Some results on modal logics having arithmetical interpretations

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Some results on modal logics having arithmetical interpretations

（論文題目和訳: 算術的解釈を持つ様相論理に関する諸結果について）

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Contents

Chapter I
Introduction

The modal system **GL** is obtained from **K** by adding an axiom $\Box(\Box\varphi \to \varphi) \to \Box\varphi$. This logic enjoys two significant properties, the arithmetical completeness and the fixed-point property.

Modal formulas can be interpreted into first-order sentences of formal arithmetic, for example, Peano Arithmetic PA. *An arithmetical interpretation* is a mapping $*$ from propositional variables to arithmetical sentences. In particular the modal operator $\Box$ is interpreted as $\mathsf{Bew}(x)$, where $\mathsf{Bew}(x)$ is the standard provability predicate of Peano Arithmetic PA. *The provability logic* of PA is the set of all modal formulas φ satisfying $\mathsf{PA} \vdash \varphi^*$ for any arithmetical interpretation $*$.

Solovay [27] established *the arithmetical completeness theorem* of **GL**. It asserts that, **GL** coincides with the logic of provability of PA, i.e., for any modal formula φ, $\mathbf{GL} \vdash \varphi$ if and only if $\mathsf{PA} \vdash \varphi^*$ for any arithmetical interpretation $*$. Thus **GL** captures some properties of the provability predicate $\mathsf{Bew}(x)$. Moreover, *the uniform arithmetical completeness theorem*, which is a stronger version of Solovay's one, also holds for **GL**. That is, there exists a fixed arithmetical interpretation $*$ such that for any modal formula φ, $\mathbf{GL} \vdash \varphi$ if and only if $\mathsf{PA} \vdash \varphi^*$ (See Artemov [1], Avron [4], Boolos [6], Montagna [16] or Visser [29]).

De Jongh and Sambin [21] independently proved *the fixed-point theorem* for **GL**. Let $\varphi(p)$ be a modal formula containing the propositional variable p. A modal formula $\varphi(p)$ is said to be *modalized in* p if all occurrences of the propositional variable p in $\varphi(p)$ are within the scope of the modal operator. The fixed-point theorem states that if $\varphi(p)$ is modalized in p then there is a modal formula ψ containing only propositional variables occurring in $\varphi(p)$ without p, and such that $\mathbf{GL} \vdash \psi \leftrightarrow \varphi(\psi)$. Moreover, effective procedures of constructing fixed-points in **GL** has been studied (See Reidhaar-Olson [18] or Lindström [13]).

In this dissertation, we investigate the following variants of **GL**: (i) Artemov's Logic of Proofs; (ii) Sacchetti's logics $\mathbf{wGL}_n$; (iii) the predicate modal logic **QGL**.

1 Logic of Proofs

A proof predicate is a formula $\mathsf{Prf}(x, y)$ which represents the explicit provability of formulas in PA. The formula $\mathsf{Prf}(x, y)$ intuitively means "there exists a proof in PA with the code (the Gödel number) x of the formula with the code y". For a proof predicate $\mathsf{Prf}(x, y)$, we call a Σ_1 formula $\mathsf{Pr}(x) \equiv \exists y \mathsf{Prf}(y, x)$ *a provability predicate.*

Artemov developed *the Logic of Proofs*, which analyzes the properties of explicit proof predicates in **PA**. The logic of proofs deals with **LP**-formulas, especially formulas of the form $t : F$, where t is called a *proof term.* An arithmetical interpretation of **LP**-formulas is defined as a collection of mapping $*$ and functions from proof terms to natural numbers. The intended meaning of $t : F$ is "t is a (code of a) proof of F".

Artemov [2] proved the arithmetical completeness theorem of $\mathbf{LP}_0$: for any **LP**-formula F, $\mathbf{LP}_0 \vdash F$ if and only if for any Δ_1 normal proof predicate $\mathsf{Prf}(x, y)$ and any arithmetical interpretation $*$ based on Prf, $\mathsf{PA} \vdash F^*$.

Technically, there is a substantial difference between Solovay's theorem and Artemov's theorem. The arithmetical completeness theorem of **GL** holds for each canonical provability predicate. On the other hand, in the case of $\mathbf{LP}_0$ the arithmetical completeness theorem does not hold with only the standard proof predicate $\mathsf{Proof}(x, y)$. Moreover, it is not known whether the uniform arithmetical completeness theorem holds for $\mathbf{LP}_0$.

In Chapter III, we examine the following two problems: (i) Does the arithmetical completeness theorem for $\mathbf{LP}_0$ hold with respect to some fixed proof predicate? (ii) Does the uniform arithmetical completeness theorem for $\mathbf{LP}_0$ hold?

For these problems, we prove the following two statements:

(i) There exists *a normal* Δ_1 *proof predicate* $\mathsf{Prf}(x, y)$ such that for any **LP**-formula F, $\mathbf{LP}_0 \vdash F$ if and only if for any arithmetical interpretation $*$ based on Prf, $\mathsf{PA} \vdash F^*$;

(ii) There exist a Σ_1 (but not normal) proof predicate $\mathsf{Prf}(x, y)$ and an arithmetical interpretation $*$ based on Prf such that for any **LP**-formula F, $\mathbf{LP}_0 \vdash F$ if and only $\mathsf{PA} \vdash F^*$.

2 Interpolation properties for Sacchetti's logics

A logic **L** is said to have *the Craig interpolation property* if for any implication $\varphi \to \psi$ which is provable in **L**, there exists a formula θ (called an interpolant of $\varphi \to \psi$) such that θ consists of common variables of φ and ψ, and satisfies $\mathbf{L} \vdash \varphi \to \theta$ and $\mathbf{L} \vdash \theta \to \psi$. A logic **L** is said to have *the Lyndon interpolation property* if for any provable implication $\varphi \to \psi$, there is a stronger interpolant θ which preserves positivity of variables, that is, every positive (negative) occurrence of a variable also occurs both in φ and ψ positively (resp. negatively).

In **GL**, there is a close connection between the fixed-point properties and the interpolation properties, since the following two facts:

(i) The fixed-point theorem for **GL** can be derived from the Craig interpolation property for the logic;

(ii) Using the effective fixed-point theorem, we can prove the effective Lyndon interpolation property for **GL**.

Proofs of the Craig interpolation property for **GL** and the fact (i) are independently given by Boolos [5] and Smoryński [25]. A comprehensive description of the fact (i) is also shown in Boolos's textbook [7].

It had been opened whether the Lyndon interpolation property posses for **GL** until Shamkanov solved in 2011. In [23] he proved the Lyndon interpolation property for **GL** by a modified version of Smoryński's semantical argument, without applying the fixed-point theorem. Later in [24] he also proved the fact (ii) by using a cut-free sequent calculus for **GL**. A benefit of Shamkanov's second proof of the Lyndon interpolation property is that, from $\varphi \to \psi$, we can effectively construct a Lyndon interpolant θ of $\varphi \to \psi$ whenever $\varphi \to \psi$ is provable in **GL**.

In the proof of Shamkanov's second result, he also introduced *a circular proof system*. A circular proof system $^\circ\mathbf{L}$ of **L** is one which has the same axioms and rules of **L** and admits "*circular proofs*". A circular proof is a derivation tree of **L** whose leaves are either axioms of **L** or identical to a sequent below that leaf. Shamkanov showed that **GL** is provably equivalent to the circular proof system $^\circ\mathbf{K4}$. He gave an effective way of constructing a Lyndon interpolant of $\varphi \to \psi$ by using $^\circ\mathbf{K4}$ and the effective fixed-point theorem.

In Chapter IV, we try to generalize Shamkanov's results of **GL** into Sacchetti's logics $\mathbf{wGL}_n$.

Sacchetti [19, 20] studied modal logics having the fixed-point property. In particular, he introduced a new modal logic $\mathbf{wGL}_n$ (the notation $\mathbf{wGL}_n$ is according to Kurahashi [11]). The logic $\mathbf{wGL}_n$ is obtained from **GL** by replacing the axiom $\Box(\Box\varphi \to \varphi) \to \Box\varphi$ by $\Box(\Box^n\varphi \to \varphi) \to \Box\varphi$, where n is a nonzero natural number, and $\Box^n\varphi$ denotes $\overbrace{\Box\cdots\Box}^{n}\varphi$.

Sacchetti's logics $\mathbf{wGL}_n$ have several properties like **GL**. Originally Sacchetti [20] showed that $\mathbf{wGL}_n$ enjoys all the Kripke completeness, the Craig interpolation property. Moreover, he proved the de Jongh-Sambin fixed-point theorem for $\mathbf{wGL}_n$. Later Kurahashi [11] proved the arithmetical completeness theorem for $\mathbf{wGL}_n$ with respect to a Σ_2 provability predicate.

It is expected that $\mathbf{wGL}_n$ posses the Lyndon interpolation property, however, this conjecture has not been clarified.

We develop two one-sided sequent calcului $\mathbf{wGL}_n^{\mathbf{G}}$ and $\mathbf{wK4}_n^{\mathbf{G}}$, and prove the following results:

(i) The calculus $\mathbf{wGL}_n^{\mathbf{G}}$ is equivalent to the circular proof system ${}^{\circ}\mathbf{wK4}_n^{\mathbf{G}}$;

(ii) Using ${}^{\circ}\mathbf{wK4}_n^{\mathbf{G}}$ and the effective fixed-point theorem for $\mathbf{wGL}_n$ (cf. Kurahashi and Okawa [12]), we can construct a Lyndon interpolant of $\varphi \to \psi$ in $\mathbf{wGL}_n$ whenever $\varphi \to \psi$ is provable.

Iemhoff [10] studied some sufficient conditions for a type of modal sequent calculus to have an equivalent circular proof system. Although the calculus $\mathbf{wGL}_n^{\mathbf{G}}$ does not enjoy Iemhoff's conditions, it has an equivalent circular proof counterpart.

3 Fixed-point properties in predicate modal logics

It is natural to extend the studies of the logic of provability to a predicate modal logic. However, the stituation of the predicate logic of provability is quite complex and most of the properties for $\mathbf{GL}$ do not hold for the predicate modal system $\mathbf{QGL}$, which is the natural predicate extension of $\mathbf{GL}$. In particular, Montagna [17] proved that $\mathbf{QGL}$ enjoys neither the Kripke completeness, nor the arithmetical completeness. He also showed the failure of the fixed-point theorem for $\mathbf{QGL}$, that is, he found a predicate modal formula $\varphi(p)$ which has no fixed-points in $\mathbf{QGL}$. Smoryński [26] gave a simpler counterexample.

On the other hand, there is a room for investigations of fixed-point properties in predicate modal logics. The logic $\mathbf{QGL}$ is not only the candidate of an extension of $\mathbf{GL}$. Recently Tanaka [28] introduced a new predicate modal logic $\mathbf{NQGL}$, which is strictly stronger than $\mathbf{QGL}$ and enjoys the Kripke completeness with respect to a proper subclass of transitive and conversely well-founded Kripke frames. There is a possibility that the fixed-point theorem holds for these natural extensions of $\mathbf{QGL}$.

Sacchetti [19] showed that the fixed-point theorem holds for the modal logic $\mathbf{K} + \Box^{n+1}\bot$. Also it has not been known that the fixed-point theorem even holds for the predicate extension of this logic.

In Chapter V we discuss some versions of the fixed-point properties for predicate modal logics. We define the following classes of Kripke frames in which all theorems of **QGL** are valid: CW (the class of transitive and conversely well-founded frames), FH (the class of transitive frames with finite height), FI (the class of finite transitive irreflexive frames) and FIFD (the class of finite transitive irreflexive frames of which domains are finite). The class FH is a proper subclass of BL (the class of transitive of which are bounded length), which is introduced by Tanaka [28]. Tanaka's system **NQGL** is Kripke complete with respect to the class BL. The class FIFD was originally investigated by Artemov and Dzhaparidze [3].

We study two semantical fixed-point properties for a class of Kripke frames, *the fixed-point property* and *the local fixed-point property*. From Montagna's result, it follows that the classes CW and BL enjoy neither the fixed-point property nor the local fixed-point property. We discuss whether the classes FH, FI and FIFD enjoy these two properties. We describe the following results:

(i) The classes FH, FI and FIFD do not enjoy the fixed-point property;

(ii) We prove the fixed-point theorem for the predicate modal system $\mathbf{QK}+\Box^{n+1}\bot$. An algorithm for calculating fixed-points in $\mathbf{QK}+\Box^{n+1}\bot$ is given in the proof. Consequently, we obtain that the classes FH, FI and FIFD enjoy the local fixed-point property.

As a consequence, we prove that Tanaka's system **NQGL** does not enjoy the Craig interpolation property.

As mentioned above, Montagna [17] showed that the fixed-point theorem does not hold for **QGL**. Although there is a possibility that the fixed-point theorem holds for some classes of formulas. It has not been known sufficient (or necessary) conditions for a formula to have a fixed-point in **QGL**. In the end of Chapter V, we investigate these conditions. We prove that if $\varphi(p)$ is a Boolean combination of Σ-*formulas*, then $\varphi(p)$ has a fixed-point in **QGL**.

Chapter II
Preliminaries

4 General definitions

For any two expressions ϵ_1 and ϵ_2 (of a certain language), $\epsilon_1 \equiv \epsilon_2$ means ϵ_1 and ϵ_2 are syntactically identical. Throughout this dissertation, we use Greek letters $\varphi, \psi, \ldots$ to express (propositional or predicate) modal formulas. Propositional variables is denoted by $p, q, \ldots$ etc.. Propositional modal formulas is constructed as the following grammar:

$$\varphi ::= \top \mid \bot \mid p \mid \neg\varphi \mid \varphi \to \varphi \mid \Box\varphi$$

where $\top$ and $\bot$ are constants, p is a propositional variable. Another Boolean connectives are defined a natural way. We put $\Diamond\varphi :\equiv \neg\Box\neg\varphi$. Formulas $\Box^n\varphi$ and $\Diamond^n\varphi$ stand for $\overbrace{\Box\cdots\Box}^{n}\varphi$ and $\overbrace{\Diamond\cdots\Diamond}^{n}\varphi$, respectively. Let $\boxdot\varphi :\equiv \Box\varphi \wedge \varphi$.

4.1 Negation normal form

While we discuss the interpolation properties for a modal logic (Chapter IV), we deal with formulas in the negation normal form to recognize positivity of propositional variables. We denote by $\overline{p}, \overline{q}, \ldots$ the complement of $p, q, \ldots$ respectively. We call propositional variables and their complements *literals*.

$$\varphi ::= p \mid \overline{p} \mid \top \mid \bot \mid \varphi \wedge \varphi \mid \varphi \vee \varphi \mid \Box\varphi \mid \Diamond\varphi.$$

We inductively define *the negation* $\overline{\varphi}$ of a formula φ in the usual way:

$$\overline{\top} :\equiv \bot, \quad \overline{\bot} :\equiv \top, \quad \overline{(p)} :\equiv \overline{p}, \quad \overline{(\overline{p})} :\equiv p,$$
$$\overline{\varphi \vee \psi} :\equiv \overline{\varphi} \wedge \overline{\psi}, \quad \overline{\varphi \wedge \psi} :\equiv \overline{\varphi} \vee \overline{\psi}, \quad \overline{\Box\varphi} :\equiv \Diamond\overline{\varphi}, \quad \overline{\Diamond\varphi} :\equiv \Box\overline{\varphi}.$$

We put $(\varphi \to \psi) :\equiv (\overline{\varphi} \vee \psi)$, and $(\varphi \leftrightarrow \psi) :\equiv (\varphi \to \psi) \wedge (\psi \to \varphi)$.

4.2 Modal systems

In this subsection we define propositional modal logics.

Definition 4.1 (Modal logic **K**, **GL**, **K4**)**.** The propositional modal system **K** consists of the following axioms and rules.

Axiom 1 All instances of tautologies of propositional logic;

Axiom 2 $\Box(\varphi \to \psi) \to (\Box\varphi \to \Box\psi)$;

Rule 1 $\dfrac{\varphi \to \psi \quad \varphi}{\psi}$ (modus ponens);

Rule 2 $\dfrac{\varphi}{\Box\varphi}$ (necessitation).

The propositional modal system **GL** and **K4** are obtained from **K** by adding the following axioms **L**, and **4**, respectively.

L $\Box(\Box\varphi \to \varphi) \to \Box\varphi$;

4 $\Box\varphi \to \Box\Box\varphi$.

Definition 4.2 (Modal logic $\mathbf{wGL}_n$ and $\mathbf{wK4}_n$)**.** Let n be a non-zero natural number. The propositional modal systems $\mathbf{wGL}_n$ (and $\mathbf{wK4}_n$) is obtained from **GL** (resp. **K4**) by replacing the axiom **L** (resp. **4**) by the following axiom $\mathbf{L}_n$ (resp. $\mathbf{4}_n$).

$\mathbf{L}_n$ $\Box(\Box^n\varphi \to \varphi) \to \Box\varphi$;

$\mathbf{4}_n$ $\Box\varphi \to \Box^{n+1}\varphi$.

4.3 Kripke semantics

We describe Kripke semantics for propositional modal logics.

A *Kripke frame* F is a pair $\langle W, \prec\rangle$, where W is a non-empty set, and $\prec$ is a binary relation on W. A *Kripke model* M is a triple $\langle W, \prec, V\rangle$, where $\langle W, \prec\rangle$ is a Kripke frame and V is a valuation function from the set of propositional variables to $\mathcal{P}(W)$. We say a propositional variable p *is true in* w (write $w \models p$) if $w \in V(p)$. The valuation of formulas is uniquely determined by V in a usual way. In particular, $w \models \Box\varphi$ iff for any $x \in W$, $w \prec x$ implies $x \models \varphi$. We say a formula φ is *valid in a Kripke model* $M = \langle W, \prec, V\rangle$ if for any $w \in W$, $w \models \varphi$. We say a formula φ is *valid in a Kripke frame* $F = \langle W, \prec\rangle$ if for any valuation V on F, φ is valid in the model $\langle W, \prec, V\rangle$.

We say $\mathcal{F}$ *is finite* if W is finite. A Kripke frame $\mathcal{F}$ is *conversely well-founded* if there is no countably infinite sequence $(w_i)_{i\in\omega}$ of worlds of W satisfying $w_i \prec w_{i+1}$ for each $i \in \omega$.

We inductively define the binary relation $\prec^n$ on W: $x \prec^0 y$ iff $x = y$, and $x \prec^{n+1} y$ iff $^\exists z \in W$ s.t. $x \prec^n z$ and $z \prec y$. A binary relation $\prec$ on W is said to be *n-transitive* if for any $x, y \in W$, $x \prec^n y$ implies $x \prec y$. A Kripke frame $F = \langle W, \prec\rangle$ (resp. a Kripke model $M = \langle W, \prec, V\rangle$) is said to be a

wGL$_n$-*frame* (resp. a **wGL**$_n$-*model*) if $\prec$ is $(n+1)$-transitive and conversely well-founded.

The following lemma will be needed in our proof of Theorem 11.6.

Lemma 4.3. Let $\prec$ be a binary relation on a set W and suppose that $\prec$ is $(n+1)$-transitive. Then for any $x, y \in W$ and $k \geq 1$, if $x \prec^{kn} y$ then $x \prec^{n} y$.

Proof. Induction on k. The case for $k = 1$ is trivial. Assume Lemma holds for $\leq k$, and $x \prec^{(k+1)n} y$. Then there exist $x_1, \ldots, x_{(k+1)n} \in W$ such that $x \prec x_1 \prec \cdots \prec x_{(k+1)n}$ and $y = x_{(k+1)n}$. In particular, $x \prec^{n+1} x_{n+1} \prec^{kn-1} y$. Since $\prec$ is $(n+1)$-transitive, $x \prec x_{n+1}$. Hence we get $x \prec^{kn} y$. By the induction hypothesis, we obtain $x \prec^{n} y$. $\square$

4.4 Provability predicates in arithmetic

In Chapter III we discuss arithmetical formulas. Let $\mathcal{L}_A$ be the first-order language of arithmetic. We assume $\mathcal{L}_A$ contains function symbols for all primitive recursive functions. The numeral for the natural number n is also denoted by n. We write $\ulcorner \varphi \urcorner$ as *the Gödel number (or simply code) of* φ. We assume that all theorems of Peano Arithmetic PA are true in the standard model $\mathbb{N}$ of arithmetic.

Definition 4.4 (Σ_1 and Δ_1 formulas)**.**

1. An $\mathcal{L}_A$-formula φ is Δ_0 if its quantifiers are all bounded.
2. An $\mathcal{L}_A$-formula φ is Σ_1 if it is PA-provably equivalent to a formula of the form $\exists \vec{x} \psi(\vec{x}, \vec{y})$ where ψ is a Δ_0 formula.
3. An $\mathcal{L}_A$-formula φ is Δ_1 if both φ and $\neg\varphi$ are Σ_1.

Definition 4.5 (Proof predicate)**.** An $\mathcal{L}_A$-formula $\mathsf{Prf}(x, y)$ is *a proof predicate* if it satisfies that for any $\mathcal{L}_A$-sentence φ,

$$\mathsf{PA} \vdash \varphi \text{ if and only if for some natural number } n,\ \mathbb{N} \models \mathsf{Prf}(n, \ulcorner \varphi \urcorner).$$

Definition 4.6 (Normal proof predicate)**.** A proof predicate $\mathsf{Prf}(x, y)$ is *normal* if it satisfies the following two conditions:

1. For any natural number k, the set $T(k) := \{n \mid \mathbb{N} \models \mathsf{Prf}(k, n)\}$ is finite. Moreover, the code of the finite set $T(k)$ is computable from k;
2. For any natural numbers k and l, there is a natural number m such that
$$T(k) \cup T(l) \subseteq T(m).$$

Definition 4.7 (Prf-functions)**.** Let $\mathsf{Prf}(x, y)$ be a proof predicate. Three computable functions $\langle \mathbf{m}(\cdot,\cdot), \mathbf{a}(\cdot,\cdot), \mathbf{c}(\cdot) \rangle$ on natural numbers are said to be Prf-*functions* if they satisfy the following conditions: For any $\mathcal{L}_A$-sentences φ and ψ and natural numbers k and l,

- $\mathsf{PA} \vdash (\mathsf{Prf}(k, \ulcorner \varphi \to \psi \urcorner) \land \mathsf{Prf}(l, \ulcorner \varphi \urcorner)) \to \mathsf{Prf}(\mathbf{m}(k, l), \ulcorner \psi \urcorner)$;
- $\mathsf{PA} \vdash (\mathsf{Prf}(k, \ulcorner \varphi \urcorner) \lor \mathsf{Prf}(l, \ulcorner \varphi \urcorner)) \to \mathsf{Prf}(\mathbf{a}(k, l), \ulcorner \varphi \urcorner)$;
- $\mathsf{PA} \vdash \mathsf{Prf}(k, \ulcorner \varphi \urcorner) \to \mathsf{Prf}(\mathbf{c}(k), \ulcorner \mathsf{Prf}(k, \ulcorner \varphi \urcorner) \urcorner)$.

Proposition 4.8. If a proof predicate $\mathsf{Prf}(x, y)$ is Δ_1 and normal, then there are Prf-functions.

Proof. See Artemov [2]. □

For example, the Gödel multi-conclusion proof predicate $\mathsf{Proof}(x, y)$ is the $\mathcal{L}_A$-formula which means the following assertion:

"x is the code of a PA-proof containing an $\mathcal{L}_A$-formula with the code y."

The formula $\mathsf{Proof}(x, y)$ is Δ_1 and normal proof predicate, therefore there are Proof-functions $\langle \otimes, \oplus, \uparrow \rangle$.

Let $\mathsf{Provable}(x)$ be the Σ_1 formula $\exists z \mathsf{Proof}(z, x)$. The formula $\mathsf{Provable}(x)$ satisfies the following propositions.

Proposition 4.9 (Derivability conditions)**.** For any $\mathcal{L}_A$-sentences φ and ψ,

1. if $\mathsf{PA} \vdash \varphi$, then $\mathsf{PA} \vdash \mathsf{Provable}(\ulcorner \varphi \urcorner)$;
2. $\mathsf{PA} \vdash \mathsf{Provable}(\ulcorner \varphi \to \psi \urcorner) \to (\mathsf{Provable}(\ulcorner \varphi \urcorner) \to \mathsf{Provable}(\ulcorner \psi \urcorner))$;
3. $\mathsf{PA} \vdash \mathsf{Provable}(\ulcorner \varphi \urcorner) \to \mathsf{Provable}(\ulcorner \mathsf{Provable}(\ulcorner \varphi \urcorner) \urcorner)$.

Clause (3) of Proposition 4.9 is a particular case of the following proposition.

Proposition 4.10 (Formalized Σ_1 completeness)**.** For any Σ_1 sentence φ, $\mathsf{PA} \vdash \varphi \to \mathsf{Provable}(\ulcorner \varphi \urcorner)$.

5 Preceding studies

5.1 Arithmetical completeness theorem

Definition 5.1. An arithmetical interpretation is a mapping from propositional modal formulas to $\mathcal{L}_A$-sentences satisfying the following conditions:

1. $*$ commutes with Boolean connectives:
2. $(\Box\varphi)^* \equiv \mathsf{Bew}(\ulcorner \varphi^* \urcorner)$.

Solovay [27] proved the following arithmetical completeness theorem of **GL**.

Theorem 5.2 (Arithmetical completeness theorem of **GL**, Solovay [27])**.** For any propositional modal formula φ, the following are equivalent:

1. $\mathbf{GL} \vdash \varphi$;
2. For any arithmetical interpretation $*$, $\mathsf{PA} \vdash \varphi^*$.

5.2 Fixed-point theorem

The fixed-point theorem was originally proved by de Jongh and Sambin [21] for the propositional logic **GL** independently. Sacchetti [19] proved the fixed-point theorem for the logic $\mathbf{K} + \Box^{n+1}\bot$.

Let $\varphi(p)$ be a propositional modal formula containing occurrences of p. We say $\varphi(p)$ *is modalized in* p if every occurrence of p in $\varphi(p)$ is in the scope of modal operators. For a propositional modal formula ψ, $\varphi(\psi)$ denotes the one obtained from φ by substituting ψ for all occurrences p in φ. To summarize the results, the fixed-point theorems are described as follows.

Theorem 5.3 (Fixed-point theorem (de Jongh, Sambin [21], and Sacchetti [19]))**.** Suppose that $\mathbf{L}$ is either **GL** or $\mathbf{K} + \Box^{n+1}\bot$. If $\varphi(p)$ is modalized in p, then there is a formula ψ containing only propositional variables occurring in $\varphi(p)$, not containing p, and such that $\mathbf{L} \vdash \psi \leftrightarrow \varphi(\psi)$.

We call such a ψ *a fixed-point of* $\varphi(p)$ *in* $\mathbf{L}$.

6 Logic of Proofs

In this section we define the logic $\mathbf{LP}_0$ that is called the Logic of Proofs. The logic $\mathbf{LP}_0$ was introduced by Artemov [2][1].

[1]In [2], Artemov also introduced the logic **LP** that contains $\mathbf{LP}_0$ and has the axiom necessitation rule, but in this paper we only discuss $\mathbf{LP}_0$.

6.1 Language of the Logic of Proofs

The language of the Logic of Proofs consists of the following symbols:

- *Propositional variables* (written $p, q, \ldots$ etc.) and *Boolean connectives*;
- *Proof variables* (written $v, w, \ldots$ etc.) and *proof constants* (written $a, b, c, \ldots$ etc.);
- Binary *function symbols* $\cdot$ and $+$, and an unary function symbol !.

Proof terms are defined by the grammar

$$t ::= \; v \mid a \mid t \cdot t \mid t + t \mid !t$$

where v is a proof variable and a is a proof constant.

LP-*formulas* are defined by the grammar:

$$F ::= \; p \mid (\neg F) \mid (F \rightarrow F) \mid (t : F)$$

where p is a propositional variable, and t is a proof term. Other Boolean connectives $\wedge$, $\vee$ and $\leftrightarrow$ are defined in a usual way.

6.2 System $\mathbf{LP}_0$

Definition 6.1 (Logic $\mathbf{LP}_0$)**.** The system $\mathbf{LP}_0$ consists of the following axioms and the rule:

Axiom 0 all instances of tautologies in the language of **LP**;

Axiom 1 $t : F \rightarrow F$;

Axiom 2 $s : (F \rightarrow G) \wedge t : F \rightarrow s \cdot t : G$;

Axiom 3 $s : F \vee t : F \rightarrow s + t : F$;

Axiom 4 $s : F \rightarrow !s : s : F$;

Rule modus ponens.

Definition 6.2 (Arithmetical interpretations of **LP**-formulas)**.** Let $\mathsf{Prf}(x, y)$ be a proof predicate, and $\langle \mathbf{m}, \mathbf{a}, \mathbf{c} \rangle$ be Prf-functions. An *arithmetical interpretation* $*$ based on $\langle \mathsf{Prf}, \mathbf{m}, \mathbf{a}, \mathbf{c} \rangle$ is an evaluation of **LP**-formulas by $\mathcal{L}_A$-sentences and an evaluation of proof terms by natural numbers, satisfying the following conditions:

1. $*$ commutes with Boolean connectives;

2. $(s \cdot t)^* = \mathbf{m}(s^*, t^*)$, $(s+t)^* = \mathbf{a}(s^*, t^*)$, $(!s)^* = \mathbf{c}(s^*)$;

3. $(t : F)^* \equiv \mathsf{Prf}(t^*, \ulcorner F^* \urcorner)$;

where s and t are proof terms, and F is an **LP**-formula.

Artemov [2] proved the arithmetical completeness theorem of $\mathbf{LP}_0$.

Theorem 6.3 (Artemov [2])**.** Let F be an **LP**-formula. The following are equivalent:

1. $\mathbf{LP}_0 \vdash F$;

2. For any Δ_1 normal proof predicate Prf, Prf-functions $\langle \mathbf{m}, \mathbf{a}, \mathbf{c} \rangle$ and arithmetical interpretation $*$ based on $\langle \mathsf{Prf}, \mathbf{m}, \mathbf{a}, \mathbf{c} \rangle$, $\mathsf{PA} \vdash F^*$.

7 Predicate modal logic

7.1 Language and formulas

The language of predicate modal logic $\mathcal{L}_Q$ consists of countably many variables $u, v, \ldots$, etc., Boolean constants $\top, \bot$, Boolean connectives $\neg, \to$, quantifier $\forall$, and countably many predicate symbols for each arity (denoted by $P, Q, \ldots$ etc.). An $\mathcal{L}_Q$-formula φ is constructed as the following manner:

$$\varphi ::= \top \mid \bot \mid P(u_1, \ldots, u_n) \mid \neg\varphi \mid \varphi \to \varphi \mid \forall u \varphi \mid \Box\varphi$$

where P is an n-ary predicate symbol, and $u_1, \ldots, u_n, u$ are variables.

Boolean constants $\top$ and $\bot$, and $\mathcal{L}_Q$-formulas of the form $P(u_1, \ldots, u_n)$ are called *atomic formulas.* We put

$$\varphi \vee \psi :\equiv \neg\varphi \to \psi,\ \varphi \wedge \psi :\equiv \neg(\varphi \to \neg\psi),\ \varphi \leftrightarrow \psi :\equiv (\varphi \to \psi) \wedge (\psi \to \varphi),$$
$$\exists u \varphi :\equiv \neg\forall u \neg\varphi,\ \Diamond\varphi :\equiv \neg\Box\neg\varphi.$$

Free variables and *bounded variables* are naturally defined. We say φ is *an $\mathcal{L}_Q$-sentence* if φ is an $\mathcal{L}_Q$-formula with no free variables.

7.2 Modal systems QK, QGL

The predicate modal system **QK** consists of the following axioms and rules:

Axiom 1 All instances of axioms of predicate logic in the language $\mathcal{L}_Q$;

Axiom 2, Rule 1, Rule 2 Same as **K**.

The predicate modal systems **QK4** and **QGL** are obtained from **QK** by adding the following axioms **4**, and **L**, respectively.

4 $\Box\varphi \to \Box\Box\varphi$;

L $\Box(\Box\varphi \to \varphi) \to \Box\varphi$.

Recall that **QK** $\subseteq$ **QK4** $\subseteq$ **QGL**.

Tanaka [28] introduced the modal proof system **NQGL** which has an infinite inference rule.

Definition 7.1. The modal system **NQGL** is obtained from **QK4** by adding the following rule:

BL If $\vdash \Box^{n+1}\bot \to A$ for all natural numbers n, then $\vdash A$.

7.3 Predicate Kripke frames

Definition 7.2. *A (predicate) Kripke frame* $\mathcal{F}$ is a triple $\langle W, \prec, \{D_w\}_{w\in W}\rangle$ where:

- W is a non-empty set;
- $\prec$ is a binary relation on W;
- Each D_w is a non-empty set, and if $w \prec w'$, then $D_w \subseteq D_{w'}$.

Definition 7.3. Let $\mathcal{F} = \langle W, \prec, \{D_w\}_{w\in W}\rangle$ be a Kripke frame. *An interpretation of* $\mathcal{F}$ is a mapping $\Vdash$ which assigns each pair $\langle w, P\rangle$, where $w \in W$ and P is an n-ary predicate symbol, into an n-ary relation on D_w. We write $w \Vdash P(a_1, \dots, a_n)$ if $(a_1, \dots, a_n)$ is a member of $\Vdash \langle w, P\rangle$. *A Kripke model* $\mathcal{M}$ is a pair $\langle \mathcal{F}, \Vdash\rangle$ where $\mathcal{F}$ is a Kripke frame and $\Vdash$ is an interpretation of $\mathcal{F}$.

Definition 7.4. Let $\mathcal{M} = \langle W, \prec, \{D_w\}_{w\in W}, \Vdash\rangle$ be a Kripke model, and φ be an $\mathcal{L}_Q$-sentence with parameters from D_w for some $w \in W$. *The truth value of* φ *in* w (we write $\mathcal{M}, w \models \varphi$ if φ is true in w) is inductively defined as follows:

- $\mathcal{M}, w \models \top$ and $\mathcal{M}, w \not\models \bot$, for every $w \in W$;
- $\mathcal{M}, w \models P(a_1, \dots a_n)$ iff $w \Vdash P(a_1, \dots a_n)$;
- $\mathcal{M}, w \models \neg\varphi$ iff $\mathcal{M}, w \not\models \varphi$;

- $\mathcal{M}, w \models \varphi \to \psi$ iff $\mathcal{M}, w \not\models \varphi$ or $\mathcal{M}, w \models \psi$;
- $\mathcal{M}, w \models \forall u \varphi(u)$ iff $\mathcal{M}, w \models \varphi(a)$ for every $a \in D_w$;
- $\mathcal{M}, w \models \Box\varphi$ iff for any $v \in W$, if $w \prec v$, then $\mathcal{M}, v \models \varphi$.

Definition 7.5. Let $\mathcal{M}$ be a Kripke model and φ be an $\mathcal{L}_Q$-sentence. We say φ *is valid in* $\mathcal{M}$ (write $\mathcal{M} \models \varphi$) if for every $w \in W$, $\mathcal{M}, w \models \varphi$.

Let $\mathcal{F}$ be a Kripke frame and φ be an $\mathcal{L}_Q$-sentence. We say φ *is valid in* $\mathcal{F}$ (write $\mathcal{F} \models \varphi$) if for any interpretation $\Vdash$ of $\mathcal{F}$, φ is valid in $\mathcal{M} = \langle \mathcal{F}, \Vdash \rangle$.

Validity of an $\mathcal{L}_Q$-formula φ is defined by the validity of the universal closure of φ.

Chapter III

Arithmetical completeness for $\mathbf{LP}_0$

8 Strong arithmetical completeness of $\mathbf{LP}_0$

8.1 Completion Algorithm

Let F be an **LP**-formula, and $\mathcal{L}(F)$ be the set of all propositional variables, proof variables and proof constants contained in F. An **LP**-formula G is called an $\mathcal{L}(F)$-*formula* if $\mathcal{L}(G) \subseteq \mathcal{L}(F)$. A proof term t is said to be an $\mathcal{L}(F)$-*term* if t is built from only proof variables and proof constants contained in $\mathcal{L}(F)$. For each **LP**-formula A, let

$$\sim A :\equiv \begin{cases} B & \text{if } A \text{ is of the form } \neg B, \\ \neg A & \text{otherwise.} \end{cases}$$

Definition 8.1. Let F be any **LP**-formula. Define $\mathcal{S}_F$ to be the finite set $\{B, \sim B \mid B \text{ is a subformula of } F\}$.

Notice that $\mathcal{S}_F$ is closed under $\sim$ and subformulas.

Let X be a set of **LP**-formulas. We say that X is $\mathbf{LP}_0$-*consistent* if $\mathbf{LP}_0 \nvdash \neg \bigwedge Y$ for all finite subsets Y of X where $\bigwedge Y$ is a conjunction of all elements of Y. The set X is called F-*maximal consistent* if X is an $\mathbf{LP}_0$-consistent subset of $\mathcal{S}_F$ and for any **LP**-formula $A \in \mathcal{S}_F$, either $A \in X$ or $\sim A \in X$.

Since the set of all theorems of $\mathbf{LP}_0$ is primitive recursive (cf. Mkrtychev [15]), we obtain the following lemma.

Lemma 8.2. For each $\mathbf{LP}_0$-unprovable formula F, we can find an F-maximal consistent set X_F of $\mathcal{L}(F)$-formulas such that $\sim F \in X_F$ in a primitive recursive way.

For each $\mathbf{LP}_0$-unprovable formula F, we define the extended set of **LP**-formulas (a completion of F) by using the following algorithm.

Lemma 8.3. Let F be an $\mathbf{LP}_0$-unprovable formula and X_F be as in Lemma 8.2. Then there is a set $\tilde{X}_F$ of **LP**-formulas (a completion of F) satisfying the following conditions:

(B1) $X_F \subseteq \tilde{X}_F$;

(B2) $\tilde{X}_F$ is $\mathbf{LP}_0$-consistent;

(B3) If $s : A \in \tilde{X}_F$, then $A \in \tilde{X}_F$;

(B4) If $s : (A \to B) \in \tilde{X}_F$ and $t : A \in \tilde{X}_F$, then $s \cdot t : B \in \tilde{X}_F$;

(B5) If $s : A \in \tilde{X}_F$, then $s + t : A \in \tilde{X}_F$ and $t + s : A \in \tilde{X}_F$ for any proof term t;

(B6) If $s : A \in \tilde{X}_F$, then $!s : s : A \in \tilde{X}_F$.

Moreover, the binary relation "$A \in \tilde{X}_F$" is primitive recursive.

Proof. We describe the algorithm $\mathcal{COM}$ which produces the sequence $(Y_i)_{i\in\omega}$ of sets of **LP**-formulas from an input X_F (the algorithm is same as in the proof of Lemma 7.5 in Artemov [2]):
(1) Let $Y_0 := X_F$;
(2)

- if $j = 3k + 1$ $(k \geq 0)$, then $\mathcal{COM}$ sets

$$Y_{j+1} := Y_j \cup \bigcup_{s,t} \{s \cdot t : B \mid s : A \to B, t : A \in Y_j\},$$

- if $j = 3k + 2$ $(k \geq 0)$, then $\mathcal{COM}$ sets

$$Y_{j+1} := Y_j \cup \bigcup_{s} \{!s : s : A \mid s : A \in Y_j\},$$

- if $j = 3k$ $(k > 0)$, then $\mathcal{COM}$ sets

$$Y_{j+1} := Y_j \cup \bigcup_{s,t} \{s + t : A, t + s : A \mid s : A \in Y_j, |t| < j\}.$$

where $|t|$ is the number of symbols occurring in t. Let

$$\tilde{X}_F := \bigcup_{i\in\omega} Y_i.$$

Since each Y_i is obviously $\mathbf{LP}_0$-consistent, $\tilde{X}_F$ is $\mathbf{LP}_0$-consistent (**B2**).The conditions **B1**, **B4**, **B5**, and **B6** clearly hold from the definition of $\mathcal{COM}$. Before proving **B3**, we show that each Y_i is closed under modus ponens. If $A \to B$ and A are in Y_i, then $A \to B \in Y_0 = X_F$ because $\mathcal{COM}$ never adds $A \to B$ in each step. Since $A \to B \in \mathcal{S}_F$, we have $A \in \mathcal{S}_F$. Since

$A \in Y_i \supseteq X_F$, $A \in X_F$ by the F-maximal consistency of X_F and the $\mathbf{LP}_0$-consistency of Y_i. Thus $B \in X_F \subseteq Y_i$ by the F-maximal consistency of X_F again.

To show **B3**, it suffices to prove that for all $i \in \omega$, if $t : A \in Y_i$ then $A \in Y_i$. We prove by induction on i.
Suppose $t : A \in Y_0 = X_F$. Since $t : A \in \mathcal{S}_F$, $A \in \mathcal{S}_F$. By the F-maximal consistency of X_F, $A \in Y_0$.
Suppose $s \cdot t : A \in Y_{j+1}$, $s : C \to A \in Y_j$ and $t : C \in Y_j$. By the induction hypothesis, $C \to A$, $C \in Y_j$. Therefore $A \in Y_j \subseteq Y_{j+1}$.
The proofs for the other cases $s + t : A$ and $!s : s : A$ are similar. □

Let F be an **LP**-formula. A proof term t is said to be an $\mathcal{L}(F)$*-term* if t contains only proof variables and proof constants in $\mathcal{L}(F)$. In the case that F is unprovable in $\mathbf{LP}_0$, we define $J(F, t) := \{G \mid t : G \in \tilde{X}_F\}$.

Proposition 8.4. For any $\mathbf{LP}_0$-unprovable formula F and **LP**-term t, $J(F, t)$ is finite, and the code of $J(F, t)$ is effectively computable from F and t.

Proof. See Lemma 7.5 in Artemov [2]. □

The following proposition will play a key role in our proof of Theorem 8.6 in the next subsection.

Proposition 8.5. Let F be any $\mathbf{LP}_0$-unprovable formula and t be any proof term. If $J(F, t)$ is nonempty, then t contains some subterm s which is an $\mathcal{L}(F)$-term (we call such a proof term s an $\mathcal{L}(F)$-subterm of t).

Proof. We prove by induction on the construction of t. Assume $G \in J(F, t)$ for some **LP**-formula G.

If t is some proof variable or constant, then $t : G$ is already contained in X_F because $X_F \subseteq \tilde{X}_F$ by **B1** and the algorithm $\mathcal{COM}$ in Lemma 8.3 does not add new formulas of the form $t : H$ in each step. Since X_F is a set of $\mathcal{L}(F)$-formulas, t is itself an $\mathcal{L}(F)$-term.

If $t \equiv s+u$, then we have either $t : G \in X_F$, or $t : G$ is added in some step by $\mathcal{COM}$. The former case, t is itself an $\mathcal{L}(F)$-term. The latter case, we have either $s : G \in \tilde{X}_F$ or $u : G \in \tilde{X}_F$ by the construction of completion by $\mathcal{COM}$. By the induction hypothesis, either s or u contains some $\mathcal{L}(F)$-subterm. In both cases, t contains an $\mathcal{L}(F)$-subterm.

The proofs for the remaining possibilities $t \equiv s \cdot u$ and $t \equiv !s$ are similar. □

8.2 A stronger version of Artemov's theorem

There is a substantial difference between Solovay's arithmetical completeness theorem of **GL** (Theorem 5.2) and Artemov's arithmetical completeness theorem of $\mathbf{LP}_0$ (Theorem 6.3). Solovay's theorem holds for each fixed appropriate provability predicate. On the other hand, the arithmetical completeness of $\mathbf{LP}_0$ does not hold with only the Gödel proof predicate Proof. Indeed, let $F :\equiv \neg v : v : p$, and $*$ be an arbitrary arithmetical interpretation based on $\langle \mathsf{Proof}, \otimes, \oplus, \uparrow \rangle$. Then F^* is $\neg \mathsf{Proof}(v^*, \ulcorner \mathsf{Proof}(v^*, \ulcorner p^* \urcorner) \urcorner)$. Since v^* cannot be a proof of the sentence $\mathsf{Proof}(v^*, \ulcorner p^* \urcorner)$ with the code larger than v^*, F^* is true in $\mathbb{N}$. Since F^* is a Δ_1 sentence, F^* is provable in PA. Thus $\mathsf{PA} \vdash F^*$ for any arithmetical interpretation $*$ based on $\langle \mathsf{Proof}, \otimes, \oplus, \uparrow \rangle$. If $\mathbf{LP}_0 \vdash F$, then the forgetful projection $\neg \Box \Box p$ of F is provable in the modal logic $\mathsf{S4}$ (see Artemov [2]), and this is not the case. Hence $\mathbf{LP}_0 \nvdash F$.

Now we prove a stronger version of the arithmetical completeness theorem of $\mathbf{LP}_0$. That is, we prove that the arithmetical completeness theorem of $\mathbf{LP}_0$ holds with a fixed appropriate Δ_1 normal proof predicate Prf and computable Prf-functions $\langle \mathbf{m}, \mathbf{a}, \mathbf{c} \rangle$.

Theorem 8.6 (A stronger version of the arithmetical completeness theorem of $\mathbf{LP}_0$)**.** There exist a Δ_1 normal proof predicate $\mathsf{Prf}(x, y)$ and computable Prf-functions $\langle \mathbf{m}, \mathbf{a}, \mathbf{c} \rangle$ such that for any **LP**-formula F, the following are equivalent:

1. $\mathbf{LP}_0 \vdash F$;

2. For any arithmetical interpretation $*$ based on $\langle \mathsf{Prf}, \mathbf{m}, \mathbf{a}, \mathbf{c} \rangle$, $\mathsf{PA} \vdash F^*$.

It suffices to show the existence of Δ_1 normal proof predicate Prf and computable Prf-functions $\langle \mathbf{m}, \mathbf{a}, \mathbf{c} \rangle$ satisfying the implication (2) $\Rightarrow$ (1). In our proof, we assume that:

- the Gödel numbering of the joint language of **LP** and $\mathcal{L}_A$ is injective, i.e., for any expressions ϵ_1 and ϵ_2, $\ulcorner \epsilon_1 \urcorner = \ulcorner \epsilon_2 \urcorner$ if and only if $\epsilon_1 \equiv \epsilon_2$;

- 0 is not a Gödel number of any expression;

- for any proof term t and for any $\mathcal{L}_A$-sentence φ, $\mathbb{N} \not\models \mathsf{Proof}(\ulcorner t \urcorner, \ulcorner \varphi \urcorner)$.

Definition 8.7. A *replacement* $r(x)$ is a substitution such that r substitutes a propositional variable, proof variable and a proof constant into each propositional variable, proof variable and proof constant, respectively.

Then for any replacement r and **LP**-formula A, the **LP**-formula $r(A)$ is uniquely determined in a usual way.

Let $\{A_i\}_{i\geq 1}$ be a primitive recursive enumeration of all $\mathbf{LP}_0$-unprovable **LP**-formulas. We can find a replacement $r(x)$ with the following conditions in a primitive recursive way:

- for any $i, j \geq 1$, if $i \neq j$, then $\mathcal{L}(r(A_i)) \cap \mathcal{L}(r(A_j)) = \emptyset$.
- There exists a primitive recursive function $f(x)$ such that:

$$f(\ulcorner \epsilon \urcorner) = \begin{cases} i & \text{if } \epsilon \text{ is in } \mathcal{L}(r(A_i)) \text{ for some } i \geq 1, \\ 0 & \text{if } \epsilon \text{ is not in } \mathcal{L}(r(A_i)) \text{ for all } i \geq 1, \end{cases}$$

  where ϵ is some propositional variable or proof variable or proof constant.

The value of f is uniquely determined by the first clause.

For each $i \geq 1$, we denote $r(A_i)$ by B_i. Each B_i is also unprovable in $\mathbf{LP}_0$.

By using the function f, we can obtain the following primitive recursive function $g(x)$:

$$g(\ulcorner G \urcorner) = \begin{cases} i & \text{if } p \in \mathcal{L}(B_i) \text{ for all propositional variables } p \in \mathcal{L}(G), \\ 0 & \text{otherwise}, \end{cases}$$

where G is an **LP**-formula. Note that the value of g is uniquely determined by the choice of the replacement r. If G is an $\mathcal{L}(B_i)$-formula, then $g(\ulcorner G \urcorner) = i$.

Recall that $\tilde{X}_{B_i}$ is a completion of B_i provided by the completion algorithm $\mathcal{COM}$. We have the following lemma.

Lemma 8.8. *Let $i \geq 1$ and F be any **LP**-formula. If $F \in \tilde{X}_{B_i}$, then $g(\ulcorner F \urcorner) = i$.*

Proof. Let $\tilde{X}_{B_i} = \bigcup_{n\in\omega} Y_n$ where $\{Y_n\}_{n\in\omega}$ is as in Lemma 8.3. We prove by induction on n that for any $n \in \omega$, if $F \in Y_n$, then $g(\ulcorner F \urcorner) = i$.

- If $F \in Y_0 = X_{B_i}$, then $\mathcal{L}(F) \subseteq \mathcal{L}(B_i)$. Thus $g(\ulcorner F \urcorner) = i$.
- Suppose $F \in Y_{n+1}$. If $F \equiv s \cdot t : B$ for some proof terms s and t and **LP**-formulas A and B such that $s : (A \to B)$, $t : A \in Y_n$. By induction hypothesis, $g(\ulcorner A \to B \urcorner) = i$. Then $g(\ulcorner F \urcorner) = g(\ulcorner B \urcorner) = i$. The other cases are obvious.

□

Recall $J(B_i, t)$ is the set $\{G \mid t : G \in \tilde{X}_{B_i}\}$. The next Lemma 8.9 directly follows from Proposition 8.4 and the effectiveness of the sequence $\{A_i\}_{i\geq 1}$ and the replacement $r(x)$.

Lemma 8.9. For any $i \geq 1$ and proof term t, the set $J(B_i, t)$ is finite. Moreover, the code of $J(B_i, t)$ is effectively computable from i and t.

Let

$$J(t) = \bigcup_{i\geq 1} J(B_i, t).$$

Lemma 8.10. For any proof term t, $J(t)$ is finite. Moreover, the code of $J(t)$ is effectively computable from t.

Proof. Let t be any proof term. First, compute the finite set

$$S(t) := \{i \geq 1 \mid t \text{ contains an } \mathcal{L}(B_i)\text{-subterm}\}.$$

By Proposition 8.5, for any $j \notin S(t)$, $J(B_j, t) = \emptyset$. Thus

$$J(t) = \bigcup_{i\in S(t)} J(B_i, t)$$

and hence this set is finite and the code of $J(t)$ is effectively computable from t by Lemma 8.9. □

By the Fixed Point Lemma (cf. Lindström [14]), we simultaneously define the auxiliary translation † of **LP**-formulas and the Δ_1 formula $\mathsf{Prf}(x, y)$ as follows:

1. $p^\dagger :\equiv \begin{cases} \ulcorner p \urcorner = \ulcorner p \urcorner & \text{if for some } i \geq 1,\ p \in \tilde{X}_{B_i}, \\ \ulcorner p \urcorner = 0 & \text{if for any } i \geq 1,\ p \notin \tilde{X}_{B_i}; \end{cases}$

2. † commutes with Boolean connectives;

3. $(t : F)^\dagger :\equiv \mathsf{Prf}(\ulcorner t \urcorner, \ulcorner F^\dagger \urcorner)$;

4. $\mathsf{PA} \vdash \mathsf{Prf}(x, y) \leftrightarrow \mathsf{Proof}(x, y)$
   $\vee$[“$x = \ulcorner t \urcorner$ & $y = \ulcorner G^\dagger \urcorner$ & $g(\ulcorner G \urcorner) = i$ & $G \in J(B_i, t)$ for some t, G, i”].

We refer Clause 4 as **FPE**. We can recover F from $F^\dagger$ effectively since † is injective. Therefore our definition of $\mathsf{Prf}(x, y)$ makes sense.

Now we define our functions $\langle \mathbf{m}, \mathbf{a}, \mathbf{c} \rangle$.

$$\mathbf{m}(x,y) := \begin{cases} \ulcorner s \cdot t \urcorner & \text{if } x = \ulcorner s \urcorner, y = \ulcorner t \urcorner \text{ for some proof terms } s, t; \\ u \otimes y & \text{if } x = \ulcorner s \urcorner \text{ for some proof term } s \\ & \quad \& \ y \neq \ulcorner t \urcorner \text{ for any proof term } t; \\ x \otimes v & \text{if } x \neq \ulcorner s \urcorner \text{ for any proof term } s \\ & \quad \& \ y = \ulcorner t \urcorner \text{ for some proof term } t; \\ x \otimes y & \text{if } x \neq \ulcorner s \urcorner, y \neq \ulcorner t \urcorner \text{ for any proof terms } s, t. \end{cases}$$

$$\mathbf{a}(x,y) := \begin{cases} \ulcorner s + t \urcorner & \text{if } x = \ulcorner s \urcorner, y = \ulcorner t \urcorner \text{ for some proof terms } s, t; \\ u \oplus y & \text{if } x = \ulcorner s \urcorner \text{ for some proof term } s \\ & \quad \& \ y \neq \ulcorner t \urcorner \text{ for any proof term } t; \\ x \oplus v & \text{if } x \neq \ulcorner s \urcorner \text{ for any proof term } s \\ & \quad \& \ y = \ulcorner t \urcorner \text{ for some proof term } t; \\ x \oplus y & \text{if } x \neq \ulcorner s \urcorner, y \neq \ulcorner t \urcorner \text{ for any proof terms } s, t. \end{cases}$$

$$\mathbf{c}(x) := \begin{cases} \ulcorner !s \urcorner & \text{if } x = \ulcorner s \urcorner \text{ for some proof term } s; \\ w \otimes \uparrow x & \text{if } x \neq \ulcorner s \urcorner \text{ for any proof term } s. \end{cases}$$

where

- u is the least natural number satisfying

$$\mathsf{Proof}(u, \ulcorner F^{\dagger} \urcorner) \text{ for all } F \in J(s);$$

- v is the least natural number satisfying

$$\mathsf{Proof}(v, \ulcorner F^{\dagger} \urcorner) \text{ for all } F \in J(t);$$

- w is the least natural number satisfying

$$\mathsf{Proof}(w, \mathsf{Proof}(x, \ulcorner \varphi \urcorner) \to \mathsf{Prf}(x, \ulcorner \varphi \urcorner)) \text{ for all } \varphi \in T(x).$$

Recall that $T(x)$ is the set $\{n \mid \mathbb{N} \models \mathsf{Prf}(x, n)\}$.

The desired arithmetical interpretation $*$ is defined as follows:

1. $p^* :\equiv p^{\dagger}$ for each propositional variable p;
2. $x^* := \ulcorner x \urcorner$, $a^* := \ulcorner a \urcorner$ for each proof variable x and proof constant a;
3. for proof terms s and t,

$$(s \cdot t)^* := \mathbf{m}(s^*, t^*),\ (s+t)^* := \mathbf{a}(s^*, t^*),\ (!s)^* := \mathbf{c}(s^*);$$

4. $(t : F)^* :\equiv \mathsf{Prf}(t^*, \ulcorner F^* \urcorner)$.

Lemma 8.11. For any proof term t and **LP**-formula F,

1. $t^* \equiv t^\dagger$;
2. $F^* \equiv F^\dagger$.

Proof. The proof is similar to the proof of Lemma 8.2 in Artemov [2]. $\square$

Lemma 8.12. Let $i \geq 1$ and F be an **LP**-formula.

1. If $F \in \tilde{X}_{B_i}$, then $\mathsf{PA} \vdash F^*$;
2. If $\sim F \in \tilde{X}_{B_i}$, then $\mathsf{PA} \vdash (\sim F)^*$.

Proof. We prove by induction on the construction of F.
Base Case (i): $F \equiv p$ for some propositional variable p.

1. Suppose that $p \in \tilde{X}_{B_i}$. Then $\mathsf{PA} \vdash p^*$ holds immediately by the definitions of $\dagger$ and $*$.
2. Suppose that $\sim p \in \tilde{X}_{B_i}$. In this case, $\sim p \equiv \neg p$. By **B2**, $p \notin \tilde{X}_{B_i}$. Since $g(\ulcorner p \urcorner) = i$, $p \notin \tilde{X}_{B_j}$ for any $j \neq i$. Therefore $p^* \equiv \ulcorner p \urcorner = 0$, and we obtain $\mathsf{PA} \vdash \neg p^*$.

Base Case (ii): $F \equiv t : G$.

1. Suppose that $t : G \in \tilde{X}_{B_i}$. Then we have $\mathsf{PA} \vdash$ "$G \in J(B_i, t)$". By Lemma 8.8, $g(\ulcorner G \urcorner) = g(\ulcorner t : G \urcorner) = i$. Thus $\mathsf{PA} \vdash$ "$g(\ulcorner G \urcorner) = i$". By **FPE**, $\mathsf{PA} \vdash \mathsf{Prf}(\ulcorner t \urcorner, \ulcorner G^\dagger \urcorner)$. By Lemma 8.11, $\mathsf{PA} \vdash \mathsf{Prf}(t^*, \ulcorner G^* \urcorner)$.
2. Suppose that $\sim t : G \in \tilde{X}_{B_i}$. In this case, $\sim t : G \equiv \neg t : G$. By **B2**, $t : G \notin \tilde{X}_{B_i}$. Then we have $\mathsf{PA} \vdash \neg$"$G \in J(B_i, t)$". By our assumption of the Gödel numbering, $\mathsf{PA} \vdash \neg\mathsf{Proof}(\ulcorner t \urcorner, \ulcorner G^\dagger \urcorner)$. By **FPE**, $\mathsf{PA} \vdash \neg\mathsf{Prf}(\ulcorner t \urcorner, \ulcorner G^\dagger \urcorner)$. By Lemma 8.11, $\mathsf{PA} \vdash \neg\mathsf{Prf}(t^*, \ulcorner G^* \urcorner)$.

Induction Case (i): $F \equiv G \to H$.

1. If $G \to H \in \tilde{X}_{B_i}$, then $G \to H \in X_{B_i}$ follows from **B1** and the description of $\mathcal{COM}$. By the B_i-maximal consistency of X_{B_i}, either $\sim G \in X_{B_i}$ or $H \in X_{B_i}$. By **B1**, either $\sim G \in \tilde{X}_{B_i}$ or $H \in \tilde{X}_{B_i}$. By the induction hypothesis, either $\mathsf{PA} \vdash \sim (G^*)$ or $\mathsf{PA} \vdash H^*$. In either cases, we obtain $\mathsf{PA} \vdash G^* \to H^*$, i.e., $\mathsf{PA} \vdash (G \to H)^*$.

2. Suppose that $\sim (G \to H) \in \tilde{X}_{B_i}$. In this case, $\sim (G \to H) \equiv \neg(G \to H)$. Then $\neg(G \to H) \in X_{B_i}$ follows from **B1** and the description of $\mathcal{COM}$. By the B_i-maximal consistency of X_{B_i}, G and $\sim H$ are elements of X_{B_i}, and by **B1**, G and $\sim H$ are elements of $\tilde{X}_{B_i}$. By the induction hypothesis, we have $\mathsf{PA} \vdash G^*$ and $\mathsf{PA} \vdash \sim H^*$. Thus $\mathsf{PA} \vdash \neg(G^* \to H^*)$, i.e., $\mathsf{PA} \vdash (\sim (G \to H))^*$.

Induction Case (ii): $F \equiv \neg G$. In this case, $\sim F \equiv G$.

1. Suppose that $\neg G \in \tilde{X}_{B_i}$. We distinguish two possibilities. Assume G is of the form $\neg H$. Then $\sim G \equiv H$ and $F \equiv \neg\neg H \in \tilde{X}_{B_i}$. By the description of $\mathcal{COM}$, $\neg\neg H \in X_{B_i}$. By B_i-maximal consistency, $H \in X_{B_i}$, and hence $\sim G \equiv H \in \tilde{X}_{B_i}$. By the induction hypothesis, $\mathsf{PA} \vdash H^*$, i.e., $\mathsf{PA} \vdash F^*$. Assume G is not of the form $\neg H$. Then $\neg G \equiv \sim G$. Since $\sim G \in \tilde{X}_{B_i}$, $\mathsf{PA} \vdash (\sim G)^*$ by the induction hypothesis. Therefore $\mathsf{PA} \vdash F^*$.

2. If $\sim F \in \tilde{X}_{B_i}$, then $G \in \tilde{X}_{B_i}$. By the induction hypothesis, $\mathsf{PA} \vdash G^*$, and hence $\mathsf{PA} \vdash (\sim F)^*$.

□

We obtain the following lemma.

Lemma 8.13. The formula $\mathsf{Prf}(x, y)$ is a proof predicate.

Proof. It suffices to show that if $\mathbb{N} \models \mathsf{Prf}(n, \ulcorner\varphi\urcorner)$ for some natural number n, then $\mathsf{PA} \vdash \varphi$. Suppose $\mathbb{N} \models \mathsf{Prf}(n, \ulcorner\varphi\urcorner)$. If $\mathbb{N} \models \mathsf{Proof}(n, \ulcorner\varphi\urcorner)$, then $\mathsf{PA} \vdash \varphi$. If $\mathbb{N} \models$ "$n = \ulcorner t \urcorner \ \&\ \ulcorner\varphi\urcorner = \ulcorner G^\dagger \urcorner \ \&\ g(\ulcorner G \urcorner) = i \ \&\ G \in J(B_i, t)$", then $t : G \in \tilde{X}_{B_i}$. By **B3**, $G \in \tilde{X}_{B_i}$. By Lemma 8.12, $\mathsf{PA} \vdash G^*$. By Lemma 8.11, $\mathsf{PA} \vdash \varphi$. □

We prove that the above three functions $\langle \mathbf{m}, \mathbf{a}, \mathbf{c} \rangle$ are Prf-functions.

Lemma 8.14. For any natural numbers k, l and $\mathcal{L}_A$-sentences φ and ψ, the following sentences are true (and hence provable in PA):

1. $(\mathsf{Prf}(k, \ulcorner\varphi \to \psi\urcorner) \land \mathsf{Prf}(l, \ulcorner\varphi\urcorner)) \to \mathsf{Prf}(\mathbf{m}(k, l), \ulcorner\psi\urcorner)$;

2. $(\mathsf{Prf}(k, \ulcorner\varphi\urcorner) \lor \mathsf{Prf}(l, \ulcorner\varphi\urcorner)) \to \mathsf{Prf}(\mathbf{a}(k, l), \ulcorner\varphi\urcorner)$;

3. $\mathsf{Prf}(k, \ulcorner\varphi\urcorner) \to \mathsf{Prf}(\mathbf{c}(k), \ulcorner\mathsf{Prf}(k, \ulcorner\varphi\urcorner)\urcorner)$.

Proof. 1. Suppose $\mathsf{Prf}(k, \ulcorner\varphi \to \psi\urcorner)$ and $\mathsf{Prf}(l, \ulcorner\varphi\urcorner)$. We distinguish the following four cases: (i) $k = \ulcorner s\urcorner$ and $l = \ulcorner t\urcorner$ for some proof terms s and t; (ii) $k = \ulcorner s\urcorner$ for some proof term s and l is not the Gödel number of any proof term; (iii) k is not the Gödel number of any proof term and $l = \ulcorner t\urcorner$ for some proof term t; (iv) k and l are not the Gödel numbers of proof term.

(i) In this case, $\mathbf{m}(k, l) = \ulcorner s \cdot t\urcorner$. By **FPE**, there are **LP**-formulas F, G and natural numbers i, $j \geq 1$ such that $\varphi \equiv F^\dagger$, $\psi \equiv G^\dagger$, $g(\ulcorner F \to G\urcorner) = i$, $g(\ulcorner F\urcorner) = j$, $s : (F \to G) \in \tilde{X}_{B_i}$ and $t : F \in \tilde{X}_{B_j}$. Then $i = j$ by the definition of g. Hence both $s : (F \to G)$ and $t : F$ are in $\tilde{X}_{B_i}$. By **B4**, $s \cdot t : G \in \tilde{X}_{B_i}$. Again by **FPE**, we have $\mathsf{Prf}(\ulcorner s \cdot t\urcorner, \ulcorner G^\dagger\urcorner)$. Therefore $\mathsf{Prf}(\mathbf{m}(k, l), \ulcorner\psi\urcorner)$.

(ii) In this case, $\mathbf{m}(k, l) = u \otimes l$, where u is as in the definition of $\mathbf{m}$. By **FPE**, there exist an **LP**-formula F and a natural number $i \geq 1$ such that $\varphi \to \psi \equiv F^\dagger$ and $s : F \in \tilde{X}_{B_i}$. In addition, $\mathsf{Proof}(l, \ulcorner\varphi\urcorner)$ holds. Compute all members of $J(s)$. Let G be one of the elements of $J(s)$. Then G is in $J(B_j, s)$ for some $j \geq 1$, i.e., $s : G \in \tilde{X}_{B_j}$. By **B3**, $G \in \tilde{X}_{B_j}$. By Lemma 8.12, $\mathsf{PA} \vdash G^*$, and by Lemma 8.11, $\mathsf{PA} \vdash G^\dagger$ (especially $F^\dagger$).

By the normality of $\mathsf{Proof}(x, y)$, we can compute the least natural number u such that $\mathsf{Proof}(u, \ulcorner G^\dagger\urcorner)$ for all $G \in J(s)$. In particular, we have $\mathsf{Proof}(u, \ulcorner F^\dagger\urcorner)$, i.e., $\mathsf{Proof}(u, \ulcorner\varphi \to \psi\urcorner)$. By the property of $\otimes$, we have $\mathsf{Proof}(u \otimes l, \ulcorner\psi\urcorner)$. By **FPE**, $\mathsf{Prf}(u \otimes l, \ulcorner\psi\urcorner)$. Therefore $\mathsf{Prf}(\mathbf{m}(k, l), \ulcorner\psi\urcorner)$.

(iii) In this case, $\mathbf{m}(k, l) = k \otimes v$ where v is as in the definition of $\mathbf{m}$. By **FPE**, there are an **LP**-formula F and a natural number $i \geq 1$ such that $\varphi \equiv F^\dagger$ and $t : F \in \tilde{X}_{B_i}$. In addition, $\mathsf{Proof}(k, \ulcorner\varphi \to \psi\urcorner)$ holds. Compute $J(t)$. Let G be one of the elements of $J(t)$. Then G is in $J(B_j, t)$ for some $j \geq 1$, i.e., $t : G \in \tilde{X}_{B_j}$. By **B3**, $G \in \tilde{X}_{B_j}$, and hence by Lemma 8.12 and Lemma 8.11, $\mathsf{PA} \vdash G^\dagger$.

By the normality of $\mathsf{Proof}(x, y)$, we can compute the least natural number v such that $\mathsf{Proof}(v, \ulcorner G^\dagger\urcorner)$ for all $G \in J(t)$. In particular, we have $\mathsf{Proof}(v, \ulcorner F^\dagger\urcorner)$, i.e., $\mathsf{Proof}(v, \ulcorner\varphi\urcorner)$. By the property of the Proof-function $\otimes$, we have $\mathsf{Proof}(k \otimes v, \ulcorner\psi\urcorner)$. By **FPE**, we obtain $\mathsf{Prf}(k \otimes v, \ulcorner\psi\urcorner)$. Therefore $\mathsf{Prf}(\mathbf{m}(k, l), \ulcorner\psi\urcorner)$.

(iv) In this case, $\mathbf{m}(k, l) = k \otimes l$. By **FPE**, $\mathsf{Proof}(k, \ulcorner\varphi \to \psi\urcorner)$ and $\mathsf{Proof}(l, \ulcorner\varphi\urcorner)$ hold. By the property of the Proof-function $\otimes$, $\mathsf{Proof}(k \otimes l, \ulcorner\psi\urcorner)$ also holds. Again by **FPE**, we obtain $\mathsf{Prf}(k \otimes l, \ulcorner\psi\urcorner)$. Therefore $\mathsf{Prf}(\mathbf{m}(k, l), \ulcorner\psi\urcorner)$.

2. We suppose $\mathsf{Prf}(k, \ulcorner\varphi\urcorner)$ holds. (The case for $\mathsf{Prf}(l, \ulcorner\varphi\urcorner)$ is similar.) In order to prove $\mathsf{Prf}(\mathbf{a}(k, l), \ulcorner\varphi\urcorner)$, we distinguish the following three cases: (i) $k = \ulcorner s\urcorner$ and $l = \ulcorner t\urcorner$ for some proof terms s and t; (ii) $k = \ulcorner s\urcorner$ for some proof

term s and $l \neq \ulcorner t \urcorner$ for any proof term t; (iii) $k \neq \ulcorner s \urcorner$ for any proof term s.

(i) In this case, $\mathbf{a}(k,l) = \ulcorner s + t \urcorner$. By **FPE**, there is an **LP**-formula F and a natural number $i \geq 1$ such that $\varphi \equiv F^\dagger$, $g(\ulcorner F \urcorner) = i$ and $s : F \in \tilde{X}_{B_i}$. By **B5**, $s + t : F \in \tilde{X}_{B_i}$. Again by **FPE**, $\mathsf{Prf}(\ulcorner s+t \urcorner, \ulcorner F^\dagger \urcorner)$. Therefore $\mathsf{Prf}(\mathbf{a}(k,l), \ulcorner \varphi \urcorner)$.

(ii) In this case, $\mathbf{a}(k,l) = u \oplus l$. By **FPE**, there is an **LP**-formula F and a natural number $i \geq 1$ such that $\varphi \equiv F^\dagger$ and $s : F \in \tilde{X}_{B_i}$. Compute the least natural number u such that $\mathsf{Proof}(u, \ulcorner G^\dagger \urcorner)$ for any $G \in J(s)$. In particular, $\mathsf{Proof}(u, \ulcorner F^\dagger \urcorner)$. By the property of Proof-function $\oplus$, we obtain $\mathsf{Proof}(u \oplus l, \ulcorner F^\dagger \urcorner)$. Again by **FPE**, $\mathsf{Prf}(u \oplus l, \ulcorner F^\dagger \urcorner)$. Therefore $\mathsf{Prf}(\mathbf{a}(k,l), \ulcorner \varphi \urcorner)$.

(iii) In this case, $\mathbf{a}(k,l)$ is either $k \oplus v$ or $k \oplus l$. By **FPE**, $\mathsf{Proof}(k, \ulcorner \varphi \urcorner)$ holds. Then we have $\mathsf{Proof}(k \oplus n, \ulcorner \varphi \urcorner)$ for any natural number n. If $l = \ulcorner t \urcorner$ for some proof term t, then let n be the least natural number v such that $\mathsf{Proof}(v, \ulcorner G^\dagger \urcorner)$ for any $G \in J(t)$. If $l \neq \ulcorner t \urcorner$ for any proof term t, then let n be l. By **FPE**, in both cases we obtain $\mathsf{Prf}(\mathbf{a}(k,l), \ulcorner \varphi \urcorner)$.

3. Suppose $\mathsf{Prf}(k, \ulcorner \varphi \urcorner)$. We distinguish the following two cases: (i) $k = \ulcorner t \urcorner$ for some proof term t; (ii) $k \neq \ulcorner t \urcorner$ for any proof term t.

(i) In this case, $\mathbf{c}(k) = \ulcorner !t \urcorner$. By **FPE**, there is an **LP**-formula F and a natural number $i \geq 1$ such that $\varphi \equiv F^\dagger$ and $F \in J(B_i, t)$, i.e., $t : F \in \tilde{X}_{B_i}$. by **B6**, $!t : t : F \in \tilde{X}_{B_i}$. By Lemma 8.12, $\mathsf{PA} \vdash (!t : t : F)^*$. By Lemma 8.11, $\mathsf{PA} \vdash (!t : t : F)^\dagger$. Since $(!t : t : F)^\dagger \equiv \mathsf{Prf}(\ulcorner !t \urcorner, \ulcorner \mathsf{Prf}(\ulcorner t \urcorner, \ulcorner F^\dagger \urcorner) \urcorner)$, we obtain $\mathsf{PA} \vdash \mathsf{Prf}(\mathbf{c}(k), \ulcorner \mathsf{Prf}(k, \ulcorner \varphi \urcorner) \urcorner)$. Thus $\mathsf{Prf}(\mathbf{c}(k), \ulcorner \mathsf{Prf}(k, \ulcorner \varphi \urcorner) \urcorner)$ holds.

(ii) In this case, $\mathbf{c}(k) = w \otimes \uparrow(k)$. By **FPE**, $\mathsf{Proof}(k, \ulcorner \varphi \urcorner)$ holds. By the property of Proof-function $\uparrow$, $\mathsf{Proof}(\uparrow(k), \ulcorner \mathsf{Proof}(k, \ulcorner \varphi \urcorner) \urcorner)$ also holds. Compute the least natural number w which satisfies

$$\mathsf{Proof}(w, \ulcorner \mathsf{Proof}(k, \ulcorner \psi \urcorner) \to \mathsf{Prf}(k, \ulcorner \psi \urcorner) \urcorner)$$

where $\psi \in T(k)$. Then we obtain

$$\mathsf{Proof}(w \otimes \uparrow(k), \ulcorner \mathsf{Prf}(k, \ulcorner \varphi \urcorner) \urcorner).$$

By **FPE**, we have $\mathsf{Prf}(\mathbf{c}(k), \ulcorner \mathsf{Prf}(k, \ulcorner \varphi \urcorner) \urcorner)$. □

Lemma 8.15. The proof predicate $\mathsf{Prf}(x,y)$ is normal.

Proof. We verify two conditions of Definition 4.6.

In order to check the condition (1), let k be a natural number. If k is not the code of any proof term, then $T(k)$ is finite since $\mathsf{Proof}(x,y)$ is normal. Suppose that $k = \ulcorner t \urcorner$ for some proof term t. Then $T(k) = \{\ulcorner G^\dagger \urcorner \mid g(\ulcorner G \urcorner) = i$ and $G \in J(B_i, t)$ for some $i\}$, specifically $T(k) = \{\ulcorner G^\dagger \urcorner \mid G \in J(t)\}$. By

Lemma 8.10, $J(t)$ is finite and the code of $J(t)$ is effectively computable from t.

Since $T(k) \cup T(l) \subseteq T(\mathbf{a}(k,l))$ by Lemma 8.12 (2), the condition (2) holds. $\square$

Proof of Theorem 8.6. If $\mathbf{LP}_0 \nvdash F$, then for some i, $F \equiv A_i$. Since $\sim B_i \in \tilde{X}_{B_i}$, we have $\mathsf{PA} \vdash \neg B_i^*$ by Lemma 8.12. Let $*$ be the arithmetical interpretation established in the above. Define the arithmetical interpretation $*_i$ as follows:

$$\epsilon^{*_i} \equiv \begin{cases} r(\epsilon)^* & \text{if } \epsilon \in \mathcal{L}(A_i); \\ \epsilon^* & \text{otherwise.} \end{cases}$$

It is easy to show that $A_i^{*_i} \equiv B_i^*$. Therefore $\mathsf{PA} \vdash \neg A_i^{*_i}$, and we conclude $\mathsf{PA} \nvdash A_i^{*_i}$. $\square$

9 Uniform arithmetical completeness of $\mathbf{LP}_0$

As in Section 8, we established a stronger version of Artemov's theorem by proving the arithmetical completeness theorem of $\mathbf{LP}_0$ with respect to a fixed Δ_1 normal proof predicate Prf and Prf-functions $\langle \mathbf{m}, \mathbf{a}, \mathbf{c} \rangle$. However, the so-called uniform arithmetical completeness theorem of $\mathbf{LP}_0$ does not hold with respect to Δ_1 proof predicates.

Proposition 9.1. There is no arithmetical interpretation $*$ based on some Δ_1 normal proof predicate $\mathsf{Prf}(x,y)$ and computable Prf-functions $\langle \mathbf{m}, \mathbf{a}, \mathbf{c} \rangle$ such that for any $\mathbf{LP}$-formula F,

$$\mathbf{LP}_0 \vdash F \text{ if and only if } \mathsf{PA} \vdash F^*.$$

Proof. Suppose, towards a contradiction, that there are such an arithmetical interpretation $*$ and a Δ_1 proof predicate $\mathsf{Prf}(x,y)$. Since two $\mathbf{LP}$-formulas $\neg v : p$ and $\neg v : \neg p$ are not provable in $\mathbf{LP}_0$, neither $\neg\mathsf{Prf}(v^*, \ulcorner p^* \urcorner)$ nor $\neg\mathsf{Prf}(v^*, \ulcorner \neg p^* \urcorner)$ is provable in PA. Since $\mathsf{Prf}(x,y)$ is a Δ_1 formula, $\mathsf{PA} \vdash \mathsf{Prf}(v^*, \ulcorner p^* \urcorner)$ and $\mathsf{PA} \vdash \mathsf{Prf}(v^*, \ulcorner \neg p^* \urcorner)$. Then $\mathsf{PA} \vdash p^*$ and $\mathsf{PA} \vdash \neg p^*$. This contradicts the consistency of PA. $\square$

Notice that Proposition 9.1 also holds when we do not fix a Δ_1 proof predicate. In the above proof, the decidability of Δ_1 formulas plays a key role. Thus for some proof predicate which is not Δ_1, the uniform arithmetical completeness theorem may hold. Indeed, in this section, we prove a version of the uniform arithmetical completeness theorem of $\mathbf{LP}_0$ with respect to some Σ_1 proof predicate.

Theorem 9.2 (The uniform arithmetical completeness theorem of $\mathbf{LP}_0$)**.** There exist a Σ_1 proof predicate $\mathsf{Prf}(x, y)$, computable Prf-functions $\langle \mathbf{m}, \mathbf{a}, \mathbf{c} \rangle$, and an arithmetical interpretation $*$ based on $\langle \mathsf{Prf}, \mathbf{m}, \mathbf{a}, \mathbf{c} \rangle$ such that for any **LP**-formula F,

$$\mathbf{LP}_0 \vdash F \text{ if and only if } \mathsf{PA} \vdash F^*.$$

First, we prove the following lemma (see Lindström [14] p.44 exercise 2.24).

Lemma 9.3. There exists a Σ_1 formula $\sigma(x)$ satisfying the following conditions:

1. $\mathsf{PA} \vdash \forall x \forall y (\sigma(x) \wedge \sigma(y) \to x = y)$,
2. for any natural number n, $\mathsf{PA} \nvdash \sigma(n)$ and $\mathsf{PA} \nvdash \neg\sigma(n)$,
3. $\mathbb{N} \models \forall x \neg\sigma(x)$.

Proof. By the Fixed Point Lemma, let $\sigma(x)$ be a Σ_1 formula satisfying the following equivalence:

$$\begin{aligned} &\mathsf{PA} \vdash \sigma(x) \leftrightarrow \\ &\quad \exists y (\mathsf{Proof}(\ulcorner \neg\sigma(\dot{x}) \urcorner, y) \wedge \forall z \forall w (\langle z, w \rangle < \langle x, y \rangle \to \neg\mathsf{Proof}(\ulcorner \neg\sigma(\dot{z}) \urcorner, w))), \end{aligned}$$

where $\langle \cdot, \cdot \rangle$ is a usual primitive recursive paring function.

1. We reason in PA. Suppose $\sigma(x)$ holds. Then there exists a proof p of $\neg\sigma(x)$ such that for any $\langle z, w \rangle$ with $\langle z, w \rangle < \langle x, p \rangle$, w is not a proof of $\neg\varphi(z)$. Let q be a proof of $\neg\sigma(y)$, then $\langle x, p \rangle \leq \langle y, q \rangle$ by the choice of $\langle x, p \rangle$. If $y \neq x$, then $\langle x, p \rangle < \langle y, q \rangle$, and hence $\neg\sigma(y)$ holds.

2. First, we prove $\mathsf{PA} \nvdash \neg\sigma(n)$ for all n. Towards a contradiction, suppose $\mathsf{PA} \vdash \neg\sigma(n)$ for some n. Let $\langle k, p \rangle = \min\{\langle n, q \rangle \mid q \text{ is a proof of } \neg\sigma(n) \text{ in } \mathsf{PA}\}$. Then $\mathsf{PA} \vdash \neg\sigma(k)$. On the other hand, since

$$\mathsf{PA} \vdash \mathsf{Proof}(\ulcorner \neg\sigma(k) \urcorner, p) \wedge \forall z \forall w (\langle z, w \rangle < \langle k, p \rangle \to \neg\mathsf{Proof}(\ulcorner \neg\sigma(\dot{z}) \urcorner, w)),$$

we have $\mathsf{PA} \vdash \sigma(k)$. This is a contradiction. Therefore there exists no n such that $\mathsf{PA} \vdash \neg\sigma(n)$.

Also, for $m \neq n$, we have $\mathsf{PA} \vdash \sigma(n) \to \neg\sigma(m)$ by 1. Hence $\mathsf{PA} \nvdash \sigma(n)$ for any n by 2.

3. If there were a natural number n such that $\mathbb{N} \models \sigma(n)$, then $\mathsf{PA} \vdash \sigma(n)$ because $\sigma(x)$ is Σ_1. This contradicts Clause 2. Therefore $\mathbb{N} \models \forall x \neg\sigma(x)$. $\square$

Let $\{A_i\}_{i\in\omega}$ be a primitive recursive enumeration of all $\mathbf{LP}_0$-unprovable formulas. For each $i \in \omega$, let X_i be a maximal $\mathbf{LP}_0$-consistent extension of $\{\neg A_i\}$. Since the set of theorems of $\mathbf{LP}_0$ is primitive recursive, we can construct such a set X_i primitive recursively. Moreover, we can define a Δ_1 formula $x \in X_u$ satisfying the following conditions: for any $n \in \omega$, $\mathbf{LP}$-formulas F, G and proof terms s, t,

(C1) $F \in X_n$ if and only if $\mathbb{N} \models \ulcorner F \urcorner \in X_n$,

(C2) $\mathsf{PA} \vdash \forall v(\ulcorner F \to G \urcorner \in X_v \leftrightarrow (\ulcorner F \urcorner \in X_v \to \ulcorner G \urcorner \in X_v))$,

(C3) $\mathsf{PA} \vdash \forall v(\ulcorner \neg F \urcorner \in X_v \leftrightarrow \neg(\ulcorner F \urcorner \in X_v))$,

(C4) $\mathsf{PA} \vdash \forall v(\ulcorner s : (F \to G) \urcorner \in X_v \wedge \ulcorner t : F \urcorner \in X_v \to \ulcorner (s \cdot t) : G \urcorner \in X_v)$,

(C5) $\mathsf{PA} \vdash \forall v(\ulcorner s : F \urcorner \in X_v \vee \ulcorner t : F \urcorner \in X_v \to \ulcorner (s + t) : F \urcorner \in X_v)$,

(C6) $\mathsf{PA} \vdash \forall v(\ulcorner t : F \urcorner \in X_v \to \ulcorner !t : (t : F) \urcorner \in X_v)$,

(C7) $\mathsf{PA} \vdash \forall v(\ulcorner t : F \urcorner \in X_v \to \ulcorner F \urcorner \in X_v)$.

We start defining a Σ_1 proof predicate $\mathsf{Prf}(x, y)$, computable Prf-functions $\mathbf{m}(x, y)$, $\mathbf{a}(x, y)$, $\mathbf{c}(x)$, and an arithmetical interpretation $*$ with the required properties. Let $\sigma(x)$ be a Σ_1 formula as in Lemma 9.3. We first define the auxiliary translation $\dagger$ of $\mathbf{LP}$-formulas and the Σ_1 formula $\mathsf{Prf}(x, y)$ simultaneously as follows:

1. $p^\dagger :\equiv \exists v(\sigma(v) \wedge \ulcorner p \urcorner \in X_v)$ for each propositional variable p,
2. $(F \to G)^\dagger :\equiv (F^\dagger \to G^\dagger)$,
3. $(\neg F)^\dagger :\equiv \neg F^\dagger$,
4. $(t : F)^\dagger :\equiv \mathsf{Prf}(\ulcorner t \urcorner, \ulcorner F^\dagger \urcorner)$,
5. $\mathsf{PA} \vdash \forall x \forall y(\mathsf{Prf}(x, y) \leftrightarrow (x = 0 \wedge \mathsf{Provable}(y)) \vee \xi(x, y))$ where $\xi(x, y)$ is the following formula:

$$\exists v(\sigma(v) \wedge \text{“}x = \ulcorner s \urcorner \ \&\ y = \ulcorner B^\dagger \urcorner \ \&\ \ulcorner s : B \urcorner \in X_v \ for\ some\ s, B\text{”}).$$

We can recover F from $F^\dagger$ in a primitive recursive way because $\dagger$ is injective, and hence the above definition makes sense.

Our formula $\mathsf{Prf}(x, y)$ is a proof predicate.

Lemma 9.4. For any $\mathcal{L}_A$-sentence φ,

$$\mathsf{PA} \vdash \varphi \text{ if and only if } \mathbb{N} \models \exists x \mathsf{Prf}(x, \ulcorner \varphi \urcorner).$$

Proof. Since $\mathbb{N} \models \forall x \neg \sigma(x)$ by Lemma 9.3, $\mathbb{N} \models \mathsf{Prf}(n, \ulcorner \varphi \urcorner) \leftrightarrow (n = 0 \wedge \mathsf{Provable}(\ulcorner \varphi \urcorner))$. Thus $\mathsf{PA} \vdash \varphi$ if and only if $\mathbb{N} \models \exists x \mathsf{Prf}(x, \ulcorner \varphi \urcorner)$. □

The following lemma plays a key role in our proof.

Lemma 9.5. For any **LP**-formula F, $\mathsf{PA} \vdash \forall u(\sigma(u) \to (\ulcorner F \urcorner \in X_u \leftrightarrow F^\dagger))$.

Proof. We prove by induction on the construction of F.
Base Case (i): $F \equiv p$ for some propositional variable p.
($\to$): $\mathsf{PA} \vdash \sigma(u) \wedge \ulcorner p \urcorner \in X_u \to p^\dagger$ by the definition of $p^\dagger$. Thus

$$\mathsf{PA} \vdash \forall u(\sigma(u) \to (\ulcorner p \urcorner \in X_u \to p^\dagger)).$$

($\leftarrow$): Since $\mathsf{PA} \vdash \forall x \forall y(\sigma(x) \wedge \sigma(y) \to x = y)$, we have

$$\mathsf{PA} \vdash \sigma(u) \wedge \neg \ulcorner p \urcorner \in X_u \to \forall v(\sigma(v) \to \neg \ulcorner p \urcorner \in X_v).$$

Hence we obtain $\mathsf{PA} \vdash \forall u(\sigma(u) \to (\neg \ulcorner p \urcorner \in X_u \to \neg p^\dagger))$ by the definition of $p^\dagger$.
Base Case (ii): $F \equiv t : G$.
($\to$): We have

$$\begin{aligned} \mathsf{PA} \vdash \sigma(u) \wedge \ulcorner t : G \urcorner \in X_u \to \exists v \big(\sigma(v) \wedge \text{“}\ulcorner t \urcorner = \ulcorner s \urcorner \ \&\ \ulcorner G^\dagger \urcorner = \ulcorner B^\dagger \urcorner \\ \&\ \ulcorner s : B \urcorner \in X_v \text{ for some } s, B\text{”}\big). \end{aligned}$$

Thus $\mathsf{PA} \vdash \sigma(u) \wedge \ulcorner t : G \urcorner \in X_u \to \xi(\ulcorner t \urcorner, \ulcorner G^\dagger \urcorner)$. Then we obtain $\mathsf{PA} \vdash \sigma(u) \wedge \ulcorner t : G \urcorner \in X_u \to \mathsf{Prf}(\ulcorner t \urcorner, \ulcorner G^\dagger \urcorner)$, and hence

$$\mathsf{PA} \vdash \forall u(\sigma(u) \to (\ulcorner t : G \urcorner \in X_u \to (t : G)^\dagger)).$$

($\leftarrow$): Since $\mathsf{PA} \vdash 0 \neq \ulcorner t \urcorner$, $\mathsf{PA} \vdash \neg(\ulcorner t \urcorner = 0 \wedge \mathsf{Provable}(\ulcorner G^\dagger \urcorner))$. Since $\mathsf{PA} \vdash \forall x \forall y(\sigma(x) \wedge \sigma(y) \to x = y)$, we have

$$\begin{aligned} \mathsf{PA} \vdash \sigma(u) \wedge \neg \ulcorner t : G \urcorner \in X_u \to \neg \exists v \big(\sigma(v) \wedge \text{“}\ulcorner t \urcorner = \ulcorner s \urcorner \ \&\ \ulcorner G^\dagger \urcorner = \ulcorner B^\dagger \urcorner \\ \&\ \ulcorner s : B \urcorner \in X_v \text{ for some } s, B\text{”}\big). \end{aligned}$$

This means $\mathsf{PA} \vdash \sigma(u) \wedge \neg \ulcorner t : G \urcorner \in X_u \to \neg \xi(\ulcorner t \urcorner, \ulcorner G^\dagger \urcorner)$. Therefore $\mathsf{PA} \vdash \sigma(u) \wedge \neg \ulcorner t : G \urcorner \in X_u \to \neg \mathsf{Prf}(\ulcorner t \urcorner, \ulcorner G^\dagger \urcorner)$ by the definition of $\mathsf{Prf}(x, y)$. Hence

$$\mathsf{PA} \vdash \forall u(\sigma(u) \to (\neg \ulcorner t : G \urcorner \in X_u \to \neg (t : G)^\dagger)).$$

Induction Case (i): $F \equiv (G \to H)$.

We suppose $\mathsf{PA} \vdash \forall u(\sigma(u) \to (\ulcorner G \urcorner \in X_u \leftrightarrow G^\dagger))$ and $\mathsf{PA} \vdash \forall u(\sigma(u) \to (\ulcorner H \urcorner \in X_u \leftrightarrow H^\dagger))$. Since $\mathsf{PA} \vdash \ulcorner G \to H \urcorner \in X_u \leftrightarrow (\ulcorner G \urcorner \in X_u \to \ulcorner H \urcorner \in X_u)$ by **C2**, we have $\mathsf{PA} \vdash \sigma(u) \to (\ulcorner G \to H \urcorner \in X_u \leftrightarrow (G^\dagger \to H^\dagger))$. Hence

$$\mathsf{PA} \vdash \forall u(\sigma(u) \to (\ulcorner G \to H \urcorner \in X_u \leftrightarrow (G \to H)^\dagger)).$$

Induction Case (ii): $F \equiv \neg G$.

We suppose $\mathsf{PA} \vdash \forall u(\sigma(u) \to (\ulcorner G \urcorner \in X_u \leftrightarrow G^\dagger))$. Since $\mathsf{PA} \vdash \ulcorner \neg G \urcorner \in X_u \leftrightarrow \neg(\ulcorner G \urcorner \in X_u)$ by **C3**, we obtain $\mathsf{PA} \vdash \sigma(u) \to (\ulcorner \neg G \urcorner \in X_u \leftrightarrow \neg G^\dagger)$. Therefore

$$\mathsf{PA} \vdash \forall u(\sigma(u) \to (\ulcorner \neg G \urcorner \in X_u \leftrightarrow (\neg G)^\dagger)).$$

□

We define computable functions $\mathbf{m}(x, y)$, $\mathbf{a}(x, y)$ and $\mathbf{c}(x)$ as follows:

$$\mathbf{m}(x, y) = \begin{cases} \ulcorner s \cdot t \urcorner & \text{if } x = \ulcorner s \urcorner \text{ and } y = \ulcorner t \urcorner \text{ for some } s \text{ and } t, \\ 0 & \text{otherwise.} \end{cases}$$

$$\mathbf{a}(x, y) = \begin{cases} \ulcorner s + t \urcorner & \text{if } x = \ulcorner s \urcorner \text{ and } y = \ulcorner t \urcorner \text{ for some } s \text{ and } t, \\ 0 & \text{otherwise.} \end{cases}$$

$$\mathbf{c}(x) = \begin{cases} \ulcorner !t \urcorner & \text{if } x = \ulcorner t \urcorner \text{ for some } t, \\ 0 & \text{otherwise.} \end{cases}$$

We define the required arithmetical interpretation $*$ as follows:

1. $p^* :\equiv p^\dagger$ for each propositional variable p,
2. $x^* := \ulcorner x \urcorner$ for each proof variable x,
3. $c^* := \ulcorner c \urcorner$ for each proof constant c,
4. for every proof terms s, t,
   - $(s \cdot t)^* := \mathbf{m}(s^*, t^*)$,
   - $(s + t)^* := \mathbf{a}(s^*, t^*)$,
   - $(!t)^* := \mathbf{c}(t^*)$.

Then as usual, we obtain the following lemma.

Lemma 9.6.

1. $t^* \equiv t^\dagger$ for each proof term t.
2. $F^* \equiv F^\dagger$ for each **LP**-formula F.

The following lemma follows from Lemma 9.5 and Lemma 9.6 immediately.

Lemma 9.7. For any **LP**-formula F, $\mathsf{PA} \vdash \forall u(\sigma(u) \to (\ulcorner F \urcorner \in X_u \leftrightarrow F^*))$.

We prove the completeness of $\mathbf{LP}_0$ with respect to the arithmetical interpretation $*$.

Lemma 9.8. For any **LP**-formula F, if $\mathbf{LP}_0 \nvdash F$, then $\mathsf{PA} \nvdash F^*$.

Proof. Suppose $\mathbf{LP}_0 \nvdash F$, then $F \equiv A_n$ for some $n \in \omega$. Since $\neg F \in X_n$, $\mathsf{PA} \vdash \ulcorner \neg F \urcorner \in X_n$ by **C1**. By Lemma 9.7, we obtain $\mathsf{PA} \vdash \sigma(n) \to \neg F^*$. Since $\mathsf{PA} \nvdash \neg\sigma(n)$ by Lemma 9.3, we conclude $\mathsf{PA} \nvdash F^*$. $\square$

Then we prove the soundness of $\mathbf{LP}_0$ with respect to $*$.

Lemma 9.9. For any $k \in \omega$ and $\mathcal{L}_A$-sentence φ, $\mathsf{PA} \vdash \mathsf{Prf}(k, \ulcorner \varphi \urcorner) \to \mathsf{Provable}(\ulcorner \varphi \urcorner)$.

Proof. First, we show that for each **LP**-formula F,

$$\mathsf{PA} \vdash \exists v(\sigma(v) \land \ulcorner F \urcorner \in X_v) \to \mathsf{Provable}(\ulcorner F^\dagger \urcorner) \tag{$\star$}$$

holds. Let F be any **LP**-formula. By Lemma 9.5, $\mathsf{PA} \vdash \exists v(\sigma(v) \land \ulcorner F \urcorner \in X_v) \to F^\dagger$. Then $\mathsf{PA} \vdash \mathsf{Provable}(\ulcorner \exists v(\sigma(v) \land \ulcorner F \urcorner \in X_v) \urcorner) \to \mathsf{Provable}(\ulcorner F^\dagger \urcorner)$ by the derivability conditions (Proposition 4.9). Since $\exists v(\sigma(v) \land \ulcorner F \urcorner \in X_v)$ is a Σ_1 sentence, by Proposition 4.10 we have $\mathsf{PA} \vdash \exists v(\sigma(v) \land \ulcorner F \urcorner \in X_v) \to \mathsf{Provable}(\ulcorner \exists v(\sigma(v) \land \ulcorner F \urcorner \in X_v) \urcorner)$. Therefore $\mathsf{PA} \vdash \exists v(\sigma(v) \land \ulcorner F \urcorner \in X_v) \to \mathsf{Provable}(\ulcorner F^\dagger \urcorner)$.

We reason in PA: Suppose $\mathsf{Prf}(k, \ulcorner \varphi \urcorner)$. If $k = 0$, then $\mathsf{Provable}(\ulcorner \varphi \urcorner)$ is obvious. If $k \neq 0$, then $\xi(k, \ulcorner \varphi \urcorner)$. In this case, $\sigma(v)$, $k = \ulcorner s \urcorner$, $\ulcorner \varphi \urcorner = \ulcorner F^\dagger \urcorner$ and $\ulcorner s : F \urcorner \in X_v$ hold for some v, proof term s and **LP**-formula F. Then $\ulcorner F \urcorner \in X_v$ by **C7**. Therefore $\exists v(\sigma(v) \land \ulcorner F \urcorner \in X_v)$, and hence $\mathsf{Provable}(\ulcorner F^\dagger \urcorner)$ holds by ($\star$). Since $\ulcorner \varphi \urcorner = \ulcorner F^\dagger \urcorner$, we obtain $\mathsf{Provable}(\ulcorner \varphi \urcorner)$. $\square$

Lemma 9.10. For any $k, l \in \omega$ and $\mathcal{L}_A$-sentences φ and ψ,

1. $\mathsf{PA} \vdash \mathsf{Prf}(k, \ulcorner \varphi \to \psi \urcorner) \land \mathsf{Prf}(l, \ulcorner \varphi \urcorner) \to \mathsf{Prf}(0, \ulcorner \psi \urcorner)$,

2. $\mathsf{PA} \vdash \mathsf{Prf}(k, \ulcorner \varphi \urcorner) \to \mathsf{Prf}(0, \ulcorner \varphi \urcorner)$,

3. $\mathsf{PA} \vdash \mathsf{Prf}(k, \ulcorner \varphi \urcorner) \to \mathsf{Prf}(0, \ulcorner \mathsf{Prf}(k, \ulcorner \varphi \urcorner) \urcorner)$.

Proof. 1. Let T be $\mathsf{PA} + \mathsf{Prf}(k, \ulcorner \varphi \to \psi \urcorner) \wedge \mathsf{Prf}(l, \ulcorner \varphi \urcorner)$. Then

$$T \vdash \mathsf{Provable}(\ulcorner \varphi \to \psi \urcorner) \wedge \mathsf{Provable}(\ulcorner \varphi \urcorner),$$

by Lemma 9.9. By the derivability conditions, $T \vdash \mathsf{Provable}(\ulcorner \psi \urcorner)$. Hence we have $T \vdash \mathsf{Prf}(0, \ulcorner \psi \urcorner)$.

2. Immediate from Lemma 9.9 and the definition of $\mathsf{Prf}(x, y)$.

3. Since $\mathsf{Prf}(k, \ulcorner \varphi \urcorner)$ is a Σ_1 sentence,

$$\mathsf{PA} \vdash \mathsf{Prf}(k, \ulcorner \varphi \urcorner) \to \mathsf{Provable}(\ulcorner \mathsf{Prf}(k, \ulcorner \varphi \urcorner) \urcorner).$$

Thus $\mathsf{PA} \vdash \mathsf{Prf}(k, \ulcorner \varphi \urcorner) \to \mathsf{Prf}(0, \ulcorner \mathsf{Prf}(k, \ulcorner \varphi \urcorner) \urcorner)$. □

Lemma 9.11. For any $k, l \in \omega$ and $\mathcal{L}_A$-sentences φ and ψ,

1. $\mathsf{PA} \vdash \mathsf{Prf}(k, \ulcorner \varphi \to \psi \urcorner) \wedge \mathsf{Prf}(l, \ulcorner \varphi \urcorner) \to \mathsf{Prf}(\mathbf{m}(k, l), \ulcorner \psi \urcorner)$,

2. $\mathsf{PA} \vdash \mathsf{Prf}(k, \ulcorner \varphi \urcorner) \vee \mathsf{Prf}(l, \ulcorner \varphi \urcorner) \to \mathsf{Prf}(\mathbf{a}(k, l), \ulcorner \varphi \urcorner)$,

3. $\mathsf{PA} \vdash \mathsf{Prf}(k, \ulcorner \varphi \urcorner) \to \mathsf{Prf}(\mathbf{c}(k), \ulcorner \mathsf{Prf}(k, \ulcorner \varphi \urcorner) \urcorner)$.

Proof. 1. If $\mathbf{m}(k, l) = 0$, it is obvious from Lemma 9.10. If $\mathbf{m}(k, l) = \ulcorner s \cdot t \urcorner$ for some proof terms s and t, then $k = \ulcorner s \urcorner$ and $l = \ulcorner t \urcorner$. Also $\mathsf{PA} \vdash \mathsf{Prf}(k, \ulcorner \varphi \to \psi \urcorner) \leftrightarrow \xi(k, \ulcorner \varphi \to \psi \urcorner)$ and $\mathsf{PA} \vdash \mathsf{Prf}(l, \ulcorner \varphi \urcorner) \leftrightarrow \xi(l, \ulcorner \varphi \urcorner)$ because $k, l \neq 0$.

We reason in PA: Suppose $\mathsf{Prf}(k, \ulcorner \varphi \to \psi \urcorner) \wedge \mathsf{Prf}(l, \ulcorner \varphi \urcorner)$, then for some v and **LP**-formulas F and G, $\sigma(v)$, $\ulcorner \varphi \urcorner = \ulcorner F^\dagger \urcorner$, $\ulcorner \psi \urcorner = \ulcorner G^\dagger \urcorner$, $\ulcorner s : (F \to G) \urcorner \in X_v$ and $\ulcorner t : F \urcorner \in X_v$ hold. We obtain $\ulcorner (s \cdot t) : G \urcorner \in X_v$ by **C4**, and $\xi(\mathbf{m}(k, l), \ulcorner \psi \urcorner)$ holds. Then we conclude $\mathsf{Prf}(\mathbf{m}(k, l), \ulcorner \psi \urcorner)$.

2. By Lemma 9.10, we may assume $\mathbf{a}(k, l) = \ulcorner s + t \urcorner$ for some proof terms s and t.

We reason in PA: Suppose $\mathsf{Prf}(k, \ulcorner \varphi \urcorner) \vee \mathsf{Prf}(l, \ulcorner \varphi \urcorner)$, then $\xi(k, \ulcorner \varphi \urcorner) \vee \xi(l, \ulcorner \varphi \urcorner)$ holds. Thus for some v and **LP**-formula F, $\sigma(v)$, $\ulcorner \varphi \urcorner = \ulcorner F^\dagger \urcorner$ and $\ulcorner s : F \urcorner \in X_v$ or $\ulcorner t : F \urcorner \in X_v$. In either case, $\ulcorner (s + t) : F \urcorner \in X_v$ holds by **C5**, and hence we have $\xi(\mathbf{a}(k, l), \ulcorner \varphi \urcorner)$. Then we conclude $\mathsf{Prf}(\mathbf{a}(k, l), \ulcorner \varphi \urcorner)$.

3. We assume $\mathbf{c}(k) = \ulcorner !t \urcorner$ for some proof term t.

We reason in PA: Suppose $\mathsf{Prf}(k, \ulcorner \varphi \urcorner)$, then $\xi(k, \ulcorner \varphi \urcorner)$ holds. Thus for some v and **LP**-formula F, $\sigma(v)$, $\ulcorner \varphi \urcorner = \ulcorner F^\dagger \urcorner$ and $\ulcorner t : F \urcorner \in X_v$. By **C6**, we have $\ulcorner !t : (t : F) \urcorner \in X_v$. Then $\xi(\mathbf{c}(k), \ulcorner \mathsf{Prf}(\ulcorner t \urcorner, \ulcorner \varphi \urcorner) \urcorner)$ holds. Therefore $\mathsf{Prf}(\mathbf{c}(k), \ulcorner \mathsf{Prf}(k, \ulcorner \varphi \urcorner) \urcorner)$ holds. □

Lemma 9.12. For any $n > 0$ and $\mathcal{L}_A$-sentence φ, $\mathsf{PA} \vdash \mathsf{Prf}(n, \ulcorner \varphi \urcorner) \to \varphi$.

Proof. Since $n \neq 0$, $\mathsf{PA} \vdash \mathsf{Prf}(n, \ulcorner \varphi \urcorner) \leftrightarrow \xi(n, \ulcorner \varphi \urcorner)$. We distinguish two cases.

Case (i): $n \neq \ulcorner s \urcorner$ for any proof term s or $\varphi \not\equiv F^\dagger$ for any **LP**-formula F.

Since $\mathsf{PA} \vdash \neg\xi(n, \ulcorner\varphi\urcorner)$, $\mathsf{PA} \vdash \neg\mathsf{Prf}(n, \ulcorner\varphi\urcorner)$. Thus $\mathsf{PA} \vdash \mathsf{Prf}(n, \ulcorner\varphi\urcorner) \to \varphi$.

Case (ii): $n = \ulcorner s \urcorner$ for some proof term s, and $\varphi \equiv F^\dagger$ for some **LP**-formula F.

In this case, we have $\mathsf{PA} \vdash \mathsf{Prf}(n, \ulcorner\varphi\urcorner) \to \exists v(\sigma(v) \wedge \ulcorner s : F \urcorner \in X_v)$. Since $\mathsf{PA} \vdash \ulcorner s : F \urcorner \in X_v \to \ulcorner F \urcorner \in X_v$ by **C7**, $\mathsf{PA} \vdash \mathsf{Prf}(n, \ulcorner\varphi\urcorner) \to \exists v(\sigma(v) \wedge \ulcorner F \urcorner \in X_v)$. By Lemma 9.5, $\mathsf{PA} \vdash \mathsf{Prf}(n, \ulcorner\varphi\urcorner) \to F^\dagger$. We conclude $\mathsf{PA} \vdash \mathsf{Prf}(n, \ulcorner\varphi\urcorner) \to \varphi$.

□

Lemma 9.13. For any **LP**-formula F, if $\mathbf{LP}_0 \vdash F$, then $\mathsf{PA} \vdash F^*$.

Proof. From Lemma 9.11 and Lemma 9.12. □

Our proof of Theorem 9.2 is completed.

Our Theorem 9.2 is not a perfect statement of the so-called uniform arithmetical completeness theorem because of the following two reasons.

Remark 9.14.

1. Our proof predicate $\mathsf{Prf}(x, y)$ is not normal because 0 is a proof of all theorems of PA.

2. The arithmetical soundness of $\mathbf{LP}_0$ does not hold with respect to our proof predicate $\mathsf{Prf}(x, y)$. For, if $\mathsf{PA} \vdash \mathsf{Prf}(0, \ulcorner\varphi\urcorner) \to \varphi$, then $\mathsf{PA} \vdash \mathsf{Provable}(\ulcorner\varphi\urcorner) \to \varphi$ by the definition of Prf. By Löb's theorem, $\mathsf{PA} \vdash \varphi$. Hence for PA-unprovable sentences φ, $\mathsf{PA} \vdash \mathsf{Prf}(0, \ulcorner\varphi\urcorner) \to \varphi$ does not hold. Let v^* be 0 and p^* be φ, then $\mathbf{LP}_0 \vdash v : p \to p$ but $\mathsf{PA} \nvdash (v : p \to p)^*$.

Chapter IV
Interpolation properties for Sacchetti's logics

10 Some propositions of $\mathbf{wGL}_n$

The next Propositions 10.1 and 10.3 state basic properties of Sacchetti's logics $\mathbf{wGL}_n$.

Proposition 10.1. Assume $n \geq 1$. For any formula φ, $\mathbf{wGL}_n \vdash \Box\varphi \to \Box^{n+1}\varphi$.

Proof. See Sacchetti [20] or Kurahashi & Okawa [12]. $\Box$

We give some notations. For $n \geq 1$ and φ, we put:

$$[n]\varphi :\equiv \Box\varphi \wedge \Box^2\varphi \wedge \cdots \wedge \Box^n\varphi, \quad [n]^+\varphi :\equiv \varphi \wedge [n]\varphi.$$

Let Γ be a set of formulas. The sets $\Box^n\Gamma$, $[n]\Gamma$ and $[n]^+\Gamma$ denote the ones obtained from Γ by replacing every formula φ in Γ by $\Box^n\varphi$, $[n]\varphi$, and $[n]^+\varphi$, respectively.

Lemma 10.2. Assume $n \geq 1$. For any φ,

1. $\mathbf{wGL}_n \vdash [n]\varphi \leftrightarrow [n+1]\varphi$.
2. $\mathbf{wGL}_n \vdash [n]\varphi \to \Box[n]\varphi$.

Proof. Clearly follows from Proposition 10.1. $\Box$

Proposition 10.3. Assume $n \geq 1$. For any sets of formulas Γ and Δ, and any formula φ,

1. $\mathbf{wGL}_n \vdash [n]^+\varphi \to \varphi$.
2. If
$$\mathbf{wGL}_n \vdash \bigwedge [n]^+\Gamma \wedge \bigwedge [n]\Delta \to \varphi,$$
then
$$\mathbf{wGL}_n \vdash \bigwedge [n]\,(\Gamma \cup \Delta) \to \Box\varphi.$$

3. If

$$\mathbf{wGL}_n \vdash \bigwedge [n]^+ \Gamma \wedge \bigwedge [n] \Delta \wedge [n]\varphi \to \varphi,$$

then

$$\mathbf{wGL}_n \vdash \bigwedge [n]^+ \Gamma \wedge \bigwedge [n] \Delta \to \varphi.$$

Proof. 1. Trivial.

2. Suppose that $\mathbf{wGL}_n \vdash \bigwedge [n]^+ \Gamma \wedge \bigwedge [n] \Delta \to \varphi$. Since $\mathbf{wGL}_n$ is normal,

$$\mathbf{wGL}_n \vdash \bigwedge [n+1] \Gamma \wedge \bigwedge \Box [n] \Delta \to \Box \varphi.$$

By Lemma 10.2, the premises can be simplified as follows:

$$\mathbf{wGL}_n \vdash \bigwedge [n] \Gamma \wedge \bigwedge [n] \Delta \to \Box \varphi.$$

That is,

$$\mathbf{wGL}_n \vdash \bigwedge [n] \, (\Gamma \cup \Delta) \to \Box \varphi.$$

3. The argument is based on Kurahashi & Okawa [12] Proposition 3.4. Suppose that $\mathbf{wGL}_n \vdash \bigwedge [n]^+ \Gamma \wedge \bigwedge [n] \Delta \wedge [n]\varphi \to \varphi$. We claim that for any k $(0 \leq k \leq n-1)$,

$$\mathbf{wGL}_n \vdash \bigwedge [n]^+ \Gamma \wedge \bigwedge [n] \Delta \wedge \left(\Box^{k+1}\varphi \wedge \cdots \wedge \Box^{n}\varphi\right) \to [n]^+\varphi.$$

We prove the claim by induction on k.

Base case ($k = 0$). It is clear that $\mathbf{wGL}_n \vdash \bigwedge [n]^+ \Gamma \wedge \bigwedge [n] \Delta \wedge [n]\varphi \to [n]\varphi$. Combining with the supposition we have

$$\mathbf{wGL}_n \vdash \bigwedge [n]^+ \Gamma \wedge \bigwedge [n] \Delta \wedge [n]\varphi \to [n]^+\varphi. \tag{1}$$

Inductive case. Suppose that the claim holds for k, i.e.,

$$\mathbf{wGL}_n \vdash \bigwedge [n]^+ \Gamma \wedge \bigwedge [n] \Delta \wedge \left(\Box^{k+1}\varphi \wedge \cdots \wedge \Box^{n}\varphi\right) \to [n]^+\varphi.$$

Since $\mathbf{wGL}_n$ is normal,

$$\mathbf{wGL}_n \vdash \bigwedge [n+1] \Gamma \wedge \bigwedge \Box [n] \Delta \wedge \left(\Box^{k+2}\varphi \wedge \cdots \wedge \Box^{n+1}\varphi\right) \to [n+1]\varphi.$$

By Lemma 10.2,

$$\mathbf{wGL}_n \vdash \bigwedge [n] \Gamma \wedge \bigwedge [n] \Delta \wedge \left(\Box^{k+2}\varphi \wedge \cdots \wedge \Box^{n+1}\varphi\right) \to [n]\varphi. \tag{2}$$

On the other hand, by the inductive hypothesis,

$$\begin{aligned}
\mathbf{wGL}_n &\vdash \bigwedge [n]^+\Gamma \wedge \bigwedge [n]\Delta \wedge \left(\Box^{k+1}\varphi \wedge \cdots \wedge \Box^n\varphi\right) \to \varphi, \\
&\vdash \bigwedge [n]^+\Gamma \wedge \bigwedge [n]\Delta \wedge \left(\Box^{k+1}\varphi \wedge \cdots \wedge \Box^{n-1}\varphi\right) \to (\Box^n\varphi \to \varphi), \\
&\vdash \bigwedge [n+1]\Gamma \wedge \bigwedge \Box[n]\Delta \wedge \left(\Box^{k+2}\varphi \wedge \cdots \wedge \Box^n\varphi\right) \to \Box(\Box^n\varphi \to \varphi), \\
&\vdash \bigwedge [n+1]\Gamma \wedge \bigwedge \Box[n]\Delta \wedge \left(\Box^{k+2}\varphi \wedge \cdots \wedge \Box^n\varphi\right) \to \Box\varphi.
\end{aligned}$$

By Proposition 10.1 and Lemma 10.2,

$$\begin{aligned}
\mathbf{wGL}_n &\vdash \bigwedge [n+1]\Gamma \wedge \bigwedge \Box[n]\Delta \wedge \left(\Box^{k+2}\varphi \wedge \cdots \wedge \Box^n\varphi\right) \to \Box^{n+1}\varphi, \\
&\vdash \bigwedge [n]\Gamma \wedge \bigwedge [n]\Delta \wedge \left(\Box^{k+2}\varphi \wedge \cdots \wedge \Box^n\varphi\right) \to \Box^{n+1}\varphi.
\end{aligned}$$

Combining with (2), we have

$$\begin{aligned}
\mathbf{wGL}_n &\vdash \bigwedge [n]\Gamma \wedge \bigwedge [n]\Delta \wedge \left(\Box^{k+2}\varphi \wedge \cdots \wedge \Box^n\varphi\right) \to [n]\varphi, \\
&\vdash \bigwedge [n]^+\Gamma \wedge \bigwedge [n]\Delta \wedge \left(\Box^{k+2}\varphi \wedge \cdots \wedge \Box^n\varphi\right) \to [n]\varphi.
\end{aligned}$$

From this and (1), we obtain

$$\mathbf{wGL}_n \vdash \bigwedge [n]^+\Gamma \wedge \bigwedge [n]\Delta \wedge \left(\Box^{k+2}\varphi \wedge \cdots \wedge \Box^n\varphi\right) \to [n]^+\varphi.$$

The proof of the claim is completed.

We return to the proof of Proposition 10.3.3. By the claim, If $k = n-1$, then

$$\begin{aligned}
\mathbf{wGL}_n &\vdash \bigwedge [n]^+\Gamma \wedge \bigwedge [n]\Delta \wedge \Box^n\varphi \to [n]^+\varphi, \\
&\vdash \bigwedge [n]^+\Gamma \wedge \bigwedge [n]\Delta \to \left(\Box^n\varphi \to [n]^+\varphi\right).
\end{aligned}$$

Since $\mathbf{wGL}_n \vdash [n]^+\varphi \to \varphi$,

$$\mathbf{wGL}_n \vdash \bigwedge [n]^+\Gamma \wedge \bigwedge [n]\Delta \to (\Box^n\varphi \to \varphi). \tag{3}$$

Then we have

$$\begin{aligned}
\mathbf{wGL}_n &\vdash \bigwedge [n+1]\Gamma \wedge \bigwedge \Box[n]\Delta \to \Box(\Box^n\varphi \to \varphi), && \text{by the normality,} \\
&\vdash \bigwedge [n+1]\Gamma \wedge \bigwedge \Box[n]\Delta \to \Box\varphi, \\
&\vdash \bigwedge [n]\Gamma \wedge \bigwedge [n]\Delta \to \Box\varphi, && \text{(by Lemma 10.2.)}
\end{aligned}$$

Moreover,

$$\begin{aligned}\mathbf{wGL}_n &\vdash \bigwedge \Box[n]\Gamma \wedge \bigwedge \Box[n]\Delta \to \Box^2\varphi,\\ &\vdash \bigwedge [n]\Gamma \wedge \bigwedge [n]\Delta \to \Box^2\varphi,\\ &\vdots\\ &\vdash \bigwedge [n]\Gamma \wedge \bigwedge [n]\Delta \to \Box^n\varphi,\\ &\vdash \bigwedge [n]^+\Gamma \wedge \bigwedge [n]\Delta \to \Box^n\varphi.\end{aligned}$$

From this and (3), we conclude $\mathbf{wGL}_n \vdash \bigwedge[n]^+\Gamma \wedge \bigwedge[n]\Delta \to \varphi$. $\Box$

11 Sequent calculi for $\mathbf{wGL}_n$

We present one-sided sequent calculi $\mathbf{wGL}_n^{\mathbf{G}}$ for Sacchetti's logics. *Sequents*, denoted by $\Gamma, \Delta, \ldots$ etc., are defined as *sets* of formulas. Let Γ and Δ be sequents and φ be a formula. We define the sequent (Γ, Δ) as the union of Γ and Δ, and (Γ, φ) as the set $\Gamma \cup \{\varphi\}$. As mentioned in Section 2, for a given Γ, $\Box^n\Gamma$ (and $\Diamond^n\Gamma$) denotes the sequents obtained from Γ by replacing every φ in Γ by $\Box^n\varphi$ (resp. $\Diamond^n\varphi$). A *derivation* in a calculus is a finite tree whose nodes are assigned by sequents that is constructed according to the rules of the calculus. A *proof* in a calculus is a derivation such that every leaf is labeled with axioms.

Definition 11.1. Assume $n \geq 1$. The one-sided sequent calculus $\mathbf{wGL}_n^{\mathbf{G}}$ consists of the following axioms and rules.

Axioms:

$$(p, \overline{p})\,, \qquad \top$$

Structural Rule:

$$\frac{\Gamma}{\Gamma, \Delta}\ (weak)$$

Propositional Rules:

$$\frac{\Gamma, \varphi, \psi}{\Gamma, \varphi \vee \psi}\ (\vee) \quad \frac{\Gamma, \varphi \quad \Gamma, \psi}{\Gamma, \varphi \wedge \psi}\ (\wedge)$$

Modal rule:

$$\frac{\Diamond^n\Gamma, \Gamma, \Diamond^n\overline{\varphi}, \varphi}{\Diamond\Gamma, \Box\varphi}\ (\Box_n)$$

The aim of this section is showing the following facts:

- For any sequent Γ, if $\mathbf{wGL}_n^{\mathbf{G}} \vdash \Gamma$, then we can construct a proof of Γ in $\mathbf{wGL}_n^{\mathbf{G}}$ effectively;
- The following rules are admissible in $\mathbf{wGL}_n^{\mathbf{G}}$:

$$\frac{\Gamma, \varphi \quad \Gamma, \overline{\varphi}}{\Gamma}\ (cut), \quad \text{and} \quad \frac{\Diamond^n \Gamma, \Gamma, \Diamond^n \overline{\varphi}, \varphi}{\Diamond^n \Gamma, \Gamma, \varphi}\ (L\ddot{o}b).$$

The argument is based on Sambin and Valentini [22].

11.1 Proof search procedure

We give an effective way of constructing a proof of Γ in $\mathbf{wGL}_n^{\mathbf{G}}$, for every sequent Γ with $\mathbf{wGL}_n^{\mathbf{G}} \vdash \Gamma$ (see Proposition 11.5 in this subsection).

Lemma 11.2. Let Γ be a sequent and φ be a formula. Then $\mathbf{wGL}_n^{\mathbf{G}} \vdash (\Gamma, \varphi, \overline{\varphi})$. Moreover, we can construct a proof of $(\Gamma, \varphi, \overline{\varphi})$ in $\mathbf{wGL}_n^{\mathbf{G}}$ effectively from φ and φ.

Proof. Induction on the construction of φ.

- Suppose that φ is one of the form $\langle p, \overline{p}, \top, \bot \rangle$. In this case, a proof of $(\Gamma, \varphi, \overline{\varphi})$ in $\mathbf{wGL}_n^{\mathbf{G}}$ is given as follows:

$$\frac{\top}{\Gamma, \top, \bot}\ (weak), \quad \frac{p, \overline{p}}{\Gamma, p, \overline{p}}\ (weak).$$

- Assume $\varphi \equiv \psi \vee \theta$ or $\varphi \equiv \psi \wedge \theta$. Consider the following derivations:

$$\dfrac{\dfrac{\Gamma, \psi, \theta, \overline{\psi}}{\Gamma, \psi \vee \theta, \overline{\psi}}\ (\vee) \quad \dfrac{\Gamma, \psi, \theta, \overline{\theta}}{\Gamma, \psi \vee \theta, \overline{\theta}}\ (\vee)}{\Gamma, \psi \vee \theta, \overline{\psi} \wedge \overline{\theta}}\ (\wedge), \quad \dfrac{\dfrac{\Gamma, \psi, \overline{\psi}, \overline{\theta}}{\Gamma, \psi, \overline{\psi} \vee \overline{\theta}}\ (\vee) \quad \dfrac{\Gamma, \theta, \overline{\psi}, \overline{\theta}}{\Gamma, \theta, \overline{\psi} \vee \overline{\theta}}\ (\vee)}{\Gamma, \psi \wedge \theta, \overline{\psi} \vee \overline{\theta}}\ (\wedge).$$

  By the induction hypothesis, we can effectively construct a proof of each assumption in the derivations. Thus we can also effectively construct a proof of $(\Gamma, \varphi, \overline{\varphi})$.

- Assume $\varphi \equiv \Box \psi$ or $\varphi \equiv \Diamond \psi$. Consider the following derivations:

$$\dfrac{\dfrac{\Diamond^n \overline{\psi}, \psi, \overline{\psi}}{\Box \psi, \Diamond \overline{\psi}}\ (\Box_n)}{\Gamma, \Box \psi, \Diamond \overline{\psi}}\ (weak), \quad \dfrac{\dfrac{\Diamond^n \psi, \psi, \overline{\psi}}{\Diamond \psi, \Box \overline{\psi}}\ (\Box_n)}{\Gamma, \Diamond \psi, \Box \overline{\psi}}\ (weak)$$

By the induction hypothesis, we can effectively construct a proof of each assumption in the derivations. Thus we can also effectively construct a proof of $(\Gamma, \varphi, \overline{\varphi})$.

$\square$

Next, we describe the proof search procedure $\mathcal{P}$. For a given Γ, $\mathcal{P}$ generates a derivation of Γ in the following variant of $\mathbf{wGL}_n^{\mathbf{G}}$.

Definition 11.3. Assume $n \geq 1$. The calculus $\mathbf{wGL}_n^{\mathbf{G}'}$ consists of the following axioms and rules:

Axioms:

$$\Gamma, \varphi, \overline{\varphi} \qquad \Gamma, \top$$

Rules: Propositional rules of $\mathbf{wGL}_n^{\mathbf{G}}$, and

$$\frac{\Diamond^n\Gamma, \Gamma, \Diamond^n\overline{\varphi_1}, \varphi_1 \quad \cdots \quad \Diamond^n\Gamma, \Gamma, \Diamond^n\overline{\varphi_m}, \varphi_m}{L, \Diamond\Gamma, \Box\varphi_1, \ldots, \Box\varphi_m}\ \Box'_n,$$

where L is a set of literals and the constant $\bot$.

The rule $(\Box'_n)$ has m assumptions of the form $(\Diamond^n\Gamma, \Gamma, \Diamond^n\overline{\varphi_i}, \varphi_i)$. The meaning of $(\Box'_n)$ is that the conclusion is provable if at least one of the assumptions is provable.

For a given sequent, the proof search procedure $\mathcal{P}$ tries to apply all applicable rules of $\mathbf{wGL}_n^{\mathbf{G}'}$ until every leaf is decomposed into an axiom of $\mathbf{wGL}_n^{\mathbf{G}'}$ or a sequent to which no more rules are applicable.

The following proposition asserts that the procedure $\mathcal{P}$ works soundly.

Proposition 11.4. For any input Γ, the proof search procedure $\mathcal{P}$ of Γ always halts.

Proof. It suffices to show that $\mathcal{P}$ never generates an infinite branch. Since any propositional rule lowers the numbers of connectives, $\mathcal{P}$ always halts as long as it applies only propositional rules. Therefore we have to show that $\mathcal{P}$ never generates infinitely many applications of $(\Box'_n)$. Suppose, for contradiction, that $\mathcal{P}$ produces an infinitely many applications of $\Box'_n$ for some sequent Γ.

$$\begin{array}{c}
\vdots \\
\dfrac{\cdots \quad \Diamond^n\Delta_2, \Delta_2, \Diamond^n\overline{\psi_2}, \psi_2 \quad \cdots}{L_2, \Diamond\Delta_2, \Box\Sigma_2 \ni \Box\psi_2}\ (\Box'_n) \\
\vdots \\
\dfrac{\cdots \quad \Diamond^n\Delta_1, \Delta_1, \Diamond^n\overline{\psi_1}, \psi_1 \quad \cdots}{L_1, \Diamond\Delta_1, \Box\Sigma_1 \ni \Box\psi_1}\ (\Box'_n) \\
\vdots \\
\Gamma
\end{array}$$

Note that every sequent in this infinite branch is non-axiomatic. Each time $\Box'_n$ is applied, we need at least one formula of the form $\Box\psi$. Since every propositional rule does not affect to any sequents of the form $\Diamond\Delta$, every $\Box\psi_{i+1}$ is obtained from either ψ_i or a formula in Δ_i by applying some propositional rules.

It is impossible that for all $i \geq 1$, $\Box\psi_{i+1}$ are obtained from ψ_i. Hence, for some natural number k, $\Box\psi_{k+1}$ is obtained from a formula θ in Δ_k by applying some propositional rules. The following table describes some formulas generated by $\mathcal{P}$ in each application of $(\Box'_n)$ from the k-th application of $(\Box'_n)$.

Table 1: Some formulas generated by $\mathcal{P}$

k	$k+1$	$k+2$	$\cdots$	$k+(n-1)$	$k+n$
	$\Diamond^n\overline{\psi_{k+1}}, \psi_{k+1}$	$\Diamond^{n-1}\overline{\psi_{k+1}}$	$\cdots$	$\Diamond^2\overline{\psi_{k+1}}$	$\Diamond\overline{\psi_{k+1}}$
$\Diamond^n\theta, \theta$	$\Diamond^{n-1}\theta$	$\Diamond^{n-2}\theta$	$\cdots$	$\Diamond\theta$	$\Diamond^n\theta, \theta$

When the $(k+n)$-th $(\Box'_n)$ is applied, the resulting sequent contains $\Diamond\overline{\psi_{k+1}}$ and θ, and $\mathcal{P}$ will decompose θ into $\Box\psi_{k+1}$ by applying propositional rules. Hence this branch contains a sequent containing $\Diamond\overline{\psi_{k+1}}$ and $\Box\psi_{k+1}$, i.e., an axiom of $\mathbf{wGL}_n^{\mathbf{G}'}$. This contradicts that the infinite branch consists of non-axiomatic sequents. $\Box$

Let π be a derivation of a sequent Γ in $\mathbf{wGL}_n^{\mathbf{G}'}$ generated by $\mathcal{P}$. A *search* of π is a subtree obtained by choosing a particular branch at each application of $(\Box_n)'$. A search π' of π is said to be *successful* if every branch of π' terminates in axioms of $\mathbf{wGL}_n^{\mathbf{G}'}$.

Proposition 11.5. For an input Γ, let π be a derivation of Γ in $\mathbf{wGL}_n^{\mathbf{G}'}$ generated by $\mathcal{P}$. If π contains a successful search, then we can construct a proof of Γ in $\mathbf{wGL}_n^{\mathbf{G}}$ from π.

Proof. Let π' be a successful search of π. Then from π' we can construct a proof of Γ in $\mathbf{wGL}_n^{\mathbf{G}}$ by Lemma 11.2 and transforming each application of $(\Box'_n)$ as follows:

$$\dfrac{\vdots \atop \Diamond^n\Gamma, \Gamma, \Diamond^n\overline{\varphi_i}, \varphi_i}{\dfrac{}{L, \Diamond\Gamma, \Box\Sigma \atop \vdots}}\,(\Box'_n) \quad\longmapsto\quad \dfrac{\dfrac{\vdots \atop \Diamond^n\Gamma, \Gamma, \Diamond^n\overline{\varphi_i}, \varphi_i}{\Diamond\Gamma, \Box\varphi_i}\,(\Box_n)}{L, \Diamond\Gamma, \Box\Sigma \atop \vdots}\,(weak)$$

.

$\Box$

11.2 Cut-admissibility

For a sequnt Γ, let $\Gamma^{\#}$ be the formula $\bigvee\{\varphi \mid \varphi \in \Gamma\}$. We prove the following theorem.

Theorem 11.6. For any sequent Γ, the following are equivalent:

1. $\mathbf{wGL}_n \vdash \Gamma^{\#}$;
2. $\Gamma^{\#}$ is valid in all finite $\mathbf{wGL}_n$-frames;
3. For the input Γ, the proof search procedure $\mathcal{P}$ generates a derivation which has a successful search of Γ;
4. $\mathbf{wGL}_n^{\mathbf{G}} \vdash \Gamma$;
5. $\mathbf{wGL}_n^{\mathbf{G}} + (cut) \vdash \Gamma$.

Proof. $(3 \Rightarrow 4)$: Clearly follows from Proposition 11.5.

$(4 \Rightarrow 5)$: Trivial.

$(5 \Rightarrow 1)$: By induction on the length of proofs in $\mathbf{wGL}_n^{\mathbf{G}} + (cut)$.

$(1 \Rightarrow 2)$: Due to Sacchetti [20].

We prove $(2 \Rightarrow 3)$. We construct a finite countermodel of Γ from the derivation of Γ which is generated by $\mathcal{P}$ and has no successful searches.

Definition 11.7. Let π be a derivation of a sequent Γ in $\mathbf{wGL}_n^{\mathbf{G}'}$. We define that π is unsuccessful inductively as follows:

- If π consists of a single sequent Γ, then π is *unsuccessful* iff $\Gamma = (L, \Diamond\Pi)$ where L is a set of literals and the constant $\perp$ satisfying that there is no propositional variable p such that $p, \overline{p} \in L$;
- If the last application of π is $(\vee)$ or $(\wedge)$, then π is unsuccessful iff for some sub-derivation of the assumption sequent is unsuccessful;
- If the last application of π is $\Box_n'$, then π is unsuccessful iff every sub-derivation of the assumption sequent is unsuccessful.

Clearly if π has no searches then π is unsuccessful. It suffices to show the following lemma.

Lemma 11.8. For any unsuccessful derivation in $\mathbf{wGL}_n^{\mathbf{G}'}$, if π is a derivation of Γ, then there is a finite $\mathbf{wGL}_n$-model $\mathcal{M}$ such that $\mathcal{M} \not\models \Gamma^{\#}$.

Proof. Induction on the height of unsuccessful π.

Suppose that π consists of a single sequent $\Gamma = (L, \Diamond\Pi)$. Define $\mathcal{M} = \langle W, \prec, V\rangle$ as follows:

- $W := \{w\}$, and $\prec := \emptyset$;
- $w \models p :\iff \overline{p} \in L$.

Then it is clear that for any $\varphi \in \Gamma$, $\mathcal{M}, w \not\models \varphi$, and hence $\mathcal{M} \not\models \Gamma^{\#}$.

Suppose that the last application of π is $(\vee)$ or $(\wedge)$. By Definition 11.7 and the induction hypothesis, for some assumption sequent Δ, there is a Kripke model $\mathcal{M}$ such that $\mathcal{M} \not\models \Delta^{\#}$. It is clear that $\mathcal{M}$ also falsifies $\Gamma^{\#}$.

Suppose that $\Gamma = (L, \Diamond\Pi, \Box\Sigma)$ and the last application of π is $(\Box'_n)$.

$$\dfrac{\begin{array}{c}\pi_1\\ \vdots \\ \Diamond^n\Pi, \Pi, \Diamond^n\overline{\varphi_1}, \varphi_1\end{array} \quad \cdots \quad \begin{array}{c}\pi_m\\ \vdots \\ \Diamond^n\Pi, \Pi, \Diamond^n\overline{\varphi_m}, \varphi_m\end{array}}{L, \Diamond\Pi, \Box\Sigma}\ \Box'_n$$

By Definition 11.7 and the induction hypothesis, each π_i $(1 \leq i \leq m)$ has a Kripke model $\mathcal{M}_i = \langle W_i, \prec_i, V_i \rangle$ such that $\mathcal{M}_i \not\models (\Diamond^n\Pi, \Pi, \Diamond^n\overline{\varphi_i}, \varphi_i)^{\#}$. We assume for any $1 \leq i, j \leq m$, if $i \neq j$ then the sets W_i and W_j are disjoint. Moreover, we may assume each $\mathcal{M}_i$ has the root $w_i \in W_i$ and $\mathcal{M}_i, w_i \not\models (\Diamond^n\Pi, \Pi, \Diamond^n\overline{\varphi_i}, \varphi_i)^{\#}$. Define a Kripke model $\mathcal{M} = \langle W, \prec, V \rangle$ as follows:

- $W := \bigcup W_i \cup \{w\}$ where w is a new object not contained in $\bigcup W_i$;
- $x \prec y :\Leftrightarrow \begin{cases} x \in W_i \text{ and } x \prec_i y \text{ for some } 1 \leq i \leq m, \text{ or} \\ x = w \text{ and } w_i \prec_i^{kn} y \text{ for some } 1 \leq i \leq m \text{ and } k \geq 0 \end{cases}$;
- For any p and $x \in W$, if $x \in W_i$ for some i, then $\mathcal{M}, x \models p :\Leftrightarrow \mathcal{M}_i, x \models p$;
- $\mathcal{M}, w \models p :\Leftrightarrow \overline{p} \in L$.

Clearly the relation $\prec$ is irreflexive and acyclic. We show that $\prec$ is $(n+1)$-transitive. Suppose that $x \prec^{n+1} y$. If $x \in W_i$ for some i, then $x \prec y$ immediately follows from the $(n+1)$-transitivity of $\prec_i$. Assume $x = w$. Then there are $x_1, \ldots, x_n \in W$ such that $x \prec x_1 \prec \cdots \prec x_n \prec y$. By the definition of $\prec$, we have $w_i \prec_i^{kn} x_1$ and $x_1 \prec_i \cdots \prec_i x_n \prec_i y$ for some $1 \leq i \leq m$ and $k \geq 0$. Thus we obtain $w_i \prec_i^{(k+1)n} y$, i.e., $x \prec y$. Thus, our $\mathcal{M}$ is a finite $\mathbf{wGL}_n$-frame.

It suffices to show that for any formula $\varphi \in \Gamma$, $\mathcal{M}, w \not\models \varphi$. If $\varphi \in L$, then $\mathcal{M}, w \not\models \varphi$ clearly follows from the definition of V. Assume $\varphi \equiv \Box\varphi_i \in \Box\Sigma$ for some $1 \leq i \leq m$. Then we have $w \prec w_i$. By the induction hypothesis, $\mathcal{M}_i, w_i \not\models \varphi_i$. Note that if $x \in W_i$, then $\mathcal{M}_i, x \models \varphi \iff \mathcal{M}, x \models \varphi$. Therefore $\mathcal{M}, w_i \not\models \varphi_i$, i.e., $\mathcal{M}, w \not\models \Box\varphi_i$. Assume $\varphi \equiv \Diamond\psi \in \Diamond\Pi$. We have to show that for every x, if $w \prec x$, then $\mathcal{M}, x \not\models \psi$. By the definition of $\prec$, we have $w_i \prec_i^{kn} x$ for some $1 \leq i \leq m$ and $k \geq 0$. If $k = 0$, then $x = w_i$. In this case, $\mathcal{M}_i, w_i \not\models \psi$ by the induction hypothesis, i.e., $\mathcal{M}, w_i \not\models \psi$. If $k \geq 1$, then by Lemma 4.3, $w_i \prec_i^n x$. By the induction hypothesis, $\mathcal{M}_i, w_i \not\models \Diamond^n \psi$, and hence $\mathcal{M}_i, x \not\models \psi$, i.e., $\mathcal{M}, x \not\models \psi$. $\square$

Now the proof of $(2 \Rightarrow 3)$ is completed. $\square$

Corollary 11.9. The rules (*cut*) and (*Löb*) are admissible in $\mathbf{wGL}_n^{\mathbf{G}}$.

Proof. The (*cut*)-admissibility immediately holds from Theorem 11.6. Suppose that $\mathbf{wGL}_n^{\mathbf{G}} \vdash (\Diamond^n\Gamma, \Gamma, \Diamond^n\overline{\varphi}, \varphi)$. Let π be a proof of this sequent. Then we can infer $(\Diamond^n\Gamma, \Gamma, \varphi)$ in $\mathbf{wGL}_n^{\mathbf{G}} + (cut)$ as follows:

$$\dfrac{\begin{array}{c}\pi\\ \vdots\\ \Diamond^n\Gamma, \Gamma, \varphi, \Diamond^n\overline{\varphi}\end{array} \qquad \dfrac{\begin{array}{c}\dfrac{\dfrac{\dfrac{\begin{array}{c}\pi\\ \vdots\\ \Diamond^n\Gamma, \Gamma, \Diamond^n\overline{\varphi}, \varphi\end{array}}{\Diamond\Gamma, \Box\varphi}\,(\Box_n)}{\Diamond^n\Diamond\Gamma, \Diamond\Gamma, \Diamond^n\overline{\Box\varphi}, \Box\varphi}\,(weak)}{\Diamond^2\Gamma, \Box^2\varphi}\,(\Box_n)\\ \vdots\\ \Diamond^n\Gamma, \Box^n\varphi\end{array}}{\Diamond^n\Gamma, \Gamma, \varphi, \Box^n\varphi}\,(weak)}{\Diamond^n\Gamma, \Gamma, \varphi}\,(cut)$$

By Theorem 11.6, $(\Diamond^n\Gamma, \Gamma, \varphi)$ is provable in $\mathbf{wGL}_n^{\mathbf{G}}$. Thus the rule (*Löb*) is admissible. $\square$

11.3 Craig interpolation for $\mathbf{wGL}_n$

In this section, we give a new proof of the Craig interpolation theorem for Sacchetti's logics via $\mathbf{wGL}_n^{\mathbf{G}}$. For a formula φ, we define $var(\varphi) := \{p \mid p$ occurs in $\varphi\} \cup \{q \mid \overline{q}$ occurs in $\varphi\}$. For a sequent Γ, we also define $var(\Gamma) := \bigcup\{var(\varphi) \mid \varphi \in \Gamma\}$. Put $\overline{\Gamma} := \{\overline{\varphi} \mid \varphi \in \Gamma\}$.

Theorem 11.10 (Craig interpolation theorem for $\mathbf{wGL}_n$)**.** Assume $n \geq 1$. If $\mathbf{wGL}_n \vdash \varphi \to \psi$, then there is a formula θ (called a Craig interpolant of $\varphi \to \psi$) such that:

1. $\mathbf{wGL}_n \vdash \varphi \to \theta$ and $\mathbf{wGL}_n \vdash \theta \to \psi$;

2. $var(\theta) \subseteq var(\varphi) \cap var(\psi)$.

Moreover, such a θ is effectively constructible from φ and ψ.

In order to prove Theorem 11.10, we introduce a *split derivation system*. A *split sequent* is one of the form $[\theta]\ \Gamma_1 \mid \Gamma_2$ where θ is a formula and Γ_1 and Γ_2 are sequents. The natural meaning of $(\Gamma_1 \mid \Gamma_2)$ is the formula $\overline{\Gamma_1^\#} \to \Gamma_2^\#$. The formula θ in the bracket is a corresponding interpolant of $(\Gamma_1 \mid \Gamma_2)$.

Definition 11.11. The split derivation system $\mathbf{wGL}_n^{\mathbf{Sp}}$ consists of the following axioms and rules.
Axioms:

$$[\bot]\ p, \overline{p} \mid \emptyset \qquad [p]\ \overline{p} \mid p \qquad [\overline{p}]\ p \mid \overline{p} \qquad [\top]\ \emptyset \mid p, \overline{p}$$
$$[\bot]\ \top \mid \emptyset \qquad [\top]\ \emptyset \mid \top$$

Rules:

$$\frac{[\theta]\ \Gamma_1 \mid \Gamma_2}{[\theta]\ \Gamma_1, \Delta_1 \mid \Gamma_2, \Delta_2}\ (weak)$$

$$\frac{[\theta]\ \Gamma_1, \varphi, \psi \mid \Gamma_2}{[\theta]\ \Gamma_1, \varphi \vee \psi \mid \Gamma_2}\ (\vee^l) \qquad \frac{[\theta]\ \Gamma_1 \mid \Gamma_2, \varphi, \psi}{[\theta]\ \Gamma_1 \mid \Gamma_2, \varphi \vee \psi}\ (\vee^r)$$

$$\frac{[\theta_1]\ \Gamma_1, \varphi \mid \Gamma_2 \quad [\theta_2]\ \Gamma_1, \psi \mid \Gamma_2}{[\theta_1 \vee \theta_2]\ \Gamma_1, \varphi \wedge \psi \mid \Gamma_2}\ (\wedge^l) \qquad \frac{[\theta_1]\ \Gamma_1 \mid \Gamma_2, \varphi \quad [\theta_2]\ \Gamma_1 \mid \Gamma_2, \psi}{[\theta_1 \wedge \theta_2]\ \Gamma_1 \mid \Gamma_2, \varphi \wedge \psi}\ (\wedge^r)$$

$$\frac{[\theta]\ \Diamond^n\Gamma_1, \Gamma_1, \Diamond^n\overline{\varphi}, \varphi \mid \Diamond^n\Gamma_2, \Gamma_2}{[\Diamond\theta]\ \Diamond\Gamma_1, \Box\varphi \mid \Diamond\Gamma_2}\ (\Box_n^l) \qquad \frac{[\theta]\ \Diamond^n\Gamma_1, \Gamma_1 \mid \Diamond^n\Gamma_2, \Gamma_2, \Diamond^n\overline{\varphi}, \varphi}{[\Box\theta]\ \Diamond\Gamma_1 \mid \Diamond\Gamma_2, \Box\varphi}\ (\Box_n^r)$$

Lemma 11.12. Assume $n \geq 1$, and let Γ_1 and Γ_2 be sequents. Then the following statements are equivalent:

1. $\mathbf{wGL}_n \vdash \overline{\Gamma_1^\#} \to \Gamma_2^\#$;
2. $\mathbf{wGL}_n^{\mathbf{G}} \vdash (\Gamma_1, \Gamma_2)$;
3. $\mathbf{wGL}_n^{\mathbf{Sp}} \vdash (\Gamma_1 \mid \Gamma_2)$.

Proof. (1$\Leftrightarrow$2): Clearly follows from Theorem 11.6.

(2$\Rightarrow$3): Let π be a proof of (Γ_1, Γ_2) in $\mathbf{wGL}_n^{\mathbf{G}}$. By bottom-up splitting of sequents, we obtain a proof of $(\Gamma_1 \mid \Gamma_2)$ in $\mathbf{wGL}_n^{\mathbf{Sp}}$.

(3$\Rightarrow$2): Let π' be a proof of $(\Gamma_1 \mid \Gamma_2)$. We can obtain a proof of (Γ_1, Γ_2) in $\mathbf{wGL}_n^{\mathbf{G}}$ by removing all splittings in π'. $\square$

Note that if a proof π of (Γ_1, Γ_2) is given, then a proof π' of $(\Gamma_1 \mid \Gamma_2)$ in $\mathbf{wGL}_n^{\mathbf{Sp}}$ is effectively constructible from π.

Lemma 11.13. Assume $n \geq 1$. Suppose that $\mathbf{wGL}_n^{\mathbf{Sp}} \vdash (\Gamma_1 \mid \Gamma_2)$, and let π be a proof of $[\theta]\ (\Gamma_1 \mid \Gamma_2)$ in $\mathbf{wGL}_n^{\mathbf{Sp}}$. Then:

1. $var\,(\theta) \subseteq var\left(\overline{\Gamma_1}\right) \cap var\,(\Gamma_2)$;

2. $\mathbf{wGL}_n^{\mathbf{G}} \vdash (\Gamma_1, \theta)$ and $\mathbf{wGL}_n^{\mathbf{G}} \vdash \left(\Gamma_2, \overline{\theta}\right)$.

Proof. By induction on the length of π.

Suppose that π consists of an axiom $[\theta]\ \Gamma_1 \mid \Gamma_2$. Then it is clear that the corresponding formula θ satisfies the conditions 1 and 2.

Suppose that π is one of the following derivations:

$$\dfrac{\begin{matrix}\pi_1\\ \vdots\\ [\theta_1]\ \Delta_{11} \mid \Delta_{12}\end{matrix}}{[\theta]\ \Gamma_1 \mid \Gamma_2}\,(R), \qquad \dfrac{\begin{matrix}\pi_1\\ \vdots\\ [\theta_1]\ \Delta_{11} \mid \Delta_{12}\end{matrix} \quad \begin{matrix}\pi_2\\ \vdots\\ [\theta_2]\ \Delta_{21} \mid \Delta_{22}\end{matrix}}{[\theta]\ \Gamma_1 \mid \Gamma_2}\,(R)$$

where π_i $(i = 1, 2)$ is the subproof of each hypothesis, and θ_i is the formula according to the rules in π_i. By the induction hypothesis for π_i, θ_i satisfies the following conditions:

- $var\,(\theta_i) \subseteq var\left(\overline{\Delta_{i1}}\right) \cup var\,(\Delta_{i2})$;
- $\mathbf{wGL}_n^{\mathbf{G}} \vdash (\Delta_{i1}, \theta)$ and $\mathbf{wGL}_n^{\mathbf{G}} \vdash \left(\Delta_{i2}, \overline{\theta}\right)$.

Then we can easily deduce from the above facts that θ enjoys the conditions 1 and 2. We only describe the case for $(\Box_n^r)$. Assume the last application of π is $(\Box_n^r)$.

$$\dfrac{\begin{matrix}\pi_1\\ \vdots\\ [\theta]\ \Diamond^n\Delta_1, \Delta_1 \mid \Diamond^n\Delta_2, \Delta_2, \Diamond^n\overline{\varphi}, \varphi\end{matrix}}{[\Box\theta]\ \Diamond\Delta_1 \mid \Diamond\Delta_2, \Box\varphi}\,(\Box_n^r) \tag{$*$}$$

By the induction hypothesis,

- $var\,(\theta) \subseteq var\left(\overline{\Diamond^n\Delta_1, \Delta_1}\right) \cap var\,(\Diamond^n\Delta_2, \Delta_2, \Diamond^n\overline{\varphi}, \varphi)$;
- $\mathbf{wGL}_n^{\mathbf{G}} \vdash (\Diamond^n\Delta_1, \Delta_1, \theta)$ and $\mathbf{wGL}_n^{\mathbf{G}} \vdash \left(\Diamond^n\Delta_2, \Delta_2, \Diamond^n\overline{\varphi}, \varphi, \overline{\theta}\right)$.

1. We have

$$\begin{aligned} var\,(\Box\theta) = var\,(\theta) &\subseteq var\left(\overline{\Diamond^n\Delta_1, \Delta_1}\right) \cap var\,(\Diamond^n\Delta_2, \Delta_2, \Diamond^n\overline{\varphi}, \varphi)\,, \\ &= var\left(\overline{\Diamond\Delta_1}\right) \cap var\,(\Diamond\Delta_2, \Box\varphi)\,. \end{aligned}$$

2. Consider the following derivations:

$$\dfrac{\dfrac{\begin{matrix}\rho_1\\ \vdots\\ \Diamond^n\Delta_1,\Delta_1,\theta\end{matrix}}{\Diamond^n\Delta_1,\Delta_1,\Diamond^n\overline{\theta},\theta}\,(weak)}{\Diamond\Delta_1,\Box\theta}\,(\Box_n)\quad,\qquad \dfrac{\dfrac{\begin{matrix}\rho_2\\ \vdots\\ \Diamond^n\Delta_2,\Delta_2,\Diamond^n\overline{\varphi},\varphi,\overline{\theta}\end{matrix}}{\Diamond^n\Delta_2,\Delta_2,\Diamond^n\overline{\varphi},\varphi,\Diamond^n\overline{\theta},\overline{\theta}}\,(weak)}{\Diamond\Delta_2,\Box\varphi,\Diamond\overline{\theta}}\,(\Box_n)\quad.$$

□

Proof of Theorem 11.10. Assume $\mathbf{wGL}_n \vdash \varphi \to \psi$. By Theorem 11.6 and Lemma 11.12, we can effectively obtain a proof π of $[\theta]\ (\overline{\varphi} \mid \psi)$ in $\mathbf{wGL}_n^{\mathbf{Sp}}$. By Lemma 11.13, we have $var(\theta) \subseteq var(\varphi) \cap var(\psi)$, $\mathbf{wGL}_n^{\mathbf{G}} \vdash \overline{\varphi}, \theta$ and $\mathbf{wGL}_n^{\mathbf{G}} \vdash \psi, \overline{\theta}$. Thus θ is indeed a Craig interpolant of $\varphi \to \psi$. □

Remark 11.14. Let φ be a formula. We define $var^+(\varphi)$ (resp. $var^-(\varphi)$) as the set of literals occurring positively (resp. negatively) in φ. Suppose that $\mathbf{wGL}_n \vdash \varphi \to \psi$. A *Lyndon interpolant* of $\varphi \to \psi$ is a formula θ satisfying the following conditions:

- $\mathbf{wGL}_n \vdash \varphi \to \theta$ and $\mathbf{wGL}_n \vdash \theta \to \psi$;
- $var^+(\theta) \subseteq var^+(\varphi) \cap var^+(\psi)$ and $var^-(\theta) \subseteq var^-(\varphi) \cap var^-(\psi)$.

It is not always true that $\mathbf{wGL}_n^{\mathbf{Sp}}$ supplies a Lyndon interpolant of $\varphi \to \psi$. Consider the derication π as in $(*)$, and suppose that θ is a Lyndon interpolant of $\overline{(\Diamond^n\Delta_1, \Delta_1)^{\#}} \to (\Diamond^n\Delta_2, \Delta_2, \Diamond^n\overline{\varphi}, \varphi)^{\#}$. Then θ may contain a literal l such that:

- l occurs in both $\overline{\Delta_1}$ and $\overline{\varphi}$;
- l does not occur in φ nor any formula in Δ_2.

Then the formula $\Box\theta$ also contains l, however, the assumption $\Diamond^n\overline{\varphi}$ is eliminated in the conclusion sequent of $(\Box_n^r)$. Thus, l is not a common literal of the conclusion sequent, and $\Box\theta$ is no longer a Lyndon interpolant of $\overline{\Diamond\Delta_1{}^{\#}} \to \Box\varphi$.

To avoid this problem, in the next section we will develop a system which preserves the positiveness of formulas, and is equivalent to $\mathbf{wGL}_n^{\mathbf{Sp}}$.

12 Lyndon interpolation property for $\mathbf{wGL}_n$

12.1 Circular proof system

We describe the calculus which admits *circular proofs*, and is equivalent to $\mathbf{wGL}_n^{\mathbf{G}}$.

Definition 12.1. The sequent calculus $\mathbf{wK4}_n^{\mathbf{G}}$ is obtained from $\mathbf{wGL}_n^{\mathbf{G}}$ by replacing the rule $\Box_n$ by the following rule

$$\frac{\Diamond^n\Gamma, \Gamma, \varphi}{\Diamond\Gamma, \Box\varphi}\ (\blacksquare_n).$$

A *circular derivation* of a calculus is a pair $\pi = (\kappa, d)$ where κ is a derivation in the calculus and d is a back-link function from some leaf x to an interior node y with an identical sequent, such that y lies on the path from the root of κ to x, and there exists at least one application of $\blacksquare_n$ between x and y. We call such an (x, y) a *circular pair*. (In other words, d is the set of circular pairs in κ.) A *circular proof* is a circular derivation such that every leaf is either marked by an axiom or connected by the back-link function. The circular proof system ${}^\circ\mathbf{wK4}_n^{\mathbf{G}}$ is obtained from $\mathbf{wK4}_n^{\mathbf{G}}$ by admitting circular proofs.

The following diagram is an example of a circular proof in ${}^\circ\mathbf{wK4}_2^{\mathbf{G}}$.

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\Diamond(\Box^2 p \wedge \overline{p}), \Box p}{\Diamond^3(\Box^2 p \wedge \overline{p}), \Diamond(\Box^2 p \wedge \overline{p}), \Box p}\ (weak)}{\Diamond^2(\Box^2 p \wedge \overline{p}), \Box^2 p}\ (\blacksquare_2)}{\Diamond^2(\Box^2 p \wedge \overline{p}), \Box^2 p, p}\ (weak) \qquad \dfrac{\overline{p}, p}{\Diamond^2(\Box^2 p \wedge \overline{p}), \overline{p}, p}\ (weak)}{\Diamond^2(\Box^2 p \wedge \overline{p}), \Box^2 p \wedge \overline{p}, p}\ (\wedge)}{\Diamond(\Box^2 p \wedge \overline{p}), \Box p}\ (\blacksquare_2)$$

Let n be an arbitrary natural number, and consider the following diagram:

$$\dfrac{\dfrac{p \vee q}{p \vee q, p, q}\ (weak)}{p \vee q}\ (\vee)$$

This diagram contains a pair of nodes labeled by the same sequent $(p \vee q)$, however, there is no application of the rule $\blacksquare_n$. Therefore this diagram is *not* a circular proof in ${}^\circ\mathbf{wK4}_n^{\mathbf{G}}$.

In the rest of this subsection we prove the following theorem.

Theorem 12.2. Assume $n \geq 1$. For any sequent Γ,

$$\mathbf{wGL}_n^{\mathbf{G}} \vdash \Gamma \iff {}^{\circ}\mathbf{wK4}_n^{\mathbf{G}} \vdash \Gamma.$$

Lemma 12.3. Assume $n \geq 1$. For any sequent Γ, $\mathbf{wGL}_n^{\mathbf{G}} \vdash \Gamma \Longrightarrow {}^{\circ}\mathbf{wK4}_n^{\mathbf{G}} \vdash \Gamma$. Moreover, if π is a proof of Γ in $\mathbf{wGL}_n^{\mathbf{G}}$, then we can construct a proof $\pi' = (\kappa, d)$ of Γ in ${}^{\circ}\mathbf{wK4}_n^{\mathbf{G}}$ from π in an effective way.

Proof. Assume $\mathbf{wGL}_n^{\mathbf{G}} \vdash \Gamma$, and let κ be a proof of Γ in $\mathbf{wGL}_n^{\mathbf{G}}$. First we introduce a complexity of formulas in κ. Let φ be a formula occurring in some sequent in κ. By the definition of $\mathbf{wGL}_n^{\mathbf{G}}$, φ is either an element of $\overline{Sub(\Gamma)}$ or of the form $\Diamond^k\psi$ where $\psi \in \overline{Sub(\Gamma)}$. The *complexity* of φ (write $c(\varphi)$) is defined as the least natural number k such that $\varphi \equiv \Diamond^k\psi$ for some $\psi \in \overline{Sub(\Gamma)}$.

In order to obtain a circular proof of Γ in ${}^{\circ}\mathbf{wK4}_n^{\mathbf{G}}$, we construct the sequence $\{\pi_i = (\kappa_i, d_i)\}_{i \leq j}$ of circular derivation satisfying the following properties:

- Each π_i is a circular derivation in the system $\mathbf{wGL}_n^{\mathbf{G}} \cup \mathbf{wK4}_n^{\mathbf{G}}$;
- For any κ_i and application of $\blacksquare_n$ in κ_i, there is no application of $\Box_n$ in the path from the root of κ_i to the conclusion of $\blacksquare_n$;
- π_j is a circular proof of Γ in ${}^{\circ}\mathbf{wK4}_n^{\mathbf{G}}$.

We construct such a sequence $\{\pi_i = (\kappa_i, d_i)\}_{i \leq j}$ from a given κ in the following steps.

1. Let $\pi_0 := (\kappa_0, d_0)$, where $\kappa_0 := \kappa$ and $d_0 := \emptyset$.
2. If π_i contains no application of $\Box_n$, then the sequence stops. Otherwise, consider the lowest application of $\Box_n$ in κ_i:

$$\begin{array}{c} \vdots \\ \dfrac{\Diamond^n\Delta, \Delta, \Diamond^n\overline{\varphi}, \varphi}{\Diamond\Delta, \Box\varphi}\ (\Box_n) \\ \vdots \end{array}$$

3. Search the path below $(\Diamond\Delta, \Box\varphi)$ for a pair (x, y) such that x and y are marked by an identical sequent and there is an application of $\blacksquare_n$ between x and y. If we find such a pair (x, y), then cut away all nodes higher than x. Let κ_{i+1} be the obtained derivation and $d_{i+1} := d_i \cup (x, y)$, and return to Step 2. Otherwise, go to Step 4.

4. Let ρ be the subproof of the sequent $(\Diamond^n\Delta, \Delta, \Diamond^n\overline{\varphi}, \varphi)$. Note that ρ contains no application of $\blacksquare_n$, and hence is a proof in $\mathbf{wGL}_n^{\mathbf{G}}$. By Corollary 11.9, there is a proof ρ' of $(\Diamond^n\Delta, \Delta, \varphi)$ in $\mathbf{wGL}_n^{\mathbf{G}}$. Replace κ_i by:

$$
\begin{array}{c}
\rho \\
\vdots \\
\dfrac{\Diamond^n\Delta, \Delta, \Diamond^n\overline{\varphi}, \varphi}{\Diamond\Delta, \Box\varphi}\,(\Box_n) \\
\vdots
\end{array}
\quad\longmapsto\quad
\begin{array}{c}
\rho' \\
\vdots \\
\dfrac{\Diamond^n\Delta, \Delta, \varphi}{\Diamond\Delta, \Box\varphi}\,(\blacksquare_n) \\
\vdots
\end{array}
$$

If $\Diamond^n\Delta$ contains no formula ψ such that $c(\psi) > n$, then let π_{i+1} be the obtained derivation, and return to Step 2. Otherwise, go to Step 5.

5. Let $\Sigma \subseteq \Delta$ be the set of all formulas ψ such that $c(\Diamond^n\psi) > n$, and put $\Pi := \Delta \backslash \Sigma$. Recall that for any $\psi \in \Sigma$, $\mathbf{wGL}_n^{\mathbf{G}} \vdash \psi, \Box^n\overline{\psi}$. By Corollary 11.9, the assumptions $\Diamond^n\Sigma$ can be eliminated. Let ρ'' be a proof of $(\Sigma, \Diamond^n\Pi, \Pi, \varphi)$ in $\mathbf{wGL}_n^{\mathbf{G}}$. Replace the derivation by:

$$
\begin{array}{c}
\rho' \\
\vdots \\
\dfrac{\Diamond^n\Sigma, \Sigma, \Diamond^n\Pi, \Pi, \varphi}{\Diamond\Delta, \Box\varphi}\,(\blacksquare_n) \\
\vdots
\end{array}
\quad\longmapsto\quad
\begin{array}{c}
\rho'' \\
\vdots \\
\dfrac{\dfrac{\Sigma, \Diamond^n\Pi, \Pi, \varphi}{\Diamond^n\Sigma, \Sigma, \Diamond^n\Pi, \Pi, \varphi}\,(weak)}{\Diamond\Delta, \Box\varphi}\,(\blacksquare_n) \\
\vdots
\end{array}
$$

(The sequent $(\Sigma, \Diamond^n\Pi, \Pi, \varphi)$ consists of formulas of which complexities are $\leq n$.) Let π_{i+1} be the obtained derivation, and return to Step 2.

In Steps 4-5 the procedure always generates a new sequent which consists of formulas having complexity $\leq n$. Since the number of such sequents is finite, the sequence must stop at some j. Suppose that the construction terminates at π_j. This is our desired circular proof of Γ in $^\circ\mathbf{wK4}_n^{\mathbf{G}}$. □

Lemma 12.4. $^\circ\mathbf{wK4}_n^{\mathbf{G}} \vdash \Gamma \Longrightarrow \mathbf{wGL}_n^{\mathbf{G}} \vdash \Gamma$.

Proof. For a circular derivation $\pi = (\kappa, d)$ in $^\circ\mathbf{wK4}_n^{\mathbf{G}}$ and a leaf a of π, a is called an *assumption leaf* if a is non-axiomatic and not connected by the back-link function d. An assumption leaf a is *boxed* if there is an application of $\blacksquare_n$ on the path from the root to a. Let $BH(\pi)$ and $H(\pi)$ be the sets of boxed, respectively not boxed assumption leaves of π. The sequent of a is denoted by Δ_a.

Claim 12.5. For any circular derivation $\pi = (\kappa, d)$ of Γ,

$$\mathbf{wGL}_n \vdash \bigwedge [n]^+ \{\Delta_a^\# \mid a \in H(\pi)\} \wedge \bigwedge [n] \{\Delta_a^\# \mid a \in BH(\pi)\} \to \Gamma^\#.$$

Proof. Induction on the construction of κ.

- Assume κ consists of a single node a. The claim clearly holds if Δ_a is an axiom of $\mathbf{wK4}_n^{\mathbf{G}}$. Otherwise, since $a \in H(\pi)$ and by Proposition 10.3.1, we have $\mathbf{wGL}_n \vdash [n]^+\Delta_a^\# \to \Delta_a^\#$.

- Consider π is one of the following derivations:

$$\dfrac{\begin{array}{c}\pi_1\\ \vdots \\ \Delta\end{array}}{\Delta, \Sigma}\ (weak), \quad \dfrac{\begin{array}{c}\pi_1\\ \vdots \\ \Delta, \varphi, \psi\end{array}}{\Delta, \varphi \vee \psi}\ (\vee), \quad \dfrac{\begin{array}{c}\pi_1\\ \vdots \\ \Delta, \varphi\end{array} \quad \begin{array}{c}\pi_2\\ \vdots \\ \Delta, \psi\end{array}}{\Delta, \varphi \wedge \psi}\ (\wedge).$$

  Suppose that Γ is not connected by d. In this case,

$$H(\pi) = \bigcup_{i=1,2} H(\pi_i), \text{ and } BH(\pi) = \bigcup_{i=1,2} BH(\pi_i).$$

  Note that

$$\mathbf{wGL}_n \vdash \Delta^\# \to (\Delta, \Sigma)^\#, \quad \mathbf{wGL}_n \vdash (\Delta, \varphi, \psi)^\# \to (\Delta, \varphi \vee \psi)^\#,$$
$$\mathbf{wGL}_n \vdash (\Delta, \varphi)^\# \wedge (\Delta, \psi)^\# \to (\Delta, \varphi \wedge \psi)^\#.$$

  By the induction hypotheses for π_1 and π_2, the claim holds.

  Suppose that Γ is connected by d. Let b be such a leaf connecting with the root. In this case,

$$H(\pi) = \bigcup_{i=1,2} H(\pi_i) \text{ and } BH(\pi) = \bigcup_{i=1,2} BH(\pi_i) \setminus \{b\}.$$

  From a similar argument as above, we obtain

$$\mathbf{wGL}_n \vdash \bigwedge [n]^+ \{\Delta_a^\# \mid a \in H(\pi)\} \wedge \bigwedge [n] \{\Delta_a^\# \mid a \in BH(\pi)\} \wedge [n]\Delta_b^\# \to \Gamma^\#,$$

  where Δ_b is exactly Γ. By Proposition 10.3.3,

$$\mathbf{wGL}_n \vdash \bigwedge [n]^+ \{\Delta_a^\# \mid a \in H(\pi)\} \wedge \bigwedge [n] \{\Delta_a^\# \mid a \in BH(\pi)\} \to \Gamma^\#.$$

- Assume the last application of π is $\blacksquare_n$:

$$\dfrac{\begin{matrix}\pi_1\\ \vdots\\ \Diamond^n\Delta, \Delta, \varphi\end{matrix}}{\Diamond\Delta, \Box\varphi}\ (\blacksquare_n).$$

Suppose that Γ is not connected by d. In this case,

$$H(\pi) = \emptyset \text{ and } BH(\pi) = H(\pi_1) \cup BH(\pi_1).$$

By the induction hypothesis for π_1,

$$\mathbf{wGL}_n \vdash \bigwedge[n]^+\{\Delta_a^\# \mid a \in H(\pi_1)\} \wedge \bigwedge[n]\{\Delta_a^\# \mid a \in BH(\pi_1)\} \\ \to (\Diamond^n\Delta, \Delta, \varphi)^\#.$$

By Proposition 10.3.2,

$$\mathbf{wGL}_n \vdash \bigwedge[n]\{\Delta_a^\# \mid a \in BH(\pi)\} \to \Box(\Diamond^n\Delta, \Delta, \varphi)^\#.$$

Note that $\mathbf{wGL}_n \vdash \Box(\Diamond^n\Delta, \Delta, \varphi)^\# \to \Box\left(\overline{(\Diamond^n\Delta, \Delta)^\#} \to \varphi\right)$.

By Proposition 10.1, $\mathbf{wGL}_n \vdash \overline{\Diamond\Delta^\#} \to \Box\overline{(\Diamond^n\Delta, \Delta)^\#}$.

Hence $\mathbf{wGL}_n \vdash \Box(\Diamond^n\Delta, \Delta, \varphi)^\# \to \left(\overline{\Diamond\Delta^\#} \to \Box\varphi\right)$, i.e.,

$\mathbf{wGL}_n \vdash \Box\,(\Diamond^n\Delta, \Delta, \varphi)^\# \to (\Diamond\Delta, \Box\varphi)^\#$. Thus we obtain

$$\mathbf{wGL}_n \vdash \bigwedge[n]\{\Delta_a^\# \mid a \in BH(\pi)\} \to \Gamma^\#.$$

Suppose that Γ is connected with b by d. In this case,

$$H(\pi) = \emptyset \text{ and } BH(\pi) = H(\pi_1) \cup BH(\pi_1)\backslash\{b\}.$$

Again by the induction hypothesis for π_1 and Proposition 10.3.2, we obtain

$$\mathbf{wGL}_n \vdash \bigwedge[n]\{\Delta_a^\# \mid a \in BH(\pi)\} \wedge [n]\Gamma^\# \to \Gamma^\#.$$

By Proposition 10.3.3, we conclude

$$\mathbf{wGL}_n \vdash \bigwedge[n]\{\Delta_a^\# \mid a \in BH(\pi)\} \to \Gamma^\#.$$

The proof of the claim is completed. $\square$

Now if π is a circular proof of $^\circ\mathbf{wK4}_n^{\mathbf{G}}$, then $H(\pi) = BH(\pi) = \emptyset$, and hence $\mathbf{wGL}_n \vdash \Gamma^{\#}$. By Theorem 11.6, we conclude $\mathbf{wGL}_n^{\mathbf{G}} \vdash \Gamma$. $\square$

Theorem 12.2 immediately follows from Lemma 12.3 and Lemma 12.4.

Remark 12.6. Iemhoff [10] studies some sufficient conditions for a type of modal sequent calculus to have an equivalent circular proof system. The calculus $\mathbf{wGL}_n^{\mathbf{G}}$ does not enjoy Iemhoff's conditions, however, has an equivalent circular proof counterpart.

12.2 Proof of Lyndon interpolation theorem

Shamkanov [23] originally showed that the standard provability logic **GL** enjoys the Lyndon interpolation property. In [24] he also gave a syntactical proof of the Lyndon interpolation theorem for **GL** by using the circular proof argument.

In this subsection, we show that Shamkanov's argument can be applied to the case for $\mathbf{wGL}_n$, i.e., for $n \geq 2$, if $\mathbf{wGL}_n \vdash \varphi \to \psi$, then we can construct a Lyndon interpolant of $\varphi \to \psi$ effectively. Before proving, we give some terminology. Definitions and Notations are according to Shamkanov [24].

For any formula φ, we define $u(\varphi)$ as the set of literals l occurring in φ out of the scope of all modal operators. We use new symbols of the form p° and $\overline{p}^\circ$ (we call them *marked literals.*) to specify literals within the scope of modal operators. We define $v(\varphi)$ as the set of marked literals l° such that l occurs in φ within the scope of a modal operator. Let $w(\varphi) := u(\varphi) \cup v(\varphi)$, and $w(\Gamma) := \bigcup\{w(\varphi) \mid \varphi \in \Gamma\}$.

Theorem 12.7 (Lyndon interpolation theorem for $\mathbf{wGL}_n$)**.** Assume $n \geq 2$. If $\mathbf{wGL}_n \vdash \varphi \to \psi$, then there is a formula θ (called a Lyndon interpolant of $\varphi \to \psi$) such that:

1. $\mathbf{wGL}_n \vdash \varphi \to \theta$ and $\mathbf{wGL}_n \vdash \theta \to \psi$;

2. $w(\theta) \subseteq w(\varphi) \cap w(\psi)$.

Moreover, such a θ is effectively constructible from φ and ψ.

First, we develop a split derivation system based on $\mathbf{wK4}_n^{\mathbf{G}}$.

Definition 12.8. The system $\mathbf{wK4}_n^{\mathbf{Sp}}$ is obtained from $\mathbf{wGL}_n^{\mathbf{Sp}}$ by replacing the rules $(\Box_n^l)$ and $(\Box_n^r)$ by:

$$\frac{[\theta]\ \Diamond^n\Gamma_1, \Gamma_1, \varphi \mid \Diamond^n\Gamma_2, \Gamma_2}{[\Diamond\theta]\ \Diamond\Gamma_1, \Box\varphi \mid \Diamond\Gamma_2}\ (\blacksquare^l_n), \text{ and } \frac{[\theta]\ \Diamond^n\Gamma_1, \Gamma_1 \mid \Diamond^n\Gamma_2, \Gamma_2, \varphi}{[\Box\theta]\ \Diamond\Gamma_1 \mid \Diamond\Gamma_2, \Box\varphi}\ (\blacksquare^r_n).$$

Similarly, the split circular proof system $^\circ\mathbf{wK4}_n^{\mathbf{Sp}}$ is obtained from $\mathbf{wK4}_n^{\mathbf{Sp}}$ by admitting circular proofs.

Proposition 12.9.

$$^\circ\mathbf{wK4}_n^{\mathbf{G}} \vdash \Gamma_1, \Gamma_2 \iff {}^\circ\mathbf{wK4}_n^{\mathbf{Sp}} \vdash \Gamma_1 \mid \Gamma_2.$$

Proof. ($\Longrightarrow$): Let π be a proof of (Γ_1, Γ_2) in $^\circ\mathbf{wK4}_n^{\mathbf{G}}$. First we expand π to an infinite derivation by adding subproofs to each leaf a connected by d. By the bottom-up splitting of sequents, we obtain an infinite derivation π_∞ of $(\Gamma_1 \mid \Gamma_2)$ in $\mathbf{wK4}_n^{\mathbf{Sp}}$. Since π consists of only finitely many different sequents, each sequent can be split into only finitely many split sequents. Therefore, π_∞ consists of finitely many different split sequents.

($\Longleftarrow$): For a given split circular proof π in $^\circ\mathbf{wK4}_n^{\mathbf{Sp}}$, we obtain a circular proof in $^\circ\mathbf{wK4}_n^{\mathbf{G}}$ by removing all splittings in π. $\Box$

Notice that for a given circular proof π of (Γ_1, Γ_2) in $^\circ\mathbf{wK4}_n^{\mathbf{G}}$, we can effectively construct a split circular proof of $(\Gamma_1 \mid \Gamma_2)$ in $^\circ\mathbf{wK4}_n^{\mathbf{Sp}}$.

We describe several facts that will be used in the proof of Theorem 12.7. Let φ be a formula and $w(\varphi) := u(\varphi) \cup v(\varphi)$ as before. For a set S of literals and marked literals, we define $S^\circ := \{l^\circ \mid l \in S \text{ or } l^\circ \in S\}$, and $w^*(\varphi) := w(\varphi) \cup w(\varphi)^\circ$. The following theorem states that $\mathbf{wGL}_n$ has effective fixed-points.

Theorem 12.10 (Fixed-point theorem for $\mathbf{wGL}_n$, Kurahashi & Okawa [12])**.** Let $\varphi(p)$ be a formula in which p occurs only within the scope of a modal operator. Then there is a formula ψ satisfying the following conditions:

1. $w(\psi) \subseteq w^*(\varphi) \cup w^*(\overline{\varphi}) \backslash \{p^\circ, \overline{p}^\circ\}$;
2. $\mathbf{wGL}_n \vdash \psi \leftrightarrow \varphi(\psi)$.

Moreover, if φ does not contain $\overline{p}$, then $w(\psi) \subseteq w^*(\varphi) \backslash \{p\}$.

The last property "$w(\psi) \subseteq w^*(\varphi) \backslash \{p\}$" will be essentially needed in our proof. Kurahashi & Okawa [12] gave an effective procedure which produces a fixed-point ψ of a given formula $\varphi(p)$. Moreover, by the construction of fixed-points in [12], such a ψ also satisfies the conditions in Theorem 12.10, and the procedure does not use any kind of interpolation. Briefly, a fixed-point of $\varphi(p)$ is obtained by multi-substituting formulas containing only literals which occur in A, for each occurrence of p. Therefore ψ enjoys the condition 1.

Moreover, if $\varphi(p)$ contains no occurrences of $\overline{p}$, then the procedure preserves the positiveness of literals in every substitution (see Lindström [13], and Kurahashi & Okawa [12]).

Lemma 12.11. For each rule of $\mathbf{wK4}_n^{\mathbf{Sp}}$, the following corresponding statement holds:

1. $$\mathbf{wGL}_n \vdash \left(\overline{\Gamma_1^{\#}} \to \theta\right) \wedge \left(\theta \to \Gamma_2^{\#}\right) \to \left(\overline{(\Gamma_1, \Delta_1)^{\#}} \to \theta\right) \wedge \left(\theta \to (\Gamma_2, \Delta_2)^{\#}\right);$$

2. $$\mathbf{wGL}_n \vdash \left(\overline{(\Gamma_1, \varphi, \psi)^{\#}} \to \theta\right) \wedge \left(\theta \to \Gamma_2^{\#}\right) \to \left(\overline{(\Gamma_1, \varphi \vee \psi)^{\#}} \to \theta\right) \wedge \left(\theta \to \Gamma_2^{\#}\right);$$

3. $$\mathbf{wGL}_n \vdash \left(\overline{\Gamma_1^{\#}} \to \theta\right) \wedge \left(\theta \to (\Gamma_2, \varphi, \psi)^{\#}\right) \to \left(\overline{\Gamma_1^{\#}} \to \theta\right) \wedge \left(\theta \to (\Gamma_2, \varphi \vee \psi)^{\#}\right);$$

4. $$\mathbf{wGL}_n \vdash \left(\overline{(\Gamma_1, \varphi)^{\#}} \to \theta_1\right) \wedge \left(\theta_1 \to \Gamma_2^{\#}\right) \wedge \left(\overline{(\Gamma_1, \psi)^{\#}} \to \theta_2\right) \wedge \left(\theta_2 \to \Gamma_2^{\#}\right) \to \left(\overline{(\Gamma_1, \varphi \wedge \psi)^{\#}} \to \theta_1 \vee \theta_2\right) \wedge \left(\theta_1 \vee \theta_2 \to \Gamma_2^{\#}\right);$$

5. $$\mathbf{wGL}_n \vdash \left(\overline{\Gamma_1^{\#}} \to \theta_1\right) \wedge \left(\theta_1 \to (\Gamma_2, \varphi)^{\#}\right) \wedge \left(\overline{\Gamma_1^{\#}} \to \theta_2\right) \wedge \left(\theta_2 \to (\Gamma_2, \psi)^{\#}\right) \to \left(\overline{\Gamma_1^{\#}} \to \theta_1 \wedge \theta_2\right) \wedge \left(\theta_1 \wedge \theta_2 \to (\Gamma_2, \varphi \wedge \psi)^{\#}\right);$$

6. $$\mathbf{wGL}_n \vdash \Box\left[\left(\overline{(\Diamond^n \Gamma_1, \Gamma_1, \varphi)^{\#}} \to \theta\right) \wedge \left(\theta \to (\Diamond^n \Gamma_2, \Gamma_2)^{\#}\right)\right] \to \left(\overline{(\Diamond \Gamma_1, \Box \varphi)^{\#}} \to \Diamond \theta\right) \wedge \left(\Diamond \theta \to (\Diamond \Gamma_2)^{\#}\right);$$

7. $$\mathbf{wGL}_n \vdash \Box\left[\left(\overline{(\Diamond^n \Gamma_1, \Gamma_1)^{\#}} \to \theta\right) \wedge \left(\theta \to (\Diamond^n \Gamma_2, \Gamma_2, \varphi)^{\#}\right)\right] \to \left(\overline{(\Diamond \Gamma_1)^{\#}} \to \Box \theta\right) \wedge \left(\Box \theta \to (\Diamond \Gamma_2, \Box \varphi)^{\#}\right).$$

Suppose that $^\circ\mathbf{wK4}_n^{\mathbf{Sp}} \vdash \Gamma_1 \mid \Gamma_2$. We define *an interpolant of* $\Gamma_1 \mid \Gamma_2$ as a Lyndon interpolant of $\overline{\Gamma_1^\#} \to \Gamma_2^\#$.

For a given split circular derivation $\pi = (\kappa, d)$ of $(\Gamma_1 \mid \Gamma_2)$, we define two sets of leaves $BH(\pi)$ and $H(\pi)$ as in Section 4. For each non-axiomatic leaf a of κ, we fix two variables x_a and w_a. The first variable x_a plays a role of the provisional interpolant of a. The second variavle w_a ranges over sets of literals and marked literals. We interpret the second variable w_a as $w(x_a)$. Let $(\Delta_1 \mid \Delta_2)$ be the split sequent of a. Define the formula I_a and the statement I'_a as follows:

$$I_a :\equiv \left(\overline{\Delta_1^\#} \to x_a\right) \land \left(x_a \to \Delta_2^\#\right);$$
$$I'_a :\Leftrightarrow w_a \subseteq w(\overline{\Delta_1}) \cap w(\Delta_2).$$

Lemma 12.12. Let π be a split circular derivation of $(\Gamma_1 \mid \Gamma_2)$. Then there is a formula θ satisfying the following conditions:

1. θ does not contain literals of the form $\overline{x_a}$;
2. If $a \in BH(\pi)$, then x_a occurs in θ only within the scope of modal operators;
3. $\mathbf{wGL}_n \vdash \bigwedge\{[n]I_a \mid a \in BH(\pi)\} \land \bigwedge\{[n]^+ I_a \mid a \in H(\pi)\}$
$$\to \left(\overline{\Gamma_1^\#} \to \theta\right) \land \left(\theta \to \Gamma_2^\#\right);$$
4. $T(\pi) \Rightarrow w_X(\theta) \cup \bigcup\{w_a^\circ \mid a \in BH(\pi)\} \cup \bigcup\{w_a \mid a \in H(\pi)\} \subseteq w(\overline{\Gamma_1}) \cap w(\Gamma_2)$, where $T(\pi)$ is the statement $\bigwedge\{I'_a \mid a \in BH(\pi) \cup H(\pi)\}$, and $w_X(\theta) := w(\theta) \backslash \{x_a, x_a^\circ \mid x_a \text{ occurs in } \theta\}$.

Proof. Induction on the construction of κ. We argue five cases.

Case 1 Assume κ consists of a single node a. If $(\Gamma_1 \mid \Gamma_2)$ is an axiom of $\mathbf{wK4}_n^{\mathbf{Sp}}$, then we take θ as the formula bracketed in the corresponding axiom. Otherwise, we put $\theta :\equiv x_a$ (Note that $H(\pi) = \{a\}$). In both cases, θ clearly satisfies the conditions 1-4.

Case 2 Assume the last application of κ is *weak* or one of the propositional rules, and $(\Gamma_1 \mid \Gamma_2)$ is not connected by d. We show that the formula θ bracketed in each conclusion satisfies Conditions 1-4.

$$\dfrac{\begin{matrix}\pi_1\\ \vdots\\ [\theta_1]\ \Delta_{11} \mid \Delta_{12}\end{matrix}}{[\theta]\ \Gamma_1 \mid \Gamma_2} \qquad \dfrac{\begin{matrix}\pi_1\\ \vdots\\ [\theta_1]\ \Delta_{11} \mid \Delta_{12}\end{matrix} \quad \begin{matrix}\pi_2\\ \vdots\\ [\theta_2]\ \Delta_{21} \mid \Delta_{22}\end{matrix}}{[\theta]\ \Gamma_1 \mid \Gamma_2}$$

Conditions 1 and 2 are clear.

Condition 3 By the induction hypotheses, we have for $i = 1, 2$,

$$\mathbf{wGL}_n \vdash \bigwedge\{[n]I_a \mid a \in BH(\pi_i)\} \wedge \bigwedge\{[n]^+ I_a \mid a \in H(\pi_i)\} \\ \to \left(\overline{\Delta_{i1}^{\#}} \to \theta_i\right) \wedge \left(\theta_i \to \Delta_{i2}^{\#}\right),$$

Recall that $BH(\pi) = \bigcup_{i=1,2} BH(\pi_i)$ and $H(\pi) = \bigcup_{i=1,2} H(\pi_i)$. By Lemma 12.3.1-5, we obtain

$$\mathbf{wGL}_n \vdash \bigwedge\{[n]I_a \mid a \in BH(\pi)\} \wedge \bigwedge\{[n]^+ I_a \mid a \in H(\pi)\} \\ \to \left(\overline{\Gamma_1^{\#}} \to \theta\right) \wedge \left(\theta \to \Gamma_2^{\#}\right).$$

Thus θ satisfies the condition *3*.

Condition 4 By the induction hypothesis, we have for $i = 1, 2$,

$$T(\pi_i) \Rightarrow w_X(\theta_i) \cup \bigcup\{w_a^\circ \mid a \in BH(\pi_i)\} \cup \bigcup\{w_a \mid a \in H(\pi_i)\} \\ \subseteq w(\overline{\Delta_{i1}}) \cap w(\Delta_{i2}).$$

Suppose $T(\pi)$ holds. (Note that $T(\pi) \Leftrightarrow T(\pi_1) \wedge T(\pi_2)$.) By $w_X(\theta) = w_X(\theta_1) \cup w_X(\theta_2)$ and $w(\overline{\Delta_{i1}}) \cap w(\Delta_{i2}) \subseteq w(\overline{\Gamma_1}) \cap w(\Gamma_2)$, we conclude

$$w_X(\theta) \cup \bigcup\{w_a^\circ \mid a \in BH(\pi)\} \cup \bigcup\{w_a \mid a \in H(\pi)\} \subseteq w(\overline{\Gamma_1}) \cap w(\Gamma_2).$$

Case 3 Assume the last application of κ is $\blacksquare_n^l$ or $\blacksquare_n^r$, and $(\Gamma_1 \mid \Gamma_2)$ is not connected by d. Again we show that the formula θ bracketed in the corresponding conclusion satisfies the conditions 1-4.

$$\dfrac{\begin{matrix}\pi_1 \\ \vdots \\ [\theta_1]\ \Delta_1 \mid \Delta_2\end{matrix}}{[\theta]\ \Gamma_1 \mid \Gamma_2}$$

The conditions 1 and 2 are clear.

Condition 3 By the induction hypotheses, we have

$$\mathbf{wGL}_n \vdash \bigwedge\{[n]I_a \mid a \in BH(\pi_1)\} \wedge \bigwedge\{[n]^+ I_a \mid a \in H(\pi_1)\} \\ \to \left(\overline{\Delta_1^{\#}} \to \theta_1\right) \wedge \left(\theta_1 \to \Delta_2^{\#}\right).$$

By Proposition 10.3.2,

$$\mathbf{wGL}_n \vdash \bigwedge\{[n]I_a \mid a \in BH(\pi_1)\} \wedge \bigwedge\{[n]I_a \mid a \in H(\pi_1)\}$$
$$\to \Box\left[\left(\overline{\Delta_1^\#} \to \theta_1\right) \wedge \left(\theta_1 \to \Delta_2^\#\right)\right].$$

Note that $BH(\pi) = BH(\pi_1) \cup H(\pi_1)$ and $H(\pi) = \emptyset$. By Lemma 12.3.6-7, we obtain

$$\mathbf{wGL}_n \vdash \bigwedge\{[n]I_a \mid a \in BH(\pi)\} \to \left(\overline{\Gamma_1^\#} \to \theta\right) \wedge \left(\theta \to \Gamma_2^\#\right).$$

Condintion 4 By the induction hypothesis,

$$T(\pi_1) \Rightarrow w_X(\theta_1) \cup \bigcup\{w_a^\circ \mid a \in BH(\pi_1)\} \cup \bigcup\{w_a \mid a \in H(\pi_1)\}$$
$$\subseteq w(\overline{\Delta_1}) \cap w(\Delta_2). \quad (4)$$

Suppose that $T(\pi)$ is true. Since $T(\pi) \Leftrightarrow T(\pi_1)$, the consequence of (4) is also true. It suffices to show that: (i) $w_X(\theta) \subseteq w(\overline{\Gamma_1}) \cap w(\Gamma_2)$ and (ii) if $a \in BH(\pi)$, then $w_a^\circ \subseteq w(\overline{\Gamma_1}) \cap w(\Gamma_2)$.

(i): By the conclusion of (4), $w_X(\theta_1) \subseteq w(\overline{\Delta_1}) \cap w(\Delta_2)$. Then $w_X(\theta_1)^\circ \subseteq w(\overline{\Delta_1})^\circ \cap w(\Delta_2)^\circ$. In this case, we have $w(\overline{\Delta_1})^\circ \subseteq w(\overline{\Gamma_1})$ and $w(\Delta_2)^\circ \subseteq w(\Gamma_2)$. Hence $w_X(\theta) = w_X(\theta_1)^\circ \subseteq w(\overline{\Gamma_1}) \cap w(\Gamma_2)$.

(ii): Let $a \in BH(\pi) = BH(\pi_1) \cup H(\pi_1)$. If $a \in BH(\pi_1)$, then $w_a^\circ \subseteq w(\overline{\Delta_1}) \cap w(\Delta_2)$ by (4). If $a \in H(\pi_1)$, then $w_a \subseteq w(\overline{\Delta_1}) \cap w(\Delta_2)$ by (4). In either case, we have $w_a^\circ \subseteq w(\overline{\Delta_1})^\circ \cap w(\Delta_2)^\circ \subseteq w(\overline{\Gamma_1}) \cap w(\Gamma_2)$.

Case 4 Assume that the last application of κ is *weak* or one of the propositional rules, and $(\Gamma_1 \mid \Gamma_2)$ is connected with b by d.

$$\dfrac{\begin{array}{c}\pi_1\\ \vdots\\ [\theta_1]\ \Delta_{11} \mid \Delta_{12}\end{array}}{[\theta]\ \Gamma_1 \mid \Gamma_2} \qquad \dfrac{\begin{array}{c}\pi_1\\ \vdots\\ [\theta_1]\ \Delta_{11} \mid \Delta_{12}\end{array} \quad \begin{array}{c}\pi_2\\ \vdots\\ [\theta_2]\ \Delta_{21} \mid \Delta_{22}\end{array}}{[\theta]\ \Gamma_1 \mid \Gamma_2}$$

(By the construction of θ, θ contains x_b.) There is at least one application of modal rules between b and the root of κ, and hence $b \in BH(\pi_1)$ or $b \in BH(\pi_2)$. By the induction hypothesis, for $i = 1, 2$,

$$\mathbf{wGL}_n \vdash \bigwedge\{[n]I_a : a \in BH(\pi_i)\} \wedge \bigwedge\{[n]^+I_a : a \in H(\pi_i)\}$$
$$\to \left(\overline{\Delta_{i1}^\#} \to \theta_i\right) \wedge \left(\theta_i \to \Delta_{i2}^\#\right).$$

Note that $BH(\pi) = \bigcup_{i=1,2} BH(\pi_i)\backslash\{b\}$, $H(\pi) = \bigcup_{i=1,2} H(\pi_i)$. By Lemma 12.3.1-5,

$$\mathbf{wGL}_n \vdash \bigwedge\{[n]I_a : a \in BH(\pi)\} \wedge \bigwedge\{[n]^+ I_a : a \in H(\pi)\} \wedge [n]I_b \\ \to \left(\overline{\Gamma_1^{\#}} \to \theta\right) \wedge \left(\theta \to \Gamma_2^{\#}\right).$$

By the construction of θ, x_b only occurs in θ positively and within the scope of modal operators. By Theorem 12.10, we can construct a formula ψ satisfying $w(\psi) \subseteq w^*(\theta)\backslash\{x_b^\circ\}$, and $\mathbf{wGL}_n \vdash \theta(\psi) \leftrightarrow \psi$. (Here $\theta(\psi)$ is the formula obtained from θ by substituting ψ for all occurrences of x_b.) Thus we have

$$\mathbf{wGL}_n \vdash \bigwedge\{[n]I_a : a \in BH(\pi)\} \wedge \bigwedge\{[n]^+ I_a : a \in H(\pi)\} \\ \wedge [n]I_b(\psi) \to \left(\overline{\Gamma_1^{\#}} \to \theta(\psi)\right) \wedge \left(\theta(\psi) \to \Gamma_2^{\#}\right). \quad (5)$$

By $\mathbf{wGL}_n^{\mathbf{G}} \vdash \theta(\psi) \leftrightarrow \psi$ and the definition of I_b,

$$\mathbf{wGL}_n \vdash \left(\overline{\Gamma_1^{\#}} \to \theta(\psi)\right) \wedge \left(\theta(\psi) \to \Gamma_2^{\#}\right) \leftrightarrow I_b(\psi). \quad (6)$$

From (5) and (6),

$$\mathbf{wGL}_n \vdash \bigwedge\{[n]I_a : a \in BH(\pi)\} \wedge \bigwedge\{[n]^+ I_a : a \in H(\pi)\} \wedge [n]I_b(\psi) \\ \to I_b(\psi).$$

By Proposition 10.3.3,

$$\mathbf{wGL}_n \vdash \bigwedge\{[n]I_a : a \in BH(\pi)\} \wedge \bigwedge\{[n]^+ I_a : a \in H(\pi)\} \to I_b(\psi).$$

Again by (6),

$$\mathbf{wGL}_n \vdash \bigwedge\{[n]I_a : a \in BH(\pi)\} \wedge \bigwedge\{[n]^+ I_a : a \in H(\pi)\} \\ \to \left(\overline{\Gamma_1^{\#}} \to \theta(\psi)\right) \wedge \left(\theta(\psi) \to \Gamma_2^{\#}\right),$$

i.e., $\theta(\psi)$ satisfies the condition 3.

Moreover, by the constructions of θ and ψ, $\theta(\psi)$ does not contain literals of the form $\overline{x_a}$, and if $a \in BH(\pi)$, then x_a occurs only within the scope of modal operators. Thus $\theta(\psi)$ enjoys the conditions 1-2.

Condition 4 By the induction hypothesis, for $i = 1, 2$,

$$T(\pi_i) \Rightarrow w_X(\theta_i) \cup \bigcup\{w_a^\circ \mid a \in BH(\pi_i)\} \cup \bigcup\{w_a \mid a \in H(\pi_i)\} \\ \subseteq w(\overline{\Delta_{i1}}) \cap w(\Delta_{i2}).$$

Suppose that $T(\pi)$ is true. Since $T(\pi) \wedge I_b'$ implies $T(\pi_1) \wedge T(\pi_2)$,

$$I_b' \Rightarrow w_X(\theta_i) \cup \bigcup\{w_a^\circ \mid a \in BH(\pi_i)\} \cup \bigcup\{w_a \mid a \in H(\pi_i)\} \\ \subseteq w(\overline{\Delta_{i1}}) \cap w(\Delta_{i2}).$$

Note that $w_X(\theta) = w_X(\theta_1) \cup w_X(\theta_2)$ and $w(\overline{\Delta_{i1}}) \cap w(\Delta_{i2}) \subseteq w(\overline{\Gamma_1}) \cap w(\Gamma_2)$. We have

$$I_b' \Rightarrow w_X(\theta) \cup \bigcup\{w_a^\circ \mid a \in BH(\pi)\} \cup w_b^\circ \cup \bigcup\{w_a \mid a \in H(\pi)\} \\ \subseteq w(\overline{\Gamma_1}) \cap w(\Gamma_2). \quad (7)$$

We show that $w_X(\theta(\psi)) \subseteq w(\overline{\Gamma_1}) \cap w(\Gamma_2)$. From (7),

$$I_b' \Rightarrow w_X(\theta) \cup w_b^\circ \subseteq w(\overline{\Gamma_1}) \cap w(\Gamma_2). \quad (8)$$

Substituting $\emptyset$ for w_b in (8), we have

$$w_X(\theta) \subseteq w(\overline{\Gamma_1}) \cap w(\Gamma_2).$$

This statement is equivalent to $I_b'(w_X(\theta))$, and hence $I_b'(w_X(\theta))$ is valid. Substituting $w_X(\theta)$ for w_b in (8), we get

$$I_b'(w_X(\theta)) \Rightarrow w_X(\theta) \cup w_X(\theta)^\circ \subseteq w(\overline{\Gamma_1}) \cap w(\Gamma_2),$$

and hence

$$w_X(\theta) \cup w_X(\theta)^\circ \subseteq w(\overline{\Gamma_1}) \cap w(\Gamma_2).$$

By the constructions of θ and ψ, $w_X(\theta(\psi)) \subseteq w_X(\theta) \cup w_X(\psi)^\circ$. On the other hand, since $w(\psi) \subseteq w^*(\theta) \backslash \{x_b^\circ\}$, we have $w(\psi)^\circ \subseteq w(\theta)^\circ \backslash \{x_b^\circ\}$, and hence $w_X(\psi)^\circ \subseteq w_X(\theta)^\circ$. Thus,

$$w_X(\theta(\psi)) \subseteq w_X(\theta) \cup w_X(\psi)^\circ \subseteq w_X(\theta) \cup w_X(\theta)^\circ \subseteq w(\overline{\Gamma_1}) \cap w(\Gamma_2). \quad (9)$$

From (4), we get

$$I_b' \Rightarrow \bigcup\{w_a^\circ \mid a \in BH(\pi)\} \cup \bigcup\{w_a \mid a \in H(\pi)\} \subseteq w(\overline{\Gamma_1}) \cap w(\Gamma_2).$$

Substituting $\emptyset$ for w_b, we obtain

$$\bigcup\{w_a^\circ \mid a \in BH(\pi)\} \cup \bigcup\{w_a \mid a \in H(\pi)\} \subseteq w(\overline{\Gamma_1}) \cap w(\Gamma_2).$$

From this and (9), we conclude that $\theta(\psi)$ satisfies the condition 4.

Case 5 Assume the last application of κ is $\blacksquare_n^l$ or $\blacksquare_n^r$, and $(\Gamma_1 \mid \Gamma_2)$ is connected with b by d.

$$\dfrac{\begin{array}{c}\pi_1\\ \vdots\\ [\theta_1]\ \Delta_1 \mid \Delta_2\end{array}}{[\theta]\ \Gamma_1 \mid \Gamma_2}$$

By the induction hypothesis,

$$\mathbf{wGL}_n \vdash \bigwedge\{[n]I_a : a \in BH(\pi_1)\} \wedge \bigwedge\{[n]^+ I_a : a \in H(\pi_1)\} \\ \to \left(\overline{\Delta_1^\#} \to \theta_1\right) \wedge \left(\theta_1 \to \Delta_2^\#\right).$$

By Proposition 10.3.2 and Lemma 12.3.6-7, we have

$$\mathbf{wGL}_n \vdash \bigwedge\{[n]I_a : a \in BH(\pi_1)\} \wedge \bigwedge\{[n]I_a : a \in H(\pi_1)\} \\ \to \left(\overline{\Gamma_1^\#} \to \theta\right) \wedge \left(\theta \to \Gamma_2^\#\right).$$

Note that $BH(\pi) = BH(\pi_1) \cup H(\pi_1) \backslash \{b\}$ and $H(\pi) = \emptyset$. We have

$$\mathbf{wGL}_n \vdash \bigwedge\{[n]I_a : a \in BH(\pi)\} \wedge [n]I_b \to \left(\overline{\Gamma_1^\#} \to \theta\right) \wedge \left(\theta \to \Gamma_2^\#\right).$$

By Theorem 12.10, we can construct the fixed-point ψ of $\theta(x_b)$. Applying a similar argument as in Case 4, we obtain

$$\mathbf{wGL}_n \vdash \bigwedge\{[n]I_a : a \in BH(\pi)\} \wedge [n]I_b(\psi) \to I_b(\psi).$$

By Propostion 10.3.3,

$$\mathbf{wGL}_n \vdash \bigwedge\{[n]I_a : a \in BH(\pi)\} \to I_b(\psi),$$

i.e.,

$$\mathbf{wGL}_n \vdash \bigwedge\{[n]I_a : a \in BH(\pi)\} \to \left(\overline{\Gamma_1^\#} \to \theta(\psi)\right) \wedge \left(\theta(\psi) \to \Gamma_2^\#\right).$$

From this and by the constructions of θ and ψ, $\theta(\psi)$ enjoys the conditions 1-3.

Condition 4 By the induction hypothesis,

$$T(\pi_1) \Rightarrow w_X(\theta_1) \cup \bigcup\{w_a^\circ \mid a \in BH(\pi_1)\} \cup \bigcup\{w_a \mid a \in H(\pi_1)\} \\ \subseteq w(\overline{\Delta_1}) \cap w(\Delta_2).$$

Assume $T(\pi)$ is true. Since $T(\pi) \wedge I'_b$ implies $T(\pi_1)$, we have

$$I'_b \Rightarrow w_X(\theta_1) \cup \bigcup\{w^\circ_a \mid a \in BH(\pi_1)\} \cup \bigcup\{w_a \mid a \in H(\pi_1)\} \subseteq w(\overline{\Delta_1}) \cap w(\Delta_2).$$

As in Case 4, we obtain

$$I'_b \Longrightarrow w_X(\theta) \cup \bigcup\{w^\circ_a \mid a \in BH(\pi)\} \cup w^\circ_b \subseteq w(\overline{\Gamma_1}) \cap w(\Gamma_2).$$

Applying a similar argument as in Case 4, we can show that $w_X(\theta(\psi)) \subseteq w(\overline{\Gamma_1}) \cap w(\Gamma_2)$ and $\bigcup\{w^\circ_a \mid a \in BH(\pi)\} \subseteq w(\overline{\Gamma_1}) \cap w(\Gamma_2)$.

The proof of Lemma 12.12 is now completed. □

Lemma 12.13. If $^\circ\mathbf{wK4}^{\mathbf{Sp}}_n \vdash \Gamma_1 \mid \Gamma_2$, then we can construct an interpolant θ of $(\Gamma_1 \mid \Gamma_2)$ in an effective way.

Proof. Let $\pi = (\kappa, d)$ be a split circular proof of $(\Gamma_1 \mid \Gamma_2)$. Then we can construct an interpolant of $(\Gamma_1 \mid \Gamma_2)$ by the following way. For each non-axiomatic leaf a, we put x_a as the provisional interpolant. From the leaf to the root, construct θ in accordance with rules of $\mathbf{wK4}^{\mathbf{Sp}}_n$. If we find a node n which is connected by a non-axiomatic leaf a, then apply Theorem 12.10, substitute the fixed-point for all occurrences of x_a, and continue to the next application. Since every non-axiomatic leaf a is connected to some interior node b by d, x_a must be eliminated. Therefore the resulting formula θ does not contain any literal of the form x_a, and satisfies that

$$\mathbf{wGL}_n \vdash \left(\overline{\Gamma_1^{\#}} \to \theta\right) \wedge \left(\theta \to \Gamma_2^{\#}\right) \text{ and}$$
$$w(\theta) \subseteq w(\overline{\Gamma_1}) \cap w(\Gamma_2),$$

by Lemma 12.12. Thus, θ is indeed an interpolant of $(\Gamma_1 \mid \Gamma_2)$. □

Theorem 12.7 follows from Lemma 12.13 immediately.

Proof of Theorem 12.7. Assume $\mathbf{wGL}_n \vdash \varphi \to \psi$. Then by Theorem 11.6, we can construct a proof π of $\overline{\varphi}, \psi$ in $\mathbf{wGL}^{\mathbf{G}}_n$. By Lemma 12.3 and Proposition 12.9, we obtain a split circular proof of $(\overline{\varphi} \mid \psi)$ in $^\circ\mathbf{wK4}^{\mathbf{Sp}}_n$. Let θ be an interpolant of $\overline{\varphi} \mid \psi$. Then θ is indeed a Lyndon interpolant of $\varphi \to \psi$. □

Chapter V

Fixed-point properties for predicate modal logics

13 Preliminaries for Chapter V

13.1 Classes of predicate Kripke frames

We specify several classes of Kripke frames. Let $\mathcal{F} = \langle W, \prec, \{D_w\}_{w\in W}\rangle$ be a Kripke frame.

Suppose that $\mathcal{F}$ is conversely well-founded. For each $w \in W$, the *height of* w (written by $h(w)$) is inductively defined by:

$$h(w) := \sup\{h(v) + 1 : w \prec v\}.$$

(In particular, $\sup \emptyset = 0$.) A Kripke frame $\mathcal{F}$ is *of bounded length* if for any $w \in W$, $h(w)$ is finite. For a Kripke frame $\mathcal{F}$, *the height of* $\mathcal{F}$ (written by $h(\mathcal{F})$) is defined by $\sup\{h(w) : w \in W\}$, and $\mathcal{F}$ is said to be *finite height* if $h(\mathcal{F})$ is finite.

We define the following five classes of Kripke frames:

1. $\mathsf{CW} := \{\mathcal{F} \mid \mathcal{F}$ is transitive and conversely well-founded$\}$;
2. $\mathsf{BL} := \{\mathcal{F} \mid \mathcal{F}$ is transitive and of bounded length$\}$;
3. $\mathsf{FH} := \{\mathcal{F} \mid \mathcal{F}$ is transitive and finite height$\}$;
4. $\mathsf{FI} := \{\mathcal{F} \mid \mathcal{F}$ is finite, transitive and irreflexive$\}$;
5. $\mathsf{FIFD} := \{\mathcal{F} \mid \mathcal{F}$ is finite, transitive and irreflexive, and for every $w \in W$, D_w is finite$\}$.

For a class C of Kripke frames, $\mathbf{MQ}(\mathsf{C})$ denotes the set of all $\mathcal{L}_Q$-formulas which are valid in any $\mathcal{F}$ in C. It is easy to show that $\mathbf{QGL} \subseteq \mathbf{MQ}(\mathsf{CW})$. Since $\mathsf{FIFD} \subseteq \mathsf{FI} \subseteq \mathsf{FH} \subseteq \mathsf{BL} \subseteq \mathsf{CW}$, we obtain $\mathbf{QGL} \subseteq \mathbf{MQ}(\mathsf{CW}) \subseteq \mathbf{MQ}(\mathsf{BL}) \subseteq \mathbf{MQ}(\mathsf{FH}) \subseteq \mathbf{MQ}(\mathsf{FI}) \subseteq \mathbf{MQ}(\mathsf{FIFD})$. The class BL is introduced by Tanaka [28].

It is easy to show $\mathbf{MQ}(\mathsf{BL}) = \mathbf{MQ}(\mathsf{FH})$. For, if $A \notin \mathbf{MQ}(\mathsf{BL})$, then there exsist a model $\mathcal{M} = \langle W, \prec, \{D_w\}_{w\in W}, \Vdash\rangle$ and $w \in W$ such that $\langle W, \prec, \{D_w\}_{w\in W}\rangle$ belongs to BL and $\mathcal{M}, w \not\models A$. Let $\mathcal{M}^*$ be the generated submodel of $\mathcal{M}$ by w. Then the frame of $\mathcal{M}^*$ is finite height, and $\mathcal{M}^*, w \not\models A$. Hence, $A \notin \mathbf{MQ}(\mathsf{FH})$.

Tanaka also showed that **NQGL** is Kripke complete with respect to BL. We obtain the following theorem:

Theorem 13.1 (Tanaka [28])**.** $\mathbf{NQGL} = \mathbf{MQ}(\mathsf{BL})\ (= \mathbf{MQ}(\mathsf{FH}))$.

By Theorem 13.1, we obtain $\mathbf{QGL} \subseteq \mathbf{NQGL}$.

13.2 Fixed point properties

To describe the semantical fixed-point properties for predicate modal logic, we need an auxiliary propositional variable to specify where to substitute fixed-points in predicate modal formulas. For this purpose, we define the following language $\mathcal{L}'_Q$. The language $\mathcal{L}'_Q$ consists of $\mathcal{L}_Q$ and one certain fixed propositional variable p. An $\mathcal{L}'_Q$-formula φ is constructed as the following manner:

$$\varphi ::= \top \mid \bot \mid p \mid P(u_1, \ldots, u_n) \mid \neg\varphi \mid \varphi \to \varphi \mid \forall u\varphi \mid \Box\varphi$$

Montagna [17] showed that the predicate version of Theorem 5.3 does not hold in **QGL**.

Theorem 13.2 (Montagna [17])**.** Let $\varphi(p) :\equiv \forall u \exists v \Box\, (p \to P(u, v))$. Then $A(p)$ has no fixed-points in **QGL**, that is, for any $\mathcal{L}_Q$-sentence ψ containing only the predicate symbol P, $\mathbf{QGL} \nvdash \psi \leftrightarrow \varphi(\psi)$.

Here we define two semantical fixed-point properties for classes of frames.

Definition 13.3. Let C be a class of Kripke frames.

1. The class C has *the fixed-point property* if for any $\mathcal{L}'_Q$-formula $\varphi(p)$ which is modalized in p, there exists an $\mathcal{L}_Q$-formula ψ such that:
   (a) The formula ψ contains only predicate symbols occurring in φ;
   (b) For any Kripke frame $\mathcal{F}$ in C, $\mathcal{F} \models \psi \leftrightarrow \varphi(\psi)$.
2. The class C has *the local fixed-point property* if for any $\mathcal{L}'_Q$-formula $\varphi(p)$ which is modalized in p, and for any Kripke frame $\mathcal{F}$ in C, there exists an $\mathcal{L}_Q$-formula ψ such that:
   (a) The formula ψ contains only predicate symbols occurring in φ;
   (b) $\mathcal{F} \models \psi \leftrightarrow \varphi(\psi)$.

Clearly if C has the fixed-point property, then C has the local fixed-point property. Montagna proved Theorem 13.2 by constructing a Kripke model $\mathcal{M}$ in BL such that for any $\mathcal{L}_Q$-sentence ψ containing only P, the formula $\psi \leftrightarrow \varphi(\psi)$ is not valid in $\mathcal{M}$. Thus we obtain the following corollary:

Corollary 13.4.

1. The classes CW and BL have neither the local fixed-point property, nor the fixed-point property.

2. The fixed-point theorem for **NQGL** does not hold.

The second clause immediately follows from the first clause and Theorem 13.1.

13.3 The substitution lemma

The following substitution lemma will be used in the later sections.

Lemma 13.5 (Substitution lemma)**.** Let $\varphi(p)$ be any $\mathcal{L}'_Q$-formula. Let α and β be $\mathcal{L}_Q$-formulas containing no free variables which are bounded in $\varphi(p)$. Then $\mathbf{QK4} \vdash \boxdot(\alpha \leftrightarrow \beta) \to (\varphi(\alpha) \leftrightarrow \varphi(\beta))$. Moreover, if $\varphi(p)$ is modalized in p, then $\mathbf{QK4} \vdash \Box(\alpha \leftrightarrow \beta) \to (\varphi(\alpha) \leftrightarrow \varphi(\beta))$.

Proof. Induction on the construction of $\varphi(p)$.

- If $\varphi(p)$ does not contain p, then Lemma trivially holds.

- Assume $\varphi(p) \equiv p$. Then $\varphi(\alpha) \equiv \alpha$ and $\varphi(\beta) \equiv \beta$, and thus Lemma holds.

- The cases $\varphi(p) \equiv \neg\psi(p)$ and $\varphi(p) \equiv \psi(p) \to \chi(p)$ are clear.

- Assume $\varphi(p) \equiv \forall u \psi(p)$ and Lemma holds for $\psi(p)$. If α and β contain no free variables which are bounded in $\varphi(p)$, then every free variable of α and β is not equal to u, and hence is not bounded in $\psi(p)$. By the induction hypothesis, $\mathbf{QK4} \vdash \boxdot(\alpha \leftrightarrow \beta) \to (\psi(\alpha) \leftrightarrow \psi(\beta))$. Since u does not occur freely in α and β, we have $\mathbf{QK4} \vdash \boxdot(\alpha \leftrightarrow \beta) \to \forall u\, (\psi(\alpha) \leftrightarrow \psi(\beta))$. Distributing $\forall$, we conclude $\mathbf{QK4} \vdash \boxdot(\alpha \leftrightarrow \beta) \to (\forall u \psi(\alpha) \leftrightarrow \forall u \psi(\beta))$. (If $\varphi(p)$ is modalized in p, then so is $\psi(p)$. By the induction hypothesis, $\mathbf{QK4} \vdash \Box(\alpha \leftrightarrow \beta) \to (\psi(\alpha) \leftrightarrow \psi(\beta))$. Applying a similar argument, we conclude $\mathbf{QK4} \vdash \Box(\alpha \leftrightarrow \beta) \to (\forall u \psi(\alpha) \leftrightarrow \forall u \psi(\beta))$.)

- Assume $\varphi(p) \equiv \Box\psi(p)$ and Lemma holds for $\psi(p)$. By the induction hypothesis, $\mathbf{QK4} \vdash \boxdot(\alpha \leftrightarrow \beta) \to (\psi(\alpha) \leftrightarrow \psi(\beta))$. By the derivation of $\mathbf{QK}$, $\mathbf{QK4} \vdash \Box \boxdot (\alpha \leftrightarrow \beta) \to (\Box\psi(\alpha) \leftrightarrow \Box\psi(\beta))$. Recall that $\mathbf{QK4} \vdash \Box\xi \to \Box\boxdot\xi$ for any ξ. Thus we conclude $\mathbf{QK4} \vdash \Box(\alpha \leftrightarrow \beta) \to (\Box\psi(\alpha) \leftrightarrow \Box\psi(\beta))$.

$\Box$

14 Semantical fixed-point properties

14.1 Failure of the fixed-point property for FIFD and NQGL

In this section, we prove that the class FIFD dos not enjoy the fixed-point property. As a consequence, we obtain that the classes FH and FI also do not have the fixed-point property.

In our proof, we borrow an idea from the following Smoryński's improvement of Montagna's theorem (Theorem 13.2).

Theorem 14.1 (Smoryński [26]). The $\mathcal{L}'$-formula $\forall u \Box (p \to P(u))$ has no fixed-points in **QGL**.

We describe the proof of Theorem 14.1. Let $\mathcal{M}_S := \langle W, \prec, \{D_n\}_{n \in W}, \Vdash \rangle$ where

- $W := \omega$;
- $m \prec n :\Leftrightarrow n < m$;
- $D_n := \{m \in \omega \mid m \geq n\}$;
- $n \Vdash P(m) :\Leftrightarrow m \neq n + 1$.

The Kripke frame $\langle W, \prec, \{D_n\}_{n \in W} \rangle$ is a member of BL. The following claim holds for $\mathcal{M}_S$.

Claim 14.2 (Smoryński [26]). Let φ be an $\mathcal{L}_Q$-sentence containing only the predicate symbol P. Then the set $\{n \in \omega \mid \mathcal{M}_S, n \models \varphi\}$ is either finite or co-finite.

Using this fact, Smoryński showed that for any $\mathcal{L}_Q$-sentence ψ containing only P, the formula $\psi \leftrightarrow \varphi(\psi)$ is not valid in $\mathcal{M}_S$, and hence $\mathbf{QGL} \nvdash \psi \leftrightarrow \varphi(\psi)$.

We prove the following lemma concerning Smoryński's model $\mathcal{M}_S$.

Lemma 14.3. Let $n \in \omega$ and $\varphi(u)$ be an $\mathcal{L}_Q$-formula with parameters from D_n containing only the predicate symbol P. Then for any $m_1, m_2 \geq n + 2$,

$$\mathcal{M}_S, n \models \varphi(m_1) \leftrightarrow \varphi(m_2).$$

Proof. Induction on the construction of $\varphi(u)$.

- The cases $\varphi(u) \equiv \top$ and $\varphi(u) \equiv \bot$ are trivial.

- Assume $\varphi(u) \equiv P(u)$. Then by the definition of $\Vdash$, for any $m_1, m_2 \geq n+2$, $\mathcal{M}_S, n \models P(m_1)$ and $\mathcal{M}_S, n \models P(m_2)$.
- The cases $\varphi(u) \equiv \neg\psi(u)$ and $\varphi(u) \equiv \psi(u) \to \chi(u)$ are clear by the induction hypothesis.
- Assume $\varphi(u) \equiv \forall v \psi(u, v)$. Then

$$\begin{aligned}\mathcal{M}_S, n \models \forall v \psi(m_1, v) &\iff \mathcal{M}_S, n \models \psi(m_1, m') \text{ for any } m' \in D_n,\\ &\iff \mathcal{M}_S, n \models \psi(m_2, m') \text{ for any } m' \in D_n, \qquad \text{(I.H.)}\\ &\iff \mathcal{M}_S, n \models \forall v \psi(m_2, v).\end{aligned}$$

- Assume $\varphi(u) \equiv \Box\psi(u)$. Then

$$\mathcal{M}_S, n \models \Box\psi(m_1) \iff \mathcal{M}_S, k \models \psi(m_1) \text{ for any } k < n.$$

By $D_n \subseteq D_k$ for any $k < n$, $\psi(u)$ is an $\mathcal{L}_Q$-formula with parameters from D_k. By the induction hypothesis (note that $k+2 < n+2 \leq m_1, m_2$),

$$\begin{aligned}\mathcal{M}_S, k \models \psi(m_1) \text{ for any } k < n &\iff \mathcal{M}_S, k \models \psi(m_2) \text{ for any } k < n,\\ &\iff \mathcal{M}_S, n \models \Box\psi(m_2).\end{aligned}$$

□

Next we define Kripke models which are finite part of Smoryński's model $\mathcal{M}_S$. For each $k \in \omega$, we define $\mathcal{M}_k := \langle W_k, \prec_k, \{D^k_n\}_{n \in W_k}, \Vdash_k \rangle$ where

- $W_k := \{0, 1, \ldots, k\}$;
- $m \prec_k n :\Leftrightarrow m \prec n (\Leftrightarrow n < m)$;
- $D^k_n := \{n, n+1, \ldots, k+2\}$;
- $n \Vdash_k P(m) :\Leftrightarrow n \Vdash P(m) (\Leftrightarrow m \neq n+1)$.

For each $k \in \omega$, the Kripke frame $\langle W_k, \prec_k, \{D^k_n\}_{n \in W_k} \rangle$ belongs to FIFD.

Lemma 14.4. Fix $k \in \omega$. For any $n \leq k$ and $\mathcal{L}_Q$-sentence φ with parameters from D^k_n containing only P,

$$\mathcal{M}_S, n \models \varphi \iff \mathcal{M}_k, n \models_k \varphi.$$

Proof. Induction on the construction of φ.

- The cases $\varphi \equiv \top$ and $\varphi \equiv \bot$ are trivial.
- Assume $\varphi \equiv P(m)$ for some $m \in D^k_n$. By the definition of $\Vdash_k$, $\mathcal{M}_S, n \models P(m) \Leftrightarrow \mathcal{M}_k, n \models P(m)$.
- The cases for $\varphi \equiv \neg\psi$, and $\varphi \equiv \psi \vee \chi$ are clear by the induction hypothesis.
- Assume $\varphi \equiv \forall u \psi(u)$. Then

$$\begin{aligned}
\mathcal{M}_S, n \models \forall u \psi(u) &\iff \mathcal{M}_S, n \models \psi(m) \text{ for all } m \in D_n, \\
&\iff \mathcal{M}_S, n \models \psi(n), \ldots, \mathcal{M}_S, n \models \psi(k+1) \text{ and} \\
&\quad\ \mathcal{M}_S, n \models \psi(m) \text{ for all } m \geq k+2. \qquad (\star)
\end{aligned}$$

By Lemma 14.3, the statement $(\star)$ is equivalent to $\mathcal{M}_S, n \models \psi(k+2)$. Thus

$$\begin{aligned}
\mathcal{M}_S, n \models \forall u \psi(u) &\iff \mathcal{M}_S, n \models \psi(n), \ldots, \mathcal{M}_S, n \models \psi(k+2), \\
&\iff \mathcal{M}_k, n \models \psi(n), \ldots, \mathcal{M}_k, n \models \psi(k+2), \quad \text{(I.H.)} \\
&\iff \mathcal{M}_k, n \models \forall u \psi(u).
\end{aligned}$$

- If $\varphi \equiv \Box\psi$, then

$$\mathcal{M}_S, n \models \Box\psi \iff \mathcal{M}_S, m \models \psi \text{ for all } m < n.$$

Since $D^k_n \subseteq D^k_m$ for any $m < n$, ψ is an $\mathcal{L}_Q$-sentence with parameters from $\bigcap_{m<n} D^k_m$, and hence

$$\begin{aligned}
\mathcal{M}_S, m \models \psi \text{ for all } m < n &\iff \mathcal{M}_k, m \models \psi \text{ for all } m < n, \quad \text{(I.H.)} \\
&\iff \mathcal{M}_k, n \models \Box\psi.
\end{aligned}$$

□

Lemma 14.5. Fix some $k \in \omega$. For any $\mathcal{L}_Q$-sentence φ, if $\mathcal{M}_k \models \varphi \leftrightarrow \forall u \Box(\varphi \to P(u))$, then for any $n \leq k$,

$$\mathcal{M}_k, n \models \varphi \iff n \text{ is even.}$$

Proof. Induction on n.

Assume $n = 0$. Since $\mathcal{M}_k, 0 \models \Box(\varphi \to P(m))$ for any $m \in D^k_0$, we have $\mathcal{M}_k, 0 \models \forall u \Box(\varphi \to P(u))$. By the assumption, $\mathcal{M}_k, 0 \models \varphi$.

(Inductive case) Assume Lemma holds for $m < n$.

($\Rightarrow$) Suppose that n is odd. Since $\mathcal{M}_k, n-1 \models \varphi$ and $\mathcal{M}_k, n-1 \not\models P(n)$, we have $\mathcal{M}_k, n \not\models \Box(\varphi \to P(n))$. This implies $\mathcal{M}_k, n \not\models \forall u \Box(\varphi \to P(u))$. By the assumption, $\mathcal{M}_k, n \not\models \varphi$.

($\Leftarrow$) Suppose that $n \neq 0$ and n is even. We claim that $\mathcal{M}_k, n \models \Box(\varphi \to P(m))$ for any $m \in D^k_n$. Take an arbitrary $l < n$. If $l < n-1$, then for every $m \in D^k_n$, $l+1 < n \leq m$, and hence $m \neq l+1$. Therefore for every $m \in D^k_n$, $\mathcal{M}_k, l \models P(m)$. This implies that for every $l < n-1$ and $m \in D^k_n$, $\mathcal{M}_k, l \models \varphi \to P(m)$.

If $l = n-1$, then l is odd. By the induction hypothesis, $\mathcal{M}_k, l \not\models \varphi$, and hence for every $m \in D^k_n$, $\mathcal{M}_k, l \models \varphi \to P(m)$.

We obtain that for every $l < n$ and $m \in D^k_n$, $\mathcal{M}_k, l \models \varphi \to P(m)$, and hence the claim is verified. Thus, $\mathcal{M}_k, n \models \forall u \Box(\varphi \to P(u))$. By the assumption, $\mathcal{M}_k, n \models \varphi$.

$\Box$

Conforming to Smoryński's argument, we prove the following theorem.

Theorem 14.6. The class FIFD does not have the fixed-point property.

Proof. Let φ be any $\mathcal{L}_Q$-sentence containing only P. It suffices to show that there is $k \in \omega$ such that $\mathcal{M}_k \not\models \varphi \leftrightarrow \forall u \Box (\varphi \to P(u))$. By Claim 14.2, the set $\{n \in \omega \mid \mathcal{M}_S, n \models \varphi\}$ is either finite or co-finite. Then for some $k \in \omega$, either

$$k \text{ is odd and } \mathcal{M}_S, k \models \varphi \quad \text{or} \quad k \text{ is even and } \mathcal{M}_S, k \not\models \varphi.$$

By Lemma 14.4, $\mathcal{M}_S, k \models \varphi \Leftrightarrow \mathcal{M}_k, k \models \varphi$. Therefore we have either

$$k \text{ is odd and } \mathcal{M}_k, k \models \varphi \quad \text{or} \quad k \text{ is even and } \mathcal{M}_k, k \not\models \varphi.$$

By Lemma 14.5, we conclude $\mathcal{M}_k \not\models \varphi \leftrightarrow \forall u \Box (\varphi \to P(u))$. $\Box$

Corollary 14.7. The classes FH and FI do not have the fixed-point property.

14.2 The fixed-point theorem for $\mathbf{QK} + \Box^{n+1}\bot$ and the local fixed-point property for FH

In this subsection, we prove the effective fixed-point theorem for $\mathbf{QK} + \Box^{n+1}\bot$. As a consequence, we show the class FH has the local fixed-point property.

Theorem 14.8. Let $n \in \omega$. Suppose that an $\mathcal{L}'_Q$-formula $\varphi(p)$ is modalized in p. Then there is an $\mathcal{L}_Q$-formula ψ such that ψ contains only predicate symbols and free variables occurring in $\varphi(p)$, and

$$\mathbf{QK} \vdash \Box^{n+1}\bot \to (\psi \leftrightarrow \varphi(\psi)).$$

Moreover, such a formula ψ is effectively calculable from $\varphi(p)$.

Before proving Theorem 14.8, we give some definitions, and prove several lemmas.

Definition 14.9.

1. Let φ be an $\mathcal{L}'_Q$-formula, and ψ be a subformula of φ. *The depth of an occurrence of* ψ *in* φ is the total number of subformulas $\Box\chi$ of φ, containing the occurrence of ψ, not ψ itself.
2. For an $\mathcal{L}'_Q$-formula φ, $\varphi^{\top(n)}$ denotes the formula obtained from φ by replacing every occurrence of the form $\Box\psi$ of depth n by $\top$.
3. For an $\mathcal{L}'_Q$-formula $\varphi(p)$, $\varphi(p)[\psi_0, \dots, \psi_n]$ denotes the formula obtained from $\varphi(p)$ by substituting ψ_i for all occurrences of p of depth i for each $i \leq n$, respectively.

For instance, put $\varphi(p) :\equiv \Box\,(p \to \forall u(Q(u) \to \Box p))$. Then the depth of φ is 0, and the depth of $\Box p$ is 1. By Definition 14.9.2,

$$\varphi^{\top(0)} \equiv \top, \quad \varphi^{\top(1)} \equiv \Box\,(p \to \forall u\,(Q(u) \to \top))\,, \text{ and } \varphi^{\top(2)} \equiv \varphi.$$

The depth of the left p is 1, and the depth of the right p is 2. By Definition 14.9.3,

$$\varphi(p)[\psi_0, \psi_1, \psi_2] \equiv \Box\,(\psi_1 \to \forall u\,(Q(u) \to \Box\psi_2))\,.$$

The following lemma immediately follows from Definition 14.9.

Lemma 14.10. Let $m, n \in \omega$ with $m \geq n$. Let $\varphi(p)$ be any $\mathcal{L}'_Q$-formula, and $\psi_0, \dots \psi_m$ be any $\mathcal{L}_Q$-formulas. Then the followings hold:

1. $\varphi^{\top(n)}$ contains only occurrences of p of depth $\leq n$. Thus $\varphi^{\top(n)}(p)\,[\psi_0, \dots, \psi_n]$ is an $\mathcal{L}_Q$-formula;
2. $\left(\varphi^{\top(m)}\right)^{\top(n)} \equiv \varphi^{\top(n)}$;
3. $(\varphi(p)\,[\psi_0, \dots, \psi_m])^{\top(n)} \equiv \varphi^{\top(n)}(p)\,[\psi_0, \dots, \psi_n]$.

Lemma 14.11. For any $n \in \omega$ and $\mathcal{L}_Q$-formula φ,

$$\mathbf{QK} \vdash \Box^{n+1}\bot \to \left(\varphi \leftrightarrow \varphi^{\top(n)}\right).$$

Proof. By the induction on the construction of φ, we show that for any $n \in \omega$, $\mathbf{QK} \vdash \Box^{n+1}\bot \to \left(\varphi \leftrightarrow \varphi^{\top(n)}\right)$.

- If φ is an atomic formula, then for any $n \in \omega$, $\varphi^{\top(n)} \equiv \varphi$. Clearly $\mathbf{QK} \vdash \varphi \leftrightarrow \varphi^{\top(n)}$, and hence $\mathbf{QK} \vdash \Box^{n+1}\bot \to \left(\varphi \leftrightarrow \varphi^{\top(n)}\right)$.
- The cases for $\varphi \equiv \neg\psi$ and $\varphi \equiv \psi \to \chi$, Lemma clearly follows from the definition of $\varphi^{\top(n)}$ and the induction hypothesis.
- Suppose that $\varphi \equiv \forall u\psi$, and Lemma holds for ψ. In this case for any $n \in \omega$, $\varphi^{\top(n)} \equiv \forall u\left(\psi^{\top(n)}\right)$. By the induction hypothesis, $\mathbf{QK} \vdash \Box^{n+1}\bot \to \left(\psi \leftrightarrow \psi^{\top(n)}\right)$ and hence $\mathbf{QK} \vdash \Box^{n+1}\bot \to \left(\forall u\psi \leftrightarrow \forall u\left(\psi^{\top(n)}\right)\right)$. Therefore $\mathbf{QK} \vdash \Box^{n+1}\bot \to \left(\varphi \leftrightarrow \varphi^{\top(n)}\right)$.
- Suppose that $\varphi \equiv \Box\psi$ and Lemma holds for ψ. We distinguish the following two cases.
  - If $n = 0$, then $\varphi^{\top(0)} \equiv \top$. Since $\mathbf{QK} \vdash \Box\bot \to (\Box\psi \leftrightarrow \top)$ for any $\mathcal{L}$-formula ψ, $\mathbf{QK} \vdash \Box\bot \to \left(\varphi \leftrightarrow \varphi^{\top(0)}\right)$.
  - Suppose that $n > 0$. By the inductive hypothesis for ψ, $\mathbf{QK} \vdash \Box^{n}\bot \to \left(\psi \leftrightarrow \psi^{\top(n-1)}\right)$. By the derivation of $\mathbf{QK}$, we have $\mathbf{QK} \vdash \Box^{n+1}\bot \to \left(\Box\psi \leftrightarrow \Box\left(\psi^{\top(n-1)}\right)\right)$. Note that each occurrence of $\Box\chi$ in $\Box\psi$ of depth $\geq n$ is the one in ψ of depth $\geq n-1$. Therefore $\varphi^{\top(n)} \equiv (\Box\psi)^{\top(n)} \equiv \Box\left(\psi^{\top(n-1)}\right)$. Thus, $\mathbf{QK} \vdash \Box^{n+1}\bot \to \left(\varphi \leftrightarrow \varphi^{\top(n)}\right)$.

$\Box$

Lemma 14.12. Suppose that $\varphi(p)$ is an $\mathcal{L}'_Q$-formula containing only occurrences of p of depth $\leq n$, and $\mathcal{L}_Q$-formulas $\alpha_0, \ldots, \alpha_n$ and $\beta_0, \ldots, \beta_n$ contain no free variables which are bounded in $\varphi(p)$. Then

$$\begin{aligned}\mathbf{QK} \vdash \Box^{n+1}\bot \wedge \bigwedge_{i \leq n} \Box^{n-i}\left(\Box^{i+1}\bot \to (\alpha_i \leftrightarrow \beta_i)\right) \\ \to \left(\varphi(p)\left[\alpha_n, \ldots, \alpha_0\right] \leftrightarrow \varphi(p)\left[\beta_n, \ldots, \beta_0\right]\right).\end{aligned}$$

Proof. Induction on the construction of $\varphi(p)$.

- Assume $\varphi(p) \equiv p$. Then for any $n \in \omega$, the depth of each occurrence of p is $\leq n$, and $\varphi(p)$ contains no free variables. For any $\mathcal{L}_Q$-formula $\alpha_0, \ldots, \alpha_n$ and $\beta_0, \ldots, \beta_n$, $\mathbf{QK} \vdash (\alpha_n \leftrightarrow \beta_n) \leftrightarrow (\alpha_n \leftrightarrow \beta_n)$, and hence

$$\mathbf{QK} \vdash \Box^{n+1}\bot \wedge \left(\Box^{n+1}\bot \to (\alpha_n \leftrightarrow \beta_n)\right) \to (\alpha_n \leftrightarrow \beta_n)\,.$$

  Adding the assumptions, we obtain

$$\mathbf{QK} \vdash \Box^{n+1}\bot \wedge \bigwedge_{i \leq n} \Box^{n-i}\left(\Box^{i+1}\bot \to (\alpha_i \leftrightarrow \beta_i)\right) \to (\alpha_n \leftrightarrow \beta_n).$$

  Since $\varphi(p)[\alpha_n, \ldots, \alpha_0] \equiv \alpha_n$ and $\varphi(p)[\beta_n, \ldots, \beta_0] \equiv \beta_n$, Lemma holds for $\varphi(p)$.

- Suppose that $\varphi(p)$ is one of the form $\neg\psi(p)$, $\psi(p) \to \chi(p)$ or $\forall u \psi(p)$. If $\varphi(p)$ contains only the occurrences of p of depth $\leq n$, then so does $\psi(p)$ and $\chi(p)$. Moreover, for any $\mathcal{L}_Q$-formula F, if all free variables occurring in F are not bounded in $\varphi(p)$, then they are not bounded in $\psi(p)$ and $\chi(p)$, too. By the induction hypothesis and the derivation of predicate logic, Lemma holds for $\varphi(p)$.

- Assume $\varphi(p) \equiv \Box\psi(p)$. If $\varphi(p)$ contains only the occurrences of p of depth $\leq n$, $\psi(p)$ contains only the occurrence of p of depth $\leq n-1$. Let $\alpha_0, \ldots, \alpha_n$ and $\beta_0, \ldots, \beta_n$ be $\mathcal{L}_Q$-formulas satisfying the assumption of Lemma. Every free variables occurring freely in α_i or β_i occur freely in $\psi(p)$. By the induction hypothesis,

$$\begin{aligned}\mathbf{QK} \vdash \Box^{n}\bot \wedge \bigwedge_{i \leq n-1} &\Box^{n-1-i}\left(\Box^{i+1}\bot \to (\alpha_i \leftrightarrow \beta_i)\right)\\ &\to (\psi(p)[\alpha_{n-1}, \ldots, \alpha_0] \leftrightarrow \psi(p)[\beta_{n-1}, \ldots, \beta_0])\,.\end{aligned}$$

  By the derivation of $\mathbf{QK}$,

$$\begin{aligned}\mathbf{QK} \vdash \Box^{n+1}\bot \wedge \bigwedge_{i \leq n-1} &\Box^{n-i}\left(\Box^{i+1}\bot \to (\alpha_i \leftrightarrow \beta_i)\right)\\ &\to (\Box\,(\psi(p)[\alpha_{n-1}, \ldots, \alpha_0]) \leftrightarrow \Box\,(\psi(p)[\beta_{n-1}, \ldots, \beta_0]))\,.\end{aligned}$$

  Since $\varphi(p)$ does not contain the occurrence of p of depth 0,

$$\begin{aligned}\Box\,(\psi(p)[\alpha_{n-1}, \ldots, \alpha_0]) &\equiv \varphi(p)[\alpha_n, \ldots, \alpha_0], \text{ and}\\ \Box\,(\psi(p)[\beta_{n-1}, \ldots, \beta_0]) &\equiv \varphi(p)[\beta_n, \ldots, \beta_0].\end{aligned}$$

Therefore

$$\mathbf{QK} \vdash \Box^{n+1}\bot \quad \wedge \bigwedge_{i \leq n-1} \Box^{n-i} \left(\Box^{i+1}\bot \to (\alpha_i \leftrightarrow \beta_i)\right)$$
$$\to (\varphi(p)[\alpha_n, \ldots, \alpha_0] \leftrightarrow \varphi(p)[\beta_n, \ldots, \beta_0]) .$$

Adding the assumptions, we obtain

$$\mathbf{QK} \vdash \Box^{n+1}\bot \wedge \bigwedge_{i \leq n} \Box^{n-i} \left(\Box^{i+1}\bot \to (\alpha_i \leftrightarrow \beta_i)\right)$$
$$\to (\varphi(p)[\alpha_n, \ldots, \alpha_0] \leftrightarrow \varphi(p)[\beta_n, \ldots, \beta_0]) .$$

□

In the remainder of this section, we fix an $\mathcal{L}'_Q$-formula $\varphi(p)$ which is modalized in p, i.e., $\varphi(p)$ contains no occurrences of p of depth 0. By replacing variables appropriately, we assume that every free variable occurring in $\varphi(p)$ does not occur in $\varphi(p)$ as a bound variable. We define the sequence $\{\Phi_n\}_{n\in\omega}$ of $\mathcal{L}_Q$-formulas recursively as follows:

1. $\Phi_0 :\equiv \varphi^{\top(0)}(p)\,[\top]\ \left(\equiv \varphi^{\top(0)}(p)\right)$;
2. $\Phi_{n+1} :\equiv \varphi^{\top(n+1)}(p)\,[\top, \Phi_n, \ldots, \Phi_0]$.

By the definition and Lemma 14.10.1, every Φ_n is an $\mathcal{L}_Q$-formula and contains only predicate symbols and free variables occurring in $\varphi(p)$.

Lemma 14.13. For any $m, n \in \omega$, if $m \geq n$, then $\mathbf{QK} \vdash \Box^{n+1}\bot \to (\Phi_m \leftrightarrow \Phi_n)$.

Proof. Induction on n.

- Assume $n = 0$, and take $m \geq 0$ arbitrarily. Then

$$\begin{aligned}
\Phi_m^{\top(0)} &\equiv \left(\varphi^{\top(m)}(p)[\top, \Phi_{m-1}, \ldots, \Phi_0]\right)^{\top(0)}, \\
&\equiv \left(\varphi^{\top(m)}\right)^{\top(0)}(p)[\top], && \text{(by Lemma 14.10.3)} \\
&\equiv \varphi^{\top(0)}(p)[\top], && \text{(by Lemma 14.10.2)} \\
&\equiv \Phi_0.
\end{aligned}$$

  By Lemma 14.11, $\mathbf{QK} \vdash \Box\bot \to \left(\Phi_m \leftrightarrow \Phi_m^{\top(0)}\right)$. Thus we have $\mathbf{QK} \vdash \Box\bot \to (\Phi_m \leftrightarrow \Phi_0)$.

- Suppose that Lemma holds for $\leq n$. Take $m+1 \geq n+1$ arbitrarily. Then by the induction hypothesis,

$$\mathbf{QK} \vdash \bigwedge_{i<n+1} \Box^{i+1}\bot \to (\Phi_{i+(m-n)} \leftrightarrow \Phi_i),$$

and hence

$$\mathbf{QK} \vdash \bigwedge_{i<n+1} \Box^{n+1-i}(\Box^{i+1}\bot \to (\Phi_{i+(m-n)} \leftrightarrow \Phi_i)).$$

Note that $\mathbf{QK} \vdash \Box^0(\Box^{n+2}\bot \to (\top \leftrightarrow \top))$,[2] and $\varphi^{\top(n+1)}(p)$ contains no free variables which is bounded in each Φ_i. From these and by Lemma 14.12, we obtain

$$\mathbf{QK} \vdash \Box^{n+2}\bot \to \big(\varphi^{\top(n+1)}(p)[\top, \Phi_m, \dots, \Phi_{m-n}] \\ \leftrightarrow \varphi^{\top(n+1)}(p)[\top, \Phi_n, \dots, \Phi_0]\big). \quad (10)$$

On the other hand, by Lemma 14.11,

$$\mathbf{QK} \vdash \Box^{n+2}\bot \to \left(\Phi_{m+1} \leftrightarrow \Phi_{m+1}^{\top(n+1)}\right).$$

Recall that

$$\begin{aligned} \Phi_{m+1}^{\top(n+1)} &\equiv \left(\varphi^{\top(m+1)}(p)[\top, \Phi_m, \dots, \Phi_0]\right)^{\top(n+1)}, \\ &\equiv \left(\varphi^{\top(m+1)}\right)^{\top(n+1)}(p)[\top, \Phi_m, \dots, \Phi_{m-n}], && \text{(by Lemma 14.10.3)} \\ &\equiv \varphi^{\top(n+1)}(p)[\top, \Phi_m, \dots, \Phi_{m-n}], && \text{(by Lemma 14.10.2)} \end{aligned}$$

Thus

$$\mathbf{QK} \vdash \Box^{n+2}\bot \to \big(\Phi_{m+1} \leftrightarrow \varphi^{\top(n+1)}(p)[\top, \Phi_m, \dots, \Phi_{m-n}]\big). \quad (11)$$

From (10) and (11), we conclude $\mathbf{QK} \vdash \Box^{n+2}\bot \to (\Phi_{m+1} \leftrightarrow \Phi_{n+1})$.

$\Box$

Let $\psi(p)$ be an $\mathcal{L}'_Q$-formula. For $n \in \omega$, we define

$$\Psi^n :\equiv \psi^{\top(n)}(p)[\Phi_n, \dots, \Phi_0].$$

By Lemma 14.10.1, the formula Ψ^n is an $\mathcal{L}_Q$-formula. Since $\varphi(p)$ is modalized in p, we obtain

$$\begin{aligned} \Phi^n &\equiv \varphi^{\top(n)}(p)[\Phi_n, \Phi_{n-1}, \dots, \Phi_0], \\ &\equiv \varphi^{\top(n)}(p)[\top, \Phi_{n-1}, \dots, \Phi_0], \\ &\equiv \Phi_n. \end{aligned}$$

[2] Here $\Box^0\varphi \equiv \varphi$.

Lemma 14.14. For any $\mathcal{L}'_Q$-formula $\psi(p)$ and $m, n \in \omega$, if $m \geq n$, then

$$\mathbf{QK} \vdash \Box^{n+1}\bot \to (\Psi^n \leftrightarrow \psi(\Phi_m))\,.$$

Proof. Induction on the construction of $\psi(p)$. Assume $m \geq n$.

- Assume $\psi(p) \equiv p$. In this case, $\Psi^n \equiv \psi^{\top(n)}(p)[\Phi_n, \ldots, \Phi_0] \equiv \Phi_n$, and $\psi(\Phi_m) \equiv \Phi_m$. By Lemma 14.13, $\mathbf{QK} \vdash \Box^{n+1}\bot \to (\Phi_m \leftrightarrow \Phi_n)$. Therefore $\mathbf{QK} \vdash \Box^{n+1}\bot \to (\Psi^n \leftrightarrow \psi(\Phi_m))$.
- The cases for $\psi(p) \equiv \neg\chi(p)$ and $\psi(p) \equiv \chi(p) \to \xi(p)$ are clear.
- Assume $\psi(p) \equiv \forall u \chi(p)$ and Lemma holds for $\chi(p)$. By the induction hypothesis, $\mathbf{QK} \vdash \Box^{n+1}\bot \to (X^n \leftrightarrow \chi(\Phi_m))$. Recall that $\forall u(X^n) \equiv (\forall u\chi)^n$. By the generalization, we have $\mathbf{QK} \vdash \Box^{n+1}\bot \to ((\forall u\chi)^n \leftrightarrow \forall u\chi(\Phi_m))$, i.e., $\mathbf{QK} \vdash \Box^{n+1}\bot \to (\Psi^n \leftrightarrow \psi(\Phi_m))$.
- Assume $\psi(p) \equiv \Box\chi(p)$ and Lemma holds for $\chi(p)$. We distinguish the following two cases.
  - If $n = 0$, then we have $\Psi^0 \equiv (\Box\chi)^0 \equiv (\Box\chi)^{\top(0)}(p)[\Phi_0] \equiv \top$. Since $\mathbf{QK} \vdash \Box\bot \to \Box\chi(\Phi_m)$, we obtain $\mathbf{QK} \vdash \Box\bot \to (\Psi^0 \leftrightarrow \psi(\Phi_m))$.
  - Suppose that $n > 0$. Take $m \geq n$ arbitrarily. Then $m > n - 1$. By the induction hypothesis for $\chi(p)$, m and $n-1$, $\mathbf{QK} \vdash \Box^n\bot \to (X^{n-1} \leftrightarrow \chi(\Phi_m))$. By the derivation of $\mathbf{QK}$, we have $\mathbf{QK} \vdash \Box^{n+1}\bot \to (\Box(X^{n-1}) \leftrightarrow \Box\chi(\Phi_m))$. Since $\psi(p)$ contains no occurrences of p of depth 0, we obtain

    $$\begin{aligned} \Box(X^{n-1}) &\equiv \Box\left(\chi^{\top(n-1)}(p)[\Phi_{n-1}, \ldots, \Phi_0]\right) \\ &\equiv (\Box\chi)^{\top(n)}(p)[\Phi_n, \Phi_{n-1}, \ldots, \Phi_0] \\ &\equiv \Psi^n. \end{aligned}$$

    Thus, $\mathbf{QK} \vdash \Box^{n+1}\bot \to (\Psi^n \leftrightarrow \psi(\Phi_m))$.

$\Box$

Here we are ready to prove Theorem 14.8.

Proof of Theorem 14.8. Let $\varphi(p)$ be the fixed $\mathcal{L}'_Q$-formula which is modalized in p, and it suffices to show that Φ_n is a fixed-point of $\varphi(p)$ in $\mathbf{QK} + \Box^{n+1}\bot$. By Lemma 14.14, we obtain $\mathbf{QK} \vdash \Box^{n+1}\bot \to (\Phi^n \leftrightarrow \varphi(\Phi_n))$. Since $\Phi^n \equiv \Phi_n$, $\mathbf{QK} \vdash \Box^{n+1}\bot \to (\Phi_n \leftrightarrow \varphi(\Phi_n))$. The formula Φ_n contains only predicate symbols and free variables occurring in φ. Thus, Φ_n is a fixed-point of $\varphi(p)$ in $\mathbf{QK} + \Box^{n+1}\bot$. $\Box$

Remark 14.15. In [19], Sacchetti proved the fixed-point theorem for propositional modal logics $\mathbf{K} + \Box^{n+1}\bot$ without giving an algorithm for calculating fixed-points in these logics. Our proof of Theorem 14.8 provides such an algorithm even for the logics $\mathbf{K} + \Box^{n+1}\bot$.

Corollary 14.16. The classes FH, FI and FIFD have the local fixed-point properties.

Proof. It suffices to prove only the case for FH. Let $\mathcal{F} = \langle W, \prec, \{D_w\}_{w \in W}\rangle$ be a Kripke frame in the class FH. Put $h(\mathcal{F}) = n$. Then for any $w \in W$, $h(w) \leq n$, i.e., $\mathcal{F} \models \Box^{n+1}\bot$. Let $\varphi(p)$ be any $\mathcal{L}'_Q$-formula which is modalized in p. From Theorem 14.8, we have $\mathbf{QK} \vdash \Box^{n+1}\bot \to (\Phi_n \leftrightarrow \varphi(\Phi_n))$. Recall that $\mathbf{QK} \subseteq \mathbf{QGL} \subseteq \mathbf{MQ}(\mathsf{FH})$. Thus we have $\mathcal{F} \models \Box^{n+1}\bot \to (\Phi_n \leftrightarrow \varphi(\Phi_n))$. From this and $\mathcal{F} \models \Box^{n+1}\bot$, we conclude $\mathcal{F} \models \Phi_n \leftrightarrow \varphi(\Phi_n)$. The formula Φ_n is indeed a local fixed-point of $\varphi(p)$ in $\mathcal{F}$. $\Box$

15 Further results

15.1 Failure of the Craig interpolation property for NQGL

In this section, we prove that the logic **NQGL** does not enjoy the Craig interpolation property.

Theorem 15.1. The system **NQGL** does not have the Craig interpolation property.

Before proving Theorem 15.1, we prepare several lemmas.

Lemma 15.2. Suppose that $\varphi(p)$ is an $\mathcal{L}'_Q$-formula not containing the unary predicate P, and not containing occurrences of u and v as bound variables. If $\mathbf{NQGL} \vdash \forall u \varphi\,(P(u))$, then for any $\mathcal{L}'_Q$-formula $\psi(v)$, $\mathbf{NQGL} \vdash \forall v \varphi\,(\psi(v))$.

Proof. Suppose that for some $\psi(v)$, $\mathbf{NQGL} \nvdash \forall v \varphi\,(\psi(v))$. By Theorem 13.1, there exists a Kripke model $\mathcal{M} = \langle \mathcal{F}, \Vdash\rangle = \langle W, \prec, \{D_w\}_{w\in W}, \Vdash\rangle$ such that $\mathcal{F} \in \mathsf{BL}$, and for some $w \in W$ and $c \in D_w$, $\mathcal{M}, w \not\models \varphi\,(\psi(c))$. We may assume w is the root of $\mathcal{F}$. Then for every $x \in W$, $c \in D_x$. We define an interpretation $\Vdash^*$ of $\mathcal{F}$ as follows:

- For any predicate symbol Q other than P, $\Vdash^* \langle w, Q\rangle = \Vdash \langle w, Q\rangle$ for every $w \in W$;
- For every $x \in W$ and $a \in D_x$, $x \Vdash^* P(a) :\Leftrightarrow x \Vdash \psi(c)$.

Let $\mathcal{M}^* := \langle \mathcal{F}, \Vdash^* \rangle$. We claim that for any $\mathcal{L}'_Q$-formula $\chi(p)$, $x \in W$ and $a \in D_x$, $\mathcal{M}, x \models \chi(\psi(c)) \iff \mathcal{M}^*, x \models \chi(P(a))$. We prove the claim by induction on the construction of $\chi(p)$.

- If $\chi(p)$ contains no occurrences of p, then the claim trivially holds.
- Assume $\chi(p) \equiv p$. Then $\chi(\psi(c)) \equiv \psi(c)$ and $\chi(P(a)) \equiv P(a)$. By the definition of $\Vdash^*$, we have $\mathcal{M}, x \models \chi(\psi(c)) \iff \mathcal{M}^*, x \models \chi(P(a))$.
- The cases $\chi(p) \equiv \neg\xi(p)$ and $\chi(p) \equiv \xi(p) \to \pi(p)$ are clear by the induction hypothesis.
- Assume $\chi(p) \equiv \forall v \xi(p)$. Then

$$\begin{aligned} \mathcal{M}, x \models \forall v \xi(\psi(c)) &\iff \mathcal{M}, x \models \xi(\psi(c))[v/b] \text{ for all } b \in D_x, \\ &\iff \mathcal{M}^*, x \models \xi(P(a))[v/b] \text{ for all } b \in D_x, \\ & \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \text{(I.H.)} \\ &\iff \mathcal{M}^*, x \models \forall v \xi(P(a)). \end{aligned}$$

- Assume $\chi(p) \equiv \Box\xi(p)$. Then

$$\begin{aligned} \mathcal{M}, x \models \Box\xi(\psi(c)) &\iff \mathcal{M}, y \models \xi(\psi(c)) \text{ for any } y \succ x, \\ &\iff \mathcal{M}^*, y \models \xi(P(a)) \text{ for any } y \succ x, \qquad \text{(I.H.)} \\ &\iff \mathcal{M}^*, x \models \Box\xi(P(a)). \end{aligned}$$

The proof of the claim is completed. From $\mathcal{M}, w \not\models \varphi(\psi(c))$ and by the claim, $\mathcal{M}^*, w \not\models \varphi(P(a))$, and hence $\mathcal{M}^*, w \not\models \forall u \varphi(P(u))$. By Theorem 13.1, $\mathbf{NQGL} \nvdash \forall u \varphi(P(u))$. $\square$

We prove the following uniqueness lemma of fixed-points in **NQGL**.

Lemma 15.3 (Uniqueness of fixed-points in **NQGL**)**.** Let $\varphi(p)$ be any $\mathcal{L}'_Q$-formula which is modalized in p. Let ψ_0 and ψ_1 be any $\mathcal{L}_Q$-formulas which contain no bounded variables occurring freely in $\varphi(p)$. Then

$$\mathbf{NQGL} \vdash \boxdot(\varphi(\psi_0) \leftrightarrow \psi_0) \land \boxdot(\varphi(\psi_1) \leftrightarrow \psi_1) \to (\psi_0 \leftrightarrow \psi_1).$$

Proof. We claim that, for any $n \in \omega$, $\mathcal{L}'$-formula $\varphi(p)$ which is modalized in p, and $\mathcal{L}$-formula ψ which contains no bounded variables occurring freely in $\varphi(p)$,

$$\mathbf{QGL} \vdash \Box^{n+1}\bot \to (\boxdot(\varphi(\psi) \leftrightarrow \psi) \to (\psi \leftrightarrow \Phi_n)),$$

where Φ_n is the $\mathcal{L}_Q$-formula defined in Section 14.2. By Lemma 13.5, $\mathbf{QK4} \vdash \Box(\psi \leftrightarrow \Phi_n) \to (\varphi(\psi) \leftrightarrow \varphi(\Phi_n))$. In particular, by Theorem 14.8, $\mathbf{QK} \vdash \Box^{n+1}\bot \to (\varphi(\Phi_n) \leftrightarrow \Phi_n)$. Thus

$$\mathbf{QK4} \vdash \Box^{n+1}\bot \to (\Box(\psi \leftrightarrow \Phi_n) \to (\varphi(\psi) \leftrightarrow \Phi_n)).$$

From this, we have

$$\begin{aligned} \mathbf{QK4} &\vdash \Box^{n+1}\bot \wedge (\varphi(\psi) \leftrightarrow \psi) \to (\Box(\psi \leftrightarrow \Phi_n) \to (\psi \leftrightarrow \Phi_n)), && (12)\\ \mathbf{QK4} &\vdash \Box^{n+2}\bot \wedge \Box(\varphi(\psi) \leftrightarrow \psi) \to \Box(\Box(\psi \leftrightarrow \Phi_n) \to (\psi \leftrightarrow \Phi_n)), \\ \mathbf{QGL} &\vdash \Box^{n+2}\bot \wedge \Box(\varphi(\psi) \leftrightarrow \psi) \to \Box(\psi \leftrightarrow \Phi_n). && (\text{by } \mathbf{L}) \end{aligned}$$

Since $\mathbf{QK4} \vdash \Box^{n+1}\bot \to \Box^{n+2}\bot$, we obtain

$$\mathbf{QGL} \vdash \Box^{n+1}\bot \wedge \Box(\varphi(\psi) \leftrightarrow \psi) \to \Box(\psi \leftrightarrow \Phi_n).$$

From this and (12), $\mathbf{QGL} \vdash \Box^{n+1}\bot \to (\boxdot(\varphi(\psi) \leftrightarrow \psi) \to (\psi \leftrightarrow \Phi_n))$. The proof of the claim is completed.

Let $\varphi(p)$, ψ_0 and ψ_1 be formulas as in the statement of Lemma. By the claim, for any $n \in \omega$,

$$\begin{aligned} \mathbf{QGL} &\vdash \Box^{n+1}\bot \to (\boxdot(\varphi(\psi_0) \leftrightarrow \psi_0) \to (\psi_0 \leftrightarrow \Phi_n)), \text{ and} \\ \mathbf{QGL} &\vdash \Box^{n+1}\bot \to (\boxdot(\varphi(\psi_1) \leftrightarrow \psi_1) \to (\psi_1 \leftrightarrow \Phi_n)). \end{aligned}$$

Therefore

$$\mathbf{QGL} \vdash \Box^{n+1}\bot \to (\boxdot(\varphi(\psi_0) \leftrightarrow \psi_0) \wedge \boxdot(\varphi(\psi_1) \leftrightarrow \psi_1) \to (\psi_0 \leftrightarrow \psi_1)).$$

Applying the rule **BL** of **NQGL**, we conclude

$$\mathbf{NQGL} \vdash \boxdot(\varphi(\psi_0) \leftrightarrow \psi_0) \wedge \boxdot(\varphi(\psi_1) \leftrightarrow \psi_1) \to (\psi_0 \leftrightarrow \psi_1).$$

□

Proof of Theorem 15.1. Let $\varphi(p) \equiv \forall u \Box(p \to P(u))$. By Lemma 15.3, for any unary predicate symbols Q and R other than P, and any variables v_0 and v_1,

$$\begin{aligned} \mathbf{NQGL} &\vdash \boxdot(\varphi(Q(v_0)) \leftrightarrow Q(v_0)) \wedge \boxdot(\varphi(R(v_1)) \leftrightarrow R(v_1)) \\ &\qquad \to (Q(v_0) \leftrightarrow R(v_1)), \\ \mathbf{NQGL} &\vdash \forall v_0 \forall v_1 (\boxdot(\varphi(Q(v_0)) \leftrightarrow Q(v_0)) \wedge \boxdot(\varphi(R(v_1)) \leftrightarrow R(v_1)) \\ &\qquad \to (Q(v_0) \leftrightarrow R(v_1))), \end{aligned}$$

and hence

$$\begin{aligned}\mathbf{NQGL} \vdash \exists v_0 \left(\boxdot \left(\varphi \left(Q(v_0)\right) \leftrightarrow Q(v_0)\right) \wedge Q(v_0)\right) \\ \to \forall v_1 \left(\boxdot \left(\varphi \left(R(v_1)\right) \leftrightarrow R(v_1)\right) \to R(v_1)\right).\end{aligned} \tag{13}$$

We show that the implication (13) has no Craig interpolants. Suppose, for the contradiction, that (13) has a Craig interpolant ψ, then ψ is an $\mathcal{L}_Q$-sentence containing only the predicate symbol P such that

$$\begin{aligned}&\mathbf{NQGL} \vdash \exists v_0 \left(\boxdot \left(\varphi \left(Q(v_0)\right) \leftrightarrow Q(v_0)\right) \wedge Q(v_0)\right) \to \psi, \text{ and} \\ &\mathbf{NQGL} \vdash \psi \to \forall v_1 \left(\boxdot \left(\varphi \left(R(v_1)\right) \leftrightarrow R(v_1)\right) \to R(v_1)\right).\end{aligned}$$

Hence

$$\mathbf{NQGL} \vdash \forall v_0 \left(\boxdot \left(\varphi \left(Q(v_0)\right) \leftrightarrow Q(v_0)\right) \to \left(Q(v_0) \to \psi\right)\right), \text{ and} \tag{14}$$

$$\mathbf{NQGL} \vdash \forall v_1 \left(\boxdot \left(\varphi \left(R(v_1)\right) \leftrightarrow R(v_1)\right) \to \left(\psi \to R(v_1)\right)\right). \tag{15}$$

We may assume ψ does not contain v_0 and v_1. By Lemma 15.2, substituting $Q(v_0)$ for $R(v_1)$ in (15), we have

$$\mathbf{NQGL} \vdash \forall v_0 \left(\boxdot \left(\varphi \left(Q(v_0)\right) \leftrightarrow Q(v_0)\right) \to \left(\psi \to Q(v_0)\right)\right).$$

From this and (14),

$$\mathbf{NQGL} \vdash \forall v_0 \left(\boxdot \left(\varphi \left(Q(v_0)\right) \leftrightarrow Q(v_0)\right) \to \left(Q(v_0) \leftrightarrow \psi\right)\right).$$

By Lemma 15.2, substituting $\varphi(\psi)$ for $Q(v_0)$, we have

$$\mathbf{NQGL} \vdash \boxdot \left(\varphi \left(\varphi(\psi)\right) \leftrightarrow \varphi(\psi)\right) \to \left(\varphi(\psi) \leftrightarrow \psi\right). \tag{16}$$

By the derivation of **QK4**, we get

$$\mathbf{NQGL} \vdash \Box \left(\varphi \left(\varphi(\psi)\right) \leftrightarrow \varphi(\psi)\right) \to \Box \left(\varphi(\psi) \leftrightarrow \psi\right).$$

By the substituion lemma (Lemma 13.5),

$$\mathbf{QK4} \vdash \Box(\varphi(\psi) \leftrightarrow \psi) \to (\varphi(\varphi(\psi)) \leftrightarrow \varphi(\psi)).$$

Thus

$$\mathbf{NQGL} \vdash \Box \left(\varphi \left(\varphi(\psi)\right) \leftrightarrow \varphi(\psi)\right) \to \left(\varphi \left(\varphi(\psi)\right) \leftrightarrow \varphi(\psi)\right).$$

Since the Löb rule is admissible in **NQGL**, we obtain $\mathbf{NQGL} \vdash \varphi\left(\varphi(\psi)\right) \leftrightarrow \varphi(\psi)$, and hence $\mathbf{NQGL} \vdash \boxdot \left(\varphi\left(\varphi(\psi)\right) \leftrightarrow \varphi(\psi)\right)$. From this and (16),

$$\mathbf{NQGL} \vdash \varphi(\psi) \leftrightarrow \psi.$$

This means that ψ would be a fixed-point of $\varphi(p)$ in **NQGL**. However, by the proof of Theorem 14.6, $\varphi(p)$ has no fixed-points in **NQGL**, contradiction. □

15.2 Formulas having a fixed-point in QGL

In this section, we investigate a sufficient condition for formulas to have a fixed-points in **QGL**. We introduce the notion of Σ-formulas[3], and then we prove that if $\varphi(p)$ is a Boolean combination of Σ-formulas and formulas without p, then $\varphi(p)$ has a fixed-point in **QGL**.

Let $\mathcal{L}''_Q$ be the language $\mathcal{L}_Q$ together with Boolean connectives $\vee, \wedge$, the existential quantifier $\exists$, and *countably infinite* propositional variables $p, q, \ldots$. We assume that an $\mathcal{L}''_Q$-formula $\varphi(p)$ may contain propositional variables other than p. Let $\mathbf{QGL}''$ be the natural extension of the system **QGL** to the language $\mathcal{L}''_Q$. It is easy to show that if an $\mathcal{L}''_Q$-formula φ is proved in $\mathbf{QGL}''$, then the $\mathcal{L}_Q$-formula obtained by substituting $\top$ for all propositional variables appearing in φ is proved in **QGL**. This shows that the system $\mathbf{QGL}''$ is a conservative extension of **QGL**. Thus in this section, we write simply **QGL** instead of $\mathbf{QGL}''$. Also it is easy to see that the substitution lemma (Lemma 13.5) is extended to the language $\mathcal{L}''$.

Definition 15.4 (Σ-formulas)**.** Σ-*formulas* are defined inductively as follows:

- An $\mathcal{L}''_Q$-formula of the form $\Box\psi$ is a Σ-formula;
- If ψ and χ are Σ-formulas, then $\psi \vee \chi$, $\psi \wedge \chi$ and $\exists u\psi$ are Σ-formulas.

If $\varphi(p)$ is a Σ-formula, then $\varphi(p)$ contains no occurrences of p of depth 0, and for any $\mathcal{L}''_Q$-formula ψ, the formula $\varphi(\psi)$ is also a Σ-formula.

Theorem 15.5. If $\varphi(p)$ is a Boolean combination of Σ-formulas and $\mathcal{L}''$-formulas containing no occurrences of p, then there exist an $\mathcal{L}''$-formula ξ such that F contains only predicate symbols, propositional variables, free variables occurring in $\varphi(p)$, not containing p, and such that $\mathbf{QGL} \vdash \xi \leftrightarrow \varphi(\xi)$.

Before proving the theorem, we give a definition and prove some lemmas.

Definition 15.6 (Self-provers)**.** An $\mathcal{L}''_Q$-formula φ is said to be *a self-prover* if $\mathbf{QGL} \vdash \varphi \to \Box\varphi$.

Lemma 15.7. The Boolean constant $\top$ and $\mathcal{L}''$-formulas of the form $\Box\varphi$ are self-provers. Moreover, the set of self-provers is closed under $\wedge, \vee, \exists$. Consequently, every Σ-formula is a self-prover.

Proof. Since $\mathbf{QGL} \vdash \top \to \Box\top$ and $\mathbf{QGL} \vdash \Box\varphi \to \Box\Box\varphi$, $\top$ and $\Box\varphi$ are self-provers. Suppose that φ and ψ are self-provers.

[3]This is a definition for predicate modal formulas, not for arithmetical formulas.

- Since φ and ψ are self-provers, $\mathbf{QGL} \vdash \varphi \wedge \psi \to \Box\varphi \wedge \Box\psi$. On the other hand, $\mathbf{QGL} \vdash \Box\varphi \wedge \Box\psi \to \Box(\varphi \wedge \psi)$. Thus we have $\mathbf{QGL} \vdash \varphi \wedge \psi \to \Box(\varphi \wedge \psi)$, and hence $\varphi \wedge \psi$ is a self-prover.
- Since $\mathbf{QGL} \vdash \varphi \to \varphi \vee \psi$, we have $\mathbf{QGL} \vdash \Box\varphi \to \Box(\varphi \vee \psi)$. Since φ is a self-prover, we get $\mathbf{QGL} \vdash \varphi \to \Box(\varphi \vee \psi)$. By a similar argument, $\mathbf{QGL} \vdash \psi \to \Box(\varphi \vee \psi)$. Thus, $\mathbf{QGL} \vdash \varphi \vee \psi \to \Box(\varphi \vee \psi)$, and hence $\varphi \vee \psi$ is a self-prover.
- Since $\mathbf{QGL} \vdash \varphi \to \Box\varphi$, we have $\mathbf{QGL} \vdash \exists u\varphi \to \exists u\Box\varphi$. On the other hand, from $\mathbf{QGL} \vdash \varphi \to \exists u\varphi$, we have $\mathbf{QGL} \vdash \Box\varphi \to \Box\exists u\varphi$, and hence $\mathbf{QGL} \vdash \exists u\Box\varphi \to \Box\exists u\varphi$. Thus, $\mathbf{QGL} \vdash \exists u\varphi \to \Box\exists u\varphi$, and hence $\exists u\varphi$ is a self-prover.

$\Box$

Lemma 15.8. Let φ and ψ be self-provers. If $\mathbf{QGL} \vdash \Box\varphi \to (\varphi \leftrightarrow \psi)$, then $\mathbf{QGL} \vdash \varphi \leftrightarrow \psi$.

Proof. Since φ is a self-prover, $\mathbf{QGL} \vdash \varphi \to \Box\varphi$. From this and the assumption, $\mathbf{QGL} \vdash \varphi \to (\varphi \leftrightarrow \psi)$, and hence $\mathbf{QGL} \vdash \varphi \to \psi$. On the other hand, by the assumption, $\mathbf{QGL} \vdash \psi \to (\Box\varphi \to \varphi)$, and hence $\mathbf{QGL} \vdash \Box\psi \to \Box(\Box\varphi \to \varphi)$. Applying the axiom $\mathbf{L}$, we get $\mathbf{QGL} \vdash \Box\psi \to \Box\varphi$. Since ψ is a self-prover, $\mathbf{QGL} \vdash \psi \to \Box\varphi$. From this and the assumption, $\mathbf{QGL} \vdash \psi \to (\varphi \leftrightarrow \psi)$, and hence $\mathbf{QGL} \vdash \psi \to \varphi$. Thus $\mathbf{QGL} \vdash \varphi \leftrightarrow \psi$. $\Box$

We assume that, by replacing variables appropriately, for any formula φ, the set of free variables of φ and the set of bound variables of φ are disjoint. (†)

Lemma 15.9. For any Σ-formula $\sigma(p)$, there is an $\mathcal{L}''_Q$-formula ξ containing only predicate symbols, propositional variables and free variables occurring in σ, not containing p, and such that $\mathbf{QGL} \vdash \xi \leftrightarrow \sigma(\xi)$.

Proof. Induction on the construction of $\sigma(p)$.

- Assume $\sigma(p) \equiv \Box\varphi(p)$. Then $\mathbf{QGL} \vdash \sigma(\top) \leftrightarrow (\top \leftrightarrow \sigma(\top))$. By the derivation of $\mathbf{QGL}$, we have

$$\mathbf{QGL} \vdash \Box\sigma(\top) \leftrightarrow \Box\left(\top \leftrightarrow \sigma(\top)\right). \tag{17}$$

Recall that $\sigma(p)$ contains no occurrences of p of depth 0, and there is no variable which occurs freely in $\sigma(\top)$ and is bounded in $\sigma(p)$. By the substitution lemma,

$$\mathbf{QGL} \vdash \Box\left(\top \leftrightarrow \sigma(\top)\right) \to \left(\sigma(\top) \leftrightarrow \sigma(\sigma(\top))\right).$$

From this and (17), we obtain $\mathbf{QGL} \vdash \Box\sigma(\top) \to (\sigma(\top) \leftrightarrow \sigma(\sigma(\top)))$. Since the formula $\sigma(p)$ is a Σ-formula, so are $\sigma(\top)$ and $\sigma(\sigma(\top))$. By Lemma 15.7, $\sigma(\top)$ and $\sigma(\sigma(\top))$ are self-provers. By Lemma 15.8, $\mathbf{QGL} \vdash \sigma(\top) \leftrightarrow \sigma(\sigma(\top))$.

- Assume $\sigma(p) \equiv \varphi(p) \wedge \psi(p)$, and let ξ and π be $\mathcal{L}''_Q$-formulas such that $\mathbf{QGL} \vdash \xi \leftrightarrow \varphi(\xi)$ and $\mathbf{QGL} \vdash \pi \leftrightarrow \psi(\pi)$. First, we have $\mathbf{QGL} \vdash (\xi \wedge \pi) \to (\xi \leftrightarrow (\xi \wedge \pi))$. By the derivation in $\mathbf{QGL}$, we get

$$\mathbf{QGL} \vdash \Box(\xi \wedge \pi) \to \Box\left(\xi \leftrightarrow (\xi \wedge \pi)\right). \tag{18}$$

  Note that all free variables occurring in ξ (or π) are free variables occurring in $\varphi(p)$ (or $\psi(p)$, resp.). By our supposition (†), no free variable occurring in ξ or $\xi \wedge \pi$ is bounded in $\sigma(p)$, i.e., bounded in $\varphi(p)$. By the substitution lemma,

$$\mathbf{QGL} \vdash \Box(\xi \leftrightarrow \xi \wedge \pi) \to (\varphi(\xi) \leftrightarrow \varphi(\xi \wedge \pi))\,.$$

  From this and (18), $\mathbf{QGL} \vdash \Box(\xi\wedge\pi) \to (\varphi(\xi) \leftrightarrow \varphi(\xi \wedge \pi))$. By $\mathbf{QGL} \vdash \xi \leftrightarrow \varphi(\xi)$, we obtain $\mathbf{QGL} \vdash \Box(\xi \wedge \pi) \to (\xi \leftrightarrow \varphi(\xi \wedge \pi))$. Similarly, we can derive $\mathbf{QGL} \vdash \Box(\xi \wedge \pi) \to (\pi \leftrightarrow \psi(\xi \wedge \pi))$. Thus, $\mathbf{QGL} \vdash \Box(\xi \wedge \pi) \to (\xi \wedge \pi \leftrightarrow \varphi(\xi \wedge \pi) \wedge \psi(\xi \wedge \pi))$, i.e., $\mathbf{QGL} \vdash \Box(\xi \wedge \pi) \to (\xi \wedge \pi \leftrightarrow \sigma(\xi \wedge \pi))$.

  We claim that ξ and π are self-provers. We show this only for ξ. Since $\varphi(\xi)$ is a Σ-formula, by Lemma 15.7, $\varphi(\xi)$ is a self-prover, and hence $\mathbf{QGL} \vdash \varphi(\xi) \to \Box\varphi(\xi)$. By the induction hypothesis, $\mathbf{QGL} \vdash \xi \leftrightarrow \varphi(\xi)$, and hence $\mathbf{QGL} \vdash \Box\xi \leftrightarrow \Box\varphi(\xi)$. Thus $\mathbf{QGL} \vdash \xi \to \Box\xi$.

  By Lemma 15.7, $\xi \wedge \pi$ is a self-prover. Since $\sigma(p)$ is a Σ-formula, and so is $\sigma(\xi \wedge \pi)$. By Lemma 15.7, $\sigma(\xi \wedge \pi)$ is a self-prover. By Lemma 15.8, $\mathbf{QGL} \vdash \xi \wedge \pi \leftrightarrow \sigma(\xi \wedge \pi)$.

- Assume $\sigma(p) \equiv \varphi(p) \vee \psi(p)$, and let ξ and π be $\mathcal{L}''_Q$-formulas such that $\mathbf{QGL} \vdash \xi \leftrightarrow \varphi(\xi)$ and $\mathbf{QGL} \vdash \pi \leftrightarrow \psi(\pi)$. First, we have $\mathbf{QGL} \vdash \xi \to (\xi \leftrightarrow \xi \vee \pi)$. Then

$$\mathbf{QGL} \vdash \Box\xi \to \Box(\xi \leftrightarrow \xi \vee \pi). \tag{19}$$

  Note that all free variables occurring in ξ (or π) are free variables occurring in $\varphi(p)$ (or $\psi(p)$, resp.). By our supposition (†), every free variable occurring in ξ or $\xi\vee\pi$ is not bounded in $\sigma(p)$, i.e., not bounded in $\varphi(p)$. By the substitution lemma,

$$\mathbf{QK4} \vdash \Box(\xi \leftrightarrow \xi \vee \pi) \to (\varphi(\xi) \leftrightarrow \varphi(\xi \vee \pi))\,.$$

From this and (19), $\mathbf{QK4} \vdash \Box\xi \to (\varphi(\xi) \leftrightarrow \varphi(\xi \vee \pi))$. By the induction hypothesis, $\mathbf{QGL} \vdash \Box\xi \to (\xi \leftrightarrow \varphi(\xi \vee \pi))$. Note that ξ and $\varphi(\xi \vee \pi)$ are self-provers. By Lemma 15.8, $\mathbf{QGL} \vdash \xi \leftrightarrow \varphi(\xi \vee \pi)$. Similarly, we can derive $\mathbf{QGL} \vdash \pi \leftrightarrow \psi(\xi \vee \pi)$. Thus $\mathbf{QGL} \vdash \xi \vee \pi \leftrightarrow \varphi(\xi \vee \pi) \vee \psi(\xi \vee \pi)$, i.e., $\mathbf{QGL} \vdash \xi \vee \pi \leftrightarrow \sigma(\xi \vee \pi)$.

- Assume $\sigma(p) \equiv \exists u\varphi(u)$, and let ξ be an $\mathcal{L}''_Q$-formula such that $\mathbf{QGL} \vdash \xi \leftrightarrow \varphi(\xi)$. Since $\mathbf{QGL} \vdash \xi \to (\xi \leftrightarrow \exists u\xi)$, we have $\mathbf{QGL} \vdash \Box\xi \to \Box(\xi \leftrightarrow \exists u\xi)$. Note that no free variable occurring in ξ or $\exists u\xi$ is bounded in $\varphi(p)$. By the substitution lemma, $\mathbf{QGL} \vdash \Box\xi \to (\varphi(\xi) \leftrightarrow \varphi(\exists u\xi))$. By the induction hypothesis, $\mathbf{QGL} \vdash \Box\xi \to (\xi \leftrightarrow \varphi(\exists u\xi))$. Recall that ξ and $\exists u\xi$ are self-provers. By Lemma 15.8, $\mathbf{QGL} \vdash \xi \leftrightarrow \varphi(\exists u\xi)$, and hence $\mathbf{QGL} \vdash \exists u\xi \leftrightarrow \exists u\varphi(\exists u\xi)$, i.e., $\mathbf{QGL} \vdash \exists u\xi \leftrightarrow \sigma(\exists u\xi)$.

$\Box$

Lemma 15.10. For any Σ-formulas $\sigma_0(p_0, \ldots, p_n), \ldots, \sigma_n(p_0, \ldots, p_n)$, there are $\mathcal{L}''_Q$-formulas $\xi_0, \ldots, \xi_n$ satisfying the desired properties such that for any $i \leq n$, $\mathbf{QGL} \vdash \xi_i \leftrightarrow \sigma_i(\xi_0, \ldots, \xi_n)$.

Proof. We prove by the induction on n. If $n = 0$, then it follows from Lemma 15.9. Assume Lemma holds for $\leq n$. Let

$$\sigma_0(p_0, \ldots, p_{n+1}), \ldots, \sigma_{n+1}(p_0, \ldots, p_{n+1})$$

be Σ-formulas. By the induction hypothesis, there are $\mathcal{L}''_Q$-formulas

$$\xi_0(p_{n+1}), \ldots, \xi_n(p_{n+1})$$

such that for any $i \leq n$, $\mathbf{QGL} \vdash \xi_i(p_{n+1}) \leftrightarrow \sigma_i\left(\xi_0(p_{n+1}), \ldots, \xi_n(p_{n+1}), p_{n+1}\right)$. Let ξ be an $\mathcal{L}'$-formula such that $\mathbf{QGL} \vdash \xi \leftrightarrow \sigma_{n+1}\left(\xi_0(\xi), \ldots, \xi_n(\xi), \xi\right)$. (The existence of such an ξ is guaranteed by Lemma 15.9.) Then for any $i \leq n$, $\mathbf{QGL} \vdash \xi_i(\xi) \leftrightarrow \sigma_i\left(\xi_0(\xi), \ldots, \xi_n(\xi), \xi\right)$. Therefore, $\langle \xi_0(\xi), \ldots, \xi_n(\xi), \xi \rangle$ are desired formulas. The proof of the case $n + 1$ is completed. $\Box$

Finally, we prove Theorem 15.5.

Proof of Theorem 15.5. Let $\varphi(p)$ be a Boolean combination of Σ-formulas and formulas containing no occurrences of p. Then there are a propositional formula $\psi(q_0, \ldots, q_{n-1}, r_0, \ldots, r_{m-1})$, Σ-formulas $\sigma_0(p), \ldots, \sigma_{n-1}(p)$, and $\mathcal{L}''_Q$-formulas $\chi_0, \ldots, \chi_{m-1}$ containing no occurrences of p, such that

$$\varphi(p) \equiv \psi\left(\sigma_0(p), \ldots, \sigma_{n-1}(p), \chi_0, \ldots, \chi_{m-1}\right).$$

For each $i < n$, put $\varphi_i(q_0, \ldots, q_{n-1}) :\equiv \sigma_i(\psi(q_0, \ldots, q_{n-1}, \chi_0, \ldots, \chi_{m-1}))$. By Lemma 15.10, there are $\xi_0, \ldots, \xi_{n-1}$ such that for each $i < n$, $\mathbf{QGL} \vdash \xi_i \leftrightarrow \varphi_i(\xi_0, \ldots, \xi_{n-1})$. Let $\xi :\equiv \psi(\xi_0, \ldots, \xi_{n-1}, \chi_0, \ldots, \chi_{m-1})$. Then we have $\mathbf{QGL} \vdash \xi_i \leftrightarrow \sigma_i(\xi)$, and hence

$$\mathbf{QGL} \vdash \xi \leftrightarrow \psi(\sigma_0(\xi), \ldots, \sigma_{n-1}(\xi), \chi_0, \ldots, \chi_{m-1}),$$

i.e., $\mathbf{QGL} \vdash \xi \leftrightarrow \varphi(\xi)$. $\square$

Chapter VI
Concluding remarks

We close this dissertation with some further problems of our studies.

In the beginning of Section 8.2, we described that the arithmetical completeness of $\mathbf{LP}_0$ does not hold with only the Gödel multi-conclusion proof predicate Proof. We showed that the arithmetical completeness of $\mathbf{LP}_0$ holds with respect to a modified version of Artemov's Δ_1 normal proof predicate Prf (Theorem 8.6). Moreover, we also proved that there exist a Σ_1 proof predicate Prf, Prf-functions $\langle \mathbf{m}, \mathbf{a}, \mathbf{c} \rangle$ and an arithmetical interpretation $*$ based on Prf such that for any $\mathbf{LP}$-formula F, $\mathbf{LP}_0 \vdash F$ if and only if $\mathsf{PA} \vdash F^*$ (Theorem 9.2). However, as mentioned in Remark 9.14, the statement of Theorem 9.2 is incomplete as compared to one of the so-called uniform arithmetical completeness theorem.

Problem 15.11.

1. Does the uniform arithmetical completeness of $\mathbf{LP}_0$ hold with respect to some Σ_1 *normal* proof predicate?

2. Does the uniform arithmetical completeness of $\mathbf{LP}_0$ hold with respect to some Σ_1 proof predicate for which arithmetical soundness of $\mathbf{LP}_0$ holds?

In Chapter IV we proved the interpolation properties for Sacchetti's logics $\mathbf{wGL}_n$, and the effectiveness of interpolants. However, we used Kripke semantics in the proof of the cut-admissibility for $\mathbf{wGL}_n^{\mathbf{G}}$. Therefore our argument is not purely syntactical at this time.

Problem 15.12. Can we prove the cut-elimination theorem for $\mathbf{wGL}_n^{\mathbf{G}}$ syntactically? In particular, for a given proof π of Γ in $\mathbf{wGL}_n^{\mathbf{G}} + (cut)$, is there an effective way to obtain a proof π' of Γ without the rule (cut)?

Shamkanov [24] used the multi-set based sequent calculus which satisfies the admissibility of structural rules. It is under consideration whether our proof works even in a multi-set setting. The proof transforming procedure described in Section 12.1 does not take care of degrees of formulas. Furthermore, the existence of the rule $(weak)$ is essential in the proof of Theorem 12.2. For these reasons, our proof transforming procedure needs to a little change in a multi-set setting.

In Chapter V we discussed semantical fixed-point properties for classes of Kripke frames. The following table summarizes the situation of these properties.

Table 2: Five classes and the fixed-point properties

class	FPP		local FPP	
FIFD	**No**	(Theorem 14.6)	**Yes**	(Corollary 14.16)
FI	**No**	(Corollary 14.7)	**Yes**	
FH	**No**		**Yes**	
BL	No	(Corollary 13.4.1)	No	(Corollary 13.4.1)
CW	No		No	

In Section 14.1, we proved that the class FIFD does not have the fixed-point property (Theorem 14.6). Corollary 14.16 shows that **MQ**(FIFD) is consistent with the fixed-point property, that is, there exists a consistent extension of **MQ**(FIFD) for which the fixed-point theorem holds. In Section 13.1, we mentioned that **MQ**(BL) equals to **MQ**(FH), and thus the classes BL and FH are not distinguished by the validity of formulas. On the other hand, BL does not have the local fixed-point property (Corollary 13.4.1), and FH has the one (Corollary 14.16). Hence we can capture some a logical difference between BL and FH through the local fixed-point property.

Montagna [17] raised some questions about fixed-points in **QGL**: (1) Can we find a procedure for deciding if a formula $\varphi(p)$ has a fixed-point in **QGL**? (2) Does it exist a procedure for calculating the possible fixed-points of a given formula $\varphi(p)$? These problems have not been settled completely yet.

Problem 15.13. Is there a formula $\varphi(p)$ satisfying the following conditions?

- $\varphi(p)$ is modalized in p;
- $\varphi(p)$ is not provably equivalent to any Boolean combination of Σ-formulas and formulas containing no occurrences of p:
- $\varphi(p)$ has a fixed-point in **QGL**.

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