



Some reflection principles in generic extensions by mixed support iteration

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博士論文
**Some reflection principles in generic
extensions by mixed support iteration**
(ミックスサポート反復強制によるジェネリック拡大
におけるいくつかの反映原理)

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Abstract

We investigate some reflection principles extending the stationary reflection and their relation with supercompact cardinals in terms of relative consistency.

We discuss about Strong Downwards Löwenheim-Skolem reflection principles (**SDLS**) for stationary logics. It is shown that a weaker version of these principles is equivalent to the Diagonal Reflection Principle (**DRP**) for internally clubness by Sean Cox and a stronger version follows from the Game Reflection Principle (**GRP**) by Bernhard König. It is also shown that such principles follow from the existence some generically supercompact cardinals for some classes of posets. The results presented in this part are needed to describe the motivation of the construction given in the next part.

In the main part of this dissertation, we modify a construction of the mixed support iteration by Krueger to show the consistency (modulo 2 supercompact cardinals) of a simultaneous reflection down to the continuum (namely the conjunction of an internal variation of the **SDLS** theorem for stationary logic with the reflection cardinal $< 2^{\aleph_0}$ and the **GRP** with reflection cardinal $\leq 2^{\aleph_0}$) with the continuum which carries a σ -saturated ideal and hence is extremely large. This construction also proves the consistency of such reflection property with a fragment of **MA** or with the Rado Conjecture with reflection cardinal $< 2^{\aleph_0}$.

Introduction

The term *large cardinal axiom* refers to axioms of the form “there exists a cardinal with a certain largeness property”, where such largeness property usually implies the existence of a model of ZFC (the Zermello-Fraenkel axiomatic system with the Axiom of Choice). A cardinal with such a largeness property is called a *large cardinal*. By Gödel’s second incompleteness theorem, if ZFC is consistent then it cannot prove the existence of a large cardinal. Large cardinal axioms provide a scale for relative consistency for many statements beyond ZFC.

Many large cardinals can be defined in terms of *elementary embeddings*: Given a language $\mathcal{L}$ and two structures $\mathcal{M} = \langle M, \dots \rangle$, $\mathcal{N} = \langle N, \dots \rangle$, an injective class function $j : M \longrightarrow N$ is called an *elementary embedding* if for every $\mathcal{L}$ -formula φ and for every $a_0, \dots, a_n \in M$ we have

$$\mathcal{M} \models \varphi(a_0, \dots, a_n) \iff \mathcal{N} \models \varphi(j(a_0), \dots, j(a_n)).$$

We shall usually consider $\mathcal{L} = \{\in\}$ and the structures we consider shall usually be models of ZFC (or of some fragment of it). Given a non-trivial elementary embedding $j : M \longrightarrow N$ (by non-trivial we mean j is not the identity function on M), we define the *critical point* of j (denote $\text{crit}(j)$) as the first ordinal $\alpha \in M$ such that $j(\alpha) \neq \alpha$.

An example of large cardinal that can be defined in terms of elementary embeddings are the *supercompact cardinals*.

Definition 0.1 (Supercompact cardinal). A non-trivial elementary embedding $j : V \longrightarrow M$ is called an *η -supercompact embedding* if ${}^\eta M \subseteq M$ and $j(\text{crit}(j)) > \eta$. An uncountable cardinal κ is a *supercompact cardinal* if for every $\eta > \kappa$ there is an η -supercompact embedding such that $\text{crit}(j) = \kappa$.

In the present thesis, we shall discuss about the consistency of some reflection principles generalizing the stationary reflection, relative to supercompact cardinals. In particular, we shall discuss about reflection principles obtained from *Löwenheim-Skolem theorems* for stationary logics.

SDLS for stationary logics

An example of a reflection principle which holds in ZFC is the famous (downward) Löwenheim-Skolem theorem:

Theorem 0.2 (Downward Löwenheim-Skolem Theorem). *Suppose $\mathfrak{A}$ is a structure of countable signature. Then given an infinite cardinal κ smaller than the cardinality of (the underlying set of) $\mathfrak{A}$, there is an elementary substructure $\mathfrak{B}$ of $\mathfrak{A}$ of cardinality κ (i.e., the underlying set of $\mathfrak{B}$ has the cardinality κ).*

Notice that by “elementary substructure” we mean it in the sense of first order logic. We can generalize the concept of elementary substructures for higher order logics and obtain versions of the statement in theorem 0.2 which are not provable in ZFC. In chapter 2 we consider SDLS reflection with respect to the *stationary logic*. The results in that chapter are presented in the joint papers [FOSa] and [FOSb] by Fuchino, Ottenbreit and Sakai.

By stationary logic we refer to the weak monadic second order logic denoted by $\mathcal{L}_{\text{stat}}^{\aleph_0}$ which is equipped with a weak second order quantifier “stat X ”, read as “for stationary many X ” (in the sense of the stationarity of countable subsets of the underlying set of a model). The interpretation of this quantifier is defined by: given a $\mathcal{L}_{\text{stat}}^{\aleph_0}$ -formula $\varphi(X)$ and a structure $\mathfrak{A} = \langle A, \dots \rangle$, let

$$\mathfrak{A} \models \text{stat } X \varphi(X) \Leftrightarrow \{B \in [A]^{\aleph_0} : \mathfrak{A} \models \varphi(B)\} \text{ is stationary in } [A]^{\aleph_0}.$$

We can generalize the definition of elementary substructure for these higher order logics and obtain different variations of SDLS. We give a general characterization for these reflection principles. Using such a characterization, we prove that a weaker version of this SDLS is equivalent to the Diagonal Reflection Principle (DRP) for internally clubness introduced by Sean Cox [Cox12]. We also show that a stronger version of these SDLS principles follows from the Game Reflection Principle (GRP) introduced by Bernhard König [Kön04].

Assuming the existence of a supercompact cardinal κ , each of the above mentioned reflection principles down to $< \kappa$ holds. We also consider generic variations of supercompact cardinals. By generic we mean that the elementary embedding witnessing the largeness of such cardinals is realized in a generic extension by some particular forcing notion. We show that some of these reflection principles holds at some generically supercompact cardinals. We also consider the *Laver-generically supercompact* cardinals, a strengthening of the generically supercompact cardinals introduced in [FOSb].

In chapter 3, we discuss about the relative consistency of the principles mentioned above under a very large continuum. We concentrate on the consistency of the conjunction of a weak version of **SDLS** with reflection cardinal $< 2^{\aleph_0}$ and a version of **GRP** with reflection cardinal $\leq 2^{\aleph_0}$. In order to do so, we generalize the construction by John Krueger [Kru08], [Kru09] of a *mixed support iteration*. In our construction, we add new reals by an arbitrary finite support iteration of small c.c.c. forcing posets, while simultaneously collapsing cardinals with an Easton support iteration of Levy collapses.

Assuming the existence of two supercompact cardinals $\kappa < \lambda$, we add κ many reals with the mixed support iteration and then collapse λ to κ^+ . We show that this construction preserves a good amount of the largeness of κ , namely we show that in the final extension $\kappa(= 2^{\aleph_0})$ carries a σ -saturated ideal.

Differently to the original construction by Krueger, where only Cohen reals were added, our construction allows us to use almost any arbitrary small c.c.c. forcing notions in the ground model as iterands. We use this to show that a fragment of **MA** holds in the obtained model. Such fragment is enough to maximize all the cardinals in Chihón's diagram. We also show that, in the version of the model where only Cohen reals are added, Rado's Conjecture with reflection cardinal $< 2^{\aleph_0}$ holds.

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Chapter 1

Preliminaries

1.1 Stationary sets

Given a set A and a cardinal μ , we shall use the notation $[A]^{<\mu}$ to denote the set $\{B \subseteq A : |B| < \mu\}$, as well as $[A]^\mu$ to denote $\{B \subseteq A : |B| = \mu\}$. We define the stationarity and clubness of subsets of $[A]^{<\kappa}$ as in Jech [Jec03]: Given an uncountable cardinal μ , set A with $|A| \geq \mu$ and some $X \subseteq [A]^{<\mu}$, we say that X is

- *unbounded* if for every $x \in [A]^{<\kappa}$ there is $y \in X$ such that $x \subseteq y$;
- *closed* if $\bigcup_{\xi < \delta} c_\xi \in X$ whenever $\langle c_\xi : \xi < \delta \rangle$ is a $\subseteq$ -increasing sequence of elements of X with $\delta < \kappa$;
- a *club* if X is closed and unbounded;
- *stationary* if $X \cap C \neq \emptyset$ has non empty intersection with every club subset of $[A]^{<\mu}$.

Notice that, for every regular cardinal μ , the set $[A]^\mu$ is a club subset of $[A]^{<\mu} := [A]^{<\mu^+}$. Therefore, we say that $\mathcal{C} \subseteq [A]^\mu$ is club subset of $[A]^\mu$ if it is club in the sense of $[A]^{<\mu^+}$. Similarly, we say that $S \subseteq [A]^\mu$ is a stationary subset of $[A]^\mu$ if it is stationary in the sense of $[A]^{<\mu^+}$.

We list here some basic facts about stationary sets. Proofs for lemmas 1.1, 1.2, 1.4 and proposition 1.3 can be found e.g in Jech [Jec03].

Lemma 1.1 (Fodor). *Suppose κ is a regular uncountable cardinal and S is a stationary subset of κ . Given any regressive¹ function f with $\text{dom}(f) = S$, there is a stationary $T \subseteq S$ and some $\zeta < \kappa$ such that $f''T = \{\zeta\}$.*

¹A function f with $\text{dom}(f) \subset \delta$ for some ordinal δ is *regressive* if $f(\alpha) \in \alpha$ for all $\alpha \in \text{dom}(f) \setminus \{0\}$.

Lemma 1.1 can be generalized as follows:

Lemma 1.2 (Jech). *Given cardinals $\kappa \leq \lambda$, with κ regular, suppose S is a stationary subset of $[\lambda]^{<\kappa}$ and f is a function with $\text{dom}(f) = S$. If $f(x) \in x$ for every nonempty $x \in S$, then there is $T \subset S$ a stationary subset of $[\lambda]^{<\kappa}$ and some $\zeta < \lambda$ such that $f''T = \{\zeta\}$.*

Proposition 1.3. *If $\mathcal{C}$ is a club subset of $[A]^{<\mu}$ for some set A and some regular cardinal $\mu < |A|$, then for any directed set $D \subseteq \mathcal{C}$ with $|D| < \mu$ we have $\bigcup D \in \mathcal{C}$.*

Lemma 1.4 (Menas). *Let κ be a regular uncountable cardinal and let A, B be sets with $A \subseteq B$ and $|A| \geq \kappa$.*

- (I) *If C is a club subset of $[A]^{<\kappa}$, then $\{x \in [B]^{<\kappa} : x \cap A \in C\}$ is a club subset of $[B]^{<\kappa}$.*
- (II) *If S is a stationary subset of $[A]^{<\kappa}$, then $\{x \in [B]^{<\kappa} : x \cap A \in S\}$ is a stationary subset of $[B]^{<\kappa}$.*
- (III) *If C is a club subset of $[B]^{<\kappa}$, then $\{x \cap A : x \in C\}$ contains a club subset of $[A]^{<\kappa}$.*
- (IV) *If S is a stationary subset of $[B]^{<\kappa}$, then $\{x \cap A : x \in S\}$ is a stationary subset of $[A]^{<\kappa}$.*

Since $[X]^{\aleph_0}$ is a club subset of $[X]^{<\aleph_1}$ for any uncountable set X , lemma 1.4 implies

Corollary 1.5. *Let A, B be uncountable sets with $A \subseteq B$.*

- (i) *If C is a club subset of $[A]^{\aleph_0}$, then $\{x \in [B]^{\aleph_0} : x \cap A \in C\}$ is a club subset of $[B]^{\aleph_0}$.*
- (ii) *If S is a stationary subset of $[A]^{\aleph_0}$, then $\{x \in [B]^{\aleph_0} : x \cap A \in S\}$ is a stationary subset of $[B]^{\aleph_0}$.*
- (iii) *If C is a club subset of $[B]^{<\aleph_0}$, then $\{x \cap A : x \in C \wedge |x \cap A| = \aleph_0\}$ contains a club subset of $[A]^{<\aleph_0}$.*
- (iv) *If S is a stationary subset of $[B]^{\aleph_0}$, then $\{x \cap A : x \in S \wedge |x \cap A| = \aleph_0\}$ is a stationary subset of $[A]^{\aleph_0}$.*

The following lemma is a well known fact.

Lemma 1.6. *Let M be an inner model of V . Suppose $A, \kappa \in M$ are such that $V \models “\kappa$ is a regular uncountable cardinal and $\kappa \leq |A|”$. For $S \in M$, if $V \models “S$ is a stationary subset of $[A]^{<\kappa}”$, then*

$$M \models “S \text{ is a stationary subset of } [A]^{<\kappa}”.$$

Definition 1.7. Let μ, κ be regular uncountable cardinals, $\mu < \kappa$, and let A be a set with $|A| \geq \kappa$. A set $\mathcal{C} \subseteq [A]^{<\kappa}$ is a μ -club if:

- $\mathcal{C}$ is unbounded in $[A]^{<\kappa}$ with respect to $\subseteq$;
- given any $\subseteq$ -increasing sequence $\langle c_\xi : \xi < \gamma \rangle$ of elements of $\mathcal{C}$ with $\mu < \text{cf}(\gamma)$ and $\gamma < \kappa$, we have $\bigcup_{\xi < \gamma} c_\xi \in \mathcal{C}$.

1.2 Forcing

We shall assume that the reader is familiar with the method of forcing. For reference, we suggest standard literature e.g. [Jec03] or [Kun11]. We now present some of the notations used in the present text.

By “poset” or “notion of forcing” we refer to a triple $\langle P, \leq_P, \mathbb{1}_P \rangle$ where P is a set, $\leq_P$ is a partial order over P and $\mathbb{1}_P$ is a maximal element of P with respect to $\leq_P$. However, we identify the triple $\langle P, \leq_P, \mathbb{1}_P \rangle$ with the underlying set P . We shall also omit the subscript P in $\leq_P$ and $\mathbb{1}_P$ when it is clear by the context which poset we are referring to.

Given a poset P and conditions $p, q \in P$, we shall denote “ p and q are incompatible” by $p \perp_P q$. On the other hand we denote “ p and q are compatible” by $p \Vdash_P q$. We shall also denote by $\tilde{G}_P$ the generic name for P , i.e. $\tilde{G}_P := \{ \langle \check{p}, p \rangle : p \in P \}$.

We say that two posets P and Q are *forcing equivalent* (we denote $P \cong Q$) if their boolean completion are isomorphic (see e.g. the discussion on chapter IV.4 in Kunen [Kun11]). Notice that the existence of a dense embedding $d : P \rightarrow Q$ (definition 1.11) is a sufficient condition for $P \cong Q$, although it is not a necessary condition.

Given a cardinal κ , we say that a poset P is κ -c.c. (alternatively, P satisfies the κ -chain condition) if every antichain of P has size $< \kappa$. For $\kappa = \aleph_1$, we shall denote $\aleph_1$ -c.c. by c.c.c. (alternatively, we say that κ satisfies the countable chain condition).

Given an ordinal δ , we say that a poset P is $\leq \delta$ -closed if any decreasing sequence of length $\leq \delta$ of conditions from P has a lower bound. We say that P is $< \delta$ -closed if it is $\leq \gamma$ -closed for all $\gamma < \delta$. We shall denote $< \omega_1$ -closed by σ -closed. If for any directed set $D \subseteq P$ with $|D| < \delta$ there is a

lower bound $d \in P$ for D , we say that P is $< \delta$ -directed closed. Since the conditions of a decreasing sequence form a directed set, we have that any $< \delta$ -directed closed poset is in particular $< \delta$ -closed.

The following weakening of $\leq \delta$ -closedness shall also be useful.

Definition 1.8. We say that a poset P is $\leq \mu$ -distributive if given any P -generic G over V and any function $f : \mu \rightarrow V$ in $V[G]$ we have $f \in V$. We say that P is $< \mu$ -distributive (also known as μ -Baire) if P is $\leq \alpha$ -distributive for all $\alpha < \mu$. We shall denote $< \aleph_1$ -distributive by σ -distributive.

We shall also use the following strengthening of $< \mu$ -closedness.

Given two sets I, J we denote by $\text{Fn}(I, J)$ the poset defined by: $p \in \text{Fn}(I, J)$ if and only if p is a function, $\text{dom}(p) \in [I]^{<\omega}$ and codomain of p is J . $\text{Fn}(I, J)$ is ordered by reverse inclusion (i.e., for $p, q \in \text{Fn}(I, J)$, $p \leq q$ if and only if $p \supseteq q$). For a set I , $\text{Fn}(I, 2)$ is the usual poset to add a Cohen subset of I , and given a regular cardinal κ $\text{Fn}(\kappa \times \omega) \cong \text{Fn}(\kappa, 2)$ is the usual poset to add κ many Cohen reals. It is well known that $\text{Fn}(I, 2)$ is a c.c.c. poset.

Definition 1.9. Given a regular cardinal μ and a set S of ordinals, we denote by $\text{Col}(\kappa, S)$ the Lévy collapse which adds surjection $f_\alpha : \mu \rightarrow \alpha$ for every $\alpha \in S$. Namely, let $p \in \text{Col}(\kappa, S)$ if and only if p is a function with $\text{dom}(p) \in [\mu \times S]^{<\mu}$ such that $p(\alpha, \beta) \in \alpha$ for all $\langle \alpha, \beta \rangle \in \text{dom}(p)$ with $\alpha > 0$. $\text{Col}(\kappa, S)$ is ordered by the reverse inclusion.

For the particular case where $S = \lambda$ for some regular cardinal $\lambda > \kappa$, we denote $\text{Col}(\kappa, S)$ by $\text{Col}(\kappa, < \lambda)$ (not to be confused with $\text{Col}(\kappa, \{\lambda\})$, which collapses just λ). It follows easily by the definition that $\text{Col}(\kappa, S)$ is a $< \kappa$ -closed poset. Furthermore, it follows from the Delta System lemma (check e.g. lemma III.6.15 in [Kun11]) that $\text{Col}(\kappa, < \lambda)$ is λ -c.c. whenever λ is a regular cardinal such that $\mu^{<\kappa} < \lambda$ for all $\mu < \lambda$. Notice that λ satisfies this property if $\lambda > \kappa$ is an inaccessible cardinal. Therefore, if κ is regular and $\lambda > \kappa$ is inaccessible, the poset $\text{Col}(\kappa, < \lambda)$ collapses λ to become κ^+ , while preserving every cardinal $\leq \kappa$ and $\geq \lambda$.

Proposition 1.10 (Fact 3.6 in Goldstern [Gol93]). *Let θ be a regular cardinal, $\mathbb{P} \in \mathcal{H}(\theta)$ a poset and $\tilde{x}$ a $\mathbb{P}$ -name such that $\Vdash_{\mathbb{P}} \tilde{x} \in \mathcal{H}(\theta)$. Then there is a $\mathbb{P}$ -name $\tilde{x}' \in \mathcal{H}(\theta)$ such that $\Vdash_{\mathbb{P}} \tilde{x} = \tilde{x}'$.*

Definition 1.11. Let P and Q be posets. A function $\iota : Q \rightarrow P$ is called a *complete embedding* if it satisfies:

- (a) $\iota(1_Q) = 1_P$;

- (b) $\forall q, q' \in Q, q' \leq_Q q \rightarrow \iota(q') \leq_P \iota(q)$;
- (c) $\forall q, q' \in Q, q' \perp_Q q \leftrightarrow \iota(q') \perp_P \iota(q)$;
- (d) given $\mathcal{A} \subseteq Q$, if $\mathcal{A}$ is a maximal antichain in Q , then $\iota''\mathcal{A}$ is a maximal antichain in P .

Given two posets P, Q , the notation $Q \leq P$ shall denote that there is a complete embedding from Q into P .

The following proposition is standard and well known. Check, for example lemma III.3.72 in [Kun11] or lemma 9 in [Cox15].

Proposition 1.12. *Suppose $\iota : Q \rightarrow P$ is a function satisfying (a), (b) and (c) from definition 1.11. Then ι is a complete embedding if and only if for every $p \in P$ there is some $p^* \in Q$ such that $\iota(q) \vee_P p$ whenever $q \leq_Q p^*$.*

Such a p^* like in proposition 1.12 is called a *reduct of p* (with respect to ι).

Definition 1.13. Let P and Q be posets. A function $\varphi : Q \rightarrow P$ is called a *dense embedding* if

- (a) $\varphi(\mathbb{1}_Q) = \mathbb{1}_P$;
- (b) $\forall q, q' \in Q, q' \leq_Q q \rightarrow \varphi(q') \leq_P \varphi(q)$;
- (c) $\forall q, q' \in Q, q' \perp_Q q \leftrightarrow \varphi(q') \perp_P \varphi(q)$;
- (d') $\varphi''P$ is a dense subset of Q .

Every dense embedding is a complete embedding. Indeed, if $\iota : Q \rightarrow P$ is a dense embedding, given $p \in P$ we can take $q \in Q$ such that $\iota(q) \leq p$ and such q is clearly a reduct of p . Dense embeddings are particularly useful to witness that two forcing posets are forcing equivalent.

Definition 1.14 (Translation of names). Let $\iota : Q \rightarrow P$ is a complete embedding.

- We define recursively the class function $\vec{\iota} : V^Q \rightarrow V^P$ by

$$\vec{\iota}(x) := \{ \langle \vec{\iota}(y), \iota(q) \rangle : \langle y, q \rangle \in x \}$$

for every $x \in V^Q$.

- Furthermore, if $\iota : Q \longrightarrow P$ is actually a dense embedding, we also define a class function $\check{\iota} : V^P \longrightarrow V^Q$ by

$$\check{\iota}(z) := \{\langle \check{\iota}(y), q \rangle : \langle y, p \rangle \in z \wedge \iota(q) \leq p\}$$

for every $z \in V^P$.

Proposition 1.15 is well known, and the reader may consult e.g. lemmas IV.4.4 and IV.4.7 in [Kun11].

Proposition 1.15. *Let $\iota : Q \longrightarrow P$ be a complete embedding.*

- *Given G a P -generic over V , define $H_G := \iota^{-1}(G)$. Then H_G is Q -generic over V and for any $\check{x} \in V^Q$ we have $\check{x}[H_G] = \check{\iota}(\check{x})[G]$. In particular, $V[H_G] \subseteq V[G]$.*
- *If ι is actually a dense embedding, we have $V[H_G] = V[G]$.*
- *Given H a Q -generic over V . Define $G_H := \{p \in P : \exists h \in H, \iota(q) \leq p\}$. If ι is a dense embedding, then G_H is P -generic over V and for any $\check{z} \in V^P$ we have $\check{z}[G_H] = \check{\iota}(\check{z})[H]$.*

We shall see in proposition 1.24 that whenever $Q \triangleleft P$, there is a Q -name $\check{R}$ such that $P \cong Q * \check{R}$. Another way of factoring a poset into a two step iteration is by using a projection.

Definition 1.16. Let P, Q be forcing posets. A function $\pi : P \longrightarrow Q$ is called a *projection* if:

- (1) $\pi(\mathbb{1}_P) = \mathbb{1}_Q$;
- (2) $\forall p, p' \in P, p' \leq_P p \rightarrow \pi(p') \leq_Q \pi(p)$;
- (3) Given any $p \in P, q \in Q$ such that $q \leq_Q \pi(p)$, there is $p' \leq_P p$ in P such that $\pi(p') \leq_Q q$.

Example 1.17. Suppose P is a poset and $\check{Q}$ is a P -name for a poset. Let $\iota : P \longrightarrow P * \check{Q}$ be the function $p \mapsto \langle p, \mathbb{1}_{\check{Q}} \rangle$ and $\pi : Q \longrightarrow P$ be the function $\langle p, q \rangle \mapsto p$. Then ι is a complete embedding and π is a projection.

Lemma 1.18. *Given posets P and Q and a projection $\pi : P \longrightarrow Q$, suppose there is a function $\iota : Q \longrightarrow P$ satisfying (a), (b) and (c) from definition 1.11 such that $p \leq_P \iota(\pi(p))$ for all $p \in P$. Then ι is a complete embedding.*

Proof. By proposition 1.12, it is enough to prove that for every $p \in P$, $\pi(p)$ is a reduct of p (with respect to the function ι). Let $p \in P$ and $q \in Q$ such that $q \leq_Q \pi(p)$. Since π is a projection, there is $p' \in P$ such that $p' \leq_P p$ and $\pi(p') \leq_Q q$. By the assumptions on ι , we have $p' \leq_P \iota(\pi(p')) \leq_P \iota(q)$, therefore p' witnesses $\iota(q) \Vdash_P p$. $\square$

Lemma 1.19. *Suppose A and B are complete boolean algebras. Then there is a complete embedding $\pi : A \setminus \{0\} \longrightarrow B \setminus \{0\}$ if and only if there is a projection $\pi : B \setminus \{0\} \longrightarrow A \setminus \{0\}$.*

Let $\vec{\mathcal{I}} = \langle \mathcal{I}_\alpha : \alpha \in \text{On} \rangle$ be a sequence of ideals such that for each $\alpha \in \text{On}$

- $[\alpha]^{<\omega} \subseteq \mathcal{I}_\alpha \subseteq \mathcal{P}(\alpha)$ and
- for every $\beta < \alpha$ we have $\mathcal{I}_\alpha \restriction \beta := \{x \cap \beta : x \in \mathcal{I}_\alpha\} = \mathcal{I}_\beta$.

We now present the definition of an $\vec{\mathcal{I}}$ -supported iteration of length δ which shall be used in the present dissertation. Our definition is similar to the one presented in [Gol93], except for (1.1).

Definition 1.20. We define recursively for all $\alpha \in \text{On}$ what is an $\vec{\mathcal{I}}$ -supported iteration of length α , and we define simultaneously the limit of such an iteration.

- A sequence $\langle P_\beta, Q_\beta : \beta < \alpha \rangle$ is an $\vec{\mathcal{I}}$ -supported iteration if and only if, for all $\beta < \alpha$, $P_\beta = \lim_{\vec{\mathcal{I}}} \langle P_\gamma, Q_\gamma : \gamma < \beta \rangle$ and Q_β is a P_β -name for a forcing poset.
- If $\langle P_\beta, Q_\beta : \beta < \alpha \rangle$ is an $\vec{\mathcal{I}}$ -supported iteration, then we define the poset $\lim_{\vec{\mathcal{I}}} \langle P_\beta, Q_\beta : \beta < \alpha \rangle$ as follows:
 - $p \in \lim_{\vec{\mathcal{I}}} \langle P_\beta, Q_\beta : \beta < \alpha \rangle$ if and only if p is a function with $\text{dom}(p) \in \mathcal{I}_\alpha$ (thus conditions of this poset are partial functions on α rather than total functions) such that, for all $\beta \in \text{dom}(p)$, $p(\beta)$ is a P_β -name of minimal rank² such that

$$\Vdash_{P_\beta} "p(\beta) \in Q_\beta". \quad (1.1)$$

²By minimal rank, we mean that $\text{rank}(p(\beta)) \leq \text{rank}(x)$ whenever x is a P_β -name such that $\Vdash_{P_\beta} "p(\beta) = x"$. This assumption ensures that the limit is indeed a set, rather than a proper class.

- For $p, q \in \lim_{\vec{\mathcal{I}}} \langle P_\beta, Q_\beta : \beta < \alpha \rangle$, $p \leq q$ if and only if for all $\beta \in \text{dom}(q)$ we have either
 - * $p \restriction \beta \Vdash_{P_\beta} "q(\beta) = \mathbb{1}_{Q_\beta}"$; or
 - * $\beta \in \text{dom}(p)$ and $p \restriction \beta \Vdash_{P_\beta} "p(\beta) \leq_{Q_\beta} q(\beta)"$.
- We define $\mathbb{1}_{\lim_{\vec{\mathcal{I}}} \langle P_\beta, Q_\beta : \beta < \alpha \rangle}$ as the empty function $\emptyset$.

Notice that in (1.1) we require that $\Vdash_{P_\beta} "p(\beta) \in Q_\beta"$ rather than $p \restriction \beta \Vdash_{P_\beta} "p(\beta) \in Q_\beta"$ as the version in [Gol93].

In the particular case when $\mathcal{I}_\alpha = [\alpha]^{<\omega}$ for all $\alpha \in \text{On}$, we shall refer to an $\vec{\mathcal{I}}$ -supported iteration as *finite support iteration*.

In chapter 3 we shall present an iteration which is not defined as definition 1.20. For that case, we shall borrow the following notation from [Cox15]:

Definition 1.21. A sequence of posets $\langle P_\alpha : \alpha < \delta \rangle$ will be called a *generalized iteration* if and only if:

1. For each $\alpha < \delta$, the conditions in P_α are partial functions on α ;
2. For $\beta \leq \alpha < \gamma$, the restriction map $P_\alpha \rightarrow P_\beta$ is a projection;
3. For $\beta \leq \alpha < \gamma$ we have $\mathbb{P}_\beta \leq \mathbb{P}_\alpha$.

It follows that iterations defined as in definition 1.20, as well as a products of posets, are generalized iterations.

Notation 1.22. Given P a forcing poset whose conditions are functions (e.g. a generalized iteration), given a set C , we denote

$$P \restriction C := \{p \in P : \text{dom}(p) \subseteq C\}.$$

Definition 1.23. Let $\langle P_\xi : \xi \leq \delta \rangle$ be a generalized iteration. For $\alpha^* < \gamma \leq \delta$, suppose $G_{\alpha^*} \subseteq P_{\alpha^*}$ is a filter. Then we define the quotient P_γ / G_{α^*} as the set $P_\gamma \restriction (\gamma \setminus \alpha^*)$ with the order $\leq_{G_{\alpha^*}}$ defined as follows: for any $p, q \in P_\gamma \restriction (\gamma \setminus \alpha^*)$, $p \leq_{G_{\alpha^*}} q$ if and only if there is some $r \in G_{\alpha^*}$ such that $r \cup p \leq_{P_\gamma} r \cup q$.

Proposition 1.24. Let P and Q be posets.

- (1) If there is a projection $\pi : P \rightarrow Q$, then $P \cong Q * \check{R}$ where $\check{R}$ is a Q -name such that

$$\Vdash_Q " \check{R} = \pi^{-1} " G_Q, \text{ with the order inherited from } \check{P}."$$

(2) If $P = P_\gamma$ and $Q = P_{\alpha^*}$ for some $\alpha^* < \gamma \leq \delta$ and some generalized iteration $\langle P_\xi : \xi < \delta \rangle$, then $P \cong Q * \underset{\sim}{R}$ where $\underset{\sim}{R}$ is a Q -name such that

$$\Vdash_Q \text{“} \underset{\sim}{R} = P_\gamma / G_{\mathbb{P}_{\alpha^*}} \text{ (as in definition 1.23)} \text{”}.$$

(3) If $Q \leq P$, then there is a Q -name $\underset{\sim}{R}$ such that $P \cong Q * \underset{\sim}{R}$.

Proof. For item (1), consider the function $\varphi : P \longrightarrow Q * \underset{\sim}{R}$ defined by $p \mapsto \langle \pi(p), \check{p} \rangle$. It can be showed that φ is a dense embedding. Similarly, for item (2) consider the function φ' defined by $p \mapsto \langle p \restriction \alpha^*, \check{p} \restriction (\gamma \setminus \alpha^*) \rangle$. Since $\langle P_\xi : \xi < \delta \rangle$ is a generalized iteration, $p \mapsto p \restriction \alpha^*$ is a projection, so it can be showed similarly that φ' is a dense embedding.

For the case (3), suppose that $Q \leq P$. Then, it follows that $B(Q) \leq B(P)$, where $B(Q)$ and $B(P)$ denote the boolean completions of Q and P respectively. By lemma 1.19, there is a projection $\pi : B(P) \setminus \{0\} \longrightarrow B(Q) \setminus \{0\}$. By item (1), there is $\underset{\sim}{R}'$ a $B(Q)$ -name such that $B(P) \setminus \{0\} \cong (B(Q) \setminus \{0\}) * \underset{\sim}{R}'$. It follows that $\underset{\sim}{R}'$ can be translated into a Q -name $\underset{\sim}{R}$ such that $P \cong Q * \underset{\sim}{R}$. (proposition 1.24) $\square$

Proposition 1.25. *Let η be some ordinal and suppose M, N are models of set theory with ${}^\eta M \subseteq M \subseteq N$. Suppose $P \in M$ is a forcing poset such that $P \subseteq M$ and $N \models \text{“} \mathbb{Q} \text{ is } \eta\text{-c.c.} \text{”}$. Then, for any P -generic G over N , we have ${}^\eta M[G] \cap N[G] \subseteq M[G]$.*

Proposition 1.26. *Let P be a forcing poset and $j : V \longrightarrow M$ an elementary embedding. Suppose G is a P -generic over V and $\hat{G}$ is a $j(P)$ -generic over M such that $j''G \subseteq \hat{G}$. Then we can extend j into an elementary embedding $J : V[G] \longrightarrow M[\hat{G}]$.*

Proof. Given $x \in V[G]$, take $\underset{\sim}{x}$ a P -name such that $\underset{\sim}{x}[G] = x$. Define $J(x) := j(\underset{\sim}{x})[\hat{G}]$. (proposition 1.26) $\square$

Lemma 1.27 (Easton). *Let κ be a regular cardinal. Let $\mathbb{P}$ be a κ -c.c. poset and $\mathbb{Q}$ be a $< \kappa$ -closed poset. Then:*

- (i) $\Vdash_{\mathbb{Q}} \text{“} \check{\mathbb{P}} \text{ is } \kappa\text{-c.c.} \text{”}$;
- (ii) $\Vdash_{\mathbb{P}} \text{“} \check{\mathbb{Q}} \text{ is } < \kappa\text{-distributive} \text{”}$.

Definition 1.28. A subset C of a poset P is *centered* if for given any $n \in \omega$ and $p_0, \dots, p_n \in C$ there is $q \in P$ such that $q \leq_P p_i$ for all $i \leq n$ (notice that the condition q need not to be in C). We say that a poset P is *σ -centered* if $P = \bigcup_{n \in \omega} C_n$ where for every $n \in \omega$ C_n is a centered subset of P .

The following fact can be found for example in [Kun11].

Proposition 1.29. *Given a set I , the poset $\text{Fn}(I, 2)$ is σ -centered if and only if $|I| \leq 2^{\aleph_0}$.*

Similarly to proposition 10.20 in Kanamori [Kan08], we have:

Theorem 1.30. *Suppose κ is regular and $\kappa < \eta$. If P is a separative poset such that $|P| = \eta$, P is $< \kappa$ -closed and*

$$\Vdash_P \text{ “ there exists a surjective function } f : \kappa \longrightarrow \eta \text{ ”},$$

then $P \cong \text{Col}(\kappa, \{\eta\})$.

Corollary 1.31. *For any $< \kappa$ -closed poset P with $|P| \leq \mu = \mu^{<\kappa} < \lambda$, we have $\text{Col}(\kappa, < \lambda) \cong P \times \text{Col}(\kappa, < \lambda) \cong P * \text{Col}(\kappa, < \lambda)^{V^P}$.*

1.3 Special trees

By *tree* we consider a partially ordered set $\langle \mathcal{T}, \leq_{\mathcal{T}} \rangle$ such that for every $t \in \mathcal{T}$, the set $\{r \in \mathcal{T} : r \leq_{\mathcal{T}} t\}$ is well ordered with respect to $\leq_{\mathcal{T}}$.

A tree $\mathcal{T}$ is *special* if it can be written as the union of countably many antichains³. In other words, $\mathcal{T}$ is special if there is a function $f : \mathcal{T} \longrightarrow \omega$ such that for any $t, t' \in \mathcal{T}$, if $f(t) = f(t')$ then t and t' are incomparable. Such a function is called a *specializing function* for $\mathcal{T}$.

We can formulate the *Rado Conjecture* (RC) in terms of reflection of non-special trees [Tod83] as the statement:

Given any tree $\mathcal{T}$, if $\mathcal{T}$ is non-special, then there is $\mathcal{T}' \in [\mathcal{T}]^{<\aleph_2}$ such that $\mathcal{T}'$ is non-special.

Given $\kappa > \aleph_1$ regular, let $\text{RC}(< \kappa)$ be the statement:

Given any tree $\mathcal{T}$, if $\mathcal{T}$ is non-special, then there is $\mathcal{T}' \in [\mathcal{T}]^{<\kappa}$ such that $\mathcal{T}'$ is non-special.

Lemma 1.32. *Suppose $\mathcal{T} \in V$ is a tree and P is one of the following:*

- (1) *a σ -distributive poset;*
- (2) *a σ -centered poset;*

³In the context of a tree, an antichain in $\mathcal{T}$ is a pairwise incomparable subset of $\mathcal{T}$.

(3) $\text{Fn}(\theta, 2)$ for some cardinal θ .

If for some P -generic G over V we have

$$V[G] \models \text{"}\mathcal{T} \text{ is special"},$$

then $\mathcal{T}$ is special in V .

Proof. Let $\mathcal{T}$ be a tree in V , P a poset, G a P -generic over V such that $\mathcal{T}$ is special in $V[G]$. First, assume (1). Suppose $f : \mathcal{T} \rightarrow \omega$ is a specializing function for $\mathcal{T}$, $f \in V[G]$.

For every node $t \in \mathcal{T}$, define $f_t := f \upharpoonright \{r \in \mathcal{T} : r \leq_{\mathcal{T}} t\}$. Since $\mathcal{T}$ is special in $V[G]$, every node of $\mathcal{T}$ has countable height. By (1), for each $t \in \mathcal{T}$ we have $f_t \in V$. Since $f := \bigcup_{t \in \mathcal{T}} f_t$ and $\mathcal{T} \in V$, we have that $f \in V$, so $\mathcal{T}$ is special in V .

Now assume (2). Let $\check{f}$ be a P -name and $p \in G$ be such that

$$p \Vdash_P \text{"}\check{f} : \check{\mathcal{T}} \rightarrow \omega \text{ is a specializing function"}.$$

In V , let $g : P \rightarrow \omega$ be a σ -centring function for P . For each $t \in \mathcal{T}$, choose $p_t \leq p$ such that p_t decides $\check{f}(\check{t})$. Then let $f' : \mathcal{T} \rightarrow \omega^2$ be the function defined in V by $f'(t) = \langle n, g(p_t) \rangle$ for each $t \in \mathcal{T}$, where $n \in \omega$ is such that $p_t \Vdash_P \text{"}\check{f}(\check{t}) = \check{n}\text{"}$. Clearly, f' is a specializing function for $\mathcal{T}$.

Finally, assume (3). If $\theta \leq (2^{\aleph_0})^V$, by proposition 1.29 P is σ -centred, so by (2) we are done. Then, suppose $\theta > (2^{\aleph_0})^V$. Suppose $f \in V[G]$ is a specializing function for $\mathcal{T}$. Let $Q := \text{Col}(2^{\aleph_0}, \{\theta\})^V$ and take H an Q -generic over $V[G]$. We have $f \in V[G][H]$.

By the product lemma, we have that H is Q -generic over V , G is P -generic over $V[H]$ and $V[G][H] = V[H][G]$. Then we have

$$V[H] \models P = \text{Fn}(2^{\aleph_0}, 2).$$

Therefore, by proposition 1.29, P is σ -centred in $V[H]$. So by (2), there is $f' \in V[H]$ a specializing function for $\mathcal{T}$. Since Q is σ -closed in V , by (1) we have that there is a specializing function for $\mathcal{T}$ in V . (lemma 1.32) $\square$

Chapter 2

Stationary logics and downward Löwenheim-Skolem theorems

The results in this chapter are included in [FOSa] and [FOSb].

2.1 Stationary logics

We shall consider some fragments of second order logic extending first order logic.

First we define the logic $\mathcal{L}^{\aleph_0}$ as follows. $\mathcal{L}^{\aleph_0}$ is equipped with first order variables denoted by lower case letters $x, y, z, \dots$ and second order variables denoted by upper case letters $X, Y, Z, \dots$. The second order variables are intended to represent countable subsets of the underlying set of the structure where the logic is interpreted. $\mathcal{L}^{\aleph_0}$ has also first order quantification $\forall x$ and $\exists x$ and a logical predicate symbol ' ε ', along with the usual '=' for the equality of first order objects. Expressions of the form $x \varepsilon X$ for any first order variable x and second order variable X are atomic formulas for the logic $\mathcal{L}^{\aleph_0}$ whose canonical interpretation in a structure $\mathfrak{A} = \langle A, \dots \rangle$ is : for any $b \in A$ and $B \in [A]^{\aleph_0}$, $\mathfrak{A} \models b \varepsilon B$ if and only if $b \in B$.

We then expand $\mathcal{L}^{\aleph_0}$ by adding the second order quantifiers $\forall X$ and $\exists X$; we denote the resulting logic by $\mathcal{L}^{\aleph_0, \text{II}}$. As it can be expected, the quantifiers $\forall X$ and $\exists X$ are interpreted as follows: Given an $\mathcal{L}^{\aleph_0, \text{II}}$ formula $\varphi(X)$ and a structure $\mathfrak{A} = \langle A, \dots \rangle$ we define

$$\mathfrak{A} \models \forall X \varphi(X) \Leftrightarrow \text{for every } B \in [A]^{\aleph_0} \mathfrak{A} \models \varphi(B);$$

and similarly

$$\mathfrak{A} \models \exists X \varphi(X) \Leftrightarrow \text{there is } B \in [A]^{\aleph_0} \text{ such that } \mathfrak{A} \models \varphi(B).$$

Now we define the logic $\mathcal{L}_{\text{stat}}^{\aleph_0}$ by adding to $\mathcal{L}^{\aleph_0}$ a second order quantifier denoted by $\text{stat } X$. This quantifier was introduced by Shelah in [She75]. The reader may consult [BF85] and [V04] for further reference.

The quantifier $\text{stat } X$ is read as “for stationary many X ” and its intended interpretation is as follows: given an $\mathcal{L}_{\text{stat}}^{\aleph_0}$ -formula $\varphi(X)$ and a structure $\mathfrak{A} = \langle A, \dots \rangle$, define:

$$\mathfrak{A} \models \text{stat } X \varphi(X) \Leftrightarrow \{B \in [A]^{\aleph_0} : \mathfrak{A} \models \varphi(B)\} \text{ is stationary in } [A]^{\aleph_0}. \quad (2.1)$$

In $\mathcal{L}_{\text{stat}}^{\aleph_0}$ we can also define “ aaX ” the dual quantifier for “ stat ”. aaX shall be read as “for almost all X ” and it is defined as an abbreviation for $\neg \text{stat } X \neg \varphi$. It then follows from (2.1) that for every $\mathcal{L}_{\text{stat}}^{\aleph_0}$ -formula φ and any structure $\mathfrak{A} = \langle A, \dots \rangle$ we have

$$\mathfrak{A} \models aaX \varphi(X) \Leftrightarrow \{B \in [A]^{\aleph_0} : \mathfrak{A} \models \varphi(B)\} \text{ contains a club subset of } [A]^{\aleph_0}. \quad (2.2)$$

We also define $\mathcal{L}_{\text{stat}}^{\aleph_0, \Pi}$ as the logic obtained by expanding $\mathcal{L}^{\aleph_0}$ simultaneously with the second order quantifiers $\exists X, \forall X$ and with $\text{stat } X$.

With these stationary logics, we can define an analogous version of the Downward Löwenheim-Skolem theorem. In order to define these principles, we shall first define the concept of elementary substructure related to these logics.

Let $\mathcal{L}$ be one of the logics above, or a family of formulas in such a logic. Let $\mathfrak{A} = \langle A, \dots \rangle$ and $\mathfrak{B} = \langle B, \dots \rangle$ be structures on the same signature, and suppose $\mathfrak{B} \subseteq \mathfrak{A}$. We say that $\mathfrak{B}$ is an $\mathcal{L}$ -*elementary substructure* of $\mathfrak{A}$ (denote $\mathfrak{B} \prec_{\mathcal{L}} \mathfrak{A}$) if for any formula $\varphi(x_0, \dots, x_n, X_0, \dots, X_m) \in \mathcal{L}$ in the same signature as $\mathfrak{A}$ and $\mathfrak{B}$, for any $c_0, \dots, c_n \in B$ and any higher order objects $C_0, \dots, C_m$ in $\mathfrak{B}$ corresponding to the variables¹ $X_0, \dots, X_m$ we have

$$\mathfrak{B} \models \varphi(c_0, \dots, c_n, C_0, \dots, C_m) \Leftrightarrow \mathfrak{A} \models \varphi(c_0, \dots, c_n, C_0, \dots, C_m).$$

Similarly, we say that $\mathfrak{B} \prec_{\mathcal{L}}^- \mathfrak{A}$ if, for any $c_0, \dots, c_n \in B$ and any formula $\varphi(x_0, \dots, x_n)$ in $\mathcal{L}$ such that *all of its free variables are of first order*, we have

$$\mathfrak{B} \models \varphi(c_0, \dots, c_n) \Leftrightarrow \mathfrak{A} \models \varphi(c_0, \dots, c_n).$$

Using the generalizations of the elementary substructure relation described above, we can define some variations of the downwards Löwenheim-Skolem theorem (theorem 0.2). Given a weak second order $\mathcal{L}$ as above and

¹In the case of $\mathcal{L}^{\aleph_0}$ and its expansions defined above, this means that $C_0, \dots, C_m$ shall be countable subsets of B .

a cardinal κ , we define the *Strong Downwards Löwenheim-Skolem Theorem* for $\mathcal{L}$ down to $< \kappa$ (denoted by $\text{SDLS}(\mathcal{L}, < \kappa)$ as the follow statement:

$\text{SDLS}(\mathcal{L}, < \kappa)$: For any structure $\mathfrak{A} = \langle A, \dots \rangle$ of countable signature, there is $B \in [A]^{< \kappa}$ such that $\mathfrak{A} \upharpoonright B \prec_{\mathcal{L}} \mathfrak{A}$.

Similarly, we also define a version of SDLS with respect to the weaker generalization of the elementary substructure relation. We denote this weaker version by $\text{SDLS}^-(\mathcal{L}, < \mu)$, which is defined as the statement:

$\text{SDLS}^-(\mathcal{L}, < \mu)$: For any structure $\mathfrak{A} = \langle A, \dots \rangle$ of countable signature, there is $B \in [A]^{< \mu}$ such that $\mathfrak{A} \upharpoonright B \prec_{\mathcal{L}}^- \mathfrak{A}$.

Example 2.1. Notice that $\text{SDLS}^-(\mathcal{L}^{\aleph_0}, < \aleph_1)$ is just the regular Downwards Löwenheim-Skolem Theorem (theorem 0.2) for first order logic, and therefore it holds in ZFC.

Example 2.2. On the other hand, $\text{SDLS}^-(\mathcal{L}_{\text{stat}}^{\aleph_0}, < \aleph_1)$ does not hold in ZFC. Indeed, let for example ϕ be the $\mathcal{L}_{\text{stat}}^{\aleph_0}$ -formula $\text{stat } X \exists x (\neg x \varepsilon X)$. Notice that for every set A , A is uncountable if and only if $A \models \phi$. Therefore, given any structure $\mathfrak{A}$ such that $\mathfrak{A} \models \phi$, there is no $B \in [A]^{< \aleph_1}$ such that $\mathfrak{A} \upharpoonright B \models \phi$. Similarly, $\text{SDLS}(\mathcal{L}^{\aleph_0}, < \aleph_1)$ and $\text{SDLS}^-(\mathcal{L}^{\aleph_0, \text{II}}, < \aleph_1)$ also fail in ZFC.

Both $\text{SDLS}(\mathcal{L}, < \mu)$ and $\text{SDLS}^-(\mathcal{L}, < \mu)$ have natural strengthenings considering stationary many reflection points:

$\text{SDLS}_+(\mathcal{L}, < \mu)$: For any structure $\mathfrak{A} = \langle A, \dots \rangle$ of countable signature, there are *stationarily many* $B \in [A]^{< \mu}$ such that $\mathfrak{A} \upharpoonright B \prec_{\mathcal{L}} \mathfrak{A}$.

$\text{SDLS}_+^-(\mathcal{L}, < \mu)$: For any structure $\mathfrak{A} = \langle A, \dots \rangle$ of countable signature, there are *stationarily many* $B \in [A]^{< \mu}$ such that $\mathfrak{A} \upharpoonright B \prec_{\mathcal{L}}^- \mathfrak{A}$.

In [Mag16], Manahem Magidor noticed that $\text{SDLS}_+^-(\mathcal{L}_{\text{stat}}^{\aleph_0}, < \aleph_2)$ implies the following reflection statement: “Every uncountable graph with uncountable coloring number has a subgraph of cardinality $\aleph_1$ with uncountable coloring number”. This reflection statement is equivalent to the Fodor-Type Reflection Principle **FRP** (see [FJS⁺10], [FSSU12]).

FRP : For every regular cardinal $\eta > \aleph_1$, stationary $S \subseteq E_{\omega}^{\eta} := \{\xi < \eta : \text{cf}(\xi) = \omega\}$, and every function $g : S \rightarrow [\eta]^{\aleph_0}$ such that $g(\alpha)$ is a cofinal subset of α with order type ω for all $\alpha \in S$, there is $\alpha^* \in E_{\omega_1}^{\eta}$ such that

$$\{x \in [\alpha^*]^{\aleph_0} : \sup(x) \in S \wedge g(\sup(x)) \subseteq x\}$$

is a stationary subset of $[\alpha^*]^{\aleph_0}$.

Therefore $\text{SDLS}^-(\mathcal{L}_{\text{stat}}^{\aleph_0}, < \aleph_2)$ implies FRP.

FRP follows from the Following the reflection principle RP defined as follows (we follow the notation in [FJS⁺10] and [FO]):

RP : For every regular cardinal $\eta > \aleph_1$, stationary set $S \subseteq [\eta]^{\aleph_0}$ and $X \in [\eta]^{\aleph_1}$, there is $Y \in [\eta]^{\aleph_1}$ such that

- (a) $\text{cf}(Y) = \omega_1$, $X \subseteq Y$; and
- (b) $S \cap [Y]^{\aleph_0}$ is a stationary subset of $[Y]^{\aleph_0}$.

Notice that the RP defined above is a strengthening of the Reflection Principle RP in Jech [Jec03] (Jech's RP is just as above but without demanding property (a)).

From the characterization of $\text{SDLS}^-(\mathcal{L}_{\text{stat}}^{\aleph_0}, < \kappa)$ in section 2.2 (lemma 2.13), it follows easily that

Theorem 2.3. $\text{SDLS}^-(\mathcal{L}_{\text{stat}}^{\aleph_0}, < \aleph_2)$ implies RP.

By a theorem of Todorćević, Jech's version of RP implies $2^{\aleph_0} \leq \aleph_2$ (see theorem 37.18 in [Jec03]). Therefore

Corollary 2.4. $\text{SDLS}^-(\mathcal{L}_{\text{stat}}^{\aleph_0}, < \aleph_2)$ implies $2^{\aleph_0} \leq \aleph_2$.

Proposition 2.5. Let κ be a cardinal with $\text{cf}(\kappa) \geq \omega_2$. Then the following are equivalent:

- (1) $\forall \mu < \kappa, \mu^{\aleph_0} < \kappa$;
- (2) $\text{SDLS}_+(\mathcal{L}^{\aleph_0}, < \kappa)$;
- (3) $\text{SDLS}_+^-(\mathcal{L}^{\aleph_0, II}, < \kappa)$;
- (4) $\text{SDLS}_+(\mathcal{L}^{\aleph_0, II}, < \kappa)$;

Proof. Notice that (4) $\Rightarrow$ (2) and (4) $\Rightarrow$ (3) are trivial.

(2) $\Rightarrow$ (1): Let $\mu < \kappa$ and suppose $\text{SDLS}_+(\mathcal{L}^{\aleph_0}, < \kappa)$ holds. Consider the structure $\mathfrak{A} := \langle \kappa \cup \mathcal{P}(\mu), \bar{\mu}^{\mathfrak{A}}, \bar{E}^{\mathfrak{A}} \rangle$ where $\bar{\mu}$ and $\bar{E}$ are respectively a unary and a binary relation symbols such that $\bar{\mu}^{\mathfrak{A}} := \mu$ and $\bar{E}^{\mathfrak{A}} := \{ \langle \alpha, a \rangle : \alpha \in a \in \mathcal{P}(\mu) \}$.

Since $\{D \in [\kappa \cup \mathcal{P}(\mu)]^{<\kappa} : \mu \subseteq D\}$ is clearly a club subset of $[\kappa \cup \mathcal{P}(\mu)]^{<\kappa}$, by $\text{SDLS}_+(\mathcal{L}^{\aleph_0}, < \kappa)$ we can take $B \in [\kappa \cup \mathcal{P}(\mu)]^{<\kappa}$ such that $\mu \subseteq B$ and $\mathfrak{A} \upharpoonright B \prec_{\mathcal{L}^{\aleph_0}} \mathfrak{A}$. Let $\varphi(X)$ be the $\mathcal{L}^{\aleph_0}$ -formula:

$$\exists x \forall y (\bar{\mu}(y) \rightarrow (y \in X \leftrightarrow \bar{E}(y, x))).$$

Since $\mu \subseteq B$, we have $\bar{\mu}^{\mathfrak{A} \upharpoonright B} = \mu$. Notice $\varphi(X)$ states that X can be translated into a first order object. Clearly, for every $U \in [\mu]^{\aleph_0}$ we have $\mathfrak{A} \models \varphi(U)$. By $\mathfrak{A} \upharpoonright B \prec_{\mathcal{L}^{\aleph_0}} \mathfrak{A}$, for every $U \in [\mu]^{\aleph_0}$ we have $\mathfrak{A} \upharpoonright B \models \varphi(U)$, so $U \in B$. Therefore $[\mu]^{\aleph_0} \subseteq B$, so $\mu^{\aleph_0} \leq |B| < \kappa$.

(3) $\Rightarrow$ (1): Let $\mu < \kappa$ and suppose $\text{SDLS}_+^-(\mathcal{L}^{\aleph_0, \Pi}, < \kappa)$. Let $\mathfrak{A} := \langle \kappa \cup \mathcal{P}(\mu), \bar{\mu}^{\mathfrak{A}}, \bar{E}^{\mathfrak{A}} \rangle$ be as above. By $\text{SDLS}_+^-(\mathcal{L}^{\aleph_0, \Pi}, < \kappa)$, we can take $B \in [\kappa \cup \mathcal{P}(\mu)]^{< \kappa}$ such that $\mathfrak{A} \upharpoonright B \prec_{\mathcal{L}^{\aleph_0, \Pi}}^- \mathfrak{A}$. Let ψ be the $\mathcal{L}^{\aleph_0, \Pi}$ -formula

$$\forall X((\forall z(z \varepsilon X \rightarrow \bar{\mu}(z)) \rightarrow \exists x \forall y(\bar{\mu}(y) \rightarrow (y \varepsilon X \leftrightarrow \bar{E}(y, x)))).$$

Notice ψ states that every weak second order object which is a subset of μ can be translated into a first order object. Then we have $\mathfrak{A} \models \psi$. Since $\mathfrak{A} \upharpoonright B \prec_{\mathcal{L}^{\aleph_0, \Pi}}^- \mathfrak{A}$, we have $\mathfrak{A} \upharpoonright B \models \psi$, thus $[\mu]^{\aleph_0} \subseteq B$. Therefore $\mu^{\aleph_0} \leq |B| < \kappa$.

(1) $\Rightarrow$ (4): Assume (1). Let $\mathfrak{A} = \langle A, \dots \rangle$ be a structure in countable signature with $|A| \geq \kappa$ and let $\mathcal{D}$ be a club subset of $\subseteq [A]^{< \kappa}$.

Let θ be some regular cardinal such that $\mathfrak{A} \in \mathcal{H}(\theta)$. Define $\tilde{\mathfrak{A}} := \langle \mathcal{H}(\theta), \in, \bar{A}^{\tilde{\mathfrak{A}}}, \bar{\mathfrak{A}}^{\tilde{\mathfrak{A}}} \rangle$, where $\bar{A}$ and $\bar{\mathfrak{A}}$ are a unary predicate symbol and a constant symbol respectively such that $\bar{A}^{\tilde{\mathfrak{A}}} := A$ and $\bar{\mathfrak{A}}^{\tilde{\mathfrak{A}}} := \mathfrak{A}$.

By (1) and since $\text{cf}(\kappa) \geq \omega_2$, we can construct $M \subseteq \mathcal{H}(\theta)$ such that $[M]^{\aleph_0} \subseteq M$, $\tilde{\mathfrak{A}} \upharpoonright M \prec \tilde{\mathfrak{A}}$ and $A \cap M \in \mathcal{D}$. Let $B := A \cap M$. The following claim allow us to prove that $\mathfrak{A} \upharpoonright B \prec_{\mathcal{L}^{\aleph_0, \Pi}} \mathfrak{A}$.

Claim 2.5.1. *Suppose $\tilde{C} \subseteq \mathcal{H}(\theta)$ is such that $\tilde{\mathfrak{A}} \upharpoonright \tilde{C} \prec \tilde{\mathfrak{A}}$ and $[C]^{\aleph_0} \subseteq \tilde{C}$, where $C := \tilde{C} \cap A$. Then given any $\mathcal{L}^{\aleph_0, \Pi}$ -formula $\varphi(x_0, \dots, x_n, Y_0, \dots, Y_m)$, any $a_0, \dots, a_n \in C$ and any $U_0, \dots, U_m \in [C]^{\aleph_0}$, we have*

$$\begin{aligned} \tilde{\mathfrak{A}} \upharpoonright \tilde{C} \models \text{“} \bar{\mathfrak{A}} \models \varphi(a_0, \dots, a_n, U_0, \dots, U_m) \text{”} \\ \Leftrightarrow \mathfrak{A} \upharpoonright C \models \varphi(a_0, \dots, a_n, U_0, \dots, U_m). \end{aligned} \quad (2.3)$$

Proof of claim 2.5.1. We prove (2.3) by induction on the complexity of the formula φ . It is trivial that (2.3) holds when φ is an atomic formula. Similarly, it is also trivial that (2.3) holds when φ is of the form $\neg\psi_0$, $\psi_0 \wedge \psi_1$, $\psi_0 \vee \psi_1$, $\forall x\psi_0(x)$ or $\exists x\psi_0(x)$ and (2.3) holds for ψ_0 and ψ_1 . Since the weak second order quantifier $\forall X$ is dual to $\exists X$, it is enough to prove that (2.3) holds for φ of the form $\exists X\psi(X, x_0, \dots, x_n, Y_0, \dots, Y_m)$ given that (2.3) holds for ψ .

First, suppose that $\tilde{\mathfrak{A}} \upharpoonright \tilde{C} \models \text{“} \bar{\mathfrak{A}} \models \exists X\psi(X, a_0, \dots, a_n, U_0, \dots, U_m) \text{”}$ holds. Then we have

$$\tilde{\mathfrak{A}} \upharpoonright \tilde{C} \models \text{“} \exists x \in [\bar{A}]^{\aleph_0}, \bar{\mathfrak{A}} \models \psi(x, a_0, \dots, a_n, U_0, \dots, U_m) \text{”}.$$

Therefore there exists $U \in \left[\overline{A}^{\tilde{\mathfrak{A}} \upharpoonright \tilde{C}} \right]^{\aleph_0} = [C]^{\aleph_0}$ such that

$$\tilde{\mathfrak{A}} \upharpoonright \tilde{C} \models \text{“} \overline{\mathfrak{A}} \models \psi(U, a_0, \dots, a_n, U_0, \dots, U_m) \text{”}.$$

By induction hypothesis, we have $\mathfrak{A} \upharpoonright C \models \psi(U, a_0, \dots, a_n, U_0, \dots, U_m)$, thus $\mathfrak{A} \upharpoonright C \models \varphi(a_0, \dots, a_n, U_0, \dots, U_m)$ holds.

Conversely, suppose $\mathfrak{A} \upharpoonright C \models \exists X \psi(X, a_0, \dots, a_n, U_0, \dots, U_m)$ holds. Then there is $U \in [C]^{\aleph_0}$ such that

$$\mathfrak{A} \upharpoonright C \models \psi(U, a_0, \dots, a_n, U_0, \dots, U_m).$$

Therefore we have

$$\tilde{\mathfrak{A}} \models \text{“} \overline{\mathfrak{A}} \upharpoonright C \models \psi(U, a_0, \dots, a_n, U_0, \dots, U_m) \text{”}.$$

Since $U \in [C]^{\aleph_0} \subseteq \tilde{C}$ and $\tilde{\mathfrak{A}} \upharpoonright \tilde{C} \prec \tilde{\mathfrak{A}}$, we have

$$\tilde{\mathfrak{A}} \upharpoonright C \models \text{“} \overline{\mathfrak{A}} \upharpoonright C \models \psi(U, a_0, \dots, a_n, U_0, \dots, U_m) \text{”}.$$

Therefore U witnesses that $\tilde{\mathfrak{A}} \upharpoonright \tilde{C} \models \text{“} \overline{\mathfrak{A}} \models \varphi(a_0, \dots, a_n, U_0, \dots, U_m) \text{”}$ holds. (claim 2.5.1) $\square$

Let $\varphi(x_0, \dots, x_n, Y_0, \dots, Y_m)$ be a $\mathcal{L}^{\aleph_0, \Pi}$ -formula. For $a_0, \dots, a_n \in B$ and $U_0, \dots, U_m \in [B]^{\aleph_0}$, by claim 2.5.1 and $\tilde{\mathfrak{A}} \upharpoonright M \prec \tilde{\mathfrak{A}}$ we have

$$\begin{aligned} \mathfrak{A} &\models \varphi(a_0, \dots, a_n, U_0, \dots, U_m) \\ &\Leftrightarrow \tilde{\mathfrak{A}} \models \text{“} \overline{\mathfrak{A}} \models \varphi(a_0, \dots, a_n, U_0, \dots, U_m) \text{”} \\ &\Leftrightarrow \tilde{\mathfrak{A}} \upharpoonright M \models \text{“} \overline{\mathfrak{A}} \models \varphi(a_0, \dots, a_n, U_0, \dots, U_m) \text{”} \\ &\Leftrightarrow \mathfrak{A} \upharpoonright B \models \varphi(a_0, \dots, a_n, U_0, \dots, U_m). \end{aligned}$$

(proposition 2.5) $\square$

Similarly to proposition 2.5, for $\mathcal{L}_{\text{stat}}^{\aleph_0}$ we have:

Lemma 2.6. *For any $\kappa > \aleph_2$, $\text{SDLS}^-(\mathcal{L}_{\text{stat}}^{\aleph_0}, < \kappa)$ implies $\kappa > 2^{\aleph_0}$.*

In order to prove lemma 2.6, we shall use the following theorem:

Theorem 2.7 (Theorem 3.2(a) in Baumgartner and Taylor [BT82]). *If $\mathcal{C}$ is a club subset of $[\omega_2]^{\aleph_0}$, then there is a countable set $A \subseteq \omega_2$ such that $|[A]^{\aleph_0} \cap \mathcal{C}| = 2^{\aleph_0}$.*

Theorem 2.7 is often a key element of a proof showing that a certain principle implies $2^{\aleph_0} \leq \aleph_2$ (see, for example, the proof of theorem 37.18 in [Jec03]).

Proof of lemma 2.6. Let $\kappa > \aleph_2$ and suppose $\text{SDLS}^-(\mathcal{L}_{\text{stat}}^{\aleph_0}, < \kappa)$. By corollary 2.4, if $\text{SDLS}^-(\mathcal{L}_{\text{stat}}^{\aleph_0}, < \aleph_2)$ holds, we have $2^{\aleph_0} \leq \aleph_2 < \kappa$. Suppose then that $\text{SDLS}^-(\mathcal{L}_{\text{stat}}^{\aleph_0}, < \aleph_2)$ fails. Let $\mathfrak{A} := \langle A, \overline{R}_0^{\mathfrak{A}}, \dots, \overline{R}_n^{\mathfrak{A}}, \dots \rangle$ be a structure of countable signature witnessing the failure of $\text{SDLS}^-(\mathcal{L}_{\text{stat}}^{\aleph_0}, < \aleph_2)$, i.e.

$$|B| \geq \aleph_2 \text{ whenever } B \subseteq A \text{ is such that } \mathfrak{A} \upharpoonright B \prec_{\mathcal{L}_{\text{stat}}^{\aleph_0}}^- \mathfrak{A}. \quad (2.4)$$

Let $\eta := |A|$. Without loss of generality, we may assume that $A = \eta$.

Let $\tilde{\mathfrak{A}} := \langle \mathcal{H}(\eta^+), \in, \eta, \overline{R}_0^{\tilde{\mathfrak{A}}}, \dots, \overline{R}_n^{\tilde{\mathfrak{A}}}, \dots \rangle$, with $\overline{R}_n^{\tilde{\mathfrak{A}}} := \overline{R}_n^{\mathfrak{A}}$ for every $n \in \omega$.

Let φ be the $\mathcal{L}_{\text{stat}}^{\aleph_0}$ -formula

$$aaX \exists x \forall y (y \varepsilon X \leftrightarrow y \in x).$$

By (2.2), for any uncountable set M we have $M \models \varphi$ if and only if M is internally club (i.e. $M \cap [M]^{\aleph_0}$ contains a club subset of $[M]^{\aleph_0}$). Since η^+ has uncountable cofinality, we have

$$\tilde{\mathfrak{A}} \models \varphi. \quad (2.5)$$

By $\text{SDLS}^-(\mathcal{L}_{\text{stat}}^{\aleph_0}, < \kappa)$, there is $M \in [\mathcal{H}(\eta^+)]^{<\kappa}$ such that

$$\tilde{\mathfrak{A}} \upharpoonright M \prec_{\mathcal{L}_{\text{stat}}^{\aleph_0}}^- \tilde{\mathfrak{A}}. \quad (2.6)$$

Let $B := M \cap \eta$. It follows that $\mathfrak{A} \upharpoonright B \prec_{\mathcal{L}_{\text{stat}}^{\aleph_0}}^- \mathfrak{A}$. Therefore, by (2.4) we have $\aleph_2 \leq |B| \leq |M|$.

By (2.5) and (2.6), we have $\tilde{\mathfrak{A}} \upharpoonright M \models \varphi$, thus there is $C \subseteq M \cap [M]^{\aleph_0}$ which is a club subset of $[M]^{\aleph_0}$. By theorem 2.7, we have $\kappa > |M| \geq |C| \geq 2^{\aleph_0}$. (lemma 2.6) $\square$

Corollary 2.8. $\text{SDLS}^-(\mathcal{L}_{\text{stat}}^{\aleph_0}, < 2^{\aleph_0})$ implies $2^{\aleph_0} = \aleph_2$.

Proof. Suppose $\text{SDLS}^-(\mathcal{L}_{\text{stat}}^{\aleph_0}, < 2^{\aleph_0})$. By lemma 2.6, it follows that $2^{\aleph_0} \leq \aleph_2$. Since $\text{SDLS}^-(\mathcal{L}_{\text{stat}}^{\aleph_0}, < \aleph_1)$ does not hold (see example 2.2), $2^{\aleph_0}$ can only be $\aleph_2$. (corollary 2.8) $\square$

It can be proved that $\text{SDLS}^-(\mathcal{L}_{\text{stat}}^{\aleph_0}, < 2^{\aleph_0})$ holds under $\text{MA}^{+\omega_1}(\sigma\text{-closed})$. (this follows corollary 2.18 and a result by Cox in [Cox12]). On the other hand, we have that

Corollary 2.9. $\text{SDLS}(\mathcal{L}_{\text{stat}}^{\aleph_0}, < 2^{\aleph_0})$ is inconsistent.

Proof. Suppose $\text{SDLS}(\mathcal{L}_{\text{stat}}^{\aleph_0}, < 2^{\aleph_0})$ holds. Then $2^{\aleph_0} = \aleph_2$ by corollary 2.8. Therefore $\text{SDLS}(\mathcal{L}_{\text{stat}}^{\aleph_0}, < \aleph_2)$ holds. By proposition 2.5 it follows that $2^{\aleph_0} = \aleph_1$, which is a contradiction. (corollary 2.9) $\square$

In section 2.3, we shall present a weakening of $\text{SDLS}(\mathcal{L}_{\text{stat}}^{\aleph_0}, < \kappa)$ which is consistent with $2^{\aleph_0} = \kappa$ for very large κ (e.g. supercompact).

2.2 Characterization of SDLS for stationary logics and the Diagonal Reflection Principle

Definition 2.10. Let κ be a regular uncountable cardinal and $\mathcal{C}$ be a class. For $\eta > \kappa$, define $(*)_{<\kappa,\eta}^{\mathcal{C}}$ as the following statement:

For any countable expansion $\mathfrak{A}$ of $\langle \mathcal{H}(\eta), \in \rangle$ and any family $\langle S_a : a \in \mathcal{H}(\eta) \rangle$ of stationary subsets of $[\mathcal{H}(\eta)]^{\aleph_0}$, there is some $N \in [\mathcal{H}(\eta)]^{<\kappa}$ such that:

- (1) $N \in \mathcal{C}$;
- (2) $\mathfrak{A} \upharpoonright N \prec \mathfrak{A}$;
- (3) for each $a \in N$, $S_a \cap [N]^{\aleph_0}$ is a stationary subset of $[N]^{\aleph_0}$.

Similarly to $\text{SDLS}(\mathcal{L}, < \kappa)$, $(*)_{<\kappa}^{\mathcal{C}}$ also has a natural strengthening which requires the existence of stationarily many reflection points.

Definition 2.11. Let κ be a regular uncountable cardinal and $\mathcal{C}$ be a class. For $\eta > \kappa$, define $(*)_{<\kappa,\eta}^{+\mathcal{C}}$ as the following statement:

For any countable expansion $\mathfrak{A}$ of $\langle \mathcal{H}(\eta), \in \rangle$ and any family $\langle S_a : a \in \mathcal{H}(\eta) \rangle$ of stationary subsets of $[\mathcal{H}(\eta)]^{\aleph_0}$, there are *stationarily many* $N \in [\mathcal{H}(\eta)]^{<\kappa}$ such that:

- (1) $N \in \mathcal{C}$;
- (2) $\mathfrak{A} \upharpoonright N \prec \mathfrak{A}$;
- (3) for each $a \in N$, $S_a \cap [N]^{\aleph_0}$ is a stationary subset of $[N]^{\aleph_0}$.

We shall consider the following classes:

$$\begin{aligned} \text{IU}_{\aleph_0} &:= \{X : [X]^{\aleph_0} \cap X \text{ is cofinal in } [X]^{\aleph_0}\}; \\ \text{IS}_{\aleph_0} &:= \{X : [X]^{\aleph_0} \cap X \text{ is a stationary subset of } [X]^{\aleph_0}\}; \\ \text{IC}_{\aleph_0} &:= \{X : [X]^{\aleph_0} \cap X \text{ contains a club subset of } [X]^{\aleph_0}\}. \end{aligned}$$

The sets in $\text{IU}_{\aleph_0}$, $\text{IS}_{\aleph_0}$, $\text{IC}_{\aleph_0}$ are called *internally unbounded*, *internally stationary* and *internally club* respectively.

Lemma 2.12. Suppose $\mathcal{C}$ is $\text{IU}_{\aleph_0}$, $\text{IS}_{\aleph_0}$ or $\text{IC}_{\aleph_0}$. For a regular cardinal κ and cardinals η, η' with $\aleph_1 < \kappa < \eta < \eta'$, $(*)_{<\kappa,\eta'}^{\mathcal{C}}$ implies $(*)_{<\kappa,\eta}^{\mathcal{C}}$ and $(*)_{<\kappa,\eta'}^{+\mathcal{C}}$ implies $(*)_{<\kappa,\eta}^{+\mathcal{C}}$.

Proof. We prove that $(*)_{<\kappa, \eta'}^{+\mathcal{C}}$ implies $(*)_{<\kappa, \eta}^{+\mathcal{C}}$. The other implications can be proved similarly.

Let κ, η, η' and $\mathcal{C}$ be as above and assume $(*)_{<\kappa, \eta'}^{+\mathcal{C}}$ holds. Let $\mathcal{D}$ be an arbitrary club subset of $[\mathcal{H}(\theta)]^{<\kappa}$. Let $\mathfrak{A} := \langle \mathcal{H}(\theta), \in, \bar{R}_0^{\mathfrak{A}}, \dots, \bar{R}_n^{\mathfrak{A}}, \dots \rangle$ be a countable extension of the structure $\langle \mathcal{H}(\theta), \in \rangle$ (where $\bar{R}_0, \dots, \bar{R}_n, \dots$ are arbitrary predicate symbols), $\langle S_a : a \in \mathcal{H}(\theta) \rangle$ be a family of stationary subsets of $[\mathcal{H}(\theta)]^{\aleph_0}$.

Define

$$\begin{aligned}\tilde{\mathfrak{A}} &:= \langle \mathcal{H}(\eta'), \in, \eta, \mathcal{H}(\eta), \bar{R}_0^{\mathfrak{A}}, \dots, \bar{R}_n^{\mathfrak{A}}, \dots \rangle, \\ \tilde{\mathcal{D}} &:= \{X \in [\mathcal{H}(\eta')]^{<\kappa} : X \cap \mathcal{H}(\eta) \in \mathcal{C}\},\end{aligned}$$

and also define

$$\tilde{S}_a := \begin{cases} \{x \in [\mathcal{H}(\eta')]^{\aleph_0} : x \cap \mathcal{H}(\eta) \in S_a\} & , \text{ if } a \in \mathcal{H}(\eta); \\ [\mathcal{H}(\eta')]^{\aleph_0} & , \text{ otherwise.} \end{cases} \quad (2.7)$$

for every $a \in \mathcal{H}(\eta')$.

We have that $\tilde{S}_a$ is a stationary subset of $[\mathcal{H}(\eta)]^{\aleph_0}$ for every $a \in \mathcal{H}(\eta')$ by corollary 1.5(ii) and that $\tilde{\mathcal{D}}$ is a club subset of $[\mathcal{H}(\eta')]^{<\kappa}$ by lemma 1.4(I).

By $(*)_{<\kappa, \eta'}^{+\mathcal{C}}$ we can take $N \in \tilde{\mathcal{D}}$ such that $N \in \mathcal{C}$, $\tilde{\mathfrak{A}} \restriction N \prec \tilde{\mathfrak{A}}$ and $\tilde{S}_a \cap [N]^{\aleph_0}$ is a stationary subset of $[N]^{\aleph_0}$ for every $a \in N$.

Let $M := \mathcal{H}(\eta) \cap N$. Since $\tilde{\mathfrak{A}} \restriction N \prec \tilde{\mathfrak{A}}$, it follows that $\mathfrak{A} \restriction M \prec \mathfrak{A}$.

Claim 2.12.1. *Given $a \in N$, if $|a| < \eta$, then $a \cap \mathcal{H}(\eta) \in M$.*

Proof of claim 2.12.1. Let $a \in N$ be such that $|a| < \eta$. Since $a \in \mathcal{H}(\eta')$ and $|a| < \eta$, we have $a \cap \mathcal{H}(\eta) \in \mathcal{H}(\eta)$. Since $\tilde{\mathfrak{A}}$ has a predicate describing $\mathcal{H}(\eta)$ and $\tilde{\mathfrak{A}} \restriction N \prec \tilde{\mathfrak{A}}$, we have that $a \cap \mathcal{H}(\eta) \in N$. We conclude that $a \in M = N \cap \mathcal{H}(\eta)$. (claim 2.12.1) $\square$

By claim 2.12.1 and corollary 1.5, it follows that $M \in \mathcal{C}$.

Let $a \in M$. Then $\tilde{S}_a \cap [N]^{\aleph_0}$ is a stationary subset of $[N]^{\aleph_0}$. By (2.7) we have:

$$S_a \cap [M]^{\aleph_0} = \{x \cap M : x \in \tilde{S}_a \cap [N]^{\aleph_0}\}.$$

Therefore, by corollary 1.5(iv) we have that $S_a \cap [M]^{\aleph_0}$ is a stationary subset of $[M]^{\aleph_0}$. (lemma 2.12) $\square$

Lemma 2.13. *Suppose $\kappa > \aleph_1$ is a regular cardinal. Then*

1. $\text{SDLS}^-(\mathcal{L}_{\text{stat}}^{\aleph_0}, < \kappa) \Leftrightarrow (*)_{<\kappa, \eta}^{\text{IC}_{\aleph_0}}$ for all cardinal $\eta \geq \kappa$.
2. $\text{SDLS}_+^-(\mathcal{L}_{\text{stat}}^{\aleph_0}, < \kappa) \Leftrightarrow (*)_{<\kappa, \eta}^{+\text{IC}_{\aleph_0}}$ holds for all cardinal $\eta \geq \kappa$.

3. $\text{SDLS}_+(\mathcal{L}_{\text{stat}}^{\aleph_0, II}, < \kappa) \Leftrightarrow (*)_{< \kappa, \eta}^{+\text{IC}_{\aleph_0}}$ holds for all cardinal $\eta \geq \kappa$ and $\mu^{\aleph_0} < \kappa$ for all $\mu < \kappa$.

Proof. We first show (2); (1) can be proved similarly to (2).

Assume $\text{SDLS}_+(\mathcal{L}_{\text{stat}}^{\aleph_0}, < \kappa)$ and let $\eta \geq \kappa$. Let $\mathfrak{A} = \langle \mathcal{H}(\eta), \in, \dots \rangle$ be a countable expansion of $\langle \mathcal{H}(\eta), \in \rangle$, $\langle S_a : a \in \mathcal{H}(\eta) \rangle$ be a sequence of stationary subsets of $[\mathcal{H}(\eta)]^{\aleph_0}$ and $\mathcal{C}$ be a club subset of $[\mathcal{H}(\eta)]^{< \kappa}$.

Let $\tilde{\mathfrak{A}} = \langle \mathcal{H}(\eta), \in, \dots, \bar{S}^{\tilde{\mathfrak{A}}} \rangle$ be the expansion of $\mathfrak{A}$ obtained by adding a binary relation symbol $\bar{S}$ which describes the the sequence $\langle S_a : a \in \mathcal{H}(\eta) \rangle$ in the structure. $\bar{S}$ shall be interpreted as follows:

$$\bar{S}^{\tilde{\mathfrak{A}}} := \{ \langle a, s \rangle \in (\mathcal{H}(\eta))^2 : s \in S_a \}.$$

By $\text{SDLS}_+(\mathcal{L}_{\text{stat}}^{\aleph_0}, < \kappa)$, we can take $M \in [\mathcal{H}(\eta)]^{< \kappa}$ such that $M \in \mathcal{C}$ and $\tilde{\mathfrak{A}} \restriction M \prec_{\mathcal{L}_{\text{stat}}^{\aleph_0}}^- \tilde{\mathfrak{A}}$.

Let φ be the sentence $\forall x \exists y \forall z (z \in X \leftrightarrow z \in y)$. By (2.2), it follows that for any uncountable set B we have $B \models \varphi \Leftrightarrow B \in \text{IC}_{\aleph_0}$. Therefore, we have $\tilde{\mathfrak{A}} \models \varphi$. Since $\tilde{\mathfrak{A}} \restriction M \prec_{\mathcal{L}_{\text{stat}}^{\aleph_0}}^- \tilde{\mathfrak{A}}$, we have $\tilde{\mathfrak{A}} \restriction M \models \varphi$, thus $M \in \text{IC}_{\aleph_0}$.

Now let ψ be the sentence $\forall x \text{ stat } X \exists y (\bar{S}(x, y) \wedge \forall z (z \in X \leftrightarrow z \in y))$. Similarly to φ , for any uncountable set B we have

$$B \models \psi \Leftrightarrow "S_b \cap [B]^{\aleph_0} \text{ is a stationary subset of } [B]^{\aleph_0} \text{ for all } b \in B".$$

Then $\tilde{\mathfrak{A}} \models \psi$. Since $\tilde{\mathfrak{A}} \restriction M \prec_{\mathcal{L}_{\text{stat}}^{\aleph_0}}^- \tilde{\mathfrak{A}}$, we have that $S_a \cap [M]^{\aleph_0}$ is a stationary subset of $[M]^{\aleph_0}$ for all $a \in M$. Therefore M witnesses $(*)_{< \kappa, \eta}^{+\text{IC}_{\aleph_0}}$.

Now assume $(*)_{< \kappa, \eta}^{+\text{IC}_{\aleph_0}}$ holds for every $\eta \geq \kappa$. We prove that $\text{SDLS}_+(\mathcal{L}_{\text{stat}}^{\aleph_0}, < \kappa)$ holds. Let $\mathfrak{A} = \langle A, \bar{R}_0^{\mathfrak{A}}, \dots, \bar{R}_n^{\mathfrak{A}}, \dots \rangle$ be a structure in countable signature with $|A| \geq \kappa$. Let $\mathcal{C} \subseteq [A]^{< \kappa}$ be a club.

Take regular a cardinal η such that $\mathfrak{A} \in \mathcal{H}(\eta)$ and define

$$\tilde{\mathfrak{A}} := \langle \mathcal{H}(\eta), \in, \bar{\mathfrak{A}}^{\tilde{\mathfrak{A}}}, \bar{A}^{\tilde{\mathfrak{A}}}, \bar{R}_0^{\tilde{\mathfrak{A}}}, \dots, \bar{R}_n^{\tilde{\mathfrak{A}}}, \dots \rangle, \quad (2.8)$$

where $\bar{\mathfrak{A}}$ and $\bar{A}$ are unary relational symbols with $\bar{\mathfrak{A}}^{\tilde{\mathfrak{A}}} = \mathfrak{A}$ and $\bar{A}^{\tilde{\mathfrak{A}}} = A$, and we also have $\bar{R}_n^{\tilde{\mathfrak{A}}} = \bar{R}_n^{\mathfrak{A}}$ for every $n \in \omega$. For each $a \in \mathcal{H}(\eta)$, define:

$$S_a := \begin{cases} \{U \in [\mathcal{H}(\eta)]^{\aleph_0} : |U \cap A| = \aleph_0 \wedge \\ \quad \mathfrak{A} \models \psi(a_0, \dots, a_m, U_0, \dots, U_l, U \cap A)\}, \\ \quad \text{if } \psi(x_0, \dots, x_m, Y_0, \dots, Y_l, X) \text{ is an } \mathcal{L}_{\text{stat}}^{\aleph_0}\text{-formula,} \\ \quad a_0, \dots, a_m \in A, U_0, \dots, U_l \in [A]^{\aleph_0}, \\ \quad \mathfrak{A} \models \text{stat } X\psi(a_0, \dots, a_m, U_0, \dots, U_l, X) \text{ and} \\ \quad a = \langle \psi, a_0, \dots, a_m, U_0, \dots, U_l \rangle; \\ [\mathcal{H}(\eta)]^{\aleph_0}, \\ \text{otherwise.} \end{cases} \quad (2.9)$$

Let

$$\tilde{\mathcal{C}} := \{U \in [\mathcal{H}(\eta)]^{<\kappa} : U \cap A \in \mathcal{C}\}. \quad (2.10)$$

By lemma 1.4(I), $\tilde{\mathcal{C}}$ is a club subset of $[\mathcal{H}(\eta)]^{<\kappa}$. By $(*)_{<\kappa, \eta}^{+\text{IC}_{\aleph_0}}$, we can take an internally club $M \in [\mathcal{H}(\eta)]^{<\kappa}$ such that

$$M \in \tilde{\mathcal{C}}; \quad (2.11)$$

$$\tilde{\mathfrak{A}} \restriction M \prec \tilde{\mathfrak{A}}; \text{ and} \quad (2.12)$$

$$S_a \cap [M]^{\aleph_0} \text{ is a stationary subset of } [M]^{\aleph_0} \text{ for all } a \in M. \quad (2.13)$$

Let $B := A \cap M$. By (2.10) and (2.11), we have $B \in \mathcal{C}$. It follows from (2.12) and the following claim that $\mathfrak{A} \restriction B \prec_{\mathcal{L}_{\text{stat}}^{\aleph_0}} \mathfrak{A}$.

Claim 2.13.1. *For any $\mathcal{L}_{\text{stat}}^{\aleph_0}$ -formula $\varphi(x_0, \dots, x_m, Y_0, \dots, Y_l)$, $a_0, \dots, a_m \in B$ and $U_0, \dots, U_l \in [B]^{\aleph_0} \cap M$, we have*

$$\tilde{\mathfrak{A}} \restriction M \models \text{“}\overline{\mathfrak{A}} \models \varphi(a_0, \dots, a_m, U_0, \dots, U_l)\text{”} \Leftrightarrow \mathfrak{A} \restriction B \models \varphi(a_0, \dots, a_m, U_0, \dots, U_l) \quad (2.14)$$

Proof of claim 2.13.1. We proceed by induction on the formula φ . We prove that (2.14) holds for φ whenever it is of the form $\text{stat } X\psi$ and (2.14) holds for ψ . As in the proof of claim 2.5.1, the remaining cases follow straightforward from the definitions.

Suppose that $\tilde{\mathfrak{A}} \restriction M \models \text{“}\overline{\mathfrak{A}} \models \varphi(a_0, \dots, a_m, U_0, \dots, U_l)\text{”}$ holds. Then, by (2.12) we have $\mathfrak{A} \models \text{stat } X\psi(a_0, \dots, a_m, U_0, \dots, U_l, X)$.

Let $a := \langle \psi, a_0, \dots, a_m, U_0, \dots, U_l \rangle$. By (2.12), it follows that $a \in M$ and by (2.9) we have

$$S_a = \{U \in [\mathcal{H}(\eta)]^{\aleph_0} : |U \cap A| = \aleph_0 \wedge \mathfrak{A} \models \psi(a_0, \dots, a_m, U_0, \dots, U_l, U \cap A)\}. \quad (2.15)$$

By (2.13), we have that $S_a \cap [M]^{\aleph_0}$ is a stationary subset of $[M]^{\aleph_0}$. Since M is internally club, we have that $S_a \cap [M]^{\aleph_0} \cap M = S_a \cap M$ is stationary. By corollary 1.5(iv), the set $\mathfrak{S} := \{U \cap B : U \in S_a \cap M \wedge |U \cap B| = \aleph_0\}$ is a stationary subset of $[M]^{\aleph_0}$. By (2.15) and (2.12), we have

$$\mathfrak{S} := \{U \cap B : U \in M \cap [M]^{\aleph_0} \wedge U \cap A = U \cap B \in A \cap [B]^{\aleph_0} \wedge \tilde{\mathfrak{A}} \restriction M \models \text{“}\overline{\mathfrak{A}} \models \psi(a_0, \dots, a_m, U_0, \dots, U_l, U \cap B)\text{”}\}$$

By induction hypothesis, we have

$$\mathfrak{S} \subseteq \{V \in [B]^{\aleph_0} : \mathfrak{A} \restriction B \models \psi(a_0, \dots, a_m, U_0, \dots, U_l, V)\}.$$

Therefore, by (2.1), we have $\mathfrak{A} \restriction B \models \text{stat } X\psi(a_0, \dots, a_m, U_0, \dots, U_l, X)$, that is, $\mathfrak{A} \restriction B \models \varphi(a_0, \dots, a_m, U_0, \dots, U_l)$.

Suppose now that $\tilde{\mathfrak{A}} \restriction M \not\models \text{“}\overline{\mathfrak{A}} \models \varphi(a_0, \dots, a_m, U_0, \dots, U_l)\text{”}$. Then we have

$$\begin{aligned} \tilde{\mathfrak{A}} \restriction M \models \text{“there is a club } \mathcal{D} \subseteq [\overline{A}]^{\aleph_0} \text{ such that} \\ \overline{\mathfrak{A}} \models \neg\psi(a_0, \dots, a_m, U_0, \dots, U_l, x) \text{ for all } x \in \mathcal{D}\text{”} \end{aligned}$$

By (2.12), there is $\mathcal{D}' \in M$ which is a club subset of $[A]^{\aleph_0}$ such that

$$\tilde{\mathfrak{A}} \restriction M \models \text{“}\overline{\mathfrak{A}} \models \neg\psi(a_0, \dots, a_m, U_0, \dots, U_l, V)\text{” for all } V \in \mathcal{D}' \cap M. \quad (2.16)$$

Since M is internally club, by (2.12) it follows that $[B]^{\aleph_0} \cap M$ contains a club subset of $[B]^{\aleph_0}$. Then it follows that the set

$$\{V \cap B : |V \cap B| = \aleph_0 \wedge V \in \mathcal{D}' \cap [B]^{\aleph_0} \cap M = \mathcal{D}' \cap M\}$$

contains a club subset of $[B]^{\aleph_0}$. Then by (2.16) it follows that the set

$$\{V \in \mathcal{D}' \cap M : \tilde{\mathfrak{A}} \restriction M \models \text{“}\overline{\mathfrak{A}} \models \neg\psi(a_0, \dots, a_m, U_0, \dots, U_l, V)\text{”}\}$$

contains a club subset of $[M \cap A]^{\aleph_0}$. By induction hypothesis, the above set is equal to

$$\{V \in \mathcal{D}' \cap M : \mathfrak{A} \restriction B \models \neg\psi(a_0, \dots, a_m, U_0, \dots, U_l, V)\}.$$

Therefore we have $\mathfrak{A} \restriction B \models \neg \text{stat } X\psi(a_0, \dots, a_m, U_0, \dots, U_l, X)$, and thus $\mathfrak{A} \restriction B \not\models \varphi(a_0, \dots, a_n, U_0, \dots, U_m)$. (claim 2.13.1) $\square$

We now prove (3). By (2) and proposition 2.5, $\text{SDLS}_+(\mathcal{L}_{\text{stat}}^{\aleph_0, \Pi}, < \kappa)$ implies “ $(*)_{< \kappa, \eta}^{+\text{IC}_{\aleph_0}}$ holds for all $\eta \geq \kappa$ and $\mu^{\aleph_0} < \kappa$ for all $\mu < \kappa$ ”.

Conversely, suppose that $(*)_{< \kappa}^{+\text{IC}_{\aleph_0}}$ holds and $\mu^{\aleph_0} < \kappa$ for all $\mu < \kappa$. Let $\mathfrak{A} = \langle A, \bar{R}_0^{\mathfrak{A}}, \dots, \bar{R}_n^{\mathfrak{A}}, \dots \rangle$ be a structure in countable signature with $|A| \geq \kappa$ and let $\mathcal{C} \subseteq [A]^{< \kappa}$ be a club. Let η be a regular cardinal such that $\mathfrak{A} \in \mathcal{H}(\eta)$ and let $\tilde{\mathfrak{A}}$ and $\langle S_a : a \in \mathcal{H}(\eta) \rangle$ be defined as in (2.8) and (2.9).

Let $\mu := 2^{\aleph_0}$. By assumption we have $\mu < \kappa$, thus by lemma 1.4(I), it follows that the set

$$\mathfrak{C} := \{M \in [\mathcal{H}(\eta)]^{< \kappa} : \mu \subseteq M \wedge M \cap A \in \mathcal{C} \wedge \{\mathcal{P}(x) : x \in M \cap [M]^{\aleph_0}\} \subseteq M\} \quad (2.17)$$

contains a club subset of $[\mathcal{H}(\eta)]^{< \kappa}$. By $(*)_{< \kappa, \eta}^{+\text{IC}_{\aleph_0}}$, we can take an internally club $M \in \mathfrak{C}$ such that

$$\mu \subseteq M, \quad (2.18)$$

$$\tilde{\mathfrak{A}} \upharpoonright M \prec \tilde{\mathfrak{A}} \text{ and} \quad (2.19)$$

$$S_a \cap [M]^{\aleph_0} \text{ is a stationary subset of } [M]^{\aleph_0} \text{ for all } a \in M. \quad (2.20)$$

Let $B := \bar{A}^{\tilde{\mathfrak{A}} \upharpoonright M} = A \cap M$. By (2.17) we have that $B \in \mathcal{C}$. We claim that

$$[M]^{\aleph_0} \subseteq M. \quad (2.21)$$

Indeed, for $x \in [M]^{\aleph_0}$, we can take $y \in [M]^{\aleph_0} \cap M$ such that $x \subseteq y$. Since y is countable, we have $V \models |\mathcal{P}(y)| = 2^{\aleph_0} = \mu$. Notice that $\mathcal{P}(y)^V \in M$ by (2.17). By (2.19), we have $\tilde{\mathfrak{A}} \upharpoonright M \models |\mathcal{P}(y)^V| = \mu$. Therefore, by (2.18) we have $x \in \mathcal{P}(y)^V \subseteq M$, therefore (2.21) holds.

As in the proof of (2), it follows that $\mathfrak{A} \upharpoonright B \prec_{\mathcal{L}_{\text{stat}}^{\aleph_0, \Pi}}^- \mathfrak{A}$. By (2.21), this implies $\mathfrak{A} \upharpoonright B \prec_{\mathcal{L}_{\text{stat}}^{\aleph_0, \Pi}} \mathfrak{A}$. So we conclude that $\text{SDLS}_+(\mathcal{L}_{\text{stat}}^{\aleph_0, \Pi}, < \kappa)$ holds. (lemma 2.13) $\square$

We now recall the Diagonal Reflection principle introduced by Sean Cox in [Cox12].

Definition 2.14 (Diagonal Reflection Principle). Given regular uncountable cardinals κ and θ ($\kappa < \theta$) and a class $\mathcal{C}$, the *Diagonal Reflection Principle* for κ , θ and $\mathcal{C}$ (denote $\text{DRP}(< \kappa, \theta, \mathcal{C})$) is the following statement:

There are stationarily many $M \in [\mathcal{H}((\theta^{\aleph_0})^+)]^{< \kappa}$ such that

- (1) $M \cap \mathcal{H}(\theta) \in \mathcal{C}$; and

- (2) for all $R \in M$ such that R is a stationary subset of $[\theta]^{\aleph_0}$,
 $R \cap [\theta \cap M]^{\aleph_0}$ is a stationary subset of $[\theta \cap M]^{\aleph_0}$.

We denote by $\text{DRP}(< \kappa, \mathcal{C})$ the statement “ $\text{DRP}(< \kappa, \theta, \mathcal{C})$ holds for all cardinal $\theta \geq \kappa$ ”.

Lemma 2.15. *Suppose $\mathcal{C}$ is $\text{IU}_{\aleph_0}$, $\text{IS}_{\aleph_0}$ or $\text{IC}_{\aleph_0}$, κ is a regular cardinal and θ, θ' are cardinals with $\kappa \leq \theta < \theta'$. Then $\text{DRP}(< \kappa, \theta', \mathcal{C})$ implies $\text{DRP}(< \kappa, \theta, \mathcal{C})$*

Proof. Suppose that $\text{DRP}(< \kappa, \theta', \mathcal{C})$ holds and let $\mathcal{D}$ be a club subset of $[\mathcal{H}((\theta^{\aleph_0})^+)]^{<\kappa}$. Define

$$\begin{aligned} \mathcal{D}' := \{N \in [\mathcal{H}((\theta'^{\aleph_0})^+)]^{<\kappa} : & \text{(a) } N \cap \mathcal{H}((\theta^{\aleph_0})^+) \in \mathcal{D} \wedge \\ & \text{(b) } \kappa, \theta \in N \wedge \\ & \text{(c) } N \prec \mathcal{H}((\theta'^{\aleph_0})^+)\}. \end{aligned} \quad (2.22)$$

It follows by lemma 1.4(I) that $\mathcal{D}'$ contains a club subset of $[\mathcal{H}((\theta'^{\aleph_0})^+)]^{<\kappa}$. Thus, by $\text{DRP}(< \kappa, \theta', \mathcal{C})$, we can take $M' \in \mathcal{D}$ such that

$$M' \cap \mathcal{H}(\eta') \in \mathcal{C}; \text{ and} \quad (2.23)$$

for all $R \in M'$ such that R is a stationary subset of $[\theta']^{\aleph_0}$,

$$R \cap [\theta' \cap M']^{\aleph_0} \text{ is a stationary subset of } [\theta' \cap M']^{\aleph_0}. \quad (2.24)$$

Let $M := M' \cap \mathcal{H}((\theta^{\aleph_0})^+)$. By (2.22)(a) we have $M \in \mathcal{D}$. We show that M satisfies (1) and (2) from definition 2.14 (with respect to θ and $\mathcal{C}$).

Let $R \in M$ be a stationary subset of $[\theta]^{\aleph_0}$. By corollary 1.5(ii), the set $R' := \{x \in [\theta']^{\aleph_0} : x \cap \theta \in R\}$ is a stationary subset of $[\theta']^{\aleph_0}$. Notice by (2.22)(b) we have $R' \in M'$, so by (2.24) $R' \cap [\theta \cap M']^{\aleph_0}$ is a stationary subset of $R' \cap [\theta \cap M']^{\aleph_0}$. Then by corollary 1.5(iv), the set

$$R \cap [\theta \cap M]^{\aleph_0} = \{y \cap \mathcal{H}(\theta) : y \in R' \cap [\theta \cap M']^{\aleph_0}\}$$

is a stationary subset of $[\theta \cap M]^{\aleph_0}$, so M satisfies (2) from definition 2.14.

Similarly to claim 2.12.1, it follows from the following claim and lemma 1.4(III),(IV) that M satisfies (1) from definition 2.14.

Claim 2.15.1. *For any $a \in M'$, if $|a| < \theta$, then $a \cap \mathcal{H}(\theta) \in M \cap \mathcal{H}(\theta)$.*

Proof of claim 2.12.1. Let $\chi(x, \theta)$ be a formula such that $\mathcal{H}(\theta') \models \chi(a, \theta)$ if and only if $a \in \mathcal{H}(\theta)$. By (2.22) (b) and (c), it follows that

$$M' \models \forall x \exists y \forall z (z \in y \leftrightarrow z \in x \wedge \chi(z, \theta)) \quad (2.25)$$

Let $a \in M'$ and suppose $|a| < \theta$. By (2.25) we have $a \cap \mathcal{H}(\theta) \in M'$ whenever $a \in M'$. By $|a| < \theta$, we have that $a \cap \mathcal{H}(\theta) \in \mathcal{H}(\theta)$. Therefore $a \cap \mathcal{H}(\theta) \in M' \cap \mathcal{H}(\theta) \subseteq M$, so $a \cap \mathcal{H}(\theta) \in M' \cap \mathcal{H}(\theta)$. (claim 2.15.1) $\square$

(lemma 2.15)□

Lemma 2.16. *Suppose $\mathcal{C}$ is one of $\text{IU}_{\aleph_0}$, $\text{IS}_{\aleph_0}$, $\text{IC}_{\aleph_0}$. Given any regular cardinal $\kappa > \aleph_1$ and any cardinal θ such that $\theta = 2^{<\theta}$ and $\text{cf}(\theta) \geq \kappa$, we have that $\text{DRP}(< \kappa, \theta, \mathcal{C})$ is equivalent to $(*)_{<\kappa, \theta}^{\mathcal{C}}$.*

Proof. First assume that $\text{DRP}(< \kappa, \theta, \mathcal{C})$ holds.

Let $\mathfrak{A} := \langle \mathcal{H}(\theta), \in, \bar{R}_0^{\mathfrak{A}}, \dots, \bar{R}_n^{\mathfrak{A}}, \dots \rangle$ be a countable extension of the structure $\langle \mathcal{H}(\theta), \in \rangle$ (where $\bar{R}_0, \dots, \bar{R}_n, \dots$ are arbitrary predicate symbols), $\langle S_a : a \in \mathcal{H}(\theta) \rangle$ be a family of stationary subsets of $[\mathcal{H}(\theta)]^{\aleph_0}$ and $\mathcal{D}$ a club subset of $[\mathcal{H}(\theta)]^{<\kappa}$.

By the assumptions about θ we have $\theta^{<\kappa} = \theta$ and $|\mathcal{H}(\theta)| = \theta$. We then fix a bijection $g : \mathcal{H}(\theta) \rightarrow \theta$. For each $a \in \mathcal{H}(\theta)$, define

$$S_a^* := \{g''x : x \in S_a\}.$$

Now define the structure

$$\tilde{\mathfrak{A}} := \langle \mathcal{H}((\theta^{\aleph_0})^+), \in, \bar{S}^{*\tilde{\mathfrak{A}}}, \bar{g}^{\tilde{\mathfrak{A}}}, \bar{H}^{\tilde{\mathfrak{A}}}, \bar{P}_0^{\tilde{\mathfrak{A}}}, \dots, \bar{P}_n^{\tilde{\mathfrak{A}}}, \dots \rangle,$$

where $\bar{P}_n^{\tilde{\mathfrak{A}}} := \bar{P}_n^{\mathfrak{A}}$ for all $n \in \omega$, $\bar{H}$ is a unary relation symbol and $\bar{S}, \bar{g}$ are unary function symbols with

$$\begin{aligned} \bar{H}^{\tilde{\mathfrak{A}}} &:= \mathcal{H}(\theta); \\ \bar{S}^{*\tilde{\mathfrak{A}}} &:= \{\langle a, S_a^* \rangle : a \in \mathcal{H}(\theta)\} \cup \{\langle a, \emptyset \rangle : a \in \mathcal{H}((\theta^{\aleph_0})^+) \setminus \mathcal{H}(\theta)\}; \\ \bar{g}^{\tilde{\mathfrak{A}}} &\supseteq g. \end{aligned}$$

Define

$$\begin{aligned} \tilde{\mathcal{D}} &:= \{M \in [\mathcal{H}((\theta^{\aleph_0})^+)]^{<\kappa} : \text{(a) } M \cap \mathcal{H}(\theta) \in \mathcal{D} \wedge \\ &\quad \text{(b) } \theta \in M \wedge \\ &\quad \text{(c) } \tilde{\mathfrak{A}} \upharpoonright M \prec \tilde{\mathfrak{A}} \wedge \\ &\quad \text{(d) } \kappa \cap M < \kappa\}. \end{aligned} \tag{2.26}$$

By lemma 1.4(I) $\tilde{\mathcal{D}}$ contains a club subset of $[\mathcal{H}((\theta^{\aleph_0})^+)]^{<\kappa}$. By $\text{DRP}(< \kappa, \theta, \mathcal{C})$, we can take $M^+ \in \tilde{\mathcal{D}}$ such that

$$M^+ \cap \mathcal{H}(\theta) \in \mathcal{C}; \text{ and} \tag{2.27}$$

$$\begin{aligned} R \cap [\theta \cap M]^{\aleph_0} &\text{ is a stationary subset of } [\theta \cap M]^{\aleph_0}, \\ \text{whenever } R \in M^+ &\text{ is a stationary subset of } [\theta]^{\aleph_0}. \end{aligned} \tag{2.28}$$

Let $M := M^+ \cap \mathcal{H}(\theta)$. We show that M witnesses that $(*)_{<\kappa, \theta}^{+C}$ holds.

Since $M^+ \in \tilde{\mathcal{D}}$, by (2.26)(c) we have $\tilde{\mathfrak{A}} \upharpoonright M \prec \tilde{\mathfrak{A}}$. Furthermore, we have $M \in \mathcal{D}$ by (2.26)(a) and $M \in \mathcal{C}$ by (2.27).

Let $a \in M$. Notice that $S_a^* = \overline{S^*}^{\tilde{\mathfrak{A}} \upharpoonright M^+}(a) \in M^+$. Since g is a bijection, it follows that S_a^* is a stationary subset of $[\theta]^{\aleph_0}$. By (2.28), we have

$$S_a^* \cap [\theta \cap M^+]^{\aleph_0} \text{ is a stationary subset of } [\theta \cap M^+]^{\aleph_0}. \quad (2.29)$$

Notice that $\bar{g}^{\tilde{\mathfrak{A}} \upharpoonright M^+} = g \upharpoonright M$ (this follows from $\bar{g}^{\tilde{\mathfrak{A}}} \supseteq g$ and $\tilde{\mathfrak{A}} \upharpoonright M^+ \prec \tilde{\mathfrak{A}}$). By $\tilde{\mathfrak{A}} \upharpoonright M^+ \prec \tilde{\mathfrak{A}}$, we have that $\bar{g}^{\tilde{\mathfrak{A}} \upharpoonright M^+} : \mathcal{H}(\theta) \cap M^+ = M \rightarrow \theta \cap M^+$ is a bijection. Therefore, by (2.29) we conclude that

$$S_a \cap [M]^{\aleph_0} = \{g^{-1}''y : y \in S_a^* \cap [\theta \cap M^+]^{\aleph_0}\}$$

is a stationary subset of $[M]^{\aleph_0}$.

Assume now that $(*)_{<\kappa, \theta}^{+C}$ holds. Let $\mathcal{D}$ be a club subset of $[\mathcal{H}((\theta^{\aleph_0})^+)]^{<\kappa}$.

Fix a bijection $h : \mathcal{H}(\theta) \rightarrow \theta$. Let $\tilde{\mathfrak{A}} := \langle \mathcal{H}((\theta^{\aleph_0})^+), \in, \bar{D}^{\tilde{\mathfrak{A}}}, \bar{h}_0^{\tilde{\mathfrak{A}}} \rangle$, where $\bar{D}$ is a unary predicate symbol, $\bar{h}_0$ is a constant symbol, $\bar{D}^{\tilde{\mathfrak{A}}} := \mathcal{D}$ and $\bar{h}_0^{\tilde{\mathfrak{A}}} := h$.

By the assumption that $\theta = 2^{<\theta}$ and $\text{cf}(\theta) \geq \kappa$, we can find $N \in [\mathcal{H}((\theta^{\aleph_0})^+)]^\theta$ such that $\tilde{\mathfrak{A}} \upharpoonright N \prec \tilde{\mathfrak{A}}$, $\mathcal{H}(\theta) \subseteq N$ and $[N]^{<\kappa} \subseteq N$.

Let $g : N \rightarrow \mathcal{H}(\theta)$ be a bijection such that

$$g \upharpoonright [\theta]^{\leq \aleph_0} = \text{Id}_{[\theta]^{\leq \aleph_0}}. \quad (2.30)$$

Let $\mathfrak{N} := \langle g''N, \bar{E}^{\mathfrak{N}}, \bar{D}^{\mathfrak{N}}, \bar{h}_0^{\mathfrak{N}} \rangle$ be the translation by g of $\tilde{\mathfrak{A}} \upharpoonright N$, i.e. $\bar{E}$ is a relational symbol with $\bar{E}^{\mathfrak{N}} := \{\langle g(x), g(y) \rangle : x \in y \in N\}$, $\bar{D}^{\mathfrak{N}} := g''(\mathcal{D} \cap N)$ and $\bar{h}_0^{\mathfrak{N}} := g(h)$.

Define also the structure $\mathfrak{A} := \langle \mathcal{H}(\theta), \in, \bar{h}_1^{\mathfrak{A}}, \bar{N}^{\mathfrak{A}}, \bar{E}^{\mathfrak{A}}, \bar{D}^{\mathfrak{A}}, \bar{h}_0^{\mathfrak{A}} \rangle$, where $\bar{h}_1$ is a unary function symbol with $\bar{h}_1^{\mathfrak{A}} := h$ and $\langle \bar{N}^{\mathfrak{A}}, \bar{E}^{\mathfrak{A}}, \bar{D}^{\mathfrak{A}}, \bar{h}_0^{\mathfrak{A}} \rangle = \mathfrak{N}$.

For each $a \in \mathcal{H}(\theta)$, define:

$$S_a := \begin{cases} \{x \in [\mathcal{H}(\theta)]^{\aleph_0} : x \cap \theta \in g^{-1}(a)\}, & \text{if } g^{-1}(a) \text{ is a stationary subset of } [\theta]^{\aleph_0}; \\ [\mathcal{H}(\theta)]^{\aleph_0}, & \text{otherwise} \end{cases} \quad (2.31)$$

Notice that, by corollary 1.5(ii), S_a is a stationary subset of $[\mathcal{H}(\theta)]^{\aleph_0}$ for all $a \in \mathcal{H}(\theta)$.

By $(*)_{<\kappa, \theta}^{+C}$ we can take $M_0 \in [\mathcal{H}(\theta)]^{<\kappa}$ such that

$$M_0 \in \mathcal{C}; \quad (2.32)$$

$$\mathfrak{A} \upharpoonright M_0 \prec \mathfrak{A}; \quad (2.33)$$

$$\kappa \cap M_0 < \kappa; \text{ and} \quad (2.34)$$

$$\forall a \in M_0, S_a \cap [M_0]^{\aleph_0} \text{ is a stationary subset of } [M_0]^{\aleph_0}. \quad (2.35)$$

Let $M = g^{-1}''M_0$. We show that M witnesses $\text{DRP}(<\kappa, \theta, \mathcal{C})$.

Applying Vaught's test, it follows that

$$\tilde{\mathfrak{A}} \upharpoonright M \prec \tilde{\mathfrak{A}} \upharpoonright N. \quad (2.36)$$

We also have:

Claim 2.16.1. $M \in \mathcal{D}$

Proof of claim 2.16.1. First we prove the following:

$$\forall D \in \mathcal{D} \cap M, D \subseteq M; \quad (2.37)$$

$$\mathcal{D} \cap M \text{ is an unbounded subset of } [M]^{<\kappa}. \quad (2.38)$$

Let $D \in \mathcal{D} \cap M$. Since $|D| < \kappa$, by (2.36), there are $\delta \in \kappa \cap M$ and a surjection $f : \delta \rightarrow D$ with $f \in M$. By (2.30) and (2.34), we have $\kappa \cap M < \kappa$, therefore $\delta \subseteq M$. Then $D \subseteq M$, so (2.37) holds. (2.38) follows from (2.36) and the fact that $\mathcal{D} \cap N$ is a club subset of $[N]^{<\kappa}$.

From (2.37), we have that $\bigcup(\mathcal{D} \cap M) \subseteq M$. On the other hand, from (2.38) we have $\bigcup(\mathcal{D} \cap M) \supseteq M$. Therefore $M = \bigcup(\mathcal{D} \cap M)$. Furthermore, it follows from (2.38) that $\mathcal{D} \cap M$ is a directed set. Since $|M| = |M_0| < \kappa$, by proposition 1.3, we conclude that $M \in \mathcal{D}$. (claim 2.16.1) $\square$

By (2.32), the next claim implies that M satisfies (1) from definition 2.14.

Claim 2.16.2. $M \cap \mathcal{H}(\theta) = M_0$.

Proof. Let $x \in M_0$. By elementarity, we have $h(x) = \overline{h_1}^{\mathfrak{A} \upharpoonright M_0}(x) \in M_0$. By (2.30), we have $h^{-1}(g^{-1}(h(x))) = x$. Since $h^{-1}, g^{-1}(h(x)) \in M$, we have $x \in M$. Since $M_0 \subseteq \mathcal{H}(\theta)$, we have $x \in M \cap \mathcal{H}(\theta)$, therefore $M_0 \subseteq M \cap \mathcal{H}(\theta)$.

Now let $y \in M \cap \mathcal{H}(\theta)$. Since $h \in M$ and $y \in \text{dom}(h)$, we have that $h(y) \in M \cap \theta = M_0 \cap \theta$. Denote $z := h(x) = g(h(x))$. Similarly to the case above, y can be described in $\mathfrak{A} \upharpoonright M_0$ as the unique element of M_0 such that $\overline{h_1}^{\mathfrak{A} \upharpoonright M_0}(y) = z$. Therefore $y \in M_0$. (claim 2.16.2) $\square$

We conclude by showing that M satisfies (2) from definition 2.14. Let $R \in M$ be a stationary subset of $[\theta]^{\aleph_0}$. By the definition of M we have $g(R) \in M_0$. By (2.31) we have

$$S_{g(R)} = \{x \in [\mathcal{H}(\theta)]^{\aleph_0} : x \cap \theta \in R\}.$$

By (2.35), we have that $S_{g(R)} \cap [M_0]^{\aleph_0}$ is a stationary subset of $[M_0]^{\aleph_0}$. Notice that

$$R \cap [M \cap \theta]^{\aleph_0} = \{x \cap M \cap \theta : x \in S_{g(R)} \cap [M_0]^{\aleph_0} \wedge |x \cap M \cap \theta| = \aleph_0\}.$$

Recall that $M \cap \theta = M_0 \cap \theta \subseteq M_0$ by (2.30). Therefore, by corollary 1.5(iv), it follows that $R \cap [M \cap \theta]^{\aleph_0}$ is a stationary subset of $[M \cap \theta]^{\aleph_0}$. (lemma 2.16) $\square$

Lemma 2.17. *Suppose $\mathcal{C}$ is one of $\text{IU}_{\aleph_0}$, $\text{IS}_{\aleph_0}$, $\text{IC}_{\aleph_0}$. Given any $\kappa > \aleph_1$, $\text{DRP}(< \kappa, \mathcal{C})$ is equivalent to $(*)_{< \kappa}^{+\mathcal{C}}$.*

Proof. Given any cardinal η above κ , let $\theta_\eta = \sup\{2^\xi : \xi < \eta^+\}$. Since $2^{< \theta_\eta} = \theta_\eta$ and $\text{cf}(\theta_\eta) \geq \kappa$, “ $\text{DRP}(< \kappa, \mathcal{C})$ is equivalent to $(*)_{< \kappa}^{+\mathcal{C}}$ ” follows from lemmas 2.16, 2.15 and 2.12. (lemma 2.17) $\square$

Corollary 2.18 follows from lemma 2.13 and lemma 2.17.

Corollary 2.18. *Suppose $\kappa > \aleph_1$ is a regular cardinal. Then*

- $\text{SDLS}_+^-(\mathcal{L}_{\text{stat}}^{\aleph_0}, < \kappa) \Leftrightarrow \text{DRP}(< \kappa, \text{IC}_{\aleph_0})$;
- $\text{SDLS}_+(\mathcal{L}_{\text{stat}}^{\aleph_0, II}, < \kappa) \Leftrightarrow \text{DRP}(< \kappa, \text{IC}_{\aleph_0})$ and $\mu^{\aleph_0} < \kappa$ for all $\mu < \kappa$.

2.3 Internal variation of SDLS

Define recursively $\models^{\text{int}}$ as follows:

- at first order, $\models^{\text{int}}$ is defined like $\models$;
- for a structure $\mathfrak{A} = \langle A, \dots \rangle$, $a \in A$ and $U \in [A]^{\aleph_0} \cap A$, define:

$$\mathfrak{A} \models^{\text{int}} a \varepsilon U \Leftrightarrow a \in U;$$

- for a structure $\mathfrak{A} = \langle A, \dots \rangle$, given an $\mathcal{L}_{\text{stat}}^{\aleph_0}$ -formula φ , suppose $\models^{\text{int}} \varphi(a_0, \dots, a_n, A_0, \dots, A_m)$ is defined for all $a_0, \dots, a_n \in A$ and all $A_0, \dots, A_m \in [A]^{\aleph_0} \cap A$. Then we define

$$\begin{aligned} \mathfrak{A} \models^{\text{int stat}} X\varphi(X, a_0, \dots, a_n, A_0, \dots, A_m) &\Leftrightarrow \\ \{B \in [A]^{\aleph_0} \cap A : \mathfrak{A} \models^{\text{int}} \varphi(B, a_0, \dots, a_n, A_0, \dots, A_m)\} &\text{ is stationary in } [A]^{\aleph_0} \cap A. \end{aligned} \quad (2.39)$$

As before, for $\mathfrak{A} = \langle A, \dots \rangle$, $\mathfrak{B} = \langle B, \dots \rangle$ ($B \subseteq A$), we say that $\mathfrak{B} \prec_{\mathcal{L}_{\text{stat}}^{\aleph_0}}^{\text{int}} \mathfrak{A}$ if and only if for all formulas $\varphi(x_0, \dots, x_n, X_0, \dots, X_m)$ in $\mathcal{L}_{\text{stat}}^{\aleph_0}$, for all $b_0, \dots, b_n \in B$, and all $B_0, \dots, B_m \in [B]^{\aleph_0} \cap B$, we have

$$\mathfrak{B} \models^{\text{int}} \varphi(b_0, \dots, b_n, B_0, \dots, B_m) \Leftrightarrow \mathfrak{A} \models^{\text{int}} \varphi(b_0, \dots, b_n, B_0, \dots, B_m).$$

Definition 2.19. For a cardinal κ , we define $\text{SDLS}_+^{\text{int}}(\mathcal{L}_{\text{stat}}^{\aleph_0}, < \kappa)$ as the statement: For any structure $\mathfrak{A} = \langle A, \dots \rangle$ of countable signature, there are stationarily many $B \in [A]^{<\kappa}$ such that $\mathfrak{A} \restriction B \prec_{\mathcal{L}_{\text{stat}}^{\aleph_0}}^{\text{int}} \mathfrak{A}$.

Definition 2.20. Let κ be a regular uncountable cardinal. For $\eta > \kappa$, define $(*)_{<\kappa, \eta}^{\text{int}}$ as the following statement:

For any countable expansion $\mathfrak{A}$ of $\langle \mathcal{H}(\eta), \in \rangle$ and any family $\langle S_a : a \in \mathcal{H}(\eta) \rangle$ of stationary subsets of $[\mathcal{H}(\eta)]^{\aleph_0}$, there is some $N \in [\mathcal{H}(\eta)]^{<\kappa}$ such that:

- (1) $\mathcal{A} \restriction N \prec \mathcal{A}$;
- (2) for each $a \in N$, $S_a \cap N$ is a stationary subset of $[N]^{\aleph_0}$.

Similarly to $(*)_{<\kappa, \eta}^{\mathcal{C}}$, we also define the following strengthening of $(*)_{<\kappa, \eta}^{\text{int}}$:

Definition 2.21. Let κ be a regular uncountable cardinal. For $\eta > \kappa$, define $(*)_{<\kappa, \eta}^{+\text{int}}$ as the following statement:

For any countable expansion $\mathfrak{A}$ of $\langle \mathcal{H}(\eta), \in \rangle$ and any family $\langle S_a : a \in \mathcal{H}(\eta) \rangle$ of stationary subsets of $[\mathcal{H}(\eta)]^{\aleph_0}$, there are *stationarily many* $N \in [\mathcal{H}(\eta)]^{<\kappa}$ such that:

- (1) $\mathcal{A} \restriction N \prec \mathcal{A}$;
- (2) for each $a \in N$, $S_a \cap N$ is a stationary subset of $[N]^{\aleph_0}$.

Lemma 2.22. For a regular cardinal κ and cardinals η, η' with $\aleph_1 < \kappa < \eta < \eta'$, $(*)_{<\kappa, \eta'}^{\text{int}}$ implies $(*)_{<\kappa, \eta}^{\text{int}}$ and $(*)_{<\kappa, \eta'}^{+\text{int}}$ implies $(*)_{<\kappa, \eta}^{+\text{int}}$.

Lemma 2.22 can be proved similarly to the proof of lemma 2.12 with no significant modifications.

Lemma 2.23. For any regular $\kappa > \aleph_1$, we have $\text{SDLS}_+^{\text{int}}(\mathcal{L}_{\text{stat}}^{\aleph_0}, < \kappa) \Leftrightarrow (*)_{<\kappa, \eta}^{+\text{int}}$ holds for all regular $\eta \geq \kappa$.

Proof. This proof is just a modification of the proof of lemma 2.13.

First assume $\text{SDLS}_+^{\text{int}}(\mathcal{L}_{\text{stat}}^{\aleph_0}, < \kappa)$. Let $\eta > \kappa$ and let $\mathcal{A}$ be a countable expansion of $\langle \mathcal{H}(\eta), \in \rangle$, $\langle S_a : a \in \mathcal{H}(\eta) \rangle$ be a sequence of stationary subsets of $[\mathcal{H}(\eta)]^{\aleph_0}$ and $\mathcal{C}$ a club subset of $[\mathcal{H}(\eta)]^{<\kappa}$.

Let $\tilde{\mathfrak{A}}$ be the expansion of $\mathfrak{A}$ obtained by adding the binary relation symbol $\bar{S}$ (used to translate the sequence $\langle S_a : a \in \mathcal{H}(\eta) \rangle$ in the structure) with the following interpretation:

$$\bar{S}^{\tilde{\mathfrak{A}}} := \{ \langle a, s \rangle \in (\mathcal{H}(\eta))^2 : s \in S_a \}.$$

By $\text{SDLS}_+^{\text{int}}(\mathcal{L}_{\text{stat}}^{\aleph_0}, < \kappa)$, we can take $M \in [\mathcal{H}(\eta)]^{<\kappa}$ such that $M \in \mathcal{C}$ and $\tilde{\mathfrak{A}} \upharpoonright M \prec_{\mathcal{L}_{\text{stat}}^{\aleph_0}}^{\text{int}} \tilde{\mathfrak{A}}$.

Let ψ be the sentence $\forall x \text{ stat } X \exists y (\bar{S}(x, y) \wedge \forall z (z \in X \leftrightarrow z \in y))$. By the definition of $\models^{\text{int}}$, for any uncountable set B we have

$$B \models^{\text{int}} \psi \Leftrightarrow "S_b \cap B \text{ is a stationary subset of } [B]^{\aleph_0} \text{ for all } b \in B".$$

Then we have $\tilde{\mathfrak{A}} \models \psi$. Since $\tilde{\mathfrak{A}} \upharpoonright M \prec_{\mathcal{L}_{\text{stat}}^{\aleph_0}}^{\text{int}} \tilde{\mathfrak{A}}$, we have that $S_a \cap M$ is a stationary subset of $[M]^{\aleph_0}$ for all $a \in M$. Therefore M witnesses $(*)_{<\kappa, \eta}^{\text{int}+}$.

Now assume $(*)_{<\kappa, \eta}^{\text{int}+}$ holds for every $\eta \geq \kappa$. We prove that $\text{SDLS}_+^{\text{int}}(\mathcal{L}_{\text{stat}}^{\aleph_0}, < \kappa)$ holds. Let $\mathfrak{A} = \langle A, \bar{R}_0, \dots, \bar{R}_n, \dots \rangle$ be a structure in countable signature with $|A| \geq \kappa$. Let $\mathcal{C} \subseteq [A]^{<\kappa}$ be a club.

Take a regular cardinal η such that $\mathfrak{A} \in \mathcal{H}(\eta)$ and define $\tilde{\mathfrak{A}} := \langle \mathcal{H}(\eta), \in, \bar{\mathfrak{A}}^{\tilde{\mathfrak{A}}}, \bar{A}^{\tilde{\mathfrak{A}}}, \bar{R}_0^{\tilde{\mathfrak{A}}}, \dots, \bar{R}_n^{\tilde{\mathfrak{A}}}, \dots \rangle$, where $\bar{\mathfrak{A}}$ and $\bar{A}$ are unary relation symbols with $\bar{\mathfrak{A}}^{\tilde{\mathfrak{A}}} = \mathfrak{A}$, $\bar{A}^{\tilde{\mathfrak{A}}} = A$ and $\bar{R}_n^{\tilde{\mathfrak{A}}} = R_n^{\mathfrak{A}}$ for every $n \in \omega$. For each $a \in \mathcal{H}(\eta)$, define:

$$S_a := \begin{cases} \{ U \in [\mathcal{H}(\eta)]^{\aleph_0} : |U \cap A| = \aleph_0 \wedge U \cap A \in A \wedge \\ \quad \mathfrak{A} \models^{\text{int}} \psi(a_0, \dots, a_m, U_0, \dots, U_l, U \cap A) \}, \\ \text{if } \psi(x_0, \dots, x_m, Y_0, \dots, Y_l, X) \text{ is an } \mathcal{L}_{\text{stat}}^{\aleph_0}\text{-formula,} \\ \quad a_0, \dots, a_m \in A, U_0, \dots, U_l \in [A]^{\aleph_0} \cap A, \\ \quad \mathfrak{A} \models^{\text{stat}} \text{stat } X \psi(a_0, \dots, a_m, U_0, \dots, U_l, X) \text{ and} \\ \quad a = \langle \psi, a_0, \dots, a_m, U_0, \dots, U_l \rangle; \\ [\mathcal{H}(\eta)]^{\aleph_0}, \\ \text{otherwise.} \end{cases} \quad (2.40)$$

Let

$$\tilde{\mathcal{C}} := \{ U \in [\mathcal{H}(\eta)]^{<\kappa} : U \cap A \in \mathcal{C} \}. \quad (2.41)$$

By lemma 1.4(I), $\tilde{\mathcal{C}}$ contains a club subset of $[\mathcal{H}(\eta)]^{<\kappa}$. By $(*)_{<\kappa,\eta}^{\text{int}+}$, we can take $M \in [\mathcal{H}(\eta)]^{<\kappa}$ such that

$$M \in \tilde{\mathcal{C}}; \quad (2.42)$$

$$\tilde{\mathfrak{A}} \restriction M \prec \tilde{\mathfrak{A}}; \text{ and} \quad (2.43)$$

$$S_a \cap M \text{ is a stationary subset of } [M]^{\aleph_0} \text{ for all } a \in M. \quad (2.44)$$

Let $B := A \cap M$. By (2.41) and (2.42), we have $B \in \mathcal{C}$. It follows from (2.43) and the following claim that $\mathfrak{A} \restriction B \prec_{\mathcal{L}_{\text{stat}}^{\aleph_0}}^{\text{int}} \mathfrak{A}$.

Claim 2.23.1. *For any $\mathcal{L}_{\text{stat}}^{\aleph_0}$ -formula $\varphi(x_0, \dots, x_m, Y_0, \dots, Y_l)$, $a_0, \dots, a_m \in B$ and $U_0, \dots, U_l \in [B]^{\aleph_0} \cap B$, we have*

$$\begin{aligned} \tilde{\mathfrak{A}} \restriction M \models \text{“} \overline{\mathfrak{A}} \models^{\text{int}} \varphi(a_0, \dots, a_m, U_0, \dots, U_l) \text{”} &\Leftrightarrow \\ \mathfrak{A} \restriction B \models^{\text{int}} \varphi(a_0, \dots, a_m, U_0, \dots, U_l) &\quad (2.45) \end{aligned}$$

Proof of claim 2.23.1. We proceed by induction on the formula φ . We prove that (2.45) holds for φ whenever it is of the form $\text{stat } X\psi$ and (2.45) holds for ψ . The remaining cases follow straightforward from the definitions.

Suppose that $\tilde{\mathfrak{A}} \restriction M \models \text{“} \overline{\mathfrak{A}} \models^{\text{int}} \varphi(a_0, \dots, a_m, U_0, \dots, U_l) \text{”}$ holds. Then, by (2.43) we have $\mathfrak{A} \models^{\text{int}} \text{stat } X\psi(a_0, \dots, a_m, U_0, \dots, U_l, X)$. Define $a := \langle \psi, a_0, \dots, a_m, U_0, \dots, U_l \rangle$. By (2.43) it follows that $a \in M$ and by (2.40) we have

$$\begin{aligned} S_a = \{U \in [\mathcal{H}(\eta)]^{\aleph_0} : |U \cap A| = \aleph_0 \wedge U \cap A \in A \wedge \\ \mathfrak{A} \models^{\text{int}} \psi(a_0, \dots, a_m, U_0, \dots, U_l, U \cap A)\}. \end{aligned} \quad (2.46)$$

By (2.44), we have that $S_a \cap M$ is a stationary subset of $[M]^{\aleph_0}$. By corollary 1.5(iv), the set $\mathfrak{S} := \{U \cap B : U \in S_a \cap M \wedge |U \cap B| = \aleph_0\}$ is a stationary subset of $[M]^{\aleph_0}$. By (2.46) and (2.43), we have

$$\begin{aligned} \mathfrak{S} := \{U \cap B : U \in M \cap [M]^{\aleph_0} \wedge U \cap A = U \cap B \in A \cap [B]^{\aleph_0} \wedge \\ \tilde{\mathfrak{A}} \restriction M \models \text{“} \overline{\mathfrak{A}} \models^{\text{int}} \psi(a_0, \dots, a_m, U_0, \dots, U_l, U \cap B) \text{”}\} \end{aligned}$$

By induction hypothesis, we have

$$\mathfrak{S} \subseteq \{V \in [B]^{\aleph_0} \cap B : \mathfrak{A} \restriction B \models^{\text{int}} \psi(a_0, \dots, a_m, U_0, \dots, U_l, V)\}.$$

Therefore, by (2.39) we have $\mathfrak{A} \restriction B \models^{\text{int}} \text{stat } X\psi(a_0, \dots, a_m, U_0, \dots, U_l, X)$, that is, $\mathfrak{A} \restriction B \models^{\text{int}} \varphi(a_0, \dots, a_m, U_0, \dots, U_l)$.

Suppose now that $\tilde{\mathfrak{A}} \restriction M \not\models \text{“}\overline{\mathfrak{A}} \models^{\text{int}} \varphi(a_0, \dots, a_m, U_0, \dots, U_l)\text{”}$. Then we have

$$\begin{aligned} \tilde{\mathfrak{A}} \restriction M &\models \text{“there is a club } \mathcal{D} \subseteq [\overline{A}]^{\aleph_0} \text{ such that} \\ &\quad \overline{\mathfrak{A}} \models^{\text{int}} \neg\psi(a_0, \dots, a_m, U_0, \dots, U_l, x) \text{ for all } x \in \mathcal{D}\text{”} \end{aligned}$$

By (2.43), there is $\mathcal{D}' \in M$ which is a club subset of $[A]^{\aleph_0}$ such that

$$\tilde{\mathfrak{A}} \restriction M \models \text{“}\overline{\mathfrak{A}} \models^{\text{int}} \neg\psi(a_0, \dots, a_m, U_0, \dots, U_l, V)\text{” for all } V \in \mathcal{D}' \cap B.$$

By induction hypothesis, we have

$$\{U \in [B]^{\aleph_0} \cap B : \mathfrak{A} \restriction B \models^{\text{int}} \psi(a_0, \dots, a_m, U_0, \dots, U_l, U)\} \cap \mathcal{D}' = \emptyset$$

Therefore we have $\mathfrak{A} \restriction B \not\models^{\text{int}} \text{stat } X\psi(a_0, \dots, a_m, U_0, \dots, U_l, X)$, i.e. $\mathfrak{A} \restriction B \not\models \varphi(a_0, \dots, a_m, U_0, \dots, U_l)$. (claim 2.23.1) $\square$

(lemma 2.23) $\square$

In section 2.5 we will show that a generically large cardinal property on κ implies $\text{SDLS}_+^{\text{int}}(\mathcal{L}_{\text{stat}}^{\aleph_0}, < \kappa)$ (theorem 2.35). It follows from this result that $\text{SDLS}_+^{\text{int}}(\mathcal{L}_{\text{stat}}^{\aleph_0}, < \kappa)$ is consistent with $2^{\aleph_0} = \kappa$ for κ supercompact. In chapter 3, we shall construct a model where $\text{SDLS}_+^{\text{int}}(\mathcal{L}_{\text{stat}}^{\aleph_0}, < 2^{\aleph_0})$ holds simultaneously with a much stronger reflection principle down to $\leq 2^{\aleph_0}$, which will be introduced in the next section.

2.4 Game Reflection principle

In this and the following sections we discuss some generalizations of results originally in [Kön04] and relate them to $\text{SDLS}_+(\mathcal{L}_{\text{stat}}^{\aleph_0, \text{II}}, < \kappa)$. Most of the ideas in these sections are already present in [Kön04].

Given an uncountable set A and some $\mathcal{A} \subseteq {}^{\leq \omega} A$, define the game $\mathcal{G}^{\leq \omega A}(\mathcal{A})$ of length ω played between players I and II as follows. Each player takes turns choosing elements of A as follows:

$$\begin{array}{c|cccc} \text{I} & a_0 \in A & \cdots & a_n \in A & \cdots \\ \hline \text{II} & b_0 \in A & \cdots & b_n \in A & \cdots \end{array} \quad (n \in \omega)$$

Player II wins if and only if

- $\langle a_n, b_n : n < m \rangle \in \mathcal{A} \wedge \langle a_n, b_n : n < m \rangle \frown \langle a_m \rangle \notin \mathcal{A}$ for some $m \in \omega$;
or

- $\langle a_n, b_n : n \in \omega \rangle \in \mathcal{A}$

where $\langle a_n, b_n : n \in \omega \rangle$ denotes the sequence $\langle c_m : m \in \omega \rangle$ given by $c_{2m} := a_m$ and $c_{2m+1} := b_m$ for every $m \in \omega$.

Definition 2.24. For a regular uncountable cardinal κ and a set A with regular cardinality $\geq \kappa$, let $\text{GRP}^\omega(< \kappa, A)$ be the following statement:

For any ω_1 -club $\mathcal{C} \subseteq [A]^{<\kappa}$ and any $\mathcal{A} \subseteq {}^\omega A$, if player II has no winning strategy in $\mathcal{G}^{\leq \omega A(\mathcal{A})}$, then there is some $B \in \mathcal{C}$ such that player II has no winning strategy in $\mathcal{G}^{\leq \omega B}(\mathcal{A} \cap {}^\omega B)$.

We denote by $\text{GRP}^\omega(< \kappa)$ the statement “ $\text{GRP}^\omega(< \kappa, A)$ holds for every set A of regular cardinality $\geq \kappa$ ”.

In definition 2.24, by *winning strategy* we mean the following: Suppose $\mathcal{G}$ is a game played by players I and II of length δ where the players take turns choosing elements of a set A . A *winning strategy* for player II is a function $\sigma : {}^{<\delta} A \rightarrow A$ such that for any $\langle a_\xi : \xi < \delta \rangle \in {}^\delta A$ player II wins the match

$$\begin{array}{c|cccc} \text{I} & a_0 & \cdots & a_\xi & \cdots \\ \hline \text{II} & \sigma(\langle a_0 \rangle) & \cdots & \sigma(\langle a_\zeta : \zeta \leq \xi \rangle) & \cdots \end{array} \quad (\xi < \delta).$$

Notice that the value of $\sigma(s)$ is not relevant for player II whenever $\text{dom}(s)$ is not a successor ordinal.

Lemma 2.25. For any regular cardinal $\kappa > \aleph_1$, $\text{GRP}^\omega(< \kappa)$ implies $2^{\aleph_0} < \kappa$.

In order to prove lemma 2.25, we shall use a *Bernstein set*. A set $B \subseteq {}^\omega 2$ is a Bernstein set if it does not contain any perfect set. The following proposition is a basic well known fact.

Proposition 2.26. There exists a Bernstein set with cardinality $2^{\aleph_0}$.

Proof lemma 2.25. Assume $\text{GRP}^\omega(< \kappa)$ holds. Towards a contradiction, assume $2^{\aleph_0} \geq \kappa$. By proposition 2.26, we can take a Bernstein set $B \subset {}^\omega 2$ with $|B| = \kappa$ (because every subset of a Bernstein set is clearly a Bernstein set too). Let $\langle r_\xi : \xi < \kappa \rangle$ be an injective enumeration of B .

Let $A := {}^{<\omega} 2 \cup \kappa$ and $\mathcal{A} := \mathcal{A}_0 \cup \mathcal{A}_1 \subseteq {}^\omega A$ where

$$\mathcal{A}_0 := \{ \langle c_m : m < k \rangle \in {}^{<\omega} A : c_m \in \kappa \text{ whenever } m \text{ is even} \};$$

$$\mathcal{A}_1 := \{ \langle a_n, b_n : n \in \omega \rangle \in {}^\omega A : a_n \in \kappa \text{ for all } n \in \omega, \langle b_n : n \in \omega \rangle \text{ is a}$$

$$\subseteq\text{-increasing sequence in } {}^{<\omega} 2 \text{ and } \bigcup_{n \in \omega} b_n = r_\xi \text{ for some}$$

$$\xi > \sup\{a_n : n \in \omega\}\}.$$

Let $\mathcal{C} := \{C \in [A]^{<\kappa} : {}^{<\omega}2 \subseteq C\}$. $\mathcal{C}$ is a ω_1 -club subset of $[A]^{<\kappa}$. Let $C \in \mathcal{C}$ be arbitrary. We can construct a winning strategy for II in the game $\mathcal{G}^{\leq \omega C}(\mathcal{A} \cap {}^{\leq \omega}C)$: Since $|C| < \kappa$, we can take $\xi < \kappa$ such that $\xi > \sup(\kappa \cap C)$. Regardless of player I moves, if player II plays $b_n := r_\xi \restriction n$ for every $n \in \omega$, then player II wins the match. Therefore, by $\text{GRP}^\omega(< \kappa)$, there is a winning strategy σ for II in the game $\mathcal{G}^{\leq \omega A}(\mathcal{A})$.

Now we construct sequences $\langle \ell_n : n \in \omega \rangle$ and $\langle t_s : s \in {}^{<\omega}2 \rangle$ such that

- (a) $\langle \ell_n : n \in \omega \rangle$ is a strictly increasing sequence in ω ;
- (b) for all $n \in \omega$ and all $s \in {}^n 2$, $t_s \in {}^{2\ell_n} A$;
- (c) for each $s \in {}^n 2$, $t_s = \langle a_k, b_k : k < \ell_n \rangle$ is an initial segment of a match of the game $\mathcal{G}^{\leq \omega A}(\mathcal{A})$ where player II plays according to σ and player I always plays avoiding instant defeat;
- (d) $t_s \subseteq t_{s'}$ whenever $s \subseteq s'$ ($s, s' \in {}^{<\omega}2$); and
- (e) for all $n \in \omega$ and $s \in {}^n 2$, $u_{s \restriction 0}$ and $u_{s \restriction 1}$ are distinct, where we define $u_s := \langle b_k : k < \ell_n \rangle$.

Notice that condition (e) is realizable because σ is a winning strategy: suppose there is some $\langle a_n, b_n : n \leq m_0 \rangle$ an initial segment of a match in $\mathcal{G}^{\leq \omega A}(\mathcal{A})$ (where II plays according to σ) such that for all possible follow up moves by player I, σ always suggests the same moves for player II. Since in this situation the moves by player II do not depend on the moves by player I after m_0 , we can define $r_\xi := \bigcup_{n > m_0} b_n$. But then player I would be able to win by playing some $a_n > \xi$ for some $n > m$.

For each $f \in {}^\omega 2$, define $\mathbf{t}_f := \bigcup_{n \in \omega} t_{f \restriction n}$ and $\mathbf{u}_f := \bigcup_{n \in \omega} u_{f \restriction n}$. By (a), (c) and (d), $\mathbf{t}_f = \langle a_n, b_n : n \in \omega \rangle$ is a match in the game $\mathcal{G}^{\leq \omega A}(\mathcal{A})$. Since player II wins the match $\mathbf{t}_f$, we have $\mathbf{u}_f \in B$. Notice that, by (b) and (e), $u_{s \restriction 0}$ and $u_{s \restriction 1}$ are incompatible for every $s \in {}^{<\omega}2$. Therefore $\{\mathbf{u}_f : f \in {}^\omega 2\} \subseteq B$ is a perfect set, which contradicts B being a Bernstein set. (lemma 2.25) $\square$

Lemma 2.27. *Suppose κ is a regular uncountable cardinal such that $\gamma^{\aleph_0} < \kappa$ for all $\gamma < \kappa$. Then $\text{GRP}^\omega(< \kappa)$ implies $(*)_{<\kappa}^{+\text{IC}_{\aleph_0}}$.*

Proof. Let $\eta > \kappa$ and let $\langle S_a : a \in \mathcal{H}(\eta) \rangle$ be such that S_a is a stationary subset of $[\mathcal{H}(\eta)]^{\aleph_0}$ for each $a \in \mathcal{H}(\eta)$. Let $A = \mathcal{H}(\eta)$ and define $\mathcal{A} \subseteq {}^{\leq \omega} A$ by

$$\mathcal{A} := {}^{<\omega} A \cup \{ \langle a_n, b_n : n \in \omega \rangle \in {}^\omega A : \{a_n : n \in \omega\} \cup \{b_n : 0 < n < \omega\} \notin S_{b_0} \}. \quad (2.47)$$

Claim 2.27.1. *Player II has no winning strategy in the game $\mathcal{G}^{\leq \omega A}(\mathcal{A})$.*

Proof of claim 2.27.1. Let $\sigma : {}^{<\omega}A \rightarrow A$. Let $a_0 \in A$ be an arbitrary first move for player I and let $b_0 = \sigma(\langle a_0 \rangle)$ be the response by player II following the strategy σ .

Since S_{b_0} is a stationary subset of $[\mathcal{H}(\eta)]^{\aleph_0}$, we can take $M \prec \mathcal{H}(\eta)$ such that $\{a_0, b_0, \sigma\} \subseteq M \in S_{b_0}$. Let $\{a_n : n \in \omega\}$ be an enumeration of M . For each $n \in \omega \setminus \{0\}$, let

$$b_n := \sigma(\langle a_i : i < n \rangle).$$

Since $\sigma \in M$, we have

$$\{a_n : n < \omega\} \cup \{b_n : 0 < n < \omega\} = M \in S_{b_0}.$$

By (2.47), we have that $\langle a_n, b_n : n < \omega \rangle \notin \mathcal{A}$. Therefore II loses the match $\langle a_n, b_n : n < \omega \rangle$, so $\langle a_n : n < \omega \rangle$ witnesses that σ is not a winning strategy for II. (claim 2.27.1) $\square$

Let $\mathfrak{A}$ be an arbitrary countable expansion of $\langle \mathcal{H}(\eta), \in \rangle$ and $\mathcal{C}$ be an arbitrary club subset of $[\mathcal{H}(\eta)]^{<\kappa}$. Define

$$\mathcal{D} := \{N \in [\mathcal{H}(\eta)]^{<\kappa} : N \in \mathcal{C} \wedge \mathfrak{A} \upharpoonright N \prec \mathfrak{A} \wedge [N]^{\aleph_0} \subseteq N\} \quad (2.48)$$

From the assumption that $\mu^{\aleph_0} < \kappa$ for all $\mu < \kappa$, it follows that $\mathcal{D}$ is an ω_1 -club. By $\text{GRP}^{\leq \omega}(< \kappa)$ there is some $N \in \mathcal{D}$ such that player II has no winning strategy in the game $\mathcal{G}^{\leq \omega N}(\mathcal{A} \cap {}^{\leq \omega}N)$. We prove that N witnesses $(*)_{<\kappa, \eta}^{+\text{IC}_{\aleph_0}}$.

By (2.48), we have $N \in \text{IC}_{\aleph_0} \cap \mathcal{C}$ and $\mathfrak{A} \upharpoonright N \prec \mathfrak{A}$. To complete the proof we still need to prove that, for all $a \in N$, $S_a \cap [N]^{\aleph_0}$ is a stationary subset of $[N]^{\aleph_0}$.

Towards a contradiction, suppose there is $a^* \in N$ such that $S_{a^*} \cap [N]^{\aleph_0}$ is not stationary in $[N]^{\aleph_0}$. We shall construct σ a winning strategy for II in the game $\mathcal{G}^{\leq \omega N}(\mathcal{A} \cap {}^{\leq \omega}N)$. Let $\mathfrak{C}$ be a club subset of $[N]^{\aleph_0}$ disjoint from S_{a^*} . Suppose player I starts a match by playing an arbitrary $a_0 \in N$. Player II then responds by playing $b_0 = \sigma(\langle a_0 \rangle) := a^*$.

Fix $\{\langle x_m, y_m \rangle : m < \omega\}$ an enumeration of $\omega \times \omega$ such that $x_m < m$ for every $m < \omega$.

Let $n \in \omega \setminus \{0\}$. At the n -th turn of the match, suppose player II has played so far $\langle a_i : i \leq n \rangle$ and player II has played $\langle b_i : i < n \rangle$, with $b_i = \sigma(\langle a_0, \dots, a_i \rangle)$ for each $i < n$. For each $k \leq n$, denote

$$B_k := \{a_i : i \leq k\} \cup \{b_i : 0 < i < k\}.$$

Suppose that for each $k \leq n$ we have chosen $C_k = \{c_{k,l} : l < \omega\}$ such that

$$\left(\bigcup_{i < k} C_i\right) \cup B_k \subseteq C_\kappa \in \mathfrak{C}.$$

This is possible since $\mathfrak{C}$ is a club subset of $[N]^{\aleph_0}$.

We can then define

$$b_n = \sigma(\langle a_i : i \leq n \rangle) := c_{x_n, y_n}.$$

Notice that $C := \bigcup_{n < \omega} C_n \in \mathfrak{C}$, thus $C \notin S_{\alpha^*}$. Clearly we have

$$\bigcup_{n < \omega} B_n = \{a_n : n < \omega\} \cup \{b_n : 0 < n < \omega\} = C$$

Therefore player II wins this match, so σ is a winning strategy for II. We thus conclude that $S_a \cap [N]^{\aleph_0}$ is a stationary subset of $[N]^{\aleph_0}$ for every $a \in N$. (lemma 2.27) $\square$

We shall now consider a version of the game $\mathcal{G}^{\leq \omega}(\mathcal{A})$ with length μ for any arbitrary regular cardinal $\mu \geq \aleph_1$.

Suppose μ is a regular uncountable cardinal. Given a set A and some $\mathcal{A} \subseteq {}^{<\mu}A$, define the game $\mathcal{G}^{<\mu A}(\mathcal{A})$ of length μ played between players I and II as follows. Each player takes turns choosing elements of A :

$$\begin{array}{c|cccc} \text{I} & a_0 \in A & \cdots & a_\xi \in A & \cdots \\ \hline \text{II} & b_0 \in A & \cdots & b_\xi \in A & \cdots \end{array} \quad (\xi < \mu)$$

Player II wins the match if and only if

- $\langle a_\xi, b_\xi : \xi < \eta \rangle \in \mathcal{A} \wedge \langle a_\xi, b_\xi : \xi < \eta \rangle \frown \langle a_\eta \rangle \notin \mathcal{A}$ for some $\eta < \mu$; or
- $\langle a_\xi, b_\xi : \xi < \mu \rangle \in [\mathcal{A}]$

where $\langle a_\xi, b_\xi : \xi < \eta \rangle$ denotes the sequence $\langle c_\iota : \iota < 2 \cdot \eta \rangle$ given by $c_{2 \cdot \iota} := a_\iota$ and $c_{2 \cdot \iota + 1} := b_\iota$ for every $\iota < 2 \cdot \eta$, and

$$[\mathcal{A}] := \{f \in {}^\mu A : \forall \alpha < \mu, f \restriction \alpha \in \mathcal{A}\}.$$

Lemma 2.28. *Let μ be a regular uncountable cardinal. Suppose P is a $< \mu$ -closed forcing poset. For any set A and any $\mathcal{A} \subseteq {}^{<\mu}A$ we have:*

(1) *if σ is a winning strategy for player II in $\mathcal{G}^{<\mu A}(\mathcal{A})$, then*

$$\Vdash_P \text{“ } \check{\sigma} \text{ is a winning strategy for player II in } \mathcal{G}^{<\mu \check{A}}(\check{\mathcal{A}}) \text{”};$$

(2) if $p \Vdash_P$ “there is a winning strategy for player II in $\mathcal{G}^{<\mu\check{A}}(\check{\mathcal{A}})$ ” for some $p \in P$, then there is $\sigma \in V$ a winning strategy for II in $\mathcal{G}^{<\mu A}(\mathcal{A})$.

Proof. (1): Since P is $<\mu$ -closed, it does not add any new initial segment of a match of $\mathcal{G}^{<\mu A}(\mathcal{A})$, therefore σ remains a winning strategy in V^P .

(2): Define $p_\emptyset := p$. Since P is $<\mu$ -closed, for each non empty $s \in {}^{<\mu}A$ we can choose $p_s \in P$ and $b_s \in A$ such that

$$p_s \Vdash_{\mathbb{P}} “\dot{\sigma}(\check{s}) = \check{b}_s”; \text{ and} \quad (2.49)$$

$$\text{for } s, s' \in {}^{<\mu}A, s \subseteq s' \text{ implies } p_{s'} \leq_{\mathbb{P}} p_s. \quad (2.50)$$

Let $\sigma' : {}^{<\gamma}A \longrightarrow A$ be such that $\sigma'(s) := b_s$ for all $s \in {}^{<\mu}A$. We prove that σ' is a winning strategy for II. Let $\langle a_\xi, b_\xi : \xi < \mu \rangle$ be a match of the game $\mathcal{G}^{<\mu A}(\mathcal{A})$ where II plays according to σ' (i.e. $b_n = \sigma'(\langle a_\iota : \iota \leq \xi \rangle)$ for each $\xi < \gamma$) and player I never makes a move which instantly leads to defeat. For each $\xi < \mu$, denote $s_\xi := \langle a_\iota : \iota \leq \xi \rangle$. By (2.49) and (2.50), for every $\xi < \mu$ we have

$$p_{s_\xi} \Vdash_P “\langle a_\iota, b_\iota : \iota < \xi \rangle \in \check{\mathcal{A}}”.$$

Therefore $\langle a_\iota, b_\iota : \iota < \xi \rangle \in \mathcal{A}$ for every $\xi < \mu$, so σ' is indeed a winning strategy for II. (lemma 2.28) $\square$

Definition 2.29. For regular uncountable cardinals μ and κ with $\mu < \kappa$ and a set A with regular cardinality $\geq \kappa$, let $\text{GRP}^{<\mu}(<\kappa, A)$ be the statement:

Given any $\mathcal{A} \subseteq {}^{<\mu}A$ and any μ -club $\mathcal{C} \subseteq [A]^{<\kappa}$, if player II has no winning strategy in $\mathcal{G}^{<\mu A}(\mathcal{A})$, then there is some $B \in \mathcal{C}$ such that player II has no winning strategy in $\mathcal{G}^{<\mu B}(\mathcal{A} \cap {}^{<\mu}B)$.

We define $\text{GRP}^{<\mu}(<\kappa)$ as the statement “ $\text{GRP}^{<\mu}(<\kappa, A)$ holds for every set A with $|A| \geq \kappa$ ”.

Lemma 2.30. Let $\mu < \mu' < \kappa$ be regular uncountable cardinals. Then $\text{GRP}^{<\mu'}(<\kappa)$ implies $\text{GRP}^{<\mu}(<\kappa)$. Furthermore, $\text{GRP}^{<\omega_1}(<\kappa)$ implies $\text{GRP}^\omega(<\kappa)$.

Proof. First we prove that $\text{GRP}^{<\mu'}(<\kappa)$ implies $\text{GRP}^{<\mu}(<\kappa)$. For any set A , $\mathcal{A} \subseteq {}^{<\mu}A$ and non-empty $B \subset A$, we shall denote

$$\tilde{\mathcal{A}}^B := \{f \in {}^{<\mu'}A : \forall \alpha < \mu, f \restriction \alpha \in \mathcal{A} \wedge \forall \beta \in \text{dom}(f) \setminus \mu, f(\beta) \in B\}. \quad (2.51)$$

Claim 2.30.1. Given any non-empty $B \subset A$, player II has a winning strategy for the game $\mathcal{G}^{<\mu A}(\mathcal{A})$ if and only if II has a winning strategy in the game $\mathcal{G}^{<\mu' A}(\tilde{\mathcal{A}}^B)$.

Proof of claim 2.30.1. Notice that, by (2.51), the outcome of a match of the game $\mathcal{G}^{<\mu' A}(\tilde{\mathcal{A}}^B)$ is already decided after μ many turns (assuming neither of the players intentionally loses). Then it is clear that for any function $\sigma : <\mu' A \rightarrow A$ we have: σ is a winning strategy for the game $\mathcal{G}^{<\mu' A}(\tilde{\mathcal{A}}^B)$ if and only if $\sigma \upharpoonright (<\mu A)$ is a winning strategy for the game $\mathcal{G}^{<\mu A}(\mathcal{A})$. (claim 2.30.1) $\square$

Assume $\text{GRP}^{<\mu'}(<\kappa)$. Let A be a set with $|A| \geq \kappa$ and $\mathcal{C} \subseteq [A]^{<\mu}$ be an arbitrary μ -club. Suppose $\mathcal{A} \subseteq <\mu A$ is such that II has no winning strategy in the game $\mathcal{G}^{<\mu A}(\mathcal{A})$. Then by claim 2.30.1 player II has no winning strategy in the game $\mathcal{G}^{<\mu' A}(\tilde{\mathcal{A}}^A)$.

Since $\mu < \mu'$ and $\mathcal{C}$ is a μ -club, we have in particular that it is also a μ' -club (see definition 1.7). Then, by $\text{GRP}^{<\mu'}(<\kappa)$, there is $B \in [A]^{<\mu'}$ such that player II has no winning strategy in the game $\mathcal{G}^{<\mu' B}(\tilde{\mathcal{A}}^A \cap <\mu' B)$. Notice that $\tilde{\mathcal{A}}^A \cap <\mu' B = \widetilde{\mathcal{A} \cap <\mu B}^B$, therefore by claim 2.30.1 player II has a winning strategy for the game $\mathcal{G}^{<\mu A}(\mathcal{A} \cap <\mu B)$.

Suppose II has a winning strategy $\sigma : <\mu B \rightarrow B$ for the game $\mathcal{G}^{<\mu B}(\mathcal{A} \cap <\mu B)$. Then any function $\sigma' : <\mu A \rightarrow A$ such that $\sigma' \upharpoonright (<\mu B) = \sigma$ is a winning strategy for II in the game $\mathcal{G}^{<\mu A}(\mathcal{A} \cap <\mu B)$. Indeed, in the game $\mathcal{G}^{<\mu A}(\mathcal{A} \cap <\mu B)$ player I is forced to always make a move in B , because otherwise player II would instantly win the match. Therefore in this case player II can always counter by playing according to σ . We conclude that player II has no winning for the game $\mathcal{G}^{<\mu B}(\mathcal{A} \cap <\mu B)$.

The proof of “ $\text{GRP}^{<\omega_1}(<\kappa)$ implies $\text{GRP}^\omega(<\kappa)$ ” is similar. (lemma 2.30) $\square$

By lemma 2.30 and lemma 2.25, it follows that:

Corollary 2.31. *For any regular uncountable μ, κ , with $\mu < \kappa$, $\text{GRP}^{<\mu}(<\kappa)$ implies $2^{\aleph_0} < \kappa$.*

Similarly, by lemma 2.30 and lemma 2.27 we have:

Corollary 2.32. *Suppose κ is a regular uncountable cardinal such that $\gamma^{\aleph_0} < \kappa$ for all $\gamma < \kappa$. Then for any regular cardinal uncountable $\mu < \kappa$, $\text{GRP}^{<\mu}(<\kappa)$ implies $\text{SDLS}_+(\mathcal{L}_{\text{stat}}^{\aleph_0, II}, <\kappa)$.*

In section 2.5, we shall show that $\text{GRP}^{<\kappa}(<\kappa^+)$ can be characterized as a *generically large* cardinal property (namely, $\text{GRP}^{<\kappa}(<\kappa^+)$ will be equivalent to “ κ^+ is generically supercompact for $<\kappa$ -closed posets”).

2.5 Generically large cardinals

Definition 2.33. Given a class $\mathcal{P}$ consisting of forcing posets, we say that an uncountable cardinal κ is

- *generically measurable* for $\mathcal{P}$ if there is $P \in \mathcal{P}$ such that for any P -generic G over V , there is a transitive class $M \subseteq V[G]$ and a class function $j : V \rightarrow M$ definable in $V[G]$ such that j is an elementary embedding with critical point κ ;
- *generically supercompact* for $\mathcal{P}$ if for any $\eta > \kappa$ there is $\mathcal{P}$ such that for any P -generic G over V , there is a transitive class $M \subseteq V[G]$ and a class function $j : V \rightarrow M$ definable in $V[G]$ such that j is an elementary embedding with critical point κ with $j(\kappa) > \eta$ and $j''\eta \in M$.

Lemma 2.34. *Let $P \in V$ be a poset and G be a P -generic filter over V . Suppose $j : V \xrightarrow{\sim} M \subseteq V[G]$ such that $\text{crit}(j) = \kappa$ and $j''\eta \in M$ for cardinals κ, η in V . Then we have*

(1) *For any set $A \in V$ with $V \models |A| \leq \eta$, we have $A \in M$.*

(2) *$j \restriction \eta, j \restriction \eta^2, j \restriction \eta^3 \in M$.*

Proof. (1): Suppose $V \models |A| \leq \eta$. Then we can take a surjection $f : \eta \rightarrow A$ in V . For each $a \in A$ there is $\xi_a < \eta$ such that $a = f(\xi_a)$. By elementarity, we have

$$j(a) = j(f(\xi_a)) = j(f)(j(\xi_a)).$$

Therefore $j''A = j(f)''(j''\eta)$. Since $j(f), j''\eta \in M$, we have $j''A \in M$.

(2): For each $\xi < \eta$, $(j \restriction \eta)(\xi)$ can be defined as the ξ -th element of $j''\eta$. Since $j''\eta \in M$, $j \restriction \eta$ is a definable subset of $\eta \times j''\eta$ in M , thus $j \restriction \eta \in M$. Since $\eta^2, \eta^3 \in M$ by (1), it follows similarly that $j \restriction \eta^2, j \restriction \eta^3 \in M$. (lemma 2.34) $\square$

Theorem 2.35. *Suppose that κ is generically supercompact for proper posets. Then $\text{SDLS}_+^{\text{int}}(\mathcal{L}_{\text{stat}}^{\aleph_0}, < \kappa)$ holds.*

Proof. Let $\eta \geq \kappa$ be a regular cardinal. By lemma 2.23, it is enough to prove that $(*)_{< \kappa, \eta}^{\text{int}+}$ holds.

Let $\eta^* := |\mathcal{H}(\eta)|$ and $\eta^{**} := (2^{\eta^*})^+$. Suppose $\mathfrak{A}$ is a countable expansion of $\langle \mathcal{H}, \in \rangle$, $\vec{S} = \langle S_a : a \in \mathcal{H}(\eta) \rangle$ is a sequence of stationary subsets of $[\mathcal{H}(\eta)]^{\aleph_0}$ and $\mathcal{C}$ is a club subset of $[\mathcal{H}(\eta)]^{< \kappa}$.

Since κ is generically supercompact for proper posets, we can take P a proper poset, G a P -generic over V , an inner model M of $V[G]$ and an

elementary embedding $j : V \longrightarrow M$ such that $\text{crit}(j) = \kappa$, $j(\kappa) > \eta^{**}$ and $j''\eta^{**} \in M$.

By elementarity, it is enough to find $X \in M$ such that

$$X \in j(\mathcal{C}) \quad (2.52)$$

$$M \models j(\mathfrak{A}) \restriction X \prec j(\mathfrak{A}) \quad (2.53)$$

$$M \models \text{"}\forall a \in X, j(\vec{S})_a \cap X \text{ is a stationary subset of } [X]^{\aleph_0} \text{"} \quad (2.54)$$

We prove that $X := j''((\mathcal{H}(\eta))^V)$ satisfies (2.52), (2.53) and (2.54). Since $V \models |\mathcal{H}(\eta)| < \eta^{**}$, by lemma 2.34(1), we have $X \in M$.

Notice that $|\mathcal{C}| < \eta^{**}$. By lemma 2.34(1), we have $j''\mathcal{C} \in M$. We have $M \models |j''\mathcal{C}| \leq |j(\mathcal{C})| < j(\kappa)$. Since $\mathcal{C}$ is an unbounded subset of $[\mathcal{H}(\eta)]^{<\kappa}$ in V , we have $M \models \text{"}j''\mathcal{C} \text{ is a directed set"}$. Furthermore, by elementarity we have $M \models \text{"}j(\mathcal{C}) \text{ is a club subset of } [j(\mathcal{H}(\eta)^V)]^{<j(\kappa)} \text{"}$. Notice also that $X = \bigcup(j''\mathcal{C})$. Therefore, by proposition 1.3, X satisfies (2.52).

By elementarity and applying Vaught's test, it follows that X satisfies (2.53).

Finally, we prove that X satisfies (2.54). Let $a \in X$. For some $b \in \mathcal{H}(\eta)$, we have $a = j(b)$. We have

$$V \models \text{"}S_b \text{ is a stationary subset of } [\mathcal{H}(\eta)]^{\aleph_0} \text{"}$$

Since $\mathbb{P}$ is proper, this implies

$$V[G] \models \text{"}S_b \text{ is a stationary subset of } [\mathcal{H}(\eta)^V]^{\aleph_0} \text{"}$$

Notice that $j''S_b = \{j''x : x \in S_a\}$ because every element of S_a has size $\aleph_0 < \text{crit}(j)$ (in V). Since j is injective, we have

$$V[G] \models \text{"}j''S_b \text{ is a stationary subset of } [X]^{\aleph_0} \text{"}$$

By lemma 2.34(1), we have that $X, j''S_b \in M$. By lemma 1.6, it follows that

$$M \models \text{"}j''S_b \text{ is a stationary subset of } [X]^{\aleph_0} \text{"}$$

Notice that $j''S_b \subseteq j(S_b) \cap X = j(\vec{S})_a \cap X$, and thus we have

$$M \models \text{"}j(\vec{S})_a \cap X \text{ is a stationary subset of } [X]^{\aleph_0} \text{"}$$

Therefore we X satisfies (2.54). (theorem 2.35)□

Lemma 2.36. *Suppose κ is a supercompact cardinal. Then*

$$V^{\text{Fn}(\kappa, 2)} \models \text{"}\kappa = 2^{\aleph_0} \text{ is generically supercompact for c.c.c. posets"}$$

Proof. Let $\eta > \kappa$. Let G be a $\text{Fn}(\kappa, 2)$ -generic over V . We show that κ is generically supercompact for c.c.c. posets in $V[G]$.

Take $j : V \rightarrow M$ an η -supercompact embedding with critical point κ . By elementarity, we have

$$j(\text{Fn}(\kappa, 2)) = \text{Fn}(j(\kappa), 2)^M = \text{Fn}(j(\kappa), 2)^V \cong \text{Fn}(\kappa, 2) \times \text{Fn}(j(\kappa) \setminus \kappa, 2).$$

Let H be an $\text{Fn}(j(\kappa) \setminus \kappa, 2)$ -generic over $V[G]$. Since $\text{crit}(j) = \kappa$ and $G \subseteq \text{Fn}(\kappa, 2)$, we have $j''G = G \hookrightarrow G \times H$. By proposition 1.26, we can extend j into an elementary embedding $J : V[G] \rightarrow M[G][H]$. Notice that $\text{Fn}(j(\kappa) \setminus \kappa, 2)$ is c.c.c. in $V[G]$ and $J''\eta = j''\eta \in M \subseteq M[G][H]$. Therefore, $\text{Fn}(j(\kappa) \setminus \kappa, 2)$, H , J and $M[G][H]$ witness that κ is generically supercompact for c.c.c. posets in $V[G]$. (lemma 2.36) $\square$

Since every c.c.c. poset is proper, the following corollary follows from theorem 2.35 and lemma 2.36.

Corollary 2.37. $\text{Con}(\text{ZFC} + \text{“there exists a supercompact cardinal”}) \Rightarrow \text{Con}(\text{ZFC} + \text{SDLS}_{+}^{\text{int}}(\mathcal{L}_{\text{stat}}^{\aleph_0}, < 2^{\aleph_0}))$.

We now return to the topic of the Game Reflection Principle. We shall characterize the principle $\text{GRP}^{<\kappa}(< \kappa^+)$ in terms of generic large cardinals, generalizing a result by König in [Kön04].

Lemma 2.38. *For any regular uncountable cardinal κ , if κ^+ is generically supercompact for $< \kappa$ -closed posets then $\text{GRP}^{<\kappa}(\leq \kappa)$ (i.e. $\text{GRP}^{<\kappa}(< \kappa^+)$) holds.*

Proof. Let $A = \theta$ for some regular cardinal $\theta \geq \kappa$. Let $\mathcal{C} \subseteq [A]^{\leq \kappa}$ be a κ -club and suppose $\mathcal{A} \subseteq {}^{<\kappa}A$ is such that II has no winning strategy in the game $\mathcal{G}^{<\kappa A}(\mathcal{A})$.

Let P be a $< \kappa$ -closed poset such that for any P -generic G over V there are M an inner model of $V[G]$ and j a class function definable in $V[G]$ such that

$$j : V \xrightarrow{\sim} M; \tag{2.55}$$

$$\text{crit}(j) = (\kappa^+)^V \wedge j((\kappa^+)^V) > \theta^\kappa; \text{ and} \tag{2.56}$$

$$j''\theta^\kappa \in M. \tag{2.57}$$

By elementarity, it is enough to find $B \in M$ such that $B \in j(\mathcal{C})$ and

$$M \models \text{“player II has no winning strategy in the game } \mathcal{G}^{<\kappa B}(j(\mathcal{A}) \cap {}^{<\kappa}B) \text{”}. \tag{2.58}$$

We show that $B := j''\theta$ satisfies (2.58) and $B \in j(\mathcal{C})$. By (2.57) and lemma 2.34(1) we have $B \in M$.

Claim 2.38.1. $B \in j(\mathcal{C})$.

Proof of claim 2.38.1. First notice that $M \models “\kappa \leq |B| = |\theta| < j((\kappa^+)^V) = \kappa^+”$. Therefore $M \models “|B| = \kappa”$. Let $\langle \alpha_\xi : \xi < \kappa \rangle \in M$ be such that $B = \{j(\alpha_\xi) : \xi < \kappa\}$. Working in $V[G]$, we construct an increasing sequence $\langle j(c_\xi) : \xi < \kappa \rangle$ such that $c_\xi \in \mathcal{C}$ and $\alpha_\xi \in c_\xi$ for every $\eta < \kappa$. For $\xi < \kappa$, suppose $\langle j(c_\zeta) : \zeta < \xi \rangle$ has been constructed. Since $\mathbb{P}$ is $< \kappa$ closed, we have that $\langle c_\zeta : \zeta < \xi \rangle \in V$ as well as $\langle \alpha_\zeta : \zeta < \xi \rangle \in V$. In V , we can take c_ξ such that

$$\left(\bigcup_{\zeta < \xi} c_\zeta\right) \cup \{\alpha_\xi\} \subseteq c_\xi \in \mathcal{C}.$$

By 2.34(1), we have that $\langle j(c_\zeta) : \zeta < \xi \rangle \in M$ for every $\xi < \kappa$. Since $\langle \alpha_\xi : \xi < \kappa \rangle \in M$, it follows that $\langle j(c_\xi) : \xi < \kappa \rangle \in M$. Notice that $B = \bigcup_{\xi < \kappa} j(c_\xi)$. By elementarity, we have $M \models “j(\mathcal{C})$ is a κ -club” and $\{j(c_\xi) : \xi < \kappa\} \subseteq j(\mathcal{C})$. Therefore we conclude that $B \in j(\mathcal{C})$. (claim 2.38.1) $\square$

To prove that B satisfies (2.58), we shall need the following claim:

Claim 2.38.2. (1) $(^{<\kappa}B)^{V[G]} = (^{<\kappa}B)^M$;

(2) $j(\mathcal{A}) \cap (^{<\kappa}B)^M = j''\mathcal{A}$.

Proof of claim 2.38.2. (1): Since $M \subseteq V[G]$, we have $(^{<\kappa}B)^M \subseteq (^{<\kappa}B)^{V[G]}$. Conversely, let $s \in (^{<\kappa}B)^{V[G]}$. In $V[G]$ we have $j^{-1}''s \in (^{<\kappa}A)^{V[G]}$. Since P is $< \kappa$ -closed, we have $j^{-1}''s \in V$. Therefore we have $s = j(j^{-1}''s) \in M$ and thus $(^{<\kappa}B)^M \supseteq (^{<\kappa}B)^{V[G]}$.

(2): First we prove $j''\mathcal{A} \subseteq j(\mathcal{A}) \cap (^{<\kappa}B)^M$. Let $t \in \mathcal{A}$. By elementarity, $j(t) \in \mathcal{A}$. Furthermore, since $\text{dom}(t) < \text{crit}(j)$, we have $\text{dom}(j(t)) = \text{dom}(t) < \kappa$. For every $\alpha \in \text{dom}(t)$, since $t(\alpha) \in A$, we have $j(t)(\alpha) = j(t(\alpha)) \in j''A$. Therefore $M \models “j(t) : \text{dom}(t) \rightarrow j''A = B”$, so $j(t) \in (^{<\kappa}B)^M$.

Now let $t^* \in j(\mathcal{A}) \cap (^{<\kappa}B)^M$. Working in $V[G]$, since $t^* \in (^{<\kappa}B)^{V[G]}$, we can construct the function $t : \text{dom}(t^*) \rightarrow A$ defined by $t(\alpha) = j^{-1}(t^*(\alpha))$ for every $\alpha \in \text{dom}(t^*)$. Since P is $< \kappa$ -closed, we have $t \in V$. Since $\text{dom}(t^*) = \text{dom}(t) < \text{crit}(j)$, we have $j(t) = j''t = t^*$. By elementarity, since $t^* \in j(\mathcal{A})$, we have $t \in \mathcal{A}$. Therefore $t^* \in j''\mathcal{A}$. (claim 2.38.2) $\square$

Suppose there is $\sigma \in M$ such that

$M \models “\sigma : ^{<\kappa}B \rightarrow B$ is a winning strategy for II
in the game $\mathcal{G}^{<\kappa B}(j(\mathcal{A}) \cap ^{<\kappa}B)”$. (2.59)

In $V[G]$, we can construct $\sigma' : {}^{<\kappa}A \longrightarrow A$ such that

$$\sigma'(\langle a_\zeta : \zeta \leq \xi \rangle) := j^{-1}(\sigma(\langle j(a_\zeta) : \zeta \leq \xi \rangle))$$

for every $\langle a_\zeta : \zeta \leq \xi \rangle \in ({}^{<\kappa}A)^{V[G]} = ({}^{<\kappa}A)^V$. From (2.59) and claim 2.38.2, it follows that σ' is a winning strategy for II in the game $\mathcal{G}^{<\kappa A}(\mathcal{A})$. By lemma 2.28, there is $\sigma'' \in V$ a winning strategy for II in the game $\mathcal{G}^{<\kappa A}(\mathcal{A})$. This contradicts the choice of $\mathcal{A}$, therefore B satisfies (2.58). (lemma 2.38) $\square$

Lemma 2.39. *Suppose κ^+ is generically measurable for $< \kappa$ -closed posets. Then we have $2^{<\kappa} = \kappa$.*

Proof. Towards a contradiction, suppose κ^+ is generically measurable by $< \kappa$ -closed and there is some $\mu < \kappa$ such that $2^\mu > \kappa$. Let $\lambda := 2^\mu$. Let P be a $< \kappa$ -closed poset such that there are a P -generic G over V , an inner model $M \subseteq V[G]$ and an elementary embedding $j : V \longrightarrow M$ with $\text{crit}(j) = \kappa^+$.

In V , let $f : \lambda \longrightarrow \mathcal{P}(\mu)$ be a bijection. By elementarity we have that

$$M \models "j(f) : j(\lambda) \longrightarrow \mathcal{P}(\mu) \text{ is a bijection}" \quad (2.60)$$

In V , since $\lambda > \kappa$ is a cardinal, in particular $\lambda \geq \kappa^+$. Therefore $\kappa^+ \in j(\lambda)$. Since P is $< \kappa$ -closed, it follows that $\mathcal{P}(\mu)^V = \mathcal{P}(\mu)^{V[G]} \supseteq \mathcal{P}(\mu)^M$. Then we have $j(f)(\kappa^+) \in \mathcal{P}(\mu)^V$. Since f is surjective, there is $\alpha < \lambda$ such that $f(\alpha) = j(f)(\kappa^+)$. Since $\mu < \text{crit}(j)$, it also follows from $j(f)(\kappa^+) \in \mathcal{P}(\mu)^V$ that

$$j(j(f)(\kappa^+)) = j''(j(f)(\kappa^+)) = j(f)(\kappa^+). \quad (2.61)$$

By elementarity, we have $j(f(\alpha)) = j(f)(j(\alpha))$. Therefore, by (2.61) we have

$$j(f)(j(\alpha)) = j(f)(\kappa^+).$$

Since $j(f)$ is injective, it follows that $j(\alpha) = \kappa^+$. But this contradicts $\text{crit}(j) = \kappa^+$. (lemma 2.39) $\square$

We now define a particular case of the game $\mathcal{G}^{<\mu A}(\mathcal{A})$ which we shall use to construct ultrafilters in theorem 2.40.

Let κ, η be regular cardinals with $\kappa \leq \eta$. Define

$$D_{\kappa,\eta} := \{u \in [\eta]^\kappa : \kappa \subseteq u\}; \quad (2.62)$$

$$\mathcal{F}_{\kappa,\eta} := \{f : D_{\kappa,\eta} \longrightarrow \eta : \forall A \in D_{\kappa,\eta}, f(A) \in A\}; \quad (2.63)$$

$$A_{\kappa,\eta} := \mathcal{F}_{\kappa,\eta} \cup \eta; \quad (2.64)$$

$$\mathcal{A}_{\kappa,\eta}^0 = \{\langle a_\alpha, b_\alpha : \alpha < \xi \rangle \frown \langle a_\xi \rangle \in {}^{<\kappa}A : a_\xi \in \mathcal{F}_{\kappa,\eta}\}; \quad (2.65)$$

$$\mathcal{A}_{\kappa,\eta}^1 = \{\langle a_\alpha, b_\alpha : \alpha < \xi \rangle \in {}^{<\kappa}A : \xi < \kappa \wedge (1) a_\alpha \in \mathcal{F}_{\kappa,\eta} \wedge \quad (2.66)$$

$$(2) b_\alpha \in \eta \wedge$$

$$(3) \forall \alpha < \xi, \bigcup_{\alpha < \xi} a_\alpha^{-1} \{b_\alpha\} = \eta \wedge$$

$$(4) \left| \bigcap_{\alpha < \xi} a_\alpha^{-1} \{b_\alpha\} \right| > 2\};$$

$$\mathcal{A}_{\kappa,\eta} := \mathcal{A}_{\kappa,\eta}^0 \cup \mathcal{A}_{\kappa,\eta}^1. \quad (2.67)$$

The goal for player II in the game $\mathcal{G}^{<\kappa A_{\kappa,\eta}}(\mathcal{A}_{\kappa,\eta})$ is to construct a filter base $\{a_\alpha^{-1} \{b_\alpha\} : \alpha < \kappa\}$ over $D_{\kappa,\eta}$ which has normality for the regressive functions $\langle a_\alpha : \alpha < \kappa \rangle$ selected by player I.

Theorem 2.40. *For a regular uncountable κ , the following are equivalent:*

- (a) $2^{<\kappa} = \kappa$ and $\text{GRP}^{<\kappa}(\leq \kappa)$ holds.
- (b) II has a winning strategy in the game $\mathcal{G}^{<\kappa A_{\kappa,\eta}}(\mathcal{A}_{\kappa,\eta})$ for every regular $\eta > \kappa$.
- (c) κ^+ is generically supercompact for $< \kappa$ -closed forcing.

Proof. (c) $\Rightarrow$ (a): It follows by lemma 2.39 and lemma 2.38.

(a) $\Rightarrow$ (b): Assume $\text{GRP}^{<\kappa}(< \kappa^+)$ and $2^{<\kappa} = \kappa$ hold. Let $\eta \geq \kappa$ be a regular cardinal and let θ be a large enough regular cardinal. By the assumption $2^{<\kappa} = \kappa$, a κ -club $\mathcal{C} \subseteq [A_{\kappa,\eta}]^{\leq \kappa}$ can be constructed such that

$$\mathcal{C} \subseteq \{A_{\kappa,\eta} \cap M : M \prec \mathcal{H}(\theta) \wedge |M| = \kappa \wedge [M]^{<\kappa} \subseteq M \wedge \kappa, \eta \in M\}.$$

Given $M \prec \mathcal{H}(\theta)$ such that $A_{\kappa,\eta} \cap M \in \mathcal{C}$, we construct a winning strategy for II in the game $\mathcal{G}^{<\kappa(A_{\kappa,\eta} \cap M)}(\mathcal{A}_{\kappa,\eta} \cap {}^{<\kappa}(A_{\kappa,\eta} \cap M))$. Let $\sigma : {}^{<\kappa}(A_{\kappa,\eta} \cap M) \longrightarrow A_{\kappa,\eta} \cap M$ be such that $\sigma(t \frown f) = f(\eta \cap M)$ for any $f \in \mathcal{F}_{\kappa,\eta} \cap M$ and any $t \in {}^{<\kappa}(A_{\kappa,\eta} \cap M)$. In other words, if $\langle a_\alpha, b_\alpha : \alpha < \kappa \rangle$ is a match of the game $\mathcal{G}^{<\kappa(A_{\kappa,\eta} \cap M)}(\mathcal{A}_{\kappa,\eta} \cap {}^{<\kappa}(A_{\kappa,\eta} \cap M))$ where player II plays according to σ and player I always plays in order not to instantly lose, then $a_\alpha \in \mathcal{F}_{\kappa,\eta} \cap M$ and $b_\alpha = a_\alpha(\eta \cap M)$ for all $\alpha < \kappa$.

Since a_α is a regressive function (by (2.63)), we have $b_\alpha \in \eta \cap M$. Since $M \prec \mathcal{H}(\theta)$, it follows that $\langle a_\alpha, b_\alpha : \alpha < \kappa \rangle$ satisfies (2.66)(3) and (2.66)(4). Therefore player II wins the match $\langle a_\alpha, b_\alpha : \alpha < \kappa \rangle$.

Since $\mathcal{C}$ is a κ -club, it follows by $\text{GRP}^{<\kappa}(<\kappa^+)$ that player II has a winning strategy in the game $\mathcal{G}^{<\kappa A_{\kappa,\eta}}(\mathcal{A}_{\kappa,\eta})$.

(b) $\Rightarrow$ (c) : Let $\eta \geq \kappa$ and suppose $\sigma : {}^{<\kappa}A_{\kappa,\eta} \rightarrow A_{\kappa,\eta}$ is a winning strategy for II in the game $\mathcal{G}^{<\kappa A_{\kappa,\eta}}(\mathcal{A}_{\kappa,\eta})$. Let $P := \text{Col}(\kappa, \{|\mathcal{F}_{\kappa,\eta}|\})$. Recall that P is a $<\kappa$ -closed poset which adds a surjection from κ onto $|\mathcal{F}_{\kappa,\eta}|$.

Let G be a P -generic over V . In $V[G]$, we can fix $\{f_\xi : \xi < \kappa\}$ an enumeration of $(\mathcal{F}_{\kappa,\eta})^V$. Since P is $<\kappa$ -closed, by lemma 2.28, σ remains a winning strategy for player II in the game $\mathcal{G}^{<\kappa(A_{\kappa,\eta})^V}((\mathcal{A}_{\kappa,\eta})^V)$ in $V[G]$. In $V[G]$, let $\langle \alpha_\xi : \xi < \kappa \rangle$ be the sequence of moves suggested for player II by σ in a match where player I plays $\langle f_\xi : \xi < \kappa \rangle$, i.e.

I	f_0	f_1	$\cdots$	f_ξ	$\cdots$	
II	α_0	α_1	$\cdots$	α_ξ	$\cdots$	$(\xi < \kappa)$

where $\alpha_\xi := \sigma(\langle f_\zeta : \zeta \leq \xi \rangle)$ for every $\xi < \kappa$. Since σ is a winning strategy for II, the above match is won by II. Let

$$\mathcal{U} := \{u \in (\mathcal{P}(D_{\kappa,\eta}))^V : \exists \xi < \kappa, f_\xi^{-1}''\{\alpha_\xi\} \subseteq u\}.$$

Claim 2.40.1. $\mathcal{U}$ is a V -ultrafilter over $D_{\kappa,\eta}$ (i.e. $\mathcal{U}$ is a filter and for every $X \in \mathcal{P}(D_{\kappa,\eta}) \cap V$ either $X \in \mathcal{U}$ or $D_{\kappa,\eta} \setminus X \in \mathcal{U}$).

Proof of claim 2.40.1. First we check that $\mathcal{U}$ is a filter. Clearly $\mathcal{U}$ is $\subseteq$ -upward closed. Let $U_0, U_1 \in \mathcal{U}$. In V , we can construct the function $f : D_{\kappa,\eta} \rightarrow 3$ given by

$$f(u) := \begin{cases} 0 & , \text{ if } u \notin U_0 \\ 1 & , \text{ if } u \in U_0 \setminus U_1 \\ 2 & , \text{ if } u \in U_0 \cap U_1 \end{cases}$$

for $u \in D_{\kappa,\eta}$. By (2.62), ${}^\kappa D_{\kappa,\eta} \subseteq \mathcal{F}_{\kappa,\eta}$, so $f \in \mathcal{F}_{\kappa,\eta} \cap V$. Therefore there is some $\xi^* < \kappa$ such that $f = f_{\xi^*}$. Since player II plays accordingly to a winning strategy, we must have $\alpha_{\xi^*} = 2$. Indeed, let $\xi_0, \xi_1 < \kappa$ be such that $U_0 \supseteq f_{\xi_0}^{-1}''\{\alpha_{\xi_0}\}$ and $U_1 \supseteq f_{\xi_1}^{-1}''\{\alpha_{\xi_1}\}$. For any δ with $\xi^*, \xi_0, \xi_1 < \delta < \kappa$, if $\alpha_{\xi^*} \neq 2$ then $\bigcap_{\xi < \delta} f_\xi^{-1}''\{\alpha_\xi\} = \emptyset$, so II would lose the match by (2.66)(4). Therefore we have $U_0 \cap U_1 \supseteq f_{\xi^*}^{-1}''\{\alpha_{\xi^*}\}$, so $U \cap V \in \mathcal{U}$.

Now, let $X \in \mathcal{P}(D_{\kappa,\eta}) \cap V$. In V define the function $f_X : D_{\kappa,\eta} \rightarrow 2$ by

$$f_X(u) := \begin{cases} 1 & , \text{ if } u \in X \\ 0 & , \text{ otherwise} \end{cases}$$

for $u \in D_{\kappa,\eta}$. Since $f_X \in \mathcal{F}_{\kappa,\eta} \cap V$, there is some $\xi^* < \kappa$ such that $f_X = f_{\xi^*}$. Since $f_X^{-1} \{ \alpha \} = \emptyset$ for every $\alpha > 1$, we have $\alpha_{\xi^*} \in \{0, 1\}$. Therefore $f_{\xi^*}^{-1} \{ \alpha_{\xi^*} \} \in \{X, D_{\kappa,\eta} \setminus X\}$, so we conclude that $\mathcal{U}$ is a V -ultrafilter. (claim 2.40.1) $\square$

Claim 2.40.2. $\mathcal{U}$ is a V -(κ^+) V -complete (i.e. $\bigcap_{\xi < \kappa} U_\xi \in \mathcal{U}$ whenever $\langle U_\xi : \xi < \kappa \rangle \in V$ is a sequence of elements of $\mathcal{U}$).

Proof of claim 2.40.2. Let $\langle U_\xi \in \mathcal{U} : \xi < \kappa \rangle \in V$ be a sequence of elements of $\mathcal{U}$. Without loss of generality, we assume $U_0 = D_{\kappa,\eta}$. In V , we construct the function $g : D_{\kappa,\eta} \longrightarrow \kappa$ by

$$g(u) := \begin{cases} 0 & , \text{ if } u \in \bigcap_{\xi < \kappa} U_\xi \\ \min\{\xi \in \kappa \setminus \{0\} : u \notin U_\xi\} & , \text{ otherwise} \end{cases}$$

for any $\alpha < \kappa$. Since $g \in \mathcal{F}_{\kappa,\eta} \cap V$, we have $g = f_{\xi^*}$ for some $\xi^* < \kappa$.

Since II plays accordingly to a winning strategy, we have $\alpha_{\xi^*} = 0$. Indeed, suppose $\alpha_{\xi^*} \neq 0$. Since $U_{\alpha_{\xi^*}} \in \mathcal{U}$, we can take $\zeta < \kappa$ such that $f_\zeta^{-1} \{ \alpha_\zeta \} \subseteq U_{\alpha_{\xi^*}}$. We have then $f_\zeta^{-1} \{ \alpha_\zeta \} \cap f_{\xi^*}^{-1} \{ \alpha_{\xi^*} \} = \emptyset$, so by (2.66)(4) player II would lose the match. Therefore we have $\alpha_{\xi^*} = 0$.

It follows that $\bigcap_{\xi < \kappa} U_\xi = g^{-1}(0) = f_{\alpha_{\xi^*}}^{-1}(b_{\alpha_{\xi^*}}) \in \mathcal{U}$. (claim 2.40.2) $\square$

As usual, we define the ultraproduct $^{D_{\kappa,\eta}}V/\mathcal{U}$ as follows.

$$^{D_{\kappa,\eta}}V/\mathcal{U} := \{[f]_{\mathcal{U}} : f \in (^{D_{\kappa,\eta}}V) \cap V\}$$

where $[f]_{\mathcal{U}}$ is the equivalence class² of the function f with respect to the binary relation $\sim_{\mathcal{U}}$ defined for every $f, g \in (^{D_{\kappa,\eta}}V) \cap V$ by:

$$f \sim_{\mathcal{U}} g \Leftrightarrow \{u \in D_{\kappa,\eta} : f(u) = g(u)\} \in \mathcal{U}.$$

We also define the relation $\in_{\mathcal{U}}$ as usual: given any $[f]_{\mathcal{U}}, [g]_{\mathcal{U}} \in ^{D_{\kappa,\eta}}V/\mathcal{U}$, we define

$$[f]_{\mathcal{U}} \in_{\mathcal{U}} [g]_{\mathcal{U}} \Leftrightarrow \{u \in D_{\kappa,\eta} : f(u) \in g(u)\} \in \mathcal{U} \quad (2.68)$$

Claim 2.40.3. The class relation $\in_{\mathcal{U}}$ is well defined and well-founded.

Proof of claim 2.40.3. Since $\mathcal{U}$ is a filter (claim 2.40.1), it follows that $\in_{\mathcal{U}}$ is well defined.

Suppose there is a sequence $\langle g_n : n < \omega \rangle \subseteq ^{D_{\kappa,\eta}}V \cap V$ such that

$$[g_0]_{\mathcal{U}} \ni_{\mathcal{U}} [g_1]_{\mathcal{U}} \ni_{\mathcal{U}} \dots [g_n]_{\mathcal{U}} \ni_{\mathcal{U}} \dots$$

²Since these equivalence classes are proper classes, we actually consider $[f]_{\mathcal{U}}$ a set of representatives of minimal rank of the equivalence class of f .

For each $n < \omega$, define

$$U_n := \{u \in D_{\kappa, \eta} : g_{n+1}(u) \in g_n(u)\}.$$

By (2.68) we have $U_n \in \mathcal{U}$ for each $n \in \omega$. Since P is $< \kappa$ -closed, we have that $\langle U_n : n \in \omega \rangle \in V$. By (2.40.2), we have $\bigcap_{n < \omega} U_n \in \mathcal{U}$. Take $u^* \in \bigcap_{n \in \omega} U_n$. Then we have an infinite chain

$$g_0(u^*) \ni g_1(u^*) \ni \dots g_n(u^*) \ni \dots,$$

a contradiction.

(claim 2.40.3) $\square$

By claim 2.40.3, we can define the Mostowski collapse $i : \langle D_{\kappa, \eta} V / \mathcal{U}, \in_{\mathcal{U}} \rangle \longrightarrow \langle M, \in \rangle$, where $M \subseteq V[G]$ is a transitive class.

Define $j : V \longrightarrow M$ by $j(x) = i([c_x]_{\mathcal{U}})$ for each $x \in V$, where $c_x : D_{\kappa, \eta} \longrightarrow V$ is the constant function $u \mapsto x$. It is enough to prove that

$$j : V \longrightarrow M \text{ is an elementary embedding;} \quad (2.69)$$

$$\text{crit}(j) = \kappa^+; \quad (2.70)$$

$$j''\eta \in M. \quad (2.71)$$

Notice that we do not require that $j(\kappa) > \eta$. It is well known that if j is an embedding satisfying (2.69), (2.70) and (2.71), then $j^{(n)} : V \longrightarrow N$ satisfies (2.69), (2.70), (2.71) and $j^{(n)}(\kappa) > \eta$ for some $n \in \omega$ and some $N \subseteq M$. See for example lemma 23.15 in [Kan08].

It follows by standard arguments that j satisfies (2.69).

Claim 2.40.4. *For every $\beta < \eta$, we have $\{u \in D_{\kappa, \eta} : \beta \in u\} \in \mathcal{U}$.*

Proof of claim 2.40.4. Let $\beta < \eta$. Denote $\delta_\beta = 1$ for $\beta = 0$ and $\delta_\beta = 0$ for $\beta > 0$. In V , consider the function $g_\beta : D_{\kappa, \eta} \longrightarrow V$ defined by

$$g_\beta(u) = \begin{cases} \beta & , \text{ if } \beta \in u \\ \delta_\beta & , \text{ otherwise} \end{cases}$$

for $u \in D_{\kappa, \eta}$. Clearly $g_\beta \in \mathcal{F}_{\kappa, \eta} \cap V$. Therefore, we can take $\xi^* < \kappa$ such that $g_\beta = f_{\xi^*}$. By (2.66)(3), II is forced to chose $\alpha_{\xi^*} = \beta$ at the ξ^* -th turn of the match. Therefore we have $\{u \in D_{\kappa, \eta} : \beta \in u\} = f_{\xi^*}^{-1} \{\alpha_{\xi^*}\} \in \mathcal{U}$. (claim 2.40.4) $\square$

We now prove that j satisfies (2.70). It follows by induction that $j(\alpha) = \alpha$ for all $\alpha < \kappa^+$: at limit steps, we use claim 2.40.2.

Now we prove that $j(\kappa^+) > \kappa^+$. Let $d : D_{\kappa,\eta} \longrightarrow V$ be function $u \mapsto \sup(u \cap \kappa^+)$. Since $\{u \in D_{\kappa,\eta} : \kappa^+ = c_{\kappa^+}(u) \ni d(u)\} = D_{\kappa,\eta} \in \mathcal{U}$, we have

$$[d]_{\mathcal{U}} \in_{\mathcal{U}} [c_{\kappa^+}]_{\mathcal{U}}. \quad (2.72)$$

Let $\alpha < \kappa^+$. By claim 2.40.4 we have

$$\{u \in D_{\kappa,\eta} : \alpha = c_{\alpha}(u) \in d(u) = \sup(u \cap \kappa^+)\} \supseteq \{u \in D_{\kappa,\eta} : \alpha + 1 \in u\} \in \mathcal{U}.$$

Therefore

$$[c_{\alpha}(u)]_{\mathcal{U}} \in_{\mathcal{U}} [d]_{\mathcal{U}}. \quad (2.73)$$

By (2.72) and (2.73) we have $\alpha = j(\alpha) < j(\kappa^+)$.

We conclude by proving that j satisfies (2.71). This follows from the following claim:

Claim 2.40.5. $i([Id]_{\mathcal{U}}) = j''\eta$, where $Id : D_{\kappa,\eta} \longrightarrow V$ is the identity function $u \mapsto u$.

Proof of claim 2.40.5. Let $\beta < \eta$. By claim 2.40.4, we have $[c_{\beta}]_{\mathcal{U}} \in_{\mathcal{U}} [Id]_{\mathcal{U}}$, thus $j(\beta) \in i([Id]_{\mathcal{U}})$.

Conversely, let $f : D_{\kappa,\eta} \longrightarrow V$ in V be such that $[f]_{\mathcal{U}} \in_{\mathcal{U}} [Id]_{\mathcal{U}}$. In V we can define $f' : D_{\kappa,\eta} \longrightarrow \eta$ by

$$f'(u) := \begin{cases} f(u) & , \text{ if } f(u) \in u \\ \min(u) & , \text{ otherwise.} \end{cases}$$

for $u \in D_{\kappa,\eta}$. By $[f]_{\mathcal{U}} \in_{\mathcal{U}} [Id]_{\mathcal{U}}$, we have $f \sim_{\mathcal{U}} f'$. Notice that $f' \in \mathcal{F}_{\kappa,\eta} \cap V$, therefore there is some $\xi^* < \kappa$ such that $f' = f_{\xi^*}$. Since $f_{\xi^*}^{-1} \{ \alpha_{\xi^*} \} \in \mathcal{U}$, it follows that $f \sim_{\mathcal{U}} f' \sim_{\mathcal{U}} c_{\alpha_{\xi^*}}$, therefore $i([f]_{\mathcal{U}}) = j(\alpha_{\xi^*}) \in j''\eta$. (claim 2.40.5) $\square$

(theorem 2.40) $\square$

Proposition 2.41. *Suppose $\kappa < \lambda$ are regular cardinals and λ is supercompact. Then*

$$V^{\text{Col}(\kappa, < \lambda)} \models \text{“} \lambda \text{ is generically supercompact for } < \kappa\text{-closed posets”}$$

Proof. Suppose G is a $\text{Col}(\kappa, < \lambda)$ -generic over V . Let $\eta > \lambda$. Since λ is supercompact in V , we can take $j : V \longrightarrow M$ an elementary embedding with critical point λ such that $j(\lambda) > \eta$ and ${}^{\eta}M \subseteq M$.

By elementarity and by ${}^{\eta}M \subseteq M$, we have

$$\begin{aligned} j(\text{Col}(\kappa, < \lambda)) &= \text{Col}(\kappa, < j(\lambda))^M = \text{Col}(\kappa, < j(\lambda))^V \\ &\cong \text{Col}(\kappa, < \lambda)^V \times \text{Col}(\kappa, j(\lambda) \setminus \lambda)^V. \end{aligned}$$

Let H be a $\text{Col}(\kappa, j(\lambda) \setminus \lambda)^V$ -generic over $V[G]$. Since $j''G \subseteq G \subseteq G * H$, we can extend j into an elementary embedding $J : V[G] \longrightarrow M[G \times H]$. Since $\text{Col}(\kappa, j(\lambda) \setminus \lambda)^V$ remains $< \kappa$ -closed in $V[G]$, we conclude that $\text{Col}(\kappa, j(\lambda) \setminus \lambda)^V$, H , J and $M[G \times H]$ witnesses that λ is generically supercompact for $< \kappa$ -closed posets in $V[G]$. (proposition 2.41) $\square$

Corollary 2.42. (1) $\text{Con}(\text{ZFC} + \text{“there exists a supercompact cardinal”}) \Rightarrow$

$$\text{Con}(\text{ZFC} + \text{GRP}^{<2^{\aleph_0}}(\leq 2^{\aleph_0})).$$

$$(2) \text{Con}(\text{ZFC} + \text{“there exists a supercompact cardinal”}) \Rightarrow \\ \text{Con}(\text{ZFC} + \text{SDLS}_+(\mathcal{L}_{\text{stat}}^{\aleph_0, II}, \leq 2^{\aleph_0})).$$

Proof. (1): Suppose there is a supercompact cardinal λ and let $\kappa < \lambda$ be a regular cardinal. Since $\text{Fn}(\kappa, 2) \in V_\lambda$, λ remains supercompact in $V^{\text{Fn}(\kappa, 2)}$. Then, by theorem 2.40 and proposition 2.41 we have

$$(V^{\text{Fn}(\kappa, 2)})^{\text{Col}(\kappa, < \lambda)} \models \text{GRP}^{< \kappa}(< \kappa^+) \wedge 2^{\aleph_0} = \kappa.$$

(2) follows from (1) and corollary 2.32. (corollary 2.42) $\square$

Chapter 3

Laver generically large cardinals and an application of mixed support iterations

3.1 Laver generically large cardinals

Definition 3.1. Let $\mathcal{P}$ be a property of posets. We say that $\mathcal{P}$ is *iteratable* if it satisfies the following:

1. If P satisfies $\mathcal{P}$ and $P' \cong P$, then P' satisfies $\mathcal{P}$;
2. Given P satisfying $\mathcal{P}$ and $p \in P$, we have that $P \restriction p$ satisfies $\mathcal{P}$, where $P \restriction p$ denotes the poset $\{q \in P : q \leq p\}$ ordered with the order inherited from P ;
3. If P satisfies $\mathcal{P}$ and $\tilde{Q}$ is a P -name such that $\Vdash_P \text{“}\tilde{Q} \text{ satisfies } \mathcal{P}\text{”}$, then $P * \tilde{Q}$ satisfies $\mathcal{P}$.

The following concept was introduced in [FOSb].

Definition 3.2. Given κ a cardinal and $\mathcal{P}$ an iteratable property of posets, we say that κ is *Laver-generically supercompact for $\mathcal{P}$* if for any $\eta > \kappa$ and any P satisfying $\mathcal{P}$, there is a poset Q satisfying $\mathcal{P}$ with $P \leq Q$ such that there are H an Q -generic filter over V , M an inner model of $V[H]$ and $j \subseteq V[H]$ a class function such that

- (1) $V[H] \models \text{“} j : V \longrightarrow M \text{ is an elementary embedding”}$;
- (2) $\text{crit}(j) = \kappa$, $j(\kappa) > \eta$;

(3) $P, H \in M$;

(4) $j''\eta \in M$.

Remark 3.3. Notice that since we consider $\mathcal{P}$ an iterable property, it is easy to see that definition 3.2 is equivalent to

κ is a Laver-generically supercompact cardinal for $\mathcal{P}$ if for every P satisfying $\mathcal{P}$ and $\eta > \kappa$ there is Q satisfying $\mathcal{P}$ with $P \triangleleft Q$ such that for every Q -generic H over V , there are M an inner model of $V[H]$ and $j \subseteq V[H]$ a class function satisfying (1) through (4) in definition 3.2.

We now introduce the following variation of the generically large cardinal property from definition 3.2.

Definition 3.4. Given $\mathcal{P}$ and $\mathcal{Q}$ iterable properties of forcing posets, we say that a cardinal κ is *Laver generically supercompact for $\langle \mathcal{P}, \mathcal{Q} \rangle$* if for any $\eta > \kappa$ and any P satisfying $\mathcal{P}$ there are $\tilde{Q}$ a $\mathbb{P}$ -name with

$$\Vdash_P \text{“}\tilde{Q} \text{ satisfies } \mathcal{Q}\text{”},$$

H a $P * \tilde{Q}$ -generic over V , M an inner model of $V[H]$ and j a class function definable in $V[H]$ such that

(1) $V[H] \models \text{“}j : V \longrightarrow M \text{ is an elementary embedding”}$;

(2) $\text{crit}(j) = \kappa$, $j(\kappa) > \eta$;

(3) $H, P, \tilde{Q} \in M$;

(4) $j''\eta \in M$.

In particular, if M satisfies

$$(5) {}^{<\omega_1}M \cap V[H] \subseteq M,$$

we say that κ is *Laver generically supercompact* for $\langle \mathcal{P}, \mathcal{Q} \rangle$* .

Lemma 3.5. Suppose κ is Laver generically supercompact* for $\langle \mathcal{P}, \mathcal{Q} \rangle$, where P is the property “is forcing equivalent to $\text{Col}(\kappa, < \theta)$ for some $\theta > \kappa$ ” and $\mathcal{Q}$ is “proper”. Then, given any $\theta > \kappa$, we have

$$V^{\text{Col}(\kappa, < \theta)} \models \text{“}\kappa \text{ is generically supercompact for proper posets”}.$$

Notice that, by corollary 1.31, it follows that κ is Laver generically supercompact* for $\langle \mathcal{P}, \mathcal{Q} \rangle$ (where $\mathcal{P}$ and $\mathcal{Q}$ are as above) if and only if κ is Laver generically supercompact* for $\langle \mathcal{P}', \mathcal{Q} \rangle$ where $\mathcal{P}'$ is “ $< \kappa$ -closed”.

Proof of lemma 3.5. Suppose κ is Laver generically supercompact* for $\langle \mathcal{P}, \mathcal{Q} \rangle$. Given $\theta > \kappa$, denote $P := \text{Col}(\kappa, < \theta)$. Fix G a P -generic over V . Let $\eta > \kappa$. Without loss of generality, we take $\eta > \theta$. Take $\tilde{Q} \in V$ a P -name, H a $P * \tilde{Q}$ -generic over V , an inner model M of $V[H]$ and a class function $j \subseteq V[H]$ such that

- (1) $V[H] \models “j : V \longrightarrow \text{ is an elementary embedding}”$;
- (2) $\text{crit}(j) = \kappa, j(\kappa) > \eta$;
- (3) $H, P, \tilde{Q} \in M$;
- (4) $j''\eta \in M$.
- (5) ${}^{<\omega_1}M \cap V[H] \subseteq M$,

By remark 3.3, we assume without loss of generality that $H = G * H^*$ for some H^* which is a $\tilde{Q}[G]$ -generic over $V[G]$.

By definition 1.9, we have $P \subseteq \theta^3$. By (4) and lemma 2.34(2), we have $j \restriction P \in M$. By (3), it follows that $j''G \in M$. Notice that, by elementarity, we have that $j(P)$ is $< j(\kappa)$ -directed closed in M . Since $M \models |j''G| < j(\kappa)$ (by item (2)), it follows that there is a master condition $m \in j(P)$, i.e. m is such that $m \leq j(p)$ for every $p \in G$ (one can take e.g. $m = \bigcup j''G$).

Let K be a $j(P)$ -generic over $V[H]$ containing the condition m . Then we have $j''G \subseteq K$, so by proposition 1.25 we can extend j into an elementary embedding $J : V[G] \longrightarrow M[K]$.

By elementarity we have $M \models “J(P) \text{ is } \sigma\text{-closed}”$. By (5), we have $V[H] \models “j(P) \text{ is } \sigma\text{-closed}”$, so $J(P)$ is proper in $V[H]$.

In $V[G]$, take $\tilde{R}$ a $\tilde{Q}[G]$ -name for $j(P)$ and define $Q^* := \tilde{Q}[G] * \tilde{R}$. Then $V[G] \models “Q^* \text{ is proper}”$.

Clearly we also have $J''\eta = j''\eta \in M[K]$. So $Q^*, H^* * K, J$ and $M[K]$ witnesses that

$$V[G] \models “\kappa \text{ is generically supercompact for proper posets}”.$$

(lemma 3.5) $\square$

3.2 Mixed support iteration

Assuming the existence of a supercompact cardinal κ and using a mixed support iteration, we shall construct a poset $\mathbb{P}_\kappa \subseteq V_\kappa$ which makes the continuum Laver-generically supercompact for $\langle \mathcal{P}, \mathcal{Q} \rangle$, where $\mathcal{P}$ is the class of all $< 2^{\aleph_0}$ -closed posets and $\mathcal{Q}$ is the class of all proper posets. Also, since $\mathbb{P}_\kappa \subseteq V_\kappa$, $\mathbb{P}_\kappa$ preserves all measurable, supercompact, superhuge, etc. cardinals above κ .

The main theorem to be proved in this section is:

Theorem 3.6.

$$\begin{aligned} &\text{Con}(\text{ZFC} + \text{“there exist 2 distinct supercompact cardinals”}) \implies \\ &\text{Con}(\text{ZFC} + \text{“}2^{\aleph_0} \text{ is weakly Mahlo and generically supercompact for proper posets”} + \\ &\quad \text{“}(2^{\aleph_0})^+ \text{ is generically supercompact for } < 2^{\aleph_0}\text{-closed posets”}) \end{aligned}$$

Actually we prove more than that, since we show that the model constructed to prove theorem 3.6 satisfies some more properties (lemmas 3.25, 3.21 and theorem 3.26).

3.2.1 Construction of the iteration

We construct the poset used for the proof of theorem 3.6. Suppose κ is a supercompact cardinal. Fix $f : \kappa \rightarrow \kappa$ a Laver function such that for any $\eta, \theta > \kappa$ there is an elementary embedding $j : V \rightarrow M$ with critical point κ such that $j(\kappa) > \eta$, ${}^\eta M \subseteq M$ and $j(f)(\kappa) = \theta$ (see, for example, theorem 20.21 in [Jec03]). We can assume without loss of generality that $f(\alpha) > \alpha$ for every $\alpha < \kappa$. We define the following set:

$$\mathcal{C} := \{\alpha < \kappa : f''\alpha \subseteq \alpha\}. \quad (3.1)$$

It is straightforward to see that $\mathcal{C}$ is a club subset of κ (check for example lemma III.6.13 in [Kun11]).

Recall that an uncountable cardinal κ is *weakly Mahlo* if and only if $\{\alpha < \kappa : \alpha \text{ is a regular cardinal}\}$ is a stationary subset of κ . Likewise, κ is *(strongly) Mahlo* if and only if $\{\alpha < \kappa : \alpha \text{ is a (strongly) inaccessible cardinal}\}$ is a stationary subset of κ .

Since κ is supercompact, the set

$$S := \{\alpha \in \mathcal{C} : \alpha \text{ is strongly Mahlo}\} \quad (3.2)$$

is a stationary subset of κ . We also define

$$T := \kappa \setminus S.$$

Notice that clearly $|T| = \kappa$.

We shall construct a *mixed support iteration* of length κ . Our construction is a modification of the iteration described by Krueger in [Kru08]. Over T we shall force with a finite support iteration of c.c.c. posets, while over S we shall force with the usual Levy collapse, guided by the Laver function f . The S part shall have Easton support.

In the original construction by Krueger, The original construction by Krueger is divided into the even part and the odd part. The even part (which corresponds to the part over the set T in the present construction) has support of size $< \mu$ for some regular cardinal μ , and at each even step of the iteration the poset $\text{Add}(\mu) := \{p : p \text{ is a function with } \text{dom}(p) \in [\mu]^{<\mu} \text{ and } \text{range}(p) = 2\}$ (ordered with reverse inclusion) is used to add a Cohen subset of μ . The poset $\text{Add}(\mu)$ is absolute between all the models in that construction. However, if we force with other posets, we may not be able to “untangle” this part from the rest of the iteration. In order to be able to force with a wider range of posets, we first fix a finite support iteration of length κ in the ground model, which will be completely embedded into the mixed support iteration.

We first fix $\vec{T} = \langle \mathbb{T}_\alpha, \mathbb{R}_\alpha : \alpha < \kappa \rangle$ a finite support iteration satisfying

$$\forall \alpha \in S, \Vdash_{\mathbb{T}_\alpha} “\mathbb{R}_\alpha = \{1\}” \text{ and} \quad (3.3)$$

$$\forall \alpha \in S, \Vdash_{\mathbb{T}_\alpha} “\mathbb{R}_\alpha \in V_{\nu_\alpha}[G_{\mathbb{T}_\alpha}] \text{ is c.c.c. and adds at least one real}” \quad (3.4)$$

where we denote $\nu_\alpha := \min(S \setminus \alpha)$.

For the main result (lemma 3.19), an iteration adding Cohen reals is be enough. However, in this construction we have the freedom to choose any finite support iteration of c.c.c. posets, as long as they are small in the sense of (3.4). By doing so, we achieve stronger results (for example lemma 3.25).

We define the support for our iteration. For each $\alpha \leq \kappa$, we denote

$$\mathcal{I}_\alpha = \{x \subseteq \alpha : |x \cap T| < \aleph_0 \wedge x \cap S \text{ is an Easton set}\}. \quad (3.5)$$

(recall that a set $x \subset \text{On}$ is Easton if for any $\beta \leq \sup(x)$, if β is a regular cardinal, then $x \cap \beta$ is a bounded subset of β). Clearly $\vec{\mathcal{I}} := \langle \mathcal{I}_\alpha : \alpha \leq \kappa \rangle$ satisfies the properties listed before definition 1.20.

We now construct a generalized iteration (in the sense of definition 1.21) $\vec{\mathbb{P}} = \langle \mathbb{P}_\alpha : \alpha \leq \kappa \rangle$ by induction on $\alpha \leq \kappa$. The support of $\vec{\mathbb{P}}$ shall be $\vec{\mathcal{I}}$, and therefore we shall say that $\vec{\mathbb{P}}$ is a *mixed support iteration*. Since we only consider the iteration $\vec{\mathbb{P}}$ with length $\kappa + 1$, we do not need to define $\mathcal{I}_\alpha$ for $\alpha > \kappa$.

For $\alpha \leq \kappa$, suppose that $\langle \mathbb{P}_\beta : \beta < \alpha \rangle$ is a generalized iteration such that for every $\beta < \alpha$

the function $\pi_\beta : \mathbb{P}_\beta \longrightarrow \mathbb{T}_\beta, p \mapsto p \restriction T$ is a projection; and (3.6)

the function $\iota_\beta : \mathbb{T}_\beta \longrightarrow \mathbb{P}_\beta, t \mapsto t \restriction T$ is a complete embedding. (3.7)

We then define $\mathbb{P}_\alpha$ as follows:

- If $\alpha \leq \kappa$ is a limit ordinal, we let $\mathbb{P}_\alpha$ be the $\mathcal{I}_\alpha$ supported limit of $\langle \mathbb{P}_\beta : \beta < \alpha \rangle$, which we define as follows:
 - $p \in \mathbb{P}_\alpha$ if and only if p is a function with $\text{dom}(p) \in \mathcal{I}_\alpha$ such that $p \restriction \beta \in \mathbb{P}_\beta$ for all $\beta < \alpha$;
 - $\mathbb{P}_\alpha$ is ordered as follows: for $p, q \in \mathbb{P}_\alpha$, $p \leq_{\mathbb{P}_\alpha} q$ if and only if $p \restriction \beta \leq_{\mathbb{P}_\beta} q \restriction \beta$ for all $\beta < \alpha$.
 - $\mathbb{1}_{\mathbb{P}_\alpha}$ is the empty function $\emptyset$.
- If $\alpha = \beta + 1$ for some $\beta \in S$, let $p \in \mathbb{P}_\alpha$ if and only if p is a function with $\text{dom}(p) \in \mathcal{I}_\alpha$, $p \restriction \beta \in \mathbb{P}_\beta$ and, whenever $\beta \in \text{dom}(p)$, $p(\beta)$ is a $\mathbb{P}_\beta$ -name of minimal rank such that

$$\Vdash_{\mathbb{P}_\beta} "p(\beta) \in \text{Col}(\beta, < f(\beta))".$$

The poset $\mathbb{P}_\alpha$ is ordered as follows: for $p, q \in \mathbb{P}_\alpha$, let $p \leq q$ if and only if $p \restriction \beta \leq_{\mathbb{P}_\beta} q \restriction \beta$ and

- $\text{dom}(p), \text{dom}(q) \subseteq \beta$; or
- $\beta \in \text{dom}(q)$ and $p \restriction \beta \Vdash_{\mathbb{P}_\beta} "q(\beta) = \mathbb{1}_{\text{Col}(\beta, < f(\beta))}"$; or
- $\beta \in \text{dom}(q) \cap \text{dom}(p)$ and $p \restriction \beta \Vdash_{\mathbb{P}_\beta} "p(\beta) \leq_{\text{Col}(\beta, < f(\beta))} q(\beta)"$.

As in the previous case, let $\mathbb{1}_{\mathbb{P}_\alpha}$ be the empty function $\emptyset$.

- If $\alpha = \beta + 1$ for some $\beta \in T$, let $p \in \mathbb{P}_\alpha$ if and only if p is a function with $\text{dom}(p) \in \mathcal{I}_\alpha$, $p \restriction \beta \in \mathbb{P}_\beta$ and, whenever $\beta \in \text{dom}(p)$, $p(\beta)$ is a $\mathbb{T}_\beta$ -name of minimal rank such that

$$\Vdash_{\mathbb{T}_\beta} "p(\beta) \in \mathbb{R}_\beta". \quad (3.8)$$

The poset $\mathbb{P}_\alpha$ is ordered as follows: for $p, q \in \mathbb{P}_\alpha$, let $p \leq q$ if and only if $p \restriction \beta \leq_{\mathbb{P}_\beta} q \restriction \beta$ and

- $\text{dom}(p), \text{dom}(q) \subseteq \beta$; or

– $\beta \in \text{dom}(q)$ and

$$\pi_\beta(p \restriction \beta) = p \restriction (\beta \cap T) \Vdash_{\mathbb{T}_\beta} "q(\beta) = \mathbb{1}_{\mathbb{R}_\beta}"; \text{ or} \quad (3.9)$$

– $\beta \in \text{dom}(q) \cap \text{dom}(p)$ and

$$\pi_\beta(p \restriction \beta) = p \restriction (\beta \cap T) \Vdash_{\mathbb{T}_\beta} "p(\beta) \leq_{\mathbb{R}_\beta} q(\beta)". \quad (3.10)$$

Once again we let $\mathbb{1}_{\mathbb{P}_\alpha}$ be the empty function $\emptyset$.

Notice that by (3.6) we have that (3.9) and (3.10) are well defined. Furthermore, by (3.7) and proposition 1.15, the statement in (3.10) implies $p \restriction (\beta \cap T) \Vdash_{\mathbb{P}_\beta} "\vec{t}_\beta(p(\beta)) \leq_{\vec{t}_\beta(\mathbb{R}_\beta)} \vec{t}_\beta(q(\beta))"$ (see definition 1.14).

Proposition 3.7. $\mathbb{P}_\alpha$ satisfies (3.6) and (3.7).

Proof. Notice that for any $p \in \mathbb{P}_\alpha$ we have $p \restriction T \in \mathbb{T}_\alpha$ because $\text{dom}(p) \cap T$ is finite and $p(\beta)$ is a $\mathbb{T}_\beta$ -name satisfying (3.8) given that $\beta \in \text{dom}(p) \cap T$.

It is straightforward that $\pi_\alpha(\mathbb{1}_{\mathbb{P}_\alpha}) = \mathbb{1}_{\mathbb{T}_\alpha}$ and that π_α preserves the order. We show that π_α satisfies (3) from definition 1.16. Let $t \in \mathbb{T}_\alpha$ and $p \in \mathbb{P}_\alpha$ such that $t \leq \pi_\alpha(p) = p \restriction T$. Define $p' := t \restriction T \cup p \restriction S$. It is easy to see that $p' \in \mathbb{P}_\alpha$. By (3.3), it follows that $p' \restriction T = t \restriction T \leq t$.

By induction on $\beta \leq \alpha$, we show that $p' \restriction \beta \leq_{\mathbb{P}_\beta} p \restriction \beta$. If β is a limit ordinal and $p' \restriction \gamma \leq_{\mathbb{P}_\gamma} p \restriction \gamma$ for all $\gamma < \beta$, then we have $p' \restriction \beta \leq_{\mathbb{P}_\beta} p \restriction \beta$. Suppose now that $p' \restriction \beta \leq_{\mathbb{P}_\beta} p \restriction \beta$. If $\beta \in S \cap \text{dom}(p')$, we have $p'(\beta) = p(\beta)$, and thus we have trivially that $p \restriction \beta \Vdash_{\mathbb{P}_\beta} "p'(\beta) \leq p(\beta)". If $\beta \in T \cap \text{dom}(p')$, notice that since $t \restriction \beta + 1 \leq_{\mathbb{P}_{\beta+1}} p \restriction (T \cap \beta)$, we have that $t \restriction \beta \Vdash_{\mathbb{T}_\beta} "t(\beta) \leq p(\beta)". Since $p'(\beta) = t(\beta)$ and $p \restriction (\beta \cap T) \leq t \restriction \beta$, we have $p \restriction (\beta \cap T) \Vdash_{\mathbb{T}_\beta} "p'(\beta) \leq p(\beta)". Therefore in both cases we have $p' \restriction (\beta + 1) \leq_{\mathbb{P}_{\beta+1}} p \restriction (\beta + 1)$. So we conclude that $p' \leq_{\mathbb{P}_\alpha} p$, thus $\mathbb{P}_\alpha$ satisfies (3.6).$$$

Similarly, we have that $t \restriction T$ is a condition in $\mathbb{P}_\alpha$ for every $t \in \mathbb{T}_\alpha$. It is clear that $\iota_\alpha(\mathbb{1}_{\mathbb{T}_\alpha}) = \mathbb{1}_{\mathbb{P}_\alpha}$ and that ι_α preserves the order. We show that the function $t \mapsto t \restriction T$ also preserves incompatibility. Let $t_0, t_1 \in \mathbb{T}_\alpha$ and suppose there is $p \in \mathbb{P}_\alpha$ such that $p \restriction T \leq_{\mathbb{P}_\alpha} t_0 \restriction T, t_1 \restriction T$. By (3.3), it follows that $p \restriction T \leq t_0, t_1$.

Since $t \leq t \restriction T = \iota_\alpha(\pi_\alpha(t))$ for any $t \in \mathbb{T}_\alpha$, by (3.6) and lemma 1.18, we have that ι_α is a complete embedding, i.e. $\mathbb{P}_\alpha$ satisfies (3.7). (proposition 3.7) $\square$

By similar arguments, we also prove the following.

Proposition 3.8. $\langle \mathbb{P}_\beta : \beta \leq \alpha \rangle$ is a generalized iteration.

Proof. Let $\beta < \alpha \leq \kappa$. We show that the restriction from $\mathbb{P}_\alpha$ into $\mathbb{P}_\beta$ is a projection. Let $p \in \mathbb{P}_\beta$ and $q \in \mathbb{P}_\alpha$ be such that $p \leq_{\mathbb{P}_\beta} q \restriction \beta$. Define $q' := p \cup (q \restriction (\alpha \setminus \beta))$. Clearly $q' \restriction \beta = p$, and by induction it follows easily that $q' \leq_{\mathbb{P}_\alpha} q$.

Now we show that the identity function $\mathbb{P}_\beta \hookrightarrow \mathbb{P}_\alpha$ is a complete embedding. Clearly $\mathbb{1}_{\mathbb{P}_\beta} = \mathbb{1}_{\mathbb{P}_\alpha}$ and $\mathbb{P}_\beta \hookrightarrow \mathbb{P}_\alpha$ preserves the order. Given $p_0, p_1 \in \mathbb{P}_\beta$ such that $q \leq_{\mathbb{P}_\alpha} p_0, p_1$ for some $q \in \mathbb{P}_\alpha$, it is easy to see that $q \restriction \beta \leq_{\mathbb{P}_\beta} p_0, p_1$. Therefore $\mathbb{P}_\beta \hookrightarrow \mathbb{P}_\alpha$ preserves incompatibility.

Similarly to the argument in the proof of proposition 3.7, it is straightforward to prove that $p \leq_{\mathbb{P}_\alpha} p \restriction \beta$ for any $p \in \mathbb{P}_\alpha$. Therefore, by lemma 1.18 we have that $\mathbb{P}_\beta \hookrightarrow \mathbb{P}_\alpha$ is a complete embedding. (proposition 3.8) $\square$

This completes the construction of $\vec{\mathbb{P}}$.

Notice that, by the definition of $\vec{\mathbb{P}}$, for every $\alpha \in S$ we have

$$\mathbb{P}_{\alpha+1} \cong \mathbb{P}_\alpha * \text{Col}(\alpha, < f(\alpha))^{V^{\mathbb{P}_\alpha}}. \quad (3.11)$$

Remark 3.9. Since $\vec{\mathbb{P}}$ is a generalized iteration (proposition 3.8), for every $\alpha < \kappa$ there is some $\mathbb{P}_\alpha$ -name $\mathbb{Q}_\alpha$ such that $\mathbb{P}_{\alpha+1} \cong \mathbb{P}_\alpha * \mathbb{Q}_\alpha$. However, for $\alpha \in T$ such $\mathbb{Q}_\alpha$ does not need to correspond to $\mathbb{R}_\alpha$.

The following lemma follows straightforwardly (by induction on $\xi \leq \alpha$) from the definition of the iteration:

Lemma 3.10. *Let $\alpha \leq \kappa$ and $p, q \in \mathbb{P}_\alpha$. Then $p \leq q$ if and only if*

(I) $\forall \beta \in \text{dom}(q) \cap S$, we have either

- $p \restriction \beta \Vdash_{\mathbb{P}_\beta} "q(\beta) = \mathbb{1}_{\text{Col}(\beta, < f(\beta))}"$ or
- $\beta \in \text{dom}(p)$ and $p \restriction \beta \Vdash_{\mathbb{P}_\beta} "p(\beta) \leq_{\text{Col}(\beta, < f(\beta))} q(\beta)";$

and

(II) $\forall \beta \in \text{dom}(q) \cap T$, we have either

- $p \restriction (\beta \cap T) \Vdash_{\mathbb{T}_\beta} "q_\beta = \mathbb{1}_{\mathbb{R}_\beta}"$ or
- $\beta \in \text{dom}(p)$ and $p \restriction (\beta \cap T) \Vdash_{\mathbb{T}_\beta} "p(\beta) \leq_{\mathbb{R}_\beta} q(\beta)".$

3.2.2 Features of $\mathbb{P}_\kappa$

We shall now present some useful properties of $\vec{\mathbb{P}}$. We begin by showing that $\mathbb{P}_\kappa$ is “small” in the following sense.

Lemma 3.11. *Let $\alpha \in S \cup \{\kappa\}$. Then $\mathbb{P}_\alpha \subseteq V_\alpha$ and $\mathbb{P}_\alpha$ is an α -c.c. poset.*

Proof. First, we prove

Claim 3.11.1. *For all $\beta < \alpha$ the following holds:*

$$\text{There is } \mu_\beta \text{ with } \beta < \mu_\beta < \alpha \text{ such that } \mathbb{P}_\beta \subseteq V_{\mu_\beta}. \quad (3.12)$$

Proof of claim 3.11.1. We proceed by induction on β . Let $\beta < \alpha$ and suppose μ_β witnesses that β satisfies (3.12). We find $\mu_{\beta+1} < \alpha$ such that $\mathbb{P}_{\beta+1} \subseteq V_{\mu_{\beta+1}}$.

First, assume $\beta \in T$. Let $p \in \mathbb{P}_{\beta+1}$ be arbitrary. We have $p \restriction \beta \subseteq V_{\mu_\beta}$, thus we only need to find $\mu_{\beta+1} < \alpha$ such that $p(\beta) \in V_{\mu_{\beta+1}}$ given that $\beta \in \text{dom}(p)$. By (3.4), we have $\Vdash_{\mathbb{T}_\beta} \text{“}\mathbb{R}_\beta \in V_{\nu_\beta}[\tilde{G}_{\mathbb{T}_\beta}] \text{”}$. Since $\mathbb{T}_\beta$ is c.c.c. and ν_β is strongly Mahlo, by (3.8) there is an inaccessible cardinal $\mu_{\beta+1} < \nu_\beta \leq \alpha$ such that $\Vdash_{\mathbb{T}_\beta} \text{“}p(\beta) \in V_{\mu_{\beta+1}}[\tilde{G}_{\mathbb{T}_\beta}] \text{”}$.

By proposition 1.10 and the assumption that $p(\beta)$ is a name of minimal rank, we have $p(\beta) \in \mathcal{H}(\mu_{\beta+1}) = V_{\mu_{\beta+1}}$. Therefore $p \in V_{\mu_{\beta+1}}$.

Now assume $\beta \in S$. Since $\alpha \in \mathcal{C}$, we have $f(\beta) < \alpha$ and thus, since α is Mahlo, we can take $\mu_{\beta+1}$ an inaccessible cardinal such that $\mu_\beta, f(\beta) < \mu_{\beta+1} < \alpha$. It follows that $\Vdash_{\mathbb{P}_\beta} \text{“}p(\beta) \in \text{Col}(\beta, < f(\beta)) \subseteq V_{\mu_{\beta+1}}[\tilde{G}_{\mathbb{T}_\beta}] \text{”}$. Similarly to the previous case (where $\beta \in T$), we have that $\mathbb{P}_{\beta+1} \subseteq V_{\mu_{\beta+1}}$ for such $\mu_{\beta+1}$.

Now, suppose $\delta < \alpha$ is a limit ordinal such that (3.12) holds for each $\beta < \delta$. For each $\beta < \delta$, let μ_β be a witness that β satisfies (3.12). Since α is a strongly Mahlo cardinal, we can take a regular cardinal $\mu_\delta < \alpha$ such that $\mu_\delta > \sup_{\beta < \delta}(\mu_\beta)$. Then we have $\mathbb{P}_\delta \subseteq V_{\mu_\delta}$. Indeed, for any $p \in \mathbb{P}_\delta$, since μ_δ is regular we have $p \in V_{\mu_\delta}$, because

$$\text{rank}(p) = \sup\{\max\{\text{rank}(p(\beta)), \beta\} + 2 : \beta \in \text{dom}(p)\} < \mu_\delta.$$

(claim 3.11.1) $\square$

Since α is a regular cardinal, by the definition of $\mathcal{I}_\alpha$ (3.5), $\mathbb{P}_\alpha$ is the direct limit of $\langle \mathbb{P}_\beta : \beta < \alpha \rangle$ and therefore $\mathbb{P}_\alpha = \bigcup_{\beta < \alpha} \mathbb{P}_\beta$. Thus it follows from claim 3.11.1 that $\mathbb{P}_\alpha \subseteq V_\alpha$.

Now we prove that $\mathbb{P}_\alpha$ is an α -c.c. poset. Let $\langle p_\xi : \xi < \alpha \rangle$ be a sequence of conditions in $\mathbb{P}_\alpha$. Take $C \subseteq \alpha$ a club set such that

$$\forall \xi_1, \xi_2 \in C, \text{ if } \xi_1 < \xi_2 \text{ then } \text{dom}(p_{\xi_1}) \subseteq \xi_2 \quad (3.13)$$

Let Y be the set of all regular cardinals in C . Since α is Mahlo, Y is a stationary subset of α . We define the function

$$\begin{aligned} h : Y &\longrightarrow \alpha \\ \xi &\longmapsto \sup(\text{dom}(p_\xi) \cap \xi) \end{aligned}$$

By (3.5), h is regressive. By the pressing down lemma (lemma 1.1), we can take $Y' \subset Y$ stationary subset of α and $\xi^* < \alpha$ such that $h''Y' = \{\xi^*\}$. By claim 3.11.1, we have that $|\mathbb{P}_{\xi^*}| < \alpha$. Therefore, we can take $\xi_1, \xi_2 \in Y'$ with $\xi_1 < \xi_2$ such that $p_{\xi_1} \restriction \xi^* = p_{\xi_2} \restriction \xi^*$.

Notice that, by (3.13) and $f(\xi_1) = f(\xi_2) = \xi^*$, we have $\text{dom}(p_{\xi_1}) \cap \text{dom}(p_{\xi_2}) \subseteq \xi^*$. Therefore, since $p_{\xi_1} \restriction \xi^* = p_{\xi_2} \restriction \xi^*$, we have that $q := p_{\xi_1} \cup p_{\xi_2}$ is a condition in $\mathbb{P}_\alpha$. Clearly q witnesses that p_{ξ_1} and p_{ξ_2} are compatible, which concludes the proof. (lemma 3.11) $\square$

Remark 3.12. For $\alpha \in S \cup \{\kappa\}$, notice that α is in particular inaccessible. Therefore, we have $V_\alpha = \mathcal{H}(\alpha)$ and $|V_\alpha| = \alpha$. Thus by lemma 3.11 we have in particular that $|\mathbb{P}_\alpha| \leq \alpha$.

Definition 3.13. Let $\alpha^* < \gamma \leq \kappa$. Let G_{α^*} be a $\mathbb{P}_{\alpha^*}$ -generic over V . In $V[G_{\alpha^*}]$ we define

$$\begin{aligned} \mathbb{S}_{\gamma, \alpha^*} &:= (\mathbb{P}_\gamma / G_{\alpha^*}) \restriction S; \\ \mathbb{T}_{\gamma, \alpha^*} &:= (\mathbb{P}_\gamma / G_{\alpha^*}) \restriction T, \end{aligned}$$

both with the order induced by $\mathbb{P}_\gamma / G_{\alpha^*}$ (see notation 1.22 and definition 1.23).

Notice that $\mathbb{T}_{\alpha, 0}$ is equivalent to the previously fixed $\mathbb{T}_\alpha$.

The next lemma shows how the quotient $\mathbb{P}_\gamma / G_{\alpha^*}$ relates to the product $\mathbb{S}_{\gamma, \alpha^*} \times \mathbb{T}_{\gamma, \alpha^*}$.

Lemma 3.14. Let $\alpha^* < \gamma \leq \kappa$ and fix G_{α^*} a $\mathbb{P}_{\alpha^*}$ -generic over V .

- (1) In $V[G_{\alpha^*}]$, the function $\pi : \mathbb{S}_{\gamma, \alpha^*} \times \mathbb{T}_{\gamma, \alpha^*} \longrightarrow \mathbb{P}_\gamma / G_{\alpha^*}$ defined by $\langle s, t \rangle \longmapsto s \cup t$ is a projection.
- (2) In $V[G_{\alpha^*}]$, the function $\pi_{\gamma, \alpha^*} : \mathbb{P}_\gamma / G_{\alpha^*} \longrightarrow \mathbb{T}_{\gamma, \alpha^*}$ defined by $\pi_{\gamma, \alpha^*}(p) = p \restriction T$ for all $p \in \mathbb{P}_\gamma / G_{\alpha^*}$ is a projection.
- (3) Let H_{γ, α^*} be a $\mathbb{T}_{\gamma, \alpha^*}$ -generic over $V[G_{\alpha^*}]$. In $V[G_{\alpha^*}][H_{\gamma, \alpha^*}]$, denote by $\tau : \pi_{\gamma, \alpha^*}^{-1} \restriction H_{\gamma, \alpha^*} \longrightarrow \mathbb{S}_{\gamma, \alpha^*}$ ¹ the function $p \mapsto p \restriction S$. Then τ is a complete embedding.

¹Recall that by proposition 1.24(1) we have $\mathbb{P}_\gamma / G_{\alpha^*} \cong \mathbb{T}_{\gamma, \alpha^*} * \pi_{\gamma, \alpha^*}^{-1} \restriction H_{\gamma, \alpha^*}$ in $V[G_{\alpha^*}]$, thus $\pi_{\gamma, \alpha^*}^{-1} \restriction H_{\gamma, \alpha^*}$ can be regarded as a quotient of $\mathbb{P}_\gamma / G_{\alpha^*}$ by $\mathbb{T}_{\gamma, \alpha^*}$.

Proof. We start by proving (1). It is easy to see that π preserves the order and $\pi(\mathbb{1}_{\mathbb{S}_{\gamma, \alpha^*} \times \mathbb{T}_{\gamma, \alpha^*}}) = \mathbb{1}_{\mathbb{P}_\gamma / G_{\alpha^*}}$. We prove that π satisfies (3) from definition 1.16, i.e. we prove that:

Given $p \in \mathbb{P}_\gamma / G_{\alpha^*}$ and $\langle s, t \rangle \in \mathbb{S}_{\gamma, \alpha^*} \times \mathbb{T}_{\gamma, \alpha^*}$ such that $p \leq_{G_{\alpha^*}} s \cup t$, there is $\langle s', t' \rangle \in \mathbb{S}_{\gamma, \alpha^*} \times \mathbb{T}_{\gamma, \alpha^*}$ such that $s' \leq_{G_{\alpha^*}} s$, $t' \leq_{G_{\alpha^*}} t$ and $s' \cup t' \leq_{G_{\alpha^*}} p$.

Let p, s, t be as above. Since $p \leq_{G_{\alpha^*}} s \cup t$, by the definition of the order $\leq_{G_{\alpha^*}}$ (definition 1.23) we can take $r \in G_{\alpha^*}$ such that

$$r \cup p \leq r \cup s \cup t. \quad (3.14)$$

We construct $s' \in \mathbb{S}_{\gamma, \alpha^*}$ and $t' \in \mathbb{T}_{\gamma, \alpha^*}$. Let $t' := p \restriction T$ and define s' as follows: let $\text{dom}(s') := \text{dom}(p) \cap S$ and for each $\alpha \in \text{dom}(s')$, let $s'(\alpha)$ be a $\mathbb{P}_\alpha$ -name of minimal rank such that

$$\Vdash_{\mathbb{P}_\alpha} "s'(\alpha) = \begin{cases} p(\alpha) & , \text{ if } (r \cup p) \restriction \alpha \in G_{\mathbb{P}_\alpha} \\ s(\alpha) & , \text{ otherwise} \end{cases} " \quad (3.15)$$

It is enough to prove:

$$r \cup s' \cup t' \leq r \cup p; \quad (3.16)$$

$$r \cup t' \leq r \cup t; \quad (3.17)$$

$$r \cup s' \leq r \cup s. \quad (3.18)$$

First we show (3.16). Let $\alpha \in \text{dom}(p)$ and assume $(r \cup s' \cup t') \restriction \alpha \leq (r \cup p) \restriction \alpha$ as an induction hypothesis. For $\alpha \in T$, we have $t'(\alpha) = p(\alpha)$, so we are done. For $\alpha \in S$, by (3.15) and the induction hypothesis we have that $(r \cup s' \cup t') \restriction \alpha \Vdash_{\mathbb{P}_\alpha} "s'(\alpha) = p(\alpha)"$.

Now we prove (3.17). Let $\alpha \in \text{dom}(t)$. By (3.14), we have

$$(r \cup p) \restriction (T \cap \alpha) \Vdash_{\mathbb{T}_\alpha} "p(\alpha) \leq_{\mathbb{R}_\alpha} t(\alpha)".$$

Since $(r \cup p) \restriction (T \cap \alpha) = (r \restriction T) \cup (t' \restriction \alpha)$ and $t'(\alpha) = p(\alpha)$, we have that (3.17) holds.

Then we prove (3.18). Let $\alpha \in \text{dom}(s)$. We show that

$$\Vdash_{\mathbb{P}_\alpha} "s'(\alpha) \leq s(\alpha)".$$

Let G_α be an arbitrary $\mathbb{P}_\alpha$ -generic over V . We have two cases:

- Case 1: $r \cup p \restriction \alpha \notin G_\alpha$. In this case, we have

$$s'(\alpha)[G_\alpha] = s(\alpha)[G_\alpha].$$

- Case 2: $r \cup p \upharpoonright \alpha \in G_\alpha$. In this case we have

$$s'(\alpha)[G_\alpha] = p(\alpha)[G_\alpha];$$

by (3.14), we have

$$r \cup p \upharpoonright \alpha \Vdash_{\mathbb{P}_\alpha} "p(\alpha) \leq s(\alpha)".$$

Therefore

$$s'(\alpha)[G_\alpha] = p(\alpha)[G_\alpha] \leq s(\alpha)[G_\alpha].$$

This finishes the proof of (3.18) and of (1).

Now prove (2). Notice that (2) is just a “modulo G_{α^*} ” version of (3.6). As in the proof of proposition 3.7, it is enough to show that given $t \in \mathbb{T}_{\gamma, \alpha^*}$ and $p \in \mathbb{P}_\gamma / G_{\alpha^*}$ with $t \leq_{G_{\alpha^*}} \tau(p) = p \upharpoonright T$ we can find $p' \in \mathbb{P}_\gamma / G_{\alpha^*}$ such that $p' \leq_{G_{\alpha^*}} p$ and $\tau(p') \leq_{G_{\alpha^*}} t$.

We show that $p' := t \cup p \upharpoonright S$ is as desired. Since $t \leq_{G_{\alpha^*}} p \upharpoonright T$, we can take $q \in G_{\alpha^*}$ such that $q \cup t \leq q \cup p$. It follows by lemma 3.10 that $q \cup p' \leq q \cup p$ and $q \cup (p' \upharpoonright T) \leq q \cup p$.

Finally we prove (3). It is trivial that τ preserves the order and the maximal condition and the order. We show that τ preserves incompatibility. Let $p_0, p_1 \in \pi_{\gamma, \alpha^*}^{-1} "H_{\gamma, \alpha^*}"$ such that $p_0 \upharpoonright S \vee_{\mathbb{S}_{\gamma, \alpha^*}} p_1 \upharpoonright S$. Since $p_0, p_1 \in \pi_{\gamma, \alpha^*}^{-1} "H_{\gamma, \alpha^*}"$, there is $t \in H_{\gamma, \alpha^*}$ such that $t \leq_{G_{\alpha^*}} p_0 \upharpoonright T, p_1 \upharpoonright T$.

Let $s \in \mathbb{S}_{\gamma, \alpha^*}$ be such that $s \leq_{G_{\alpha^*}} p_0 \upharpoonright S, p_1 \upharpoonright S$. We claim that $q := s \cup t \in \pi_{\gamma, \alpha^*}^{-1} "H_{\gamma, \alpha^*}"$ is such that $q \leq_{G_{\alpha^*}} p_0, p_1$. Let $u \in G_\alpha^*$ be such that

$$u \cup t \leq u \cup p_0 \upharpoonright T; \text{ and} \tag{3.19}$$

$$u \cup s \leq u \cup p_0 \upharpoonright S. \tag{3.20}$$

Notice that we trivially have

$$u \cup s \cup t \leq s. \tag{3.21}$$

Let $\alpha \in \text{dom}(p_0)$. Without loss of generality, assume that $\alpha \in \text{dom}(q)$.

- Suppose $\alpha \in T$. By (3.19), we have

$$(u \cup t) \upharpoonright (\alpha \cap T) = (u \cup q) \upharpoonright (\alpha \cap T) \Vdash_{\mathbb{T}_\alpha} "q(\alpha) = t(\alpha) \leq p_0(\alpha)".$$

- Suppose $\alpha \in S$. By (3.20) and (3.21), we have

$$u \cup q \upharpoonright \alpha \Vdash_{\mathbb{P}_\alpha} "q(\alpha) = s(\alpha) \leq p_0".$$

Therefore we have $q \leq_{G_{\alpha^*}} p_0$. Similarly, we can also prove that $q \leq_{G_{\alpha^*}} p_1$.

Notice that for any $s \in \mathbb{S}_{\gamma, \alpha^*}$ we have $s \in \pi_{\gamma, \alpha^*}^{-1} "H_{\gamma, \alpha^*}$ because $\pi_{\gamma, \alpha^*}(s) = \emptyset \in \pi_{\gamma, \alpha^*}^{-1} "H_{\gamma, \alpha^*}$. It follows trivially that every $s \in \mathbb{S}_{\gamma, \alpha^*}$ is a reduct of itself with respect to τ . Therefore, by proposition 1.12, τ is a complete embedding. (lemma 3.14) $\square$

The following lemma shows the amount of closedness of the poset $\mathbb{S}_{\gamma, \alpha^*}$.

Lemma 3.15. *Let $\alpha^*, \gamma, G_{\alpha^*}$ and $\mathbb{S}_{\gamma, \alpha^*}$ be as in definition 3.13. Denote $\nu_{\alpha^*} := \min(S \setminus \alpha^*)$. Then $\mathbb{S}_{\gamma, \alpha^*}$ is $< \nu_{\alpha^*}$ -closed in $V[G_{\alpha^*}]$.*

Proof. Working in $V[G_{\alpha^*}]$, let $\langle p_\xi : \xi < \delta \rangle$ be a decreasing sequence in $\mathbb{S}_{\gamma, \alpha^*}$ for some $\delta < \nu_{\alpha^*}$. We shall construct a lower bound $q \in \mathbb{S}_{\gamma, \alpha^*}$ (with respect to the order of $\mathbb{S}_{\gamma, \alpha^*}$) for this sequence. Recall that the conditions in $\mathbb{S}_{\gamma, \alpha^*}$ are actually conditions from the poset $\mathbb{P}_\gamma^V$. Even though the conditions are taken in V , the sequence itself is taken in $V[G_{\alpha^*}]$, so we need to work with a name for the sequence. Let $\tilde{h} \in V^{\mathbb{P}_{\alpha^*}}$ be such that $\tilde{h}[G_{\alpha^*}] = \langle p_\xi : \xi < \delta \rangle$ and

$$V \models \Vdash_{\mathbb{P}_{\alpha^*}} \text{"}\tilde{h} : \delta \longrightarrow \mathbb{S}_{\gamma, \alpha^*} \text{ is a decreasing sequence in } \mathbb{S}_{\gamma, \alpha^*} \text{"}. \quad (3.22)$$

In V , we define the set

$$B := \{\alpha < \gamma : \exists r \in \mathbb{P}_{\alpha^*}, r \Vdash_{\mathbb{P}_{\alpha^*}} \text{"}\alpha \in \bigcup_{\xi < \delta} \text{dom}(\tilde{h}(\xi)) \text{"}\}.$$

We now construct the condition q by defining $\text{dom}(q) = B$ and, for each $\alpha \in B$, choosing $q(\alpha)$ a $\mathbb{P}_\alpha$ -name of minimal rank such that

$$V \models \Vdash_{\mathbb{P}_\alpha} \text{"}q(\alpha) = \bigcup \{\tilde{h}(\xi)(\alpha) : \xi < \delta \wedge \tilde{h}(\xi) \restriction \alpha \in G_{\mathbb{P}_\alpha}\} \text{"}. \quad (3.23)$$

To conclude the proof, it is enough to prove the following claims:

Claim 3.15.1. $q \in \mathbb{S}_{\gamma, \alpha^*}$.

Claim 3.15.2. q is a lower bound for $\langle p_\xi : \xi < \delta \rangle$.

Proof of claim 3.15.1. First, we have to prove that B is an Easton set. Let $\beta = \text{cf}(\beta) \leq \gamma$. Notice that for every $\beta < \nu_{\alpha^*}$ we have $B \cap \beta = \emptyset$. Therefore we assume $\beta \geq \nu_{\alpha^*}$. Notice that, since $\mathbb{P}_{\alpha^*}$ is ν_{α^*} -c.c. (by lemma 3.11), we have $\Vdash_{\mathbb{P}_{\alpha^*}} \text{"}\check{\beta} = \text{cf}(\check{\beta}) \text{"}$.

Since $\delta < \nu_{\alpha^*} \leq \beta$, by (3.22) we have

$$\Vdash_{\mathbb{P}_{\alpha^*}} \text{"}\beta \cap \bigcup_{\xi < \delta} \text{dom}(\tilde{h}(\xi)) \text{ is bounded in } \beta \text{"}.$$

Therefore, we can take a maximal antichain $\mathcal{A} \subseteq \mathbb{P}_{\alpha^*}$ such that for each $r \in \mathcal{A}$ there is $\beta_r < \beta$ with

$$r \Vdash_{\mathbb{P}_{\alpha^*}} \text{“} \beta \cap \bigcup_{\xi < \delta} \text{dom}(\tilde{h}(\xi)) \subseteq \beta_r \text{”}.$$

Since $\mathbb{P}_{\alpha^*}$ is ν_{α^*} -c.c. by lemma 3.11, we have $|\mathcal{A}| < \nu_{\alpha^*} \leq \beta = \text{cf}(\beta)$. Therefore

$$B \cap \beta \subseteq \sup_{r \in \mathcal{A}} \beta_r < \beta,$$

thus indeed $B \cap \beta$ is bounded in β .

Now, we show that for every $\alpha \in B$ we have

$$V \models \Vdash_{\mathbb{P}_{\alpha}} \text{“} q(\alpha) \in \text{Col}(\alpha, < f(\alpha)) \text{”}. \quad (3.24)$$

Let $\alpha \in B$. Clearly, we have $\alpha \geq \nu_{\alpha^*} > \delta$. Since the poset $\text{Col}(\alpha, < f(\alpha))$ is $< \alpha$ -closed, it is enough to prove that

$$\Vdash_{\mathbb{P}_{\alpha}} \text{“} \langle \tilde{h}(\xi)(\alpha) : \xi < \delta \wedge \tilde{h}(\xi) \restriction \alpha \in G_{\mathbb{P}_{\alpha}} \wedge \alpha \in \text{dom}(\tilde{h}(\xi)) \rangle \text{ is } \subseteq\text{-increasing”}.$$

Let G_{α} be an arbitrary $\mathbb{P}_{\alpha}$ -generic over V . Since $\mathbb{P}_{\alpha^*} \leq \mathbb{P}_{\alpha}$ (proposition 3.8), we have that $\tilde{h}$ is in particular a $\mathbb{P}_{\alpha}$ -name, so we can compute $\tilde{h}[G_{\alpha}]$. Furthermore, it follows that $G_{\alpha} \cap \mathbb{P}_{\alpha^*}$ is a $\mathbb{P}_{\alpha^*}$ -generic over V (not necessarily the same as the previously fixed G_{α^*}) and $\tilde{h}[G_{\alpha} \cap \mathbb{P}_{\alpha^*}] = \tilde{h}[G_{\alpha}]$. Let $\xi_1 < \xi_2 < \delta$ such that $\tilde{h}[G_{\alpha}](\xi_1) \restriction \alpha, \tilde{h}[G_{\alpha}](\xi_2) \restriction \alpha \in G_{\alpha}$ and $\alpha \in \text{dom}(\tilde{h}[G_{\alpha}](\xi_1)) \cap \text{dom}(\tilde{h}[G_{\alpha}](\xi_2))$. We show that

$$\tilde{h}[G_{\alpha}](\xi_1)(\alpha) \subseteq \tilde{h}[G_{\alpha}](\xi_2)(\alpha) \quad (3.25)$$

By (3.22), the sequence $\langle \tilde{h}[G_{\alpha}](\xi) : \xi < \delta \rangle$ is $\leq_{G_{\alpha} \cap \mathbb{P}_{\alpha^*}}$ -decreasing. Therefore, $\tilde{h}[G_{\alpha}](\xi_2) \leq_{G_{\alpha} \cap \mathbb{P}_{\alpha^*}} \tilde{h}[G_{\alpha}](\xi_1)$, so we can take $r \in G_{\alpha} \cap \mathbb{P}_{\alpha^*}$ such that

$$r \cup \tilde{h}[G_{\alpha}](\xi_2) \leq r \cup \tilde{h}[G_{\alpha}](\xi_1).$$

Since $\alpha \in S$, we have

$$r \cup \tilde{h}[G_{\alpha}](\xi_2) \restriction \alpha \Vdash_{\mathbb{P}_{\alpha}} \text{“} \tilde{h}(\xi_1)(\alpha) \subseteq \tilde{h}(\xi_2)(\alpha) \text{”}. \quad (3.26)$$

Since $r \in G_{\alpha}$ and by the assumption $\tilde{h}[G_{\alpha}](\xi_2) \restriction \alpha \in G_{\alpha}$, we have that $r \cup \tilde{h}[G_{\alpha}](\xi_2) \restriction \alpha \in G_{\alpha}$. Then (3.26) implies (3.25). So we conclude that (3.24) holds for α . (claim 3.15.1) $\square$

Proof of claim 3.15.2. Let $\xi < \delta$. We can take $r \in G_{\alpha^*}$ such that

$$V \models r \Vdash_{\mathbb{P}_{\alpha^*}} \text{“} \check{h}(\xi) = \check{p}_\xi \text{”}. \quad (3.27)$$

We show that r witnesses $q \leq_{G_{\alpha^*}} p_\xi$, i.e. $r \cup q \leq r \cup p_\xi$.

First, notice that

$$r \Vdash_{\mathbb{P}_{\alpha^*}} \text{“} \text{dom}(\check{p}_\xi) \subseteq \check{B} \text{”}.$$

Therefore, since $p_\xi, B \in V$, we have $\text{dom}(r \cup q) \supseteq \text{dom}(r \cup p_\xi)$.

Now let $\alpha \in \text{dom}(p_\xi)$ and suppose

$$r \cup q \restriction \alpha \leq r \cup p_\xi \restriction \alpha. \quad (3.28)$$

Let G_α be an arbitrary $\mathbb{P}_\alpha$ -generic over V containing $r \cup q \restriction \alpha$. In particular we have $r \in G_\alpha \cap \mathbb{P}_{\alpha^*}$, so by (3.27) we have $\check{h}[G_\alpha](\xi) = p_\xi$. By (3.28), we have $\check{h}[G_\alpha](\xi) \restriction \alpha \in G_\alpha$. Thus, by (3.23) we have $q(\alpha)[G_\alpha] \supseteq p_\xi(\alpha)[G_\alpha]$. This proves that

$$V \models r \cup q \restriction \alpha \Vdash_{\mathbb{P}_\alpha} \text{“} q(\alpha) \supseteq p_\xi(\alpha) \text{”}.$$

(claim 3.15.2) $\square$

(lemma 3.15) $\square$

Corollary 3.16. *Let $\alpha^* < \gamma \leq \kappa$, G_{α^*} be a $\mathbb{P}_{\alpha^*}$ -generic over V and H_{γ, α^*} be a $\mathbb{T}_{\gamma, \alpha^*}$ -generic over $V[G_{\alpha^*}]$. Then $\pi_{\gamma, \alpha}^{-1} \text{“} H_{\gamma, \alpha^*} \text{”}$ is $< \nu_{\alpha^*}$ -distributive in $V[G_{\alpha^*}][H_{\gamma, \alpha^*}]$, where $\nu_{\alpha^*} := \min(S \setminus \alpha^*)$.*

Proof. Let K be a $\pi_{\gamma, \alpha}^{-1} \text{“} H_{\gamma, \alpha^*} \text{”}$ -generic over $V[G_{\alpha^*}][H_{\gamma, \alpha^*}]$ and let $f : \delta \longrightarrow V[G_{\alpha^*}][H_{\gamma, \alpha^*}]$ with $\delta < \nu_{\alpha^*}$ and $f \in V[H_{\gamma, \alpha^*}][K]$. By lemma 3.14(3) and proposition 1.24(3), there is some $\pi_{\gamma, \alpha}^{-1} \text{“} H_{\gamma, \alpha^*} \text{”}$ -name $\check{Q}$ such that $\mathbb{S}_{\gamma, \alpha^*} \cong (\pi_{\gamma, \alpha}^{-1} \text{“} H_{\gamma, \alpha^*} \text{”}) * \check{Q}$. Therefore we can take an $\mathbb{S}_{\gamma, \alpha^*}$ -generic K' over $V[G_{\alpha^*}][H_{\gamma, \alpha^*}]$ such that $V[G_{\alpha^*}][H_{\gamma, \alpha^*}][K] \subseteq V[G_{\alpha^*}][H_{\gamma, \alpha^*}][K']$.

By lemma 3.15, $\mathbb{S}_{\gamma, \alpha^*}$ is $< \nu_{\alpha^*}$ -closed in $V[G_{\alpha^*}]$. Then, by Easton lemma (lemma 1.27(ii)), $\mathbb{S}_{\gamma, \alpha^*}$ is $< \nu_{\alpha^*}$ -distributive in $V[G_{\alpha^*}][H_{\gamma, \alpha^*}]$. Therefore, since $f \in V[G_{\alpha^*}][H_{\gamma, \alpha^*}][K']$, we have $f \in V[G_{\alpha^*}][H_{\gamma, \alpha^*}]$. So indeed $\pi_{\gamma, \alpha}^{-1} \text{“} H_{\gamma, \alpha^*} \text{”}$ is $< \nu_{\alpha^*}$ -distributive in $V[H_{\gamma, \alpha^*}]$. (corollary 3.16) $\square$

Since $\nu_0 = \min(S) > \aleph_1$, corollary 3.16 implies that

$$\Vdash_{\mathbb{T}_\kappa} \text{“} \pi_{\kappa, 0}^{-1} \text{“} H_{\kappa, 0} \text{” is } \sigma\text{-distributive”}. \quad (3.29)$$

In other words, $\mathbb{P}_\kappa$ is equivalent to a two step iteration of a c.c.c. poset and a σ -distributive poset. In particular, all of the new reals added by $\mathbb{P}_\kappa$ are added exclusively by the $\mathbb{T}_\kappa$ part.

While lemma 3.15 describes an useful property of $\mathbb{S}_{\gamma, \alpha^*}$, the next lemma similarly describes an useful property of $\mathbb{T}_{\gamma, \alpha^*}$.

Lemma 3.17. *Let $\alpha^* < \gamma \leq \kappa$, and let G_{α^*} be a $\mathbb{P}_{\alpha^*}$ -generic over V . Then $\mathbb{T}_{\gamma, \alpha^*}$ is c.c.c. in $V[G_{\alpha^*}]$.*

Proof. Denote $\mathbb{T}_{\alpha^*}^* := \iota_{\alpha^*}'' \mathbb{T}_{\alpha^*} = \mathbb{P}_{\alpha^*} \restriction T$. Notice that by proposition 3.7 we have $\mathbb{T}_{\alpha^*} \cong \mathbb{T}_{\alpha^*}^* \leq \mathbb{P}_{\alpha^*}$. Working in $V[G_{\alpha^*}]$, define $H_{\alpha^*} := G_{\alpha^*} \cap \mathbb{T}_{\alpha^*}^*$. It follows that H_{α^*} is a $\mathbb{T}_\gamma^*$ -generic over V .

Notice that, by the definition of quotient (definition 1.23), both posets $\mathbb{T}_{\gamma, \alpha^*}$ and $\mathbb{T}_\gamma^*/H_{\alpha^*}$ have the same set of conditions (i.e. $\mathbb{P}_\gamma \restriction (T \cap (\gamma \setminus \alpha^*))$).

Actually, their orders ($\leq_{G_{\alpha^*}}$ and $\leq_{H_{\alpha^*}}$) also coincide. Indeed, let $t, t' \in \mathbb{P}_\gamma \restriction (T \cap (\gamma \setminus \alpha^*))$. Clearly $t' \leq_{H_{\alpha^*}} t$ implies $t' \leq_{G_{\alpha^*}} t$ because $H_{\alpha^*} \subseteq G_{\alpha^*}$. For the converse, suppose $t' \leq_{G_{\alpha^*}} t$. Then there is $q \in G_{\alpha^*}$ such that

$$q \cup t' \leq q \cup t. \quad (3.30)$$

Since $q \leq q \restriction T$, we have $q \restriction T \in G_{\alpha^*}$, thus $q \restriction T \in H_{\alpha^*}$. It follows from the next claim that $q \restriction T$ witnesses $t' \leq_{H_{\alpha^*}} t$.

Claim 3.17.1. $(q \restriction T) \cup t' \leq (q \restriction T) \cup t$.

Proof of claim 3.17.1. Without loss of generality, assume $\text{dom}(t') \supseteq \text{dom}(t)$. By lemma 3.10, it is enough to prove that

$$((q \restriction T) \cup t') \restriction (\beta \cap T) \Vdash_{\mathbb{T}_\beta} "t'(\beta) \leq_{\tilde{R}_\beta} t(\beta)" \quad (3.31)$$

for all $\beta \in \text{dom}(t) \subseteq T$. Since $(q \cup t') \restriction (\beta \cap T) = ((q \restriction T) \cup t') \restriction (\beta \cap T)$, we have that (3.31) follows from (3.30) and lemma 3.10. (claim 3.17.1) $\square$

Since $\mathbb{T}_\gamma$ and $\mathbb{T}_{\alpha^*}$ are c.c.c. in V and $\mathbb{T}_\gamma \cong \mathbb{T}_\gamma^* \cong \mathbb{T}_{\alpha^*}^* * \mathbb{T}_\gamma^*/G_{\mathbb{T}_{\alpha^*}^*}$, we have

$$V[H_{\alpha^*}] \models " \mathbb{T}_\gamma^*/H_{\alpha^*} \text{ is a c.c.c. poset } ". \quad (3.32)$$

By (3.29), we have that $V[G_{\alpha^*}]$ is a σ -distributive extension of $V[H_{\alpha^*}]$. Therefore it follows from (3.32) that

$$V[G_{\alpha^*}] = V[H_{\alpha^*} * H'] \models " \mathbb{T}_\gamma/H_{\alpha^*} \text{ is a c.c.c. poset } ".$$

(lemma 3.17) $\square$

Corollary 3.18. $\Vdash_{\mathbb{P}_\kappa} " \tilde{\kappa} \text{ is weakly Mahlo } ".$

Proof. Let $\alpha \in S$. We can write $\mathbb{P}_\kappa \cong \mathbb{P}_\alpha * \underset{\sim}{A}_\alpha * \underset{\sim}{B}_\alpha$, where $\mathbb{P}_\alpha$ is α -c.c. (lemma 3.11), $\underset{\sim}{A}_\alpha$ is a $\mathbb{P}_\alpha$ -name for a c.c.c. poset (by lemma 3.17) and $\underset{\sim}{B}_\alpha$ is a $\mathbb{P}_\kappa \cong \mathbb{P}_\alpha * \underset{\sim}{A}_\alpha$ -name for a $< \alpha$ -distributive poset (by corollary 3.16). Therefore, α remains regular after forcing with $\mathbb{P}_\kappa$.

Since $\mathbb{P}_\kappa$ is κ -c.c. (lemma 3.11), S remains a stationary subset of κ in $V^{\mathbb{P}_\kappa}$, so κ remains weakly Mahlo after forcing with $\mathbb{P}_\kappa$. (corollary 3.18) $\square$

Actually, we shall prove in lemma 3.21 that $\mathbb{P}_\kappa * \text{Col}(\kappa, < \theta)^{V^{\mathbb{P}_\kappa}}$ forces the continuum to carry a σ -saturated ideal for any large enough regular θ .

3.2.3 Main result

Now we present the main lemma which characterises the poset $\mathbb{P}_\kappa$.

Lemma 3.19. *Let $\mathcal{P}$ and $\mathcal{Q}$ be as in Then*

$$V^{\mathbb{P}_\kappa} \models "2^{\aleph_0} = \kappa \text{ is Laver generically supercompact* for } \langle \mathcal{P}, \mathcal{Q} \rangle".$$

Proof. Fix G_κ a $\mathbb{P}_\kappa$ -generic over V and let $\eta > \kappa$. We will show that κ is Laver generically supercompact* for $\langle \mathcal{P}, \mathcal{Q} \rangle$ in $V[G_\kappa]$. Let $\theta > \kappa$. Without loss of generality, we assume

$$\eta > (2^\theta)^{V[G_\kappa]}. \quad (3.33)$$

Since f is a Laver function, we can take $j : V \longrightarrow M$ an elementary embedding with critical point κ such that $j(\kappa) > \eta$,

$${}^\eta M \subseteq M \subseteq V \text{ and} \quad (3.34)$$

$$j(f)(\kappa) = \theta. \quad (3.35)$$

By elementarity, $j(\overrightarrow{\mathbb{P}})$ is a mixed support iteration of length $j(\kappa)$ in M . We denote $\langle \mathbb{P}_\alpha^* : \alpha \leq j(\kappa) \rangle := j(\overrightarrow{\mathbb{P}})$. Since $j \restriction V_\kappa$ is the identity, by lemma 3.11 we have that $\mathbb{P}_\alpha^* = \mathbb{P}_\alpha$ for all $\alpha < \kappa$. By the construction of $\overrightarrow{\mathbb{P}}$, since κ is regular, $\mathbb{P}_\kappa$ is the direct limit of $\overrightarrow{\mathbb{P}} \restriction \kappa$. By elementarity, κ is regular in M , so $\mathbb{P}_\kappa^*$ is the direct limit of $\langle \mathbb{P}_\alpha^* : \alpha < \kappa \rangle = \overrightarrow{\mathbb{P}} \restriction \kappa$, therefore $\mathbb{P}_\kappa^* = \mathbb{P}_\kappa$.

Claim 3.19.1. $\kappa \in j(S)$.

Proof of claim 3.19.1. By elementarity, (3.1) and (3.2) we have

$$j(S) = \{\alpha < j(\kappa) : j(f)''\alpha \subseteq \alpha \wedge M \models "\alpha \text{ is strongly Mahlo}"\}.$$

Since $\text{crit}(j) = \kappa$, we have $j(f)''\kappa = f''\kappa \subseteq \kappa$. We just need to prove that κ is strongly Mahlo in M . Since κ is supercompact (in V), we have in particular

$$V \models \text{“}\{\alpha < \kappa : \alpha \text{ is inaccessible}\} \text{ is a stationary subset of } \kappa\text{”}.$$

Since κ is the critical point of j , we have that α is inaccessible in M if and only if it is inaccessible in V . Therefore, since $M \subset V$, we have

$$M \models \text{“}\{\alpha < \kappa : M \models \text{“}\alpha \text{ is inaccessible”}\} \text{ is a stationary subset of } \kappa\text{”}.$$

So we conclude that $\kappa \in j(S)$. (claim 3.19.1) $\square$

By claim 3.19.1, (3.11), (3.35) and by elementarity it follows that $\mathbb{P}_{\kappa+1}^* \cong \mathbb{P}_{\kappa}^* * \text{Col}(\kappa, < \theta)^{M^{\mathbb{P}_{\kappa}}}$. Since $\mathbb{P}_{\kappa}$ is κ -c.c. (lemma 3.11), by (3.34) and proposition 1.25 we have ${}^{<\kappa}M[G_{\kappa}] \cap V[G_{\kappa}] \subseteq M[G_{\kappa}]$. It follows that $\text{Col}(\kappa, < \theta)^{V[G_{\kappa}]} = \text{Col}(\kappa, < \theta)^{M[G_{\kappa}]}$. Therefore we conclude that $\mathbb{P}_{\kappa+1}^* \cong \mathbb{P}_{\kappa}^* * \text{Col}(\kappa, < \theta)^{V^{\mathbb{P}_{\kappa}}}$. In particular, by proposition 3.8, we have

$$\mathbb{P}_{\kappa}^* * \text{Col}(\kappa, < \theta)^{V^{\mathbb{P}_{\kappa}}} \cong \mathbb{P}_{\kappa+1}^* \triangleleft \mathbb{P}_{j(\kappa)}^* = j(\mathbb{P}_{\kappa}). \quad (3.36)$$

Working in $V[G_{\kappa}]$, since G_{κ} is a $\mathbb{P}_{\kappa}^*$ -generic filter we can consider the quotient $\mathbb{P}_{j(\kappa)}^*/G_{\kappa}$ as in definition 1.23. It follows from (3.36) that $\text{Col}(\kappa, < \theta) \triangleleft \mathbb{P}_{j(\kappa)}^*/G_{\kappa}$, thus by proposition 1.24(3) we can take a $\text{Col}(\kappa, < \theta)$ -name $\tilde{Q}$ such that

$$V[G_{\kappa}] \models \text{Col}(\kappa, < \theta) * \tilde{Q} \cong \mathbb{P}_{j(\kappa)}^*/G_{\kappa}. \quad (3.37)$$

We show that $\tilde{Q}$ witnesses that κ is Laver generically supercompact* for $\langle \mathcal{P}, \mathcal{Q} \rangle$ in $V[G_{\kappa}]$.

Let $G(\kappa)$ be an arbitrary $\mathbb{P}_{\kappa+1}^*/G_{\kappa}$ -generic over $V[G_{\kappa}]$ and let $G_{\kappa+1}$ be a $\mathbb{P}_{\kappa+1}^*$ -generic over V such that $G_{\kappa+1} \cong G_{\kappa} * G(\kappa)$. By elementarity and lemmas 3.14(1), 3.15 and 3.17, we have:

$$M[G_{\kappa+1}] \models \text{“}\mathbb{P}_{j(\kappa)}^*/G_{\kappa+1} \text{ is the projection of the product of a c.c.c. poset and a } \sigma\text{-closed poset”}.$$

By lemma 3.11 and (3.33) it follows that $\mathbb{P}_{\kappa+1}^*$ is an η -c.c. poset (if θ is inaccessible, then it is θ -c.c.). Therefore, by proposition 1.25, we have

$${}^{\eta}M[G_{\kappa+1}] \cap V[G_{\kappa+1}] \subseteq M[G_{\kappa+1}].$$

Since $\eta > \omega_1$, it then follows that

$$V[G_{\kappa+1}] \models \text{“ } \mathbb{P}_{j(\kappa)}^*/G_{\kappa+1} \text{ is the projection of the product of a c.c.c. poset and a } \sigma\text{-closed poset ”.} \quad (3.38)$$

Since $\text{Col}(\kappa, < \theta)^{V[G_\kappa]}$ is proper in $V[G_\kappa]$ and $G(\kappa)$ is arbitrary, it follows from (3.38) that $V[G_\kappa] \models \text{“ } \Vdash_{\text{Col}(\kappa, < \theta)} \text{“ } \tilde{Q} \text{ is proper ” ”.}$

Let K be an arbitrary $\text{Col}(\kappa, < \theta)^{V[G_\kappa]} * \tilde{Q}$ -generic over $V[G_\kappa]$. It follows from (3.36) and (3.37) that there is $G_{j(\kappa)}$ a $\mathbb{P}_{j(\kappa)}^*$ -generic over V such that $G_{j(\kappa)} \cong G_\kappa * K$. Since $\mathbb{P}_\kappa \subseteq V_\kappa$ (lemma 3.11), we have $j''G_\kappa \subseteq G_\kappa \subseteq G_{j(\kappa)}$. Therefore, by proposition 1.26, we can extend j into an elementary embedding $J : V[G_\kappa] \longrightarrow M[G_{j(\kappa)}] = M[G_\kappa * K]$. We have $\tilde{Q}, G_\kappa \in M[G_\kappa]$. Furthermore, we also have $J''\eta = j''\eta \in M$. So we conclude that $\tilde{Q}, K, M[G_{j(\kappa)}]$ and J witness that

$$V[G_\kappa] \models \text{“ } \kappa \text{ is Laver generically supercompact for } \langle \mathcal{P}, \mathcal{Q} \rangle \text{ ”.}$$

To complete the proof, we show that

$$\omega_1 M[G_{j(\kappa)}] \cap V[G_{j(\kappa)}] \subseteq M[G_{j(\kappa)}]. \quad (3.39)$$

By lemma 3.14(2), corollary 3.16 and elementarity, we have that there is $\tilde{\mathbb{S}}^* \in M$ a $j(\mathbb{T}_\kappa^*)$ -name for a poset such that

$$M \models \text{“ } \mathbb{P}_{j(\kappa)}^* \cong j(\mathbb{T}_\kappa^*) * \tilde{\mathbb{S}}^*, j(\mathbb{T}_\kappa^*) \text{ is c.c.c. and } \Vdash_{j(\mathbb{T}_\kappa^*)} \text{“ } \tilde{\mathbb{S}}^* \text{ is } \sigma\text{-distributive ” ”.} \quad (3.40)$$

Since ${}^\eta M \subset M$ and $\eta > \omega_1$, it follows that

$$V \models \text{“ } \mathbb{P}_{j(\kappa)}^* \cong j(\mathbb{T}_\kappa^*) * \tilde{\mathbb{S}}^*, j(\mathbb{T}_\kappa^*) \text{ is c.c.c. and } \Vdash_{j(\mathbb{T}_\kappa^*)} \text{“ } \tilde{\mathbb{S}}^* \text{ is } \sigma\text{-distributive ” ”.} \quad (3.41)$$

By (3.41) we can take A a $j(\mathbb{T}_\kappa^*)$ -generic over V and B a $\tilde{\mathbb{S}}^*[A]$ -generic over $V[A]$ such that $A * B \cong G_{j(\kappa)}$.

Since $j(\mathbb{T}_\kappa^*)$ is c.c.c., by proposition 1.25, (3.41) and (3.34) we have

$${}^{<\omega_1} M[A] \cap V[A] \subseteq M[A]. \quad (3.42)$$

By (3.41), we have that $\tilde{\mathbb{S}}^*[A]$ is σ -distributive in $V[A]$. Since $V[G_{j(\kappa)}] = V[A][B]$ and $M[G_{j(\kappa)}] = M[A][B]$, we have that (3.42) implies (3.39). (lemma 3.19) $\square$

We recall the definition of a σ -saturated ideal.

Definition 3.20. An ideal $\mathfrak{I} \subseteq \mathcal{P}(\kappa)$ is σ -saturated if whenever $\langle U_\xi : \xi < \omega_1 \rangle$ is a sequence of elements of $\mathcal{P}(\kappa) \setminus \mathfrak{I}$, there are distinct $\xi_0, \xi_1 < \omega_1$ such that $U_{\xi_0} \cap U_{\xi_1} \notin \mathfrak{I}$.

We have already proved that the cardinal κ is still a Mahlo cardinal in $V^{\mathbb{P}_\kappa}$ (corollary 3.18). The following lemma shows that $\mathbb{P}_\kappa$ preserves a larger amount of the largeness of κ .

Lemma 3.21. Assume $V \models GCH$. Let $\theta > \kappa$ be an inaccessible cardinal and suppose G is a $\mathbb{P}_\kappa$ -generic over V and H is a $\text{Col}(\kappa, < \theta)^{V[G]}$ -generic over $V[G]$. Then

$$V[G * H] \models \text{“} \kappa \text{ carries a } \sigma\text{-saturated } \kappa\text{-closed ideal”}.$$

Proof. Let $U \subseteq \mathcal{P}([\lambda]^{<\kappa})$ be a normal measure such that $j(f)(\kappa) = \theta$, where $j := j_U \longrightarrow M$ is the elementary embedding associated with U .

As in the proof of lemma 3.19, $\langle \mathbb{P}_\alpha^* : \alpha \leq j(\kappa) \rangle := j(\overrightarrow{\mathbb{P}})$. As before, we have $\langle \mathbb{P}_\alpha^* : \alpha \leq \kappa \rangle = \overrightarrow{\mathbb{P}}$ and $\mathbb{P}_{\kappa+1}^* \cong \mathbb{P}_\kappa * \underset{\sim}{Q}$, where $\underset{\sim}{Q}$ is a $\mathbb{P}_\kappa$ -name for $\text{Col}(\kappa, < \lambda)^{V^{\mathbb{P}_\kappa}}$. Denote $\mathbb{P}_{j(\kappa)+1}^* := j(\mathbb{P}_{\kappa+1}^*) \cong \mathbb{P}_{j(\kappa)}^* * j(\underset{\sim}{Q})$. We can take a $\mathbb{P}_{\kappa+1}^*$ -generic $G_{\kappa+1}$ over V such that $G_{\kappa+1} \cong G * H$.

Similarly to definition 3.13, define in $M[G_{\kappa+1}]$

$$\mathbb{S}_{j(\kappa)+1, \kappa+1} = (\mathbb{P}_{j(\kappa)+1}^*) / (G_{\kappa+1}) \restriction S,$$

$$\mathbb{T}_{j(\kappa)+1, \kappa+1} = (\mathbb{P}_{j(\kappa)+1}^*) / (G_{\kappa+1}) \restriction T,$$

both ordered by $\leq_{G_{\kappa+1}}$.

Recall that j is a θ -supercompact embedding, so we have ${}^\theta M \subseteq M$. Then it follows that $(\theta^+)^M = (\theta^+)^V$. Since $\mathbb{P}_\kappa$ is κ -c.c. (lemma 3.11) and θ is inaccessible, we have that $\mathbb{P}_{\kappa+1}^*$ is θ -c.c. in V , thus $(\theta^+)^V = (\theta^+)^{V[G_{\kappa+1}]} = (\theta^+)^{M[G_{\kappa+1}]}$. We can just refer to θ^+ without specifying the model where the successor operator is considered.

By lemma 3.15 and elementarity, we have that $\mathbb{S}_{j(\kappa)+1, \kappa+1}$ is $< \nu_{\kappa+1}$ -closed in $M[G_{\kappa+1}]$ for $\nu_{\kappa+1} := j(S) \setminus \kappa + 1$.

By elementarity and (3.1), it follows that $j(\mathcal{C}) = \{\alpha < j(\kappa) : j(f)''\alpha \subseteq \alpha\}$. Since $j(f)(\kappa) = \theta$, κ witnesses that $\theta \notin j(\mathcal{C})$. Therefore we have $\nu_{\kappa+1} > \theta$. Furthermore, we also have $M \models \text{“} \theta^+ \text{ is not Mahlo”}$, so $\theta^+ \notin j(S)$. Therefore $\nu_{\kappa+1} > \theta^+$, thus

$$M[G_{\kappa+1}] \models \text{“} \mathbb{S}_{j(\kappa)+1, \kappa+1} \text{ is } < \theta^+\text{-closed”}. \quad (3.43)$$

It is straightforward to see that $M[G_{\kappa+1}] \models |\mathbb{S}_{j(\kappa)+1, \kappa+1}| = j(\kappa)$. Recall that we assumed that GCH holds in V . By elementarity, $M \models \text{GCH}$. Since $|\mathbb{P}_{\kappa+1}^*|^M = \theta$, we have that GCH holds above θ in $M[G_{\kappa+1}]$, thus

$$M[G_{\kappa+1}] \models |\mathcal{P}(\mathbb{S}_{j(\kappa)+1, \kappa+1})| = j(\theta)^+ = j(\theta^+)$$

Since $j = j_U$ is the elementary embedding associated to a normal measure $U \subseteq \mathcal{P}([\theta]^{<\kappa})$, by a direct calculation (using GCH), we have $V \models |j(\theta^+)| = \theta^+$. Therefore, we have

$$V[G_{\kappa+1}] \models |\mathcal{P}(\mathbb{S}_{j(\kappa)+1, \kappa+1}) \cap M[G_{\kappa+1}]| = \theta^+. \quad (3.44)$$

Working in M , take $\tilde{m}$ a $\mathbb{P}_{j(\kappa)}^*$ -name of minimal rank such that

$$M \models \text{“} \Vdash_{\mathbb{P}_{j(\kappa)}^*} \text{“} \tilde{m} = \bigcup \{j(p(\kappa)) : p \in G_{\mathbb{P}_{j(\kappa)}^*} \wedge \kappa \in \text{dom}(p)\} \text{””}. \quad (3.45)$$

Working in $V[G_{\kappa+1}]$, by (3.44) we can take $\langle D_\xi : \xi < \theta^+ \rangle$ an enumeration of all the dense subsets of $\mathbb{S}_{j(\kappa)+1, \kappa+1}$ lying in $M[G_{\kappa+1}]$. Since ${}^\theta M[G_{\kappa+1}] \cap V[G_{\kappa+1}] \subseteq M[G_{\kappa+1}]$ (by proposition 1.25) and by (3.43), we can construct in $V[G_{\kappa+1}]$ a decreasing sequence $\langle d_\xi : \xi < \theta^+ \rangle$ of conditions in $\mathbb{S}_{j(\kappa)+1, \kappa+1}$ such that $d_0(j(\kappa)) = \tilde{m}$ and

$$\forall \xi < \theta^+, d_{\xi+1} \in D_\xi. \quad (3.46)$$

In $V[G_{\kappa+1}]$, let K_S be the filter generated in $\mathbb{S}_{j(\kappa)+1, \kappa+1}$ by the sequence $\langle d_\xi : \xi < \theta^+ \rangle$. By (3.46), K_S is a $\mathbb{S}_{j(\kappa)+1, \kappa+1}$ -generic filter over $M[G_{\kappa+1}]$.

In $M[G_{\kappa+1}]$, let $\pi : \mathbb{S}_{j(\kappa)+1, \kappa+1} \times \mathbb{T}_{j(\kappa)+1, \kappa+1} \longrightarrow \mathbb{P}_{j(\kappa)+1}^*/(G_{\kappa+1})$ be the function defined by $\langle s, t \rangle \mapsto s \cup t$ for every $\langle s, t \rangle \in \mathbb{S}_{j(\kappa)+1, \kappa+1} \times \mathbb{T}_{j(\kappa)+1, \kappa+1}$. By elementarity and lemma 3.14(1), it follows that π is a projection.

Now, let K_T be an arbitrary $\mathbb{T}_{j(\kappa)+1, \kappa+1}$ -generic filter over $V[G_{\kappa+1}]$. In $V[G_{\kappa+1} * K_T]$, let K be the filter in $\mathbb{P}_{j(\kappa)+1}^*/(G_{\kappa+1})$ generated by $\pi''(K_S \times K_T)$. We have that K is a $\mathbb{P}_{j(\kappa)+1}^*/(G_{\kappa+1})$ -generic over $M[G_{\kappa+1}]$. Let $G_{j(\kappa)+1}$ be a $\mathbb{P}_{j(\kappa)+1}^*$ -generic over V such that $G_{j(\kappa)+1} \cong G_{\kappa+1} * K$. We claim that

$$j''G_{\kappa+1} \subseteq G_{j(\kappa)+1} \quad (3.47)$$

Indeed, let $p \in G_{\kappa+1}$. Recall that $\mathbb{P}_\kappa \subseteq V_\kappa$ (lemma 3.11), so

$$j(p \restriction \kappa) = j(p) \restriction \kappa = p \restriction \kappa \in G_{\kappa+1}.$$

If $\kappa \in \text{dom}(p)$, we have

$$j(p) = p \restriction \kappa \cup \langle j(\kappa), j(p(\kappa)) \rangle.$$

By (3.45), we have

$$M \models p \cup j(p) \restriction \{j(\kappa)\} \leq p \cup d_0,$$

therefore we have $j(p) \leq_{G_{\kappa+1}} d_0$. Thus $j(p) \in G_{j(\kappa)+1}$, so indeed (3.47) holds.

By (3.47) and proposition 1.26, we can extend j into an elementary embedding $J : V[G_{\kappa+1}] \longrightarrow M[G_{j(\kappa)+1}] = M[G_{\kappa+1} * K]$. Since $K \in V[G_{\kappa+1} * K_T]$, we have that J is definable in $V[G_{\kappa+1} * K_T]$.

Since K_T is arbitrary, J can be constructed in any $\mathbb{T}_{j(\kappa)+1, \kappa+1}$ -extension of $V[G_{\kappa+1}]$, so for any $x \in V[G_{\kappa+1}]$ we can take $\tilde{x}'$ a $\mathbb{T}_{j(\kappa)+1, \kappa+1}$ -name for $J(x)$. Therefore in $V[G_{\kappa+1}]$, we can define

$$\mathfrak{I} := \{x \in \mathcal{P}(\kappa) \cap V[G_{\kappa+1}] : V[G_{\kappa+1}] \models \Vdash_{\mathbb{T}_{j(\kappa)+1, \kappa+1}} \text{“}\kappa \notin \tilde{x}'\text{”}\}. \quad (3.48)$$

The next claim completes the proof of lemma 3.21.

Claim 3.21.1. *In $V[G_{\kappa+1}]$, $\mathfrak{I}$ is a σ -saturated ideal over κ .*

Proof of claim 3.21.1. Let $\langle U_\xi : \xi < \omega_1 \rangle$, where $U_\xi \in \mathcal{P}(\kappa) \setminus \mathfrak{I}$ for each $\xi < \omega_1$. By (3.48), for each $\xi < \omega_1$ we can take a condition $p_\xi \in \mathbb{T}_{j(\kappa)+1, \kappa+1}$ such that $p_\xi \Vdash_{\mathbb{T}_{j(\kappa)+1, \kappa+1}} \text{“}\kappa \in J(U_\xi)\text{”}$. Since $\mathbb{T}_{j(\kappa)+1, \kappa+1}$ is c.c.c. there are distinct $\xi_0, \xi_1 < \omega_1$ such that p_{ξ_0} and p_{ξ_1} are compatible. Let $q \in \mathbb{T}_{j(\kappa)+1, \kappa+1}$ be such that $q \leq p_{\xi_0}, p_{\xi_1}$. Then we have

$$q \Vdash_{\mathbb{T}_{j(\kappa)+1, \kappa+1}} \text{“}\kappa \in J(U_{\xi_0}) \cap J(U_{\xi_1}) = J(U_{\xi_0} \cap U_{\xi_1})\text{”}.$$

Therefore q witnesses that $U_{\xi_0} \cap U_{\xi_1} \in \mathfrak{I}$. Then $\mathfrak{I}$ is indeed σ -saturated. (claim 3.21.1) $\square$

(lemma 3.21) $\square$

3.2.4 Further results

We prove that some fragment of $\mathbf{MA}^{++}$ holds in the model we constructed with mixed support iteration. We recall the definition of $\mathbf{MA}^{++}$: Suppose P is a poset and $\tilde{S}$ is a P -name for a set of ordinals. Given any filter $F \subseteq P$ (not necessarily generic), we define

$$\tilde{S}(F) := \{\alpha \in \text{On} : \exists p \in F, p \Vdash_P \text{“}\check{\alpha} \in \tilde{S}\text{”}\}.$$

Notice that, whenever F is generic, $\tilde{S}(F)$ coincides with $\tilde{s}[F]$, the usual interpretation of the name $\tilde{s}$ by the generic filter F .

Definition 3.22. Let $\mathcal{P}$ be a class of posets and κ, μ be regular uncountable cardinals with $\kappa > \aleph_1$. Define $\text{MA}^{+\mu}(\mathcal{P}, < \kappa)$ as the following statement: Given any $P \in \mathcal{P}$, any family $\mathcal{D}$ of dense subsets of P with $|\mathcal{D}| < \kappa$ and any family $\mathcal{S}$ of P -names with $|\mathcal{S}| \leq \mu$ such that $\Vdash_P \check{S} \subseteq \omega_1$ is stationary" for every $\check{S} \in \mathcal{S}$, there is G a $\mathcal{D}$ -generic filter over P such that $\check{S}(G)$ is a stationary subset of ω_1 for every $\check{S} \in \mathcal{S}$.

Similarly to MA , $\text{MA}^{+\mu}(\mathcal{P})$ denotes $\text{MA}^{+\mu}(\mathcal{P}, < 2^{\aleph_0})$. Furthermore, the forcing axiom $\text{MA}^{+\omega_1}(\text{c.c.c.})$ is sometimes referred to as MA^{++} .

In connection with this stronger forcing axiom, we use the following lemma in order to pull names down from some extension.

Lemma 3.23. *Let P be a c.c.c. poset and Q a σ -distributive poset. Let H be a Q -generic over V . For any P -name $\check{x} \in V[H]$, there is a P -name $\check{x}' \in V$ such that $V[H] \models \Vdash_P \check{x}' = \check{x}$.*

Proof. We prove this by induction on the rank of $\check{x}$. Let β be an ordinal and suppose that for every P -name $\check{z} \in V[H]$ with $\text{rank}(\check{z}) < \beta$ there is $\check{z}' \in V$ such that $V[H] \models \Vdash_P \check{z}' = \check{z}$. We show that for any P -name $\check{x} \in V[H]$ with $\text{rank}(\check{x}) = \beta$ there is $\check{x}' \in V$ such that $V[H] \models \Vdash_P \check{x}' = \check{x}$.

Working in $V[H]$, for each P -name $\check{y} \in V$ with $\text{rank}(\check{y}) < \beta$ take $\mathcal{A}_{\check{y}}$ an antichain which is a maximal subset of $\{p \in P : V[H] \models p \Vdash_P \check{y} \in \check{x}\}$. Since Q is σ -distributive, it follows that P is c.c.c. in $V[H]$. Therefore each $\mathcal{A}_{\check{y}}$ is countable. Since Q is σ -distributive, each $\mathcal{A}_{\check{y}}$ belongs to V .

In V , we can define

$$\check{x}' := \bigcup \{ \{\check{y}\} \times \mathcal{A}_{\check{y}} : \check{y} \in V_\beta \text{ is a } P\text{-name} \}. \quad (3.49)$$

Clearly we have $\text{rank}(\check{x}') \leq \beta$. We prove that $V[H] \models \Vdash_P \check{x}' = \check{x}$. Let G be a P -generic over $V[H]$. We show that $\check{x}'[G] = \check{x}[G]$. Let $y \in \check{x}'[G]$. By (3.49), there is $\check{y} \in V^P$ and $p \in G \cap \mathcal{A}_{\check{y}}$ such that $y = \check{y}[G]$. Since $p \in \mathcal{A}_{\check{y}}$, we have $p \Vdash_P \check{y} \in \check{x}$. Therefore $y = \check{y}[G] \in \check{x}[G]$.

Conversely, let $z \in \check{x}[G]$. By induction hypothesis, there is a P -name $\check{z} \in V$ with $\text{rank}(\check{z}) < \beta$ such that $\check{z}[G] = z$. Since $\check{z}[G] \in \check{x}[G]$, there is $p \in G$ such that $p \Vdash_P \check{z} \in \check{x}$. By the maximality of $\mathcal{A}_{\check{z}}$, we can assume without loss of generality that $p \in \mathcal{A}_{\check{z}}$. Then, by (3.49), we have $\langle \check{z}, p \rangle \in \check{x}'$. Therefore $z = \check{z}[G] \in \check{x}'[G]$. (lemma 3.23) $\square$

Proposition 3.24. *Assume GCH. Let κ be a measurable cardinal and suppose $S \subset \kappa$ is an stationary set whose every element is a Mahlo cardinal and $T := \kappa \setminus S$. Then we can construct $\langle \mathbb{T}_\alpha, \mathbb{R}_\alpha : \alpha < \kappa \rangle$ satisfying (3.3) and (3.4) such that $\mathbb{T}_\kappa$ forces $2^{\aleph_0} = \kappa$ and $\mathbf{MA}^{++}$.*

$\langle \mathbb{T}_\alpha, \mathbb{R}_\alpha : \alpha < \kappa \rangle$ is just a slight modification of the construction by Solovay and Tennenbaum of a finite support iteration of c.c.c. posets forcing MA while adding reals (check e.g. Theorem 16.13 on [Jec03]). We modify this construction so that the finite support iteration satisfies (3.3) and (3.4).

Proof of proposition 3.24. Recall we denote $\nu_\alpha := \min(S \setminus \alpha)$ for every $\alpha < \kappa$. Since $|T| = \kappa$, we fix $\{\langle \beta_\alpha, \gamma_\alpha \rangle : \alpha \in T\}$ a bijective enumeration of the set $\{\langle \beta, \gamma \rangle \in \kappa \times \kappa : \gamma < \nu_\beta\}$ such that $\beta_\alpha < \alpha$ for all $\alpha \in T$. We shall use this enumeration for book-keeping names for c.c.c. posets.

For $\alpha < \kappa$, suppose $\langle \mathbb{T}_\beta, \mathbb{R}_\beta : \beta < \alpha \rangle$ already constructed and such that for each $\beta < \alpha$ $\mathbb{R}_\beta$ satisfies (3.3), (3.4) and we have fixed $\langle \mathbb{Q}_{\beta, \gamma} : \gamma < \nu_\beta \rangle$ an enumeration of all $\mathbb{T}_\beta$ -names for c.c.c. posets with rank $< \nu_\beta$ (modulo isomorphisms).

Let $\mathbb{T}_\alpha$ be the direct limit of $\langle \mathbb{T}_\beta, \mathbb{R}_\beta : \beta < \alpha \rangle$. By (3.3) and (3.4), we have that $\mathbb{T}_\alpha$ is c.c.c. and $\mathbb{T}_\alpha \subseteq V_{\nu_\alpha}$, so we can fix $\langle \mathbb{Q}_{\alpha, \gamma} : \gamma < \nu_\alpha \rangle$ an enumeration of all $\mathbb{T}_\alpha$ -names for c.c.c. posets with rank $< \nu_\alpha$ modulo isomorphisms.

If $\alpha \in S$, let $\mathbb{R}_\alpha$ be a $\mathbb{T}_\alpha$ -name for the trivial poset $\{1\}$. If $\alpha \in T$, let $\mathbb{R}_\alpha$ be a $\mathbb{T}_\alpha$ name such that

$$\Vdash_{\mathbb{T}_\alpha} \mathbb{R}_\alpha = \begin{cases} \mathbb{Q}_{\beta_\alpha, \gamma_\alpha} & , \text{ if } \mathbb{Q}_{\beta_\alpha, \gamma_\alpha} \text{ is c.c.c. } \\ \{1\} & , \text{ otherwise} \end{cases} \quad (3.50)$$

Clearly $\mathbb{R}_\alpha$ satisfies (3.3) and (3.4). We finish the proof by showing that $\mathbb{T}_\kappa$ forces $\mathbf{MA}^{++}$, where $\mathbb{T}_\kappa$ is the finite support limit of $\langle \mathbb{T}_\alpha, \mathbb{R}_\alpha : \alpha < \kappa \rangle$.

Let H be a $\mathbb{T}_\kappa$ -generic over V . It is enough to proof that $V[H] \models \mathbf{MA}^{+\omega_1}(\{P\})$ for all $P \in V[H]$ such that $V[H] \models "P \text{ is c.c.c. and } |P| < \kappa"$ (see e.g lemma 16.12 in [Jec03]). In $V[H]$, let P be a c.c.c. poset with $|P| < \kappa$, $\mathcal{D}$ a family of dense subsets of P with $|\mathcal{D}| < \kappa$ and $\mathcal{S}$ a family of P -names with $|\mathcal{S}| \leq \aleph_1$ such that $\Vdash_P \mathbb{S}$ is a stationary subset of ω_1 for every $\mathbb{S} \in \mathcal{S}$.

For each $\xi < \omega_1$ and $\mathcal{S} \in \mathcal{S}$, take an antichain $A_{\xi, \mathcal{S}}$ which is a maximal subset of $\{p \in P : p \Vdash_P \check{\xi} \in \check{\mathcal{S}}\}$. Let $\mathcal{A} = \{A_{\xi, \mathcal{S}} : \xi < \omega_1 \wedge \mathcal{S} \in \mathcal{S}\}$. Since P is c.c.c. in $V[H]$, each $A_{\xi, \mathcal{S}}$ is countable, thus $|\mathcal{A}| \leq \aleph_1$.

For each $\alpha < \kappa$, let $H_\alpha := H \cap \mathbb{T}_\alpha$. As in the original construction, we can find $\beta^* < \kappa$ such that $\text{rank}(Q) < \nu_{\beta^*}$ and $P, \mathcal{D}, \mathcal{A} \in V[H_{\beta^*}]$. Notice that H_{β^*} is a $\mathbb{T}_{\beta^*}$ -generic over V .

Clearly, the rank of P in $V[H_{\beta^*}]$ is smaller than ν_{β^*} , so we can take $\gamma^* < \nu_{\beta^*}$ such that $P = Q_{\beta^*, \gamma^*}$. Then we can take $\alpha^* \in T \setminus \beta^*$ such that $\langle \beta^*, \gamma^* \rangle = \langle \beta_{\alpha^*}, \gamma_{\alpha^*} \rangle$. Since P is c.c.c. in $V[H]$, it is also c.c.c. in $V[H_{\alpha^*}]$, therefore by (3.50) we have $P = \mathbb{R}_{\alpha^*}[H_{\alpha^*}]$. Since $\mathcal{A} \in V[H_{\beta^*}] \subseteq V[H_{\alpha^*}]$, for each $\mathcal{S} \in \mathcal{S}$ we can construct a P -name $\mathcal{S}' \in V[H_{\alpha^*}]$ such that $V[H] \models \Vdash_P \check{\mathcal{S}} = \check{\mathcal{S}}'$. Let

$$F := H(\alpha^*) = \{t(\alpha^*)[H_{\alpha^*}] : t \in H_{\alpha^*+1}\}.$$

Notice that F is a P -generic filter over $V[H_{\alpha^*}]$ such that $H_{\alpha^*+1} \cong H_{\alpha^*+1} * F$. It follows that F is a $\mathcal{D}$ -generic filter and $\mathcal{S}'[F] = \mathcal{S}[F]$ is a stationary subset of ω_1 for all $\mathcal{S} \in \mathcal{S}$. Since $F \in V[H_{\alpha^*}] \subseteq V[H]$, F witnesses $V[H] \models \text{MA}^{+\omega_1}(\{P\})$. (proposition 3.24) $\square$

Lemma 3.25. *Suppose κ is supercompact, $\{S, T\}$ is a partition of κ such that S is stationary and $\min(S) \geq \omega_2$. Suppose also that $\langle \mathbb{T}_\alpha, \mathbb{R}_\beta : \alpha \leq \kappa, \beta < \alpha \rangle$ is a finite support iteration satisfying (3.3) and (3.4) such that $\mathbb{T}_\kappa$ forces $2^{\aleph_0} = \kappa \wedge \text{MA}^{++}$. Let $\mathbb{P}_\kappa$ be constructed as in subsection 3.2.1 with respect to $\langle \mathbb{T}_\alpha, \mathbb{R}_\alpha : \alpha < \kappa \rangle$. Then the following fragment of MA^{++} holds in $V^{\mathbb{P}_\kappa}$:*

There is an inner model N of V such that V is an extension of N by a σ -distributive κ -c.c. poset $Q \in N$ such that for every $P \in N$ which is a c.c.c. poset in V we have $\text{MA}^{+\aleph_1}(\{P\}, < \kappa)$.

Notice that such an iteration $\langle \mathbb{T}_\alpha, \mathbb{R}_\beta : \alpha \leq \kappa, \beta < \alpha \rangle$ is constructed in proposition 3.24. Notice also that the fragment of MA^{++} in lemma 3.25 is strong enough to imply that all cardinal invariants in Chihon's diagram are equal to $2^{\aleph_0}$.

Proof. Let G be a $\mathbb{P}_\kappa$ -generic over V . Let $H := G \cap \mathbb{T}_\kappa^*$ ($\mathbb{T}_\kappa^* \subseteq \mathbb{P}_\kappa$ is defined as in the proof of lemma 3.17). By (3.29), there is $\mathbb{S}^*$ a $\mathbb{T}_\kappa^*$ -name such that $\mathbb{P}_\kappa \cong \mathbb{T}_\kappa^* * \mathbb{S}^*$ and $\Vdash_{\mathbb{T}_\kappa^*} \check{\mathbb{S}}^*$ is $< \min(S)$ -distributive". Therefore we can

take G' an $\mathbb{S}^*[H]$ -generic over $V[H]$ such that $G \cong H * G'$. We show that $N := V[H]$ is an inner model of $V[G]$ which witnesses the desired fragment of $\mathbf{MA}^{++}$. By corollary 3.16 and lemma 3.11, it follows that $V[G]$ is a κ -c.c. and σ -distributive extension of N .

In $V[G]$, let $P \in W$ be a c.c.c. poset. Let $\mathcal{D}$ be a family of dense subsets of P with $|\mathcal{D}| < \kappa$ and let also $\mathcal{S}$ be a family of P -names with $|\mathcal{S}| \leq \aleph_1$ such that $\Vdash_P \text{“}\dot{G} \text{ is a stationary subset of } \omega_1\text{”}$ for every $\dot{G} \in \mathcal{S}$.

For each $D \in \mathcal{D}$ we choose $\mathcal{A}_D \subseteq D$ a maximal antichain. Since P is c.c.c. in $V[G]$, each $\mathcal{A}_D$ is countable. Since $\mathbb{S}^*[H]$ is σ -distributive in $V[H]$, we have $\{\mathcal{A}_D : D \in \mathcal{D}\} \subseteq V[H]$.

In $V[H]$, denote by B_P the set of all maximal antichains of P . Since $\mathbb{P}_\kappa$ is κ -c.c. (3.11) and $\mathbb{T}_\kappa^*$ is c.c.c., we have that $\mathbb{S}^*[H]$ is κ -c.c. in $V[H]$. Therefore we can take $\mathcal{D}^* \in [B_P]^{<\kappa} \cap V[H]$ such that

$$\{\mathcal{A}_D : D \in \mathcal{D}\} \subseteq \mathcal{D}^*.$$

Back in $V[G]$, for each $\dot{G} \in \mathcal{S}$ we can, by lemma 3.23, take $\dot{G}' \in V[H]$ such that $\Vdash_P \text{“}\dot{G}' = \dot{G}\text{”}$. Since $|\mathcal{S}| \leq \omega_1$ (in $V[G]$), by corollary 3.16 we have $\{\dot{G}' : \dot{G} \in \mathcal{S}\} \in V[H]$.

Since $V[H] \models \mathbf{MA}^{++}$, we can take $G^* \in V[H]$ a $\mathcal{D}^*$ -generic filter in P such that $\dot{G}(G^*)$ is a stationary subset of ω_1 for every $\dot{G} \in \mathcal{S}$. Notice that G^* is also $\mathcal{D}$ -generic in $V[G]$. Furthermore, notice that $\mathbb{S}^*[H]$ is $\leq \omega_1$ -distributive and thus does not add any new club subset of ω_1 . Therefore $\dot{G}(G^*)$ remains stationary in $V[G]$ for every $\dot{G} \in \mathcal{S}$, so we conclude that $V[G] \models \mathbf{MA}^{+\aleph_1}(\{P\}, < \kappa)$. (lemma 3.25) $\square$

Theorem 3.26. *Suppose κ is supercompact, $\{S, T\}$ is a partition of κ such that S is stationary and $\min(S) \geq \omega_2$. Suppose also that $\langle \mathbb{T}_\alpha, \mathbb{R}_\beta : \alpha \leq \kappa, \beta < \kappa \rangle$ is a finite support iteration satisfying (3.3) and $\mathbb{R}_\alpha$ is a check-name for ${}^{<\omega}2$ for every $\alpha \in T$. Then for any inaccessible $\theta > \kappa$, we have*

$$\Vdash_{\mathbb{P}_\kappa * \text{Col}(\kappa, < \theta)^{V^{\mathbb{P}_\kappa}}} \text{“RC}(< \kappa)\text{”}.$$

For the definition of the Rado conjecture RC and the lemmas used for this proof, check section 1.3.

Proof. Suppose G is a $\mathbb{P}_\kappa$ -generic over V and suppose H is a $\text{Col}(\kappa, < \theta)^{V[G]}$ -generic over $V[G]$. In $V[G][H]$, let $\mathcal{T}$ be a non-special tree. Let $\eta := |\mathcal{T}|$.

Without loss of generality, assume $\eta > \kappa$. In V , let $j : V \rightarrow M \subseteq V$ be an elementary embedding with critical point κ such that $j(\kappa) > \eta$, ${}^\eta M \cap V \subseteq M$ and $j(f)(\kappa) = \theta$.

As in the proof of lemma 3.19, we can take $\hat{G}$ a $j(\mathbb{P}_\kappa)$ -generic over V with $G * H \subseteq \hat{G}$ and extend j into $j^* : V[G] \rightarrow M[\hat{G}]$. Then, as in the proof of proposition 2.41, we can take $\hat{H}$ a $j^*(\text{Col}(\kappa, < \theta)^{V[G]})$ -generic over $V[\hat{G}]$ such that j^* can be extended to an elementary embedding $J : V[G * H] \rightarrow M[\hat{G} * \hat{H}]$.

By elementarity, it is enough to find $X \in M[\hat{G} * \hat{H}]$ such that

$$M[\hat{G} * \hat{H}] \models X \in [J(\mathcal{T})]^{< j(\kappa)} \text{ is non-special.} \quad (3.51)$$

By lemma 2.34(1), we have $J''\mathcal{T} \in M[\hat{G} * \hat{H}]$. We show that $X := J''\mathcal{T}$ satisfies (3.51). Towards a contradiction, suppose there is $f \in M[\hat{G} * \hat{H}]$ a specializing function for $J''\mathcal{T}$.

Notice that $f \in V[\hat{G} * \hat{H}]$ and $J''\mathcal{T}$ is isomorphic to $\mathcal{T}$ in $V[\hat{G} * \hat{H}]$. Therefore $\mathcal{T}$ is special in $V[\hat{G} * \hat{H}]$.

Claim 3.26.1. *There are A a $\text{Fn}(j(\kappa), 2)$ -generic over $V[G * H]$, $\mathbb{Q}$ a σ -distributive poset in $V[G * H * A]$ and B a $\mathbb{Q}$ -generic over $V[G * H * A]$ such that $V[\hat{G} * \hat{H}] = V[G * H * A * B]$.*

By claim 3.26.1, $V[\hat{G} * \hat{H}]$ is a σ -distributive extension of $V[G * H * A]$. Therefore by lemma 1.32(1), we have that $\mathcal{T}$ is special in $V[G * H * A]$. Then, since A is an $\text{Fn}(j(\kappa), 2)$ -generic over $V[G * H]$, by lemma 1.32(3) we have that $\mathcal{T}$ is special in $V[G * H]$, which is a contradiction.

We complete the proof of lemma 3.26 by proving claim 3.26.1.

Proof of claim 3.26.1. As in the proof of lemma 3.21, we can define $\langle \mathbb{P}_\alpha^* : \alpha \leq j(\kappa) + 1 \rangle$, $\mathbb{S}_{j(\kappa)+1, \kappa+1}$, $\mathbb{T}_{j(\kappa)+1, \kappa+1}$ and $G_{\kappa+1}$ such that $j(\vec{\mathbb{P}}) = \langle \mathbb{P}_\alpha^* : \alpha \leq j(\kappa) \rangle$, $\vec{\mathbb{P}} = \langle \mathbb{P}_\alpha^* : \alpha \leq \kappa \rangle$, $\mathbb{P}_{\kappa+1}^* \cong \mathbb{P}_\kappa * \text{Col}(\kappa, < \theta)^{V^{\mathbb{P}_\kappa}}$, $G_{\kappa+1} \cong G * H$, $\mathbb{S}_{j(\kappa)+1, \kappa+1} = (\mathbb{P}_{j(\kappa)+1}^* / G_{\kappa+1}) \restriction S$, $\mathbb{T}_{j(\kappa)+1, \kappa+1} = (\mathbb{P}_{j(\kappa)+1}^* / G_{\kappa+1}) \restriction T$.

By elementarity, lemma 3.14(2) and proposition 1.24(1), there is $\mathbb{Q} \in M[G_{\kappa+1}] = M[G * H]$ a $\mathbb{T}_{j(\kappa)+1, \kappa+1}$ -name such that

$$M[G_{\kappa+1}] \models \mathbb{P}_{j(\kappa)+1}^* / G_{\kappa+1} \cong \mathbb{T}_{j(\kappa)+1, \kappa+1} * \mathbb{Q}.$$

It follows that

$$V[G_{\kappa+1}] \models \mathbb{P}_{j(\kappa)+1}^* / G_{\kappa+1} \cong \mathbb{T}_{j(\kappa)+1, \kappa+1} * \mathbb{Q}.$$

We can then take A' be a $\mathbb{T}_{j(\kappa)+1, \kappa+1}$ -generic over $V[G_{\kappa+1}]$ and B' a $\mathbb{Q}[A']$ -generic over $V[G_{\kappa+1} * A']$ such that $\hat{G} \cong G_{\kappa+1} * A' * B'$.

By elementarity and corollary 3.16, we have

$$M[G_{\kappa+1} * A'] \models \text{“}\mathbb{Q}[A'] \text{ is } \sigma\text{-distributive”}. \quad (3.52)$$

As in the proof of lemma 3.19, we have ${}^{\omega_1}M[G_{\kappa+1}] \cap V[G_{\kappa+1}] \subseteq M[G_{\kappa+1}]$. By elementarity and lemma 3.17, we have that $\mathbb{T}_{j(\kappa)+1, \kappa+1}$ is c.c.c. in $M[G_{\kappa+1}]$. By proposition 1.25 we have

$${}^{\omega_1}M[G_{\kappa+1} * A'] \cap V[G_{\kappa+1} * A'] \subseteq M[G_{\kappa+1} * A']. \quad (3.53)$$

By (3.52) and (3.53), we have

$$V[G_{\kappa+1} * A'] \models \text{“}\mathbb{Q}[A'] \text{ is } \sigma\text{-distributive”}. \quad (3.54)$$

In particular, from (3.53), (3.54) and proposition 1.25, we have

$${}^{\omega_1}M[G_{\kappa+1} * A' * B'] \cap V[G_{\kappa+1} * A' * B'] \subseteq M[G_{\kappa+1} * A' * B'].$$

Therefore

$$V[\hat{G}] \models j^*(\text{Col}(\kappa, < \theta)^{V[G]}) = \text{Col}(j(\kappa), < j(\theta))^{M[\hat{G}]} \text{ is } \sigma\text{-distributive}. \quad (3.55)$$

Since $\mathbb{R}_\alpha$ is a check-name for $<^{\omega}2$ for all $\alpha \in T$, we can construct a dense embedding $d : \text{Fn}(j(\kappa), 2) \longrightarrow \mathbb{T}_{j(\kappa)+1, \kappa+1}$. So we can take an $\text{Fn}(j(\kappa), 2)$ -generic A over $V[G * H]$ such that $A' \cong A$. Let $\mathbb{Q} := \mathbb{Q}[A']$. Then we can take a $\mathbb{Q}$ -generic B over $V[G * H * A]$ such that $B \cong A' * \hat{H}$. Then $\mathbb{Q}$, A and B are as desired. (claim 3.26.1) $\square$

(theorem 3.26) $\square$

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