Some results on modal logics having arithmetical interpretations

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論文内容の要旨

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Some results on modal logics having arithmetical interpretations

(算術的解釈を持つ様相論理に関する諸結果について)

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Abstract

In this dissertation, we investigate several properties for the following logics having arithmetical interpretations, the modal logic **GL**, Sacchetti's logics $\mathbf{wGL}_n$, and Artemov's logic of proofs $\mathbf{LP}_0$. The dissertation is divided into three parts. In the first part, we prove stronger versions of the arithmetical completeness theorem of $\mathbf{LP}_0$. In the second part, we study interpolation properties for Sacchetti's logic $\mathbf{wGL}_n$. An effective procedure which calculates Lyndon interpolants in $\mathbf{wGL}_n$ is also given. In the final part, we discuss semantical fixed-point properties for predicate modal logics containing **QGL**, which is the natural predicate extension of **GL**.

The modal logic **GL** is obtained from **K** by adding an axiom $\Box(\Box\varphi \to \varphi) \to \Box\varphi$. This logic enjoys two significant properties, the arithmetical completeness and the fixed-point property.

Modal formulas can be interpreted into first-order arithmetical sentences of formal arithmetic, for example, Peano Arithmetic PA. An arithmetical interpretation of modal formula is defined as a mapping $*$ from all propositional variables into arithmetical sentences. In particular, the modal operator $\Box$ is interpreted as $\mathsf{Bew}(x)$ where $\mathsf{Bew}(x)$ is the standard provability predicate of PA. The provability logic of PA is the set of all modal formulas φ satisfying $\mathsf{PA} \vdash \varphi^*$ for any arithmetical interpretation $*$.

Solovay (1976) established the arithmetical completeness theorem of **GL**. It asserts that, **GL** coincides with the logic of provability of PA, i.e., for any modal formula φ, $\mathbf{GL} \vdash \varphi$ if and only if $\mathsf{PA} \vdash \varphi^*$ for any arithmetical interpretation $*$. Thus **GL** captures some properties of the provability predicate $\mathsf{Bew}(x)$. Moreover, the uniform arithmetical completeness theorem also holds for **GL**. That is, there is a fixed arithmetical interpretation $*$ such that for any modal formula φ, $\mathbf{GL} \vdash \varphi$ if and only if $\mathsf{PA} \vdash \varphi^*$.

De Jongh and Sambin (1976) independently proved the fixed-point theorem for **GL**. Let $\varphi(p)$ be a modal formula containing p. We say $\varphi(p)$ is modalized in p if all occurrences of the propositional variable p in $\varphi(p)$ are within the scope of the modal operator. The fixed-point theorem states that if $\varphi(p)$ is modalized in p, then there is a modal formula ψ containing only propositional variables occurring in $\varphi(p)$ without p and such that $\mathbf{GL} \vdash \psi \leftrightarrow \varphi(\psi)$. Moreover, effective procedures of constructing

fixed-points in **GL** has been studied by several researchers.

1 Logic of Proofs

A proof predicate is a formula $\mathsf{Prf}(x, y)$ which represents the explicit provability of formulas in PA. The formula $\mathsf{Prf}(x, y)$ intuitively means "there exists a proof in PA with the code (the Gödel number) x of the formula with the code y." For a proof predicate $\mathsf{Prf}(x, y)$, we call a Σ_1 formula $\mathsf{Pr}(x) \equiv \exists y \mathsf{Prf}(y, x)$ a provability predicate.

Artemov (2001) developed the Logic of Proofs, which analyzes proof predicates in **PA**. The logic of proofs deals with **LP**-formulas, especially formulas of the form $t : F$, where t is called a proof term. An arithmetical interpretation of **LP**-formulas is defined as a collection of mapping $*$ and functions from proof terms to natural numbers. The intended meaning of $t : F$ is "t is a (code of a) proof of F."

Artemov proved the arithmetical completeness theorem of $\mathbf{LP}_0$: for any **LP**-formula F, $\mathbf{LP}_0 \vdash F$ if and only if for any Δ_1 normal proof predicate $\mathsf{Prf}(x, y)$ and any arithmetical interpretation $*$ based on Prf, $\mathsf{PA} \vdash F^*$.

Technically, there is a substantial difference between Solovay's theorem and Artemov's theorem. The arithmetical completeness theorem of **GL** holds for each canonical provability predicate. On the other hand, in the case of $\mathbf{LP}_0$ the arithmetical completeness theorem does not hold with only the standard proof predicate $\mathsf{Proof}(x, y)$. Moreover, it is not known whether the uniform arithmetical completeness theorem holds for $\mathbf{LP}_0$.

We examine the following problems: (i) Does the arithmetical completeness theorem for $\mathbf{LP}_0$ hold with respect to some fixed proof predicate? (ii) Does the uniform arithmetical completeness theorem for $\mathbf{LP}_0$ hold?

For these problems, we prove the following two statements:

(i) There exists a normal Δ_1 proof predicate $\mathsf{Prf}(x, y)$ such that for any **LP**-formula F, $\mathbf{LP}_0 \vdash F$ if and only if for any arithmetical interpretation $*$ based on Prf, $\mathsf{PA} \vdash F^*$;

(ii) There exist a Σ_1 (but not normal) proof predicate $\mathsf{Prf}(x, y)$ and an arithmetical interpretation $*$ based on Prf such that for any **LP**-formula F, $\mathbf{LP}_0 \vdash F$ if and only $\mathsf{PA} \vdash F^*$.

2 Interpolation properties

A logic **L** is said to have the Craig interpolation property if for any formula $\varphi \to \psi$ which is provable in **L**, there exists a formula θ (called an interpolant of $\varphi \to \psi$) such that θ consists of common variables of φ and ψ, and satisfies $\mathbf{L} \vdash \varphi \to \theta$ and $\mathbf{L} \vdash \theta \to \psi$. A logic **L** is said to have the Lyndon interpolation property if for any provable implication $\varphi \to \psi$, there is a stronger interpolant θ which preserves the positivity of variables, that is, every positive (negative) occurrence of a variable also occurs both in φ and ψ positively (resp. negatively).

In **GL**, there is a close connection between the fixed-point properties and the interpolation properties, since the following facts:

(i) The fixed-point theorem for **GL** can be derived from the Craig interpolation property for the logic;

(ii) Using the effective fixed-point theorem, we can prove the effective Lyndon interpolation property for **GL**.

Proofs of the Craig interpolation property for **GL** and the fact (i) are independently given by Boolos (1979) and Smoryński (1978).

It had been opened whether the Lyndon interpolation property posses for **GL** until Shamkanov solved in 2011. He proved the Lyndon interpolation property for **GL** by a modified version of Smoryński's semantical argument, without applying the fixed-point theorem. Later in 2014 he also proved the fact (ii) by using a cut-free sequent calculus for **GL**. A benefit of Shamkanov's second proof of the Lyndon interpolation property is that, from $\varphi \to \psi$, we can effectively construct a Lyndon interpolant θ of $\varphi \to \psi$ whenever $\varphi \to \psi$ is provable in **GL**.

In the proof of Shamkanov's second result, he also introduced a circular proof system. A circular proof system $^\circ\mathbf{L}$ of **L** is one which has the same axioms and rules of **L** and admits "circular proofs." A circular proof is a derivation tree of **L** whose leaves are either axioms of **L** or identical to a sequent below that leaf. Shamkanov showed that **GL** is provably equivalent to the circular proof system $^\circ\mathbf{K4}$. He gave an effective way of constructing a Lyndon interpolant of $\varphi \to \psi$ by using $^\circ\mathbf{K4}$ and the effective fixed-point theorem for **GL**.

In Chapter IV, we discuss the interpolation properties for Sacchetti's logics $\mathbf{wGL}_n$, which are proper fragments of $\mathbf{GL}$.

Sacchetti (2001) studied modal logics having the fixed-point property. In particular, he introduced a new modal logic $\mathbf{wGL}_n$. The logic $\mathbf{wGL}_n$ is obtained from $\mathbf{GL}$ by replacing the axiom $\Box(\Box\varphi \to \varphi) \to \Box\varphi$ by $\Box(\Box^n\varphi \to \varphi) \to \Box\varphi$, where n is a nonzero natural number, and $\Box^n\varphi$ denotes $\overbrace{\Box\cdots\Box}^{n}\varphi$.

Sacchetti's logics $\mathbf{wGL}_n$ have several properties like $\mathbf{GL}$. Originally Sacchetti (2001) showed that $\mathbf{wGL}_n$ enjoys all the Kripke completeness, the Craig interpolation property. Moreover, he proved the fixed-point theorem for $\mathbf{wGL}_n$. Later Kurahashi (2018) proved the arithmetical completeness theorem for $\mathbf{wGL}_n$ with respect to a Σ_2 provability predicate.

It is expected that $\mathbf{wGL}_n$ posses the Lyndon interpolation property, however, this conjecture has not been clarified.

We develop two one-sided sequent calculi $\mathbf{wGL}_n^{\mathbf{G}}$ and $\mathbf{wK4}_n^{\mathbf{G}}$, and prove the following results:

(i) The calculus $\mathbf{wGL}_n^{\mathbf{G}}$ is provably equivalent to the circular proof system $^\circ\mathbf{wK4}_n^{\mathbf{G}}$;

(ii) Using $^\circ\mathbf{wK4}_n^{\mathbf{G}}$ and the effective fixed-point theorem for $\mathbf{wGL}_n$ (Kurahashi and Okawa), we can construct a Lyndon interpolant of $\varphi \to \psi$ in $\mathbf{wGL}_n$ whenever $\varphi \to \psi$ is provable.

Iemhoff (2016) studied some sufficient conditions for a type of modal sequent calculus to have an equivalent circular proof system. Although the calculus $\mathbf{wGL}_n^{\mathbf{G}}$ does not enjoy Iemhoff's conditions, it has an equivalent circular proof counterpart.

3 Fixed-point properties in predicate logic of provability

It is natural to extend the studies of the logic of provability to a predicate modal logic. However, the situation of the predicate logic of provability is quite complex and most of the properties for $\mathbf{GL}$ do not hold for the predicate modal system $\mathbf{QGL}$, which is the natural predicate extension of $\mathbf{GL}$. In particular, Montagna (1984) proved that $\mathbf{QGL}$ enjoys neither the Kripke completeness, nor the arithmetical completeness.

He also showed the failure of the fixed-point theorem for $\mathbf{QGL}$, that is, he found a predicate modal formula $\varphi(p)$ which has no fixed-points in $\mathbf{QGL}$.

On the other hand, there is a room for investigations of the fixed-point properties in predicate modal logics. The logic $\mathbf{QGL}$ is not only the candidate of an extension of $\mathbf{GL}$. Recently Tanaka (2018) introduced a new predicate modal logic $\mathbf{NQGL}$, which is strictly stronger than $\mathbf{QGL}$ and enjoys the Kripke completeness with respect to a proper subclass of transitive and conversely well-founded Kripke frames. There is a possibility that the fixed-point theorem holds for some extension of $\mathbf{QGL}$.

Sacchetti (1999) showed the fixed-point theorem for the modal logic $\mathbf{K} + \Box^{n+1}\bot$. Also it has not been known that the fixed-point theorem even holds for the predicate extension of this logic.

In Chapter V we discuss some versions of the fixed-point properties for predicate modal logics. We define the following classes of Kripke frames in which all theorems of $\mathbf{QGL}$ are valid: CW (the class of transitive and conversely well-founded frames), FH (the class of transitive frames with finite height), FI (the class of finite transitive irreflexive frames) and FIFD (the class of finite transitive irreflexive frames of which domains are finite). The class FH is a proper subclass of BL (the class of transitive of which are bounded length), which is introduced by Tanaka (2018). The Tanaka's system $\mathbf{NQGL}$ is Kripke complete with respect to the class BL. The class FIFD was originally investigated by Artemov and Dzhaparidze (1990).

We study two semantical fixed-point properties for a class of Kripke frames, the fixed-point property and the local fixed-point property. By Montagna's result, it follows that the classes CW and BL enjoy neither the fixed-point property nor the local fixed-point property. We discuss whether the classes FH, FI and FIFD enjoy these two properties. We prove the following results:

(i) The classes FH, FI and FIFD do not enjoy the fixed-point property;

(ii) We prove the fixed-point theorem for the predicate modal logic $\mathbf{QK} + \Box^{n+1}\bot$. An algorithm for calculating fixed-points in $\mathbf{QK} + \Box^{n+1}\bot$ is given in the proof. Consequently, we obtain that the classes FH, FI and FIFD enjoy the local fixed-point property.

As a consequence, we prove that Tanaka's system $\mathbf{NQGL}$ does not enjoy the Craig interpolation property.

As mentioned above, Montagna showed that the fixed-point theorem does not hold for **QGL**. Although there is a possibility that the fixed-point theorem holds for some classes of formulas. It has not been known sufficient (or necessary) conditions for a formula to have a fixed-point in **QGL**. In the end of Chapter V, we investigate these conditions. We prove that if $\varphi(p)$ is a Boolean combination of Σ-formulas, then $\varphi(p)$ has a fixed-point in **QGL**.

(別紙１)

論文審査の結果の要旨

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論文題目	Some results on modal logics having arithmetical interpretations (算術的解釈を持つ様相論理に関する諸結果について)		
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要　　旨

GL は形式的算術(例えばペアノ算術 PA)で定義可能な証明可能性述語を□とみなす様相論理である．様相論理式から算術の論理式への，証明可能性述語を介した変換を算術的解釈と呼ぶ．Solovay は 1976 年，GL で φ が証明できることと，φ をどのように算術的解釈によって解釈してもそれが PA で証明できることは同値であることを示した（算術的完全性）．これにより，GL が証明可能性述語を扱う様相論理として妥当なことが示された．

また，GL は不動点性質と呼ばれる論理的性質も持っている．φ(p) において，p の出現がすべて□の内部に現れているならば，φ の命題変数から p を除いたものからなる ψ で GL⊢ψ↔φ(ψ) を満たす自己言及的な論理式（不動点）が存在することが，de Jongh や Sambin らによって証明された（GL の不動点定理）．これらの結果を踏まえ，本論文では GL と同様に算術と関わりのある 3 つの論理についてそれぞれ分析がなされた．

まず第 III 章では， Logic of proofs LP_0 の算術的完全性について考察された．Logic of proofs は，論理式の証明可能性だけでなく，具体的にどのように証明されるかを構成的に捉えるために Artemov が導入した特殊な様相論理である．Logic of proofs は □ の代わりに 3 種のオペレータで構成される証明項を用い，証明項つきの論理式(LP 論理式)を扱う．PA で「x は y を証明する」を意味する証明述語を任意に選び，その下で LP 論理式を解釈したとき，LP_0 は GL と同様に算術的完全性が成り立つことが Artemov によって示された．今回，Artemov の結果を更に改良し，ある種の証明述語を固定したときにも LP_0 が算術的完全性を持つことを示す．更に Artemov の議論での証明述語の条件を緩めると，LP_0 で一様的な算術的完全性が成り立つことが示された．

第 IV 章では，Sacchetti によって発見された新しい様相論理 wGL_n の補完性質についての結果を述べられている．wGL_n は GL の部分論理でありながら GL といくつかの類似する性質を持つ．Sacchetti や Kurahashi らにより，wGL_n が不動点性質や算術的完全性を満たすことが示された．一方で，リンドン補間性と呼ばれる強い補間性質を持つことが，GL の場合には Shamkanov によって示されたが，wGL_n の場合にはまだ明らかになっていない．そこで，Shamkanov のリンドン補間性の証明法を応用し，wGL_n と同じ証明能力を持つ circular proof system という特殊な証明体系を用いることで，wGL_n がリンドン補間性を持つことが証明された．更に具体的な補間論理式を構成する方法が与えられた．

第 V 章では，GL の述語拡大での不動点性質について議論されている．GL を述語様相論理に拡張した QGL では，GL で成り立っていたほとんどの性質が失われてしまう．QGL は算術的完全性も不動点性質も満たさないことが Montagna によって示されている．そこで，QGL よりも広い，クリプキ意味論によって特徴づけられる論理式のクラスを考え，そこで不動点性質が成り立つかどうかが分析された．いくつかのクラスにおいて意味論的不動点性質が成り立たないが，一方で局所的には不動点の性質を持つことが示された．最後に，QGL で φ(p) が不動点を持つための十分条件についても述べられている．

本研究は，算術的解釈を持つ様相論理関する幾つかの定理を証明し，数理論理学について重要な知見を得たものとして価値のあるものである．提出された論文はシステム情報学研究科学位論文評価基準を満たしており，学位申請者の岩田荘平は，博士（学術）の学位を得る資格があると認める．