)} = \ell (\ell + n - 1) - 1 \quad (2.158)$$

$$\text{and } \gamma_{\mathcal{S}}^{(n)} = \ell (\ell + n - 1). \quad (2.159)$$

Here, N , A_3 and A_4 are defined by

$$\begin{aligned}
N &= \frac{1}{r^{\frac{n}{2}}} \left(n(n+1)(1-f) + 2\gamma_{\mathcal{S}}^{(n)} - 2n \right) \\
A_1 &= \frac{1}{nf} \left(r^2 f'' + nr f' - \gamma_{\mathcal{S}}^{(n)} \right) \\
A_2 &= 1 - \frac{rf'}{2f} + A_1 \\
A_3 &= 2nf + rf' + \frac{1}{A_2} \left(\gamma_{\mathcal{S}}^{(n)} - nf - \frac{1}{2}(n+2)rf' \right) \\
A_4 &= (n-1) \frac{f}{r} \left(n + \frac{rf'}{f} \right) + \left(\gamma_{\mathcal{S}}^{(n)} - nf - \frac{1}{2}(n+2)rf' \right) \frac{(n-1-A_1)}{rA_2}
\end{aligned} \tag{2.160}$$

So, even on the spherical black hole in arbitrary dimensions, we can derive the master equation of the perturbed metric. Therefore, we can study the stability and quasi-normal modes.

Chapter 3

Stability analysis by S -deformation

3.1 Summary of S -deformation method

In this section, we give a short review of the analysis of the stability by the S -deformation method [21, 24–26, 36]. This method is useful to show the stability of fields on a black hole.

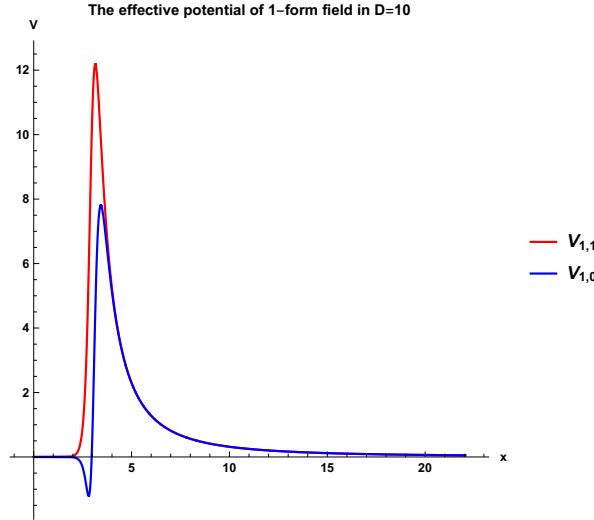


Figure3.1: The effective potentials of a 1-form field in $D = 10$

We plot the effective potentials for a vector field (1-form field) in 10-dimensions. The effective potential for the vector component $V_{1,1}$ is positive definite. On the other hand, the effective potential for the scalar component $V_{1,0}$ has a negative region. Here, we took $\ell = 1$ and $\mu = 1$.

Let us start with the master equation

$$\omega^2 \Psi = A \Psi, \quad (3.1)$$

where we defined the operator A as

$$A \equiv -\frac{d^2}{dx^2} + V. \quad (3.2)$$

We assume the time-dependence of Ψ as

$$\Psi \propto e^{-i\omega t}. \quad (3.3)$$

From Eq. (3.1), we can deduce

$$\begin{aligned} \omega^2 \int_{-\infty}^{\infty} dx |\Psi|^2 &= \int_{-\infty}^{\infty} dx \Psi^* A \Psi \\ &= - \left[\Psi^* \frac{d\Psi}{dx} \right]_{-\infty}^{\infty} + \int_{-\infty}^{\infty} dx \left[\left| \frac{d\Psi}{dx} \right|^2 + V |\Psi|^2 \right]. \end{aligned} \quad (3.4)$$

Here, we assumed the boundary conditions for Ψ such that Ψ and $d\Psi/dx$ become 0 at $x \rightarrow \pm\infty$, that is, Ψ is localized and compact out side of the horizon of a black hole.

If the potential V is positive definite in $-\infty < x < \infty$, the surface term in the right hand side of Eq.(3.4) vanishes. Then, ω^2 satisfies

$$\omega^2 > 0.$$

This indicates that the field is stable because the amplitude of Ψ does not grow. On the other hand, if the effective potential has negative regions, the positivity of ω^2 becomes nontrivial. If $\omega^2 < 0$, this solution is unstable because the amplitude of Ψ exponentially grows. But even in the case of the effective potential with negative regions, we can show the positivity of ω^2 for some cases by using the S -deformation method.

Let us explain the S -deformation method. First, we define a new operator using the new operator D_x as

$$D_x \equiv \frac{d}{dx} + S(x), \quad (3.5)$$

where the function S will be determined later. We can rearrange Eq. (3.4) as

$$\omega^2 \int_{-\infty}^{\infty} dx |\Psi|^2 = - [\Psi^* D_x \Psi]_{-\infty}^{\infty} + \int_{-\infty}^{\infty} dx \left[|D_x \Psi|^2 + \bar{V} |\Psi|^2 \right]. \quad (3.6)$$

The surface term vanishes too. Here, the modified effective potential $\bar{V}$ is defined by

$$\bar{V} \equiv V + \frac{dS}{dx} - S^2.$$

If we can find the continuous function, S , so that $\bar{V} > 0$ holds for $-\infty < x < \infty$. We have the inequality below

$$\omega^2 \int_{-\infty}^{\infty} dx |\Psi|^2 = \int_{-\infty}^{\infty} dx \left[|D_x \Psi|^2 + \bar{V} |\Psi|^2 \right] > 0. \quad (3.7)$$

So we can show the positivity of ω^2 , that is, there is no growing mode for Ψ . This is the S -deformation method.

In 4-dimensions, the effective potential of perturbation of the gravitational field on the Schwarzschild black hole is always positive for $\ell \geq 2$. So, there is no unstable configuration for the Regge-Wheeler and Zerilli perturbations. In higher dimensions, the positivity is nontrivial. But using the S -deformation method, we can show the positivity of the effective potential for all perturbation on the Schwarzschild-Tangherlini solution, see [21, 24–26, 34]. This method is also useful for studying the stability of differential form fields on a spherical black hole.

3.2 Examples of S -deformation method

In this section, we review the known results of the function S [34] using the gauge fixing method. In the gauge invariant approach, Ishibashi and Kodama gave them [21, 24–26, 34]. Takahashi got the expression S for each perturbation mode.

■ **Tensor perturbation** In the case of the tensor perturbation, the effective potential in Eq.(2.156) is given by

$$\begin{aligned} V_{\mathcal{T}}^{(n)} &= \frac{f}{r^2} \left(\ell(\ell + n - 1) + 2(n - 1) + \frac{1}{4}(n^2 - 10n + 8)f + \frac{1}{2}(n - 4)rf' \right) \\ &= \frac{f}{r^2} \left(\ell(\ell + n - 1) + \frac{1}{4}n(n - 2) + \frac{\mu n^2}{4r^{n-1}} \right). \end{aligned}$$

Then, n satisfies $n \geq 3$, and ℓ satisfies $\ell \geq 2$. So, the Schwarzschild-Tangherlini solution is stable under the tensor perturbations because the effective potential is positive definite.

■ **Vector perturbation** In the case of the vector perturbations, the effective potential in Eq.(2.156) reads

$$V_{\mathcal{V}}^{(n)} = \frac{f}{r^2} \left(\frac{1}{4} n(n+2) f - \frac{1}{2} (n+4) r f' + (n-1)(1-2f) + \gamma_{\mathcal{V}}^{(n)} \right).$$

Here, we can choose S as

$$S = -\frac{d}{dx} \ln \left(\frac{1}{r^{\frac{n}{2}}} \right).$$

Then, the modified effective potential $\bar{V}$ becomes

$$\bar{V}_{\mathcal{V}}^{(n)} = (\ell(\ell+n-1) - n) \frac{f}{r^2},$$

where ℓ and n are 2 or more. Since this modified potential is positive definite, this black hole is stable under the vector perturbations.

■ **Scalar perturbation** In the case of the scalar perturbations, the effective potential in Eq.(2.156) is

$$V_{\mathcal{S}}^{(n)} = -\frac{f}{rN} (rfN'' + A_3N' + A_4N),$$

and N, A_3 and A_4 are defined in Eq. (2.160). Here, we can choose S as

$$S = \frac{d}{dx} (\ln (r^{n+\ell-1}N)).$$

The modified potential $\bar{V}_{\mathcal{S}}^{(n)}$ becomes

$$\bar{V}_{\mathcal{S}}^{(n)} = \frac{\ell\mu f}{r^{3n/2+1}N} \left(\frac{\ell\mu n(n+1)}{r^{n-1}} + 2(\ell-1)(n^2 + (3\ell-2)n + (\ell-1)^2) \right) > 0.$$

So, this potential is positive definite.

From the above results, the Schwarzschild-Tangherlini black hole is stable in arbitrary dimensions. The key point is that we must find a function for the good S to show the positivity of the effective potential.

Chapter 4

Quasinormal modes

4.1 Summary of quasinormal modes

In this section, we describe quasinormal modes. From the previous analysis, it was found that the master equation determines the dynamics of various fields. Generally, the effective potential of the master equation includes all kinds of physical information such as the type of the field, and the mass, the angular momentum, and the charge of a black hole. The quasinormal modes are discrete damped oscillations determined by the effective potential in the master equation and physically natural boundary conditions which we will define later. Conversely, if the spectrum of the quasinormal modes is observed through some experiments, it is possible to specify the physical quantities of the black hole and the type of the field. So, the quasinormal modes are fundamental and important vibration modes in the physics of black holes. Historically, S. Chandrasekhar et. al. [13] did the first calculation of the quasinormal mode of gravitational perturbations on the Schwarzschild black hole using the shooting method. E. W. Leaver [37] derived an analytical expression of the quasinormal modes of the Schwarzschild black hole and Kerr black hole by the method of continued fractions. Another useful method is the WKB method [14–18, 35, 38–40]. There are other methods for calculating the quasinormal modes [18]. Here, we summarize the main points for calculating the quasinormal modes.

Firstly, we can define the two regions outside a black hole in Fig.4.1. The region I ($-\infty < x < x_0$) is the one ranging from the top of the effective potential x_0 to the horizon of the black hole. The region II ($x_0 < x < \infty$) is the one ranging from the top of the potential to the far outside of the black hole, i.e., the radial infinity.

The wave traveling to the top of the effective potential is called the ingoing wave,

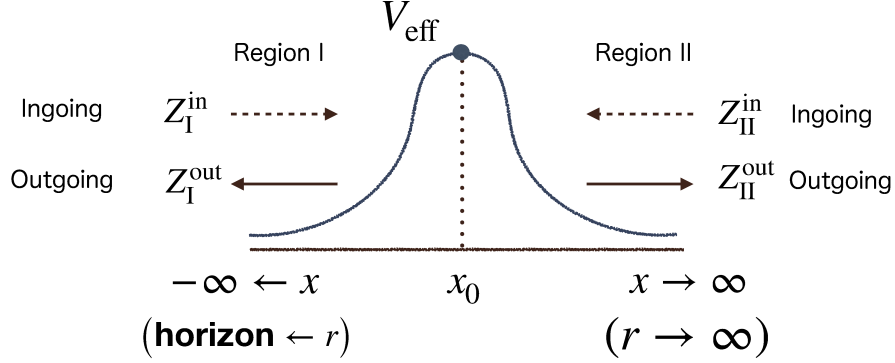


Figure4.1: Example of the effective potential

and the wave traveling from the top of the potential is called the outgoing wave. In each region, the solutions of the master equation can be expressed as

$$\Psi = \begin{cases} \Psi_{\text{I}} = Z_{\text{I}}^{\text{in}} \Psi_{\text{I}}^{\text{in}} + Z_{\text{I}}^{\text{out}} \Psi_{\text{I}}^{\text{out}} \\ \Psi_{\text{II}} = Z_{\text{II}}^{\text{in}} \Psi_{\text{II}}^{\text{in}} + Z_{\text{II}}^{\text{out}} \Psi_{\text{II}}^{\text{out}} \end{cases}, \quad (4.1)$$

where Ψ^{in} and Ψ^{out} represent ingoing and outgoing waves, respectively. $Z_{\text{I/II}}^{\text{in/out}}$ means the amplitude of $\Psi_{\text{I/II}}^{\text{in/out}}$. The boundary condition for obtaining quasi-normal modes is that there is no ingoing waves:

$$Z_{\text{I}}^{\text{in}} = Z_{\text{II}}^{\text{in}} = 0. \quad (4.2)$$

By this condition, only discrete complex modes are allowed.

In the N th order WKB-method, we approximate the function $Q(x)$ defined by

$$Q(x) \equiv \omega^2 - V(x) \quad (4.3)$$

in terms of the $2N$ th order Taylor expansion around the maximum of the potential as

$$\left. \frac{dQ(x)}{dx} \right|_{x=x_0} = Q_0^{(1)} = 0 \quad \text{and} \quad Q(x) \simeq \sum_{k=0}^{2N} \frac{1}{k!} Q_0^{(k)} (x - x_0)^k. \quad (4.4)$$

Expressing the wave function using the WKB approximation, we can calculate the scattering matrix. Thus, we can get the formula for the quasinormal modes [14–18] as

$$\omega \simeq \sqrt{V_0 + \sqrt{2V_0^{(2)}} \left(n_{\text{tone}} + \frac{1}{2} + \sum_{k=1}^{N-1} \Omega_k \right)}, \quad (4.5)$$

where $V_0 \equiv V(x_0)$, and $V_0^{(k)}$ is the k th order derivative of the potential

$$V_0^{(k)} \equiv \left. \frac{d^k V(x)}{dx^k} \right|_{x=x_0}. \quad (4.6)$$

Here, Ω_k is the collection terms for quasinormal modes, and the first and second of Ω_k which the collection term are given by

$$\Omega_1 = (-30\beta_1^2 + 6\beta_2)\beta^2 - \frac{7}{2}\beta_1^2 + \frac{3}{2}\beta_2 \quad \text{and} \quad (4.7)$$

$$\begin{aligned} \Omega_2 = & (-2820\beta_1^4 + 1800\beta_1^2\beta_2 - 280\beta_1\beta_3 - 68\beta_2^2 + 20\beta_4)\beta^3 \\ & + (-1155\beta_1^4 + 918\beta_1^2\beta_2 - 190\beta_1\beta_3 - 67\beta_2^2 + 25\beta_4)\beta. \end{aligned} \quad (4.8)$$

Here, β is defined by

$$\beta \equiv n_{\text{tone}} + \frac{1}{2} \quad (4.9)$$

and β_k ($k \geq 1$) is defined by

$$\beta_k \equiv \frac{V_0^{(k+2)}}{(k+2)!} \left(\frac{1}{2V_0^{(2)}} \right)^{\frac{k}{4}+1}. \quad (4.10)$$

The higher Ω_k can be derived explicitly, but the expression of Ω_k ($k \geq 3$) is too long to write down here. So, we write them in appendix F. The integer n_{tone} is called the tone number. This method is often called the N th order WKB approximation. It is known that in the case of $n_{\text{tone}} < \ell$ this approximation is good. So we focus on the case $n_{\text{tone}} = 0$ in this paper. To calculate the quasinormal mode we write the code in Maple, and it is in the appendix E. The detail is given [14–16, 18, 39, 40] and our thesis [41].

4.2 Examples of quasinormal modes

Here, we list the value of the quasinormal modes of the gravitational perturbations in arbitrary dimensions for several ℓ s. We use the potentials in Eq. (2.156).

^{*1} In TABLE I of [39], those quasinormal modes for $D = 5$ and 6 are wrong. They are the results of 3rd order WKB method.

$D \setminus \ell$	2	3	4
4	-	-	-
5	$1.06809 - 0.252931 i$	$1.41979 - 0.251599 i$	$1.77222 - 0.251020 i$
6	$1.59648 - 0.398701 i$	$2.04710 - 0.395876 i$	$2.49843 - 0.394692 i$
7	$2.09980 - 0.531734 i$	$2.61915 - 0.527167 i$	$3.13871 - 0.525596 i$
8	$2.59103 - 0.653602 i$	$3.16211 - 0.646701 i$	$3.73161 - 0.645373 i$
9	$3.07606 - 0.766252 i$	$3.68913 - 0.755481 i$	$4.29595 - 0.755436 i$
10	$3.55736 - 0.871699 i$	$4.20824 - 0.854062 i$	$4.84240 - 0.856686 i$

Table4.1: Tensor component of the gravitational perturbation

$D \setminus \ell$	2	3	4
4	$0.373618 - 0.0888910 i$	$0.599443 - 0.0927028 i$	$0.809180 - 0.0941642 i$
5	$0.801368 - 0.237556 i^{*1}$	$1.21972 - 0.236151 i$	$1.61254 - 0.240418 i$
6	$1.22570 - 0.382029 i^{*1}$	$1.73565 - 0.374211 i$	$2.24079 - 0.375358 i$
7	$1.65419 - 0.510980 i$	$2.22180 - 0.505532 i$	$2.79533 - 0.501894 i$
8	$2.08648 - 0.628290 i$	$2.69928 - 0.626190 i$	$3.31773 - 0.621195 i$
9	$2.52314 - 0.736413 i$	$3.17226 - 0.736798 i$	$3.82489 - 0.732588 i$
10	$2.96396 - 0.836952 i$	$3.64262 - 0.839071 i$	$4.32326 - 0.836265 i$

Table4.2: Vector component of the gravitational perturbation

$D \setminus \ell$	2	3	4
4	$0.373708 - 0.0889242 i$	$0.599441 - 0.0927027 i$	$0.809178 - 0.0941638 i$
5	$1.50059 - 0.185179 i$	$1.13450 - 0.221132 i$	$1.55023 - 0.232924 i$
6	$1.12408 - 0.290613 i$	$1.50850 - 0.302909 i$	$2.08346 - 0.346285 i$
7	$1.21757 - 0.391512 i$	$1.87670 - 0.454790 i$	$2.52024 - 0.451102 i$
8	$1.54821 - 0.489096 i$	$2.28867 - 0.551747 i$	$2.95096 - 0.569245 i$
9	$1.90577 - 0.580833 i$	$2.67811 - 0.643471 i$	$3.39662 - 0.672835 i$
10	$2.27804 - 0.668274 i$	$3.07832 - 0.731986 i$	$3.83570 - 0.765250 i$

Table4.3: Scalar component of the gravitational perturbation

Part II

Main result

In this part, we introduce the main result of the stability analysis of a p -form field on a spherical black hole. A p -form (or differential form) field is a general totally antisymmetric tensor field with a rank p . This field can exist generally in the theory in higher dimensions, for example, in string theory. The theory of the p -form field is an extension of the Klein-Gordon equation and the Maxwell equation. In this part, we summarize the theory of p -form fields, that is, we derive the master equations, study the stability analysis, and calculate the quasinormal modes.

Chapter 5

p -form fields on a spherical black hole

In this chapter, we study the decomposition of a p -form field in a black hole spacetime. Then, we eliminate some components using the gauge degrees of freedom. We also discuss the eigenvalues of the spherical harmonics for the p -form field. This is the first step and half of the second step is to derive the master equation of the p -form field.

5.1 Basics of p -form fields

In this section, we introduce the action for a p -form field and check the gauge transformation. We also calculate the physical degrees of freedom of a p -form field. We refer the reader to the textbook [42] for more detailed explanation of the theory of differential forms. In appendix G, we note the important operators $*$, d , δ , and Δ .

A p -form field $\mathbf{A}_p$ is defined as

$$\mathbf{A}_p \equiv \frac{1}{p!} A_{\mu_1 \dots \mu_p} dx^{\mu_1} \wedge dx^{\mu_2} \wedge \dots \wedge dx^{\mu_p}, \quad (5.1)$$

where p is a rank of this tensor, and $\wedge$ is the wedge product satisfying

$$dx^2 \wedge dx^1 = -dx^1 \wedge dx^2.$$

Here, dx^μ is the basis of this tensor field. Due to the property of this basis, the coefficient of $\mathbf{A}_p$ should be totally antisymmetric. The action of a p -form field is given by

$$S = -\frac{1}{2} \int \mathbf{F} \wedge * \mathbf{F} = -\frac{1}{2(p+1)!} \int d^D x \sqrt{-g} F_{\mu_1 \dots \mu_{p+1}} F^{\mu_1 \dots \mu_{p+1}}, \quad (5.2)$$

where we used the Hodge operator $*$. The Hodge operator maps a p -form field to a $(D - p)$ -form field by the epsilon tensor $\epsilon^{\mu_1 \cdots \mu_D}$ defined by Eq. (G.4). Here, D is the dimension of the spacetime. The field strength $\mathbf{F}$ is defined by

$$\mathbf{F} \equiv d\mathbf{A}. \quad (5.3)$$

This operator d is equivalent to the rotation in the vector analysis in three dimensions. The operator d is called the exterior derivative satisfying the identity

$$d^2 = 0. \quad (5.4)$$

Using the Hodge operator, we can define the coderivative δ as

$$\delta \equiv (-1)^{D(p+1)+1} * d *. \quad (5.5)$$

This operator is equivalent to the divergence of the vector calculus. Note that the coderivative also satisfies the identity

$$\delta^2 = 0, \quad (5.6)$$

because of the identity Eq.(5.4). The equations of motion for the form field can be deduced as

$$d\mathbf{F} = 0 \quad \text{and} \quad \delta\mathbf{F} = 0. \quad (5.7)$$

This system has the symmetry under the dual transformation

$$\tilde{\mathbf{F}} = *\mathbf{F}. \quad (5.8)$$

This symmetry will appear in the master equation for general form fields. Because of this symmetry, we do not need to consider p -form field with the rank higher than

$$p_{\max} = \left\lfloor \frac{D}{2} \right\rfloor - 1 = \left\lfloor \frac{n}{2} \right\rfloor. \quad (5.9)$$

Here, $\lfloor \cdot \rfloor$ denotes the Gauss symbol. So, we concentrate on the form fields $\mathbf{A}_p$ ($0 \leq p \leq p_{\max}$) in arbitrary dimensions.

The p -form field has the invariance under the gauge transformation

$$\mathbf{A}_p \rightarrow \underline{\mathbf{A}}_p = \mathbf{A}_p + d\mathbf{\Xi}_{p-1}. \quad (5.10)$$

with an arbitrary $(p-1)$ -form field $\mathbf{\Xi}_{p-1}$. This is because the field strength is defined by (5.3). The gauge parameter $\mathbf{\Xi}_{p-1}$ itself has the degeneracy

$$\mathbf{\Xi}_{p-1} \rightarrow \underline{\mathbf{\Xi}}_{p-1} = \mathbf{\Xi}_{p-1} + d\mathbf{\Xi}_{p-2}. \quad (5.11)$$

with an arbitrary $(p-2)$ -form field Ξ_{p-2} . Hence, in order to count the physical degrees of freedom of the p -form field $\mathbf{A}_p$, we need to take into account these degrees generated by the transformations $\Xi_{p-1}, \Xi_{p-2}, \dots, \Xi_0$. Taking into account that components $A_{0\mu_2\dots\mu_p}$ are not dynamical, the formula for physical degrees of freedom is given by

$${}_{D-1}C_p - {}_{D-1}C_{p-1} + {}_{D-1}C_{p-2} - \dots = {}_{D-2}C_p. \quad (5.12)$$

For example, in $D = 4$, a 2-form field has one physical degree of freedom.

5.2 Decomposition of p -form field in terms of coexact form fields on the sphere

To derive the master equation of a p -form field, we need to take three steps. The first step is the choice of the variable of a p -form field. We use the notation $\hat{\mathbf{A}}_p$ to denote a p -form field on the sphere, that is, $\hat{\mathbf{A}}_p$ is written by

$$\hat{\mathbf{A}}_p = A_{A_1\dots A_p} \mathcal{D}^{A_1\dots A_p}, \quad (5.13)$$

where we define the basis

$$\mathcal{D}^{A_1\dots A_p} \equiv dx^{A_1} \wedge \dots \wedge dx^{A_p}. \quad (5.14)$$

We can write down the components of $\mathbf{A}_p$ as

$$\begin{aligned} \mathbf{A}_p = & \frac{1}{(p-2)!} A_{trA_1\dots A_{p-2}} dt \wedge dr \wedge \mathcal{D}^{A_1\dots A_{p-2}} + \frac{1}{(p-1)!} A_{tA_1\dots A_{p-1}} dt \wedge \mathcal{D}^{A_1\dots A_{p-1}} \\ & + \frac{1}{(p-1)!} A_{rA_1\dots A_{p-1}} dr \wedge \mathcal{D}^{A_1\dots A_{p-1}} + \frac{1}{p!} A_{A_1\dots A_p} \mathcal{D}^{A_1\dots A_p}. \end{aligned} \quad (5.15)$$

If we define the components of $\mathbf{A}_p$ as

$$U_{A_1\dots A_{p-2}} \equiv A_{trA_1\dots A_{p-2}}, \quad (5.16)$$

$$V_{A_1\dots A_{p-1}} \equiv A_{tA_1\dots A_{p-1}}, \quad (5.17)$$

$$W_{A_1\dots A_{p-1}} \equiv A_{rA_1\dots A_{p-1}}, \quad (5.18)$$

$$\text{and } X_{A_1\dots A_p} \equiv A_{A_1\dots A_p}. \quad (5.19)$$

Thus, $\mathbf{A}_p$ is written by

$$\mathbf{A}_p = dt \wedge dr \wedge \hat{\mathbf{U}}_{p-2} + dt \wedge \hat{\mathbf{V}}_{p-1} + dr \wedge \hat{\mathbf{W}}_{p-1} + \hat{\mathbf{X}}_p. \quad (5.20)$$

Here, the p -form can be expressed by the one $(p-2)$ -form $\hat{U}_{p-2}$, the two $(p-1)$ -forms $\hat{V}_{p-1}$ and $\hat{W}_{p-1}$ and the one p -form $\hat{X}_p$. The identity

$${}_D C_p = {}_{D-2} C_{p-2} + 2 {}_{D-2} C_{p-1} + {}_{D-2} C_p \quad (5.21)$$

guarantees the matching of the mathematical degrees of freedom $\mathbf{A}_p$.

Using the Hodge decomposition on the sphere S^n , a p -form field $\hat{\mathbf{A}}_p$ can be decomposed as

$$\hat{\mathbf{A}}_p = \hat{d}\hat{\mathbf{A}}_{p-1} + \hat{\mathcal{A}}_p, \quad \text{for } 1 \leq p < n-1, \quad (5.22)$$

where $\hat{\mathcal{A}}_p$ is coexact form, that is, $\hat{\mathcal{A}}_p \equiv \hat{\delta}\hat{\mathbf{B}}_{p+1}$. From this decomposition theorem on the sphere, we can write down a more useful expansion. In fact, for $p \geq 2$, we can express $\hat{\mathbf{A}}_p$ by coexact forms,

$$\begin{aligned} \hat{\mathbf{A}}_p &= \hat{d}\hat{\mathbf{A}}_{p-1} + \hat{\mathcal{A}}_p \\ &= \hat{d}\left(\hat{d}\hat{\mathbf{A}}_{p-2} + \hat{\mathcal{A}}_{p-1}\right) + \hat{\mathcal{A}}_p \\ &= \hat{d}\hat{\mathcal{A}}_{p-1} + \hat{\mathcal{A}}_p. \end{aligned} \quad (5.23)$$

This result shows that all of the components of a form field on S^n are expressed by 0-forms and coexact form fields. Thus, we can represent $\mathbf{A}_p$ which is given by

$$\begin{aligned} \mathbf{A}_p &= dt \wedge dr \wedge \hat{U}_{p-2} + dt \wedge \hat{V}_{p-1} + dr \wedge \hat{W}_{p-1} + \hat{X}_p \\ &= dt \wedge dr \wedge \left(\hat{d}\hat{U}_{p-3} + \hat{U}_{p-2}\right) + dt \wedge \left(\hat{d}\hat{V}_{p-2} + \hat{V}_{p-1}\right) \\ &\quad + dr \wedge \left(\hat{d}\hat{W}_{p-2} + \hat{W}_{p-1}\right) + \left(\hat{d}\hat{X}_{p-1} + \hat{X}_p\right). \end{aligned} \quad (5.24)$$

This general expression contains the decomposition of a 1-form field in Eq. (2.45).

5.3 Gauge fixing of p -form field $\mathbf{A}_p$

The p -form field $\mathbf{A}_p$ has the gauge invariance under the transformation by $\mathbf{B}_{p-1}$ which is a general $(p-1)$ -form field, that is,

$$\mathbf{A}_p \rightarrow \underline{\mathbf{A}}_p = \mathbf{A}_p + d\mathbf{B}_{p-1}. \quad (5.25)$$

Now starting from the general expression for $\mathbf{A}_p$ in D dimensions, we can deduce the following result

$$\mathbf{A}_p = dt \wedge dr \wedge \hat{U}_{p-2} + dt \wedge \hat{V}_{p-1} + dr \wedge \hat{W}_{p-1} + \hat{X}_p \quad (5.26)$$

by using the gauge transformation for $\mathbf{A}_p$ in the appendix H. This is the end of the first step for the derivation.

5.4 Spherical harmonics for coexact p -form field

We review the spherical harmonics of the p -form field [43]. This is the second step to derive the master equation. The Laplace-Beltrami operator is defined by

$$\hat{\Delta} \equiv \hat{\delta}\hat{d} + \hat{d}\hat{\delta}. \quad (5.27)$$

The spherical harmonics of the coexact p -form field $\hat{\mathbf{Y}}_p$ is defined by

$$\hat{\mathbf{Y}}_p = \frac{1}{p!} \mathcal{Y}_{A_1 \dots A_p} \mathcal{D}^{A_1 \dots A_p} \quad (5.28)$$

which satisfies

$$\hat{\delta}\hat{\mathbf{Y}}_p = 0 \quad (5.29)$$

and

$$\hat{\Delta}\hat{\mathbf{Y}}_p = \lambda_p \hat{\mathbf{Y}}_p. \quad (5.30)$$

The last equation becomes

$$\hat{\delta}\hat{d}\hat{\mathbf{Y}}_p = \lambda_p \hat{\mathbf{Y}}_p \quad (5.31)$$

by using the identity (5.6). Here, λ_p is given by

$$\lambda_p \equiv (\ell + p)(\ell + n - p - 1), \quad (5.32)$$

and ℓ is a positive integer, $\ell = 1, 2, \dots, \infty$. On the sphere S^n , the left hand side of equation becomes

$$\hat{\delta}\hat{d}\hat{\mathbf{Y}}_p = \frac{1}{p!} \left(- \left(\hat{\Delta} - p(n - p) \right) \mathcal{Y}_{A_1 \dots A_p} \right) \mathcal{D}^{A_1 \dots A_p}. \quad (5.33)$$

Here, we defined $\hat{\Delta}$ as

$$\hat{\Delta}\mathcal{Y}_{A_1 \dots A_p} = \mathcal{Y}_{A_1 \dots A_p}{}^{:A}{}_{:A}. \quad (5.34)$$

Then, the spectrum of the coefficient of the p -form harmonics $\hat{\mathbf{Y}}_p$ is

$$- \left(\hat{\Delta} - p(n - p) \right) \mathcal{Y}_{A_1 \dots A_p} = \lambda_p \mathcal{Y}_{A_1 \dots A_p}. \quad (5.35)$$

We rewrite this equation as follows

$$\hat{\Delta}\mathcal{Y}_{A_1 \dots A_p} = -\gamma_p^{(n)} \mathcal{Y}_{A_1 \dots A_p}, \quad (5.36)$$

where

$$\gamma_p^{(n)} \equiv \lambda_p - p(n - p) . \quad (5.37)$$

With the harmonics $\hat{\mathcal{Y}}_p$, the coefficient of the general coexact p -form field $\hat{\mathcal{A}}_p$ can be expanded as

$$\mathcal{A}_{A_1 \dots A_p}(x) = \sum_{l, \sigma} \mathcal{A}_{l, \sigma}(t, r) \mathcal{Y}_{A_1 \dots A_p}^{l, \sigma}(x^A) , \quad (5.38)$$

where σ denotes other indices to characterize the degeneracy. For simplicity, we denote $\mathcal{A}_{l, \sigma}(t, r)$ just as $\mathcal{A}$.

Chapter 6

Master equations for p -form field

In this chapter, we complete the other half of the second step and third step. Finally, we will get the master equation for a p -form field.

6.1 Equation of motion for a p -form field

We derive the master equations for a general p -form field as

$$\omega^2 \Psi = -\frac{d^2 \Psi}{dx^2} + V \Psi, \quad (6.1)$$

where V is the effective potential and x is the tortoise coordinate defined by

$$\frac{d}{dx} = f \frac{d}{dr}. \quad (6.2)$$

In Eq.(5.7), the first equation is trivially satisfied from the identity (5.4), and the second equation $\delta \mathbf{F} = 0$ in the coordinate basis is expressed as follows

$$\frac{1}{\sqrt{-g}} \partial_\mu (\sqrt{-g} F^{\mu\mu_1 \cdots \mu_p}) = 0. \quad (6.3)$$

This equation can be decomposed into four patterns.

The first pattern we consider is

$$\frac{1}{\sqrt{-g}} \partial_\mu (\sqrt{-g} F^{\mu tr A_1 \cdots A_{p-2}}) = 0. \quad (6.4)$$

Substituting the components into the above equation, we obtain

$$-\frac{1}{r^{2(p-1)}} \left(\hat{\Delta} - (p-2)(n - (p-2)) \right) \mathcal{U}^{A_1 \cdots A_{p-2}} = 0 \quad (6.5)$$

This yields

$$(\ell + p - 2)(\ell + 1 + n - p) \mathcal{U}_{\ell, \sigma} = 0. \quad (6.6)$$

Since we are considering $p \geq 2$, and generally $n > p$, the quantity $(\ell + p - 2)(\ell + 1 + n - p)$ is always positive. Hence, the coefficient $\mathcal{U}_{\ell,\sigma}$ must vanish for all ℓ , namely,

$$\hat{\mathcal{U}}_{p-2} = 0. \quad (6.7)$$

Thus, the $(p-2)$ -form $\hat{\mathcal{U}}_{p-2}$ in Eq.(5.26) is not dynamical.

The second pattern is the following:

$$\frac{1}{\sqrt{-g}} \partial_\mu (\sqrt{-g} F^{\mu t A_1 \dots A_{p-1}}) = 0. \quad (6.8)$$

It is easy to get

$$\frac{1}{r^{n-2(p-1)}} f \partial_r \left(r^{n-2(p-1)} (\dot{\mathcal{W}} - \mathcal{V}') \right) + \frac{1}{r^2} \left(\gamma_{p-1}^{(n)} + (p-1)(n-p+1) \right) \mathcal{V} = 0. \quad (6.9)$$

The third pattern is given by

$$\frac{1}{\sqrt{-g}} \partial_\mu (\sqrt{-g} F^{\mu r A_1 \dots A_{p-1}}) = 0. \quad (6.10)$$

This leads to

$$\ddot{\mathcal{W}} - \dot{\mathcal{V}}' + \frac{f}{r^2} \left(\gamma_{p-1}^{(n)} + (p-1)(n-p+1) \right) \mathcal{W} = 0. \quad (6.11)$$

The final pattern

$$\frac{1}{\sqrt{-g}} \partial_\mu (\sqrt{-g} F^{\mu A_1 \dots A_p}) = 0 \quad (6.12)$$

generates two equations for the $p-1$ -forms $\mathcal{V}^{A_1 \dots A_{p-1}}$, $\mathcal{W}^{A_1 \dots A_{p-1}}$ and a p -form $\mathcal{X}^{A_1 \dots A_p}$ as

$$\dot{\mathcal{V}} - \frac{1}{r^{n-2p}} f \partial_r (r^{n-2p} f \mathcal{W}) = 0 \quad (6.13)$$

and

$$-\ddot{\mathcal{X}} + \frac{1}{r^{n-2p}} f \partial_r (r^{n-2p} f \mathcal{X}') - \frac{f}{r^2} \left(\gamma_p^{(n)} + p(n-p) \right) \mathcal{X} = 0. \quad (6.14)$$

6.2 Master equations

This is the final step, that is, we derive the master equations of a p -form field. These general results reproduce the effective potential in Eq. (15) and (16) [44] and in Eq. (99) [45] as special cases and agree with the known results [46, 47].

6.2.1 Coexact p -form

From Eq.(6.14), we can read off the master variable for the p -form component as

$$\Psi_p = \frac{1}{r^a} \mathcal{X} \quad \text{and} \quad a \equiv \frac{2p-n}{2}. \quad (6.15)$$

Assuming the time dependence of Ψ_p as

$$e^{-i\omega t},$$

we obtain the master equation

$$-\frac{d^2 \Psi_p}{dx^2} + V_{p,p}^{(n)} \Psi_p = \omega^2 \Psi_p \quad (6.16)$$

with the effective potential $V_{p,p}^{(n)}$

$$V_{p,p}^{(n)} = \frac{f}{r^2} \left((\ell + p)(\ell + n - p - 1) + \frac{n-2p}{2} \left(r f' + \frac{n-2p-2}{2} f \right) \right). \quad (6.17)$$

6.2.2 Coexact $(p-1)$ -form

The $(p-1)$ -form is contained in Eq.(6.9), (6.11) and (6.13), but the master equation is derived by Eq.(6.11) and (6.13). Using the master variable Ψ_{p-1} for the $(p-1)$ -form

$$\Psi_{p-1} \equiv \frac{f}{r^a} \mathcal{W} \quad \text{and} \quad \mathcal{V} = r^{2a} f \partial_r \left(\frac{f}{r^{2a}} \int \mathcal{W} dt \right), \quad (6.18)$$

we can deduce the master equation as

$$-\frac{d^2 \Psi_{p-1}}{dx^2} + V_{p,p-1}^{(n)} \Psi_{p-1} = \omega^2 \Psi_{p-1}. \quad (6.19)$$

Here, we assumed the same time dependence as before. The effective potential $V_{p,p-1}^{(n)}$ is given by

$$V_{p,p-1}^{(n)} = \frac{f}{r^2} \left((\ell + p - 1)(\ell + n - p) + \frac{n-2p}{2} \left(\frac{n-2p+2}{2} f - r f' \right) \right). \quad (6.20)$$

Note that Eq.(6.9) is trivially satisfied.

6.3 Degrees of freedom and dual relations

6.3.1 Degrees of freedom

We have shown that D -dimensional p -form fields can be represented by coexact p -form and coexact $(p-1)$ -form fields.

Firstly, we define the coexact p -form field as

$$\hat{\mathcal{A}}_p \equiv \delta \hat{\mathbf{B}}_{p+1}. \quad (6.21)$$

However, $\hat{\mathbf{B}}_{p+1}$ has a freedom $\hat{\mathbf{B}}_{p+1} + \delta \hat{\mathbf{B}}_{p+2}$. This argument continues up to the maximum value n . Hence, the degrees of freedom of the coexact p -form is given by

$${}_nC_{p+1} - {}nC_{p+2} + {}nC_{p+3} + \cdots = {}nC_p - {}nC_{p-1} + {}nC_{p-2} + \cdots = {}_{n-1}C_p. \quad (6.22)$$

Similarly, we obtain ${}_{n-1}C_{p-1}$ for the degrees of freedom of coexact $(p-1)$ -form. Note that the identity

$${}_{n-1}C_p + {}_{n-1}C_{p-1} = {}nC_p \quad (6.23)$$

exactly give rise to the physical degrees of freedom of a p -form field.

6.3.2 Dual relations

The duality of the field strength plays an important role in form fields. Indeed, we found the following duality relations

$$V_{n-p, n-p}^{(n)} = V_{p, p-1}^{(n)}, \quad (6.24)$$

$$V_{n-p, n-p-1}^{(n)} = V_{p+1, p+1}^{(n)}. \quad (6.25)$$

In particular, in even dimensions, we have the degeneracy

$$V_{\frac{n}{2}, \frac{n}{2}}^{(n)} = V_{\frac{n}{2}, \frac{n}{2}-1}^{(n)}. \quad (6.26)$$

Later, we will see this degeneracy in the quasinormal mode spectrum.

6.4 Examples of effective potentials

In 4 dimensions where $n = 2$, the master equation for the $p = 0$ -form $\mathbf{A}_0$ becomes

$$V_{0,0}^{(2)} = \frac{f}{r^2} (\ell(\ell+1) + rf'). \quad (6.27)$$

From the master equations for the $p = 1$ -form $\mathbf{A}_1$, we see the effective potential of the coexact 1-form is

$$V_{1,1}^{(2)} = \ell(\ell + 1) \frac{f}{r^2} \quad (6.28)$$

and that of coexact 0-form reads

$$V_{1,0}^{(2)} = \ell(\ell + 1) \frac{f}{r^2}. \quad (6.29)$$

The effective potentials for the the coexact 1-form and the coexact 0-form in $\mathbf{A}_1$ are the same. Our results agree with the expression (99) in [45]. Moreover, since the coexact 2-forms on the sphere S^2 does not exist, the master equation for the $p = 2$ -form $\mathbf{A}_2$ becomes single. The effective potential of the coexact 1-form is given by

$$V_{2,1}^{(2)} = \frac{f}{r^2} (\ell(\ell + 1) + rf') . \quad (6.30)$$

As is expected, this effective potential is the same expression as that for the 0-form field in Eq.(6.27). This confirms we can consider only the form fields with the rank larger than p_M .

In 5 dimensions, where $n = 3$, the master equation for the $p = 0$ -form $\mathbf{A}_0$ becomes

$$V_{0,0}^{(3)} = \frac{f}{r^2} \left(\ell(\ell + 2) + \frac{3}{2} \left(rf' + \frac{f}{2} \right) \right). \quad (6.31)$$

We can also read off the effective potentials in the master equations for the 1-form $\mathbf{A}_1$. The effective potential of the coexact 1-form reads

$$V_{1,1}^{(3)} = \frac{f}{r^2} \left((\ell + 1)^2 + \frac{1}{2} \left(rf' - \frac{1}{2}f \right) \right) \quad (6.32)$$

and that of the coexact 0-form is given by

$$V_{1,0}^{(3)} = \frac{f}{r^2} \left(\ell(\ell + 2) + \frac{1}{2} \left(\frac{3}{2}f - rf' \right) \right). \quad (6.33)$$

In 5-dimensions, we need not consider a 2-form field because of the duality. Here, for the check, we look at the master equation for the 2-form $\mathbf{A}_2$. The effective potentials of the coexact 2-form is given by

$$V_{2,2}^{(3)} = \frac{f}{r^2} \left(\ell(\ell + 2) + \frac{1}{2} \left(\frac{3}{2}f - rf' \right) \right) \quad (6.34)$$

and that of the coexact 1-form becomes

$$V_{2,1}^{(3)} = \frac{f}{r^2} \left((\ell + 1)^2 + \frac{1}{2} \left(r f' - \frac{1}{2} f \right) \right). \quad (6.35)$$

The effective potential (6.35) is the same as Eq.(6.32), and the effective potential (6.34) is the same as Eq.(6.33). These results just reflect the duality relations (6.24) and (6.25).

Chapter 7

Proof of stability of p -form field

In the present case, we found an appropriate S -deformation as follows

$$S = -\frac{d}{dx} \left(\ln \left(\frac{1}{r^\alpha} \right) \right). \quad (7.1)$$

This is similar to the case of the vector gravitational perturbations of the Schwarzschild-Tangherlini solution [34]. Here, α is the parameter we should choose appropriately.

7.1 Effective potential of coexact p -form

If we choose

$$\alpha = -\frac{n-2p}{2}, \quad (7.2)$$

the effective potential of the coexact p -form on the sphere becomes

$$\begin{aligned} V_{p,p}^{(n)} &\rightarrow \bar{V}_{p,p}^{(n)} = V_{p,p}^{(n)} + f \frac{dS}{dr} - S^2 \\ &= \frac{f}{r^2} (\ell + p) (\ell + n - p - 1). \end{aligned} \quad (7.3)$$

This modified potential is positive definite outside the event horizon, because a rank p satisfies $p \leq p_{\max} < n$.

7.2 Effective potential of coexact $(p-1)$ -form

If we choose

$$\alpha = \frac{n-2p}{2}, \quad (7.4)$$

the effective potential of the coexact $(p-1)$ -form on the sphere becomes

$$\begin{aligned}
V_{p,p-1}^{(n)} \rightarrow \bar{V}_{p,p-1}^{(n)} &= V_{p,p-1}^{(n)} + f \frac{dS}{dr} - S^2 \\
&= \frac{f}{r^2} ((\ell + p)(\ell + n - p - 1) - (n - 2p)) \\
&\geq \frac{f}{r^2} ((1 + p)(1 + n - p - 1) - (n - 2p)) \\
&= \frac{f}{r^2} p(n - p + 1) > 0.
\end{aligned} \tag{7.5}$$

This modified potential is also positive definite outside the event horizon.

From the above analysis, we see p -form fields in arbitrary dimensions are stable in the spherical black hole even for other gravity theories as long as f satisfies the positivity $f > 0$ outside the horizon $r > r_h$. Here, r_h which satisfies

$$f = 0$$

is the horizon of a black hole.

Chapter 8

Quasinormal modes of p -form fields

D	QNM: ω	D	QNM: ω
6	$1.2618 - 0.4616i$	6	$1.2618 - 0.4616i$
7	$1.7509 - 0.5920i$	7	$1.5387 - 0.5652i$
8	$2.2231 - 0.7192i$	8	$1.8352 - 0.7345i$
9	$2.6681 - 0.8555i$	9	$2.2630 - 0.8521i$
10	$3.0791 - 1.0080i$	10	$2.6910 - 0.9444i$

(a) 2-form component (b) 1-form component

Table8.1: Quasinormal modes of 2-form field $\mathbf{A}_2$

D	QNM: ω	D	QNM: ω
8	$2.0779 - 0.6754i$	8	$2.0779 - 0.6754i$
9	$2.6018 - 0.7640i$	9	$2.3795 - 0.7729i$
10	$3.1539 - 0.8307i$	10	$2.6947 - 0.9197$

(a) The 3-form component (b) The 2-form component

Table8.2: Quasinormal modes of 3-form field $\mathbf{A}_3$

Now, we show the quasinormal modes of the p -form fields up to $D = 10$ -dimensions. In this case, we need to consider form fields up to $p = 4$. In this study we used the 6th order WKB method [40], so we need the $\Omega_1, \Omega_2, \dots, \Omega_5$. We choose a set of

D	QNM: ω	D	QNM: ω
10	$2.9227 - 0.8595i$	10	$2.9227 - 0.8595i$

(a) The 4-form component (b) The 3-form component

Table8.3: Quasinormal modes of 4-form field $\mathbf{A}_4$

parameters,

$$(\ell, \mu, n_{\text{tone}}) = (1, 1, 0). \quad (8.1)$$

In Table 8.1, we showed the quasinormal modes of a 2-form field. As you can see a quasinormal mode of coexact 2-form and a quasinormal mode of coexact 1-form in 6-dimension coincide. This comes from the duality relations (6.24) and (6.25) for the field strength. Except for $D = 8$, the coexact 2-form decays faster than coexact 1-form. In Table 8.2, we listed quasinormal modes of a 3-form field. In $D = 8$ dimensions, we can see duality relations hold. In other dimensions, the coexact 2-form decays faster than coexact 3-form. In Table 8.3, we displayed QNMs of a 4-form field. In this case, only $D = 10$ is relevant. Here, we see the duality relations again. In all cases, we see, as D increases, real and imaginary parts of quasinormal modes increase.

Chapter 9

Conclusion

We investigated the stability of various tensor fields on a black hole. We studied the high-dimensional black hole physics, and in particular, we studied p -form fields. We formulated a method for investigating massless differential form fields. They are generally expressed in terms of the coexact p -form and coexact $(p - 1)$ -form on the sphere. These variables allowed us to find the master equations. We revealed some relations between the effective potentials in the master equations under the duality relation of p -form fields. By examining the master equation of differential form fields, we investigated the stability and quasinormal modes of p -form fields. As for the stability, we showed that any massless p -form field is stable on a symmetric black hole in any dimensions using the S -deformation method. Then, we investigated the quasinormal modes, which are important physical quantities on a black hole, using the WKB approximation. We clarified the duality relation and the dependence of dimensions of p -form fields in $D(\leq 10)$ -dimensions.

It is interesting to include rotating black holes in our analysis. Recently, Lunin found the ansatz for the 1-form field in arbitrary dimensions to get the separable equations of motion [48]. After our work [49], he proposed the conjecture of the ansatz for the general p -form field [50]. So, it is interesting to investigate p -form fields in higher dimensional rotational black holes. We can also consider higher spin fields in arbitrary dimensions. Resolving the above problems must have implications for the string theory. We leave these problems for future work.

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Appendix

Appendix A

Conventions

Here, we define some quantities which are used in this thesis.

■Unit We use the natural unit. c is the velocity of light. $\hbar$ is the Dirac constant. G is the Newton constant. They are set to

$$c = \hbar = G = 1. \quad (\text{A.1})$$

■Inverse metric $g^{\mu\nu}$ The inverse metric $g^{\mu\nu}$ satisfies

$$g^{\mu\rho} g_{\rho\nu} = \delta_\nu^\mu.$$

Here, δ_ν^μ is the Kronecker delta.

■Christoffel symbol $\Gamma_{\mu\nu}^\rho$ The Christoffel symbol is defined by

$$\Gamma_{\mu\nu}^\rho \equiv \frac{1}{2} g^{\rho\alpha} (\partial_\mu g_{\alpha\nu} + \partial_\nu g_{\alpha\mu} - \partial_\alpha g_{\mu\nu}). \quad (\text{A.2})$$

■Riemann tensor $R^\alpha_{\mu\beta\nu}$ The Riemann tensor $R^\alpha_{\mu\beta\nu}$ is defined by

$$R^\alpha_{\mu\beta\nu} \equiv \partial_\beta \Gamma_{\nu\mu}^\alpha - \partial_\nu \Gamma_{\beta\mu}^\alpha + \Gamma_{\beta\lambda}^\alpha \Gamma_{\nu\mu}^\lambda - \Gamma_{\nu\lambda}^\alpha \Gamma_{\beta\mu}^\lambda. \quad (\text{A.3})$$

■Ricci tensor $R_{\mu\nu}$ The Ricci tensor is defined by

$$R_{\mu\nu} \equiv R^\alpha_{\mu\alpha\nu}. \quad (\text{A.4})$$

■Einstein tensor $G_{\mu\nu}$ The Einstein tensor is defined by

$$G_{\mu\nu} \equiv R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu}. \quad (\text{A.5})$$

Appendix B

Black hole solutions

B.1 Schwarzschild solution

The Schwarzschild solution is given by

$$ds^2 = -f(r) dt^2 + \frac{1}{f(r)} dr^2 + r^2 d\Omega^2, \quad (\text{B.1})$$

where $d\Omega^2$ is the infinitesimal spatial displacement of S^2 . It is given by

$$\begin{aligned} d\Omega^2 &= q_{AB} dx^A dx^B \\ &= d\theta^2 + \sin^2 \theta d\phi^2. \end{aligned} \quad (\text{B.2})$$

Here, q_{AB} is the metric of S^2 . This metric has the spherical symmetry, and this metric is time-independent, that is, this solution is static. This solution was found by K. Schwarzschild in 1916. From the Einstein equation f is given by

$$f = 1 - \frac{2M}{r}. \quad (\text{B.3})$$

M is the mass of a black hole. The Schwarzschild radius r_h which is defined by $f = 0$ corresponds to the size of the event horizon of a black hole.

B.2 Schwarzschild-Tangherlini solutions

The general relativity in higher dimensions is also an interesting target to study its mathematical properties. F. R. Tangherlini found a natural extension of the Schwarzschild solution in Einstein's theory in higher dimensions [19]. This solution can be written in a similar expression as

$$ds^2 = -f(r) dt^2 + \frac{1}{f(r)} dr^2 + r^2 d\Omega^2. \quad (\text{B.4})$$

Here, $d\Omega^2$ is the metric with respect to S^n . If we write the spherical coordinate as $x^A = (\theta_1, \dots, \theta_n)$, this is written by

$$\begin{aligned} d\Omega^2 &= q_{AB} dx^A dx^B \\ &= d\theta_1^2 + \sin^2 \theta_1 (d\theta_2^2 + \sin^2 \theta_2 (\dots (d\theta_n^2))) . \end{aligned} \quad (\text{B.5})$$

This metric has the spherical symmetry in n -dimension, and this metric is time-independent. From the Einstein equation in higher dimension the function f is given by

$$f = 1 - \frac{\mu}{r^{n-1}}. \quad (\text{B.6})$$

μ is related to ADM mass and n is defined by

$$n \equiv D - 2.$$

B.3 Kerr solution

R. Kerr found a rotational solution of the Einstein equation [51]. This solution is often written on the Boyer-Lindquist coordinate in [52]. In this coordinate, this black hole solution is given by

$$ds^2 = -\frac{\Delta_r}{\rho^2} (dt - a \sin^2 \theta d\phi)^2 + \frac{\rho^2}{\Delta_r} dr^2 + \rho^2 d\theta^2 + \frac{\sin^2 \theta}{\rho^2} (a dt - (r^2 + a^2) d\phi)^2. \quad (\text{B.7})$$

Here, the Δ_r and ρ are defined by

$$\Delta_r \equiv r^2 - 2Mr + a^2 \quad \text{and} \quad \rho^2 \equiv \rho^2(r, \theta) = r^2 + a^2 \cos^2 \theta. \quad (\text{B.8})$$

The coefficients of the metric are independent of t, ϕ . This solution is time independent solution. The coordinate ϕ expresses the azimuthal angle. If $M \rightarrow 0$, the metric becomes

$$ds^2 = -\frac{r^2 + a^2}{\rho^2} (dt - a \sin^2 \theta d\phi)^2 + \frac{\rho^2}{r^2 + a^2} dr^2 + \rho^2 d\theta^2 + \frac{\sin^2 \theta}{\rho^2} (a dt - (r^2 + a^2) d\phi)^2. \quad (\text{B.9})$$

This spacetime is equivalent to the Minkowski spacetime in the spheroidal coordinates. We can define a coordinate transformation

$$x = \sqrt{r^2 + a^2} \sin \theta \cos \phi, \quad (\text{B.10})$$

$$y = \sqrt{r^2 + a^2} \sin \theta \sin \phi, \quad (\text{B.11})$$

$$\text{and} \quad z = r \cos \theta. \quad (\text{B.12})$$

In this coordinate, the metric in Eq.(B.9) becomes

$$ds^2 = -dt^2 + dx^2 + dy^2 + dz^2, \quad (\text{B.13})$$

that is, the Minkowski spacetime.

B.4 Myers-Perry solution

R. C. Myers and M. J. Perry found an extension of the Kerr solution in higher dimensions in [20], and good reviews of the Myers-Perry solution are given [48, 53]. This black hole solution has a slightly different expression in odd and even dimensions. In even dimensions, the metric is given by

$$ds^2 = -dt^2 + \frac{Mr}{FR} \left(dt + \sum_{i=1}^n a_i \mu_i^2 d\phi_i \right)^2 + \frac{FR}{R - Mr} dr^2 + \sum_{i=1}^n (r^2 + a_i^2) (d\mu_i^2 + \mu_i^2 d\phi_i^2) + r^2 d\alpha^2. \quad (\text{B.14})$$

There is the relation between α and μ_i ($i = 1, \dots, n$) is given by

$$\alpha^2 + \sum_{i=1}^n \mu_i^2 = 1, \quad (\text{B.15})$$

and F and R are defined by

$$F = 1 - \sum_{k=1}^n \frac{(a_k \mu_k)^2}{r^2 + a_k^2} \quad \text{and} \quad R = \prod_{k=1}^n (r^2 + a_k^2). \quad (\text{B.16})$$

In odd dimension, the metric is given by

$$ds^2 = -dt^2 + \frac{Mr^2}{FR} \left(dt + \sum_{i=1}^n a_i \mu_i^2 d\phi_i \right)^2 + \frac{FR}{R - Mr^2} dr^2 + \sum_{i=1}^n (r^2 + a_i^2) (d\mu_i^2 + \mu_i^2 d\phi_i^2). \quad (\text{B.17})$$

There is also a constraint to the coordinates

$$\sum_{i=1}^n \mu_i^2 = 1. \quad (\text{B.18})$$

F and R have same expressions in Eq. (B.16) each other.

Appendix C

Geometrical quantities on spherical black hole

C.1 Metric

The Schwarzschild-Tangherlini solution is given by

$$ds^2 = \bar{g}_{\mu\nu} dx^\mu dx^\nu \quad (\text{C.1})$$

$$= -f(r) dt^2 + \frac{1}{f(r)} dr^2 + r^2 q_{AB} dx^A dx^B. \quad (\text{C.2})$$

Here, we define the metric q_{AB} of S^2 . We represent the spherical coordinates as $x^A = (\theta^1, \dots, \theta^n)$.

C.2 properties of S^n

S^n is a constant curvature spacetime. So, we can derive the Riemann tensor of S^n as

$$\hat{R}_{ABCD} = k (q_{AC} q_{BD} - q_{AD} q_{BC}). \quad (\text{C.3})$$

k is a constant parameter and usually normalized to $|k| = 0, 1$. Then, the Ricci tensor and the Ricci scalar become

$$\hat{R}_{AB} = (n-1) k q_{AB} \quad \text{and} \quad (\text{C.4})$$

$$\hat{R} = n(n-1) k. \quad (\text{C.5})$$

k corresponds to

$$k = \begin{cases} k = 1 & S^n \\ k = 0 & \mathbb{E}^n \\ k = -1 & H^n \end{cases}. \quad (\text{C.6})$$

C.3 Christoffel symbol

The components of the Christoffel symbol are given by

$$\Gamma_{JK}^I = \frac{f'}{2f} \delta_t^I \delta_J^t \delta_K^r + \frac{f'}{2f} \delta_t^I \delta_J^r \delta_K^t + \frac{ff'}{2} \delta_r^I \delta_J^t \delta_K^t + \left(-\frac{f'}{2f}\right) \delta_r^I \delta_J^r \delta_K^r, \quad (\text{C.7})$$

$$\Gamma_{JA}^I = 0, \quad (\text{C.8})$$

$$\Gamma_{AB}^I = (-rf) \delta_r^I q_{AB}, \quad (\text{C.9})$$

$$\Gamma_{IJ}^A = 0, \quad (\text{C.10})$$

$$\Gamma_{IB}^A = \frac{1}{r} \delta_I^r \delta_B^A, \quad (\text{C.11})$$

$$\text{and } \Gamma_{AB}^C = \hat{\Gamma}_{AB}^C(q). \quad (\text{C.12})$$

Here, ' represents the derivative with respect to r .

C.4 Riemann tensor

The components of the Riemann tensor are given by

$$R^I{}_{JKL} = -\frac{f''}{f} \delta_t^I \delta_J^r \delta_{[K}^t \delta_{L]}^r - ff'' \delta_r^I \delta_J^t \delta_{[K}^t \delta_{L]}^r, \quad (\text{C.13})$$

$$R^I{}_{JKA} = 0, \quad (\text{C.14})$$

$$R^I{}_{JAB} = 0, \quad (\text{C.15})$$

$$R^I{}_{AJK} = 0, \quad (\text{C.16})$$

$$R^I{}_{AJB} = -\frac{rf'}{2} \delta_t^I \delta_J^t q_{AB} - \frac{rf'}{2} \delta_r^I \delta_J^r q_{AB}, \quad (\text{C.17})$$

$$R^I{}_{ABC} = 0, \quad (\text{C.18})$$

$$R^A{}_{IJK} = 0, \quad (\text{C.19})$$

$$R^A{}_{IJB} = -\frac{ff'}{2r} \delta_I^t \delta_J^t \delta_B^A + \frac{f'}{2rf} \delta_I^r \delta_J^r \delta_B^A, \quad (\text{C.20})$$

$$R^A{}_{IBC} = 0, \quad (\text{C.21})$$

$$R^A{}_{BIJ} = 0, \quad (\text{C.22})$$

$$R^A{}_{BIC} = 0, \quad (\text{C.23})$$

$$\text{and } R^A{}_{BCD} = \hat{R}^A{}_{BCD} - 2f\delta_{[C}^A q_{D]B}. \quad (\text{C.24})$$

C.5 Ricci tensor

The components of the Ricci tensor are given by

$$R_{IJ} = \frac{f}{2} \left(f'' + \frac{nf'}{r} \right) \delta_I^t \delta_J^t - \frac{1}{2f} \left(f'' + \frac{nf'}{r} \right) \delta_I^r \delta_J^r, \quad (\text{C.25})$$

$$R_{IA} = 0, \quad \text{and} \quad (\text{C.26})$$

$$R_{AB} = ((n-1)(k-f) - rf') q_{AB}. \quad (\text{C.27})$$

C.6 Ricci scalar

The Ricci scalar is given by

$$R = -\frac{1}{r^2} (r^2 f'' + 2nr f' - n(n-1)(k-f)). \quad (\text{C.28})$$

C.7 Solution of Einstein equation

In vacuum, the Einstein equation is equivalent to

$$R_{\mu\nu} = 0. \quad (\text{C.29})$$

So, the solution of this metric can be derived as

$$f = k - \frac{\mu}{r^{n-1}}. \quad (\text{C.30})$$

C.8 Covariant derivative on spherical black hole

We summarize the operation of the covariant derivative to a vector on a spherical black hole.

■ Covariant derivative with respect to a vector: $\nabla_\mu V_\nu$ We can decompose $\nabla_\mu V_\nu$ into the four patterns $\bar{\nabla}_I V_J, \bar{\nabla}_I V_A, \bar{\nabla}_A V_I$ and $\bar{\nabla}_A V_B$. We write them following,

$$\bar{\nabla}_I V_J = \partial_I V_J - \frac{f'}{2f} (\delta_I^t \delta_J^r + \delta_I^r \delta_J^t) V_t - \frac{1}{2} f' \left(f \delta_I^t \delta_J^t - \frac{1}{f} \delta_I^r \delta_J^r \right) V_r, \quad (\text{C.31})$$

$$\bar{\nabla}_I V_A = \partial_I V_A - \frac{1}{r} \delta_I^r V_A, \quad (\text{C.32})$$

$$\bar{\nabla}_I V_A = \partial_I V_A - \frac{1}{r} \delta_I^r V_A, \quad (\text{C.33})$$

$$\text{and } \bar{\nabla}_A V_B = V_{B:A} + r f V_r q_{AB}. \quad (\text{C.34})$$

■ **Covariant derivative with respect to a 2nd-rank tensor:** $\nabla_\mu T_{\nu\rho}$ We can decompose $\nabla_\mu T_{\nu\rho}$ into the six patterns $\bar{\nabla}_I T_{JK}, \bar{\nabla}_I T_{JA}, \bar{\nabla}_I T_{AB}, \bar{\nabla}_A T_{IJ}, \bar{\nabla}_A T_{IB}$ and $\bar{\nabla}_A T_{BC}$. We write them following,

$$\begin{aligned} \bar{\nabla}_I T_{JK} = & \partial_I T_{JK} - \frac{f'}{2f} \delta_I^t (\delta_J^r T_{Kt} + \delta_K^r T_{Jt}) - \frac{1}{2} f f' \delta_I^t (\delta_J^t T_{Kr} + \delta_K^t T_{Jr}) \\ & - \frac{f'}{2f} \delta_I^r (\delta_J^t T_{Kt} + \delta_K^t T_{Jt}) + \frac{f'}{2f} \delta_I^r (\delta_J^r T_{Kr} + \delta_K^r T_{Jr}), \end{aligned} \quad (\text{C.35})$$

$$\bar{\nabla}_I T_{JA} = \partial_I T_{JA} - \frac{f'}{2f} (\delta_I^t \delta_J^r + \delta_I^r \delta_J^t) T_{tA} - \frac{1}{2} f f' \delta_I^t \delta_J^t T_{rA} + \frac{f'}{2f} \delta_I^r \delta_J^r T_{rA} - \frac{1}{r} \delta_I^r T_{JA}, \quad (\text{C.36})$$

$$\bar{\nabla}_I T_{AB} = \partial_I T_{AB} - \frac{2}{r} \delta_I^r T_{AB}, \quad (\text{C.37})$$

$$\bar{\nabla}_A T_{IJ} = T_{IJ:A} - \frac{1}{r} (\delta_I^r T_{JA} + \delta_J^r T_{IA}), \quad (\text{C.38})$$

$$\bar{\nabla}_A T_{IB} = T_{IB:A} - \frac{1}{r} \delta_I^r T_{AB} + r f T_{Ir} q_{AB}, \quad (\text{C.39})$$

$$\text{and } \bar{\nabla}_A T_{BC} = T_{BC:A} + 2r f q_{A(B} T_{C)r}. \quad (\text{C.40})$$

Here, $'$ represents the derivative with respect to r , and $\cdot$ represents the derivative with respect to t .

Appendix D

Gravitational perturbation on spherical black hole

This chapter gives the first-order perturbed quantities of the metric. The perturbed metric can be decomposed into three types, that is, a scalar, vector and tensor with respect to the spherical coordinate. We will note the perturbed quantity for each perturbations one by one. Especially, we note the results in arbitrary dimensions on a spherical black hole.

D.1 General formulae for the first order perturbation of the metric

D.1.1 Perturbed metric

We define a perturbation, or linearization, of a metric as

$$g_{\mu\nu} \equiv \bar{g}_{\mu\nu} + h_{\mu\nu}. \quad (\text{D.1})$$

Here, $g_{\mu\nu}$ is a whole metric of the spacetime. $\bar{g}_{\mu\nu}$ is a background spacetime. For example, in our case $\bar{g}_{\mu\nu}$ corresponds to the Schwarzschild metric. $h_{\mu\nu}$ is a perturbation with respect to the metric $\bar{g}_{\mu\nu}$. We assume the amplitude of $h_{\mu\nu}$ is sufficiently small so that we will neglect the second and higher order perturbed quantities of $h_{\mu\nu}$. In the following, a bar indicates a quantity with respect to the background metric $\bar{g}_{\mu\nu}$.

D.1.2 Inverted metric

The inverted metric is given by

$$g^{\mu\nu} = \bar{g}^{\mu\nu} - h^{\mu\nu} + \mathcal{O}(h^2). \quad (\text{D.2})$$

D.1.3 Christoffel symbol

The linearized Christoffel symbol is given by

$$\delta\Gamma_{\mu\nu}^{\rho} \equiv \Gamma_{\mu\nu}^{\rho}(g_{\mu\nu}) - \bar{\Gamma}_{\mu\nu}^{\rho}(\bar{g}_{\mu\nu}) \quad (\text{D.3})$$

$$= \frac{1}{2} (\bar{\nabla}_{\mu} h^{\rho}_{\nu} + \bar{\nabla}_{\nu} h^{\rho}_{\mu} - \bar{\nabla}^{\rho} h_{\mu\nu}) + \mathcal{O}(h^2). \quad (\text{D.4})$$

D.1.4 Riemann tensor

The linearized Riemann tensor is given by

$$\delta R^{\alpha}_{\mu\beta\nu} \equiv R^{\alpha}_{\mu\beta\nu} - \bar{R}^{\alpha}_{\mu\beta\nu} \quad (\text{D.5})$$

$$= \bar{\nabla}_{\beta} \delta\Gamma^{\alpha}_{\mu\nu} - \bar{\nabla}_{\nu} \delta\Gamma^{\alpha}_{\beta\mu} + \mathcal{O}(h^2). \quad (\text{D.6})$$

D.1.5 Ricci tensor

The linearized Ricci tensor is given by

$$\delta R_{\mu\nu} \equiv \delta R^{\alpha}_{\mu\alpha\nu} \quad (\text{D.7})$$

$$= \frac{1}{2} (\bar{\nabla}_{\alpha} \bar{\nabla}_{\mu} h^{\alpha}_{\nu} + \bar{\nabla}_{\alpha} \bar{\nabla}_{\nu} h^{\alpha}_{\mu} - \bar{\nabla}_{\alpha} \bar{\nabla}^{\alpha} h_{\mu\nu} - \bar{\nabla}_{\nu} \bar{\nabla}_{\mu} h^{\alpha}_{\alpha}). \quad (\text{D.8})$$

D.1.6 Ricci scalar

The linearized Ricci scalar is given by

$$\begin{aligned} \delta R &\equiv R - \bar{R} \\ &= \bar{g}^{\mu\nu} \delta R_{\mu\nu} - h^{\mu\nu} \bar{R}_{\mu\nu}. \end{aligned} \quad (\text{D.9})$$

D.1.7 Einstein tensor

The linearized Einstein tensor is given by

$$\delta G_{\mu\nu} \equiv G_{\mu\nu} - \bar{G}_{\mu\nu} \quad (\text{D.10})$$

$$\begin{aligned} &= R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R - \left(\bar{R}_{\mu\nu} - \frac{1}{2}\bar{g}_{\mu\nu}\bar{R} \right) \\ &= \delta R_{\mu\nu} - \frac{1}{2}\bar{g}_{\mu\nu}\delta R - \frac{1}{2}h_{\mu\nu}\bar{R} + \mathcal{O}(h^2). \end{aligned} \quad (\text{D.11})$$

D.2 Perturbations of the spherical black hole in arbitrary dimensions

Here, we summarize the formulae to study the gravitational perturbations on a spherical black hole in arbitrary dimensions. We will use the so-called complete gauge fixing method. In Sec.2.3.3.1 and 2.3.3.2, we treated the complete decomposition of the perturbed metric on the spherical coordinates. Here, we will use the same notation in those sections. The capital Latin indices $(I, J, K, \dots)$ represent the (t, r) coordinates, and $(A, B, C, \dots)$ represent the spherical coordinates. Here, we assume all components of $h_{\mu\nu}$ have the time dependence written by $e^{-i\omega t}$.

D.2.1 Einstein equation of tensor perturbations in vacuum

In this section, we give the master equation of the tensorial components of the perturbed metric. Firstly, the tensorial components of the perturbed metric, $h_{\mu\nu}^{(\text{T})}$, are given by

$$h_{\mu\nu}^{(\text{T})} = \begin{pmatrix} \mathbf{O} & \mathbf{0} \\ \text{sym.} & S_{AB} \end{pmatrix}. \quad (\text{D.12})$$

Here, for the simplicity, we rescale the tensor S_{AB} as

$$h_{AB}^{(\text{T})} = S_{AB} = r^2 s_{AB}. \quad (\text{D.13})$$

s_{AB} satisfies

$$s_A{}^A = 0 \quad (\text{D.14})$$

and

$$s_{AB}{}^{;A} = 0. \quad (\text{D.15})$$

D.2.1.1 Covariant derivative of perturbed metric

The components of the derivative of the perturbed metric on the spherical coordinates are given by

$$\bar{\nabla}_I h_{JK} = 0, \quad (D.16)$$

$$\bar{\nabla}_I h_{JA} = 0, \quad (D.17)$$

$$\bar{\nabla}_I h_{AB} = r^2 \delta_I^t \dot{s}_{AB} + r^2 \delta_I^r s'_{AB}, \quad (D.18)$$

$$\bar{\nabla}_A h_{IJ} = 0, \quad (D.19)$$

$$\bar{\nabla}_A h_{IB} = -r \delta_I^r s_{AB}, \quad (D.20)$$

$$\text{and } \bar{\nabla}_A h_{BC} = r^2 s_{BC:A}. \quad (D.21)$$

D.2.1.2 Perturbed Christoffel symbol

The components of the linearized Christoffel symbol are given by

$$\delta\Gamma_{KL}^I = 0, \quad (D.22)$$

$$\delta\Gamma_{KA}^I = 0, \quad (D.23)$$

$$\delta\Gamma_{IJ}^A = 0, \quad (D.24)$$

$$\delta\Gamma_{AB}^I = \frac{r^2}{2f} \delta_I^t \dot{s}_{AB} - \frac{r^2}{2} f \delta_r^I \left(s'_{AB} + \frac{2}{r} s_{AB} \right), \quad (D.25)$$

$$\delta\Gamma_{IC}^A = \frac{1}{2} \left(\delta_I^t \dot{s}_C^A + \delta_I^r s_C'^A \right), \quad (D.26)$$

$$\text{and } \delta\Gamma_{CD}^A = \frac{1}{2} \left(s^A_{D:C} + s^A_{C:D} - s_{CD}{}^{;A} \right). \quad (D.27)$$

D.2.1.3 Perturbed Ricci tensor

The components of the linearized Ricci tensor are given by

$$\delta R_{IJ} = 0, \quad (D.28)$$

$$\delta R_{IA} = 0, \quad (D.29)$$

$$\text{and } \delta R_{AB} = \frac{r^2}{2f} \ddot{s}_{AB} - \frac{1}{2} r^2 f s''_{AB} - \frac{1}{2} (r f' + n f) r s'_{AB} - \left((n-1) f + r f' - \frac{1}{2} (2nk - \hat{\Delta}) \right) s_{AB}. \quad (D.30)$$

The equation $\delta R_{\mu\nu} = 0$ is equivalent to the linearized Einstein equation in vacuum.

D.2.1.4 Master equation for tensorial component

From the Einstein equations in vacuum, s_{AB} satisfies

$$0 = \delta R_{AB} = \frac{r^2}{2f} \ddot{s}_{AB} - \frac{1}{2} r^2 f s''_{AB} - \frac{1}{2} (r f' + n f) r s'_{AB} - \left((n-1) f + r f' - \frac{1}{2} (2nk - \hat{\Delta}) \right) s_{AB}. \quad (\text{D.31})$$

We can decompose s_{AB} by the spherical harmonics $\mathcal{Y}_{AB}^{(\text{T})}$ as

$$s_{AB} = \sum s_\ell \mathcal{Y}_{AB}^{(\text{T})} \supset s_\ell \mathcal{Y}_{AB}^{(\text{T})}.$$

Here, the spherical harmonics satisfy

$$q^{AB} \mathcal{Y}_{AB}^{(\text{T})} = 0 \text{ and } \mathcal{Y}_{AB}^{(\text{T}):A} = 0. \quad (\text{D.32})$$

Additionally, $\mathcal{Y}_{AB}^{(\text{T})}$ are the eigenfunctions of the Laplacian $\hat{\Delta}$ in the spherical coordinates,

$$\hat{\Delta} \mathcal{Y}_{AB}^{(\text{T})} = \mathcal{Y}_{AB}^{(\text{T}):C}{}_{:C} \quad (\text{D.33})$$

$$= -\gamma_{\mathcal{S}}^{(n)} \mathcal{Y}_{AB}^{(\text{T})}. \quad (\text{D.34})$$

Here, a spectrum $\gamma_{\mathcal{S}}^{(n)}$ is given [22] by

$$\gamma_{\mathcal{S}}^{(n)} = \ell(\ell + n - 1) - 2. \quad (\text{D.35})$$

In the following, we will omit the index ℓ for simplicity. The nontrivial components of the Einstein equation become

$$0 = \delta R_{AB} \supset 0 = \frac{r^2}{2f} \ddot{s} - \frac{1}{2} r^2 f s'' - \frac{1}{2} (r f' + n f) r s' - \left((n-1) f + r f' - \frac{1}{2} (2nk + \gamma_{\mathcal{S}}^{(n)}) \right) s. \quad (\text{D.36})$$

We can introduce the master variable Ψ which is defined by

$$\Psi \equiv r^{\frac{n}{2}} s. \quad (\text{D.37})$$

We convert the radial coordinate r into the tortoise coordinate x which is defined by

$$\frac{dx}{dr} = \frac{1}{f}. \quad (\text{D.38})$$

Thus, we can derive the master equation for the tensorial components as

$$\omega^2 \Psi = - \frac{d^2 \Psi}{dx^2} + V_{\mathcal{T}}^{(n)} \Psi. \quad (\text{D.39})$$

Here, the effective potential $V_{\mathcal{T}}^{(n)}$ is

$$V_{\mathcal{T}}^{(n)} = \frac{f}{r^2} \left(\ell(\ell + n - 1) + 2(nk - 1) + \frac{1}{4}(n^2 - 10n + 8)f + \frac{1}{2}(n - 4)rf' \right). \quad (\text{D.40})$$

D.2.2 Einstein equation of vector perturbation in vacuum

In this section, we give the master equation of the vectorial components of the perturbed metric. Firstly, the perturbed metric $h_{\mu\nu}^{(\text{V})}$ is

$$h_{\mu\nu}^{(\text{V})} = \begin{pmatrix} 0 & 0 & v_A \\ & 0 & w_A \\ & & 0 \end{pmatrix}. \quad (\text{D.41})$$

v_A and w_A satisfy the transverse condition, that is,

$$v_A{}^{;A} = w_A{}^{;A} = 0. \quad (\text{D.42})$$

D.2.2.1 Covariant derivative of the perturbed metric

The components of the derivative of the perturbed metric on the spherical coordinates are given by

$$\bar{\nabla}_t h_{tA} = \dot{v}_A - \frac{1}{2} f' f w_A, \quad (\text{D.43})$$

$$\bar{\nabla}_t h_{rA} = \dot{w}_A - \frac{f'}{2f} v_A, \quad (\text{D.44})$$

$$\bar{\nabla}_r h_{tA} = v'_A - \left(\frac{1}{r} + \frac{f'}{2f} \right) v_A, \quad (\text{D.45})$$

$$\bar{\nabla}_r h_{rA} = w'_A - \left(\frac{1}{r} - \frac{f'}{2f} \right) w_A, \quad (\text{D.46})$$

$$\bar{\nabla}_A h_{tr} = -\frac{1}{r} v_A, \quad (D.47)$$

$$\bar{\nabla}_A h_{rr} = -\frac{2}{r} w_A, \quad (D.48)$$

$$\bar{\nabla}_A h_{tB} = v_{B:A}, \quad (D.49)$$

$$\bar{\nabla}_A h_{rB} = w_{B:A}, \quad (D.50)$$

$$\text{and } \bar{\nabla}_A h_{BC} = r f (q_{AB} w_C + q_{AC} w_B). \quad (D.51)$$

D.2.2.2 Perturbed Christoffel symbol

The components of the linearized Christoffel symbol are given by

$$\delta\Gamma_{JK}^I = 0, \quad (D.52)$$

$$\delta\Gamma_{tA}^t = 0, \quad (D.53)$$

$$\delta\Gamma_{rA}^t = -\frac{1}{2f} \left(v_A' - \dot{w}_A - \frac{2}{r} v_A \right), \quad (D.54)$$

$$\delta\Gamma_{tA}^r = \frac{1}{2} f (\dot{w}_A - v_A'), \quad (D.55)$$

$$\delta\Gamma_{rA}^r = -\frac{f}{r} w_A, \quad (D.56)$$

$$\delta\Gamma_{tt}^A = \frac{1}{r^2} \left(\dot{v}^A - \frac{1}{2} f' f w^A \right), \quad (D.57)$$

$$\delta\Gamma_{tr}^A = \frac{1}{2r^2} \left(\dot{w}^A + v'^A - \frac{f'}{f} v^A \right), \quad (D.58)$$

$$\delta\Gamma_{rr}^A = \frac{1}{r^2} \left(w'^A + \frac{f'}{2f} w^A \right), \quad (D.59)$$

$$\delta\Gamma_{AB}^t = -\frac{1}{2f} (v_{B:A} + v_{A:B}), \quad (D.60)$$

$$\delta\Gamma_{AB}^r = \frac{1}{2} f (w_{B:A} + w_{A:B}), \quad (D.61)$$

$$\delta\Gamma_{tB}^A = \frac{1}{2r^2} (v^A{}_{:B} - v_B{}^{;A}), \quad (D.62)$$

$$\delta\Gamma_{rB}^A = \frac{1}{2r^2} (w^A{}_{:B} - w_B{}^{;A}), \quad (D.63)$$

$$\text{and } \delta\Gamma_{BC}^A = \frac{f}{r} w^A q_{BC}. \quad (D.64)$$

D.2.2.3 Perturbed Ricci tensor

The components of the linearized Ricci tensor are given by

$$\delta R_{tt} = \frac{1}{r^2} \left(\dot{v}^A - \frac{1}{2} f' f w^A \right)_{:A} = 0, \quad (\text{D.65})$$

$$\delta R_{tr} = \frac{1}{2r^2} \left(\dot{w}^A + v'^A - \frac{f'}{f} v^A \right)_{:A} = 0, \quad (\text{D.66})$$

$$\delta R_{rr} = \frac{1}{r^2} \left(w'^A + \frac{f'}{2f} w^A \right)_{:A} = 0, \quad (\text{D.67})$$

$$\delta R_{tA} = \frac{1}{2} f \left(\dot{w}'_A - v''_A + \frac{2}{r} \dot{w}_A \right) + \frac{1}{2r^2} \left((n-1)k - \hat{\Delta} - 2rf' \right) v_A, \quad (\text{D.68})$$

$$\delta R_{rA} = -\frac{1}{2f} \partial_t \left(v'_A - \dot{w}_A - \frac{2}{r} v_A \right) + \frac{1}{2r^2} \left((n-1)k - 2((n-1)f + rf') - \hat{\Delta} \right) w_A, \quad (\text{D.69})$$

$$\text{and } \delta R_{AB} = -\frac{1}{f} \left(\dot{v}_{(A:B)} - \frac{f}{r^{n-2}} \partial_r (r^{n-2} f w_{(A:B)}) \right). \quad (\text{D.70})$$

The equation $\delta R_{\mu\nu} = 0$ is equivalent to the linearized Einstein equation in vacuum.

D.2.2.4 Master equation for vectorial component

The set of equation becomes

$$\dot{w}' - v'' + \frac{2}{r} \dot{w} + \frac{1}{r^2 f} \left((n-1)k + \gamma_{\mathcal{V}}^{(n)} - 2rf' \right) v = 0, \quad (\text{D.71})$$

$$\ddot{w} - \dot{v}' + \frac{2}{r} \dot{v} + \frac{f}{r^2} \left((n-1)(k-2f) - 2rf' + \gamma_{\mathcal{V}}^{(n)} \right) w = 0, \quad (\text{D.72})$$

$$\text{and } \dot{v} - \frac{f}{r^{n-2}} \partial_r (r^{n-2} f w) = 0. \quad (\text{D.73})$$

Here, we use the spherical vector harmonics given in Sec. 5.4. The master variable is defined by

$$\Psi = r^{(n-4)/2} f w. \quad (\text{D.74})$$

Then, the master equation is given by

$$0 = \ddot{w} - \dot{v}' + \frac{2}{r} \dot{v} + \frac{f}{r^2} \left((n-1)(k-2f) - 2rf' + \gamma_{\mathcal{V}}^{(n)} \right) w \quad (\text{D.75})$$

$$= \frac{1}{r^k f} \left(-\omega^2 \Psi - \frac{d^2 \Psi}{dx^2} + V_{\mathcal{V}}^{(n)} \Psi \right). \quad (\text{D.76})$$

Here, the effective potential for the vectorial perturbation $V_{\mathcal{V}}^{(n)}$ becomes

$$V_{\mathcal{V}}^{(n)} = \frac{f}{r^2} \left(\frac{1}{4} n(n+2) f - \frac{1}{2} (n+4) r f' + (n-1)(K-2f) + \gamma_{\mathcal{V}}^{(n)} \right). \quad (\text{D.77})$$

D.2.3 Einstein equation of scalar perturbation in vacuum

In this subsection, we give the master equation of the scalar components of the perturbed metric. Firstly, the scalar components of the perturbed metric, $h_{\mu\nu}^{(S)}$, are

$$h_{\mu\nu}^{(S)} = \begin{pmatrix} f\bar{H}Y & H_1Y & \mathbf{0} \\ \text{sym.} & f^{-1}HY & \mathbf{0} \\ \text{sym.} & \text{sym.} & r^2KYq_{AB} \end{pmatrix}. \quad (\text{D.78})$$

D.2.3.1 Covariant derivative of the perturbed metric

The components of the derivative of the perturbed metric on the spherical coordinates are given by

$$\bar{\nabla}_t h_{tt} = f \left(\dot{\bar{H}} - f' H_1 \right) Y, \quad (\text{D.79})$$

$$\bar{\nabla}_t h_{tr} = \left(\dot{H}_1 - \frac{f'}{2} (\bar{H} + H) \right) Y, \quad (\text{D.80})$$

$$\bar{\nabla}_t h_{rr} = \frac{1}{f} \left(\dot{H} - f' H_1 \right) Y, \quad (\text{D.81})$$

$$\bar{\nabla}_r h_{tt} = f \bar{H}' Y, \quad (\text{D.82})$$

$$\bar{\nabla}_r h_{tr} = H_1' Y, \quad (\text{D.83})$$

$$\bar{\nabla}_r h_{rr} = \frac{1}{f} H' Y, \quad (\text{D.84})$$

$$\bar{\nabla}_I h_{JA} = 0, \quad (\text{D.85})$$

$$\bar{\nabla}_t h_{AB} = r^2 \dot{K} Y q_{AB}, \quad (\text{D.86})$$

$$\bar{\nabla}_r h_{AB} = r^2 K' Y q_{AB}, \quad (\text{D.87})$$

$$\bar{\nabla}_A h_{tt} = f \bar{H} Y_{;A}, \quad (\text{D.88})$$

$$\bar{\nabla}_A h_{tr} = H_1 Y_{;A}, \quad (\text{D.89})$$

$$\bar{\nabla}_A h_{rr} = f^{-1} H Y_{;A}, \quad (\text{D.90})$$

$$\bar{\nabla}_A h_{tB} = r f H_1 q_{AB} Y, \quad (\text{D.91})$$

$$\bar{\nabla}_A h_{rB} = r (H - K) q_{AB} Y, \quad (\text{D.92})$$

$$\text{and } \bar{\nabla}_A h_{BC} = r^2 K Y_{;A} q_{BC}. \quad (\text{D.93})$$

D.2.3.2 Perturbed Christoffel symbol

The components of the linearized Christoffel symbol are given by

$$\delta\Gamma_{tt}^t = -\frac{1}{2} \left(\dot{\bar{H}} - f' H_1 \right) Y, \quad (\text{D.94})$$

$$\delta\Gamma_{tr}^t = -\frac{1}{2} \bar{H}' Y, \quad (\text{D.95})$$

$$\delta\Gamma_{rr}^t = -\frac{1}{2f^2} \left(-\dot{H} + 2fH_1' + f'H_1 \right) Y, \quad (\text{D.96})$$

$$\delta\Gamma_{tt}^r = -\frac{1}{2} f \left(-2\dot{H}_1 + (f\bar{H})' + f'H \right) Y, \quad (\text{D.97})$$

$$\delta\Gamma_{tr}^r = \frac{1}{2} \left(\dot{H} - f' H_1 \right) Y, \quad (\text{D.98})$$

$$\delta\Gamma_{rr}^r = \frac{1}{2} H' Y, \quad (\text{D.99})$$

$$\delta\Gamma_{tA}^t = -\frac{1}{2} \bar{H} Y_{:A}, \quad (\text{D.100})$$

$$\delta\Gamma_{rA}^t = -\frac{1}{2f} H_1 Y_{:A}, \quad (\text{D.101})$$

$$\delta\Gamma_{tA}^r = \frac{1}{2} f H_1 Y_{:A}, \quad (\text{D.102})$$

$$\delta\Gamma_{rA}^r = \frac{1}{2} H Y_{:A}, \quad (\text{D.103})$$

$$\delta\Gamma_{tt}^A = -\frac{f}{2r^2} \bar{H} Y^{:A}, \quad (\text{D.104})$$

$$\delta\Gamma_{tr}^A = -\frac{1}{2r^2} H_1 Y^{:A}, \quad (\text{D.105})$$

$$\delta\Gamma_{rr}^A = -\frac{1}{2r^2 f} H Y^{:A}, \quad (\text{D.106})$$

$$\delta\Gamma_{AB}^t = r \left(\frac{r}{2f} \dot{K} - H_1 \right) q_{AB} Y, \quad (\text{D.107})$$

$$\delta\Gamma_{AB}^r = \frac{1}{2} f \left(2rH - (r^2 K)' \right) q_{AB} Y, \quad (\text{D.108})$$

$$\delta\Gamma_{tB}^A = \frac{1}{2} \dot{K} \delta_B^A Y, \quad (\text{D.109})$$

$$\delta\Gamma_{rB}^A = \frac{1}{2} K' \delta_B^A Y, \quad (\text{D.110})$$

$$\text{and } \delta\Gamma_{BC}^A = \frac{1}{2} K \left(\delta_B^A Y_{:C} + Y_{:B} \delta_C^A - Y^{:A} q_{BC} \right). \quad (\text{D.111})$$

D.2.3.3 Perturbed Ricci tensor

The components of the linearized Ricci tensor are given by

$$\begin{aligned}
\delta R_{tt} = & \frac{f^2}{4r^2} \left(\frac{4r}{f} \left(r\dot{H}_1' + \left(n + \frac{rf'}{2f} \right) \dot{H}_1 \right) \right. \\
& - \left(2r^2 \bar{H}'' + \left(2n + 3\frac{rf'}{f} \right) r\bar{H}' + \frac{2}{f} \left(r^2 f'' + nr f' + \hat{\Delta} \right) \bar{H} \right) \\
& \left. - \left(\frac{2r^2}{f^2} \ddot{H} + \frac{rf'}{f} rH' + 2 \left(\frac{r^2 f''}{f} + n \frac{rf'}{f} \right) H \right) - 2n \left(\frac{r^2}{f^2} \ddot{K} - \frac{rf'}{2f} rK' \right) \right) Y,
\end{aligned} \tag{D.112}$$

$$\delta R_{tr} = -\frac{1}{2r^2} \left(r^2 f'' + nr f' + \hat{\Delta} \right) H_1 Y + \frac{n}{2r} \dot{H} Y - \frac{n}{2} \left(\partial_r + \frac{1}{r} \left(1 - \frac{rf'}{2f} \right) \right) \dot{K} Y, \tag{D.113}$$

$$\begin{aligned}
\delta R_{rr} = & \frac{1}{2} \left(\left(\bar{H}'' + \frac{3f'}{2f} \bar{H}' \right) + \left(\frac{1}{f^2} \ddot{H} + \left(\frac{n}{r} + \frac{f'}{2f} \right) H' + \frac{\gamma_{\mathcal{S}}^{(n)}}{r^2 f} H \right) \right. \\
& \left. - \frac{2}{f} \left(\dot{H}_1' + \frac{f'}{2f} \dot{H}_1 \right) - n \left(K'' + \left(\frac{2}{r} + \frac{f'}{2f} \right) K' \right) \right) Y,
\end{aligned} \tag{D.114}$$

$$\delta R_{tA} = \frac{1}{2} \left((fH_1)' + \frac{n-2}{r} fH_1 - \dot{H} - (n-1) \dot{K} \right) Y_{;A}, \tag{D.115}$$

$$\delta R_{rA} = \frac{1}{2} \left(-\frac{1}{f} \dot{H}_1 + \bar{H}' - \frac{1}{r} \left(1 - \frac{rf'}{2f} \right) \bar{H} + \frac{1}{r} \left(n-1 + \frac{rf'}{2f} \right) H - (n-1) K' \right) Y_{;A}, \tag{D.116}$$

$$\begin{aligned}
\text{and } \delta R_{AB} = & \frac{1}{2} \left(-2r\dot{H}_1 + rf\bar{H}' + rfH' + 2f \left(n-1 + \frac{rf'}{f} \right) H \right. \\
& + \frac{r^2}{f} \ddot{K} - r^2 f K'' - rf \left(2n + \frac{rf'}{f} \right) K' + \left(\gamma_{\mathcal{S}}^{(n)} - 2f \left(n-1 + \frac{rf'}{f} \right) \right) K \Big) q_{AB} Y \\
& + \frac{1}{2} (\bar{H} - H - (n-2) K) Y_{;AB}
\end{aligned} \tag{D.117}$$

The equation $\delta R_{\mu\nu} = 0$ is equivalent to the linearized Einstein equation in vacuum.

D.2.3.4 Master equation for tensorial component

From the Einstein equation, the set of the equations of motion becomes

$$\begin{aligned}
0 = & \frac{4r}{f} \left(r\dot{H}_1' + \left(n + \frac{rf'}{2f} \right) \dot{H}_1 \right) \\
& - \left(2r^2 \bar{H}'' + \left(2n + 3\frac{rf'}{f} \right) r\bar{H}' + \frac{2}{f} \left(r^2 f'' + nr f' + \hat{\Delta} \right) \bar{H} \right) \\
& - \left(\frac{2r^2}{f^2} \ddot{H} + \frac{rf'}{f} rH' + 2 \left(\frac{r^2 f''}{f} + n \frac{rf'}{f} \right) H \right) - 2n \left(\frac{r^2}{f^2} \ddot{K} - \frac{rf'}{2f} rK' \right),
\end{aligned} \tag{D.118}$$

$$0 = -\frac{1}{2r^2} \left(r^2 f'' + nr f' - \gamma_{\mathcal{S}}^{(n)} \right) H_1 + \frac{n}{2r} \dot{H} - \frac{n}{2} \left(\partial_r + \frac{1}{r} \left(1 - \frac{rf'}{2f} \right) \right) \dot{K}, \quad (\text{D.119})$$

$$0 = \bar{H}'' + \frac{3f'}{2f} \bar{H}' + \frac{1}{f^2} \ddot{H} + \left(\frac{n}{r} + \frac{f'}{2f} \right) H' + \frac{\gamma_{\mathcal{S}}^{(n)}}{r^2 f} H - \frac{2}{f} \left(\dot{H}_1' + \frac{f'}{2f} \dot{H}_1 \right) - n \left(K'' + \left(\frac{2}{r} + \frac{f'}{2f} \right) K' \right), \quad (\text{D.120})$$

$$0 = (fH_1)' + \frac{n-2}{r} fH_1 - \dot{H} - (n-1) \dot{K}, \quad (\text{D.121})$$

$$0 = -\dot{H}_1 + f\bar{H}' - \frac{f}{r} \left(1 - \frac{rf'}{2f} \right) \bar{H} + \frac{f}{r} \left(n-1 + \frac{rf'}{2f} \right) H - (n-1) fK', \quad (\text{D.122})$$

$$0 = -2\dot{H}_1 + f\bar{H}' + fH' + 2\frac{f}{r} \left(n-1 + \frac{rf'}{f} \right) H + \frac{r}{f} \ddot{K} - rfK'' - f \left(2n + \frac{rf'}{f} \right) K' + \frac{1}{r} \left(\gamma_{\mathcal{S}}^{(n)} - 2f \left(n-1 + \frac{rf'}{f} \right) \right) K, \quad (\text{D.123})$$

$$\text{and } 0 = \bar{H} - H - (n-2) K. \quad (\text{D.124})$$

The non-scaled master variable ψ is defined by

$$H_1 = \frac{r}{f} \left(\dot{\psi} + \dot{K} \right). \quad (\text{D.125})$$

Using Eq. (D.124), $\bar{H}$ is expressed by H and K as

$$\bar{H} = H + (n-2) K. \quad (\text{D.126})$$

From the second equation (D.119), H can be expressed by ψ and K

$$H = \frac{1}{nf} \left(r^2 f'' + nr f' - \gamma_{\mathcal{S}}^{(n)} \right) \psi + rK' + \left(1 - \frac{rf'}{2f} + \frac{1}{nf} \left(r^2 f'' + nr f' - \gamma_{\mathcal{S}}^{(n)} \right) \right) K \quad (\text{D.127})$$

$$= A_1 \psi + rK' + A_2 K. \quad (\text{D.128})$$

Here, we define A_1 and A_2 by

$$A_1 \equiv \frac{1}{nf} \left(r^2 f'' + nr f' - \gamma_{\mathcal{S}}^{(n)} \right) \quad (\text{D.129})$$

$$\text{and } A_2 \equiv 1 - \frac{rf'}{2f} + \frac{1}{nf} \left(r^2 f'' + nr f' - \gamma_{\mathcal{S}}^{(n)} \right). \quad (\text{D.130})$$

$\bar{H}$ can also be expressed by ψ and K as following,

$$\begin{aligned}\bar{H} &= H + (n-2)K \\ &= A_1\psi + rK' + A_2K + (n-2)K \\ &= A_1\psi + rK' + (A_2 + n-2)K.\end{aligned}\tag{D.131}$$

Then, K can be expressed by ψ only by the fourth equation (D.121),

$$\begin{aligned}A_2\dot{K} &= r\dot{\psi}' + (n-1-A_1)\dot{\psi} \\ \Rightarrow K &= \frac{r\psi' + (n-1-A_1)\psi}{A_2}.\end{aligned}\tag{D.132}$$

Thus, H , $\bar{H}$ and H_1 are expressed by ψ only.

The master equation is given by subtracting the fifth expression (D.122) from the sixth expression (D.123),

$$\begin{aligned}0 &= -2\dot{H}_1 + f\bar{H}' + fH' + 2\frac{f}{r}\left(n-1 + \frac{rf'}{f}\right)H \\ &\quad + \frac{r}{f}\ddot{K} - rfK'' - f\left(2n + \frac{rf'}{f}\right)K' + \frac{1}{r}\left(\gamma_{\mathcal{S}}^{(n)} - 2f\left(n-1 + \frac{rf'}{f}\right)\right)K \\ &\quad - \left(-\dot{H}_1 + f\bar{H}' - \frac{f}{r}\left(1 - \frac{rf'}{2f}\right)\bar{H} + \frac{f}{r}\left(n-1 + \frac{rf'}{2f}\right)H - (n-1)fK'\right)\end{aligned}\tag{D.133}$$

$$= -\frac{r}{f}\ddot{\psi} + rf\psi'' + A_3\psi' + A_4\psi.\tag{D.134}$$

Here, A_3 and A_4 are defined by

$$A_3 \equiv 2nf + rf' + \frac{1}{A_2}\left(\gamma_{\mathcal{S}}^{(n)} - nf - \frac{1}{2}(n+2)rf'\right) \quad \text{and} \tag{D.135}$$

$$A_4 \equiv \frac{1}{r}\left((n-1)f\left(n + \frac{rf'}{f}\right) + \left(\gamma_{\mathcal{S}}^{(n)} - nf - \frac{1}{2}(n+2)rf'\right)\frac{(n-1-A_1)}{A_2}\right), \tag{D.136}$$

and we used the relation between ψ and K given by

$$A_1\psi + A_2K = A_1\psi + A_2\frac{r\psi' + (n-1-A) \psi}{A_2} \tag{D.137}$$

$$= (n-1)\psi + r\psi'. \tag{D.138}$$

Then, the master variable Ψ which is defined by

$$\Psi \equiv N^{-1}(r)\psi \tag{D.139}$$

gives the master equation. N is a function of r . By this scaling, the master equation becomes

$$\begin{aligned} 0 &= -\frac{r}{f}N\ddot{\Psi} + rf(N\Psi)'' + A_3(N\Psi)' + A_4\Psi \\ &= -\frac{r}{f}N\ddot{\Psi} + \frac{rN}{f}\frac{d^2\Psi}{dx^2} + 2r\left(N' + \frac{A_3 - rf'}{2rf}N\right)\partial_x\Psi + (rfN'' + A_3N' + A_4N)\Psi. \end{aligned} \quad (\text{D.140})$$

To derive the master equation, we impose that N satisfies a differential equation as

$$N' + \frac{A_3 - rf'}{2rf}N = 0. \quad (\text{D.141})$$

We can find the solution as

$$N = (n(n+1)(1-f) + 2(\lambda - n))r^{-n/2}. \quad (\text{D.142})$$

Then, the master equation becomes

$$0 = \frac{r}{f}N\left(-\ddot{\Psi} + \frac{d^2\Psi}{dx^2} + \frac{f}{rN}(rfN'' + A_3N' + A_4N)\Psi\right) \quad (\text{D.143})$$

$$= -\frac{r}{f}N\left(-\omega^2\Psi - \frac{d^2\Psi}{dx^2} + V_{\mathcal{S}}\Psi\right). \quad (\text{D.144})$$

The effective potential $V_{\mathcal{S}}^{(n)}$ is given by

$$V_{\mathcal{S}}^{(n)} = -\frac{f}{rN}(rfN'' + A_3N' + A_4N) \quad (\text{D.145})$$

$$N = \frac{1}{r^{\frac{n}{2}}}\left(n(n+1)(1-f) + 2\gamma_{\mathcal{S}}^{(n)} - 2n\right) \quad (\text{D.146})$$

$$A_1 = \frac{1}{nf}\left(r^2f'' + nrf' - \gamma_{\mathcal{S}}^{(n)}\right) \quad (\text{D.147})$$

$$A_2 = 1 - \frac{rf'}{2f} + A_1 \quad (\text{D.148})$$

$$A_3 = 2nf + rf' + \frac{1}{A_2}\left(\gamma_{\mathcal{S}}^{(n)} - nf - \frac{1}{2}(n+2)rf'\right) \quad (\text{D.149})$$

$$A_4 = (n-1)\frac{f}{r}\left(n + \frac{rf'}{f}\right) + \left(\gamma_{\mathcal{S}}^{(n)} - nf - \frac{1}{2}(n+2)rf'\right)\frac{(n-1-A_1)}{rA_2} \quad (\text{D.150})$$

Astonishingly, this potential is equivalent to the following expression,

$$V_{\mathcal{S}} = 2\gamma_{\mathcal{S}}f\frac{(r^nN)'}{nNr^{n+1}} + r^{n-1}N\partial_x^2\left(\frac{1}{r^{n-1}N}\right). \quad (\text{D.151})$$

Appendix E

Code to calculate quasinormal modes by N th order WKB method

Here, we would like to give the code to calculate the quasinormal modes on the Schwarzschild black hole. The fundamental theory is established using the WKB method [14–17, 38–40], and its review is given by Ref. [18]. If the potential has a single peak and is positive definite, we can apply this code. This code is written in the language of Maple. This code is used by Refs. [35, 49]. This code can calculate Ω_k in arbitrary orders.

Algorithm E.1 Code to calculate quasinormal mode by WKB method (1/3)

#Reset calculation

restart :

#Orders for WKB-approximation

$B := 6:$

#Mass of black hole

$M := 2:$

#angular parameter of spherical harmonics

$l := 2:$

#tone number

$n := 0:$

Algorithm E.2 Code to calculate quasinormal mode by WKB method (2/3)

#quadratic quasinormal mode

$\Sigma := V - i \frac{Q_2}{k^{1/2}} (\text{add}(\Lambda_p, p = 1..B - 1) + \beta) :$

#Derivation of collection terms Ω_k

for q to $B - 1$ do

$S_{2q-1} := \text{add}(s_{(2q-1) \cdot 100 + 2p} t^{2(q-p)}, p = 0..q) :$

$S_{2q} := \text{add}(s_{2q \cdot 100 + 2p} t^{2(q-p)+1}, p = 0..q) :$

end do:

$g_0 := t + \text{add}(\alpha^p S_p, p = 1..2B - 2) :$

$g_1 := \text{diff}(g_0, t) :$

$g_2 := \text{diff}(g_1, t) :$

$g_3 := \text{diff}(g_2, t) :$

$z := \beta - \frac{1}{4} t^2 + \text{add}(\alpha^p Z_p, p = 1..2 * B - 2) :$

for q to $B - 1$ do

$Z_{2q-1} := -b_{2q-1} t^{2q+1} :$

$Z_{2q} := \Lambda_q - b_{2q} t^{2q+2} :$

end do;

$R_0 := g_1^4 * (\beta - \frac{1}{4} g_0^2) + \frac{1}{2} g_3 g_1 - \frac{3}{4} g_2^2 - z g_1^2 = 0 :$

$R_1 := \text{diff}(R_0, \alpha) :$

for q from 2 to $2 * B - 2$ do

$R[q] := \text{diff}(R_{q-1}, \alpha)$

end do:

$\alpha := 0 :$

for q to $B - 1$ do

$\text{solutions}_{2q-1} := \text{solve}(\text{identity}(R_{2q-1}, t), \{\text{seq}(s_{(2q-1) \cdot 100 + 2p}, p = 0..q)\}) :$

$\text{solutions}_{2q} := \text{solve}(\text{identity}(R_{2q}, t), \{\Lambda_q, \text{seq}(s_{2q \cdot 100 + 2p}, p = 0..q)\}) :$

$\text{assign}(\text{solutions}_{2q-1}) :$

$\text{assign}(\text{solutions}_{2q}) :$

end do:

for q to $2B - 2$ do

$b_q := \frac{1}{4} \cdot \frac{2}{(q+2)!} \cdot \frac{Q_{q+2}}{Q_2} \cdot (4k)^{-q/4} e^{i\pi q/4}$

end do:

Algorithm E.3 Code to calculate quasinormal mode by WKB method (3/3)

Effective potential $h_1 := l(l+1) :$ $h_2 := -3 : \# \text{scalar:1, vector:0, tensor:-3}$ $f := 1 - \frac{M}{r}$ $V := \text{simplify} \left(f \left(\frac{l(l+1)}{r^2} + \frac{h_2}{r} \text{diff}(f, r) \right) \right) :$ # **Derivatives of potential** $Q_0 := \omega^2 - V :$ $Q_1 := \text{simplify} (f \cdot \text{diff}(Q_0, r)) :$ for q from 2 to $2B$ do $Q_q := \text{simplify} (f \cdot \text{diff}(Q_{q-1}, r)) :$

end do:

 $k := \frac{1}{2} Q_2 :$ # **Insert the value of the coordinate at top of effective potential** $r := \text{simplify} \left(\frac{3}{2h_1} \left(h_1 - h_2 + \left(h_1^2 + \frac{14}{9} h_1 h_2 + h_2^2 \right)^{1/2} \right) \right) :$ # **Quasinormal mode σ** $\sigma := \text{evalf} \left((\text{simplify}(\Sigma))^{1/2}, 10 \right) ;$

Appendix F

Correction terms of WKB-method

In this section, we would like to give the expressions of the correction terms Ω_k . To perform the 6-th order WKB-method, we need Ω_3, Ω_4 and Ω_5 in addition to Ω_1 and Ω_2 . Here, we display them for completeness:

$$\begin{aligned}\Omega_3 = & (-463020\beta_1^6 + 465300\beta_1^4\beta_2 - 78120\beta_1^3\beta_3 - 99780\beta_1^2\beta_2^2 + 10860\beta_1^2\beta_4 \\ & + 19320\beta_1\beta_2\beta_3 - 1260\beta_1\beta_5 + 1500\beta_2^3 - 660\beta_2\beta_4 - 630\beta_3^2 + 70\beta_6) \beta^4 \\ & + (-418110\beta_1^6 + 479970\beta_1^4\beta_2 - 95460\beta_1^3\beta_3 - 124026\beta_1^2\beta_2^2 + 17070\beta_1^2\beta_4 \\ & + 29340\beta_1\beta_2\beta_3 + 3414\beta_2^3 - 2730\beta_1\beta_5 - 1770\beta_2\beta_4 - 1085\beta_3^2 + 245\beta_6) \beta^2 \\ & - \frac{101479}{4}\beta_1^6 + \frac{131817}{4}\beta_1^4\beta_2 - \frac{14777}{2}\beta_1^3\beta_3 - \frac{40261}{4}\beta_1^2\beta_2^2 + \frac{6055}{4}\beta_1^2\beta_4 \\ & + \frac{5667}{2}\beta_1\beta_2\beta_3 - \frac{1155}{4}\beta_1\beta_5 + \frac{1539}{4}\beta_2^3 - \frac{945}{4}\beta_2\beta_4 - \frac{1107}{8}\beta_3^2 + \frac{315}{8}\beta_6, \quad (\text{F.1})\end{aligned}$$

$$\begin{aligned}
\Omega_4 = & (-95872644\beta_1^8 + 130619664\beta_1^6\beta_2 - 22467312\beta_1^5\beta_3 - 51067800\beta_1^4\beta_2^2 \\
& + 3454920\beta_1^4\beta_4 + 13073760\beta_1^3\beta_2\beta_3 + 5418000\beta_1^2\beta_2^3 - 493920\beta_1^3\beta_5 \\
& - 1285200\beta_1^2\beta_2\beta_4 - 732480\beta_1^2\beta_3^2 - 1140720\beta_1\beta_2^2\beta_3 - 42756\beta_2^4 \\
& + 59472\beta_1^2\beta_6 + 98784\beta_1\beta_2\beta_5 + 110544\beta_1\beta_3\beta_4 + 25032\beta_2^2\beta_4 \\
& + 49392\beta_2\beta_3^2 - 5544\beta_1\beta_7 - 3024\beta_2\beta_6 - 5544\beta_3\beta_5 - 1572\beta_4^2 + 252\beta_8) \beta^5 \\
& + (-154601370\beta_1^8 + 231728040\beta_1^6\beta_2 - 45019560\beta_1^5\beta_3 - 101714460\beta_1^4\beta_2^2 \\
& + 8269260\beta_1^4\beta_4 + 29638800\beta_1^3\beta_2\beta_3 + 12782760\beta_1^2\beta_2^3 - 1456560\beta_1^3\beta_5 \\
& - 3618360\beta_1^2\beta_2\beta_4 - 1870400\beta_1^2\beta_3^2 - 3138600\beta_1\beta_2^2\beta_3 - 178330\beta_2^4 \\
& + 223720\beta_1^2\beta_6 + 361200\beta_1\beta_2\beta_5 + 354120\beta_1\beta_3\beta_4 + 118220\beta_2^2\beta_4) \beta^3 \\
& + 150360\beta_2\beta_3^2 - 28140\beta_1\beta_7 - 17640\beta_2\beta_6 - 21420\beta_3\beta_5 - 8290\beta_4^2 + 1890\beta_8) \beta^3 \\
& + \left(-\frac{129443349}{4}\beta_1^8 + 53574549\beta_1^6\beta_2 - 11535783\beta_1^5\beta_3 - \frac{53000175}{2}\beta_1^4\beta_2^2 \right. \\
& + \frac{4785249}{2}\beta_1^4\beta_4 + 8708550\beta_1^3\beta_2\beta_3 + 3909285\beta_1^2\beta_2^3 - 482622\beta_1^3\beta_5 \\
& - 1246797\beta_1^2\beta_2\beta_4 - 632340\beta_1^2\beta_3^2 - 1119415\beta_1\beta_2^2\beta_3 - \frac{305141}{4}\beta_2^4 \\
& + 88753\beta_1^2\beta_6 + 149478\beta_1\beta_2\beta_5 + 145417\beta_1\beta_3\beta_4 + \frac{117281}{2}\beta_2^2\beta_4 \\
& + 64731\beta_2\beta_3^2 - \frac{28077}{2}\beta_1\beta_7 - 10521\beta_2\beta_6 - \frac{22029}{2}\beta_3\beta_5 \\
& \left. - \frac{19277}{4}\beta_4^2 + \frac{5607}{4}\beta_8 \right) \beta, \tag{F.2}
\end{aligned}$$

$$\begin{aligned}
\text{and } \Omega_5 = & (-22598568720\beta_1^{10} + 38797354512\beta_1^8\beta_2 - 6749494080\beta_1^7\beta_3 - 22002812832\beta_1^6\beta_2^2 \\
& + 1080123744\beta_1^6\beta_4 + 6257684160\beta_1^5\beta_2\beta_3 + 4660027680\beta_1^4\beta_2^3 - 165561984\beta_1^5\beta_5 \\
& - 766795680\beta_1^4\beta_2\beta_4 - 413669760\beta_1^4\beta_3^2 - 1443234240\beta_1^3\beta_2^2\beta_3 - 291804240\beta_1^2\beta_2^4 \\
& + 23163840\beta_1^4\beta_6 + 85128960\beta_1^3\beta_2\beta_5 + 90350400\beta_1^3\beta_3\beta_4 + 108679200\beta_1^2\beta_2^2\beta_4 \\
& + 126329280\beta_1^2\beta_2\beta_3^2 + 64196160\beta_1\beta_2^3\beta_3 + 1400784\beta_2^5 - 2919840\beta_1^3\beta_7 \\
& - 7531776\beta_1^2\beta_2\beta_6 - 8618400\beta_1^2\beta_3\beta_5 - 3939600\beta_1^2\beta_4^2 - 6233472\beta_1\beta_2^2\beta_5 \\
& - 13981632\beta_1\beta_2\beta_3\beta_4 - 2849280\beta_1\beta_3^3 - 1023456\beta_2^3\beta_4 - 3116736\beta_2^2\beta_3^2 \\
& + 307440\beta_1^2\beta_8 + 487872\beta_1\beta_2\beta_7 + 583520\beta_1\beta_3\beta_6 + 544320\beta_1\beta_4\beta_5 \\
& + 129472\beta_2^2\beta_6 + 487872\beta_2\beta_3\beta_5 + 134736\beta_2\beta_4^2 + 272160\beta_3^2\beta_4 \\
& - 24024\beta_1\beta_9 - 13440\beta_2\beta_8 - 24024\beta_3\beta_7 - 14224\beta_4\beta_6 - 12012\beta_5^2 + 924\beta_{10}) \beta^6 \\
& + (-57626387280\beta_1^{10} + 106553134800\beta_1^8\beta_2 - 20386144800\beta_1^7\beta_3 - 65772661920\beta_1^6\beta_2^2 \\
& + 3750215280\beta_1^6\beta_4 + 20631693600\beta_1^5\beta_2\beta_3 + 15479738400\beta_1^4\beta_2^3 - 675647280\beta_1^5\beta_5 \\
& - 2947719600\beta_1^4\beta_2\beta_4 - 1496632200\beta_1^4\beta_3^2 - 5324508000\beta_1^3\beta_2^2\beta_3 - 1135963920\beta_1^2\beta_2^4 \\
& + 113720040\beta_1^4\beta_6 + 390240480\beta_1^3\beta_2\beta_5 + 381413280\beta_1^3\beta_3\beta_4 + 487029840\beta_1^2\beta_2^2\beta_4 \\
& + 513631440\beta_1^2\beta_2\beta_3^2 + 283029600\beta_1\beta_2^3\beta_3 + 9396240\beta_2^5 - 17603880\beta_1^3\beta_7 \\
& - 42912240\beta_1^2\beta_2\beta_6 - 43022280\beta_1^2\beta_3\beta_5 - 19968240\beta_1^2\beta_4^2 - 34807920\beta_1\beta_2^2\beta_5 \\
& - 69789120\beta_1\beta_2\beta_3\beta_4 - 12597200\beta_1\beta_3^3 - 7583760\beta_2^3\beta_4 - 14953960\beta_2^2\beta_3^2 \\
& + 2324700\beta_1^2\beta_8 + 3618720\beta_1\beta_2\beta_7 + 3614520\beta_1\beta_3\beta_6 + 3353280\beta_1\beta_4\beta_5 \\
& + 1142120\beta_2^2\beta_6 + 2859360\beta_2\beta_3\beta_5 + 1092720\beta_2\beta_4^2 + 1464400\beta_3^2\beta_4 - 237930\beta_1\beta_9 \\
& + -147000\beta_2\beta_8 - 182490\beta_3\beta_7 - 135380\beta_4\beta_6 - 82005\beta_5^2 + 12705\beta_{10}) \beta^4
\end{aligned}$$

$$\begin{aligned}
& + \left(-26541790065\beta_1^{10} + 53237904993\beta_1^8\beta_2 - 11123381220\beta_1^7\beta_3 - 36045764154\beta_1^6\beta_2^2 \right. \\
& + 2279955006\beta_1^6\beta_4 + 12440307420\beta_1^5\beta_2\beta_3 + 9481289682\beta_1^4\beta_2^3 - 461383776\beta_1^5\beta_5 \\
& - 2012614434\beta_1^4\beta_2\beta_4 - 999867660\beta_1^4\beta_3^2 - 3630132780\beta_1^3\beta_2^2\beta_3 - 809619141\beta_1^2\beta_2^4 \\
& + 89013120\beta_1^4\beta_6 + 304548384\beta_1^3\beta_2\beta_5 + 292426020\beta_1^3\beta_3\beta_4 + 388974714\beta_1^2\beta_2^2\beta_4 \\
& + 392853060\beta_1^2\beta_2\beta_3^2 + 230221620\beta_1\beta_2^3\beta_3 + 9317949\beta_2^5 - 16119726\beta_1^3\beta_7 \\
& - 39660264\beta_1^2\beta_2\beta_6 - 38201310\beta_1^2\beta_3\beta_5 - 18168321\beta_1^2\beta_4^2 - 33071472\beta_1\beta_2^2\beta_5 \\
& - 63944892\beta_1\beta_2\beta_3\beta_4 - 10841880\beta_1\beta_3^3 - 8518614\beta_2^3\beta_4 - 14034096\beta_2^2\beta_3^2 + 2582685\beta_1^2\beta_8 \\
& + 4096764\beta_1\beta_2\beta_7 + 3870930\beta_1\beta_3\beta_6 + 3671892\beta_1\beta_4\beta_5 + 1518048\beta_2^2\beta_6 + 3168732\beta_2\beta_3\beta_5 \\
& + 1390869\beta_2\beta_4^2 + 1530210\beta_3^2\beta_4 - \frac{671517}{2}\beta_1\beta_9 - 236460\beta_2\beta_8 - \frac{476973}{2}\beta_3\beta_7 \\
& \left. + -204771\beta_4\beta_6 - \frac{444381}{4}\beta_5^2 + \frac{101409}{4}\beta_{10} \right) \beta^2 \\
& - \frac{2375536317}{2}\beta_1^{10} + \frac{5112354429}{2}\beta_1^8\beta_2 - 570170440\beta_1^7\beta_3 - 1875235809\beta_1^6\beta_2^2 \\
& + 125451228\beta_1^6\beta_4 + 697320300\beta_1^5\beta_2\beta_3 + 542138237\beta_1^4\beta_2^3 - 27429003\beta_1^5\beta_5 \\
& - 122723430\beta_1^4\beta_2\beta_4 - \frac{121918445}{2}\beta_1^4\beta_3^2 - 226646440\beta_1^3\beta_2^2\beta_3 - \frac{104283313}{2}\beta_1^2\beta_2^4 \\
& + \frac{11623829}{2}\beta_1^4\beta_6 + 20366894\beta_1^3\beta_2\beta_5 + 19607424\beta_1^3\beta_3\beta_4 + 27070372\beta_1^2\beta_2^2\beta_4 \\
& + 27194427\beta_1^2\beta_2\beta_3^2 + 16533060\beta_1\beta_2^3\beta_3 + \frac{1456569}{2}\beta_2^5 - \frac{2336663}{2}\beta_1^3\beta_7 - 2995587\beta_1^2\beta_2\beta_6 \\
& - \frac{5703723}{2}\beta_1^2\beta_3\beta_5 - \frac{2729425\beta_1^2\beta_4^2}{2} - 2594391\beta_1\beta_2^2\beta_5 - 5045766\beta_1\beta_2\beta_3\beta_4 - 854685\beta_1\beta_3^3 \\
& - 735210\beta_2^3\beta_4 - \frac{2301381}{2}\beta_2^2\beta_3^2 + \frac{848925}{4}\beta_1^2\beta_8 + 358344\beta_1\beta_2\beta_7 + \frac{674037}{2}\beta_1\beta_3\beta_6 \\
& + 315150\beta_1\beta_4\beta_5 + \frac{292005}{2}\beta_2^2\beta_6 + 289908\beta_2\beta_3\beta_5 + \frac{269325}{2}\beta_2\beta_4^2 + 143370\beta_3^2\beta_4 \\
& - \frac{259875}{8}\beta_1\beta_9 - \frac{51975}{2}\beta_2\beta_8 - \frac{203931}{8}\beta_3\beta_7 - \frac{89775}{4}\beta_4\beta_6 - \frac{180675}{16}\beta_5^2 + \frac{51975}{16}\beta_{10}.
\end{aligned} \tag{F.3}$$

We can check them using codes in Alg. E.1 and E.2.

Appendix G

Foundation of differential form fields

Here, we would like to give some formulae for form fields.

G.1 p -form field

A p -form field $\mathbf{A}_p$ is defined by

$$\mathbf{A}_p \equiv \frac{1}{p!} A_{\mu_1 \dots \mu_p} dx^{\mu_1} \wedge dx^{\mu_2} \wedge \dots \wedge dx^{\mu_p}. \quad (\text{G.1})$$

Here, the wedge product $\wedge$ is defined by the tensor product $\otimes$ as

$$dx^1 \wedge dx^2 \equiv \frac{1}{2} (dx^1 \otimes dx^2 - dx^2 \otimes dx^1).$$

Then, the wedge product is antisymmetric under the permutation of the basis;

$$dx^1 \wedge dx^2 = -dx^2 \wedge dx^1.$$

A p -form field is equivalent to a totally antisymmetric tensor with a rank p .

G.2 Exterior derivative d

The exterior derivative d of $\mathbf{A}_p$ is defined by

$$d\mathbf{A}_p \equiv \frac{1}{p!} dA_{\mu_1 \dots \mu_p} \wedge dx^{\mu_1} \wedge dx^{\mu_2} \wedge \dots \wedge dx^{\mu_p}.$$

Here, we define $dA_{\mu_1 \dots \mu_p}$ as

$$dA_{\mu_1 \dots \mu_p} \equiv (\nabla_{\mu_{p+1}} A_{\mu_1 \dots \mu_p}) dx^{\mu_{p+1}}.$$

Hense, $d\mathbf{A}_p$ becomes

$$\begin{aligned} d\mathbf{A}_p &\equiv \frac{1}{p!} dA_{\mu_1 \dots \mu_p} \wedge dx^{\mu_1} \wedge \dots \wedge dx^{\mu_p} \\ &= \frac{1}{(p+1)!} (p+1) \nabla_{\mu_1} A_{\mu_2 \dots \mu_{p+1}} dx^{\mu_1} \wedge \dots \wedge dx^{\mu_{p+1}}. \end{aligned} \quad (\text{G.2})$$

The exterior derivative d maps a p -form field to a $(p+1)$ -form field. For general form fields α_p and β_q , the exterior derivative acts on $\alpha_p \wedge \beta_q$ as

$$d(\alpha_p \wedge \beta_q) = d\alpha_p \wedge \beta_q + (-)^p \alpha_p \wedge d\beta_q. \quad (\text{G.3})$$

Here, q means a rank of a form fields β_q .

G.3 Hodge operator

The Hodge operator $*$ maps a p -form field to a $(D-p)$ -form field. It is defined by

$$*(dx^{\mu_1} \wedge \dots \wedge dx^{\mu_p}) \equiv \frac{1}{(D-p)!} \epsilon^{\mu_1 \dots \mu_p}_{\mu_{p+1} \dots \mu_D} dx^{\mu_{p+1}} \wedge \dots \wedge dx^{\mu_D}.$$

$\epsilon_{\mu_1 \dots \mu_D}$ is the pseudo totally antisymmetric tensor which is defined by

$$\epsilon_{\mu_1 \dots \mu_D} = \sqrt{-g} \tilde{\epsilon}_{\mu_1 \dots \mu_D}. \quad (\text{G.4})$$

We define $\tilde{\epsilon}_{\mu_1 \dots \mu_D}$ as

$$\tilde{\epsilon}_{\mu_1 \mu_2 \dots \mu_D} = \begin{cases} +1 & \text{if } \mu_1 \mu_2 \dots \mu_D \text{ is an even permutation of } 01 \dots D-1 \\ -1 & \text{if } \mu_1 \mu_2 \dots \mu_D \text{ is an odd permutation of } 01 \dots D-1 \\ 0 & \text{otherwise} \end{cases}.$$

Especially, for the case of the basis of the 0-form field $*$ maps 1 to

$$*1 = \frac{1}{D!} \epsilon_{\mu_1 \dots \mu_D} dx^{\mu_1} \wedge \dots \wedge dx^{\mu_D}.$$

For the case of the D -form field it maps $dx^{\mu_1} \wedge \dots \wedge dx^{\mu_D}$ to

$$*(dx^{\mu_1} \wedge \dots \wedge dx^{\mu_D}) = \epsilon^{\mu_1 \dots \mu_D}.$$

Then, $*\mathbf{A}_p$ becomes

$$\begin{aligned} *\mathbf{A} &= \frac{1}{p!} A_{\mu_1 \dots \mu_p} *(dx^{\mu_1} \wedge \dots \wedge dx^{\mu_p}) \\ &= \frac{1}{(D-p)!} (*A)_{\mu_{p+1} \dots \mu_D} dx^{\mu_{p+1}} \wedge \dots \wedge dx^{\mu_D}. \end{aligned}$$

Here, we defined $(*A)_{\mu_{p+1}\dots\mu_D}$ as

$$(*A)_{\mu_{p+1}\dots\mu_D} \equiv \frac{1}{p!} \epsilon^{\mu_1\dots\mu_p}{}_{\mu_{p+1}\dots\mu_D} A_{\mu_1\dots\mu_p}.$$

If we use $*$ twice, a basis with a rank p behaves

$$** (dx^{\mu_1} \wedge \dots \wedge dx^{\mu_p}) = (-1)^{s+p(D-p)} dx^{\mu_1} \wedge \dots \wedge dx^{\mu_p},$$

that is, a p -form field behaves

$$** \mathbf{A} = (-1)^{s+p(D-p)} \mathbf{A}.$$

Here, s is the number of minus signature in the spacetime metric.

G.4 Coderivative operator δ

In the vector calculus on a flat space in three dimensions, there are two derivatives, the divergence and the rotation. Roughly speaking, the exterior derivative d is an extension of the rotation. Similaly, we can define the coderivative δ as an extension of the divergence. It is defined by

$$\delta \equiv (-1)^{s+D(p+1)+1} * d *.$$

If δ acts on $\mathbf{A}_p$, $\delta \mathbf{A}_p$ becomes

$$\delta \mathbf{A}_p = \frac{-1}{(p-1)!} \nabla^\mu A_{\mu\mu_1\dots\mu_{p-1}} dx^{\mu_1} \wedge \dots \wedge dx^{\mu_{p-1}}.$$

So, the operator δ maps a p -form field to a $(p-1)$ -form field.

G.5 Laplace-Beltrami operator Δ

Here, we assume the spacetime with a constant curvature k . The Laplace-Beltrami operator is defined by

$$\Delta \equiv \delta d + d\delta.$$

The operation of Δ to $\mathbf{A}_p$ becomes

$$\Delta \mathbf{A}_p = \frac{1}{p!} \left(- \left(\tilde{\Delta} - p(n-p)k \right) \right) A_{\mu_1\dots\mu_p} dx^{\mu_1} \wedge \dots \wedge dx^{\mu_p}.$$

Here, we defined $\tilde{\Delta} \equiv \nabla_\mu \nabla^\mu$ which is called the Laplace operator.

Appendix H

Gauge fixing of p -form field

We show the gauge fixing of a p -form field on a spherical black hole becomes in Eq. (5.26). In Sec. 5.3, we wrote the expression of a p -form field as

$$\begin{aligned}\mathbf{A}_p &= dt \wedge dr \wedge \hat{\mathbf{U}}_{p-2} + dt \wedge \hat{\mathbf{V}}_{p-1} + dr \wedge \hat{\mathbf{W}}_{p-1} + \hat{\mathbf{X}}_p \\ &= dt \wedge dr \wedge \left(\hat{d}\hat{\mathbf{U}}_{p-3} + \hat{\mathbf{U}}_{p-2} \right) + dt \wedge \left(\hat{d}\hat{\mathbf{V}}_{p-2} + \hat{\mathbf{V}}_{p-1} \right) \\ &\quad + dr \wedge \left(\hat{d}\hat{\mathbf{W}}_{p-2} + \hat{\mathbf{W}}_{p-1} \right) + \left(\hat{d}\hat{\mathbf{X}}_{p-1} + \hat{\mathbf{X}}_p \right).\end{aligned}$$

in Eq. (5.24). $\mathbf{A}_p$ has the gauge degrees of freedom of $d\mathbf{B}_{p-1}$. In addition, from Hodge's decomposition theorem $\mathbf{B}_{p-1}$ is also represented by the 0-forms or coexact forms.

Let us examine how the gauge degrees of freedom of $\mathbf{B}_p$ are eliminated. Firstly, $\mathbf{B}_1$ can be expressed by

$$\begin{aligned}\mathbf{B}_1 &= dt\hat{\mathbf{B}}_{t0} + dr\hat{\mathbf{B}}_{r0} + \hat{\mathbf{B}}_1 \\ &= dt\hat{\mathbf{B}}_{t,0} + dr\hat{\mathbf{B}}_{r,0} + \left(\hat{d}\hat{\mathbf{B}}_0 + \hat{\mathbf{B}}_1 \right).\end{aligned}$$

Here, $\hat{\mathbf{B}}_1$ is the coexact form, that is, $\hat{\mathbf{B}}_1 = \hat{\delta}\hat{\beta}_2$. We wrote an arbitrary 2-form field on the spherical coordinates as $\hat{\beta}_2$. t and r of the subscript of form fields indicate the components with respect to the corresponding basis, and the number represents a rank of a form field. Then, $\mathbf{B}_1$ has the gauge degree of freedom by $\mathbf{C}_0$ as following,

$$\begin{aligned}\mathbf{B}_1 &\rightarrow \underline{\mathbf{B}}_1 = \mathbf{B}_1 + d\mathbf{C}_0 \\ &= dt \left(\hat{\mathbf{B}}_{t,0} + \hat{\mathbf{C}}_0 \right) + dr \left(\hat{\mathbf{B}}_{r,0} + \hat{\mathbf{C}}'_0 \right) + \hat{\mathbf{B}}_1 + \hat{d} \left(\hat{\mathbf{B}}_0 + \hat{\mathbf{C}}_0 \right).\end{aligned}$$

Then, we can choose the gauge

$$\hat{\mathbf{B}}_0 + \hat{\mathbf{C}}_0 = 0.$$

Thus, $\mathbf{B}_1$ can be written as

$$\mathbf{B}_1 = dt \hat{\mathbf{B}}_{t,0} + dr \hat{\mathbf{B}}_{r,0} + \hat{\mathbf{B}}_1.$$

This is equivalent to the expression in Sec.(2.3.2).

Next, let us consider $\mathbf{B}_2$. Firstly, it can be decomposed as follows,

$$\begin{aligned} \mathbf{B}_2 &= dt \wedge dr \hat{\mathbf{B}}_{tr,0} + dt \wedge \hat{\mathbf{B}}_{t,1} + dr \wedge \hat{\mathbf{B}}_{r,1} + \hat{\mathbf{B}}_2 \\ &= dt \wedge dr \hat{\mathbf{B}}_{tr,0} + dt \wedge \left(\hat{d} \hat{\mathbf{B}}_{t,0} + \hat{\mathbf{B}}_{t,1} \right) + dr \wedge \left(\hat{d} \hat{\mathbf{B}}_{r,0} + \hat{\mathbf{B}}_{r,1} \right) + \left(\hat{d} \hat{\mathbf{B}}_1 + \hat{\mathbf{B}}_2 \right). \end{aligned}$$

Of course, $\mathbf{B}_2$ has gauge degrees of freedom due to the exterior derivative of $\mathbf{C}_1$. We have seen before that the differential form field $\mathbf{C}_1$ can be gauge-fixed as follows,

$$\mathbf{C}_1 = dt \hat{\mathbf{C}}_{t0} + dr \hat{\mathbf{C}}_{r0} + \hat{\mathbf{C}}_1.$$

The transformation of $\mathbf{B}_2$ by $\mathbf{C}_1$ gives

$$\begin{aligned} \mathbf{B}_2 &\rightarrow \underline{\mathbf{B}}_2 = \mathbf{B}_2 + d\mathbf{C}_1 \\ &= dt \wedge dr \left(\hat{\mathbf{B}}_{tr0} + \hat{\mathbf{C}}_{r0} - \hat{\mathbf{C}}'_{t0} \right) + dt \wedge \left(\hat{d} \left(\hat{\mathbf{B}}_{t0} - \hat{\mathbf{C}}_{t0} \right) + \hat{\mathbf{B}}_{t1} + \hat{\mathbf{C}}_1 \right) \\ &\quad + dr \wedge \left(\hat{d} \left(\hat{\mathbf{B}}_{r0} - \hat{\mathbf{C}}_{r0} \right) + \hat{\mathbf{B}}_{r1} + \hat{\mathbf{C}}_1 \right) + \hat{d} \left(\hat{\mathbf{B}}_1 + \hat{\mathbf{C}}_1 \right) + \hat{\mathbf{B}}_2. \end{aligned}$$

Therefore, we can choose the gauge following,

$$\hat{\mathbf{B}}_{t0} - \hat{\mathbf{C}}_{t0} = \hat{\mathbf{B}}_{r0} - \hat{\mathbf{C}}_{r0} = \hat{\mathbf{B}}_1 + \hat{\mathbf{C}}_1 = 0.$$

This gauge is also completely fixed. So, we can express $\mathbf{B}_2$ as

$$\mathbf{B}_2 = dt \wedge dr \hat{\mathbf{B}}_{tr0} + dt \wedge \hat{\mathbf{B}}_{t1} + dr \wedge \hat{\mathbf{B}}_{r1} + \hat{\mathbf{B}}_2.$$

Therefore, we can assume that the gauge degrees of freedom $\mathbf{C}_p$ can be expressed in the following form,

$$\mathbf{C}_p = dt \wedge dr \wedge \hat{\mathbf{C}}_{trp-2} + dt \wedge \hat{\mathbf{C}}_{tp-1} + dr \wedge \hat{\mathbf{C}}_{rp-1} + \hat{\mathbf{C}}_p.$$

Then, a form field $\mathbf{B}_{p+1}$ is transformed by $\mathbf{C}_p$, as follows,

$$\begin{aligned} \mathbf{B}_{p+1} &\rightarrow \underline{\mathbf{B}}_{p+1} = \mathbf{B}_{p+1} + d\mathbf{C}_p \\ &= dt \wedge dr \wedge \left(\hat{\mathbf{B}}_{trp-1} + \hat{\mathbf{C}}_{rp-1} - \hat{\mathbf{C}}'_{tp-1} + \hat{d} \left(\hat{\mathbf{C}}_{trp-2} + \hat{\mathbf{B}}_{trp-2} \right) \right) \\ &\quad + dt \wedge \left(\hat{d} \left(\hat{\mathbf{B}}_{tp-1} - \hat{\mathbf{C}}_{tp-1} \right) + \hat{\mathbf{B}}_{tp} + \hat{\mathbf{C}}_p \right) \\ &\quad + dr \wedge \left(\hat{d} \left(\hat{\mathbf{B}}_{rp-1} - \hat{\mathbf{C}}_{rp-1} \right) + \hat{\mathbf{B}}_{rp} + \hat{\mathbf{C}}_p \right) + \hat{d} \left(\hat{\mathbf{B}}_p + \hat{\mathbf{C}}_p \right) + \hat{\mathbf{B}}_{p+1}. \end{aligned}$$

Here, we choose the next gauge condition,

$$\hat{\mathcal{C}}_{trp-2} + \hat{\mathcal{B}}_{trp-2} = \hat{\mathcal{B}}_{tp-1} - \hat{\mathcal{C}}_{tp-1} = \hat{\mathcal{B}}_{rp-1} - \hat{\mathcal{C}}_{rp-1} = \hat{\mathcal{B}}_p + \hat{\mathcal{C}}_p = 0.$$

Then, we can express $\mathbf{B}_{p+1}$ as

$$\mathbf{B}_{p+1} = dt \wedge dr \wedge \hat{\mathcal{B}}_{trp-1} + dt \wedge \hat{\mathcal{B}}_{tp} + dr \wedge \hat{\mathcal{B}}_{rp} + \hat{\mathcal{B}}_{p+1}.$$

Therefore, by subtracting gauge degrees of freedom $\mathbf{B}_{p-1}$ from a form field $\mathbf{A}_p$, we could fix the gauge of $\mathbf{A}_p$. In this way, by mathematical induction, we showed that all of the components of $\mathbf{A}_p$ are expressed by the zero forms or coexact forms on the spherical coordinates.

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