Nef cone of a generalized Kummer 4-fold

Mori, Akira

(Degree)
博士（理学）
(Date of Degree)
2021-03-25
(Date of Publication)
2022-03-01
(Resource Type)
doctoral thesis
(Report Number)
甲第8000号
(URL)
https://hdl.handle.net/20.500.14094/D1008000

博　士　論　文

Nef cone of a generalized Kummer 4-fold

(一般化されたクンマー 4 次元多様体のネフ錐について)

令和 3 年 3 月

神戸大学大学院理学研究科

森 章

Nef cone of a generalized Kummer 4-fold

Akira Mori

March 2, 2021

Abstract

Let A be an abelian surface of Picard number 1 In [Yo6], Yoshioka gave a lattice theoretic description of the movable cone $\mathrm{Mov}(\mathrm{Km}^{l-1}(A))$ and the nef cone $\mathrm{Nef}(\mathrm{Km}^{l-1}(A))$ of a generalized Kummer variety $\mathrm{Km}^{l-1}(A)$ $(l \geq 3)$. In this thesis, we shall give a more concrete description of $\mathrm{Nef}(\mathrm{Km}^2(A))$ and $\mathrm{Mov}(\mathrm{Km}^2(A))$. We also describe the chamber decomposition for $\mathrm{Mov}(\mathrm{Km}^2(A))$.

Contents

0 Introduction.

The higher dimensional varieties with trivial canonical bundle over $\mathbb{C}$ are important class in birational classification theory. They are important varieties not only in algebraic geometry, but also mathematical physics. For example, we can see importance in the theory of elementary particles and mirror symmetry, and so on. By Bogomolov's decomposition theory ([Bo]), such varieties consist of the three classes, which are a complex torus, a Calabi-Yau manifold and an irreducible symplectic manifold. In this thesis, we focus on an irreducible symplectic manifold. Let M be an irreducible symplectic manifold, i.e., M is a compact Kähler manifold which satisfies 1) M is simply connected, 2) there is a nondegenerate holomorphic 2-form $\sigma \in H^0(M, \Omega_M^2)$ such that $H^0(M, \Omega_M^2)$ is one dimensional, spanned by σ. By the existence of a nondegenerate holomorphic 2-form, we note that M is even dimensional. An irreducible symplectic manifold M has been introduced as a higher dimensional analogue of $K3$ surface. Actually, it is known that M is a $K3$ surface if $\dim M = 2$. What kind of irreducible symplectic manifolds are there in dimension 4 ? For them, a Hilbert scheme of two points on a $K3$ surface and a generalized Kummer 4-fold are known. In general, a Hilbert scheme of points on a $K3$ surface, a generalized Kummer manifold ([Be]) and O'Grady's examples ([O2, O3]) are known as the examples of an irreducible symplectic manifold.

Definition 0.1. ([Be]) Let A be an abelian surface over $\mathbb{C}$ and $S^l(A)$ be the l-th symmetric product of A. Then a generalized Kummer manifold $\mathrm{Km}^{l-1}(A)$ is defined by the fiber of the composite map $\mathrm{Hilb}_A^l \overset{\varphi}{\to} S^l(A) \overset{\phi}{\to} A$, where φ is the Hilbert-Chow map and ϕ is the additive map $(x_1, ..., x_l) \mapsto \sum x_i$.

Since a $K3$ surface is the important class which has many beautiful properties, it is natural to study an irreducible symplectic manifold M. In fact, M shares many of the well known properties of $K3$ surfaces ([Hu1]). In particular, the second cohomology $H^2(M, \mathbb{Z})$ carries a natural weight 2 Hodge structure and, due to [Be], $H^2(M, \mathbb{Z})$ can be endowed with a nondegenerate quadratic form B_M generalizing the intersection pairing of a $K3$ surface.

On the other hand, there are relations between an irreducible symplectic manifold and a moduli space of stable sheaves on a surface with a trivial canonical bundle. Let X be an (algebraic) $K3$ surface or an abelian surface and H be an ample divisor on X. For a coherent sheaf $E \in \mathrm{Coh}(X)$, we take a Mukai vector $v(E) = \mathrm{ch}(E)\sqrt{\mathrm{td}(X)}$. This Mukai vector $v(E)$ is the element of an (algebraic) Mukai lattice $H^*(X, \mathbb{Z})_{\mathrm{alg}} := \mathbb{Z} \oplus \mathrm{NS}(X) \oplus \mathbb{Z}$ with a symmetric bilinear form $\langle\ ,\ \rangle$ on $H^*(X, \mathbb{Z})_{\mathrm{alg}}$ which is called a Mukai pairing, where $\mathrm{NS}(X)$ is a Néron-Severi group of X ([Mu4]). Here a Mukai pairing $\langle\ ,\ \rangle$ is defined by

$$\langle x, y\rangle := (x_1, y_1) - x_0 y_2 - x_2 y_0$$

for $x = (x_0, x_1, x_2), y = (y_0, y_1, y_2) \in H^*(X, \mathbb{Z})_{\mathrm{alg}}$. Moreover we write $\langle v^2\rangle := \langle v, v\rangle$. If a Mukai vector v satisfies $\langle v^2\rangle = 0$, then we call v isotropic.

Definition 0.2. A Mukai vector $v = (r, \xi, a)$ is primitive if $\gcd(r, \xi, a) = 1$. A Mukai vector v is positive (we write $v > 0$) if $r > 0$, or $r = 0, \xi \neq 0$ is effective and $a \neq 0$ or $r = \xi = 0$ and $a > 0$.

A moduli space of stable sheaves on X is constructed by using the Gieseker-Maruyama stability and the geometric invariant theory (cf. [HL]). We denote the moduli space of stable sheaves E of $v(E) = v$ by $M_H(v)$. The moduli space $M_H(v)$ is studied by many authors. In particular, Mukai showed that $M_H(v)$ is a smooth variety of dimension $\langle v^2\rangle + 2$ and has a symplectic structure in [Mu3]. Let v be a primitive Mukai vector with $v > 0$. For $K3$ surface S, $M_H(v)$ is deformation equivalent to the Hilbert scheme of points $\mathrm{Hilb}_S^{\langle v^2\rangle/2+1}$ on S ([O1, Yo1]). In particular, $M_H(v)$ is an irreducible symplectic manifold. However, for an abelian surface A, the situation is different. If a primitive Mukai vector v satisfies $v > 0$ and $\langle v^2\rangle \geq 2$, then the moduli space $M_H(v)$ is deformation equivalent to $\hat{A} \times \mathrm{Hilb}_A^{\langle v\rangle^2/2}$, where $\hat{A}$ is the dual of A ([Yo2]). Hence $M_H(v)$ is not irreducible. A morphism

$$\begin{array}{cccc} \mathfrak{a}_v : & M_H(v) & \to & A \times \hat{A} \\ & E & \mapsto & (\det(\Phi_{\mathcal{P}}(E)), \det(E)) \end{array}$$

is the Albanese map ([Yo2]), where $\mathcal{P}$ is the Poincaré line bundle on $A \times \hat{A}$ and $\Phi_{\mathcal{P}}$ is the Fourier-Mukai transform ([Mu2]).

Theorem 0.3. ([Yo2, Theorem 0.2]) We assume that $\langle v^2\rangle \geq 6$ and H is a general ample divisor with respect to v. Let $K_H(v)$ be a fiber of the Albanese map $\mathfrak{a}_v$. Then $K_H(v)$ is deformation equivalent to the generalized Kummer manifold with dimension $\langle v^2\rangle - 2$ which is constructed by Beauville (Definition 0.1). Moreover the Mukai homomorphism $\theta_v : (v^\perp, \langle\ ,\ \rangle) \to (H^2(K_H(v),\mathbb{Z}), B_{K_H(v)})$ is a Hodge isometry.

Bridgeland introduced the notation of a stability condition on a triangulated category $\mathcal{T}$ and proved the set $\mathrm{Stab}_\Gamma(\mathcal{T})$ consisting of stability conditions such that they satisfy the support property and central charge factors through a finitely generated free module Γ is a complex manifold ([Br1]). Hence, for a smooth projective variety X over $\mathbb{C}$, we can treat the stability on a bounded derived category $\mathcal{D}(X) := \mathcal{D}^b(\mathrm{Coh}(X))$ of coherent sheaves on X. We set $\mathrm{Stab}(X) := \mathrm{Stab}_\Gamma(\mathcal{D}(X))$. For a stability condition $\sigma \in \mathrm{Stab}(X)$, σ is a pair of a bounded t-structure on $\mathcal{D}(X)$ and a stability function Z on its heart $\mathcal{A}$ with the Harder-Narasimhan property ([Br1]). In general, to construct Bridgeland's stability condition and to study the structure of $\mathrm{Stab}(X)$ are difficult problems. But it is interesting to study them from the point of view of mathematical physics and generalization of stability of sheaves. In fact, Bridgeland's stability on Calabi-Yau manifold is related to mirror symmetry, and twisted stability is the special case of Bridgeland's stability at the large volume limit. In [Br2], by tilting with respect to the torsion pair, Bridgeland constructed the stability condition $\sigma_{(\beta,\omega)} = (Z_{(\beta,\omega)}, \mathcal{A}_{(\beta,\omega)})$ on a $K3$ surface and an abelian surface X associated with $(\beta,\omega) \in \mathrm{NS}(X)_\mathbb{R} \times \mathrm{Amp}(X)_\mathbb{R}$. Then for $\beta \in \mathrm{NS}(X)_\mathbb{R}$, β-twisted stability conditions give many examples of a bounded t-structure. Moreover he defined the wall-chamber structure for $\mathrm{Stab}(X)$. Then wall-crossing behavior is studied in [BM, MYY1, MYY2]. In particular, Minamide, Yanagida and Yoshioka characterized the walls for Bridgeland's stability conditions on an abelian surface A.

Definition 0.4. ([MYY1]) Let v be a Mukai vector. For a Mukai vector u satisfying the inequalities

$$\langle u, v-u\rangle > 0,\ \langle u^2\rangle \geq 0,\ \langle (v-u)^2\rangle \geq 0,\ \langle v,u\rangle^2 > \langle v^2\rangle\langle u^2\rangle,$$

we define the wall W_u as

$$W_u := \{(\beta,\omega) \in \mathrm{NS}(A)_\mathbb{R} \times \mathrm{Amp}(A)_\mathbb{R} \mid \mathbb{R}Z_{(\beta,\omega)}(u) = \mathbb{R}Z_{(\beta,\omega)}(v)\}.$$

Γ denotes the set of Mukai vectors u satisfying the above inequalities. We call a connected component

$$\mathcal{C} \subset (\mathrm{NS}(A)_\mathbb{R} \times \mathrm{Amp}(A)_\mathbb{R}) \setminus \bigcup_{u\in\Gamma} W_u$$

chamber for stabilities.

For a Mukai vector v, we denote the moduli space of $\sigma_{(\beta,\omega)}$-semistable objects E on $\mathcal{A}_{(\beta,\omega)}$ with $v(E) = v$ by $M_{(\beta,\omega)}(v)$, if it exists. The moduli space $M_{(\beta,\omega)}(v)$ has a symplectic structure ([MYY1]). For a $K3$ surface S, $M_{(\beta,\omega)}(v)$ is an irreducible symplectic manifold deformation equivalent to $\mathrm{Hilb}_S^{\langle v^2\rangle/2+1}$ if v is a primitive Mukai vector with $\langle v^2\rangle \geq 2$. Let A be an abelian surface and v be a primitive Mukai vector. We assume that $(\beta,\omega) \in \mathrm{NS}(A)_\mathbb{R} \times \mathrm{Amp}(A)_\mathbb{R}$ is general with respect to v. We fix $E_0 \in M_{(\beta,\omega)}(v)$. Then

$$\begin{array}{rccc} \mathfrak{a}: & M_{(\beta,\omega)}(v) & \to & A \times \hat{A} \\ & E & \mapsto & (\det(\Phi_\mathcal{P}(E-E_0)), \det(E-E_0)) \end{array}$$

is the Albanese map which is an étale locally trivial fibration.

Theorem 0.5. ([MYY1]) We assume that $\langle v^2\rangle \geq 6$. Let $K_{(\beta,\omega)}(v)$ be a fiber of $\mathfrak{a}$. Then $M_{(\beta,\omega)}(v)$ is a smooth projective symplectic manifold which is deformation equivalent to $\mathrm{Hilb}_A^{\langle v^2\rangle/2} \times \hat{A}$ and $K_{(\beta,\omega)}(v)$ is an irreducible symplectic manifold with dimension $\langle v^2\rangle - 2$ which is deformation equivalent to the generalized Kummer manifold.

To study the nef cone of a complex smooth projective variety X has relations with the studying of birational geometry of X. For example, the cone theorem is an important tool in the minimal model ([BM, HT, KM]). A divisor D on X is ample if some nonnegative multiple mD is in the class of a hyperplane section of some projective embedding of X. D is numerically effective (nef, for short) if the intersection pairing is nonnegative for any curves C on X : $(C \cdot D) \geq 0$. Ample divisors are clearly nef. D is movable if the base locus of the linear system $|D|$ has codimension ≥ 2. We denote an ample cone (resp. nef cone and movable cone) by $\mathrm{Amp}(X)$ (resp. $\mathrm{Nef}(X)$ and $\mathrm{Mov}(X)$). Nef cone $\mathrm{Nef}(X)$ is identified with the dual of Kleiman-Mori cone of X by the nondegenerate bilinear form $(C, D) \mapsto (C \cdot D)$, where $D \in \mathrm{NS}(X) \otimes_{\mathbb{Z}} \mathbb{R}$ and C is a 1-cycle on X with $\mathbb{R}$-coefficient.

For an irreducible symplectic manifold M, Bayer, Macri and Yoshioka gave the description of the nef cone and the movable cone ([BM, Yo6]). In particular, Yoshioka gave the description of them for $K_{(\beta,\omega)}(v)$ which is deformation equivalent to the generalized Kummer manifold (Theorem 0.5). In this thesis, we shall give a more concrete description of $\mathrm{Nef}(\mathrm{Km}^2(A))$ and $\mathrm{Mov}(\mathrm{Km}^2(A))$ for an abelian surface A with Picard number 1.

Let A be an abelian surface with Picard number 1. We assume that H is an ample generator and $n := (H^2)/2 \in \mathbb{N}$. Bridgeland showed that $\mathrm{Stab}(A)$ carries the actions of the universal cover $\widetilde{\mathrm{GL}^+}(2, \mathbb{R})$ of $\mathrm{GL}^+(2, \mathbb{R})$ and the autoequivalence group $\mathrm{Aut}(\mathcal{D}(A))$ of $\mathcal{D}(A)$ ([Br1]). Let $\mathrm{Stab}^\dagger(A)$ be the subset of $\mathrm{Stab}(A)$ consisting of stability conditions $\sigma_{(\beta,\omega)}$ such that for each point $x \in A$, the skyscraper sheaf $\mathcal{O}_x$ is stable in $\sigma_{(\beta,\omega)}$ of the same phase. Then we have

$$\mathrm{Stab}^\dagger(A)/\widetilde{\mathrm{GL}^+}(2, \mathbb{R}) \cong \mathbb{H},$$

where $\mathbb{H} := \{(s,t) \in \mathbb{R}^2 \mid t > 0\}$. Hence stability conditions in $\mathrm{Stab}^\dagger(A)$ are parametrized by an upper half plane up to the action of $\widetilde{\mathrm{GL}^+}(2, \mathbb{R})$. We set Mukai vectors $v = (1, 0, -3), h = (0, H, 0)$ and $\delta = (1, 0, 3)$. If $s < 0$ and $t \gg 0$, then we have

$$M_{(sH,tH)}(v) = M_H(v) = \mathrm{Hilb}^3_A \times \hat{A}.$$

Hence $K_{(sH,tH)}(v) = \mathrm{Km}^2(A)$. In this situation, we also have $v^\perp \cong \mathbb{Z}h \oplus \mathbb{Z}\delta$ and $v^\perp \cong \mathrm{NS}(\mathrm{Km}^2(A))$ ([Yo2]). We set the positive cone

$$P^+ := \{x \in v^\perp \mid \langle x^2 \rangle > 0, \langle x, h \rangle > 0\}.$$

If $u \in \Gamma$, then $u^\perp$ is not empty. Moreover u is an isotropic vector.

Lemma 0.6. (Lemma 4.5) *Let $v = (1, 0, -l)$ and $l \leq 4$. Then $u \in \Gamma$ satisfies one of the following conditions:*

1. *$\langle u^2 \rangle = 0$ and $0 < \langle u, v \rangle \leq l$.*
2. *$\langle (v - u)^2 \rangle = 0$ and $0 < \langle v - u, v \rangle \leq l$.*

The connected component $\mathcal{C}$ of $P^+ \setminus \cup_{u \in \Gamma} u^\perp$ containing $h - \varepsilon\delta$ $(0 < \varepsilon \ll 1)$ is the ample cone $\mathrm{Amp}(\mathrm{Km}^2(A))$ of $\mathrm{Km}^2(A)$. For

$$\Gamma_M := \{u \in \Gamma \mid \langle u^2 \rangle = 0,\ \langle u, v \rangle = 1 \text{ or } 2\},$$

let $\mathcal{C}'$ be the connected component of $P^+ \setminus \cup_{u \in \Gamma_M} u^\perp$ containing $\mathcal{C}$. Then

$$\mathrm{Nef}(\mathrm{Km}^2(A)) = \overline{\mathcal{C}},\ \mathrm{Mov}(\mathrm{Km}^2(A)) = \overline{\mathcal{C}'} \tag{0.1}$$

by [Yo6]. Here, $\overline{\mathcal{C}}$ is a closure of $\mathcal{C}$. We note that the Mukai vector $u = (0, 0, 1)$ defines the trivial boundary of a nef / movable cone. Thus we can describe the boundary of the movable cone and the nef cone of $\mathrm{Km}^2(A)$ as a half line in $h\delta$-plane.

By using the orthogonal decomposition of u with respect to v, we have the expression

$$\frac{6}{\langle u, v \rangle} u = v + Xh + Y\delta,$$

where X and Y are integers. Since u is an isotropic vector by Lemma 0.6, we have a Diophantine equation

$$3Y^2 - nX^2 = 3. \tag{0.2}$$

We note that $3 \mid X$ if $3 \nmid n$. Hence we set $Z := X/3$ if $3 \nmid n$. The equation (0.2) is equivalent to

$$\begin{cases} Y^2 - \frac{n}{3}X^2 = 1, & \text{if } 3 \mid n \\ Y^2 - 3nZ^2 = 1, & \text{if } 3 \nmid n. \end{cases} \tag{0.3}$$

These equations are called Pell equations. If $3 \mid n$ and $\sqrt{n/3} \notin \mathbb{Q}$, then let (X_1, Y_1) $(X_1, Y_1 > 0)$ be a fundamental solution of (0.3), that is, (X_1, Y_1) is a solution of (0.3) minimizing X in the solution (X, Y) of (0.3) with $X, Y > 0$. We define (X_k, Y_k) by

$$Y_k + \sqrt{n/3}X_k = (Y_1 + \sqrt{n/3}X_1)^k.$$

We also set $(X_0, Y_0) := (0, 1)$. Then X_k, Y_k are positive integers satisfying (0.2) and the set of all solutions of (0.2) is $\{\pm(X_k, Y_k), \pm(X_k, -Y_k) \mid k \geq 0\}$. If $3 \nmid n$, let (Z_1, Y_1) be a fundamental solution of (0.3) and $(Z_0, Y_0) := (0, 1)$. We define (Z_k, Y_k) by $Y_k + \sqrt{3n}Z_k = (Y_1 + \sqrt{3n}Z_1)^k$. We also set $X_k := 3Z_k$. Then (X_k, Y_k) is the solution of (0.2).

In particular, we treat $\{\pm(X_k, -Y_k) \mid k \geq 0\}$ to describe the boundary of cones.

Theorem 0.7. (Theorem 4.7). The movable cone $\mathrm{Mov}(\mathrm{Km}^2(A))$ and the nef cone $\mathrm{Nef}(\mathrm{Km}^2(A))$ of $\mathrm{Km}^2(A)$ are characterized by the solution of (0.2) and n as the following table:

<table>
<tr><th>type of n</th><th colspan="4">type of (X_1, Y_1)</th><th>$\mathrm{Nef}(\mathrm{Km}^2(A))$</th><th>$\mathrm{Mov}(\mathrm{Km}^2(A))$</th></tr>
<tr><td>$3 \nmid n$</td><td colspan="4">$3 \mid X_1$</td><td>$\mathbb{R}_{\geq 0}h + \mathbb{R}_{\geq 0}(h - \frac{nX_1}{3Y_1}\delta)$</td><td>$\mathbb{R}_{\geq 0}h + \mathbb{R}_{\geq 0}(h - \frac{nX_1}{3Y_1}\delta)$</td></tr>
<tr><td rowspan="6">$n = 3m$</td><td rowspan="5">m is not square</td><td colspan="3">$3 \mid X_1$</td><td>$\mathbb{R}_{\geq 0}h + \mathbb{R}_{\geq 0}(h - \frac{nX_1}{3Y_1}\delta)$</td><td>$\mathbb{R}_{\geq 0}h + \mathbb{R}_{\geq 0}(h - \frac{nX_1}{3Y_1}\delta)$</td></tr>
<tr><td rowspan="4">$3 \nmid X_1$</td><td rowspan="2">X_1 even</td><td>$3 \mid Y_1$</td><td>$\mathbb{R}_{\geq 0}h + \mathbb{R}_{\geq 0}(h - \frac{nX_1}{3Y_1}\delta)$</td><td>$\mathbb{R}_{\geq 0}h + \mathbb{R}_{\geq 0}(h - \frac{nX_2}{3Y_2}\delta)$</td></tr>
<tr><td>$3 \nmid Y_1$</td><td>$\mathbb{R}_{\geq 0}h + \mathbb{R}_{\geq 0}(h - \frac{nX_1}{3Y_1}\delta)$</td><td>$\mathbb{R}_{\geq 0}h + \mathbb{R}_{\geq 0}(h - \frac{nX_3}{3Y_3}\delta)$</td></tr>
<tr><td rowspan="2">X_1 odd</td><td>$3 \mid Y_1$</td><td>$\mathbb{R}_{\geq 0}h + \mathbb{R}_{\geq 0}(h - \frac{nX_2}{3Y_2}\delta)$</td><td>$\mathbb{R}_{\geq 0}h + \mathbb{R}_{\geq 0}(h - \frac{nX_2}{3Y_2}\delta)$</td></tr>
<tr><td>$3 \nmid Y_1$</td><td>$\mathbb{R}_{\geq 0}h + \mathbb{R}_{\geq 0}(h - \frac{nX_2}{3Y_2}\delta)$</td><td>$\mathbb{R}_{\geq 0}h + \mathbb{R}_{\geq 0}(h - \frac{nX_3}{3Y_3}\delta)$</td></tr>
<tr><td>m is square</td><td colspan="3"></td><td>$\mathbb{R}_{\geq 0}h + \mathbb{R}_{\geq 0}(h - \sqrt{\frac{n}{3}}\delta)$</td><td>$\mathbb{R}_{\geq 0}h + \mathbb{R}_{\geq 0}(h - \sqrt{\frac{n}{3}}\delta)$</td></tr>
</table>

As we already know the trivial boundary of $\mathrm{Nef}(\mathrm{Km}^2(A))$ defining Hilbert-Chow contraction, we shall describe the other boundary.

By the proof of Theorem 0.7, we have a chamber decomposition of $\mathrm{Mov}(\mathrm{Km}^2(A))$ such that each chamber is an ample cone of a minimal model of $\mathrm{Km}^2(A)$. Then we study some flops which appear as minimal models of $\mathrm{Km}^2(A)$. We also study the dependence of $\mathrm{Km}^2(A)$ on A.

0.1 Construction

Let us explain the organization of this thesis. In section 1, we introduce some definitions. We define the generalized Kummer manifold $\mathrm{Km}^{l-1}(A)$ and explain the description of Néron-Severi group $\mathrm{NS}(\mathrm{Km}^{l-1}(A))$ of $\mathrm{Km}^{l-1}(A)$. In section 2, we explain a Bridgeland's stability condition and several properties. In order to describe the boundary of the nef/movable cone of $\mathrm{Km}^{l-1}(A)$, we define the moduli space of σ-semistable objects for $\sigma \in \mathrm{Stab}(A)$. In section 3, we explain that the boundary of the nef/movable cone of $\mathrm{Km}^{l-1}(A)$ are described as a half line in $h\delta$-plane. In section 4, we prove the main theorem. Then the theory of a Pell equation is useful. It is explained in subsection 4.1. For the main theorem, we also give the examples. In section 5, we discuss the chamber decomposition of $\mathrm{Mov}(\mathrm{Km}^2(A))$. We also discuss some flops which appear as the minimal models of the generalized Kummer manifold. In section 6, We calculate the boundary of the nef/movable cone of a generalized Kummer 6-fold $\mathrm{Km}^3(A)$. As an appendix, we calculate $\mathrm{Mov}(\mathrm{Km}^{l-1}(A))$ in section 7.

0.2 Acknowledgements.

I would like to express my gratitude to my adviser, Professor Kōta Yoshioka. He gave me many advices for studying algebraic geometry. I would also like to thank Iue Memorial Foundation for the financial support.

Notations

All varieties and schemes used in this thesis paper are over a complex field $\mathbb{C}$. For a smooth projective variety X, $\mathrm{NS}(X)$: Néron-Severi group of X, $\rho(X)$: Picard number of X, $\mathrm{Amp}(X)$: ample cone of X, $\mathrm{Nef}(X)$: nef cone of X, $\mathrm{Mov}(X)$: movable cone of X. We set $\mathrm{NS}(X)_k := \mathrm{NS}(X) \otimes_{\mathbb{Z}} k$ for $k = \mathbb{Q}, \mathbb{R}$. We also define $\mathrm{Amp}(X)_k := \mathrm{Amp}(X) \otimes_{\mathbb{Z}} k$, $\mathrm{Nef}(X)_k := \mathrm{Nef}(X) \otimes_{\mathbb{Z}} k$ and $\mathrm{Mov}(X)_k := \mathrm{Mov}(X) \otimes_{\mathbb{Z}} k$ for $k = \mathbb{Q}, \mathbb{R}$. $\mathrm{Coh}(X)$: category of coherent sheaves on X, $\mathcal{D}(X) := \mathcal{D}^b(\mathrm{Coh}(X))$: bounded derived category of coherent sheaves on X. For a triangulated category $\mathcal{T}$, $\mathcal{K}(\mathcal{T})$: Grothendieck group of $\mathcal{T}$,

1 Generalized Kummer Manifold.

We introduce some definitions used in this thesis. In particular, we define a generalized Kummer manifold and explain the relation between a moduli space of stable sheaves and generalized Kummer manifold. We also describe the Néron-Severi group of a generalized Kummer manifold as the orthogonal space of Mukai vector.

1.1 Mukai vector and moduli space.

Let X be an abelian surface or a $K3$ surface. If X is an abelian surface (or a $K3$ surface), then the Euler character $\chi(\mathcal{O}_X)/2$ is 0 (or 1). Hence we set $\varepsilon = 0, 1$ according as X is of type abelian or $K3$ surface. First of all, we define the Mukai lattice [Mu4]. Mukai lattice is an essential tool in analysis of each surface. We can see the examples in [Mu4], [Or1] and [Yo2].

Definition 1.1. The Mukai lattice is a pair of a cohomology subring

$$H^*(X, \mathbb{Z}) = \bigoplus_{i=0}^{2} H^{2i}(X, \mathbb{Z}),$$

and a symmetric bilinear form on $H^*(X, \mathbb{Z})$

$$\langle x_0 + x_1 + x_2\rho_X, y_0 + y_1 + y_2\rho_X \rangle = (x_1, y_1) - x_0y_2 - x_2y_0$$

where $x_1, y_1 \in H^2(X, \mathbb{Z}),\ x_0, x_2, y_0, y_2 \in \mathbb{Z}$ and $\rho_X \in H^4(X, \mathbb{Z})$ is the fundamental class of X. We call this bilinear form Mukai pairing. For $x \in H^*(X, \mathbb{Z})$, we write $\langle x^2 \rangle := \langle x, x \rangle$. The algebraic Mukai lattice (we often call this Mukai lattice) is a pair of

$$H^*(X, \mathbb{Z})_{\mathrm{alg}} = \mathbb{Z} \oplus \mathrm{NS}(X) \oplus \mathbb{Z}$$

and the Mukai pairing $\langle\ ,\ \rangle$ on $H^*(X, \mathbb{Z})_{\mathrm{alg}}$.

The structure of Mukai lattice is well known. For an abelian surface, Mukai lattice is isomorphic to $U^{\oplus 4}$ where U is the hyperbolic lattice. For a $K3$ surface, Mukai lattice is isomorphic to $U^{\oplus 4} \oplus E_8^{\oplus 2}$ where E_8 is a negative definite even lattice defined by Dynkin diagram of type E_8.

Mukai lattice has the weight 2 Hodge structure:

$$\begin{cases} H^{0,2}(H^*(X, \mathbb{C})) = H^{0,2}(X) \\ H^{1,1}(H^*(X, \mathbb{C})) = H^{0,0}(X) \oplus H^{1,1}(X) \oplus H^{2,2}(X) \\ H^{2,0}(H^*(X, \mathbb{C})) = H^{2,0}(X). \end{cases}$$

An isometry $\varphi : H^*(X, \mathbb{Z}) \to H^*(X, \mathbb{Z})$ is called a Hodge isometry if $\varphi \otimes \mathbb{C}$ preserves this subspace. Moreover we can get

$$H^{1,1}(H^*(X, \mathbb{C})) \cap H^*(X, \mathbb{Z}) = \mathbb{Z} \oplus \mathrm{NS}(X) \oplus \mathbb{Z} = H^*(X, \mathbb{Z})_{\mathrm{alg}}.$$

Definition 1.2. For a coherent sheaf $E \in \mathrm{Coh}(X)$, $v(E) = \mathrm{ch}(E)\sqrt{\mathrm{td}(X)} = \mathrm{ch}(E)(1+\varepsilon\rho_X) \in H^*(X,\mathbb{Z})_{\mathrm{alg}}$ is the Mukai vector of E, where $\mathrm{ch}(E)$ is the Chern character of E and $\mathrm{td}(X)$ is the Todd class of X.

A Mukai vector $v = (r, \xi, a)$ is called primitive if $\gcd(r, \xi, a) = 1$. A Mukai vector v is called isotropic if $\langle v^2 \rangle = 0$. A Mukai vecor v is positive (we write $v > 0$) if $r > 0$, or $r = 0, \xi \neq 0$ is effective and $a \neq 0$ or $r = \xi = 0$ and $a > 0$.

Remark 1.3. Let X be a smooth projective variety. A Mukai vector $v = \mathrm{ch}(E)\sqrt{\mathrm{td}(X)}$ is a map from the Grothendieck group $\mathcal{K}(X)$ of X to the cohomology ring $H^*(X,\mathbb{Q})$ of X. Mukai vector is compatible with an integral functor by the Grothendieck-Riemann-Roch formula. In general, Mukai vector is not surjective except special varieties. But if an integral functor is a derived equivalence, then the induced cohomological Fourier-Mukai transform on cohomorogy rings is a bijection of rational vector spaces ([Hu2]).

We introduce the notation of the moduli space of stable sheaves on X. Let us define the Gieseker's stability.

Definition 1.4. Let E be a torsion free sheaf on X. Then E is (semi)stable if for any subsheaf $F \subset E$ one has

$$(\mu(F), H) < (\mu(E), H) \text{ or } (\mu(F), H) = (\mu(E), H) \text{ and } \frac{\chi(F)}{\mathrm{rk}(F)} \underset{(=)}{<} \frac{\chi(E)}{\mathrm{rk}(E)},$$

where $\mu(E) := c_1(E)/\mathrm{rk}(E) \in H^2(X,\mathbb{Q})$).

Let H be an ample divisor on X and v be a Mukai vector.

Definition 1.5. We denote the moduli space of stable sheaves E of $v(E) = v$ by $M_H(v)$. We also denote its Gieseker compactification (by adding S-equivalent classes of semistable sheaves to $M_H(v)$) by $\overline{M}_H(v)$.

For the construction of the moduli space, we should refer to [HL]. There are several properties for the moduli space $M_H(v)$. In [Mu3], Mukai showed that $M_H(v)$ is a smooth variety of dimension $\langle v^2 \rangle + 2$ (if $M_H(v)$ is not empty) and has a symplectic structure, i.e., a nowhere degenerate holomorphic 2-form. Moreover, if H is general in $\mathrm{Amp}(X)_{\mathbb{R}}$ (i.e., there are hyperplanes called walls in $\mathrm{NS}(A)$ and H does not lie on these walls) and v is primitive, then $\overline{M}_H(v) = M_H(v)$. In particular, $M_H(v)$ is a projective scheme. Yoshioka showed that $\overline{M}_H(v)$ is a normal variety if a Mukai vector $v > 0$ satisfies $\langle v^2 \rangle > 0$ and H is general with respect to v (i.e., for every μ-semistable sheaf E with $v(E) = v$, if $F \subset E$ satisfies $(\mu(F), H) = (\mu(E), H)$, then $\mu(F) = \mu(E)$. He also showed that for a primitive Mukai vector v with $v > 0$, there is a semistable sheaf E with $v(E) = v$ if and only if $\langle v^2 \rangle \geq -2\varepsilon$ ([Yo3, Yo4]).

If a Mukai vector v is primitive and H is general, $M_H(v)$ is deformation equivalent to a moduli space of torsion free sheaves of rank 1 ([Yo2]). Moreover if $\langle v^2 \rangle \geq 0$ and X is an abelian surface, then $M_H(v)$ is deformation equivalent to $\hat{X} \times \mathrm{Hilb}_X^{\langle v^2 \rangle/2}$ (see Theorem 1.12). On the other hand, if $\langle v^2 \rangle \geq -2$ and X is a $K3$ surface, $M_H(v)$ is deformation equivalent to $\mathrm{Hilb}_X^{\langle v^2 \rangle/2+1}$ ([O1, Yo1]). For a $K3$ surface X, $\mathrm{Hilb}_X^{\langle v^2 \rangle/2+1}$ is an irreducible symplectic manifold, but $\mathrm{Hilb}_X^{\langle v^2 \rangle/2}$ is not an irreducible symplectic manifold for an abelian surface X. In next subsection, we see that some submanifold $K_H(v)$ of $M_H(v)$ is deformation equivalent to an irreducible symplectic manifold (see Theorem 1.13).

Remark 1.6. Let v be a primitive isotropic Mukai vector with positive rank. We assume that H is a general ample divisor with respect to v. Then $M_H(v)$ is an abelian surface (resp. a $K3$ surface), if X is an abelian surface (resp. a $K3$ surface).

1.2 Irreducible symplectic manifold.

In this subsection, we define a generalized Kummer manifold and describe the Néron-Severi group of the generalized Kummer manifold for an abelian surface with Picard number 1.

First of all, we explain an irreducible symplectic manifold (compact hyperKähler manifold). Let M be a complex manifold of even dimension $2n$. We assume that $\sigma \in H^0(M, \Omega_M^2)$ is a non-degenerate holomorphic 2-form such that $\wedge^n \sigma$ is not zero everywhere. The pair (M, σ) is called a symplectic manifold. We call σ holomorphic symplectic form.

By the following Bogomolov's decomposition theorem, we can decompose a smooth projective variety with trivial canonical bundle into the following 3 types of varieties.

Theorem 1.7. ([Bo]) Let X be a compact Kähler manifold with trivial first Chern class $c_1(X)$ in $H^2(X, \mathbb{R})$. Then there is an unramified covering X' of finite order of X such that it has the decomposition of manifolds with a trivial canonical bundle

$$X' = T \times \Pi C_i \times \Pi M_j,$$

where

(1) T is a complex torus,

(2) each C_i is simply connected projective manifold with $H^0(C_i, \Omega^p_{C_i}) = 0$ for all $p \neq 0, \dim C_i$,

(3) each M_j is simply connected symplectic Kähler manifold with $H^0(M_j, \Omega^2_{M_j}) = 1$.

We call the manifold satisfying Theorem 1.7-(3) an irreducible symplectic manifold. Let M be an irreducible symplectic manifold with dimension 2n. Then Beauville constructed the nondegenerate symmetric bilinear form on $H^2(M, \mathbb{Z})$ (hence on Néron-Severi group $\mathrm{NS}(M)$)

$$B_M : H^2(M, \mathbb{Z}) \times H^2(M, \mathbb{Z}) \to \mathbb{Z}$$

in [Be]. The signature of B_M is $(3, b_2(M) - 3)$ where $b_2(M)$ is the second Betti number. We set $q_M(x) := B_M(x, x)$. Let $\sigma \in H^0(M, \Omega^2_M)$ such that $\int_M \sigma^n \overline{\sigma}^n = 1$. $q_M(x)$ satisfies

$$q_M(x) = \frac{n}{2} \int_M \sigma^{n-1} \overline{\sigma}^{n-1} x^2 + (1-n) \int_M \sigma^n \overline{\sigma}^{n-1} x \int_M \sigma^{n-1} \overline{\sigma}^n x,$$

and if we write $x = \lambda\sigma + \beta + \overline{\lambda}\overline{\sigma}$ ($\lambda \in \mathbb{C}, \beta \in H^{1,1}(M)$) according to the Hodge decomposition on M, then

$$q_M(x) = \frac{n}{2} \int_M \sigma^{n-1} \overline{\sigma}^{n-1} \beta^2 + \lambda\overline{\lambda}$$

holds ([Be], [Hu1]).

Example 1.8. A complex surface is an irreducible symplectic manifold if and only if it is a $K3$ surface. For $K3$ surface S, the Hilbert scheme Hilb^n_S of n points on S is an irreducible symplectic manifold.

Example 1.9. For an abelian surface A, Hilb^n_A is a symplectic manifold, but is not irreducible. We explain an irreducible symplectic manifold constructed by Beauville [Be].

Let $S^l(A)$ be the l-th symmetric product of A and $\mathrm{Hilb}^l(A)$ the Hilbert scheme of l points on A ($l \geq 2$). Thus $\mathrm{Hilb}^l(A)$ parameterizes 0-dimension subschemes Z of A of $\chi(\mathcal{O}_Z) = l$. We have the Hilbert-Chow morphism $\varphi : \mathrm{Hilb}^l(A) \to S^l(A)$ sending a subscheme Z of length l to the 0-cycle $[Z]$ defined by Z. Since A is an abelian surface, we have the morphism $\mathrm{Hilb}^l(A) \to S^l(A) \xrightarrow{\sigma} A$, where $\sigma(x_1, x_2, ..., x_l) = \sum_i x_i$. Then the fiber $\mathrm{Km}^{l-1}(A)$ of this morphism is an irreducible symplectic manifold of dimension $2(l-1)$. Thus we have the following commutative diagram:

$$\begin{array}{ccccc} \mathrm{Km}^{l-1}(A) & \longrightarrow & \mathrm{Hilb}^l(A) & & \\ {\scriptstyle\varphi}\downarrow & & \downarrow{\scriptstyle\varphi} & & \\ \sigma^{-1}(0) & \longrightarrow & S^l(A) & \xrightarrow{\sigma} & A \end{array} \tag{1.1}$$

If $l = 2$, then $\mathrm{Km}^1(A)$ is nothing but the Kummer K3 surface of A, that is, $\sigma^{-1}(0) \cong A/\iota$ and $\varphi : \mathrm{Km}^1(A) \to A/\iota$ is the minimal resolution where $\iota : A \to A$ is the involution $a \mapsto -a$. Hence $\mathrm{Km}^{l-1}(A)$ ($l \geq 3$) is called a generalized Kummer manifold.

Example 1.10. ([O2, O3, Ra]) We set $v = (2, 0, -2)$. If S is a $K3$ surface, then there exists a symplectic desingularization $\widetilde{\overline{M}_H(v)} \to \overline{M}_H(v)$. This $\widetilde{\overline{M}_H(v)}$ is a 10-dimensional irreducible symplectic variety and $b_2(\widetilde{\overline{M}_H(v)}) = 24$. We assume that A is an abelian surface. Let $\overline{K}_H(v)$ be a fiber of $\mathfrak{a}_v$ which is defined below this example. Then there exists a symplectic desingularization $\widetilde{\overline{K}_H(v)} \to \overline{K}_H(v)$. This $\widetilde{\overline{K}_H(v)}$ is a 6-dimensional irreducible symplectic variety and $b_2(\widetilde{\overline{K}_H(v)}) = 8$.

Remark 1.11. It is not known another irreducible symplectic manifold which is not deformation equivalent to the above examples.

Let A be an abelian surface and H be an ample divisor on A. Let $\hat{A}$ be the dual abelian surface of A and $\mathcal{P}$ be the Poincaré line bundle on $A \times \hat{A}$. We assume that v is a positive Mukai vector with $c_1(v) \in \mathrm{NS}(A)$. Fix an element $E_0 \in \overline{M}_H(v)$. We define the morphism $\alpha : \overline{M}_H(v) \to A$ by

$$\alpha(E) := \det p_{\hat{X}!}((E - E_0) \otimes (\mathcal{P} - \mathcal{O}_{\hat{A}\times A})) \in \mathrm{Pic}^0(\hat{A}) = A.$$

Moreover $\det : \overline{M}_H(v) \to \hat{A}$ be the morphism sending E to $\det E \otimes \det E_0^{\vee} \in \hat{A}$. If we set $\mathfrak{a}_v := \alpha \times \det$, then the following is claimed.

Theorem 1.12. ([Yo2, Theorem 0.1]) Let v be a primitive Mukai vector such that $v > 0$, $\langle v^2 \rangle \geq 2$ and H be a general element in the ample cone $\mathrm{Amp}(A)$. Then

(1) $\mathfrak{a}_v : M_H(v) \to A \times \hat{A}$ is the Albanese map,

(2) $M_H(v)$ is deformation equivalent to $\hat{A} \times \mathrm{Hilb}_A^{\langle v^2 \rangle/2}$.

Let $K_H(v)$ be a fiber of $\mathfrak{a}_v$. Then we have the following theorem.

Theorem 1.13. ([Yo2, Theorem 0.2]) Let v be a primitive Mukai vector with $v > 0$ and $\langle v^2 \rangle \geq 6$.

(1) For a general ample divisor H with respect to v, $K_H(v)$ is deformation equivalent to a generalized Kummer manifold with dimension $\langle v^2 \rangle - 2$ which is constructed by Beauville. In particular, $K_H(v)$ is an irreducible symplectic manifold.

(2) We have the Hodge isometry

$$\theta_v : (v^{\perp}, \langle\ ,\ \rangle) \to (H^2(K_H(v), \mathbb{Z}), B_{K_H(v)}).$$

Remark 1.14. Let $v = (1, 0, -l)$ and $l \in \mathbb{Z}_{>0}$. We assume that A is an abelian surface with $\rho(A) = 1$. Every stable sheaf $E \in M_H(v)$ is written as $E \cong \mathcal{L} \otimes \mathcal{I}_Z$, where $\mathcal{L} \in \mathrm{Pic}^0(A) = \hat{A}$ and $\mathcal{I}_Z$ is an ideal sheaf of 0-dimensional subscheme $Z \subset A$ of length l. Conversely, an ideal sheaf $\mathcal{I}_Z$ satisfies $v(\mathcal{I}_Z) = (1, 0, -l)$. Hence we have $M_H(1, 0, -l) \cong \hat{A} \times \mathrm{Hilb}_A^l$. In particular, $K_H(v) = \mathrm{Km}^{l-1}(A)$.

Assume that A is an abelian surface with $\rho(A) = 1$. Let H be an ample generator of $\mathrm{NS}(A)$ and set $n := H^2/2 \in \mathbb{N}$. Then $\mathrm{NS}(\mathrm{Km}^{l-1}(A))$ $(l \geq 3)$ is described as

$$\mathrm{NS}(\mathrm{Km}^{l-1}(A)) \cong \mathbb{Z}h \oplus \mathbb{Z}\delta \tag{1.2}$$

and the Beauville-Bogomolov bilinear form satisfies

$$B_{\mathrm{Km}^{l-1}(A)}(h, h) = 2n,\ B_{\mathrm{Km}^{l-1}(A)}(\delta, \delta) = -2l,\ B_{\mathrm{Km}^{l-1}(A)}(h, \delta) = 0,$$

where h is the pull-back of an ample divisor on $S^l(A)$ by $\mathrm{Km}^{l-1}(A) \to S^l(A)$ (cf. [Yo2, Proposition 4.11]). Let D be the exceptional divisor of $\mathrm{Km}^{l-1}(A) \to \sigma^{-1}(0)$. Then $D \in |2\delta|$ and $kh - \delta$ is ample for $k \gg 0$.

We set $v = (1, 0, -l)$. It is easy see that

$$v^{\perp} = \mathbb{Z}(0, H, 0) \oplus \mathbb{Z}(1, 0, l)$$

and we get an isometry

$$\begin{array}{cccc} \theta_v : & v^{\perp} & \to & \mathbb{Z}h \oplus \mathbb{Z}\delta \\ & (0, H, 0) & \mapsto & h \\ & (1, 0, l) & \mapsto & \delta \end{array} \tag{1.3}$$

By θ_v, we shall identify $v^{\perp}$ with $\mathrm{NS}(\mathrm{Km}^{l-1}(A))$. We define a positive cone by

$$P^+ := \{x \in v^{\perp} \mid \langle x^2 \rangle > 0, \langle x, h \rangle > 0\}.$$

2 Bridgeland's Stability Condition.

Bridgeland introduced the notions of a stability condition on a triangulated category $\mathcal{T}$ and showed that the set of stability conditions $\mathrm{Stab}(\mathcal{T})$ on $\mathcal{T}$ is a complex manifold ([Br1]). Hence we can treat the stability of a derived category $\mathcal{D}(X)$ for a smooth projective variety X. Studying Bridgeland's stability conditions is interesting but difficult. For example, the construction of stability condition is. For a $K3$ surface and an abelian surface X, he studied $\mathrm{Stab}(\mathcal{D}(X))$. In particular, he constructed the stability condition on $\mathcal{D}(X)$ by tilting with respect to the torsion pair.

2.1 Definitions and properties.

In this subsection, we introduce the definition of Bridgeland's stability condition and several properties ([Br1]). Let $\mathcal{T}$ be a triangulated category.

Definition 2.1. A stability condition $\sigma = (Z_\sigma, \mathcal{P}_\sigma)$ on $\mathcal{T}$ consists of a linear map $Z_\sigma : \mathcal{K}(\mathcal{T}) \to \mathbb{C}$ and a full additive subcategory $\mathcal{P}_\sigma(\phi) \subset \mathcal{T}$ for each $\phi \in \mathbb{R}$, satisfying the following axioms:

(a) if $0 \neq E \in \mathcal{P}_\sigma(\phi)$, then $Z_\sigma(E) = m(E) \exp i\pi\theta$ for some $m(E) \in \mathbb{R}_{>0}$,

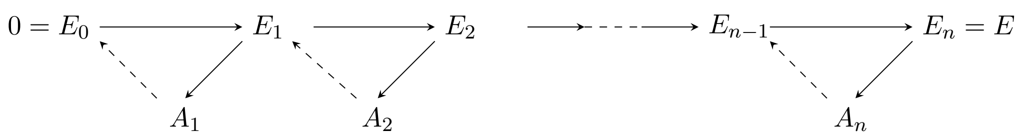
(b) for all $\phi \in \mathbb{R}$, $\mathcal{P}_\sigma(\phi + 1) = \mathcal{P}_\sigma(\phi)[1]$,

(c) if $A_j \in \mathcal{P}_\sigma(\phi_j)$ and $\phi_1 > \phi_2$, then $\mathrm{Hom}_X(A_1, A_2) = 0$,

(d) for $0 \neq E \in \mathcal{T}$ there is a finite sequence of real numbers

$$\phi_1 > \phi_2 > \cdots > \phi_n$$

and a collection of triangles



with $A_j \in \mathcal{P}_\sigma(\phi_j)$ for all j.

Remark 2.2.
- We call a linear map Z_σ central charge.
- Each subcategory $\mathcal{P}_\sigma(\phi)$ is an abelian category([Br1, Lemma 5.2]).
- The non-zero object of $\mathcal{P}_\sigma(\phi)$ is said to be σ-semistable of phase ϕ with respect to σ.
- The simple object of $\mathcal{P}_\sigma(\phi)$ is said to be stable.
- The decomposition of a non-zero object $E \in \mathcal{T}$ given by axiom (d) is uniquely defined up to isomorphism.
- Each object A_j is called the semistable factor of E with respect to σ.

We can rewrite Definition 2.1 by using a t-structure and a group homomorphism on its heart.

Definition 2.3. Let $\mathcal{T}$ be a triangulated category. The subcategory $\mathcal{T}^{\leq 0} \subset \mathcal{T}$ is a t-structure on $\mathcal{T}$ if the following conditions hold:

(1) $\mathcal{T}^{\leq 0}[1] \subset \mathcal{T}^{\leq 0}$.

(2) We define a full subcategory $\mathcal{T}^{\geq 1} := \{F \in \mathcal{T} \mid \mathrm{Hom}(\mathcal{T}^{\leq 0}, F) = 0\} \subset \mathcal{T}$. Then for every object $E \in \mathcal{T}$, there are objects $E_0 \in \mathcal{T}^{\leq 0}$ and $E_1 \in \mathcal{T}^{\geq 1}$ which satisfy the following triangle:

$$E_0 \to E \to E_1 \to E_0[1].$$

For a t-structure $\mathcal{T}^{\leq 0}$, $\mathcal{T}^{\leq 0} \cap (\mathcal{T}^{\geq 1}[1]) \subset \mathcal{T}$ is an abelian category. We call this the heart of a t-structure $\mathcal{T}^{\leq 0}$. A t-structure $\mathcal{T}^{\leq 0}$ is said to be bounded if

$$\mathcal{T} = \cup_{i,j \in \mathbb{Z}} \mathcal{T}^{\leq 0}[i] \cap \mathcal{T}^{\geq 1}[j].$$

Example 2.4. Let $\mathcal{A}$ be an abelian category and $\mathcal{T} := \mathcal{D}^b(\mathcal{A})$. We define the subcategory $\mathcal{T}^{\leq 0} \subset \mathcal{T}$ by

$$\mathcal{T}^{\leq 0} := \{E \in \mathcal{T} \mid H^i(E) = 0 \text{ for all } i > 0\}.$$

This defines a bounded t-structure on $\mathcal{T}$ which is called the canonical t-structure. Then we have $\mathcal{T}^{\geq 1} := \{E \in \mathcal{T} \mid H^i(E) = 0 \text{ for all } i \leq 0\}$. Moreover the functor $\mathcal{A} \to \mathcal{T}^{\leq 0} \cap (\mathcal{T}^{\geq 1}[1])$ which sends $E \in \mathcal{A}$ to the complex $\cdots \to 0 \to E \to 0 \to \cdots$ (E is located at degree 0) is a category equivalence.

Definition 2.5. Let $\mathcal{A}$ be an abelian category and $Z : \mathcal{K}(\mathcal{A}) \to \mathbb{C}$ be a group homomorphism. Z has the Harder-Narasimhan property if each object $E \in \mathcal{A}$ has a finite filtration

$$0 = E_0 \subset E_1 \subset \cdots \subset E_{n-1} \subset E_n = E$$

whose factors $F_i = E_i/E_{i-1}$ are Z-semistable objects of $\mathcal{A}$ and satisfy $\arg(Z(F_i)) > \arg(Z(F_{i+1}))$ for all i. Here $E \in \mathcal{A}$ is Z-semistable if for a non-zero subobject $F \subset E$, $\arg(Z(F)) \leq \arg(Z(E))$ holds in $(0, \pi]$.

For a stability condition σ, let $\mathcal{A}_\sigma = \mathcal{P}_\sigma(0, 1]$, which is the extension closed full subcategory of $\mathcal{T}$ generated by $E \in \mathcal{P}_\sigma(\phi)$ with $\phi \in (0, 1]$. Then $\mathcal{A}_\sigma$ is an abelian category and the decomposition of an object in $\mathcal{A}$ which is given by Definition 2.1-(d) is the Harder-Narasimhan filtration.

Conversely let $\mathcal{A}$ be the heart of a bounded t-structure (it is an abelian category), and a group homomorphism $Z : \mathcal{K}(\mathcal{T}) \to \mathbb{C}$ such that

$$Z(\mathcal{A}\backslash\{0\}) \subset \{\mathbb{R}_{>0} \exp(\pi i \phi) \mid \phi \in (0, 1]\} = \mathbb{H} \cup \mathbb{R}_{<0},$$

where $\mathbb{H} = \{z \in \mathbb{C} \mid \mathrm{Im} z > 0\}$ is an upper half plane (We call this Z the stability function). We also assume that Z satisfies the Harder-Narasimhan property. We set

$$\mathcal{P}(\phi) = \{E \in \mathcal{A} \mid E \text{ is } Z\text{-semistable and } Z(E) \in \mathbb{R}_{>0} e^{i\pi\phi}\} \cup \{0\}.$$

Then we have a stability condition $\sigma = (Z_\sigma, \mathcal{P}_\sigma)$ with $Z_\sigma = Z$ and $\mathcal{A} = \mathcal{P}_\sigma(0, 1]$.

Therefore we have one to one correspondence:

$$\sigma = (Z_\sigma, \mathcal{P}_\sigma) \longleftrightarrow (Z_\sigma, \mathcal{A}_\sigma).$$

Proposition 2.6. ([Br1, Proposition 5.3]) To give a stability condition on a triangulated category $\mathcal{T}$ is equivalent to giving a bounded t-structure on $\mathcal{T}$ and a stability function on its heart with the Harder-Narasimhan property.

Remark 2.7. By above argument, a σ-semistable object of phase ϕ is Z-semistable.

Example 2.8. ([Br1]) Let C be a smooth projective curve. We set $\mathcal{A} = \mathrm{Coh}(C)$. Then $\mathcal{A}$ is the heart of the canonical t-structure (Example 2.4). We define a group homomorphism $Z : \mathcal{K}(C) \to \mathbb{C}$ by the formula

$$Z(E) := -\deg(E) + \sqrt{-1}\mathrm{rk}(E).$$

Since $\mathrm{rk}(E) \geq 0$ and $\deg(E) > 0$ for a non-trivial coherent sheaf E, we can define $\arg(Z(E)) = \mathrm{Im}(\log Z(E)) \in (0, \pi]$. Moreover Z satisfies the Harder-Narasimhan property. By Proposition 2.6, we have a stability condition on the bounded derived category $\mathcal{D}(C)$.

For a given triangulated category $\mathcal{T}$, we fix a finitely generated free module Γ and a group homomorphism

$$\mathrm{cl} : \mathcal{K}(\mathcal{T}) \to \Gamma.$$

Moreover we fix the norm $||\cdot||$ on $\Gamma_{\mathbb{R}} := \Gamma \otimes_{\mathbb{Z}} \mathbb{R}$. Since $\Gamma_{\mathbb{R}}$ is a finite-dimensional vector space, we note that all norm on $\Gamma_{\mathbb{R}}$ is equivalent.

Example 2.9. Let X be a smooth projective variety and $\mathcal{T} = \mathcal{D}(X)$. Then we can set $\mathrm{cl} := \mathrm{ch}$ and $\Gamma := \mathrm{Im}(\mathrm{ch} : \mathcal{K}(X) \to H^*(X, \mathbb{Q}))$.

Definition 2.10. We define $\mathrm{Stab}_\Gamma(\mathcal{T})$ by the set of a pair $(Z, \mathcal{P})$ which satisfies the following conditions:

- $Z : \Gamma \to \mathbb{C}$ is a group homomorphism and $(Z \circ \mathrm{cl}, \mathcal{P})$ is a stability condition on $\mathcal{T}$. In other words, there is a stability condition $\sigma = (Z_\sigma, \mathcal{P}_\sigma)$ such that $Z_\sigma : \mathcal{K}(X) \to \mathbb{C}$ factors through Γ and $\mathcal{P}_\sigma = \mathcal{P}$.
- The support property holds:

$$\sup\left\{ \frac{||\mathrm{cl}(E)||}{|Z(E)|} \ : \ 0 \neq E \in \bigcup_{\phi \in \mathbb{R}} \mathcal{P}(\phi) \right\} < \infty.$$

The set $\mathrm{Stab}_\Gamma(\mathcal{T})$ of stability conditions has a structure of complex manifold ([Br1]). We define the topology for $\mathrm{Stab}_\Gamma(\mathcal{T})$. For $\sigma \in \mathrm{Stab}_\Gamma(\mathcal{T})$ and $E \in \mathcal{T}$, we take the collection of triangles in Definition 2.1 and define $\phi^+_\sigma(E) := \phi_1$ and $\phi^-_\sigma(E) := \phi_n$. For $\varepsilon_1, \varepsilon_2 > 0$ we define $B_{\varepsilon_1,\varepsilon_2}(\sigma) \subset \mathrm{Stab}_\Gamma(\mathcal{T})$ as following:

$$B_{\varepsilon_1,\varepsilon_2}(\sigma) = \left\{ (W, \mathcal{Q}) \ \middle| \ \begin{array}{c} ||W - Z|| < \varepsilon_1, \\ \text{for any } E \in \cup_{\phi \in \mathbb{R}} \mathcal{Q}(\phi) \backslash \{0\}, \ \phi^+_\sigma(E) - \phi^-_\sigma(E) < \varepsilon_2. \end{array} \right\}. \tag{2.1}$$

Theorem 2.11. ([Br1]) $\mathrm{Stab}_\Gamma(\mathcal{T})$ has the topology generated by the basis of open sets $B_{\varepsilon_1,\varepsilon_2}(\sigma)$. Moreover the forgettable map

$$\begin{array}{ccccc} \mathrm{Stab}_\Gamma(\mathcal{T}) & \to & \Gamma^\vee_\mathbb{C} & = & \mathrm{Hom}_\mathbb{Z}(\Gamma, \mathbb{C}) \\ (Z, \mathcal{P}) & \mapsto & Z & & \end{array}$$

is a locally homeomorphism. In particular, $\mathrm{Stab}_\Gamma(\mathcal{T})$ is a complex manifold.

We introduce the notation for a derived category of coherent sheaves on a smooth projective variety.

Definition 2.12. Let X be a smooth projective variety. We take Γ and cl as in Example 2.9. Then we define the complex manifold $\mathrm{Stab}(X)$ by

$$\mathrm{Stab}(X) := \mathrm{Stab}_\Gamma(\mathcal{D}(X)).$$

Remark 2.13. We consider the same situation as Example 2.9. Since $\mathcal{K}(\mathcal{D}(X)) \to \Gamma$ is surjective, the dual map

$$\Gamma^\vee_\mathbb{C} \to \mathrm{Hom}(\mathcal{K}(\mathcal{D}(X)), \mathbb{C}) \tag{2.2}$$

is injective. Hence we consider the central charge of the element in $\mathrm{Stab}(X)$ as the group homomorphism from $\mathcal{K}(\mathcal{D}(X))$ to $\mathbb{C}$ through an embedding (2.2).

2.2 Group actions on $\mathrm{Stab}_\Gamma(\mathcal{T})$.

In this subsection, we explain the group actions on $\mathrm{Stab}_\Gamma(\mathcal{T})$([Br1]). In particular, we see that $\mathrm{Stab}(X)$ has the group actions.

Let $\mathcal{T}$ be a triangulated category. We take Γ and cl as in the previous subsection. $\mathrm{Stab}_\Gamma(\mathcal{T})$ carries a right action of the group $\widetilde{\mathrm{GL}^+}(2, \mathbb{R})$, the universal cover of $\mathrm{GL}^+(2, \mathbb{R})$, and a left action by isometries of the group $\mathrm{Aut}(\mathcal{T})$ of exact autoequivalences of $\mathcal{T}$ ([Br1, Lemma 8.2]).

First the group $\widetilde{\mathrm{GL}^+}(2, \mathbb{R})$ can be thought of as the set of pairs (A, f) where $A \in \mathrm{GL}^+(2, \mathbb{R})$ and a monotonically increasing function $f : \mathbb{R} \to \mathbb{R}$, and they satisfy the following:

- for all $\phi \in \mathbb{R}$, $f(\phi + 1) = f(\phi) + 1$,

- the induced maps on
$$S^1 \cong \mathbb{R}/2\mathbb{Z} \cong (\mathbb{R}^2/\{0\})/\mathbb{R}_{>0}$$
are the same.

Let $\sigma = (Z_\sigma, \mathcal{P}_\sigma) \in \mathrm{Stab}(X)$. We define the right action of $(A, f) \in \widetilde{\mathrm{GL}^+}(2,\mathbb{R})$ on $\mathrm{Stab}_\Gamma(\mathcal{T})$ by

$$(\sigma,(A,f)) \to \sigma' = (Z_{\sigma'}, \mathcal{P}_{\sigma'}),\ (Z_{\sigma'} = A^{-1}\circ Z_\sigma,\ \mathcal{P}_{\sigma'}(\phi) = \mathcal{P}_\sigma(f(\phi))).$$

Here we identify $\mathbb{C}$ with $\mathbb{R}^2$. We note that the semistable objects of the stability conditions σ and σ' are the same, but the phases have been relabeled.

Next, we define the left action of $\mathrm{Aut}(\mathcal{T})$ on $\mathrm{Stab}_\Gamma(\mathcal{T})$. We assume the following: There is a group homomorphism

$$\begin{array}{ccc} \mathrm{Aut}(\mathcal{T}) & \to & \mathrm{Aut}(\Gamma) \\ \Phi & \mapsto & \tilde{\Phi} \end{array}$$

such that the following diagram is commutative:

$$\begin{array}{ccc} \mathcal{T} & \xrightarrow{\Phi} & \mathcal{T} \\ {\scriptstyle \mathrm{cl}}\downarrow & & \downarrow{\scriptstyle \mathrm{cl}} \\ \Gamma & \xrightarrow[\tilde{\Phi}]{} & \Gamma \end{array} \tag{2.3}$$

Under this assumption, we define the left action of $\Phi \in \mathrm{Aut}(\mathcal{D}(X))$ on $\mathrm{Stab}(X)$ by

$$(\Phi,\sigma) \to \sigma' = (Z'_\sigma, \mathcal{P}_{\sigma'}),\ (Z_{\sigma'} = Z_\sigma \circ \tilde{\Phi}^{-1},\ \mathcal{P}_{\sigma'} = \Phi(\mathcal{P}_\sigma(\phi))).$$

Remark 2.14. (1) We can check that these two actions satisfy the axioms of a group action and commute.

(2) Let X be a smooth projective variety and $\mathcal{T} = \mathcal{D}(X)$. We take Γ and cl as in the example 2.9. By Orlov's theorem ([Or1]), for any autoequivalence Φ, there is an object $\mathcal{E} \in \mathcal{D}(X \times X)$ such that $\Phi \cong \Phi_\mathcal{E}$ where $\Phi_\mathcal{E}$ is a Fourier-Mukai transform with Fourier-Mukai kernel $\mathcal{E}$. We set $\tilde{\Phi} = \Phi^H_{v(\mathcal{E})}$, where $\Phi^H_{v(\mathcal{E})}$ is a cohomological Fourier-Mukai transform corresponding to $\Phi_\mathcal{E}$. Then the assumption (2.3) holds for $\mathcal{T}$ (cf. Remark 1.3). That is, $\mathrm{Stab}(X)$ carries a right action of the group $\widetilde{\mathrm{GL}^+}(2,\mathbb{R})$, the universal cover of $\mathrm{GL}^+(2,\mathbb{R})$, and a left action by isometries of the group $\mathrm{Aut}(\mathcal{D}(X))$.

Example 2.15. ([Br1, Section 9]) Let C be an elliptic curve. Then there is a stability condition $\sigma \in \mathrm{Stab}(C)$ (the stability condition in Example 2.8 satisfies the conditions of Definition 2.10). The map which is defined by a right action of $\widetilde{\mathrm{GL}^+}(2,\mathbb{R})$

$$\begin{array}{ccc} \widetilde{\mathrm{GL}^+}(2,\mathbb{R}) & \to & \mathrm{Stab}(C) \\ (A,f) & \mapsto & \sigma\cdot(A,f) \end{array}$$

is a homeomorphism. On the other hand, we have $\Gamma = H^0(C,\mathbb{Z}) \oplus H^2(C,\mathbb{Z}) \cong \mathbb{Z}^{\oplus 2}$ and the image of $\mathrm{Aut}(\mathcal{D}(C)) \to \mathrm{Aut}(\Gamma)$ is identified with $\mathrm{SL}(2,\mathbb{Z})$ (cf. [Or2, Section 4]). Hence we have the double quotient

$$\mathrm{Aut}(\mathcal{D}(C))\backslash\mathrm{Stab}(C)/\mathbb{C} \cong \mathrm{SL}(2,\mathbb{Z})\backslash\widetilde{\mathrm{GL}^+}(2,\mathbb{R})/\mathbb{C} \cong \mathrm{SL}(2,\mathbb{Z})\backslash\mathrm{GL}^+(2,\mathbb{R})/\mathbb{C}^* \cong \mathrm{SL}(2,\mathbb{Z})\backslash\mathbb{H},$$

where $\mathrm{SL}(2,\mathbb{Z})$ acts on $\mathbb{H}$ as following : For $\tau \in \mathbb{H}$,

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \cdot \tau = \frac{a\tau + b}{c\tau + d}.$$

The double quotient is identified with the moduli space of elliptic curves.

2.3 Construction of stability conditions $\sigma_{(\beta,\omega)}$ and moduli space of complexes.

For smooth projective curve C, we can construct the stability condition with the heart $\mathrm{Coh}(C)$ of bounded t-structure (Example 2.8) and this stability condition belongs to $\mathrm{Stab}(C)$. However, for a smooth projective variety X with $\dim(X) \geq 2$, there is no stability condition $(Z, \mathcal{A}) \in \mathrm{Stab}(X)$ with $\mathcal{A} = \mathrm{Coh}(X)$ ([To, Lemma 2.7]).

Let X be a $K3$ surface or an abelian surface. In this subsection, we describe the construction of stability condition on $\mathcal{D}(X)$ by tilting with respect to the torsion pair via [Br2]. We also give the definition of wall and chamber on $\mathrm{Stab}(X)$. We set $\mathcal{K}(X) := \mathcal{K}(\mathcal{D}(X))$.

In order to construct the stability condition on $\mathcal{D}(X)$, we introduce the twisted stability. By using such stability, we can construct a bounded t-structure.

Definition 2.16. Let ω be an ample divisor on X and $\beta \in \mathrm{NS}(X)_{\mathbb{Q}}$. For a torsion free sheaf E on X, we set

$$\mu_{\beta,\omega}(E) := (\mu(E) - \beta, \omega),\ \ \chi_{\beta,\omega}(E) := (\chi(E) - \mathrm{rk}(E)c_1(E), \beta).$$

E is β-twisted (semi)stable if E satisfies

$$\mu_{\beta,\omega}(F) < \mu_{\beta,\omega}(E) \text{ or } \mu_{\beta,\omega}(F) = \mu_{\beta,\omega}(E) \text{ and } \frac{\chi_{\beta,\omega}(F)}{\mathrm{rk}(F)} \underset{(=)}{<} \frac{\chi_{\beta,\omega}(E)}{\mathrm{rk}(E)}$$

for all non-trivial subsheaf F of E.

Definition 2.17. Let $v \in H^*(X, \mathbb{Z})_{\mathrm{alg}}$ be a Mukai vector. $\overline{M}^{\beta}_{\omega}(v)$ denotes the moduli space of β-twisted semistable sheaves E with $v(E) = v$. M^{β}_{ω} denotes denotes the open subscheme consisting of β-twisted stable sheaves.

Remark 2.18. For the construction of $\overline{M}^{\beta}_{\omega}(v)$, we should refer to [Yo4]. If $\beta = 0$, then β-twisted semistability is nothing but the semistability of Giseker, and we denote the moduli space $\overline{M}^{\beta}_{\omega}(v)$ by $\overline{M}_{\omega}(v)$. Moreover, if H is general in $\mathrm{Amp}(X)_{\mathbb{R}}$, then $\overline{M}^{\beta}_{\omega}(v)$ does not depend on the choice of β.

Let $(\beta, \omega) \in \mathrm{NS}(X)_{\mathbb{R}} \times \mathrm{Amp}(X)_{\mathbb{R}}$. For $E \in \mathcal{K}(X)$, we define a group homomorphism $Z_{(\beta,\omega)} : \mathcal{K}(X) \to \mathbb{C}$ by

$$Z_{(\beta,\omega)}(E) = \langle \exp{(\beta + i\omega)}, v(E) \rangle$$

where $\langle\ ,\ \rangle$ is Mukai pairing. Then there is a unique torsion pair $(\mathcal{T}_{(\beta,\omega)}, \mathcal{F}_{(\beta,\omega)})$ on $\mathrm{Coh}(X)$:

- $\mathcal{T}_{(\beta,\omega)}$: a full subcategory of $\mathrm{Coh}(X)$ generated by β-twisted stable sheaves with $Z_{(\beta,\omega)}(E) \in \mathbb{H} \cup \mathbb{R}_{<0}$,
- $\mathcal{F}_{(\beta,\omega)}$: a full subcategory of $\mathrm{Coh}(X)$ generated by β-twisted stable sheaves with $-Z_{(\beta,\omega)}(E) \in \mathbb{H} \cup \mathbb{R}_{<0}$.

Tilting with respect to the torsion pair $(\mathcal{T}_{(\beta,\omega)}, \mathcal{F}_{(\beta,\omega)})$ gives a bounded t-structure on $\mathcal{D}(X)$ with heart

$$\mathcal{A}_{(\beta,\omega)} = \{E \in \mathcal{D}(X) \mid H^i(E) = 0 \text{ for } i \notin \{-1, 0\}, H^{-1}(E) \in \mathcal{F}_{(\beta,\omega)} \text{ and } H^0(E) \in \mathcal{T}_{(\beta,\omega)}\}.$$

We define the phase $\phi_{(\beta,\omega)}(E) \in (0, 1]$ of $0 \neq E \in \mathcal{A}_{(\beta,\omega)}$ by

$$Z_{(\beta,\omega)}(E) = |Z_{(\beta,\omega)}(E)| \exp{(i\pi\phi_{(\beta,\omega)}(E))}.$$

If X is a $K3$ surface, we assume that $Z_{(\beta,\omega)}(E) \notin \mathbb{R}_{\geq 0}$ for all spherical objects E. Then it is shown that $Z_{(\beta,\omega)}$ is the stability function on the heart $\mathcal{A}_{(\beta,\omega)}$ with Harder-Narasimhan property. Therefore we can show that

$$\sigma_{(\beta,\omega)} = (Z_{(\beta,\omega)}, \mathcal{A}_{(\beta,\omega)})$$

is the stability condition on $\mathcal{D}(X)$ by Proposition 2.6. Moreover, for any point $x \in X$, the skyscraper sheaf $\mathcal{O}_x$ is stable in $\sigma_{(\beta,\omega)}$ of phase one. In this way, we can construct a stability condition $\sigma_{(\beta,\omega)}$ associated to $(\beta, \omega) \in \mathrm{NS}(X)_{\mathbb{R}} \times \mathrm{Amp}(X)_{\mathbb{R}}$. $\sigma_{(\beta,\omega)}$ satisfies the support property (Definition 2.10). Hence $\sigma_{(\beta,\omega)} \in \mathrm{Stab}(X)$.

In what follows, we assume that A is an abelian surface.

Remark 2.19. ([Br2], [Yo5])

(1) A bounded derived category $\mathcal{D}(A)$ has no spherical objects.

(2) We assume that $\mathrm{Stab}^\dagger(A) \subset \mathrm{Stab}(A)$ is the subset consisting of stability conditions σ constructed above such that for each point $x \in A$, the sheaf $\mathcal{O}_x$ is stable in σ. Then we have

$$\mathrm{Stab}^\dagger(A)/\widetilde{\mathrm{GL}^+}(2,\mathbb{R}) \cong \mathrm{NS}(A)_\mathbb{R} \times \mathrm{Amp}(A)_\mathbb{R}.$$

Moreover if $\rho(A) = 1$ (i.e., $\mathrm{NS}(A) = \mathbb{Z}H$ where H is an ample divisor),

$$\mathrm{Stab}^\dagger(A)/\widetilde{\mathrm{GL}^+}(2,\mathbb{R}) \cong \mathbb{H}.$$

If A is a principally polarized abelian surface with $\rho(A) = 1$, the action of $\mathrm{Aut}(\mathcal{D}(A))$ on $\mathbb{H}$ factors through the natural action of $\mathrm{SL}(2,\mathbb{Z})$. Hence we have

$$\mathrm{SL}(2,\mathbb{Z})\backslash\mathrm{Stab}^\dagger(A)/\widetilde{\mathrm{GL}^+}(2,\mathbb{R}) \cong \mathrm{SL}(2,\mathbb{Z})\backslash\mathbb{H} \cong \mathbb{C}.$$

Definition 2.20. A non-trivial object $E \in \mathcal{A}_{(\beta,\omega)}$ is $\sigma_{(\beta,\omega)}$-semistable if $\phi_{(\beta,\omega)}(F) \le \phi_{(\beta,\omega)}(E)$ for all non-trivial proper subobjects F of E. If the inequality is strict, then E is $\sigma_{(\beta,\omega)}$-stable (cf. Remark 2.7). A non-trivial object $E \in \mathcal{D}$ is $\sigma_{(\beta,\omega)}$-semistable if there is an integer n such that $E[-n] \in \mathcal{A}_{(\beta,\omega)}$ and $E[-n]$ is $\sigma_{(\beta,\omega)}$-semistable.

$\sigma_{(\beta,\omega)} \in \mathrm{Stab}^\dagger(A)$ satisfies the following:

Proposition 2.21. ([MYY2]) Let $E \in \mathcal{D}(A)$.

(1) (Large volume limit) Assume that E satisfies $\mathrm{rk}(E) \ge 0$ and $(\omega^2) \gg \langle v(E)^2\rangle$. Then E is $\sigma_{(\beta,\omega)}$-semistable if and only if E is a coherent sheaf on A and β-twisted semistable.

(2) Every derived equivalence $\Phi : \mathcal{D}(A) \xrightarrow{\sim} \mathcal{D}(A')$ preserves the stability, that is, if E is $\sigma_{(\beta,\omega)}$-semistable, then there is $(\beta',\omega') \in \mathrm{Stab}(A')$ such that $\Phi(E)$ is $\sigma_{(\beta',\omega')}$-semistable.

Remark 2.22. Proposition 2.21 holds for a $K3$ surface.

For the dependence of the moduli space of stable sheaves on an ample divisor H, it is important to study the wall-crossing behavior on $\mathrm{Amp}(A)_\mathbb{R}$. If we consider the analogy for Bridgeland's stability $\sigma_{(\beta,\omega)}$, then there must be the subspace of $\mathrm{NS}(A)_\mathbb{R} \times \mathrm{Amp}(A)_\mathbb{R}$ that it changes the stability. For the space $\mathrm{NS}(A)_\mathbb{R} \times \mathrm{Amp}(A)_\mathbb{R}$, we give the definition of wall and chamber.

Definition 2.23. (cf. [MYY1, Proposition 5.7]) Let v be a Mukai vector. For a Mukai vector u satisfying the inequality

$$\langle u, v-u\rangle > 0,\ \langle u^2\rangle \ge 0,\ \langle (v-u)^2\rangle \ge 0, \tag{2.4}$$

we define the wall $W_u \subset \mathrm{NS}(A)_\mathbb{R} \times \mathrm{Amp}(A)_\mathbb{R}$ as

$$W_u := \{(\beta,\omega) \in \mathrm{NS}(A)_\mathbb{R} \times \mathrm{Amp}(A)_\mathbb{R} \mid \mathbb{R}Z_{(\beta,\omega)}(u) = \mathbb{R}Z_{(\beta,\omega)}(v)\}. \tag{2.5}$$

Γ is denoted the set of Mukai vectors u satisfying the above inequalities. We call a connected component

$$\mathcal{C} \subset (\mathrm{NS}(A)_\mathbb{R} \times \mathrm{Amp}(A)_\mathbb{R})\backslash \bigcup_{u\in\Gamma} W_u$$

chamber for stabilities.

In general, W_u may be an empty set. The following proposition gives the characterization for the non-emptiness of the wall.

Proposition 2.24. ([Yo6, Proposition 1.3]) Let u be a Mukai vector which defines the wall with respect to v. Then

$$W_u \cap (\mathrm{NS}(A)_\mathbb{R} \times \mathrm{Amp}(A)_\mathbb{R}) \ne \phi \iff \langle v,u\rangle^2 > \langle v^2\rangle\langle u^2\rangle.$$

Corollary 2.25. Let w be an isotropic Mukai vector with $0 < \langle v, w\rangle < \langle v^2\rangle/2$. Then w defines the wall with respect to v and $W_w \cap (\mathrm{NS}(A)_{\mathbb{R}} \times \mathrm{Amp}(A)_{\mathbb{R}})$ is non-empty.

Example 2.26. Let A be an abelian surface with $\rho(A) = 1$ and we assume $\mathrm{NS}(A) \cong \mathbb{Z}H$ with $(H^2) = 2n$. Then we can write $(\beta, \omega) = (sH, tH)$ where $s, t \in \mathbb{R}$ and $t > 0$. Hence we have

$$\mathrm{NS}(A)_{\mathbb{R}} \times \mathrm{Amp}(A)_{\mathbb{R}} = \{(sH, tH) \mid s \in \mathbb{R}, t \in \mathbb{R}_{>0}\}.$$

We write down the equation of a wall. Let $v = (r, dH, a)$ and $u = (r', d'H, a')$. We note that $\exp(sH + \sqrt{-1}tH) = (1, (s + \sqrt{-1}t)H, \frac{1}{2}(s + \sqrt{-1}t)^2(H^2))$. Since

$$\begin{aligned} Z_{(sH,tH)}(v) &= \langle \exp(sH + \sqrt{-1}tH), (r, dH, a)\rangle \\ &= 2n((\tfrac{1}{2}r(t^2 - s^2)ds - \tfrac{a}{2n}) + \sqrt{-1}(d - rs)t), \end{aligned}$$

we have the equation of semi-circle

$$\begin{aligned} & \mathbb{R}Z_{(sH,tH)}(v) = \mathbb{R}Z_{(sH,tH)}(u) \\ \Leftrightarrow\ & (\tfrac{1}{2}r(t^2 - s^2)ds - \tfrac{a}{2n})(d' - r's)t = (\tfrac{1}{2}r'(t^2 - s^2)d's - \tfrac{a'}{2n})(d - rs)t \\ \Leftrightarrow\ & \tfrac{1}{2}(rd' - r'd)t^2 + \tfrac{1}{2}(rd' - r'd)s^2 + \tfrac{ar' - a'r}{2n}s = \tfrac{ad' - a'd}{2n}. \end{aligned}$$

We set $p := d/r, q := d^2/r^2 - 2a/2nr$ and $\delta = (a'r - ar')/2n(rd' - r'd)$. If $rd' - r'd = 0$, then we have $W_u = \{(sH, tH) \mid s = p, t > 0\}$. If $rd' - r'd \neq 0$, then we have $W_{u,\delta} := W_u = \{(sH, tH) \mid (s - \delta)^2 + t^2 = (p - \delta)^2 - q, t > 0\}$. Moreover if $\delta \neq \delta'$, then $W_{u,\delta} \cap W_{u,\delta'} = \phi$.

Next we define the moduli space of Bridgeland's semistable objects.

Definition 2.27. For a Mukai vector v, $M_{(\beta,\omega)}(v)$ denotes the moduli space of $\sigma_{(\beta,\omega)}$-semistable objects E on $\mathcal{A}_{(\beta,\omega)}$ with $v(E) = v$, if it exists.

Remark 2.28. If $\langle v^2\rangle \geq 0$, $M_{(\beta,\omega)}(v)$ depends only on the chamber C to which (β, ω) belongs. Then we denote this moduli space by $M_C(v)$.

There are several properties for the moduli space $M_{(\beta,\omega)}$ ([MYY1]). If (β, ω) is general (i.e., (β, ω) belongs to a chamber), then $M_{(\beta,\omega)}(v)$ is a projective scheme. For a $K3$ surface S, if v is a primitive Mukai vector with $\langle v^2\rangle \geq 2$, then $M_{(\beta,\omega)}$ is an irreducible symplectic manifold deformation equivalent to $\mathrm{Hilb}_S^{\langle v^2\rangle/2+1}$. For an abelian surface, we state the symplectic structure of $M_{(\beta,\omega)}(v)$ in the next section.

There are the relations between the twisted stability and Bridgeland's stability. As we have already seen, Bridgeland's semistability is twisted semistability at the large volume limit (Proposition 2.21). If v is positive, then $\overline{M}_\omega^\beta(v) = M_{(\beta+s\omega,t\omega)}(v)$ for some (s, t). That is, twisted stability is the special case of Bridgeland's stability.

3 The Description of Movable Cone and Nef Cone.

Recent years, a nef cone and a movable cone is a tool for studying the birational geometry. For example, it is well known that the cone theorem claims extremal rays correspond to contractions (cf. [KM]). Moreover the cone theorem and the minimal model program induce a locally polyhedral chamber decomposition of the movable cone of an irreducible symplectic variety (see [HT]). In this section, we characterize the movable cone and the nef cone of the Albanese fiber which is deformation equivalent to a generalized Kummer manifold ([Yo6]). In our case, we can concretely describe the boundary of the movable cone and the nef cone. We summarize the descriptions at the last of this section.

3.1 $M_{(\beta,\omega)}(v)$ and Albanese map.

Let A be an abelian surface. We assume that v is a primitive Mukai vector and $(\beta,\omega) \in \mathrm{NS}(A)_{\mathbb{R}} \times \mathrm{Amp}(A)_{\mathbb{R}}$ is general with respect to v. We fix $E_0 \in M_{(\beta,\omega)}(v)$. Let

$$\Phi^{\mathcal{P}}_{A\to\hat{A}} : \mathcal{D}(A) \to \mathcal{D}(\hat{A})$$

be the Fourier-Mukai transform by Poincaré line bundle $\mathcal{P}$ on $A \times \hat{A}$ where $\hat{A} := \mathrm{Pic}^0(A)$ is the dual of A. Then we have an Albanese map $\mathfrak{a} : M_{(\beta,\omega)}(v) \to A \times \hat{A}$ by

$$\mathfrak{a}(E) := (\det(\Phi^{\mathcal{P}}_{A\to\hat{A}}(E - E_0)), \det(E - E_0)) \in A \times \hat{A},$$

which $\mathfrak{a}$ is an étale locally trivial fibration.

Definition 3.1. Let v be a primitive Mukai vector with $\langle v^2 \rangle \geq 6$.

(1) We denote a fiber of the Albanese map $\mathfrak{a} : M_{(\beta,\omega)}(v) \to A \times \hat{A}$ by $K_{(\beta,\omega)}(v)$. If v is positive, then we also denote a fiber $K_{(\beta,\omega)}(v)$ by $K^{\beta}_{H}(v)$.

(2) We denote the Mukai's homomorphism by

$$\theta_{v,\beta,\omega} : v^{\perp} \to H^2(M_{(\beta,\omega)}(v), \mathbb{Z}) \to H^2(K_{(\beta,\omega)}(v), \mathbb{Z}).$$

If there is an universal family $\mathbf{E}$ on $M_{(\beta,\omega)}(v)$ (e.g., there is a Mukai vector w with $\langle v, w\rangle = 1$), then we can express the Mukai's homomorphism as

$$\theta_{v,\beta,\omega}(x) = c_1(p_{M_{(\beta,\omega)}(v)*}(\mathrm{ch}(E)p_A^*(x^{\vee})))|_{K_{(\beta,\omega)}(v)},$$

where p_A and $p_{M_{(\beta,\omega)}(v)}$ are projections from $A \times M_{(\beta,\omega)}(v)$ to A and $M_{(\beta,\omega)}(v)$ respectively.

For the moduli space of complexes on $\mathcal{D}(A)$, there are results which are similar to the moduli space of stable sheaves (cf. Theorem 1.13).

Theorem 3.2. ([MYY1, Proposition 5.16]) We assume that v is a primitive Mukai vector with $\langle v^2\rangle \geq 6$ and ω is general.

(1) $M_{(\beta,\omega)}(v)$ is a smooth projective symplectic manifold which is deformation equivalent to $\mathrm{Hilb}^{\langle v^2\rangle/2}_A \times \hat{A}$.

(2) $K_{(\beta,\omega)}(v)$ is an irreducible symplectic manifold with dimension $\langle v^2\rangle - 2$ which is deformation equivalent to the generalized Kummer variety constructed by Beauville.

(3)
$$\theta_{v,\beta,\omega} : (v^{\perp}, \langle\ ,\ \rangle) \to (H^2(K_{(\beta,\omega)}(v), \mathbb{Z}), B_{K_{(\beta,\omega)}(v)})$$
is an isometry of Hodge structure, where $B_{K_{(\beta,\omega)}(v)}$ is Beauville-Fujiki form on $H^2(K_{(\beta,\omega)}(v), \mathbb{Z})$.

3.2 Movable cone and nef cone of $K_{(\beta,\omega)}(v)$.

In this subsection, we assume that A is an abelian surface with $\rho(A) = 1$ and H is the ample generator with $(H^2) = 2n, n \in \mathbb{N}$. Then since we have

$$\mathrm{NS}(A)_{\mathbb{R}} \times \mathrm{Amp}(A)_{\mathbb{R}} = \mathbb{R}H \times \mathbb{R}_{>0}H,$$

we can express $(\beta,\omega) = (sH, tH)$ by using $s, t \in \mathbb{R}$ with $t > 0$ as stated in Example 2.26. Also we can express the Mukai homomorphism $\theta_{v,\beta,\omega}$ in the form of $\theta_{v,\beta,\omega} = \theta_{v,sH,tH}$. Moreover we get the form of a Mukai vector as $v = (r, dH, a)$.

We prepare the following two definitions for the later proposition.

Definition 3.3. Let $v = (r, dH, a)$ be a Mukai vector with $r > 0$ and set

$$l := \frac{\langle v^2 \rangle}{2} = d^2 n - ra.$$

We set

$$s_{\pm} := \frac{d}{r} \pm \frac{1}{r}\sqrt{\frac{l}{n}} \in \mathbb{R}.$$

Definition 3.4. We set

$$\xi(s,t) := (r(s^2 + t^2)n - a)(H + \frac{2dn}{r}\rho_A) - 2n(d - rs)(1 - \frac{a}{r}\rho_A).$$

Now let $V \subset \mathbb{R}^m$ be a cone and $C(V) := (V \backslash \{0\})/\mathbb{R}_{>0}$. The following proposition gives the characterization of a nef cone.

Proposition 3.5. ([Yo6, Proposition 4.11])

(1) If (s,t) belongs to a chamber and $s, t^2 \in \mathbb{Q}$, then $\theta_{v,sH,tH}(\xi(s,t))$ is an ample $\mathbb{Q}$-divisor of $K_{(sH,tH)}(v)$.

(2) We have a bijective map

$$\varphi : [s_-, s_+] \to C(\overline{P^+(K_{(sH,tH)}(v))_{\mathbb{R}}})$$

such that

$$\varphi(\lambda) := \mathbb{R}_{>0}\theta_{v,sH,tH}(\xi(\lambda, 0)).$$

(3) We have

$$\mathrm{Nef}(K_{(sH,tH)}(v))_{\mathbb{R}} = \varphi(\overline{D(sH,tH)} \cap [s_-, s_+]),$$

where $D(sH,tH)$ is the chamber including (sH,tH).

The following proposition gives the characterization of movable cone.

Theorem 3.6. ([Yo6, Theorem 3.31]) Let v be a primitive Mukai vector with $v^2 \geq 6$. Γ is denoted the set of Mukai vectors which define the wall with respect to v. We assume that $(s,t) \in \overline{\mathbb{H}}$ satisfies with $\xi(s,t) \notin \cup_{u \in \Gamma} u^{\perp}$ and Γ_M is the set of primitive isotropic Mukai vectors $\tilde{u}$ with $\langle \tilde{u}, v \rangle = 0, 1, 2$. Moreover let $\mathcal{D}(sH,tH)$ be the connected component of $P^+(v^{\perp})_{\mathbb{R}} \backslash \cup_{\tilde{u} \in \Gamma_M} \tilde{u}^{\perp}$ including $\xi(s,t)$. Then we have

$$\overline{\mathrm{Mov}(K_{(sH,tH)}(v))_{\mathbb{R}}} = \theta_{v,sH,tH}(\overline{\mathcal{D}(sH,tH)}).$$

Moreover

$$\theta_{v,sH,tH}(H^*(A,\mathbb{Z})_{\mathrm{alg}} \cap \overline{\mathcal{D}(sH,tH)}) \subset \mathrm{Mov}(K_{(sH,tH)}(v)).$$

We set Mukai vectors $v = (1, 0, -l)$, $h = (0, H, 0)$ and $\delta = (1, 0, l)$. We note that $K_H(v) = \mathrm{Km}^{l-1}(A)$ (Remark 1.14). Then we have

$$\theta_v : v^{\perp} = \mathbb{Z}h + \mathbb{Z}\delta \xrightarrow{\sim} \mathrm{NS}(K_{(sH,tH)}(v))$$

(See (1.3)). Moreover if $s < 0$, $t \gg 0$, we get

$$M_{(sH,tH)}(v) = M_H(v) = \mathrm{Hilb}^l_A \times \hat{A}$$

by Theorem 1.12 and Proposition 2.21. Then by Proposition 3.5, $h - \varepsilon\delta$ $(0 < \varepsilon \ll 1)$ becomes the ample divisor of $K_H(v)$.

Therefore we can describe the boundary of movable cone and nef cone of $\mathrm{Km}^{l-1}(A)$ as a half line in $h\delta$-plane.

Summary

We recall a description of $\mathrm{Mov}(\mathrm{Km}^{l-1}(A))$ and $\mathrm{Nef}(\mathrm{Km}^{l-1}(A))$: We consider the set Γ of Mukai vector u satisfying the inequalities

$$\langle u, v-u\rangle > 0,\ \langle u^2\rangle \geq 0,\ \langle (v-u)^2\rangle \geq 0,\ \langle v, u\rangle^2 > \langle v^2\rangle\langle u^2\rangle.$$

If $u \in \Gamma$, then $u^\perp \cap P^+ \neq \phi$ The connected component $\mathcal{C}$ of $P^+ \setminus \cup_{u\in\Gamma} u^\perp$ containing $h - \varepsilon\delta$ $(0 < \varepsilon \ll 1)$ is the ample cone $\mathrm{Amp}(\mathrm{Km}^{l-1}(A))$ of $\mathrm{Km}^{l-1}(A)$. For

$$\Gamma_M := \{u \in \Gamma \mid \langle u^2\rangle = 0,\ \langle u, v\rangle = 1 \text{ or } 2\},$$

let $\mathcal{C}'$ be the connected component of $P^+ \setminus \cup_{u\in\Gamma_M} u^\perp$ containing $\mathcal{C}$. Then

$$\mathrm{Nef}(\mathrm{Km}^{l-1}(A)) = \overline{\mathcal{C}},\ \mathrm{Mov}(\mathrm{Km}^{l-1}(A)) = \overline{\mathcal{C}'}. \tag{3.1}$$

We note that the Mukai vector $u = (0, 0, 1)$ defines the trivial boundary of a nef / movable cone. Therefore we can describe the boundary of the movable cone and the nef cone of $\mathrm{Km}^2(A)$ as a half line in $h\delta$-plane.

4 The Calculation of Boundary of Cones.

We saw the description of cones in section 3. In particular, we saw that the boundaries of cones are described as the half line in $h\delta-$plane. In this section, we give the proof of Theorem 0.7 (see Theorem 4.7). That is, we calculate the half lines concretely. In order to show it, we should determine the vector $u \in \Gamma$. Then a Pell equation is a very useful tool for determining the boundaries. So we see the properties of a Pell equation. We also show that the Mukai vector which determines the boundary is an isotropic vector if $l \leq 4$. Then the boundaries are characterized by solutions of Pell equation and the number of self intersection of the ample generator in $\mathrm{NS}(A)$, and we can show Theorem 0.7 by using classical number theory.

4.1 Pell equation.

In this subsection, we explain the properties of a Pell equation (cf. [AAC, Section I.3]).

4.1.1 Fundamental properties of a Pell equation

The Pell equation is the Diophantine equation of the form

$$y^2 - nx^2 = 1, \tag{4.1}$$

where n is a positive integer and integer solutions are sought for x and y. For example, for $n = 6$ one can take $(x, y) = \pm(2, 5), \pm(2, -5)$. If n is a square number, then there is only one solution up to sign: $(x, y) = (0, 1)$. We assume that $\sqrt{n} \notin \mathbb{Q}$. Then (4.1) has infinitely many integral solutions. Let (x_1, y_1) be the fundamental solution of (4.1), that is, (x_1, y_1) is a solution of (4.1) minimizing x in the solution (x, y) of (0.2) with $x, y > 0$. We define $(x_k, y_k)(k \geq 1)$ by

$$y_k + x_k\sqrt{n} = (y_1 + x_1\sqrt{n})^k.$$

Then x_k, y_k are positive integers satisfying (4.1) and $\{\pm(x_k, y_k), \pm(x_k, -y_k) | k \geq 0\}$ is the set of all solutions of (4.1). Moreover we have

$$x_k = \frac{1}{2\sqrt{n}}((y_1 + x_1\sqrt{n})^k - (y_1 - x_1\sqrt{n})^k),\ y_k = \frac{1}{2}((y_1 + x_1\sqrt{n})^k + (y_1 - x_1\sqrt{n})^k),$$

and

$$\frac{x_k}{y_k} < \frac{x_{k+1}}{y_{k+1}}\ ,\ \lim_{k\to\infty} \frac{x_k}{y_k} = \sqrt{\frac{1}{n}}. \tag{4.2}$$

Hence it is natural to calculate the fundamental solution of (4.1). In this thesis, we calculate a few solutions in order to explain the examples of Theorem 4.7. One method to calculate the fundamental solution is to use the continued fraction of $\sqrt{n}$ and its periodic sequence. However since we used Mathematica for doing, we do not explain it.

Remark 4.1. Let (X_1, Y_1) be the fundamental solution of

$$Y^2 - nX^2 = -1 \tag{4.3}$$

and define $(X_k, Y_k)(k \geq 1)$ by $Y_k + X_k\sqrt{n} = (Y_1 + X_1\sqrt{n})^k$. Then $(X_{2k}, Y_{2k}) = (x_k, y_k)$ satisfy (4.1) and (X_{2k-1}, Y_{2k-1}) satisfy (4.3). Thus we can see all solutions of $y^2 - nx^2 = \pm 1$ if we find the fundamental solution of (4.3).

We characterize the solution of (4.1) by evenness. If n is even, then y is odd. If n is odd, then either x or y is even. Moreover if x is even, then y is odd.

Lemma 4.2. We assume that $n' = 3n$ and $2 \mid x_1$. Then for solutions $(X_1, Y_1) = \pm(x_1, y_1)$ of (4.1), the following does not hold: $Y_1 \equiv -1 \mod p$ for all prime divisors $p > 2$ of n' and $Y_1 \equiv -1 \mod 4$ if $2 \mid n'$.

Proof. We suppose that it holds. We note that $Y_1 \not\equiv 1 \mod p$ and $\frac{Y_1 - 1}{2}$ is odd if $2 \mid n'$. Since $\frac{Y_1+1}{2n}\frac{Y_1-1}{2} = (\frac{X_1}{2})^2$ and $\gcd(\frac{Y_1+1}{2n}, \frac{Y_1-1}{2}) = 1$, there are positive integers a, b such that

$$\pm a^2 = \frac{Y_1 + 1}{2n}, \ \pm b^2 = \frac{Y_1 - 1}{2}, \ \pm ab = \frac{X_1}{2}.$$

We note that $3 \mid a^2$. We assume $(X_1, Y_1) = (x_1, y_1)$. Then we get $b^2 - na^2 = -1$, but this does not hold. Next, we assume $(X_1, Y_1) = -(x_1, y_1)$. Then $(x, y) = (a, b)$ satisfies the Pell equation (4.1). We have $(b + a\sqrt{n})^2 = y_1 + x_1\sqrt{n}$, and this contradicts that (x_1, y_1) is the fundamental solution of (4.1). □

We set $s_k := y_k + x_k\sqrt{n}$ and $s_{-k} := y_k - x_k\sqrt{n}$. We note that $s_{-1} = s_1^{-1}$. Then $\langle s_1 \rangle \subset \mathbb{Z}(\sqrt{n})$ is a cyclic group by multiplication which is isomorphic to $\{(x_k, \pm y_k) \mid k \in \mathbb{Z}\}$. We identify $\{(x_k, \pm y_k) \mid k \in \mathbb{Z}\}$ with $\langle s_1 \rangle$. We also set $P_a := \{(x_k, y_k) \mid a \mid x_k\}$. For $s_k, s_l \in P_a$, since we have $s_k s_l = (y_k y_l + n x_k x_l) + (x_k y_l + x_l y_k)\sqrt{n}$, P_a is a subgroup of $\langle s_1 \rangle$.

4.1.2 Generalized Pell equation

For a positive integer n with $\sqrt{n} \notin \mathbb{Q}$ and $k \in \mathbb{Z}$, let us consider the following equation

$$x^2 - ny^2 = k. \tag{4.4}$$

We have a decomposition $n = f^2 m$, where $f \in \mathbb{Z}$ and $m \in \mathbb{Z}$ is square free. Let $\mathcal{O}$ be the ring of integers of $\mathbb{Q}(\sqrt{m})$. Thus

$$\mathcal{O} = \begin{cases} \mathbb{Z}[\frac{1+\sqrt{m}}{2}], & m \equiv 1 \mod 4, \\ \mathbb{Z}[\sqrt{m}], & m \equiv 2, 3 \mod 4. \end{cases} \tag{4.5}$$

For $\alpha := x + y\sqrt{n}$ $(x, y \in \mathbb{Q})$, $N(\alpha) := x^2 - y^2 n$ is the norm of α. Then we have a bijective correspondence

$$\begin{array}{ccc} \{(x, y) \mid x^2 - ny^2 = k\} & \to & \{\alpha \in \mathbb{Z}[\sqrt{n}] \mid N(\alpha) = k\} \\ (x, y) & \mapsto & x + y\sqrt{n}. \end{array} \tag{4.6}$$

We first solve the equation

$$N(\alpha) = k, \ \alpha \in \mathcal{O}. \tag{4.7}$$

If $\alpha \in \mathcal{O}$ satisfies $N(\alpha) = k$, then the ideal (α) has a prime ideal decomposition $(\alpha) = P_1^{e_1} P_2^{e_2} \cdots P_t^{e_t}$ with $k\mathbb{Z} = N(\alpha)\mathbb{Z} = N(P_1)^{e_1} N(P_2)^{e_2} \cdots N(P_t)^{e_t}$, where $N(I)$ is the ideal norm of an ideal I. For each prime ideal P_i, there is a prime number p_i such that $N(P_i) = p_i\mathbb{Z}, p_i^2\mathbb{Z}$ and $(p) = P_i P_i'$, P_i or P_i^2. Hence

$$\{(\alpha) \mid N(\alpha) = k\mathbb{Z}\}$$

is a finite set. For a principal ideal (β) with $N(\beta) = k\mathbb{Z}$, a generator $\gamma \in (\beta)$ satisfies $N(\gamma) = \pm k$. If $N(\gamma) = k$, then γ is a solution of (4.7). We set

$$\mathcal{O}^{\#} := \{\alpha \in \mathcal{O} \mid N(\alpha) = 1\}. \tag{4.8}$$

Then $\mathcal{O}^{\#}$ acts on

$$E_k := \{\alpha \in \mathcal{O} \mid N(\alpha) = k\}$$

by the multiplication and there are finitely many orbits. We set

$$\mathbb{Z}[\sqrt{n}]^{\#} := \{\alpha \in \mathbb{Z}[\sqrt{n}] \mid N(\alpha) = 1\}. \tag{4.9}$$

Since $\mathcal{O}^{\#}/\mathbb{Z}[\sqrt{n}]^{\#}$ is a finite group,

$$\{\alpha \in \mathbb{Z}[\sqrt{n}] \mid N(\alpha) = k\} = E_k \cap \mathbb{Z}[\sqrt{n}] \tag{4.10}$$

has a finitely many orbits of the $\mathbb{Z}[\sqrt{n}]^{\#}$-action.

Lemma 4.3. Assume that $\gcd(2f, k) = 1$. For each orbit $\mathcal{O}^{\#}\alpha$, $\mathcal{O}^{\#}\alpha \cap \mathbb{Z}[\sqrt{n}]$ is an orbit of $\mathbb{Z}[\sqrt{n}]^{\#}$-action or an empty set.

Proof. We only treat the case where $m \equiv 1 \mod 4$. We set $\omega := \frac{1+\sqrt{m}}{2}$. For $\alpha = p + q\omega \in \mathcal{O}$, α satisfies $N(\alpha) = k$ and $\alpha = x + y\sqrt{n} \in \mathbb{Z}[\sqrt{n}]$ if and only if $2f \mid q$. Assume that $\alpha = p + q\omega \in \mathbb{Z}[\sqrt{n}]$ and $\beta = p' + q'\omega \in \mathbb{Z}[\sqrt{n}]$ satisfy $N(\alpha) = N(\beta) = k$ and $\beta/\alpha \in \mathcal{O}$. We set $\beta/\alpha = a + b\omega$. Then $\beta = ap + bq\frac{(m-1)}{4} + (bp + (a+b)q)\omega$. By $\gcd(2f, k) = 1$, we have $(2f, p + q\omega) = \mathcal{O}$, and hence $\gcd(2f, p) = 1$. Since $q' = bp + (a+b)q \equiv bp \mod 2f$, $2f \mid b$. Therefore $\beta/\alpha \in \mathbb{Z}[\sqrt{n}]^{\times}$. □

In order to solve the case of $n \equiv 1 \mod 3$ in Section 6, we introduce the following lemma.

Lemma 4.4. Assume that $n \equiv 1 \mod 3$ and $k = 9$. Then there are 1 or 3 orbits of the solutions.

Proof. We have a prime ideal decomposition $(3) = PP'$, where

$$P := \begin{cases} (3, 1 + \sqrt{m}), & m \equiv 2, 3 \mod 4 \\ (3, \frac{1+\sqrt{m}}{2}), & m \equiv 1 \mod 4 \end{cases} \tag{4.11}$$

and P' is the conjugate of P. Then $I := P^2, P'^2, (3)$ are the ideals satisfying $N(I) = 9\mathbb{Z}$. If P^2 contains a generator α satisfying $N(\alpha) = 9$, then the conjugate α' also satisfies $N(\alpha') = 9$. By using Lemma 4.3, we see that there are 1 or 3 orbits of the solutions. □

4.2 The calculation of boundary of cones.

We first prove that the boundaries of $\mathrm{Nef}(\mathrm{Km}^2(A))$ are defined by an isotropic vector $u \in \Gamma$ (For the definition of Γ, see the summary in the last of the section 3).

Lemma 4.5. Let $v = (1, 0, -l)$ and $l \leq 4$. Then $u \in \Gamma$ satisfies one of the following conditions:

1. $\langle u^2 \rangle = 0$ and $0 < \langle u, v \rangle \leq l$.
2. $\langle (v - u)^2 \rangle = 0$ and $0 < \langle v - u, v \rangle \leq l$.

Proof. We set $w = v - u$. Since $w \in \Gamma$, we may assume that $\langle u^2 \rangle \leq \langle w^2 \rangle$. We shall prove $\langle u^2 \rangle = 0$ and $0 < \langle u, v \rangle \leq l$. Since $\langle u^2 \rangle \leq \langle w^2 \rangle$, we have $\langle u, v \rangle \leq l$. Since $\langle u, w \rangle > 0$ and $\langle u^2 \rangle \geq 0$, we get

$$0 \leq \langle u^2 \rangle < \langle u, v \rangle \leq l.$$

Next we prove $\langle u^2 \rangle = 0$. Since $\langle v^2 \rangle \geq 2(\langle u^2 \rangle + \langle u, w \rangle)$ and $l \leq 4$, we have $4 \geq \langle u^2 \rangle + \langle u, w \rangle$. w lies in Γ, so we get $4 > \langle u^2 \rangle + \sqrt{\langle u^2 \rangle \langle w^2 \rangle} > 2\langle u^2 \rangle$. Since $\langle u^2 \rangle$ is even, the statement follows. □

From now on, we assume that $l = 3$, that is, $v = (1, 0, -3)$. We take $u \in \Gamma$ defining a wall. We may assume that $\langle u^2 \rangle = 0$ and $\langle u, v \rangle = 1, 2$ or 3 by Lemma 4.5. If we assume $u - \lambda v \in v^{\perp}$ where $\lambda \in \mathbb{R}$, then we have $\lambda = \langle u, v \rangle / 6$. Hence, u is represented by

$$u = \frac{\langle u, v \rangle}{6} v + xh + y\delta, \text{ where } x, y \in \mathbb{Q}. \tag{4.12}$$

Since $\frac{6}{\langle u,v\rangle}u - v \in v^\perp \cap H^*(X,\mathbb{Z})_{\mathrm{alg}}$, we get

$$\frac{6}{\langle u,v\rangle}u = v + Xh + Y\delta, \text{ where } X, Y \in \mathbb{Z}. \tag{4.13}$$

Since u is isotropic, (X,Y) is a solution of the equation

$$3Y^2 - nX^2 = 3 \tag{4.14}$$

satisfying

$$\gcd(Y+1, X, 6) = 2, 3, 6. \tag{4.15}$$

Conversely for an integral solution of (4.14) satisfying (4.15), we have a primitive isotropic Mukai vector u satisfying (4.13). If $3 \nmid n$, then $3 \mid X$. Hence we set $Z := X/3$ if $3 \nmid n$. By (4.14), we have Pell equations

$$\begin{cases} Y^2 - \frac{n}{3}X^2 = 1, & \text{if } 3 \mid n \\ Y^2 - 3nZ^2 = 1, & \text{if } 3 \nmid n. \end{cases} \tag{4.16}$$

If $3 \mid n$ and $\sqrt{n/3} \notin \mathbb{Q}$, let (X_1, Y_1) be a fundamental solution of (4.16) and $(X_0, Y_0) := (0, 1)$. We define (X_k, Y_k) by $Y_k + \sqrt{n/3}X_k = (Y_1 + \sqrt{n/3}X_1)^k$. Then (X_k, Y_k) is the solution of (4.14). If $3 \nmid n$, let (Z_1, Y_1) be a fundamental solution of (4.16) and $(Z_0, Y_0) := (0, 1)$. We define (Z_k, Y_k) by $Y_k + \sqrt{3n}Z_k = (Y_1 + \sqrt{3n}Z_1)^k$. We also set $X_k := 3Z_k$. Then (X_k, Y_k) is the solution of (4.14).

Since we consider the connected component of $P^+ \setminus \cup_{u\in\Gamma} u^\perp$ containing $h - \varepsilon\delta$ $(0 < \varepsilon \ll 1)$, we treat $(X,Y) = \pm(X_k, -Y_k)$. Then for the isotropic vector u, the wall $u^\perp$ in P^+ is $\mathbb{R}_{>0}(h - \frac{nX_k}{3Y_k}\delta)$ and

$$\frac{6}{\langle u,v\rangle}u = v \pm X_k h \mp Y_k \delta. \tag{4.17}$$

The following lemma shows that the slope converges monotonically when a Pell equation has infinitely many solutions.

Lemma 4.6. Assume that $\sqrt{n/3} \notin \mathbb{Q}$. Then

$$0 = \frac{X_0}{Y_0} < \frac{X_1}{Y_1} < \frac{X_2}{Y_2} < \frac{X_3}{Y_3} < \cdots, \quad \lim_{k\to\infty} \frac{X_k}{Y_k} = \sqrt{\frac{3}{n}}.$$

Proof. This follows from (4.2). □

Theorem 4.7. ([Mo, Theorem 0.1]) The movable cone $\mathrm{Mov}(\mathrm{Km}^2(A))$ and the nef cone $\mathrm{Nef}(\mathrm{Km}^2(A))$ of $\mathrm{Km}^2(A)$ are characterized by the solution of (4.14) and n as the following table:

<table>
<tr><th colspan="2">type of n</th><th colspan="3">type of (X_1, Y_1)</th><th>$\mathrm{Nef}(\mathrm{Km}^2(A))$</th><th>$\mathrm{Mov}(\mathrm{Km}^2(A))$</th></tr>
<tr><td colspan="2">$3 \nmid n$</td><td colspan="3">$3 \mid X_1$</td><td>$\mathbb{R}_{\geq 0}h + \mathbb{R}_{\geq 0}(h - \frac{nX_1}{3Y_1}\delta)$</td><td>$\mathbb{R}_{\geq 0}h + \mathbb{R}_{\geq 0}(h - \frac{nX_1}{3Y_1}\delta)$</td></tr>
<tr><td rowspan="6">$n = 3m$</td><td rowspan="5">m is not square</td><td colspan="3">$3 \mid X_1$</td><td>$\mathbb{R}_{\geq 0}h + \mathbb{R}_{\geq 0}(h - \frac{nX_1}{3Y_1}\delta)$</td><td>$\mathbb{R}_{\geq 0}h + \mathbb{R}_{\geq 0}(h - \frac{nX_1}{3Y_1}\delta)$</td></tr>
<tr><td rowspan="4">$3 \nmid X_1$</td><td rowspan="2">X_1 even</td><td>$3 \mid Y_1$</td><td>$\mathbb{R}_{\geq 0}h + \mathbb{R}_{\geq 0}(h - \frac{nX_1}{3Y_1}\delta)$</td><td>$\mathbb{R}_{\geq 0}h + \mathbb{R}_{\geq 0}(h - \frac{nX_2}{3Y_2}\delta)$</td></tr>
<tr><td>$3 \nmid Y_1$</td><td>$\mathbb{R}_{\geq 0}h + \mathbb{R}_{\geq 0}(h - \frac{nX_1}{3Y_1}\delta)$</td><td>$\mathbb{R}_{\geq 0}h + \mathbb{R}_{\geq 0}(h - \frac{nX_3}{3Y_3}\delta)$</td></tr>
<tr><td rowspan="2">X_1 odd</td><td>$3 \mid Y_1$</td><td>$\mathbb{R}_{\geq 0}h + \mathbb{R}_{\geq 0}(h - \frac{nX_2}{3Y_2}\delta)$</td><td>$\mathbb{R}_{\geq 0}h + \mathbb{R}_{\geq 0}(h - \frac{nX_2}{3Y_2}\delta)$</td></tr>
<tr><td>$3 \nmid Y_1$</td><td>$\mathbb{R}_{\geq 0}h + \mathbb{R}_{\geq 0}(h - \frac{nX_2}{3Y_2}\delta)$</td><td>$\mathbb{R}_{\geq 0}h + \mathbb{R}_{\geq 0}(h - \frac{nX_3}{3Y_3}\delta)$</td></tr>
<tr><td>m is square</td><td colspan="3"></td><td>$\mathbb{R}_{\geq 0}h + \mathbb{R}_{\geq 0}(h - \sqrt{\frac{n}{3}}\delta)$</td><td>$\mathbb{R}_{\geq 0}h + \mathbb{R}_{\geq 0}(h - \sqrt{\frac{n}{3}}\delta)$</td></tr>
</table>

Proof. We divide the proof into two cases.
(1) We assume that $3 \nmid n$. In this case, $\sqrt{3n} \notin \mathbb{Q}$. Then since $3 \mid X$, we get $(X_1, Y_1) = (3Z_1, Y_1)$, where (Z_1, Y_1) is the fundamental solution of Pell equation

$$Y^2 - (3n)Z^2 = 1.$$

By $3 \mid X_1$, we get $Y_1 \equiv \pm 1 \mod 3$. Hence $\gcd(Y+1, X, 6) = 3, 6$ for $(X, Y) = (X_1, -Y_1), (-X_1, +Y_1)$. Moreover $\gcd(Y+1, X, 6) = 6$ if and only if $2 \mid X_1$. Therefore

$$u = \begin{cases} (\frac{\pm Y_1+1}{3}, \mp \frac{X_1}{3} H, \pm Y_1 - 1), & \text{if } 2 \nmid X_1 \\ (\frac{\pm Y_1+1}{6}, \mp \frac{X_1}{6} H, \frac{\pm Y_1 - 1}{2}), & \text{if } 2 \mid X_1 \end{cases}$$

Then $\langle u, v \rangle = 1$ or 2. Hence we get $\mathrm{Nef}(\mathrm{Km}^2(A)) = \mathrm{Mov}(\mathrm{Km}^2(A))$.

(2) We consider the case of $n = 3m$.

(2-1) We assume that m is not a square number. Then (4.14) is equivalent to the Pell equation

$$Y^2 - (n/3)X^2 = 1.$$

We divide this case into three cases.

(i) We assume that $3 \mid X_1$. Then we can get the statement as with the case of $3 \nmid n$.

(ii) We assume that $2 \mid X_1$ and $3 \nmid X_1$. Since $Y_1^2 \equiv 1 \mod 2$, $\gcd(1 \mp Y_1, \pm X_1, 6) = 2$ and $\langle u, v \rangle = 3$ for $u = (\frac{\pm Y_1+1}{2}, \mp \frac{X_1}{2} H, \frac{3}{2}(\pm Y_1 - 1))$. In order to determine a movable cone, we must consider the next solution $(X_2, Y_2) = (2X_1Y_1, Y_1^2 + mX_1^2)$ by Lemma 4.6.

- Assume that $3 \mid Y_1$. Since $Y_2 = Y_1^2 + mX_1^2 = 2Y_1^2 - 1$, $\gcd(Y_2 + 1, X_2, 6) = 6$ and $\langle v, u \rangle = 1$ for $u = (\frac{Y_1^2}{3}, -\frac{Y_1X_1}{3} H, mX_1^2)$.
- If $3 \nmid Y_1$, then $\gcd(1 \mp Y_2, \pm X_2, 6) = 2$. Thus, we consider the next solution $(X_3, Y_3) = (3X_1Y_1^2 + mX_1^3, Y_1(Y_1^2 + 3mX_1^2))$. Note that $3|m$ since $3 \nmid X_1, 3 \nmid Y_1$ and $Y_1^2 - mX_1^2 = 1$. Since either $Y_3 + 1$ or $Y_3 - 1$ is a multiple of 54, we have $\langle u, v \rangle = 1$ for $u = (\frac{\pm Y_3+1}{6}, \mp \frac{X_3}{6} H, \frac{\pm Y_3 - 1}{2})$.

(iii) We assume that $2 \nmid X_1$ and $3 \nmid X_1$. Since $\gcd(1 \mp Y_1, \pm X_1, 6) = 1$, we consider the next solution (X_2, Y_2).

- If $3 \mid Y_1$, then we also see that $\gcd(Y_2 + 1, -X_2, 6) = 6$ and $u = (\frac{Y_1^2}{3}, -\frac{Y_1X_1}{3} H, mX_1^2)$.
- If $3 \nmid Y_1$, then $\gcd(Y_2 + 1, -X_2, 6) = 2$ and $\langle u, v \rangle = 3$ for $u = (Y_1^2, -Y_1X_1H, 3mX_1^2)$. We consider the next solution (X_3, Y_3). Since $X_3 = X_1(3Y_1^2 + mX_1^2) = X_1(4Y_1^2 - 1)$, $2 \nmid X_3$. By $Y_1^2 \equiv 1 \mod 3$ and $3 \nmid X_1$, $3 \mid m$. Hence we also have $3 \mid X_3$. Therefore $\gcd(Y + 1, -X, 6) = 3$ and $u = (\frac{Y+1}{3}, -\frac{X}{3} H, Y - 1)$ for $(X, Y) = (X_3, -Y_3)$ or $(X, Y) = (-X_3, Y_3)$.

(2-2) We assume that m is square number. Then the statement is showed by [Yo6, Proposition 4.16] and the fact that (4.14) has only trivial solutions $(0, \pm 1)$. □

Remark 4.8. We state the characterization of solutions of Pell equation $Y^2 - mX^2 = 1$ by m:

$$3|m \implies 3 \nmid Y, \quad m = 3k+1 \implies 3|X \text{ and } 3 \nmid Y,$$

$$m = 3k+2 \implies 3|X \text{ and } 3 \nmid Y, \text{ or } 3 \nmid X \text{ and } 3|Y.$$

Actually, they follow by solving the Pell equation in the residue field $\mathbb{F}_3$. In particular, we can see that $3|X_1$ if $m = 3k+1$.

Corollary 4.9. If $3 \nmid n$ or $n \equiv 3 \mod 9$, then $\mathrm{Nef}(\mathrm{Km}^2(A)) = \mathrm{Mov}(\mathrm{Km}^2(A))$.

Example 4.10. We see the case of $3 \nmid n$ and $2 \nmid X_1$. Let $n = 1$. Then (X, Y) satisfies the Pell equation $Y^2 - 3(X/3)^2 = 1$. The fundamental solution of this equation is $(\frac{X_1}{3}, Y_1) = (1, 2)$. Since $3 \mid (Y_1 + 1)$, we have

$$\frac{6}{\langle u, v \rangle} u = v - 3h + 2\delta = (3, -3H, 3) = 3(1, -H, 1),$$

and $\langle u, v \rangle = 2$ by (4.17). That is, the boundaries of $\mathrm{Mov}(\mathrm{Km}^2(A))$ and $\mathrm{Nef}(\mathrm{Km}^2(A))$ are determined for $u = (1, -H, 1)$. Moreover, the slope of $u^{\perp}$ monotonically converges to $-\sqrt{3}/3$. We illustrate the movement that the walls monotonically converge to the boundary of the positive cone (see Figure 1).

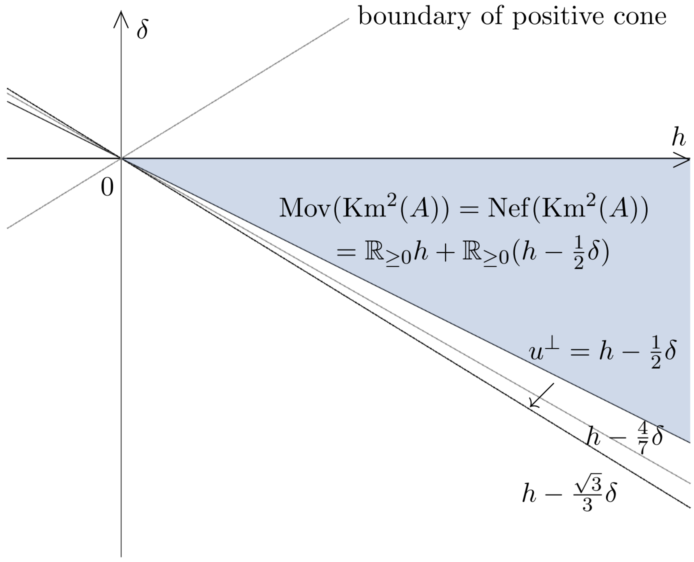


Figure 1: The walls monotonically converge to the boundary of positive cone

Next we see the case of $3 \nmid n$ and $2 \mid X_1$. Let $n = 2$. Then $(\frac{X_1}{3}, Y_1) = (2, 5)$. Since $6 \mid (Y_1 + 1)$, we have

$$\frac{6}{\langle u, v\rangle} = (6, -6H, 12) = 6(1, -H, 2)$$

and $\langle u, v\rangle = 1$ for $u = (1, -H, 2)$. Hence

$$\mathrm{Nef}(\mathrm{Km}^2(A)) = \mathrm{Mov}(\mathrm{Km}^2(A)) = \mathbb{R}_{\geq 0}h + \mathbb{R}_{\geq 0}(h - \tfrac{4}{5}\delta).$$

Example 4.11. We see the case of $3 \mid n, 3 \mid X_1$ and $2 \nmid X_1$. Let $n = 21$. Then $(X_1, Y_1) = (3, 8)$. Since $3 \mid (Y_1 + 1)$, we have

$$\frac{6}{\langle u, v\rangle} = (9, -3H, 21) = 3(3, -H, 7)$$

and $\langle u, v\rangle = 2$ for $u = (3, -H, 7)$. Hence

$$\mathrm{Nef}(\mathrm{Km}^2(A)) = \mathrm{Mov}(\mathrm{Km}^2(A)) = \mathbb{R}_{\geq 0}h + \mathbb{R}_{\geq 0}(h - \tfrac{21}{8}\delta).$$

Next We see the case of $3 \mid n$ and $6 \mid X_1$. Let $n = 63$. Then $(X_1, Y_1) = (12, 55)$. Since $6 \mid (1 - Y_1)$, We have

$$\frac{6}{\langle u, v\rangle} = (-54, 12H, -168) = -6(9, -2H, 28)$$

and $\langle u, v\rangle = 1$ for $u = -(9, -2H, 28)$. Hence

$$\mathrm{Nef}(\mathrm{Km}^2(A)) = \mathrm{Mov}(\mathrm{Km}^2(A)) = \mathbb{R}_{\geq 0}h + \mathbb{R}_{\geq 0}(h - \tfrac{252}{55}\delta).$$

Example 4.12. We see the case of $3 \mid n, 3 \nmid X_1, 2 \mid X_1$ and $3 \mid Y_1$. Let $n = 15$. Then $(X_1, Y_1) = (4, 9)$. We have

$$\frac{6}{\langle u, v\rangle} = (10, -4H, 24), -(8, -4H, 30) = 2(5, -2H, 12), -2(4, -2H, 15)$$

and $\langle u, v\rangle = 3$ for $u = (5, -2H, 12), -(4, -2H, 15)$. Hence $\mathrm{Nef}(\mathrm{Km}^2(A)) = \mathbb{R}_{\geq 0}h + \mathbb{R}_{\geq 0}(h - \frac{10}{9}\delta)$. We consider $(X_2, Y_2) = (72, 161)$. Since $6 \mid (Y_2 + 1)$, we have

$$\frac{6}{\langle u, v\rangle} = (162, -72H, 480) = 6(27, -12H, 80)$$

and $\langle u, v\rangle = 1$ for $u = (27, -12H, 80)$. Hence $\mathrm{Mov}(\mathrm{Km}^2(A)) = \mathbb{R}_{\geq 0}h + \mathbb{R}_{\geq 0}(h - \frac{360}{161}\delta)$.

Next we see the case of $3 \mid n, 3 \nmid X_1, 2 \mid X_1$ and $3 \nmid Y_1$. Let $n = 18$. Then $(X_1, Y_1) = (2, 5)$. We have

$$\frac{6}{\langle u, v\rangle} = v \pm 2h \mp 5\delta = 2(3, -H, 6), -2(2, -H, 9)$$

and $\langle u, v\rangle = 3$ for $u = (3, -H, 6), -(2, -H, 9)$. Hence $\mathrm{Nef}(\mathrm{Km}^2(A)) = \mathbb{R}_{\geq 0}h + \mathbb{R}_{\geq 0}(h - \frac{12}{5}\delta)$. We consider $(X_2, Y_2) = (20, 49)$. We have

$$\frac{6}{\langle u, v\rangle} = v \pm 20h \mp 49\delta = 2(25, -10H, 72), -2(24, -10H, 75)$$

and $\langle u, v\rangle = 3$ for $u = (25, -10H, 72), -(24, -10H, 75)$. Hence we consider $(X_3, Y_3) = (198, 485)$. Since $6 \mid (Y_1 + 1)$, we have

$$\frac{6}{\langle u, v\rangle} = (486, -198H, 1452) = 6(81, -33H, 242)$$

and $\langle u, v\rangle = 1$ for $u = (81, -33H, 242)$. Hence $\mathrm{Mov}(\mathrm{Km}^2(A)) = \mathbb{R}_{\geq 0}h + \mathbb{R}_{\geq 0}(h - \frac{1188}{485}\delta)$.

Example 4.13. We see the case of $3 \mid n, 3 \nmid X_1, 2 \nmid X_1$ and $3 \mid Y_1$. Let $n = 24$. Then $(X_1, Y_1) = (1, 3)$. We have

$$\frac{6}{\langle u, v\rangle} = v \pm h \mp 3\delta = (4, -H, 6), -(2, -H, 12).$$

Since $\langle u, v\rangle = 1, 2$ or 3, we consider $(X_2, Y_2) = (6, 17)$. Since $6 \mid (Y_1 + 1)$, we have

$$\frac{6}{\langle u, v\rangle} = (18, -6H, 48) = 6(3, -H, 8)$$

and $\langle u, v\rangle = 1$ for $u = (3, -H, 8)$. Hence

$$\mathrm{Nef}(\mathrm{Km}^2(A)) = \mathrm{Mov}(\mathrm{Km}^2(A)) = \mathbb{R}_{\geq 0}h + \mathbb{R}_{\geq 0}(h - \tfrac{48}{17}\delta).$$

Next We see the case of $3 \mid n, 3 \nmid X_1, 2 \nmid X_1$ and $3 \nmid Y_1$. Let $n = 45$. Then $(X_1, Y_1) = (1, 4)$. We have

$$\frac{6}{\langle u, v\rangle} = v \pm h \mp 4\delta = (5, -H, 9), -(3, -H, 15).$$

Hence we consider $(X_2, Y_2) = (8, 31)$. Since $6 \mid (Y_2 + 1)$, we have

$$\frac{6}{\langle u, v\rangle} = (32, -8H, 90) = 2(16, -4H, 45)$$

and $\langle u, v\rangle = 3$ for $u = (16, -4H, 45)$. Hence $\mathrm{Nef}(\mathrm{Km}^2(A)) = \mathbb{R}_{\geq 0}h + \mathbb{R}_{\geq 0}(h - \frac{120}{31}\delta)$. We consider $(X_3, Y_3) = (63, 244)$. Since $3 \mid (1 - Y_3)$, we have

$$\frac{6}{\langle u, v\rangle} = -(243, -63H, 245) = -3(81, -21H, 245)$$

and $\langle u, v\rangle = 2$ for $u = -(81, -21H, 245)$. Hence $\mathrm{Mov}(\mathrm{Km}^2(A)) = \mathbb{R}_{\geq 0}h + \mathbb{R}_{\geq 0}(h - \frac{945}{244}\delta)$.

By the proof of Theorem 4.7, we have the following.

Corollary 4.14. Assume that $\sqrt{3n} \notin \mathbb{Q}$. In the following cases, the non-trivial boundary of $\mathrm{Mov}(\mathrm{Km}^2(A))$ is given by Hilbert-Chow contractions:

1. $2 \mid X_1$.
2. $3 \mid n$, $X_1 \equiv \pm 1 \mod 6$ and $3 \mid Y_1$.

5 Chamber Decomposition and Birational Correspondences.

We calculate the boundary of movable cones and nef cones of $\mathrm{Km}^2(A)$ in section 4. We can see them as half lines in $h\delta$-plane. For $\mathrm{Mov}(\mathrm{Km}^2(A))$, we have a chamber decomposition such that each chamber is an ample cone of a minimal model of $\mathrm{Km}^2(A)$. In this section, we shall describe the decomposition and some flops which appears for the minimal models of $\mathrm{Km}^2(A)$. We also see how minimal models can be described.

5.1 Some birational correspondences of the minimal models.

We shall describe some flops which appear for the minimal models of $\mathrm{Km}^l(A)$. For this purpose, we need to describe other minimal models of $\mathrm{Km}^l(A)$ by using the moduli of stable sheaves.

Let $M_H(v)$ be the moduli space of semi-stable sheaves E on A with $v(E) = v$. If v is primitive and $\langle v^2 \rangle \geq 6$, then we have albanese map $\mathfrak{a} : M_H(v) \to \mathrm{Pic}^0(A) \times A$, which is locally trivial. Let $K_H(v)$ be a fiber of $\mathfrak{a}$. Then $K_H(v)$ is an irreducible symplectic manifold deformation equivalent to $\mathrm{Km}^l(A)$, where $\langle v^2 \rangle = 2l + 2$ [Yo2].

5.1.1 Uhlenbeck compactification of the moduli space of μ-stable vector bundles

Let $N_H(v)$ be the Uhlenbeck compactification of the moduli space of μ-stable vector bundles E on A with $v(E) = v$ [HL, 8.2]. Thus $N_H(v)$ parameterizes pairs (E, Z) of a polystable vector bundle E and a 0-cycle Z with $v(E) - (0, 0, \deg Z) = v$.

Remark 5.1. In [HL], determinant line bundle is fixed. So we apply the construction to the family $M_H(v) \to \mathrm{Pic}(A)$ of moduli spaces, and get a family of moduli spaces $N_H(v) \to \mathrm{Pic}(A)$.

For a μ-semi-stable sheaf E, there is a filtration

$$0 \subset F_1 \subset F_2 \subset \cdots \subset F_s = E \tag{5.1}$$

such that $E_i := F_i/F_{i-1}$ are μ-stable sheaves. Let $Z(E_i)$ be the 0-cycle defined as the support of $E_i^{\vee\vee}/E_i$. Then $S(E) := (\oplus_i E_i^{\vee\vee}, \sum_i Z(E_i))$ is independent of the choice of the filtration (5.1), where $E_i^{\vee} := \mathcal{H}om_{\mathcal{O}_A}(E_i, \mathcal{O}_A)$. Hence we have a map

$$\begin{array}{ccc} M_H(v) & \to & N_H(v) \\ E & \mapsto & S(E) \end{array} \tag{5.2}$$

which is a projective morphism. Assume that H is general with respect to v. Then there is a line bundle $\mathcal{L}_v$ on $M_H(v)$ such that $c_1(\mathcal{L}_v) = \theta_v((0, rH, (H, \xi)))$ and the linear system $H^0(\mathcal{L}_v^{\otimes m})$ $(m \gg 0)$ defines a morphism $M_H(v) \to \mathbb{P}^N$ whose image is $N_H(v)$, where $v = (r, \xi, a)$. Since $M_H(v)$ is normal, $N_H(v)$ is also normal.

The following claim is a consequence of [Yo6].

Proposition 5.2. Assume that $\rho(A) = 1$. If $r \geq 3$ and $\gcd(r, d) = 1$, then we have an isomorphism $N_H(r, dH, a) \cong N_H(r, -dH, a)$ with a commutative diagram

$$\begin{array}{ccc} M_H(r, dH, a) & \cdots\to & M_H(r, -dH, a) \\ {\scriptstyle S}\downarrow & & \downarrow{\scriptstyle S} \\ N_H(r, dH, a) & \longrightarrow & N_H(r, -dH, a) \end{array} \tag{5.3}$$

Proof. By taking the dual, we have a birational map $M_H(r, dH, a) \cdots\to M_H(r, -dH, a)$ which is regular up to codimension 2. Moreover we have an identification $H^0(\mathcal{L}_v^{\otimes m}) \cong H^0(\mathcal{L}_{v'}^{\otimes m})$, where $v' = (r, -dH, a)$. Hence we have a natural identification $N_H(v) \cong N_H(v')$ with the diagram (5.3). □

Assume that $v = (r, dH, a)$ satisfies $\langle v^2 \rangle = 2r$ and $\gcd(r, d) = 1$. We set

$$M_H(r, dH, a)_1 := \{E \in M_H(r, dH, a) \mid \ E \text{ is not locally free}\}.$$

For $E \in M_H(r, dH, a)_1$, we have an exact sequence

$$0 \to E \to F \to \mathbb{C}_x \to 0 \tag{5.4}$$

where $F \in M_H(r, dH, a+1)$ is semi-homogeneous ([Mu1]) and $x \in A$. $M_H(r, dH, a)_1$ is a $\mathbb{P}^{r-1}$-bundle over $M_H(r, dH, a+1) \times A$. We set

$$M_H(r, dH, a)^* := \{E \in \mathcal{D}(A) \mid D(E) \in M_H(r, -dH, a)\}. \tag{5.5}$$

where

$$D(E) := \mathbf{R}\mathcal{E}xt_{\mathcal{O}_A}(E, \mathcal{O}_A) \in \mathcal{D}(A)$$

is the derived dual of E. We also have a contraction map

$$S : M_H(r, dH, a)^* \to M_H(r, -dH, a) \to N_H(r, -dH, a) \to N_H(r, dH, a).$$

By Proposition 5.2, we have a flopping diagram along $M_H(r, dH, a)_1$:

$$\begin{array}{ccc} M_H(r, dH, a) & \cdots\to & M_H(r, dH, a)^* \\ {\scriptstyle S}\downarrow & & \downarrow{\scriptstyle S} \\ N_H(r, dH, a) & = & N_H(r, dH, a) \end{array} \tag{5.6}$$

Assume that $(r, dH, a+k)$ is primitive for all $k \geq 0$. Then $M_H(r, dH, a)$ consists of μ-stable sheaves.

Let $M_H(r, dH, a) \to \mathrm{Pic}^0(A) \times A$ be the albanese map. It factors through the Uhlenbeck compactification:

$$M_H(r, dH, a) \xrightarrow{S} N_H(r, dH, a) \xrightarrow{\psi} \mathrm{Pic}^0(A) \times A.$$

In order to describe the flop of a fiber, let us study the morphism

$$\psi \circ S : M_H(r, dH, a)_1 \to M_H(r, dH, a+1) \times A \to \mathrm{Pic}^0(A) \times A.$$

We set

$$\psi((F, x)) := ((\det F) \otimes (\det F_0)^\vee, x + \psi_2(F)) \in \mathrm{Pic}^0(A) \times A,$$

where F_0 is a fixed member of $M_H(r, dH, a+1)$. Then we have

$$\psi^{-1}((0,0)) = \{(F, -\psi_2(F)) \mid F \in M_H(r, dH, a+1), \det F = \det F_0\}.$$

By Lemma 5.3, we get $\#\psi^{-1}((0,0)) = r^2$. Hence the intersection of $M_H(r, dH, a)_1$ and a fiber of the albanese map is r^2 copies of $\mathbb{P}^{r-1}$.

Lemma 5.3. Let $u = (r, dH, a)$ be a primitive and isotropic Mukai vector with $r > 0$. Let $\det : M_H(u) \to \mathrm{Pic}^{dH}(A)$ be the map by $E \mapsto \det E$. Then $\det$ is a finite map of degree r^2.

Proof. We take $E \in M_H(u)$. Then we have a surjective morphism $A \to M_H(u)$ by sending $x \in A$ to $T_x^* E$. We set $K(E) := \{x \in A \mid T_x^*(E) \cong E\}$. By [Mu1, Cor. 7.9],

$$\frac{\#K(\det E)}{\#K(E)} = \frac{\chi(\det E)^2}{\chi(E)^2} = r^2.$$

Hence the claim holds. □

Remark 5.4. A related result is contained in [Yo2, 4.4].

We set $v = (1, 0, -r)$ and $\langle v, u \rangle = r \geq 3$ for an isotropic vector u. We assume that $w := v - u$ is primitive. Then $M_H(u)$ is fine or $M_H(w)$ is fine. Assume that $A' := M_H(u)$ is fine. Let $\mathcal{C}_\pm$ be two chambers separated by u and $M_\pm$ the associated minimal models. By the description of the movable cone in Theorem 3.6 and Proposition 3.5, they are the albanese fibers of moduli of Bridgeland stable objects.

Proposition 5.5. We set $v = (1, 0, -r)$ and $\langle v, u\rangle = r \geq 3$ for an isotropic vector u. We assume that $A' := M_H(u)$ is fine. Let $\mathcal{C}_\pm$ be two chambers separated by u and $M_\pm$ the associated minimal models. Then the birational map $M_- \cdots \to M_+$ is the flop along r^2-copies of $\mathbb{P}^{r-1}$.

Proof. Let $\mathbf{P}$ be a universal family on $A \times A'$ and $\Phi_{A\to A'}^{\mathbf{P}^\vee[1]} : \mathcal{D}(A) \to \mathcal{D}(A')$ the Fourier-Mukai transform associated to $\mathbf{P}$. Then by [MYY1, Thm. 4.9], there are integers d', a' and

$$\{\Phi_{A\to A'}^{\mathbf{P}^\vee[1]}(M_+), \{\Phi_{A\to A'}^{\mathbf{P}^\vee[1]}(M_-)\} = \{K_{H'}(r, d'H', a'), K_{H'}(r, d'H', a')^*\}$$

where H' is an ample generator of $\mathrm{NS}(A')$. Hence $M_- \cdots \to M_+$ is the flop along r^2-copies of $\mathbb{P}^{r-1}$. □

5.1.2 The dependence of $\mathrm{Km}^l(A)$ on A

In this subsection, we shall study the dependence of $\mathrm{Km}^l(A)$ on A. Let D_A be the exceptional divisor of the Hilbert-Chow morphism $\mathrm{Hilb}_A^l \to S^l A$. We shall describe a smooth model of D_A. For the ideal sheaf I_Δ of the diagonal, we have $p_{1*}(I_\Delta/I_\Delta^2) \cong \Omega_A$, where $p_1 : A \times A \to A$ is the first projection. We set $P := \mathbb{P}(\Omega_A)$. Then $f : P \to A$ is a $\mathbb{P}^1$-bundle over A and there is a surjective homomorphism

$$f^*\Omega_A \to \mathcal{O}_P(\lambda),$$

where $\mathcal{O}_P(\lambda)$ is the tautological line bundle on P. Let I'_Δ be the pull-back of I_Δ by $P \times A \to A \times A$. Then $\mathcal{O}_{P\times A}/I'_\Delta \cong \mathcal{O}_\Gamma$, where $\iota : \Gamma \hookrightarrow P \times A$ is the graph of f. Since $I'_\Delta/(I'_\Delta)^2 \cong \iota_*(f^*\Omega_A)$, we have a surjective homomorphism

$$\varphi : I'_\Delta \to I'_\Delta/(I'_\Delta)^2 \to \mathcal{O}_\Gamma(\lambda). \tag{5.7}$$

Then $I_{\mathcal{Z}_1} := \ker \varphi$ is a family of ideal sheaves of colength 2.

Let $I_{\mathcal{Z}_2}$ be the universal ideal sheaf on $\mathrm{Hilb}_A^{l-2} \times A$. Let $I'_{\mathcal{Z}_1}$ and $I'_{\mathcal{Z}_2}$ be the pull-backs of $I_{\mathcal{Z}_1}$ and $I_{\mathcal{Z}_2}$ to $(A \times \mathrm{Hilb}_A^{l-2}) \times A$ and set $I := I'_{\mathcal{Z}_1} \cap I'_{\mathcal{Z}_2}$. Then I is a family of ideal sheaves of colength l on an open subset U of $A \times \mathrm{Hilb}_A^{l-2}$. Hence we have a birational map $A \times \mathrm{Hilb}_A^{l-2} \cdots \to D_A$. We next describe a desingularization of $d_A := D_A \cap \mathrm{Km}^{l-1}(A)$. The restriction of the albanese map of Hilb_A^l to D_A induces a map

$$\begin{array}{ccccc} A \times \mathrm{Hilb}_A^{l-2} & \to & A \times A & \to & A, \\ (x, Z_2) & \mapsto & (x, a(Z_2)) & \mapsto & 2x + a(Z_2) \end{array} \tag{5.8}$$

where $a : \mathrm{Hilb}_A^{l-2} \to A$ is the albanese map. We define $\psi : P \to A$ by $\psi(t) := -2f(t)$. Let us consider the following fiber product:

$$\begin{array}{ccc} \mathbf{H}_A := P \times_A \mathrm{Hilb}_A^{l-2} & \longrightarrow & \mathrm{Hilb}_A^{l-2} \\ {\scriptstyle 1_P \times a}\big\downarrow & & \big\downarrow{\scriptstyle a} \\ P & \xrightarrow{\psi} & A \end{array} \tag{5.9}$$

Then the rational map $\mathbf{H}_A \cdots \to D_A$ induces a birational map $\mathbf{H}_A \cdots \to d_A$ and $\mathbf{H}_A$ gives a desingularization of d_A. Since a is a locally trivial fibration with the fiber $\mathrm{Km}^{l-3}(A)$, Lemma 5.7 implies $\mathbf{H}_A \to P \xrightarrow{f} A$ is the albanese map of $\mathbf{H}_A$.

Proposition 5.6. If $\mathrm{Km}^l(A) \cong \mathrm{Km}^l(B)$ and the isomorphism preserves the Hilbert-Chow contractions. Then $A \cong B$.

Proof. We note that $d_A \cong d_B$. Then $\mathbf{H}_A$ and $\mathbf{H}_B$ are birationally equivalent. By Lemma 5.7, $A \cong B$. □

Lemma 5.7. Let $f : X \to Y$ be a smooth morphism of smooth projective varieties. Assume that $H^1(f^{-1}(y), \mathcal{O}_{f^{-1}(y)}) = 0$ for all $y \in Y$. Then $\mathrm{Alb}(X) \to \mathrm{Alb}(Y)$ is isomorphic.

Proof. Let L be a line bundle on X such that $c_1(L) = 0$. Since $f^{-1}(y)$ is a smooth projective variety such that $H^1(f^{-1}(y), \mathcal{O}_{f^{-1}(y)}) = 0$, $\mathrm{Pic}^0(f^{-1}(y)) = 0$. Hence $c_1(L) = 0$ implies $L_{|f^{-1}(y)} \cong \mathcal{O}_{f^{-1}(y)}$. Hence $H^1(f^{-1}(y), L_{|f^{-1}(y)}) = 0$. By the base change theorem, $f_*(L)$ is a locally free sheaf of rank 1 and $f^*f_*(L) \to L$ is isomorphic. Hence $\mathrm{Pic}^0(Y) \to \mathrm{Pic}^0(X)$ is surjective. Since $f_*(\mathcal{O}_X) = \mathcal{O}_Y$, $\mathrm{Pic}^0(Y) \to \mathrm{Pic}^0(X)$ is an isomorphism. Therefore $\mathrm{Alb}(X) \to \mathrm{Alb}(Y)$ is isomorphic. □

5.2 Chamber Decomposition of $\mathrm{Mov}(\mathrm{Km}^2(A))$.

For $\mathrm{Mov}(\mathrm{Km}^2(A))$, we have a chamber decomposition such that each chamber is an ample cone of a minimal model of $\mathrm{Km}^2(A)$. In this subsection, we shall describe the decomposition. By Theorem 4.7, it is sufficient to treat the following 3 cases:

(1) $3 \nmid X_1$, $2 \mid X_1$ and $3 \mid Y_1$.

(2) $3 \nmid X_1$, $2 \mid X_1$ and $3 \nmid Y_1$.

(3) $3 \nmid X_1$, $2 \nmid X_1$ and $3 \nmid Y_1$.

For other case, $\mathrm{Nef}(\mathrm{Km}^2(A)) = \mathrm{Mov}(\mathrm{Km}^2(A))$. Before explaining chamber decompositions, we prepare the following lemma.

Lemma 5.8. For an abelian surface A, $\mathrm{End}(A) \cong \mathbb{Z}$ if and only if $\rho(A) = 1$.

Proof. We set $D := \mathrm{End}(A)_{\mathbb{Q}}$. We shall prove that $\mathrm{End}(A) \cong \mathbb{Z}$ if $\rho(A) = 1$. Since $\mathrm{End}(A)$ is a free abelian group of finite rank, we shall prove that $D \cong \mathbb{Q}$. We may assume that A is simple, that is, D is a division algebra. By [BL, Exercise 9.10 (1), (4)], D is neigher a totally definite quaternion algebra over $\mathbb{Q}$ nor an imaginary quadratic number field. By [BL, Prop. 5.5.7], $\rho(A) = 1$ implies $D = \mathbb{Q}$. □

Case (1). Since $Y_1^2 - mX_1^2 = 1$, $m \equiv -1 \mod 3$. Then $\mathrm{Mov}(\mathrm{Km}^2(A))$ has two chambers

$$\begin{aligned} \mathcal{C}_1 :=& \mathbb{R}_{>0}h + \mathbb{R}_{>0}(h - \tfrac{nX_1}{3Y_1}\delta), \\ \mathcal{C}_2 :=& \mathbb{R}_{>0}(h - \tfrac{nX_1}{3Y_1}\delta) + \mathbb{R}_{>0}(h - \tfrac{nX_2}{3Y_2}\delta). \end{aligned} \tag{5.10}$$

We set $M_1 := \mathrm{Km}^2(A)$. Let M_2 be the minimal model of M_1 such that $\mathrm{Amp}(M_2) = \mathcal{C}_2$. Then $\langle u, (1, 0, -3)\rangle = 1$ for $u = (\frac{Y_1^2}{3}, -\frac{Y_1X_1}{3}H, mX_1^2)$ and $M_2 \cong \mathrm{Km}^2(A')$, where $A' := M_H(u)$. Moreover u corresponds to the Hilbert-Chow contraction of $\mathrm{Km}^2(A')$ by the proof of [Yo2, Proposition 3.27 (2)]. Since the birational map between 4-dimensional hyperKähler manifold can be decomposed into Mukai flops [WW, Theorem 1.2], M_2 is a flop of M_1 along copies of $\mathbb{P}^2$. Then Lemma 5.10 implies $A' \ncong A$ and $M_1 \ncong M_2$ by $3 \mid Y_1$.

Case (2). In this case, $3 \mid m$ and $\mathrm{Mov}(\mathrm{Km}^2(A))$ is divided into 3 chambers

$$\begin{aligned} \mathcal{C}_1 :=& \mathbb{R}_{>0}h + \mathbb{R}_{>0}(h - \tfrac{nX_1}{3Y_1}\delta), \\ \mathcal{C}_2 :=& \mathbb{R}_{>0}(h - \tfrac{nX_1}{3Y_1}\delta) + \mathbb{R}_{>0}(h - \tfrac{nX_2}{3Y_2}\delta), \\ \mathcal{C}_3 :=& \mathbb{R}_{>0}(h - \tfrac{nX_2}{3Y_2}\delta) + \mathbb{R}_{>0}(h - \tfrac{nX_3}{3Y_3}\delta). \end{aligned} \tag{5.11}$$

We set $M_1 = \mathrm{Km}^2(A)$. Let M_2, M_3 be the minimal models of M_1 with $\mathrm{Amp}(M_i) = \mathcal{C}_i$ $(i = 2, 3)$. By Corollary 4.14, $M_3 \cong \mathrm{Km}^2(A')$, where $A' = M_H(u)$. Since $Y_3 \equiv Y_1 \mod p$ and $Y_3 \equiv Y_1 \mod 4$ for all prime divisors $p > 2$ of n, we have $\pm Y_3 \not\equiv -1 \mod p$ and $\pm Y_3 \not\equiv -1 \mod 4$ if $2 \mid n$ by Lemma 4.2. Then Lemma 5.10 implies $A' \ncong A$ and $M_1 \ncong M_3$.

Case (3). In this case, $3 \mid m$ and $\mathrm{Mov}(\mathrm{Km}^2(A))$ has two chambers

$$\begin{aligned} \mathcal{C}_1 :=& \mathbb{R}_{>0}h + \mathbb{R}_{>0}(h - \tfrac{nX_2}{3Y_2}\delta), \\ \mathcal{C}_2 :=& \mathbb{R}_{>0}(h - \tfrac{nX_2}{3Y_2}\delta) + \mathbb{R}_{>0}(h - \tfrac{nX_3}{3Y_3}\delta). \end{aligned} \tag{5.12}$$

We set $M_1 := \mathrm{Km}^2(A)$. Let M_2 be the minimal model of M_1 such that $\mathrm{Amp}(M_2) = \mathcal{C}_2$. By the proof of Theorem 4.7 case (iii), $\langle u, (1, 0, -3)\rangle \neq \pm 1$. Therefore $M_1 \ncong M_2$.

By Case (1), Case (2) and Case (3), we can see the number of minimal models.

Corollary 5.9. Let (X_1, Y_1) be the fundamental solution of (4.16). Then we have the following table:

	the conditions of (X_1, Y_1)	the number of minimal models
(1)	$3 \nmid X_1$, $2 \mid X_1$ and $3 \mid Y_1$	2
(2)	$3 \nmid X_1$, $2 \mid X_1$ and $3 \nmid Y_1$	3
(3)	$3 \nmid X_1$, $2 \nmid X_1$ and $3 \nmid Y_1$	2
(4)	Otherwise	1

Lemma 5.10. (1) For a solution (X,Y) of (0.2), assume that $u=(\frac{Y+1}{6},\frac{X}{6}H,\frac{Y-1}{2})$ is a primitive vector. Then $u=(na^2,abH,b^2)$ if and only if $Y\equiv -1 \mod p$ for all prime divisors $p>2$ of n and $Y\equiv -1 \mod 4$ if n is even.

(2) $M_H(u)\cong A$ if and only if $(s,t)=(n,1)$.

(3) We assume that $\mathrm{Nef}(\mathrm{Km}^2(A))\neq \mathrm{Mov}(\mathrm{Km}^2(A))$. Then $\mathrm{Km}^2(M_H(u))\cong \mathrm{Km}^2(A)$ for u in (1) if and only if $Y\equiv -1 \mod p$ for all prime divisors $p>2$ of n and $Y\equiv -1 \mod 4$ if n is even.

Proof. (1) We write $u=\pm(sa^2,abH,tb^2)$ with $st=n$. Then $\pm(3tb^2-sa^2)=\langle v,u\rangle=1$. If $s=n$, then $3na^2-b^2=\pm 1$ implies $p\nmid (Y-1)$ for all prime divisors $p>2$ of n. Moreover if $2\mid n$, then $(Y-1)/2$ is odd.

Conversely if the conditions hold, then $\gcd(s,t)=1$. Hence $p\nmid (Y-1)$ for all prime divisors $p>2$ of n. Moreover if $2\mid n$, then $(Y-1)/2$ is odd. Therefore $s=n$.

(2) The claim is a consequence of [YY, Lemma 7.3]. (3) If $\mathrm{Km}^2(M_H(u))\cong \mathrm{Km}^2(A)$, then the isomorphism preserves the Hilbert-Chow contractions. In particular the isomorphism induces an isomorphism of the exceptional divisors. Since the Albanese varieties are $M_H(u)$ and A respectively [Na] (or Proposition 5.6), we have $M_H(u)\cong A$. □

Remark 5.11. There is the assumption $\mathrm{End}(A)\cong\mathbb{Z}$ in [YY, Lemma 7.3]. By Lemma 5.8, we can remove the assumption.

For M_1,M_2 and M_3 in Case (1), (2), (3), there are birational maps $M_1\cdots\to M_2$ and $M_2\cdots\to M_3$. By [WW], we can see M_2 (resp. M_3) is a flop of M_1 (resp. M_2) along copies $\mathbb{P}^2$. In what follows, we shall explain M_2 (resp. M_3) is the flop of M_1 (resp. M_2) along 9 copies of $\mathbb{P}^2$.

If $\langle v,u\rangle=3$ for an isotropic vector u, then $w:=v-u$ is also an isotropic vector with $\langle v,w\rangle=3$. Since $\mathrm{rk} v=1$, $3\nmid \mathrm{rk} u$ or $3\nmid \mathrm{rk} w$. Therefore $\langle u,z\rangle=1$ or $\langle w,z\rangle=1$ for a Mukai vector z. Thus $M_H(u)$ is fine or $M_H(w)$ is fine. We may assume that $A':=M_H(u)$ is fine. If w is not primitive, then $w=3w'$ for a primitive isotropic vector w' and $v=u+3w'$. So we may assume that w is primitive. If $E\in M_1\setminus M_2$, then we have an exact triangle

$$E_1\to E\to E_2\to E_1[1] \tag{5.13}$$

such that E_1 and $E_2[-1]$ are semi-homogeneous sheaves with $\{v(E_1),v(E_2)\}=\{u,w\}$. For $E'\in M_2\setminus M_1$, we also have an exact triangle

$$E_2\to E'\to E_1\to E_2[1] \tag{5.14}$$

Assume that $v(E_1)=u$. By the Fourier-Mukai transform $\Phi_{A\to A'}^{\mathbf{P}^\vee[1]}:\mathcal{D}(A)\to\mathcal{D}(A')$ defined by a universal family $\mathbf{P}$ on $A\times A'$, we have an isomorphism $M_1\cong K_{H'}(3,d'H',a')$ and $\widehat{E}:=\Phi_{A\to A'}^{\mathbf{P}^\vee[1]}(E)$ fits in an exact sequence

$$0\to\widehat{E}\to F\to\mathbb{C}_x\to 0, \tag{5.15}$$

where $\mathbb{C}_x:=\Phi_{A\to A'}^{\mathbf{P}^\vee}(E_1)[2]\in M_{H'}(0,0,1)$ and $F:=\Phi_{A\to A'}^{\mathbf{P}^\vee}(E_2)[2]\in M_{H'}(3,d'H',a'+1)$. For $\widehat{E'}:=\Phi_{A\to A'}^{\mathbf{P}^\vee[1]}(E')$, we have an exact triangle

$$0\to F\to\widehat{E'}\to\mathbb{C}_x[-1]\to F[1] \tag{5.16}$$

and an exact sequence

$$0\to D(\widehat{E'})\to D(F)\to\mathbb{C}_x\to 0, \tag{5.17}$$

where $D(F)\in M_{H'}(3,-d'H',a'+1)$. We set

$$M_{H'}(3,d'H',a')^*:=\{E\in\mathcal{D}(A')\mid D(E)\in M_{H'}(3,-d'H',a')\}.$$

Then M_2 is isomorphic to $K_{H'}(3,d'H',a')^*$ by $\Phi_{A\to A'}^{\mathbf{P}^\vee[1]}$, where $K_{H'}(3,d'H',a')^*$ is a fiber of the albanese map of $M_{H'}(3,d'H',a')$.

For Case (1), we have a decomposition

$$(1,0,-3) = (\tfrac{Y_1+1}{2}, -\tfrac{X_1}{2}H, \tfrac{3}{2}(Y_1-1)) - (\tfrac{Y_1-1}{2}, -\tfrac{X_1}{2}H, \tfrac{3}{2}(Y_1+1)). \tag{5.18}$$

We set $M_- := M_1, M_+ := M_2, \mathcal{C}_- := \mathcal{C}_1$ and $\mathcal{C}_+ := \mathcal{C}_2$. We also set $u = (\frac{Y_1+1}{2}, -\frac{X_1}{2}H, \frac{3}{2}(Y_1-1))$ and $w = (\frac{Y_1-1}{2}, -\frac{X_1}{2}H, \frac{3}{2}(Y_1+1))$. Then $M_H(u)$ and $M_H(w)$ are fine by $3 \mid Y_1$. By Proposition 5.5, the birational map $M_1 \cdots \to M_2$ is the flop along 9-copies of $\mathbb{P}^2$.

For Case (2), we have a decomposition

$$(1,0,-3) = (\tfrac{Y_1+1}{2}, -\tfrac{X_1}{2}H, \tfrac{3}{2}(Y_1-1)) - (\tfrac{Y_1-1}{2}, -\tfrac{X_1}{2}H, \tfrac{3}{2}(Y_1+1)). \tag{5.19}$$

We set $M_- := M_1, M_+ := M_2, \mathcal{C}_- := \mathcal{C}_1$ and $\mathcal{C}_+ := \mathcal{C}_2$. We also set $u = (\frac{Y_1+1}{2}, -\frac{X_1}{2}H, \frac{3}{2}(Y_1-1))$ and $w = (\frac{Y_1-1}{2}, -\frac{X_1}{2}H, \frac{3}{2}(Y_1+1))$. Since $3 \nmid Y_1$, $3 \nmid \frac{Y_1+1}{2}$ or $3 \nmid \frac{Y_1-1}{2}$. Thus $M_H(u)$ is fine or $M_H(w)$ is fine. By Proposition 5.5, the birational map $M_1 \cdots \to M_2$ is the flop along 9-copies of $\mathbb{P}^2$. Next we set $M_- := M_2, M_+ := M_3, \mathcal{C}_- := \mathcal{C}_2$ and $\mathcal{C}_+ := \mathcal{C}_3$. We also have a decomposition

$$(1,0,-3) = (Y_1^2, -X_1Y_1H, 3mX_1^2) - (mX_1^2, -X_1Y_1H, 3Y_1^2). \tag{5.20}$$

Since $3 \mid m$ and $3 \nmid Y_1$, $M_H(u)$ is fine if and only if $u = (Y_1^2, -X_1Y_1H, 3mX_1^2)$. Hence the birational map $M_2 \cdots \to M_3$ is the flop along 9-copies of $\mathbb{P}^2$ by Proposition 5.5.

For Case (3), we have a decomposition

$$(1,0,-3) = (Y_1^2, -X_1Y_1H, 3mX_1^2) - (mX_1^2, -X_1Y_1H, 3Y_1^2). \tag{5.21}$$

We set $M_- := M_1, M_+ := M_2, \mathcal{C}_- := \mathcal{C}_1$ and $\mathcal{C}_+ := \mathcal{C}_2$. Since $3 \mid m$ and $3 \nmid Y_1$, $M_H(u)$ is fine if and only if $u = (Y_1^2, -X_1Y_1H, 3mX_1^2)$. Hence the birational map $M_1 \cdots \to M_2$ is the flop along 9-copies of $\mathbb{P}^2$ by Proposition 5.5.

Example 5.12. For $m = 6$, $(X_1, Y_1) = (2, 5)$. Then $\mathrm{Mov}(\mathrm{Km}^2(A))$ is divided into 3 chambers

$$\begin{aligned} \mathcal{C}_1 :=& \mathbb{R}_{>0}h + \mathbb{R}_{>0}(h - \tfrac{12}{5}\delta), \\ \mathcal{C}_2 :=& \mathbb{R}_{>0}(h - \tfrac{12}{5}\delta) + \mathbb{R}_{>0}(h - \tfrac{120}{49}\delta), \\ \mathcal{C}_3 :=& \mathbb{R}_{>0}(h - \tfrac{120}{49}\delta) + \mathbb{R}_{>0}(h - \tfrac{1188}{485}\delta). \end{aligned} \tag{5.22}$$

We set $u = (3, -H, 6), w = (-2, H, -9), u' = (25, -10H, 72)$ and $w' = (-24, 10H, -75)$. Then $v = u + w = u' + w'$. Since $3 \nmid \mathrm{rk} w, \mathrm{rk} u'$, $M_H(w)$ and $M_H(u')$ are fine moduli spaces. By Proposition 5.5, the birational maps $M_1 \cdots \to M_2$ and $M_2 \cdots \to M_3$ are the flop along 9-copies of $\mathbb{P}^2$.

6 Generalized Kummer 6-fold

Due to Lemma 4.5, we can compute the boundary of $\mathrm{Nef}(\mathrm{Km}^2(A))$ and $\mathrm{Mov}(\mathrm{Km}^2(A))$. We recall that there is an assumption $l \leq 4$ in Lemma 4.5. Hence Mukai vectors which determine the wall with respect to $v = (1, 0, -4)$ are isotropic vectors. In this section, we try to determine the boundary of $\mathrm{Nef}(\mathrm{Km}^3(A))$ and $\mathrm{Mov}(\mathrm{Km}^3(A))$.

We shall compute the boundary of nef/movable cones of a generalized Kummer 6-fold for an abelian surface A with Picard number 1. Hence we consider the case of $l = 4$. Then we have $v = (1, 0, -4)$ and $\langle v^2 \rangle/2 = 4$. Moreover Lemma 4.5 implies that $\langle u^2 \rangle = 0$ and $\langle u, v \rangle = 1, 2, 3$ or 4 for $u \in \Gamma$. By considering the orthogonal decomposition of u with respect to v, we have

$$\frac{8}{\langle u, v \rangle} u = v + Xh + Y\delta, \tag{6.1}$$

where $X, Y \in \mathbb{Q}$. We assume that $\langle u, v \rangle = 3$. We set $a := 3X, b := 3Y$. We note that $a, b \in \mathbb{Z}$. Since u is an isotropic vector, we have

$$4b^2 - na^2 = 36. \tag{6.2}$$

Since $n \equiv (b/a)^2 \mod 3$, we have $n \not\equiv 2 \mod 3$. So if $n \equiv 2 \mod 3$, then $\langle u, v \rangle \neq 3$ and $X, Y \in \mathbb{Z}$.

If $3 \mid n$, then we have $3 \mid 3Y$ by (6.2), that is, $Y \in \mathbb{Z}$. Since both a and b are not divided by 3, we get $3 \nmid 3X$. $9 \cdot 8 + 2n(3X)^2 - 9 \cdot 8Y^2 = 0$ holds from (6.2). Hence we have $9 \mid n$. Therefore if $9 \nmid n$, then we can assume that $\langle u, v \rangle \neq 3$ and $X, Y \in \mathbb{Z}$.

6.1 The case of $n \equiv 2 \mod 3$.

In this subsection, we assume that $n \equiv 2 \mod 3$. Then n is not a square number. If $2 \nmid X$, then $\gcd(\pm Y+1, X, 4(\pm Y-1)) = 1$. Hence we can assume that $2 \mid X$ and set $Z := X/2$. So we have a Pell equation

$$Y^2 - nZ^2 = 1 \tag{6.3}$$

by (6.1). Then we also consider the fundamental solution (Z_1, Y_1) of (6.3) (see subsection 4.1). Moreover we treat $(Z, Y) = \pm(Z_k, -Y_k)$ where (Z_k, Y_k) is given by $Y_k + \sqrt{n}Z_k = (Y_1 + \sqrt{n}Z_1)^k$. We set $X_k := 2Z_k$ and $m_k^{\pm} := \gcd(\pm Y_k + 1, \mp X_k, 4(\pm Y_k - 1))$. We note that $m_k^{\pm} = 1, 2, 4$ or 8.

Lemma 6.1. If $2 \mid Y_1$, then $\mathrm{Nef}(\mathrm{Km}^3(A)) = \mathrm{Mov}(\mathrm{Km}^3(A))$.

Proof. Since $\pm Y_1 + 1$ is odd, we have $m_1^{\pm} = 1$. Thus we consider the next solution (X_2, Y_2). Since $2 \mid Y_1, \gcd(Y_1^2/4, Y_1^2 - 1) = 1$ and $4 \nmid (1 - Y_2)$, we have $m_2^{+} = 8$. Therefore $\langle u, v\rangle = 1$ for $u = (\frac{Y_1^2}{4}, -\frac{X_1Y_1}{4}H, Y_1^2 - 1)$. □

Remark 6.2. We assume that $2 \nmid Y_1$. Replacing Y_1 by $-Y_1$ if necessary, we can suppose $Y_1 \equiv -1 \mod 4$. Since $Y_1^2 - 1 = n(\frac{X_1}{2})^2$, we have $2 \mid n(\frac{X_1}{2})^2$.

Lemma 6.3. If $2 \nmid Y_1$ and $2 \mid \frac{X_1}{2}$, then $m_1^{+} = 4$ or 8. In particular, $\mathrm{Nef}(\mathrm{Km}^3(A)) = \mathrm{Mov}(\mathrm{Km}^3(A))$.

Proof. We assume that $8 \mid X_1$. Note that we can assume $Y_1 \equiv -1 \mod 4$ by Remark 6.2. Since $\frac{Y_1+1}{4}(\frac{Y_1+1}{2} - 1) = 2n(\frac{X_1}{8})^2$, we have $2 \mid \frac{Y_1+1}{4}$. Hence we get $m_1^{+} = 8$ and $\langle u, v\rangle = 1$ for $u = (\frac{Y_1+1}{8}, -\frac{X_1}{8}H, \frac{Y_1-1}{2})$. If $8 \nmid X_1$, then we can see $m_1^{+} = 4$ and $\langle u, v\rangle = 2$ for $u = (\frac{Y_1+1}{4}, -\frac{X_1}{4}H, Y_1 - 1)$. □

Lemma 6.4. If $2 \nmid Y_1$ and $2 \nmid \frac{X_1}{2}$, then $m_2^{-} = 4$. In particular, $\mathrm{Nef}(\mathrm{Km}^3(A)) \neq \mathrm{Mov}(\mathrm{Km}^3(A))$

Proof. By our assumption, we have $m_1^{\pm} = 2$ and $\langle u, v\rangle = 4$ for $u = (\frac{\pm Y_1+1}{2}, \mp\frac{X_1}{2}H, 2(\pm Y_1 - 1))$. Since $(\frac{X_2}{2}, Y_2) = (2Y_1\frac{X_1}{2}, Y_1^2 + n(\frac{X_1}{2})^2)$ and $4 \mid (1 - Y_2)$, we have $m_2^{-} = 4$ and $\langle u, v\rangle = 2$ for $u = (\frac{1-Y_1^2}{2}, \frac{X_1Y_1}{2}H, -2Y_1^2)$. □

We summarize the above results in the following theorem.

Theorem 6.5. We assume that $n \equiv 2 \mod 3$. Then $\mathrm{Nef}(\mathrm{Km}^3(A))$ and $\mathrm{Mov}(\mathrm{Km}^3(A))$ are characterized by the solution of (6.3) and n as the following table:

type of (X_1, Y_1)		$\mathrm{Nef}(\mathrm{Km}^3(A))$	$\mathrm{Mov}(\mathrm{Km}^3(A))$
$2 \mid Y_1$		$\mathbb{R}_{\geq 0}h + \mathbb{R}_{\geq 0}(h - \frac{nX_2}{4Y_2}\delta)$	$\mathbb{R}_{\geq 0}h + \mathbb{R}_{\geq 0}(h - \frac{nX_2}{4Y_2}\delta)$
$2 \nmid Y_1$	$2 \mid \frac{X_1}{2}$	$\mathbb{R}_{\geq 0}h + \mathbb{R}_{\geq 0}(h - \frac{nX_1}{4Y_1}\delta)$	$\mathbb{R}_{\geq 0}h + \mathbb{R}_{\geq 0}(h - \frac{nX_1}{4Y_1}\delta)$
	$2 \nmid \frac{X_1}{2}$	$\mathbb{R}_{\geq 0}h + \mathbb{R}_{\geq 0}(h - \frac{nX_1}{4Y_1}\delta)$	$\mathbb{R}_{\geq 0}h + \mathbb{R}_{\geq 0}(h - \frac{nX_2}{4Y_2}\delta)$

By the proof of Theorem 6.5, we have the following.

Corollary 6.6. Assume that $n \equiv 2 \mod 3$ and $2 \nmid \frac{X_1}{2}, Y_1$. Then $\mathrm{Mov}(\mathrm{Km}^3(A))$ is divided into the chambers such that each chamber is an ample cone of a minimal model of $\mathrm{Km}^3(A)$.

$$\mathcal{C}_1 := \mathbb{R}_{>0}h + \mathbb{R}_{>0}(h - \frac{nX_1}{4Y_1}\delta),\ \mathcal{C}_2 := \mathbb{R}_{>0}(h - \frac{nX_1}{4Y_1}\delta) + \mathbb{R}_{>0}(h - \frac{nX_2}{4Y_2}\delta). \tag{6.4}$$

Example 6.7. We see the case of $2 \mid Y_1$. Let $n = 11$. Then the fundamental solution $(\frac{X_1}{2}, Y_1)$ of the Pell equation (6.3) is $(3, 10)$. We have

$$\frac{8}{\langle u, v\rangle}u = v \pm 6h \mp 10\delta = (11, -6H, 36), -(9, -6H, 44)$$

from (6.1), and $\langle u,v\rangle = 8$ for $u = (11, 6H, 36), -(9, 6H, 44)$. Since $\langle u,v\rangle = 1, 2, 3$ or 4, we consider $(\frac{X_2}{2}, Y_2) = (60, 199)$. Since $8 \mid (Y_2+1)$, then we have

$$\frac{8}{\langle u,v\rangle}u = (200, -120H, 792) = 8(25, -15H, 99)$$

and $\langle u,v\rangle = 1$ for $u = (25, -15H, 99)$. Hence

$$\mathrm{Nef}(\mathrm{Km}^3(A)) = \mathrm{Mov}(\mathrm{Km}^3(A)) = \mathbb{R}_{\geq 0}h + \mathbb{R}_{\geq 0}(h - \tfrac{330}{199}\delta).$$

Example 6.8. We see the case of $2 \nmid Y_1$ and $2 \mid \frac{X_1}{2}$. Let $n = 14$. Then $(\frac{X_1}{2}, Y_1) = (4, 15)$. Since $8 \mid (Y_1 + 1)$, we have

$$\frac{8}{\langle u,v\rangle}u = (16, -8H, 56) = 8(2, -H, 7)$$

and $\langle u,v\rangle = 1$ for $u = (2, -H, 7)$. Hence

$$\mathrm{Nef}(\mathrm{Km}^3(A)) = \mathrm{Mov}(\mathrm{Km}^3(A)) = \mathbb{R}_{\geq 0}h + \mathbb{R}_{\geq 0}(h - \tfrac{28}{15}\delta).$$

Let $n = 26$. Then $(\frac{X_1}{2}, Y_1) = (10, 51)$. Since $4 \mid (Y_1 + 1)$, we have

$$\frac{8}{\langle u,v\rangle}u = (52, -20H, 200) = 4(13, -5H, 50)$$

and $\langle u,v\rangle = 2$ for $u = (13, -5H, 50)$. Hence

$$\mathrm{Nef}(\mathrm{Km}^3(A)) = \mathrm{Mov}(\mathrm{Km}^3(A)) = \mathbb{R}_{\geq 0}h + \mathbb{R}_{\geq 0}(h - \tfrac{130}{51}\delta).$$

Example 6.9. We see the case of $2 \nmid Y_1$ and $2 \nmid \frac{X_1}{2}$. Let $n = 8$. Then $(\frac{X_1}{2}, Y_1) = (1, 3)$. We have

$$\frac{8}{\langle u,v\rangle}u = (4, -2H, 8), -(2, -2H, 16) = 2(2, -H, 4), -2(1, -H, 8)$$

and $\langle u,v\rangle = 4$ for $u = (2, -H, 4), -(1, -H, 8)$. Hence $\mathrm{Nef}(\mathrm{Km}^3(A)) = \mathbb{R}_{\geq 0}h + \mathbb{R}_{\geq 0}(h - \frac{4}{3}\delta)$. We consider $(\frac{X_2}{2}, Y_2) = (6, 17)$. Since $4 \mid (1 - Y_2)$, we have

$$\frac{8}{\langle u,v\rangle}u = (-16, 12H, -72) = -4(4, -3H, 18)$$

and $\langle u,v\rangle = 2$ for $u = -(4, -3H, 18)$. Hence $\mathrm{Mov}(\mathrm{Km}^3(A)) = \mathbb{R}_{\geq 0}h + \mathbb{R}_{\geq 0}(h - \frac{24}{17}\delta)$.

6.2 The case of $3 \mid n$.

In this subsection, we assume that $3 \mid n$. If $2 \nmid X$, then $\gcd(\pm Y + 1, X, 4(\pm Y - 1)) = 1$. Hence we can assume that $2 \mid X$. If $\langle u,v\rangle \neq 3$, n is not a square and we have a Pell equation

$$Y^2 - n\left(\frac{X}{2}\right)^2 = 1. \tag{6.5}$$

We define (X_k, Y_k) in the same way as subsection 6.1. Therefore we get the same results as in $n \equiv 2 \mod 3$.

Theorem 6.10. We assume that $n \equiv 3, 6 \mod 9$. Then we have the following table:

type of (X_1, Y_1)		$\mathrm{Nef}(\mathrm{Km}^3(A))$	$\mathrm{Mov}(\mathrm{Km}^3(A))$
$2 \mid Y_1$		$\mathbb{R}_{\geq 0}h + \mathbb{R}_{\geq 0}(h - \frac{nX_2}{4Y_2}\delta)$	$\mathbb{R}_{\geq 0}h + \mathbb{R}_{\geq 0}(h - \frac{nX_2}{4Y_2}\delta)$
$2 \nmid Y_1$	$2 \mid \frac{X_1}{2}$	$\mathbb{R}_{\geq 0}h + \mathbb{R}_{\geq 0}(h - \frac{nX_1}{4Y_1}\delta)$	$\mathbb{R}_{\geq 0}h + \mathbb{R}_{\geq 0}(h - \frac{nX_1}{4Y_1}\delta)$
	$2 \nmid \frac{X_1}{2}$	$\mathbb{R}_{\geq 0}h + \mathbb{R}_{\geq 0}(h - \frac{nX_1}{4Y_1}\delta)$	$\mathbb{R}_{\geq 0}h + \mathbb{R}_{\geq 0}(h - \frac{nX_2}{4Y_2}\delta)$

By the proof of Theorem 6.10, we have the following.

Corollary 6.11. Assume that $n \equiv 3, 6 \mod 9$ and $2 \nmid \frac{X_1}{2}, Y_1$. Then $\mathrm{Mov}(\mathrm{Km}^3(A))$ is divided into the chambers such that each chamber is an ample cone of a minimal model of $\mathrm{Km}^3(A)$.

$$\mathcal{C}_1 := \mathbb{R}_{>0}h + \mathbb{R}_{>0}(h - \tfrac{nX_1}{4Y_1}\delta),\ \mathcal{C}_2 := \mathbb{R}_{>0}(h - \tfrac{nX_1}{4Y_1}\delta) + \mathbb{R}_{>0}(h - \tfrac{nX_2}{4Y_2}\delta). \tag{6.6}$$

Example 6.12. We see the case of $2 \mid Y_1$. Let $n = 15$. Then the fundamental solution is $(\frac{X_1}{2}, Y_1) = (1, 4)$. We have

$$\frac{8}{\langle u, v\rangle}u = v \pm 2h \mp 4\delta = (5, -2H, 12), -(3, -2H, 20)$$

and $\langle u, v\rangle = 8$ for $u = (5, -2H, 12), -(3, -2H, 20)$. Since $\langle u, v\rangle = 1, 2, 3$ and 4, we consider $(\frac{X_2}{2}, Y_2) = (8, 31)$. Since $8 \mid (Y_2 + 1)$, we have

$$\frac{8}{\langle u, v\rangle}u = (32, -16H, 120) = 8(4, -2H, 15)$$

and $\langle u, v\rangle = 1$ for $u = (4, -2H, 15)$. Hence

$$\mathrm{Nef}(\mathrm{Km}^3(A)) = \mathrm{Mov}(\mathrm{Km}^3(A)) = \mathbb{R}_{\geq 0}h + \mathbb{R}_{\geq 0}(h - \tfrac{60}{31}\delta).$$

Example 6.13. We see the case of $2 \nmid Y_1$ and $2 \mid \frac{X_1}{2}$. Let $n = 21$. Then $(\frac{X_1}{2}, Y_1) = (12, 55)$. Since $8 \mid (Y_1 + 1)$, we have

$$\frac{8}{\langle u, v\rangle}u = (56, -24H, 216) = 8(7, -3H, 27)$$

and $\langle u, v\rangle = 1$ for $u = (7, -3H, 27)$. Hence

$$\mathrm{Nef}(\mathrm{Km}^3(A)) = \mathrm{Mov}(\mathrm{Km}^3(A)) = \mathbb{R}_{\geq 0}h + \mathbb{R}_{\geq 0}(h - \tfrac{126}{55}\delta).$$

Let $n = 6$. Then $(\frac{X_1}{2}, Y_1) = (2, 5)$. Since $4 \mid (1 - Y_1)$, we have

$$\frac{8}{\langle u, v\rangle}u = -(4, -4H, 24) = -4(1, -H, 6)$$

and $\langle u, v\rangle = 2$ for $u = (1, -H, 6)$. Hence

$$\mathrm{Nef}(\mathrm{Km}^3(A)) = \mathrm{Mov}(\mathrm{Km}^3(A)) = \mathbb{R}_{\geq 0}h + \mathbb{R}_{\geq 0}(h - \tfrac{6}{5}\delta).$$

Example 6.14. We see the case of $2 \nmid Y_1$ and $2 \nmid \frac{X_1}{2}$. Let $n = 24$. Then $(\frac{X_1}{2}, Y_1) = (1, 5)$. We have

$$\frac{8}{\langle u, v\rangle}u = (6, -2H, 16), -(4, -2H, 24) = 2(3, -H, 8), -2(2, -H, 12)$$

and $\langle u, v\rangle = 4$ for $u = (3, -H, 8), -(2, -H, 12)$. Hence $\mathrm{Nef}(\mathrm{Km}^3(A)) = \mathbb{R}_{\geq 0}h + \mathbb{R}_{\geq 0}(h - \frac{12}{5}\delta)$. We consider $(\frac{X_2}{2}, Y_2) = (10, 49)$. Since $4 \mid (1 - Y_2)$, we have

$$\frac{8}{\langle u, v\rangle}u = -(48, -20H, 200) = -4(12, -5H, 50)$$

and $\langle u, v\rangle = 2$ for $u = -(12, -5H, 50)$. Hence $\mathrm{Mov}(\mathrm{Km}^3(A)) = \mathbb{R}_{\geq 0}h + \mathbb{R}_{\geq 0}(h - \frac{120}{49}\delta)$.

We assume that $9 \mid n$ and $\sqrt{n/9} \notin \mathbb{Q}$. We consider the following Pell equation

$$y^2 - \left(\frac{n}{9}\right)x^2 = 1, \tag{6.7}$$

where $x, y \in \mathbb{Z}$. Let (x_k, y_k) be the solutions of (6.7). We also set $(X_k, Y_k) = (\frac{2x_k}{3}, y_k)$ and

$$\frac{8}{\langle u, v\rangle}u = v \pm X_k h \mp Y_k \delta. \tag{6.8}$$

Lemma 6.15. We assume that $9 \mid n$.

(1) If $3 \nmid x_k$ and $4 \mid x_k$, then $\langle u, v\rangle = 3$.

(2) If $3 \mid x_k$ and y_k is odd, then $\langle u, v\rangle = 1, 2$ or 4. Moreover, if $2 \mid x_k$, then $\langle u, v\rangle = 1$ or 2.

Proof. (1) We note that $X_k = \frac{2x_k}{3} \in \mathbb{Q}$. We have $Y_k \equiv \pm 1 \mod 8$ by $16 \mid (Y_k+1)(Y_k-1)$. Hence we can assume that $8 \mid (1+Y_k)$ by replacing Y_k by $-Y_k$ if necessary. We have

$$\frac{1}{\langle u, v\rangle} u = \Big(\frac{1+Y_k}{8}, -\frac{X_k}{8}H, \frac{Y_k-1}{2}\Big)$$

by (6.8). Since $u \in H^*(A, \mathbb{Z})_{\mathrm{alg}}$ and $\gcd(3, \frac{3X_k}{8}) = 1$, we get $\langle u, v\rangle = 3$ for $u = (\frac{3(Y_k+1)}{8}, -\frac{3X_k}{8}H, \frac{3(Y_k-1)}{2})$.

(2) Since $X_k \in \mathbb{Z}$ and $2 \mid (1 \pm y_k)$, the second claim holds. □

Lemma 6.16. We assume that $9 \mid n$.

(1) If $3 \mid x_1$, then we have the same consequence of Theorem 6.10.

(2) If $3 \nmid x_1$ and $4 \mid x_1$, then $\mathrm{Nef}(\mathrm{Km}^3(A)) = \mathbb{R}_{\geq 0}h + \mathbb{R}_{\geq 0}(h - \frac{nX_1}{4Y_1}\delta)$ and

$$\mathrm{Mov}(\mathrm{Km}^3(A)) = \begin{cases} \mathbb{R}_{\geq 0}h + \mathbb{R}_{\geq 0}(h - \frac{nX_2}{4Y_2}\delta), & \text{if } 3 \mid y_1, \\ \mathbb{R}_{\geq 0}h + \mathbb{R}_{\geq 0}(h - \frac{nX_3}{4Y_3}\delta), & \text{if } 3 \nmid y_1. \end{cases}$$

Proof. (1) By using induction on k, we can see that $3 \mid x_k$. Since $X_k \in \mathbb{Z}$, we have the Pell equation (6.5) for $k \in \mathbb{Z}$. Thus the first claim holds.

(2) We note that y_1 is odd. By Lemma 6.15-(1), we have $\langle u, v\rangle = 3$ for $u = (\frac{3(Y_1+1)}{8}, -\frac{3X_1}{8}H, \frac{3(Y_1-1)}{2})$. We consider $(x_2, y_2) = (2x_1y_1, 2y_1^2 - 1)$. We assume that $3 \mid y_1$. Since $8 \mid (1 - y_2)$, we have

$$\frac{8}{\langle u, v\rangle} u = -8\Big(\frac{Y_1^2-1}{4}, -\frac{X_1Y_1}{4}H, Y_1^2\Big)$$

and $\langle u, v\rangle = 1$ for $u = -(\frac{Y_1^2-1}{4}, -\frac{X_1Y_1}{4}H, Y_1^2)$. If $3 \nmid y_1$, then we have $3 \nmid x_2$ and $4 \mid x_2$. Hence $\langle u, v\rangle = 3$ holds by Lemma 6.15-(1). Thus we consider $(x_3, y_3) = (y_1x_2 + x_1y_2, y_1y_2 + \frac{n}{9}x_1x_2)$. Since $x_3 = x_1(4y_1^2 - 1)$ and $y_3 = 2y_1(y_1^2 + \frac{n}{9}x_1y_1) - y_1$, we have $12 \mid x_3$ and $2 \nmid y_3$. Hence we can assume that $8 \mid (y_3 + 1)$. Then we get $\langle u, v\rangle = 1$ for $u = (\frac{1+Y_3}{8}, -\frac{X_3}{8}H, \frac{Y_3-1}{2})$. □

By the proof of Lemma 6.16, we have the following.

Corollary 6.17. Assume that $9 \mid n, 3 \nmid x_1$ and $4 \mid x_1$. In the following cases, $\mathrm{Mov}(\mathrm{Km}^3(A))$ is divided into the chambers such that each chamber is an ample cone of a minimal model of $\mathrm{Km}^3(A)$.

(1) If $2 \mid y_1$, then

$$\mathcal{C}_1 := \mathbb{R}_{>0}h + \mathbb{R}_{>0}(h - \tfrac{nX_1}{4Y_1}\delta),\ \mathcal{C}_2 := \mathbb{R}_{>0}(h - \tfrac{nX_1}{4Y_1}\delta) + \mathbb{R}_{>0}(h - \tfrac{nX_2}{4Y_2}\delta). \tag{6.9}$$

(2) If $2 \nmid y_1$, then

$$\begin{aligned} &\mathcal{C}_1 := \mathbb{R}_{>0}h + \mathbb{R}_{>0}(h - \tfrac{nX_1}{4Y_1}\delta),\ \mathcal{C}_2 := \mathbb{R}_{>0}(h - \tfrac{nX_1}{4Y_1}\delta) + \mathbb{R}_{>0}(h - \tfrac{nX_2}{4Y_2}\delta), \\ &\mathcal{C}_3 := \mathbb{R}_{>0}(h - \tfrac{nX_2}{4Y_2}\delta) + \mathbb{R}_{>0}(h - \tfrac{nX_3}{4Y_3}\delta). \end{aligned} \tag{6.10}$$

Example 6.18. We see the case of $3 \mid x_1$. Let $n = 63$. Then $(x_1, y_1) = (3, 8)$ and $(X_1, Y_1) = (2, 8)$. Since y_1 is even, we consider $(x_2, y_2) = (48, 127)$. Then $(X_2, Y_2) = (32, 127)$. Since $8 \mid (Y_2 + 1)$, we have

$$\frac{8}{\langle u, v\rangle} u = (128, -48H, 504) = 8(16, -6H, 63)$$

and $\langle u, v\rangle = 1$ for $u = (16, -6H, 63)$. Hence

$$\mathrm{Nef}(\mathrm{Km}^3(A)) = \mathrm{Mov}(\mathrm{Km}^3(A)) = \mathbb{R}_{\geq 0}h + \mathbb{R}_{\geq 0}(h - \tfrac{504}{127}\delta).$$

Example 6.19. We see the case of $3 \nmid x_1, 4 \mid x_1$ and $3 \mid y_1$. Let $n = 45$. Then $(x_1, y_1) = (4, 9)$ and $(X_1, Y_1) = (\frac{8}{3}, 9)$. Since $3 \nmid x_1, 4 \mid x_1$ and $8 \mid (1 - Y_1)$, we have

$$\frac{8}{\langle u, v\rangle} u = -(8, -\tfrac{8}{3}H, 40) = -8(1, -\tfrac{1}{3}H, 5).$$

Hence we get $\langle u, v\rangle = 3$ for $u = -(3, -H, 15)$ and $\mathrm{Nef}(\mathrm{Km}^3(A)) = \mathbb{R}_{\geq 0}h + \mathbb{R}_{\geq 0}(h - \frac{10}{3}\delta)$. Next we consider $(x_2, y_2) = (72, 161)$. Then $(X_2, Y_2) = (48, 161)$. Since $3 \mid x_2$ and $8 \mid (1 - Y_2)$, we have

$$\frac{8}{\langle u, v\rangle} u = -(160, -48H, 648) = -8(20, -6H, 81)$$

and $\langle u, v\rangle = 1$ for $u = -(20, -6H, 81)$. Hence $\mathrm{Mov}(\mathrm{Km}^3(A)) = \mathbb{R}_{\geq 0}h + \mathbb{R}_{\geq 0}(h - \frac{2415}{64}\delta)$.

Example 6.20. We see the case of $3 \nmid x_1, 4 \mid x_1$ and $3 \nmid y_1$. Let $n = 162$. Then $(x_1, y_1) = (4, 17)$ and $(X_1, Y_1) = (\frac{8}{3}, 17)$. Since $8 \mid (1 - Y_1)$, we have

$$\frac{8}{\langle u, v\rangle} u = -(16, -\tfrac{8}{3}H, 72) = -8(2, -\tfrac{1}{3}H, 9)$$

and $\langle u, v\rangle = 3$ for $u = -(6, -H, 27)$. Hence $\mathrm{Nef}(\mathrm{Km}^3(A)) = \mathbb{R}_{\geq 0}h + \mathbb{R}_{\geq 0}(h - \frac{108}{17}\delta)$. We consider $(x_2, y_2) = (136, 577)$ and $(X_2, Y_2) = (\frac{272}{3}, 577)$. Since $8 \mid (1 - Y_2)$, we have

$$\frac{8}{\langle u, v\rangle} u = -(576, -\tfrac{272}{3}H, 2312) = -8(72, -\tfrac{34}{3}H, 289)$$

and $\langle u, v\rangle = 3$ for $u = -(216, -34H, 867)$. Thus we consider the next solution $(x_3, y_3) = (4620, 19601)$. Then $(X_3, Y_3) = (3080, 19601)$. Since $8 \mid (1 - Y_3)$, we have

$$\frac{8}{\langle u, v\rangle} u = -(19600, -3080H, 78408) = -8(2450, -385H, 9801)$$

and $\langle u, v\rangle = 1$ for $u = -(2450, -385H, 9801)$. Hence $\mathrm{Mov}(\mathrm{Km}^3(A)) = \mathbb{R}_{\geq 0}h + \mathbb{R}_{\geq 0}(h - \frac{124740}{19601}\delta)$.

Lemma 6.21. We assume that $9 \mid n, 3 \nmid x_1$ and $4 \nmid x_1$.

(1) If $3 \mid y_1$, then $\mathrm{Nef}(\mathrm{Km}^3(A)) = \mathrm{Mov}(\mathrm{Km}^3(A)) = \mathbb{R}_{\geq 0}h + \mathbb{R}_{\geq 0}(h - \frac{nX_2}{4Y_2}\delta)$.

(2) We suppose that $3 \nmid y_1$.

(i) If $2 \mid x_1$, then

$$\mathrm{Nef}(\mathrm{Km}^3(A)) = \mathbb{R}_{\geq 0}h + \mathbb{R}_{\geq 0}(h - \tfrac{nX_2}{4Y_2}\delta),$$
$$\mathrm{Mov}(\mathrm{Km}^3(A)) = \mathbb{R}_{\geq 0}h + \mathbb{R}_{\geq 0}(h - \tfrac{nX_3}{4Y_3}\delta).$$

(ii) If $2 \nmid x_1$ and $2 \mid y_1$, then

$$\mathrm{Nef}(\mathrm{Km}^3(A)) = \mathbb{R}_{\geq 0}h + \mathbb{R}_{\geq 0}(h - \tfrac{nX_2}{4Y_2}\delta),$$
$$\mathrm{Mov}(\mathrm{Km}^3(A)) = \mathbb{R}_{\geq 0}h + \mathbb{R}_{\geq 0}(h - \tfrac{nX_6}{4Y_6}\delta).$$

(iii) If $2 \nmid x_1$ and $2 \nmid y_1$, then

$$\mathrm{Nef}(\mathrm{Km}^3(A)) = \mathbb{R}_{\geq 0}h + \mathbb{R}_{\geq 0}(h - \tfrac{nX_3}{4Y_3}\delta),$$
$$\mathrm{Mov}(\mathrm{Km}^3(A)) = \mathbb{R}_{\geq 0}h + \mathbb{R}_{\geq 0}(h - \tfrac{nX_6}{4Y_6}\delta).$$

Proof. By Lemma 6.15-(1), we consider $(x_2, y_2) = (2x_1y_1, 2y_1^2 - 1)$.

(1) Since $3 \mid y_1$, we get $3 \mid x_2$. If y_1 is even, then $8 \mid (Y_2 + 1)$. Hence we have $\langle u, v\rangle = 1$ for $u = (\frac{Y_2+1}{8}, -\frac{X_2}{8}H, \frac{Y_2-1}{2})$. We assume that y_1 is odd. Then we have $8 \mid (1 - Y_2)$. If $2 \nmid x_1$, then $\langle u, v\rangle = 2$ for $u = -(\frac{Y_2-1}{4}, -\frac{X_2}{4}H, Y_2 + 1)$. If $2 \mid x_1$, then $\langle u, v\rangle = 1$ for $u = -(\frac{Y_2-1}{8}, -\frac{X_2}{8}H, \frac{Y_2+1}{2})$.

(2) Since $3 \nmid x_1, y_1$, we get $3 \nmid x_2$.

(i) We assume that $2 \mid x_1$. Then we have $2 \nmid y_1, 3 \nmid x_2$ and $4 \mid x_2$. Thus we have $\langle u, v\rangle = 3$ for $u = (\frac{3(Y_2+1)}{8}, -\frac{3X_2}{8}H, \frac{3(Y_2-1)}{2})$ by Lemma 6.15-(1). We consider $(x_3, y_3) = (x_1y_2 + x_2y_1, y_1y_2 + \frac{n}{9}x_1x_2)$. Since $x_3 = x_1(4y_1^2 - 1)$, we have $6 \mid x_3$ and $4 \nmid x_3$. We can assume that $4 \mid (y_3 + 1)$ by remark 6.2. Then $\langle u, v\rangle = 2$ for $u = (\frac{Y_3+1}{4}, -\frac{X_3}{4}H, Y_3 - 1)$.

(ii) We assume that $2 \nmid x_1$ and $2 \mid y_1$. Then we have $P_3 = \langle s_3\rangle$ and $P_2 = P_4 = \langle s_2\rangle$ (for a notation, see the subsection 4.1). Thus we have $\langle u, v\rangle = 3$ for $u = (\frac{3(Y_2+1)}{8}, \frac{3X_2}{8}H, \frac{3(Y_2-1)}{2})$ by Lemma 6.15-(1). For (x_3, y_3), since y_3 is even by $y_1 \mid y_3$, we have $\gcd(\pm y_3 + 1, x_3, 8)$=1. For (x_4, y_4), we have $\langle u, v\rangle = 3$ by Lemma 6.15-(1). For (x_6, y_6), we can assume that $4 \mid (y_6 + 1)$ by $(y_6 + 1)(y_6 - 1) = 16\frac{n}{9}(\frac{x_6}{4})^2$. Then we have $\langle u, v\rangle = 1$ for $u = (\frac{Y_6+1}{8}, -\frac{X_6}{8}H, \frac{Y_6-1}{2})$.

(iii) We assume that $2 \nmid x_1, y_1$. Then we have $P_2 = \langle s_2\rangle, P_3 = \langle s_3\rangle$ and $P_4 = \langle s_4\rangle$. For (x_3, y_3), since y_3 is odd, we have $\langle u, v\rangle = 4$ for $u = (\frac{\pm Y_3+1}{2}, \mp\frac{X_3}{2}H, \pm 2(Y_3 - 1))$ by Lemma 6.15-(2). For (x_4, y_4), we have $\langle u, v\rangle = 3$ for $u = (\frac{3(Y_4+1)}{8}, \frac{3X_4}{8}H, \frac{3(Y_4-1)}{2})$. For (x_6, y_6), we can assume that $4 \mid (y_6 + 1)$. Then we have $\langle u, v\rangle = 2$ for $u = (\frac{Y_6+1}{4}, -\frac{X_6}{4}H, Y_6 - 1)$.

□

By the proof of Lemma 6.21, we have the following.

Corollary 6.22. Assume that $9 \mid n$ and $3, 4 \nmid x_1$. In the following cases, $\mathrm{Mov}(\mathrm{Km}^3(A))$ is divided into the chambers such that each chamber is an ample cone of a minimal model of $\mathrm{Km}^3(A)$.

(1) If $2 \mid x_1$, then

$$\mathcal{C}_1 := \mathbb{R}_{>0}h + \mathbb{R}_{>0}(h - \tfrac{nX_2}{4Y_2}\delta),\ \mathcal{C}_2 := \mathbb{R}_{>0}(h - \tfrac{nX_2}{4Y_2}\delta) + \mathbb{R}_{>0}(h - \tfrac{nX_3}{4Y_3}\delta). \tag{6.11}$$

(2) If $2 \nmid x_1$ and $2 \mid y_1$, then

$$\begin{aligned} \mathcal{C}_1 :=& \mathbb{R}_{>0}h + \mathbb{R}_{>0}(h - \tfrac{nX_2}{4Y_2}\delta),\ \mathcal{C}_2 := \mathbb{R}_{>0}(h - \tfrac{nX_2}{4Y_2}\delta) + \mathbb{R}_{>0}(h - \tfrac{nX_4}{4Y_4}\delta), \\ \mathcal{C}_3 :=& \mathbb{R}_{>0}(h - \tfrac{nX_4}{4Y_4}\delta) + \mathbb{R}_{>0}(h - \tfrac{nX_6}{4Y_6}\delta). \end{aligned} \tag{6.12}$$

(3) If $2 \nmid x_1, y_1$, then

$$\begin{aligned} \mathcal{C}_1 :=& \mathbb{R}_{>0}h + \mathbb{R}_{>0}(h - \tfrac{nX_3}{4Y_3}\delta),\ \mathcal{C}_2 := \mathbb{R}_{>0}(h - \tfrac{nX_3}{4Y_3}\delta) + \mathbb{R}_{>0}(h - \tfrac{nX_4}{4Y_4}\delta), \\ \mathcal{C}_3 :=& \mathbb{R}_{>0}(h - \tfrac{nX_4}{4Y_4}\delta) + \mathbb{R}_{>0}(h - \tfrac{nX_6}{4Y_6}\delta). \end{aligned} \tag{6.13}$$

Example 6.23. We see the case of $3, 4 \nmid x_1, 3 \mid y_1$ and $2 \nmid x_1$. Let $n = 72$. Then $(x_1, y_1) = (1, 3)$ and $(X_1, Y_1) = (\frac{2}{3}, 3)$. Since $4 \nmid x_1$, we consider the next solution $(x_2, y_2) = (6, 17)$. Then $(X_2, Y_2) = (4, 17)$. Since $8 \mid (1 - Y_2)$, we have

$$\frac{8}{\langle u, v\rangle}u = -(16, -4H, 72) = -4(4, -H, 18)$$

and $\langle u, v\rangle = 2$ for $u = -(4, -H, 18)$. Hence

$$\mathrm{Nef}(\mathrm{Km}^3(A)) = \mathrm{Mov}(\mathrm{Km}^3(A)) = \mathbb{R}_{\geq 0}h + \mathbb{R}_{\geq 0}(h - \tfrac{72}{17}\delta).$$

Next we see the case of $3, 4 \nmid x_1, 3 \mid y_1$ and $2 \mid x_1$. Let $n = 18$. Then $(x_1, y_1) = (2, 3)$ and $(X_1, Y_1) = (\frac{4}{3}, 3)$. Since $4 \nmid x_1$, we consider $(x_2, y_2) = (12, 17)$. Then $(X_2, Y_2) = (8, 17)$. Since $8 \mid (1 - Y_2)$, we have

$$\frac{8}{\langle u, v\rangle}u = -(16, -8H, 72) = -8(2, -H, 9)$$

and $\langle u, v\rangle = 1$ for $u = -(2, -H, 9)$. Hence

$$\mathrm{Nef}(\mathrm{Km}^3(A)) = \mathrm{Mov}(\mathrm{Km}^3(A)) = \mathbb{R}_{\geq 0}h + \mathbb{R}_{\geq 0}(h - \tfrac{36}{17}\delta).$$

Example 6.24. We see the case of $3,4 \nmid x_1, 3 \nmid y_1$ and $2 \mid x_1$. Let $n = 54$. Then $(x_1, y_1) = (2,5)$ and $(X_1, Y_1) = (\frac{4}{3}, 5)$. Since $4 \nmid x_1$, we consider $(x_2, y_2) = (20, 49)$. Then $(X_2, Y_2) = (\frac{40}{3}, 49)$. Since $8 \mid (1 - Y_2)$, we have

$$\frac{8}{\langle u,v\rangle}u = -(48, -\tfrac{40}{3}H, 200) = -8(6, -\tfrac{5}{3}H, 25)$$

and $\langle u,v\rangle = 3$ for $u = -(18, -5H, 75)$. Hence $\mathrm{Nef}(\mathrm{Km}^3(A)) = \mathbb{R}_{\geq 0}h + \mathbb{R}_{\geq 0}(h - \frac{180}{49}\delta)$. We consider $(x_3, y_3) = (198, 485)$. Then $(X_3, Y_3) = (132, 485)$. Since $4 \mid (1 - Y_3)$, we have

$$\frac{8}{\langle u,v\rangle}u = -(484, -132H, 1944) = -4(121, -33H, 486)$$

and $\langle u,v\rangle = 2$ for $u = -(121, -33H, 486)$. Hence $\mathrm{Mov}(\mathrm{Km}^3(A)) = \mathbb{R}_{\geq 0}h + \mathbb{R}_{\geq 0}(h - \frac{1782}{485}\delta)$.

Example 6.25. We see the case of $2,3 \nmid x_1, 3 \nmid y_1$ and $2 \mid y_1$. Let $n = 27$. Then $(x_1, y_1) = (1,2)$ and $(X_1, Y_1) = (\frac{2}{3}, 2)$. Since $4 \nmid x_1$, we consider $(x_2, y_2) = (4, 7)$. Then $(X_2, Y_2) = (\frac{8}{3}, 7)$. Since $8 \mid (Y_2 + 1)$, we have

$$\frac{8}{\langle u,v\rangle}u = -(8, -\tfrac{8}{3}H, 24) = 8(1, -\tfrac{1}{3}H, 3)$$

and $\langle u,v\rangle = 3$ for $u = (3, -H, 9)$. Hence $\mathrm{Nef}(\mathrm{Km}^3(A)) = \mathbb{R}_{\geq 0}h + \mathbb{R}_{\geq 0}(h - \frac{18}{7}\delta)$. We consider $(x_3, y_3) = (15, 26)$. Then $(X_3, Y_3) = (10, 26)$. Since Y_3 is even, we consider $(x_4, y_4) = (56, 97)$. Then $(X_4, Y_4) = (\frac{112}{3}, 97)$. Since $8 \mid (1 - Y_4)$, we have $\langle u,v\rangle = 3$ for $u = -(36, -14H, 147)$. We consider $(x_5, y_5) = (209, 362)$. Then $(X_5, Y_5) = (\frac{418}{3}, 362)$. Since $4 \nmid x_5$, we consider $(x_6, y_6) = (780, 1351)$. Then $(X_6, Y_6) = (520, 1351)$. Since $8 \mid (Y_6 + 1)$, we have

$$\frac{8}{\langle u,v\rangle}u = (1352, -520H, 5400) = 8(169, -65H, 675)$$

and $\langle u,v\rangle = 1$ for $u = (169, -65H, 675)$. Hence $\mathrm{Mov}(\mathrm{Km}^3(A)) = \mathbb{R}_{\geq 0}h + \mathbb{R}_{\geq 0}(h - \frac{3510}{1351}\delta)$.

Example 6.26. We see the case of $2,3 \nmid x_1$ and $2,3 \nmid y_1$. Let $n = 216$. Then $(x_1, y_1) = (1,5)$ and $(X_1, Y_1) = (\frac{2}{3}, 5)$. Since $4 \nmid x_1$, we consider $(x_2, y_2) = (10, 49)$. Since $4 \nmid x_2$ also holds, we consider $(x_3, y_3) = (99, 485)$. Then $(X_3, Y_3) = (66, 485)$. Then we have

$$\frac{8}{\langle u,v\rangle}u = 2(243, -33H, 968), -2(242, -33H, 972).$$

Thus $\langle u,v\rangle = 4$ for $u = (243, -33H, 968), -(242, -33H, 972)$. Hence $\mathrm{Nef}(\mathrm{Km}^3(A)) = \mathbb{R}_{\geq 0}h + \mathbb{R}_{\geq 0}(h - \frac{4365}{11}\delta)$. We consider $(x_4, y_4) = (980, 4801)$. Then $(X_4, Y_4) = (\frac{1960}{3}, 4801)$. Since $8 \mid (1 - Y_4)$, we have $\langle u,v\rangle = 3$ for $u = -(1800, -245H, 7203)$. We consider $(x_5, y_5) = (9701, 47525)$. Since $4 \nmid x_5$, we consider $(x_6, y_6) = (96030, 470449)$. Then $(X_6, Y_6) = (64020, 470449)$. Since $4 \mid (1 - Y_6)$, we have

$$\frac{8}{\langle u,v\rangle}u = -(470448, -64020H, 1881800) = -4(117612, -16005H, 470450).$$

and $\langle u,v\rangle = 2$ for $u = -(117612, -16005H, 470450)$. Hence $\mathrm{Mov}(\mathrm{Km}^3(A)) = \mathbb{R}_{\geq 0}h + \mathbb{R}_{\geq 0}(h - \frac{3457080}{470449}\delta)$.

Lemma 6.27. If $\sqrt{n/9} \in \mathbb{Q}$, then we have

$$\mathrm{Nef}(\mathrm{Km}^3(A)) = \mathrm{Mov}(\mathrm{Km}^3(A)) = \mathbb{R}_{\geq 0}h + \mathbb{R}_{\geq 0}(h - \tfrac{\sqrt{n}}{2}\delta).$$

Proof. This is the consequence from [Yo6, Proposition 4.16] (cf. Proof of Theorem 4.7-(2-2)). ☐

In summary, we get the following theorem.

Theorem 6.28. We assume that $3 \mid n$.

(1) If $n \equiv 3, 6 \mod 9$, then we have the same consequence as Theorem 6.5.

(2) We suppose that $9 \mid n$.

(i) If $3 \mid x_1$, then we have the same consequence as Theorem 6.5.

(ii) If $3 \nmid x_1$, then we have the following table:

type of (x_1, y_1)				$\mathrm{Nef}(\mathrm{Km}^3(A))$	$\mathrm{Mov}(\mathrm{Km}^3(A))$
$4 \mid x_1$	$3 \mid y_1$			$\mathbb{R}_{\geq 0}h + \mathbb{R}_{\geq 0}(h - \frac{nX_1}{4Y_1}\delta)$	$\mathbb{R}_{\geq 0}h + \mathbb{R}_{\geq 0}(h - \frac{nX_2}{4Y_2}\delta)$
	$3 \nmid y_1$			$\mathbb{R}_{\geq 0}h + \mathbb{R}_{\geq 0}(h - \frac{nX_1}{4Y_1}\delta)$	$\mathbb{R}_{\geq 0}h + \mathbb{R}_{\geq 0}(h - \frac{nX_3}{4Y_3}\delta)$
$4 \nmid x_1$	$3 \mid y_1$			$\mathbb{R}_{\geq 0}h + \mathbb{R}_{\geq 0}(h - \frac{nX_2}{4Y_2}\delta)$	$\mathbb{R}_{\geq 0}h + \mathbb{R}_{\geq 0}(h - \frac{nX_2}{4Y_2}\delta)$
	$3 \nmid y_1$	$2 \mid x_1$		$\mathbb{R}_{\geq 0}h + \mathbb{R}_{\geq 0}(h - \frac{nX_2}{4Y_2}\delta)$	$\mathbb{R}_{\geq 0}h + \mathbb{R}_{\geq 0}(h - \frac{nX_3}{4Y_3}\delta)$
		$2 \nmid x_1$	$2 \mid y_1$	$\mathbb{R}_{\geq 0}h + \mathbb{R}_{\geq 0}(h - \frac{nX_2}{4Y_2}\delta)$	$\mathbb{R}_{\geq 0}h + \mathbb{R}_{\geq 0}(h - \frac{nX_6}{4Y_6}\delta)$
			$2 \nmid y_1$	$\mathbb{R}_{\geq 0}h + \mathbb{R}_{\geq 0}(h - \frac{nX_3}{4Y_3}\delta)$	$\mathbb{R}_{\geq 0}h + \mathbb{R}_{\geq 0}(h - \frac{nX_6}{4Y_6}\delta)$

(3) If $\sqrt{n/9} \in \mathbb{Q}$, then we have

$$\mathrm{Nef}(\mathrm{Km}^3(A)) = \mathrm{Mov}(\mathrm{Km}^3(A)) = \mathbb{R}_{\geq 0}h + \mathbb{R}_{\geq 0}(h - \tfrac{\sqrt{n}}{2}\delta).$$

6.3 The case of $n \equiv 1 \mod 3$.

In this subsection, we assume that $n \equiv 1 \mod 3$ and $\sqrt{n} \notin \mathbb{Q}$. We consider the following generalized Pell equation:

$$y^2 - nx^2 = 9. \tag{6.14}$$

We set $(X, Y) = (\frac{2x}{3}, \frac{y}{3})$ and

$$\frac{8}{\langle u, v \rangle} u = v \pm Xh \mp Y\delta.$$

By Lemma 4.4, there are 1 or 3 orbits of the solutions of (6.14). We calculate the boundaries of $\mathrm{Nef}(\mathrm{Km}^3(A))$ and $\mathrm{Mov}(\mathrm{Km}^3(A))$.

Let $(u_1, v_1)(u_1, v_1 > 0)$ be the fundamental solution of $y^2 - nx^2 = 1$.

Lemma 6.29. We assume that (6.14) has one orbit of solutions. Then we have the same consequence as Theorem 6.5.

Proof. By the assumption,

$$\pm 3(v_1 + u_1\sqrt{n})^k, \ k \in \mathbb{Z} \tag{6.15}$$

give the solutions of (6.14) via the correspondence (4.6). We set $v_k + u_k\sqrt{n} := (v_1 + u_1\sqrt{n})^k$. Then we have $v_k^2 - nu_k^2 = 1$. If we set $(X_k, Y_k) = (2u_k, v_k)$, then (X_k, Y_k) are integral solutions of

$$Y^2 - n\Big(\frac{X}{2}\Big)^2 = 1.$$

Hence we have the same consequence as Theorem 6.5. □

Example 6.30. Let $n = 79$. Then $(u_1, v_1) = (9, 80)$. Hence $2 \mid v_1$. Let $n = 34$. Then $(u_1, v_1) = (6, 35)$. Hence $2 \nmid v_1$ and $2 \mid u_1$. Let $n = 136$. Then $(u_1, v_1) = (3, 35)$. Hence $2 \nmid v_1$ and $2 \nmid u_1$.

We assume that (6.14) has three orbits of solutions. Let (a, b) be the solution satisfying $a, b > 0$ and $a/b \leq a'/b'$ for all solutions (a', b') of $a', b' > 0$. In particular, $a/b < u_1/v_1$. Then

$$\pm 3(v_1 + u_1\sqrt{n})^k, \ \pm(b + a\sqrt{n})(v_1 + u_1\sqrt{n})^k, \ \pm(b - a\sqrt{n})(v_1 + u_1\sqrt{n})^k, \ k \in \mathbb{Z} \tag{6.16}$$

give the solutions of (6.14) via the correspondence (4.6). Let us describe the chamber decomposition of $\mathrm{Mov}(\mathrm{Km}^3(A))$ in terms of (6.16). We set

$$\begin{aligned} b_k + a_k\sqrt{n} &:= (b + a\sqrt{n})(v_1 + u_1\sqrt{n})^k, \\ b'_k + a'_k\sqrt{n} &:= (b - a\sqrt{n})(v_1 + u_1\sqrt{n})^k. \end{aligned} \tag{6.17}$$

Then $(b'_k, a'_k) = (b_{-k}, -a_{-k})$ and we have a sequence of rational numbers

$$\cdots < \tfrac{nu_{-1}}{2v_{-1}} < \tfrac{na_{-1}}{2b_{-1}} < \tfrac{na'_0}{2b'_0} < 0 < \tfrac{na_0}{2b_0} < \tfrac{na'_1}{2b'_1} < \tfrac{nu_1}{2v_1} < \tfrac{na_1}{2b_1} < \tfrac{na'_2}{2b'_2} < \tfrac{nv_2}{2u_2} < \cdots . \tag{6.18}$$

For the solution (x, y) of (6.14), we have the following lemma.

Lemma 6.31. (1) If $3 \nmid x, y$ and $4 \mid x$, then we have $\langle u, v\rangle = 3$.

(2) If $3 \mid x, y$ and y is odd, then we have $\langle u, v\rangle = 1, 2$ or 4. Moreover, if $2 \mid x$, then $\langle u.v\rangle = 1$ or 2.

Proof. (1) Since $y^2 - nx^2 = 9$, we have $(y + 3)(y - 3) = nx^2$. Replacing y by $-y$ if necessary, we can assume that $8 \mid (y + 3)$ by $4 \mid x$. Then we have

$$\frac{8}{\langle u, v\rangle} u = \frac{8}{3}\Big(\frac{3 + y}{8}, -\frac{x}{4} H, \frac{y - 3}{2}\Big).$$

Since $u \in H^*(A, \mathbb{Z})_{\text{alg}}$ and $\gcd(3, \frac{3X}{8}, \frac{3(Y+1)}{8}) = 1$, we have $\langle u, v\rangle = 3$ for $u = (\frac{3(Y+1)}{8}, -\frac{3X}{8} H, \frac{3(Y-1)}{2})$.

(2) Since $Y^2 - n(\frac{X}{2})^2 = 1$ and Y is odd by the assumptions, we have $\langle u, v\rangle = 1, 2$ or 4. If $2 \mid x$, then $4 \mid X$. Hence we can assume that $4 \mid (1 + Y)$. Thus we have $\langle u, v\rangle = 1$ or 2. □

Lemma 6.32. For (a, b) in (6.16), we have $3 \nmid a, b$. Moreover, we have $3 \nmid a_k, b_k, a'_k, b'_k$.

Proof. If $3 \nmid a$ and $3 \mid b$, then considering $b^2 - na^2 = 9$ in $\mathbb{F}_3$ leads the contradiction. $3 \mid a$ and $3 \nmid b$ do not occur as well. We assume that $3 \mid a, b$. We set $a = 3\alpha$ and $b = 3\beta (\alpha, \beta \in \mathbb{Z})$. Then we have $\beta^2 - n\alpha^2 = 1$. Since $\beta + \alpha\sqrt{n} = (v_1 + u_1\sqrt{n})^k$ for some $k \in \mathbb{Z}$, we have $b + a\sqrt{n} = 3(v_1 + u_1\sqrt{n})^k$. Since (6.14) has three orbits of solutions, this is contradiction.

By using induction on k, we show the last claim. For $k = 0$, this is our assumption. We assume that $3 \nmid a_{k-1}, b_{k-1}, a'_{k-1}, b'_{k-1}$ for $k \geq 1$. Since $n \equiv 1 \mod 3$ and $v_1^2 - nu_1^2 = 1$, we have $3 \mid u_1$ and $3 \nmid v_1$. Then $3 \nmid (v_1 b_{k-1} + na_{k-1}u_1), (v_1 a_{k-1} + u_1 b_{k-1})$. Thus $3 \nmid a_k, b_k$. This also holds for (a'_k, b'_k). □

We consider the following two cases: Case(1). $4 \mid a$, Case(2). $4 \nmid a$.

Case(1).

We assume that $4 \mid a$. We note that b is odd. Then we have $\langle u, v\rangle = 3$ for $u = (\frac{3\pm b}{8}, \mp\frac{a}{4} H, \frac{\pm b - 3}{2})$ and

$$\text{Nef}(\text{Km}^3(A)) = \mathbb{R}_{\geq 0} h + \mathbb{R}_{\geq 0}(h - \tfrac{na}{2b}\delta)$$

by Lemma 6.31. In order to describe the chamber decomposition of $\text{Mov}(\text{Km}^3(A))$, we consider $(x, y) = (a'_1, b'_1)$.

Lemma 6.33. If $4 \mid u_1$, then $\text{Mov}(\text{Km}^3(A)) = \mathbb{R}_{\geq 0} h + \mathbb{R}_{\geq 0}(h - \frac{nu_1}{2v_1}\delta)$.

Proof. We note that v_1 is odd by $v_1^2 - nu_1^2 = 1$. Since $a'_1 = bu_1 - av_1$, we have $4 \mid a'_1$. Since $3 \nmid a'_1, b'_1$ by Lemma 6.32, we have $\langle u, v\rangle = 3$ for $u = (\frac{3\pm b'_1}{8}, \mp\frac{a'_1}{4} H, \frac{\pm b'_1 - 3}{2})$ by Lemma 6.31. Moreover, since $4 \mid u_1$, $8 \mid (v_1 \pm 1)$. Hence we have $\langle u, v\rangle = 1$ for $u = (\frac{1\pm v_1}{8}, \mp\frac{u_1}{4} H, \frac{\pm v_1 - 1}{2})$. □

Example 6.34. Let $n = 13$. Then $(a, b) = (8, 29)$ and $(u_1, v_1) = (180, 649)$. Hence $4 \mid a$ and $4 \mid u_1$.

By the proof of Lemma 6.33, we have the following.

Corollary 6.35. Assume that $n \equiv 1 \mod 3, 4 \mid a$ and $4 \mid u_1$. Then $\text{Mov}(\text{Km}^3(A))$ is divided into the chambers such that each chamber is an ample cone of a minimal model of $\text{Km}^3(A)$.

$$\begin{aligned} \mathcal{C}_1 :=& \mathbb{R}_{>0} h + \mathbb{R}_{>0}(h - \tfrac{na}{2b}\delta),\ \mathcal{C}_2 := \mathbb{R}_{>0}(h - \tfrac{na}{2b}\delta) + \mathbb{R}_{>0}(h - \tfrac{na'_1}{2b'_1}\delta), \\ \mathcal{C}_3 :=& \mathbb{R}_{>0}(h - \tfrac{na'_1}{2b'_1}\delta) + \mathbb{R}_{>0}(h - \tfrac{nu_1}{2v_1}\delta). \end{aligned} \tag{6.19}$$

Lemma 6.36. We assume that $4 \nmid u_1$.

(1) If $2 \mid u_1$, then $\mathrm{Mov}(\mathrm{Km}^3(A)) = \mathbb{R}_{\geq 0}h + \mathbb{R}_{\geq 0}(h - \frac{nu_1}{2v_1}\delta)$.

(2) If $2 \nmid u_1$, then $\mathrm{Mov}(\mathrm{Km}^3(A)) = \mathbb{R}_{\geq 0}h + \mathbb{R}_{\geq 0}(h - \frac{nu_2}{2v_2}\delta)$.

Proof. Since b is odd, $4 \nmid a_1'$. Hence we consider $(x, y) = (3u_1, 3v_1)$.

(1) If $2 \mid u_1$, then v_1 is odd. We have $4 \mid (1 \pm v_1)$. Since $2 \mid u_1$ and $4 \nmid u_1$, we have $\langle u, v\rangle = 2$ for $u = (\frac{1\pm v_1}{4}, \mp\frac{u_1}{2}H, \pm v_1 - 1)$.

(2) We assume that $2 \nmid u_1$. We divide the proof into two cases. (i) We assume that v_1 is even. Since $\gcd(1 \pm v_1, 2) = 1$, we consider $(x, y) = (a_1, b_1)$. We have $4 \nmid a_1$. Hence we consider $(x, y) = (a_2', b_2')$. Since $4 \mid a_2' = 2bu_1v_1 - a(2v_1^2 - 1)$, we have $\langle u, v\rangle = 3$ for $u = (\frac{3\pm b_2'}{8}, \mp\frac{a_2'}{4}H, \frac{\pm b_2' - 3}{2})$. Thus we consider $(x, y) = (3u_2, 3v_2)$. Since $v_2 = 2v_1^2 - 1$ is odd and $4 \mid u_2 = 2u_1v_1$, we have $\langle u, v\rangle = 1$ for $u = (\frac{1\pm v_2}{8}, \mp\frac{u_2}{4}H, \frac{\pm v_2 - 1}{2})$. (ii) We assume that v_1 is odd. Since $2 \nmid u_1$, we have $\langle u, v\rangle = 4$ for $u = (\frac{1\pm v_1}{2}, \mp u_1 H, 2(\pm v_1 - 1))$. As seen in the case (i), we consider $(x, y) = (a_2', b_2')$. Since $2 \nmid v_1$, we have $4 \nmid a_2'$. Hence we consider $(x, y) = (u_2', v_2')$. Since $2 \mid u_2$ and $4 \nmid u_2$, we have $\langle u, v\rangle = 2$ for $u = (\frac{1\pm v_2}{4}, \mp\frac{u_2}{2}H, \pm v_2 - 1)$. □

Example 6.37. Let $n = 22$. Then $(a, b) = (4, 19)$ and $(u_1, v_1) = (42, 197)$. Hence $4 \mid a, 4 \nmid u_1$ and $2 \mid u_1$. Let $n = 67$. Then $(a, b) = (16, 131)$ and $(u_1, v_1) = (5967, 48842)$. Hence $4 \mid a$ and $2 \nmid u_1$.

By the proof of Lemma 6.36, we have the following.

Corollary 6.38. Assume that $n \equiv 1 \mod 3$ and $4 \nmid u_1$. In the following cases, $\mathrm{Mov}(\mathrm{Km}^3(A))$ is divided into the chambers such that each chamber is an ample cone of a minimal model of $\mathrm{Km}^3(A)$.

(1) If $2 \mid u_1$, then

$$\mathcal{C}_1 := \mathbb{R}_{>0}h + \mathbb{R}_{>0}(h - \tfrac{na}{2b}\delta),\ \mathcal{C}_2 := \mathbb{R}_{>0}(h - \tfrac{na}{2b}\delta) + \mathbb{R}_{>0}(h - \tfrac{nu_1}{2v_1}\delta). \tag{6.20}$$

(2) If $2 \nmid u_1$ and $2 \mid v_1$, then

$$\begin{aligned}\mathcal{C}_1 &:= \mathbb{R}_{>0}h + \mathbb{R}_{>0}(h - \tfrac{na}{2b}\delta),\ \mathcal{C}_2 := \mathbb{R}_{>0}(h - \tfrac{na}{2b}\delta) + \mathbb{R}_{>0}(h - \tfrac{na_2'}{2b_2'}\delta),\\ \mathcal{C}_3 &:= \mathbb{R}_{>0}(h - \tfrac{na_2'}{2b_2'}\delta) + \mathbb{R}_{>0}(h - \tfrac{nu_2}{2v_2}\delta).\end{aligned} \tag{6.21}$$

(3) If $2 \nmid u_1, v_1$, then

$$\begin{aligned}\mathcal{C}_1 &:= \mathbb{R}_{>0}h + \mathbb{R}_{>0}(h - \tfrac{na}{2b}\delta),\ \mathcal{C}_2 := \mathbb{R}_{>0}(h - \tfrac{na}{2b}\delta) + \mathbb{R}_{>0}(h - \tfrac{nu_1}{2v_1}\delta),\\ \mathcal{C}_3 &:= \mathbb{R}_{>0}(h - \tfrac{nu_1}{2v_1}\delta) + \mathbb{R}_{>0}(h - \tfrac{nu_2}{2v_2}\delta).\end{aligned} \tag{6.22}$$

Case(2).

We assume that $4 \nmid a$. Then a Mukai vector u which determines the walls is not determined for $(x, y) = (a, b)$, i.e., $u \notin H^*(A, \mathbb{Z})_{\mathrm{alg}}$. In order to determine the wall of $\mathrm{Nef}(\mathrm{Km}^3(A))$, we consider $(a_1', b_1') = (bu_1 - av_1, bv_1 - nau_1)$ by (6.18).

Lemma 6.39. We assume that a is even and $2 \nmid v_1$.

(1) If $2 \nmid u_1$, then

$$\mathrm{Nef}(\mathrm{Km}^3(A)) = \mathbb{R}_{\geq 0}h + \mathbb{R}_{\geq 0}(h - \tfrac{nu_1}{2v_1}\delta),$$
$$\mathrm{Mov}(\mathrm{Km}^3(A)) = \mathbb{R}_{\geq 0}h + \mathbb{R}_{\geq 0}(h - \tfrac{nu_2}{2v_2}\delta).$$

(2) If $4 \mid u_1$, then

$$\mathrm{Nef}(\mathrm{Km}^3(A)) = \mathrm{Mov}(\mathrm{Km}^3(A)) = \mathbb{R}_{\geq 0}h + \mathbb{R}_{\geq 0}(h - \tfrac{nu_1}{2v_1}\delta).$$

(3) If $2 \mid u_1$ and $4 \nmid u_1$, then

$$\mathrm{Nef}(\mathrm{Km}^3(A)) = \mathbb{R}_{\geq 0}h + \mathbb{R}_{\geq 0}(h - \tfrac{na_1'}{2b_1'}\delta),$$
$$\mathrm{Mov}(\mathrm{Km}^3(A)) = \mathbb{R}_{\geq 0}h + \mathbb{R}_{\geq 0}(h - \tfrac{nu_1}{2v_1}\delta).$$

Proof. We note that b is odd by $b^2 - na^2 = 9$. Moreover $2 \mid a$ and $4 \nmid a$. Since $a^2 \equiv 4 \mod 8$ and $b^2 \equiv 1 \mod 8$, $2 \mid n$. In particular, $2 \nmid v_1$.

We assume that $2 \mid a$ and $2 \nmid v_1$.

(1) We suppose that u_1 is odd. Since $4 \nmid a'_1$, we consider $(x, y) = (u_1, v_1)$. Since $2 \nmid u_1, v_1$, we have $\langle u, v\rangle = 4$ for $u = (\frac{1 \pm v_1}{2}, \mp u_1 H, 2(\pm v_1 - 1))$. Since $4 \nmid a_1$, we consider $(x, y) = (a'_2, b'_2)$. $a'_2 = 2 - 2 = 0$ in $\mathbb{Z}/4\mathbb{Z}$. Hence we have $\langle u, v\rangle = 3$ for $u = (\frac{3 \pm b'_2}{8}, \mp \frac{a'_2}{4} H, \frac{\pm b'_2 - 3}{2})$. For $(x, y) = (u_2, v_2)$, $2 \mid u_2, 4 \nmid u_2$ and v_2 is odd. Thus we have $\langle u, v\rangle = 2$ for $u = (\frac{1 \pm v_2}{4}, \mp \frac{u_2}{2} H, \pm v_2 - 1)$.

(2) In this case, we have $4 \nmid a'_1$. Since $2 \nmid v_1$ and $4 \mid u_1$, we have $\langle u, v\rangle = 1$ for $u = (\frac{1 \pm v_1}{8}, \mp \frac{u_1}{4} H, \frac{\pm v_1 - 1}{2})$.

(3) In this case, we have $4 \mid a'_1$. Thus we have $\langle u, v\rangle = 3$ for $u = (\frac{3 \pm b'_1}{8}, \mp \frac{a'_1}{4} H, \frac{\pm b'_1 - 3}{2})$. Since $2 \mid u_1$ and $4 \nmid u_1$, then we have $\langle u, v\rangle = 2$ for $u = (\frac{1 \pm v_1}{4}, \mp \frac{u_1}{2} H, \pm v_1 - 1)$. □

Example 6.40. Let $n = 28$. Then $(a, b) = (2, 11)$ and $(u_1, v_1) = (24, 127)$. Hence $4 \nmid a, 2 \mid a, 2 \nmid v_1$ and $4 \mid u_1$. Let $n = 10$. Then $(a, b) = (2, 7)$ and $(u_1, v_1) = (6, 19)$. Hence $4 \nmid a, 2 \mid a, 2 \nmid v_1, 4 \nmid u_1$ and $2 \mid u_1$. Let $n = 88$. Then $(a, b) = (2, 19)$ and $(u_1, v_1) = (21, 197)$. Hence $4 \nmid a, 2 \mid a, 2 \nmid v_1$ and $2 \nmid u_1$.

By the proof of Lemma 6.39, we have the following.

Corollary 6.41. Assume that $n \equiv 1 \mod 3, 2 \mid a$ and $2 \nmid v_1$. In the following cases, $\mathrm{Mov}(\mathrm{Km}^3(A))$ is divided into the chambers such that each chamber is an ample cone of a minimal model of $\mathrm{Km}^3(A)$.

(1) If $2 \nmid u_1$, then

$$\begin{aligned} \mathcal{C}_1 :=& \mathbb{R}_{>0} h + \mathbb{R}_{>0}(h - \tfrac{nu_1}{2v_1}\delta),\ \mathcal{C}_2 := \mathbb{R}_{>0}(h - \tfrac{nu_1}{2v_1}\delta) + \mathbb{R}_{>0}(h - \tfrac{na'_2}{2b'_2}\delta), \\ \mathcal{C}_3 :=& \mathbb{R}_{>0}(h - \tfrac{na'_2}{2b'_2}\delta) + \mathbb{R}_{>0}(h - \tfrac{nu_2}{2v_2}\delta). \end{aligned} \tag{6.23}$$

(2) If $2 \mid u_1$ and $4 \nmid u_1$, then

$$\mathcal{C}_1 := \mathbb{R}_{>0} h + \mathbb{R}_{>0}(h - \tfrac{na'_1}{2b'_1}\delta),\ \mathcal{C}_2 := \mathbb{R}_{>0}(h - \tfrac{na'_1}{2b'_1}\delta) + \mathbb{R}_{>0}(h - \tfrac{nu_1}{2v_1}\delta). \tag{6.24}$$

Lemma 6.42. We assume that $2 \nmid a, 2 \mid v_1$ and $4 \nmid v_1$. Then

$$\mathrm{Nef}(\mathrm{Km}^3(A)) = \mathbb{R}_{\geq 0} h + \mathbb{R}_{\geq 0}(h - \tfrac{na'_1}{2b'_1}\delta),$$

$$\mathrm{Mov}(\mathrm{Km}^3(A)) = \mathbb{R}_{\geq 0} h + \mathbb{R}_{\geq 0}(h - \tfrac{nu_2}{2v_2}\delta).$$

Proof. We note that u_1 and n are odd. Hence $2 \mid b$. Since $a^2 \equiv u_1^2 \equiv 1 \mod 8$, we have $b^2 \equiv v_1^2 \mod 8$. Thus $4 \nmid b$.

Since $a'_1 = 2 - 2 = 0$ in $\mathbb{Z}/4\mathbb{Z}$, $4 \mid a'_1$. Thus we have $\langle u, v\rangle = 3$ for $u = (\frac{3 \pm b'_1}{8}, \mp \frac{a'_1}{4} H, \frac{\pm b'_1 - 3}{2})$. Since $2 \mid v_1$, we consider $(x, y) = (a_1, b_1)$. Since $4 \mid a_1$, we have $\langle u, v\rangle = 3$ for $u = (\frac{3 \pm b_1}{8}, \mp \frac{a_1}{4} H, \frac{\pm b_1 - 3}{2})$. Since $a'_2 = a$ in $\mathbb{Z}/4\mathbb{Z}$, $4 \nmid a'_2$. For (u_2, v_2), we have $4 \mid u_2$ and v_2 is odd. Thus we have $\langle u, v\rangle = 1$ for $u = (\frac{1 \pm v_2}{8}, \mp \frac{u_2}{4} H, \frac{\pm v_2 - 1}{2})$. □

Example 6.43. Let $n = 19$. Then $(a, b) = (5, 22)$ and $(u_1, v_1) = (39, 170)$. Hence $2 \nmid a, 2 \mid v_1, 4 \nmid v_1, 2 \mid b$ and $4 \nmid b$.

By the proof of Lemma 6.42, we have the following.

Corollary 6.44. Assume that $n \equiv 1 \mod 3, 2 \nmid a, 2 \mid v_1$ and $4 \nmid v_1$. Then $\mathrm{Mov}(\mathrm{Km}^3(A))$ is divided into the chambers such that each chamber is an ample cone of a minimal model of $\mathrm{Km}^3(A)$.

$$\begin{aligned} \mathcal{C}_1 :=& \mathbb{R}_{>0} h + \mathbb{R}_{>0}(h - \tfrac{na'_1}{2b'_1}\delta),\ \mathcal{C}_2 := \mathbb{R}_{>0}(h - \tfrac{na'_1}{2b'_1}\delta) + \mathbb{R}_{>0}(h - \tfrac{na_1}{2b_1}\delta), \\ \mathcal{C}_3 :=& \mathbb{R}_{>0}(h - \tfrac{na_1}{2b_1}\delta) + \mathbb{R}_{>0}(h - \tfrac{nu_2}{2v_2}\delta). \end{aligned} \tag{6.25}$$

Lemma 6.45. We assume that $2 \nmid a$ and $4 \mid v_1$. Then

$$\mathrm{Nef}(\mathrm{Km}^3(A)) = \mathbb{R}_{\geq 0}h + \mathbb{R}_{\geq 0}(h - \tfrac{na'_1}{2b'_1}\delta),$$

$$\mathrm{Mov}(\mathrm{Km}^3(A)) = \mathbb{R}_{\geq 0}h + \mathbb{R}_{\geq 0}(h - \tfrac{nu_2}{2v_2}\delta).$$

Proof. We note that u_1 and n are odd. Hence $2 \mid b$. Since $a^2 \equiv u_1^2 \equiv 1 \mod 8$, we have $b^2 \equiv v_1^2 \mod 8$. Thus $4 \mid b$.

In this case, we have $4 \mid a'_1$. Hence we have $\langle u, v\rangle = 3$ for $u = (\frac{3\pm b'_1}{8}, \mp\frac{a'_1}{4}H, \frac{\pm b'_1-3}{2})$. Since $2 \mid v_1$ and $4 \mid a_1$, we have $\langle u, v\rangle = 3$ for $u = (\frac{3\pm b_1}{8}, \mp\frac{a_1}{4}H, \frac{\pm b_1-3}{2})$. Since $a'_2 = a$ in $\mathbb{Z}/4\mathbb{Z}$, $4 \nmid a'_2$. Since $4 \mid u_2$, we have $\langle u, v\rangle = 1$ for $u = (\frac{1\pm v_2}{8}, \mp\frac{u_2}{4}H, \frac{\pm v_2-1}{2})$. □

Example 6.46. Let $n = 7$. Then $(a, b) = (1, 4)$ and $(u_1, v_1) = (3, 8)$. Hence $2 \nmid a, 4 \mid v_1$ and $4 \mid b$.

By the proof of Lemma 6.45, we have the following.

Corollary 6.47. Assume that $n \equiv 1 \mod 3, 2 \nmid a$ and $4 \mid v_1$. We also assume that $4 \mid b$. Then $\mathrm{Mov}(\mathrm{Km}^3(A))$ is divided into the chambers such that each chamber is an ample cone of a minimal model of $\mathrm{Km}^3(A)$:

$$\begin{aligned} \mathcal{C}_1 :=& \mathbb{R}_{>0}h + \mathbb{R}_{>0}(h - \tfrac{na'_1}{2b'_1}\delta),\ \mathcal{C}_2 := \mathbb{R}_{>0}(h - \tfrac{na'_1}{2b'_1}\delta) + \mathbb{R}_{>0}(h - \tfrac{na_1}{2b_1}\delta), \\ \mathcal{C}_3 :=& \mathbb{R}_{>0}(h - \tfrac{na_1}{2b_1}\delta) + \mathbb{R}_{>0}(h - \tfrac{nu_2}{2v_2}\delta). \end{aligned} \tag{6.26}$$

Lemma 6.48. We assume that $2 \nmid a, v_1$.

(1) If $2 \mid u_1$, then $\mathrm{Nef}(\mathrm{Km}^3(A)) = \mathrm{Mov}(\mathrm{Km}^3(A)) = \mathbb{R}_{\geq 0}h + \mathbb{R}_{\geq 0}(h - \frac{nu_1}{2v_1}\delta)$.

(2) We assume that $2 \nmid u_1$.

(i) If $bu_1 = 1, av_1 = 3$ or $bu_1 = 3, av_1 = 1$ in $\mathbb{Z}/4\mathbb{Z}$, then

$$\mathrm{Nef}(\mathrm{Km}^3(A)) = \mathbb{R}_{\geq 0}h + \mathbb{R}_{\geq 0}(h - \tfrac{nu_1}{2v_1}\delta),$$

$$\mathrm{Mov}(\mathrm{Km}^3(A)) = \mathbb{R}_{\geq 0}h + \mathbb{R}_{\geq 0}(h - \tfrac{nu_2}{2v_2}\delta).$$

(ii) If $bu_1 = av_1 = 1$ or $bu_1 = av_1 = 3$ in $\mathbb{Z}/4\mathbb{Z}$, then

$$\mathrm{Nef}(\mathrm{Km}^3(A)) = \mathbb{R}_{\geq 0}h + \mathbb{R}_{\geq 0}(h - \tfrac{na'_1}{2b'_1}\delta),$$

$$\mathrm{Mov}(\mathrm{Km}^3(A)) = \mathbb{R}_{\geq 0}h + \mathbb{R}_{\geq 0}(h - \tfrac{nu_2}{2v_2}\delta).$$

Proof. (1) In this case, $4 \nmid a'_1$. If $4 \mid u_1$, then we have $\langle u, v\rangle = 1$ for $u = (\frac{1\pm v_1}{8}, \mp\frac{u_1}{4}H, \frac{\pm v_1-1}{2})$. If $2 \mid u_1$ and $4 \nmid u_1$, then we have $\langle u, v\rangle = 2$ for $u = (\frac{1\pm v_1}{4}, \mp\frac{u_1}{2}H, \pm v_1 - 1)$.

(2) By our assumption, we have $2 \mid n$. Hence $2 \nmid b$. (i) In this case, $4 \nmid a'_1$. Since v_1 is odd, we have $\langle u, v\rangle = 4$ for $u = (\frac{1\pm v_1}{2}, \mp u_1 H, 2(\pm v_1 - 1))$. $a_1 = bu_1 + av_1 = 0$ in $\mathbb{Z}/4\mathbb{Z}$. Hence $4 \mid a_1$ and we have $\langle u, v\rangle = 3$ for $u = (\frac{3\pm b_1}{8}, \mp\frac{a_1}{4}H, \frac{\pm b_1-3}{2})$. Since $4 \nmid a'_2, 2 \mid u_2$ and $4 \nmid u_2$, we have $\langle u, v\rangle = 2$ for $u = (\frac{1\pm v_2}{4}, \mp\frac{u_2}{2}H, \pm v_2 - 2)$. (ii) In this case, $4 \mid a'_1$. Hence we have $\langle u, v\rangle = 3$ for $u = (\frac{3\pm b'_1}{8}, \mp\frac{a'_1}{4}H, \frac{\pm b'_1-3}{2})$. Since v_1 is odd, we have $\langle u, v\rangle = 4$ for $u = (\frac{1\pm v_1}{2}, \mp u_1 H, 2(\pm v_1 - 1))$. $a_1 = bu_1 + av_1 = 2$ in $\mathbb{Z}/4\mathbb{Z}$. Thus $4 \nmid a_1$. Since $4 \nmid a'_2, 2 \mid u_2$ and $4 \nmid u_2$, we have $\langle u, v\rangle = 2$ for $u = (\frac{1\pm v_2}{4}, \mp\frac{u_2}{2}H, \pm v_2 - 2)$. □

Example 6.49. Let $n = 55$. Then $(a, b) = (1, 8)$ and $(u_1, v_1) = (12, 89)$. Hence $2 \nmid a, v_1$ and $4 \mid u_1$. Let $n = 40$. Then $(a, b) = (1, 7)$ and $(u_1, v_1) = (3, 19)$. Hence $2 \nmid a, b, v_1, u_1$ and $bu_1 = 1, av_1 = 3$ in $\mathbb{Z}/4\mathbb{Z}$. Let $n = 280$. Then $(a, b) = (1, 17)$ and $(u_1, v_1) = (15, 251)$. Hence $2 \nmid a, b, v_1, u_1$ and $bu_1 = av_1 = 3$ in $\mathbb{Z}/4\mathbb{Z}$.

By the proof of Lemma 6.48, we have the following.

Corollary 6.50. Assume that $n \equiv 1 \mod 3$ and $2 \nmid a, v_1$. In the following cases, $\mathrm{Mov}(\mathrm{Km}^3(A))$ is divided into the chambers such that each chamber is an ample cone of a minimal model of $\mathrm{Km}^3(A)$.

(1) Assume that $2 \nmid u_1, b$. If $bu_1 = 1, av_1 = 3$ or $bu_1 = 3, av_1 = 1$ in $\mathbb{Z}/4\mathbb{Z}$, then

$$\begin{aligned} \mathcal{C}_1 :=& \mathbb{R}_{>0}h + \mathbb{R}_{>0}(h - \tfrac{nu_1}{2v_1}\delta),\ \mathcal{C}_2 := \mathbb{R}_{>0}(h - \tfrac{nu_1}{2v_1}\delta) + \mathbb{R}_{>0}(h - \tfrac{na_1}{2b_1}\delta), \\ \mathcal{C}_3 :=& \mathbb{R}_{>0}(h - \tfrac{na_1}{2b_1}\delta) + \mathbb{R}_{>0}(h - \tfrac{nu_2}{2v_2}\delta). \end{aligned} \tag{6.27}$$

(2) Assume that $2 \nmid u_1, b$. If $bu_1 = av_1 = 1$ or $bu_1 = av_1 = 3$ in $\mathbb{Z}/4\mathbb{Z}$, then

$$\begin{aligned} \mathcal{C}_1 :=& \mathbb{R}_{>0}h + \mathbb{R}_{>0}(h - \tfrac{na_1'}{2b_1'}\delta),\ \mathcal{C}_2 := \mathbb{R}_{>0}(h - \tfrac{na_1'}{2b_1'}\delta) + \mathbb{R}_{>0}(h - \tfrac{nu_1}{2v_1}\delta), \\ \mathcal{C}_3 :=& \mathbb{R}_{>0}(h - \tfrac{nu_1}{2v_1}\delta) + \mathbb{R}_{>0}(h - \tfrac{nu_2}{2v_2}\delta). \end{aligned} \tag{6.28}$$

Lemma 6.51. We assume that $n \equiv 1 \mod 3$ and $\sqrt{n} \in \mathbb{Q}$.

(1) If $n \neq 1$, then
$$\mathrm{Nef}(\mathrm{Km}^3(A)) = \mathrm{Mov}(\mathrm{Km}^3(A)) = \mathbb{R}_{\geq 0}h + \mathbb{R}_{\geq 0}(h - \tfrac{\sqrt{n}}{2}\delta).$$

(2) If $n = 1$, then
$$\begin{aligned} \mathrm{Nef}(\mathrm{Km}^3(A)) &= \mathbb{R}_{\geq 0}h + \mathbb{R}_{\geq 0}(h - \tfrac{2}{5}\delta), \\ \mathrm{Mov}(\mathrm{Km}^3(A)) &= \mathbb{R}_{\geq 0}h + \mathbb{R}_{\geq 0}(h - \tfrac{1}{2}\delta). \end{aligned}$$

Proof. If $y^2 - nx^2 = 9$ and $x \neq 0$, then $(n, x) = (1, \pm 4), (4, \pm 2), (16, \pm 1)$. For $n = 1$, $4 \mid x$. Hence we have $\langle u, v \rangle = 3$ for $u = (1, -H, 1)$. That is, for $n = 1$, $\mathrm{Nef}(\mathrm{Km}^3(A))$ is determined by $(X, Y) = (\frac{8}{3}, \frac{5}{3})$. If $y^2 - nx^2 = 9$ and $x = 0$, then $y = \pm 3$. Then the statement is showed by [Yo6, Proposition 4.16] and the fact that $y^2 - nx^2 = 1$ has only trivial solutions $(0, \pm 1)$. In particular, for $n = 1$, $\mathrm{Mov}(\mathrm{Km}^3(A))$ is determined.

□

In summary, we get the following theorem.

Theorem 6.52. We assume that $n \equiv 1 \mod 3$.

(1) If $\sqrt{n} \notin \mathbb{Q}$, then we have the following:

(1-1) If (6.14) has one orbit of solutions, then we have the same consequence as Theorem 6.5.

(1-2) We assume that (6.14) has three orbits of solutions. Then we have the following:

(1-2-1) If $4 \mid a$, then we have the following table:

type of u_1	$\mathrm{Nef}(\mathrm{Km}^3(A))$	$\mathrm{Mov}(\mathrm{Km}^3(A))$
$2 \mid u_1$	$\mathbb{R}_{\geq 0}h + \mathbb{R}_{\geq 0}(h - \frac{na}{2b}\delta)$	$\mathbb{R}_{\geq 0}h + \mathbb{R}_{\geq 0}(h - \frac{nu_1}{2v_1}\delta)$
$2 \nmid u_1$	$\mathbb{R}_{\geq 0}h + \mathbb{R}_{\geq 0}(h - \frac{na}{2b}\delta)$	$\mathbb{R}_{\geq 0}h + \mathbb{R}_{\geq 0}(h - \frac{nu_2}{2v_2}\delta)$

(1-2-2) If $2 \mid a$ and $4 \nmid a$, then we have the following table:

type of (u_1, v_1)		$\mathrm{Nef}(\mathrm{Km}^3(A))$	$\mathrm{Mov}(\mathrm{Km}^3(A))$
$2 \nmid v_1$	$4 \mid u_1$	$\mathbb{R}_{\geq 0}h + \mathbb{R}_{\geq 0}(h - \frac{nu_1}{2v_1}\delta)$	$\mathbb{R}_{\geq 0}h + \mathbb{R}_{\geq 0}(h - \frac{nu_1}{2v_1}\delta)$
	$2 \mid u_1, 4 \nmid u_1$	$\mathbb{R}_{\geq 0}h + \mathbb{R}_{\geq 0}(h - \frac{na_1'}{2b_1'}\delta)$	$\mathbb{R}_{\geq 0}h + \mathbb{R}_{\geq 0}(h - \frac{nu_1}{2v_1}\delta)$
	$2 \nmid u_1$	$\mathbb{R}_{\geq 0}h + \mathbb{R}_{\geq 0}(h - \frac{nu_1}{2v_1}\delta)$	$\mathbb{R}_{\geq 0}h + \mathbb{R}_{\geq 0}(h - \frac{nu_2}{2v_2}\delta)$

(1-2-3) If $2 \nmid a$, then we have the following table:

type of (u_1, v_1) and (a, b)			$\mathrm{Nef}(\mathrm{Km}^3(A))$	$\mathrm{Mov}(\mathrm{Km}^3(A))$
$4 \mid v_1$			$\mathbb{R}_{\geq 0}h + \mathbb{R}_{\geq 0}(h - \frac{na'_1}{2b'_1}\delta)$	$\mathbb{R}_{\geq 0}h + \mathbb{R}_{\geq 0}(h - \frac{nu_2}{2v_2}\delta)$
$2 \mid v_1, 4 \nmid v_1$			$\mathbb{R}_{\geq 0}h + \mathbb{R}_{\geq 0}(h - \frac{na'_1}{2b'_1}\delta)$	$\mathbb{R}_{\geq 0}h + \mathbb{R}_{\geq 0}(h - \frac{nu_2}{2v_2}\delta)$
$2 \nmid v_1$	$2 \mid u_1$		$\mathbb{R}_{\geq 0}h + \mathbb{R}_{\geq 0}(h - \frac{nu_1}{2v_1}\delta)$	$\mathbb{R}_{\geq 0}h + \mathbb{R}_{\geq 0}(h - \frac{nu_1}{2v_1}\delta)$
	$2 \nmid u_1$	(A)	$\mathbb{R}_{\geq 0}h + \mathbb{R}_{\geq 0}(h - \frac{nu_1}{2v_1}\delta)$	$\mathbb{R}_{\geq 0}h + \mathbb{R}_{\geq 0}(h - \frac{nu_2}{2v_2}\delta)$
		(B)	$\mathbb{R}_{\geq 0}h + \mathbb{R}_{\geq 0}(h - \frac{na'_1}{2b'_1}\delta)$	$\mathbb{R}_{\geq 0}h + \mathbb{R}_{\geq 0}(h - \frac{nu_2}{2v_2}\delta)$

where the condition (A) is $bu_1 = 1, av_1 = 3$ or $bu_1 = 3, av_1 = 1$ in $\mathbb{Z}/4\mathbb{Z}$ and (B) is $bu_1 = av_1 = 1$ or $bu_1 = av_1 = 3$ in $\mathbb{Z}/4\mathbb{Z}$.

(2) If $\sqrt{n} \in \mathbb{Q}$, then we have the following table:

type of n	$\mathrm{Nef}(\mathrm{Km}^3(A))$	$\mathrm{Mov}(\mathrm{Km}^3(A))$
$n \neq 1$	$\mathbb{R}_{\geq 0}h + \mathbb{R}_{\geq 0}(h - \frac{\sqrt{n}}{2}\delta)$	$\mathbb{R}_{\geq 0}h + \mathbb{R}_{\geq 0}(h - \frac{\sqrt{n}}{2}\delta)$
$n = 1$	$\mathbb{R}_{\geq 0}h + \mathbb{R}_{\geq 0}(h - \frac{2}{5}\delta)$	$\mathbb{R}_{\geq 0}h + \mathbb{R}_{\geq 0}(h - \frac{1}{2}\delta)$

7 Appendix.

We calculate $\mathrm{Mov}(\mathrm{Km}^{l-1}(A))$ for an abelian surface A with $\rho(A) = 1$. Let u be the Mukai vector which determines the non-trivial boundary of $\mathrm{Mov}(\mathrm{Km}^{l-1}(A))$. Then as explained in section 3, u is an isotropic Mukai vector. By using the same way as in the case of $l = 3$, we have

$$\frac{2l}{\langle u, v\rangle}u = v + Xh + Y\delta = (1 + Y, XH, l(Y - 1)),$$

where $X, Y \in \mathbb{Z}$. Since u is an isotropic vector, we have $lY^2 - nX^2 = l$. Moreover, since u satisfies $\langle u, v\rangle = 1$ or 2, we have $l \mid X$. Let $X = lZ$. Then we have

$$Y^2 - lnZ^2 = 1. \tag{7.1}$$

Let (Z_1, Y_1) be the minimum solution of (7.1). Then it satisfies

$$Y_1^2 - ln{Z_1}^2 = 1 \Longleftrightarrow (Y_1 + 1)(Y_1 - 1) = lnZ_1^2.$$

(1) If $l|(Y_1 + 1)$ or $l|(Y_1 - 1)$, then we have $Y_1 \pm 1 = kl$, where $k \in \mathbb{N}$. The vectors

$$(1 \pm Y_1, \mp X_1H, l(\pm Y_1 - 1)) = (\pm kl, \mp lZ_1H, l(\pm kl - 2)) = \pm l(k, -Z_1H, kl \mp 2)$$

are divided by l. Then $\gcd(k, Z_1, kl \mp 2) = \gcd(k, Z_1, 2) = 1$ or 2. If $\gcd(k, Z_1, 2) = 1$, we have $\langle u, v\rangle = 2$ for $u = (\frac{1 \pm Y_1}{l}, \mp\frac{X_1}{l}H, \pm Y_1 - 1)$. If $\gcd(k, Z_1, 2) = 2$, we have $\langle u, v\rangle = 1$ for $u = (\frac{1 \pm Y_1}{2l}, \mp\frac{X_1}{2l}H, \frac{\pm Y_1 - 1}{2})$.

(2) If $l \nmid (Y_1 \pm 1)$, we consider the next solution $(Z_2, Y_2) = (2Y_1Z_1, Y_1^2 + lnZ_1^2)$. Since $Y_1^2 - lnZ_1^2 = 1$, we see that

$$(1 - Y_2, X_2H, l(-Y_2 - 1)) = -2l(nZ_1^2, -Z_1Y_1H, Y_1^2).$$

Hence $\langle u, v\rangle = 1$ for $u = -(nZ_1^2, -Z_1Y_1H, Y_1^2)$.

Thus, we have the following theorem.

Theorem 7.1. 1. Assume that $\sqrt{ln} \notin \mathbb{Q}$. Let u be the Mukai vector which determines the boundary of movable cones of $\mathrm{Km}^{l-1}(A)$.

(1) Assume that $l \mid Y + 1$ for $Y = Y_1$ or $Y = -Y_1$. We set $X = -X_1$ or X_1 according as $Y = Y_1$ or $Y = -Y_1$. Then,

$\langle u,v\rangle = 2$ for $u = (\frac{Y+1}{l}, -\frac{X}{l}H, Y-1)$ if $\gcd(\frac{Y+1}{l}, -\frac{X}{l}, Y-1) = 1$.
$\langle u,v\rangle = 1$ for $u = (\frac{Y+1}{2l}, -\frac{X}{2l}H, \frac{Y-1}{2})$ if $\gcd(\frac{Y+1}{l}, -\frac{X}{l}, Y-1) = 2$.

(2) Let $l \nmid (Y_1 \pm 1)$. Then we have $\langle u,v\rangle = 1$ for $u = -(nZ_1^2, Z_1Y_1H, Y_1^2)$.

2. Assume that $\sqrt{ln} \in \mathbb{Q}$. Then $\mathrm{Mov}(\mathrm{Km}^{l-1}(A)) = \mathbb{R}_{\geq 0}h + \mathbb{R}_{\geq 0}(h - \sqrt{\frac{n}{l}}\delta)$.

References

[AAC] T. Andreescu, D. Andrica, I. Cucurezeanu, An introduction to Diophantine equations: A problem-based approach, Springer-Verlag, New York, 2010.

[BM] A. Bayer, E. Macri, MMP for moduli of sheaves on $K3$s via : nef and movable cones, Lagrangian fibrations, Invent Math. **198** (2014), no. 3, 505–590.

[Be] A. Beauville, Variétés Kähleriennes dont la première classe de Chern est nulle, J. Diff. Geom. **18** (1983), 755–782.

[BL] C. Birkenhake, H. Lange, Complex abelian varieties, Second edition. Grundlehren der Mathematischen Wissenschaften, 302. Springer-Verlag, Berlin, 2004. xii+635 pp.

[Bo] F. A. Bogomolov, On the decomposition of Kähler manifolds with trivial canonical class, Mat. Sb., **93** (1974), 580–583.

[Br1] T. Bridgeland, Stability conditions on triangulated categories, Ann. of Math. (2) **166** (2007), no. 2, 317–345.

[Br2] T. Bridgeland, Stability conditions on $K3$ surfaces, Duke Math. J. **141** (2008), 241–291.

[HT] B. Hassett, Y. Tschinkel, Moving and ample cones of holomorphic symplectic fourfolds, Geom. Funct. Anal. **19**(4), 1065–1080(2009).

[Hu1] D. Huybrechts, Compact hyperkähler manifolds: Basic results, Inv. Math. **135** (1999), 63–113.

[Hu2] D. Huybrechts, Fourier-Mukai Transforms in Algebraic Geometry, Oxford Univ. Press, 2006.

[HL] D. Huybrechts, M. Lehn, The Geometry of Moduli Spaces of Sheaves, 2nd edit., Cambridge University Press, 2010.

[KM] J. Kollár, S. Mori, Birational geometry of algebraic varieties, Cambridge Tracts in Math. **134**, Cambridge Univ. Press, 1998.

[Mo] A. Mori, Nef cone of a generalized Kummer 4-fold, to appear in Hokkaido Mathematical Journal.

[Mu1] S. Mukai, Semi-homogeneous vector bundles on an abelian variety, J. Math. Kyoto Univ. **18** (1978), 239–272.

[Mu2] S. Mukai, Duality between $\mathcal{D}(X)$ and $\mathcal{D}(\hat{X})$ with its application to Picard sheaves, Nagoya Math. J. **81**, 153–175 (1981).

[Mu3] S. Mukai, Symplectic structure of the moduli space of sheaves on an abelian or $K3$ surface, Invent. math. **77**, 101–116 (1984).

[Mu4] S. Mukai, On the moduli space of bundles on $K3$ surfaces I, Vector bundles on Algebraic Varieties, pp. 341–413. Oxford 1987.

[MYY1] H. Minamide, S. Yanagida, K. Yoshioka, Some moduli spaces of Bridgeland's stability conditions, Int. Math. Res. Not. **19** (2014) 5264–5327.

[MYY2] H. Minamide, S. Yanagida, K. Yoshioka, The wall-crossing behavior for Bridgeland's stability conditions on abelian and K3 surfaces, J. reine angew. Math. **735** (2018), 1–107.

[Na] Y. Namikawa, Counter-example to global Torelli problem for irreducible symplectic manifolds, Math. Ann. **324** (2002), no. 4, 841–845.

[O1] K. O'Grady, The weight-two Hodge structure of moduli spaces of sheaves on a $K3$ surface, J. Algebraic Geom., **6** (1997), 599–644.

[O2] K. O'Grady, Desingularized moduli spaces of sheaves on a $K3$, J. reine angew. Math. **512** (1999), 49–117.

[O3] K. O'Grady, A new six-dimensional irreducible symplectic variety, J. Algebraic Geom., **12** (2003), 435–505.

[Or1] D. Orlov, On equivalences of derived categories and $K3$ surfaces, J. Math. Sci. (New York) **84** (1997), 1361–1381.

[Or2] D. Orlov, Derived categories of coherent sheaves on abelian varieties and equivalences between them, Izv. Ross. Akad. Nauk Ser. Mat., **66**(2002), no. 3, 131–158.

[Ra] A. Rapagnetta, On the Beauville form of the known irreducible symplectic varieties, Math. Ann. **340** (2008), 77–95.

[To] Y. Toda, Limit stable objects on Calabi-Yau 3-folds, Duke Math. J. Vol. 149 (2009), 157–208.

[WW] J. Wierzba, J.A. Wiśniewski, Small contractions of symplectic 4-folds, Duke Math. J. 120:1 (2003), 65–95.

[YY] S. Yanagida, K. Yoshioka, Semi-homogeneous sheaves, Fourier-Mukai transforms and moduli of stable sheaves on Abelian surfaces, J. reine angew. Math. **684** (2013), 31–86.

[Yo1] K. Yoshioka, Some examples of Mukai's reflections on $K3$ surfaces, J. reine angew. Math. **515**, (1999), 97–123.

[Yo2] K. Yoshioka, Moduli spaces of stable sheaves on abelian surfaces, Math. Ann. **321** (2001), 817–884.

[Yo3] K. Yoshioka, Twisted stability and Fourier-Mukai transform I, Comp. Math., **138** (2003), 261–288.

[Yo4] K. Yoshioka, Twisted stability and Fourier-Mukai transform II, Manuscripta math., **110** (2003), 433–465.

[Yo5] K. Yoshioka, Vector bundles on algebraic surfaces, in the proceeding of Daisūgaku Symposium (Tokyo university, Sep 2014), 53–67.

[Yo6] K. Yoshioka, Bridgeland's stability and the positive cone of the moduli spaces of stable objects on an abelian surface, Adv. Stub. Pure Math. **69** (2016), 473–537.