



Hidden supersymmetry and fermion generation structure in extra dimensional models

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Doctoral Dissertation

**Hidden supersymmetry
and fermion generation structure
in extra dimensional models**

(余剰次元模型における隠れた超対称性とフェルミオン世代構造)

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Abstract

In this dissertation, we clarify hidden supersymmetries and the fermion generation structure in extra dimensional models.

Extra dimensional models could solve problems left in the Standard Model (SM) and are promising candidates of beyond the SM. We decompose higher dimensional fields in extra dimensional models into Kaluza–Klein (KK) mode functions, and derive 4D effective theories with an infinite number of 4D fields. In these effective theories, we regard massless 4D fields as the ones in the SM and massive fields correspond to new particles. Although many proposals have been made, we have not obtained the sufficient models which unifiedly explain the mysteries in the SM yet. In extra dimensional models, 4D effective theories greatly depend on a geometry and boundary conditions of extra dimensions. Therefore, to construct phenomenological models with extra dimensions, we should reveal the theoretical structure in extra dimensional models from the viewpoint of a geometry and boundary conditions, and clarify when the SM is realized and what new particles appear in the 4D effective theories. These are the main purposes in this dissertation.

Here we focus on the structure of the supersymmetric quantum mechanics (SUSY QM) in the 4D mass spectrum obtained from higher dimensional Dirac fields. We show that the structure of $\mathcal{N} = 2$ SUSY QM exists in any extra dimensional models with Dirac fields, which results from the requirement that the mass terms of 4D spinors should consist of the pair of 4D left and right-handed chiral fields. The supercharge in this SUSY QM relates the KK mode functions for 4D left and right-handed chiral fields and explains the 2-fold degeneracy of them. In addition to this 2-fold degeneracy, more degeneracies generally exist in the 4D mass spectrum due to the degrees of freedom of the extra space. We will reveal the structure of $\mathcal{N}$ -extended SUSY QM hidden in these degeneracies, and show that the supercharges relate additionally degenerate KK mode functions.

Furthermore, the $\mathcal{N} = 2$ SUSY QM indicates that the number of massless 4D chiral fields is related to the topology in extra dimensions. Then, we study the 5D Dirac fermion on quantum graph to investigate the topology and construct the phenomenological models from the viewpoint of general 1D extra spaces. We clarify the allowed boundary conditions and show that the chiral fermions appear from the topology of the boundary conditions on quantum graph. It is also found that this model could naturally solve the problems of the fermion generation, fermion mass hierarchy, the origin of the flavor mixing and CP -violating phase.

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- [3] Y. Fujimoto, T. Inoue, M. Sakamoto, K. Takenaga, and I. Ueba, “5D Dirac fermion on quantum graph,” *J. Phys. A* **52** no. 45, (2019) 455401, [arXiv:1904.12458 \[hep-th\]](#).
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Chapter 1

Introduction

1.1 Mysteries in the Standard Model

The Standard Model (SM) is four-dimensional (4D) $SU(3)_C \times SU(2)_L \times U(1)_Y$ gauge theory with the quarks, leptons, gauge bosons and Higgs boson, and known as the most established model which describes the nature of elementary particles so far. It is completed by the discovery of the Higgs boson [1, 2] and successfully reproduces experimental results, *e.g.* the electron g -factor with the marvelous accuracy of $O(10^{-12})$.

However, there are many mysteries left in the SM. Examples are as follows:

- **generations:**

The SM includes three copies of the quarks and the leptons so called the generation and they have same quantum numbers except for their masses. Although it is known that more than or equal to three generations are required to realize the CP -violation which is confirmed experimentally [3], a theoretical origin of three generations of the quarks and the leptons cannot be explained in the SM.

- **chiral asymmetry:**

In the SM, the left-handed fermions are $SU(2)_L$ doublet while the right-handed ones are $SU(2)_L$ singlet since only the left-handed fermions interact with the $SU(2)_L$ gauge bosons from the parity violation. Why this chiral asymmetry exists in the SM?

- **mass hierarchy:**

There are huge mass differences among the generations of fermions. For example, the mass ratio of the top quark and the up quark m_t/m_u is given as $O(10^5)$. Since the mass terms of the quarks and leptons are obtained by the Higgs mechanism, the mass hierarchies imply that the Yukawa coupling constants greatly differ with each other. In the SM, the Yukawa coupling constants are the free parameters and the SM cannot answer why the nature chooses such unnatural Yukawa couplings with huge hierarchies.

- **flavor mixing and CP -violating phase:**

The flavor mixing and the CP -violation are induced through the weak interaction. The strength of the flavor mixing and the CP -violation are described by the three mixing angles and CP -violating phase in the Cabibbo–Kobayashi–Maskawa (CKM) matrix for the quarks. What does theoretically determine this mixing matrix in the SM?

- **gauge hierarchy:**

The energy scale of the electroweak interaction and also the Higgs mass in the SM are given by $O(100)$ GeV due to the vacuum expectation value (VEV) of the Higgs $v = 246$ GeV. However, scalar fields does not have symmetries which ensure they are massless such as the chiral symmetry for spinors and the gauge symmetry for gauge fields. Therefore, the quantum correction for the Higgs mass includes a term in proportion to the square of a cut off scale. Then if we assume that the SM is applicable up to GUT scale $O(10^{15})$ GeV or Planck scale $O(10^{18})$ GeV, we require a fine tuning between the bare mass and the quantum correction to obtain the physical Higgs mass 125 GeV. This is known as the gauge hierarchy problem which also asks that why the energy scale of the weak interaction is much smaller than the GUT scale or the Planck scale.

Therefore, we expect that there exists a more fundamental theory than the SM and it would answer these mysteries. To unveil this beyond the SM is an important task in the particle physics.

1.2 Beyond the Standard Model with extra dimensions

One of promising candidates of beyond the SM is extra dimensional model. Extra dimensional models have extra spacial dimensions in addition to the ordinary 4D spacetime and we assume that an extra space is compactified to a finite volume due to the consistency of experiments.¹ Then the SM is realized as the effective theory of extra dimensional models.

The idea that we live in higher dimensions is firstly proposed by Kaluza and Klein with a motivation of the unification of the electromagnetic and gravitational interactions [6, 7]. The superstring theories, which describe the quantum gravity and the candidates for a theory of everything, is also based on the notion of higher dimensions. These days, many extra dimensional models are considered and play important roles to solve the mysteries in the SM.

1.2.1 Properties of extra dimensional models

The most remarkable feature of extra dimensional models is that there appear infinite 4D fields in the effective 4D action by the Kaluza–Klein (KK) decomposition of higher dimensional fields. Higher dimensional fields are decomposed by KK mode functions which are eigenfunctions of the Laplacian on the extra space and span complete sets. Then, coefficients of KK mode functions are identified as 4D fields whose masses are give by eigenvalues of an extra dimensional Laplacian. For example, when we consider a 5D scalar field $\Phi(x, y)$ on an S^1 -extra dimension with the periodic boundary condition, we can consider the Fourier series expansion

$$\Phi(x, y) = \sum_n \phi_n(x) \sqrt{\frac{1}{2\pi R}} e^{i\frac{n}{R}y}, \quad (1.2.1)$$

where x and y denote the coordinates of the 4D spacetime and the extra space respectively, and R is the radius of S^1 . Then, the higher dimensional equation of motion $[-\partial_\mu^2 - \partial_y^2]\Phi(x, y) = 0$ leads to 4D equations of motion

$$\left[-\partial_\mu^2 + \left(\frac{n}{R}\right)^2\right]\phi_n(x) = 0, \quad (1.2.2)$$

¹Although models with non-compact extra dimensions are also known [4, 5], we focus on compact cases in this dissertation.

and we can regard $\phi_n(x)$ as the 4D fields with the mass n/R .

Massive particles and their mode functions are called the KK particle and massive KK modes respectively. On the other hand, mode functions of massless particles are referred as KK zero modes. As well as the above example, masses of KK particles are generally in inverse proportion to the size of a extra dimensional space. Therefore, if we consider the case that the size of extra dimensions are sufficiently small, the KK particles have huge masses and invisible at the low energy scale in the current experiments. Then the low energy effective theory is described by the KK zero modes. We can identify these zero modes as the fields in the SM and the massive KK modes as new particles which are not observed so far.

In extra dimensional models, a geometry and boundary conditions of extra dimensions greatly affect the existence of KK zero modes and what KK particles appear. Therefore we should pay much attention to the geometry and boundary conditions of extra dimensions when we construct phenomenological models.

It should be also noted that the structure of the $\mathcal{N} = 2$ supersymmetric quantum mechanics (SUSY QM) is hidden in the 4D mass spectrum of higher dimensional spinor, gauge, gravity theories respectively, due to chiral symmetry, higher dimensional gauge symmetry, higher dimensional general covariance symmetry [8–13]. For example, mass terms of 4D spinors should consist of the pair of 4D left and right-handed chiral fields. This implies that the corresponding massive KK mode functions from a higher dimensional spinor should have same mass eigenvalues and be related with each other. The structure of the SUSY QM appears in this relation. The massive KK mode functions of 4D left and right-handed chiral fields form supermultiplets of the $\mathcal{N} = 2$ SUSY QM.

In addition to this 2-fold degeneracy, more degeneracies generally exist in the 4D mass spectrum due to the degrees of freedom of the extra space. This implies that effective 4D theories includes many KK particles with same masses. If degeneracies appear, it is expected that some symmetries are hidden behind them. It is important to reveal the symmetries which explain these additional mass degeneracies. Therefore, in this dissertation, we try to clarify the symmetries hidden in the additional degeneracies.

1.2.2 Resolution of the mysteries in the Standard Model

Next, let us discuss how to resolve the mysteries in the SM by extra dimensional models. There are many models which address the mysteries respectively or simultaneously, *e.g.* gauge hierarchy problem [14–16], fermion flavour structure [17–43] and so on. Here, we briefly see the basic ideas and models.

• gauge hierarchy

For example, the gauge hierarchy problem is addressed by the Arkani-Hamed–Dimopoulos–Dvali (ADD) model [14], the Randall-Sundrum (RS) model [15] and the gauge-Higgs unification scenario [16].

The ADD model supposes the situation that only the gravity propagates in flat extra dimensions while the matters and gauge fields are confined on the 4D object what is called the brane. Then the matters and gauge fields have no KK particles. This model says that the hierarchy does not exist since the genuine cut off scale is the higher dimensional Planck mass $M_{4+d} \sim 1$ TeV. In this model, the size of d -dimensional extra space L is given by $L \sim 10^{13}$ cm, 10^{-2} cm, 10^{-7} cm, $\dots$ for $d = 1, 2, 3, \dots$, from the relation between the 4D Planck mass M_p and the higher dimensional Planck mass M_{4+d} , $M_p^2 = (M_{4+d})^{2+d} L^d$. In the case of $d = 1$, the size L is too big and the inconsistency

with Newton's law of gravitation occurs due to the effect of the propagation of the gravitation into the extra space. However, the case of $d \geq 2$ are not excluded by the experiments and the size of extra dimensions might be given by $L \sim 10^{-2}$ cm surprisingly. Therefore this model is also referred to the Large Extra Dimension model. It should be noted that a new hierarchy problem will arise between the electroweak scale and the large size of the extra dimension $1/L \sim 10^{-3}$ eV in the case of $d = 2$.

The RS model considers the 5D Anti-de Sitter spacetime whose extra space is compactified to the S^1/Z_2 orbifold by the identification $y \sim -y$ ($-\pi R \leq y < \pi R$). There are two branes so called UV brane and IR brane at the fixed points $y = 0, \pi R$ respectively and the metric is given by $-ds^2 = e^{-2k|y|} \eta_{\mu\nu} dx^\mu dx^\nu + dy^2$, where k is defined by the ratio of the bulk cosmological constant and the tensions of the branes. Therefore, if we consider that the particles in the SM live in the IR brane, the quantity on the IR brane such as the Higgs VEV is rescaled by the warp factor $e^{-2\pi k R}$. Then, even if the radius R , k and the non-rescaled Higgs VEV are of the order of the Planck scale, the exponential suppression by the warp factor can naturally leads to the electroweak scale and we can explain gauge hierarchy without the fine tuning. This model also causes no new hierarchy between the electroweak scale and the large size of the extra dimension unlike the ADD model.

The gauge-Higgs unification scenario is based on the idea that the 4D Higgs field is identified as the KK zero mode of an extra-dimensional component of a higher dimensional gauge field A_y . It does not transform under the 4D Lorentz transformation and therefore, we can regard A_y as a scalar field from a 4D perspective. In this scenario, the higher dimensional gauge symmetry prohibits the quadratic UV divergence of the Higgs mass, since the local operators which break the gauge symmetry cannot exist. The compactification of the extra space leads to the Feynman diagrams which wind around the extra space and cannot shrink to a point. Then, this contribution has no UV divergence and gives the finite Higgs mass through the quantum correction.

We can also naturally relate the origin of generations, chiral asymmetry, mass hierarchy, flavor mixing and CP -violating phase to extra dimensional models.

• generations and chiral asymmetry

The chiral asymmetry results from the geometry, background fields and boundary conditions in extra dimensions. It is also known that the number of generations is related to the topology of the extra space and given by the topological quantity so-called Witten index in the SUSY QM.

The suitable geometry, background fields and boundary conditions of the extra space lead to the degenerate KK zero mode functions for a higher dimensional spinor and we can obtain the multiple 4D massless fields. Even if there are massless 4D Dirac spinors, they can become massive through quantum corrections and decouple from massless 4D chiral fields. This leads to the chiral asymmetry. Furthermore, we can identify remaining massless chiral fields as the chiral fermions in the SM. Then the generations are obtained from one higher dimensional spinor.

• mass hierarchy, flavor mixing and CP -violating phase

The mass hierarchies of the quarks and leptons are explained by overlap integrals of KK mode functions. For example, we consider the 5D Yukawa interaction for a 5D scalar field Φ and 5D Dirac fields Ψ and Ψ' with a 5D Yukawa coupling constant g_{5D}

$$S_{\text{Yukawa}} = g_{5D} \int d^4x \int dy \Phi(x, y) \bar{\Psi}(x, y) \Psi'(x, y) + h.c. \quad (1.2.3)$$

Furthermore, we assume that the scalar field has a y -dependent VEV $\langle \Phi(y) \rangle$ and the Dirac fields

can be decomposed as

$$\Psi(x, y) = \psi_L^{(0)}(x)f^{(0)}(y) + (\text{massive KK modes}), \quad (1.2.4)$$

$$\Psi'(x, y) = \psi_R'^{(0)}(x)g^{(0)}(y) + (\text{massive KK modes}), \quad (1.2.5)$$

where $\psi_L^{(0)}(x)$ and $\psi_R'^{(0)}(x)$ indicate a 4D massless left-handed and a right-handed chiral fields respectively, and $f^{(0)}(y)$ and $g^{(0)}(y)$ denote the zero mode functions. Then, the 5D Yukawa interaction leads to the following 4D mass term

$$S_{\text{Yukawa}} = \int d^4x m_{4\text{D}} \bar{\psi}_L^{(0)}(x)\psi_R'^{(0)}(x) + h.c. + (\text{massive KK modes}), \quad (1.2.6)$$

where the 4D mass $m_{4\text{D}}$ is given as an overlap integral of the mode functions

$$m_{4\text{D}} = g_{5\text{D}} \int dy \langle \Phi(y) \rangle f^{(0)\dagger}(y)g^{(0)}(y). \quad (1.2.7)$$

Therefore, if the scalar VEV and the KK mode functions are localized at some positions such as the Gaussian function, we can easily obtain large mass hierarchies without a fine tuning of 5D Yukawa couplings. The closely localized functions give a huge mass while the separated functions give a tiny mass. The schematic figure of the overlap integral is described in Figure 1.1.

Furthermore, if we identify the degenerate chiral zero modes are generations of the fermions in the SM and there are overlaps between the mode functions of different fermion generations, the overlap integrals lead to off-diagonal components of mass matrices through the higher dimensional Higgs mechanism. Then, we obtain the CKM matrix by diagonalizing the mass matrices for the quarks and the flavor mixing occurs. Moreover, if the degenerate mode functions are genuine complex functions, the CP -violating phase can also appear in the CKM matrix without introducing the different complex Yukawa couplings for each generations like 4D case.

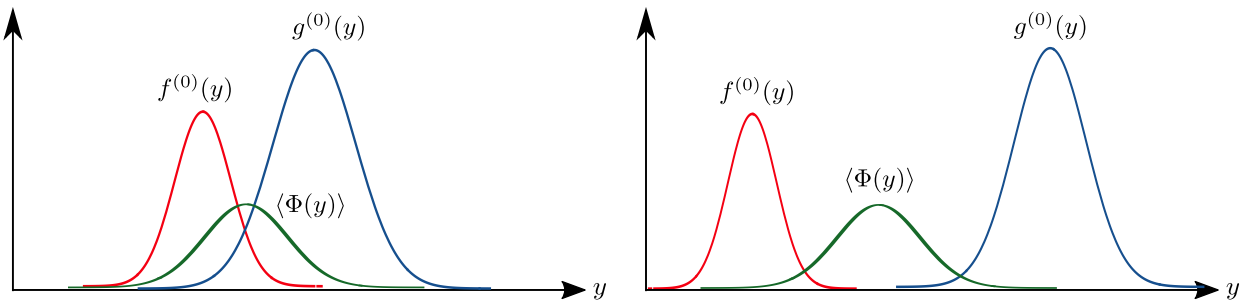


Figure 1.1: The overlap integrals of localized mode functions. The left figure illustrates the closely localized configuration and it gives a huge mass through the overlap integral. On the other hand, the right one corresponds to the separated configuration and gives a tiny mass.

The models which will explain the generations of chiral fermions, mass hierarchy, realistic flavor mixing and CP -violating phase simultaneously are less known. One of them is the model of torus-extra dimensions with or without orbifolding in a background homogeneous magnetic flux [25–40]. In this model, the magnetic flux is quantized by the condition that fields should be

single valued on the torus. Then, this quantized value of magnetic flux is related to the number of KK chiral zero modes or equivalently the generation of fermions. It is known that the KK zero modes are localized and also complex functions due to the shape of the torus and a Wilson line phase of a background gauge potential. These give the mass hierarchies, flavor mixing and CP -violating phase through the higher dimensional Yukawa interactions.

Another is the model whose extra space is an interval or S^1 with point interactions [41–43]. The point interaction is the zero-thickness interaction such as a delta function potential, and it is known that the possible point interactions in one-dimensional quantum mechanics are described by boundary conditions whose parameter space is $U(2)$ [44, 45]. By applying this idea to extra dimensional models, we can add “extra boundary conditions” for all kinds of 5D fields in the bulk space, in addition to the boundary points. In the model [41–43], they discuss the case of the point interactions with Dirichlet boundary conditions for 5D fermions. The number of generation for the fermions in the SM corresponds to the number of the point interactions for 5D spinors. Furthermore, KK zero modes become localized at positions of each point interaction. The point interactions for each 5D spinors can be placed at different positions, and the overlap of the KK zero modes are induced. This gives the mass hierarchies and the flavor mixing. However, they are not genuine complex functions since the KK zero modes for the 5D spinors can be taken as real functions by a redefinition of normalization constants. Actually, the CP -violating phase arises from a twisted boundary condition for the 5D Higgs field. This boundary condition leads to the y -dependent complex Higgs VEV and the components of mass matrices become complex through the overlap integral. The mass hierarchies of the quarks and leptons depend on the positions of the point interactions and this model realizes mass hierarchies by adjusting their positions “by hand”. Recently, the idea is proposed that a minimization of a Casimir energy determines the positions of the point interactions. Then, we may “dynamically” realize the observed quark/lepton mass hierarchy.

Although we have seen some examples, the most of models have considered only the simplest extra spaces and there are few works from the viewpoint of general geometry and boundary conditions of extra dimensions. It is important to reveal possible configurations to constructing the realistic models, and we can expect that there are extra dimensional models which can unifiedly explain the fermion generation structure and also other mysteries in the SM.

1.3 Motivation and Outline

As we have seen in the above, a geometry and boundary conditions of extra dimensions are much important phenomenologically. Therefore, we should reveal the theoretical structure in extra dimensional models from the viewpoint of general geometry and boundary conditions, and also clarify when the SM is realized and what new particles appear in the 4D effective theories.

In this dissertation, we will focus on two topics: hidden SUSY QM in the 4D mass spectrum of a higher dimensional Dirac spinor and the fermion generation structure in extra dimensional models. In addition to the 2-fold degeneracy from the $\mathcal{N} = 2$ SUSY QM, more degeneracies generally exist in the 4D mass spectrum due to the degrees of freedom of the extra space. These lead to many KK particles with same masses in an effective 4D theory. Then, we can expect the further symmetries are hidden in these additional degeneracies. Here, we will reveal the structure of $\mathcal{N}$ -extended SUSY QM with central charges hidden in these degeneracies, and show that the supercharges relate additionally degenerate KK mode functions.

Furthermore, we study the 5D Dirac fermion on the quantum graph from the viewpoint of general 1D extra spaces. The quantum graph is known as a quantum mechanical system on a one dimensional graph which consists of edges and vertices connected with each other, and boundary conditions are imposed on each vertex. Then various graphs can be obtained depending on how to connect edges and vertices, in addition to an interval and S^1 with and without point interactions. We clarify the allowed boundary conditions and show that this model could naturally solve the problems of the fermion generation, chiral asymmetry, the fermion mass hierarchy and the origin of the flavor mixing and CP -violating phase. Moreover, we also consider the case of an interval with point interactions and confirm whether this model can dynamically give the realistic fermion mass hierarchy by the minimization of the Casimir energy.

This dissertation is organized as follows: In the next chapter, we briefly review the $\mathcal{N} = 2$ SUSY QM, $\mathcal{N}$ -extended SUSY QM, its central extension and see their characteristic properties. Then, in Chapter 3, we consider the KK decomposition of higher dimensional Dirac fields. Since a 4D mass term consists of 4D left and right chiral fields, it is turned out that massive KK mode functions for them should be related with each other. The structure of $\mathcal{N} = 2$ SUSY QM is hidden in this relation and the KK mode functions form supermultiplets. Furthermore, we show that the number of massless chiral fields are given as the Witten index in $\mathcal{N} = 2$ SUSY QM and related to a topology in extra dimensions. In Chapter 4, we investigate the 5D Dirac fermion on the quantum graph from the viewpoint of general 1D extra dimensional models. We will show that the generation of fermions is obtained by the topology of boundary conditions on the quantum graph and also the mass hierarchy, the origin of the flavor mixing and CP -violating phase in the SM can be explained qualitatively. In Chapter 5, we focus on the model of the interval extra dimension with point interactions which corresponds to the quantum graph with the Dirichlet boundary condition and discuss whether the mass hierarchy of fermions can be numerically and dynamically obtained by minimizing the Casimir energy. In Chapter 6, we study the structures of $\mathcal{N}$ -extended SUSY QM with central charges in the 4D mass spectrum of the higher dimensional Dirac field. We will show that this extended SUSY results from the symmetries in the extra space. Chapter 7 is devoted to summary and discussion.

Chapter 2

Supersymmetry in quantum mechanics

In this chapter, we review the SUSY QM, which is well known as the supersymmetry realized in quantum mechanics.

SUSY QM was firstly introduced by Witten to investigate the supersymmetry breaking [46]. One of the characteristic properties in SUSY QM is that nonzero energy states are always doubly degenerate and they are related by a supercharge which corresponds to a square root of Hamiltonian. This property are applied to exactly solvable quantum mechanics [47–50]. These days, SUSY QM is also applied to a wide range of research areas, *e.g.* Berry phase [51–53], black holes and AdS/CFT [54–58], Sachdev–Ye–Kitaev model [59–63], extra dimensional models [8–13, 64] and so on.

Its extensions are also investigated. The $\mathcal{N}$ -extended supersymmetry is an extension which includes $\mathcal{N}$ independent supercharges in the supersymmetry algebra [65–72], and they explain the further degeneracy in the energy spectrum. In addition, a central extension which introduces central charges in the algebra is also studied [73–76].

This chapter is organized as follows: In the next section, we briefly review the $\mathcal{N} = 2$ supersymmetric quantum mechanics. Then, in Section 2.2, we consider the $\mathcal{N}$ -extended supersymmetry and the central extension.

2.1 $\mathcal{N} = 2$ supersymmetric quantum mechanics

In this section, we review the $\mathcal{N} = 2$ SUSY QM and see its characteristic properties. As an example, we see the Witten model which realizes the $\mathcal{N} = 2$ SUSY QM. Furthermore, we discuss the relation between the Witten index and the supersymmetry breaking.

2.1.1 Definition and properties

$\mathcal{N} = 2$ SUSY QM consists of the Hamiltonian H , supercharge Q , “fermion number” operator $(-1)^F$ which are Hermitian operators and satisfy the following relations

$$H = Q^2, \tag{2.1.1}$$

$$\{Q, (-1)^F\} = 0, \tag{2.1.2}$$

$$[(-1)^F]^2 = 1. \tag{2.1.3}$$

From the above relations, we can obtain the $\mathcal{N} = 2$ SUSY algebra

$$\{Q_i, Q_j\} = 2H\delta_{ij}, \quad [H, Q_i] = 0 \quad (i, j = 1, 2), \quad (2.1.4)$$

where we define $Q_1 \equiv Q$, $Q_2 \equiv i(-1)^F Q$. This algebra corresponds to the quantum-mechanical version of supersymmetry algebra in quantum field theory, so that the supercharges dose not have spinor indices.

Let us then consider characteristic properties of SUSY QM. The relations (2.1.1), (2.1.2) indicate H and $(-1)^F$ commute with each other. Furthermore, (2.1.3) implies the eigenvalues of $(-1)^F$ are given by ± 1 . Therefore we can consider simultaneous eigenstates

$$H|E_n, \pm\rangle = E_n|E_n, \pm\rangle, \quad (-1)^F|E_n, \pm\rangle = \pm|E_n, \pm\rangle \quad (n = 0, 1, \dots). \quad (2.1.5)$$

The Hermiticity of Q and the relation (2.1.1) imply that the Hamiltonian H is semi-positive definite

$$\langle\psi|H|\psi\rangle = \langle\psi|Q^2|\psi\rangle = \langle\psi|Q^\dagger Q|\psi\rangle = |Q|\psi\rangle|^2 \geq 0, \quad (2.1.6)$$

for any state $|\psi\rangle$. Furthermore, since the Hamiltonian H commutes with Q , the states $Q|E_n, \pm\rangle$ have the same energy eigenvalue of $|E_n, \pm\rangle$

$$H(Q|E_n, \pm\rangle) = QH|E_n, \pm\rangle = E_n Q|E_n, \pm\rangle. \quad (2.1.7)$$

In addition, Q flips the eigenvalues of $(-1)^F$ due to the anti-commutation relation (2.1.2)

$$(-1)^F(Q|E_n, \pm\rangle) = -Q(-1)^F|E_n, \pm\rangle = \mp Q|E_n, \pm\rangle. \quad (2.1.8)$$

Therefore we can find that

$$Q|E_n, \pm\rangle = \alpha_\pm|E_n, \mp\rangle, \quad (2.1.9)$$

where $\alpha_\pm$ are some constants. From the relation

$$\begin{aligned} E_n &= \langle E_n, +|H|E_n, +\rangle = \langle E_n, +|Q^2|E_n, +\rangle \\ &= |Q|E_n, +\rangle|^2 = |\alpha_+|^2, \end{aligned} \quad (2.1.10)$$

the constant α_+ is given by $\alpha_+ = \sqrt{E_n}$ up to a phase. Then the relation

$$E_n|E_n, +\rangle = Q^2|E_n, +\rangle = \alpha_+ Q|E_n, -\rangle = \alpha_+ \alpha_-|E_n, -\rangle \quad (2.1.11)$$

leads to $\alpha_- = (\alpha_+)^* = \sqrt{E_n}$. We also find that

$$Q|0, \pm\rangle = 0 \quad (2.1.12)$$

for zero energy states. Therefore, they are not related by the supercharge Q .

Then, we can summarize the properties of SUSY QM as follows:

1. The Hamiltonian H is positive-semidefinite

$$\langle \psi | H | \psi \rangle \geq 0 \quad \text{for any state } |\psi\rangle, \quad (2.1.13)$$

or equivalently

$$E_n \geq 0 \quad \text{for all the energy eigenvalues.} \quad (2.1.14)$$

2. The positive energy eigenstates with $(-1)^F = 1$ and $(-1)^F = -1$ are doubly degenerate and related by the supercharge Q

$$Q|E_n, \pm\rangle = \sqrt{E_n}|E_n, \mp\rangle. \quad (2.1.15)$$

3. Zero energy eigenstates are not related by the supercharges Q and not necessarily degenerate

$$Q|0, \pm\rangle = 0. \quad (2.1.16)$$

A typical energy spectrum of SUSY QM is described as Figure 2.1.

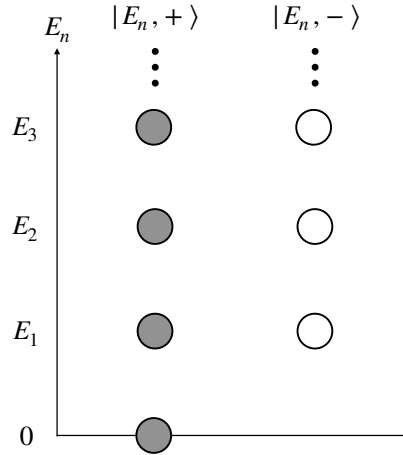


Figure 2.1: The typical energy spectrum of SUSY QM

2.1.2 Witten index and supersymmetry breaking

The Witten index Δ_W is defined by

$$\Delta_W = \mathcal{N}_{E=0,+} - \mathcal{N}_{E=0,-}, \quad (2.1.17)$$

where $\mathcal{N}_{E=0,\pm}$ denote the number of the zero energy states with $(-1)^F = \pm 1$. This index is a topological quantity which is invariant under continuous deformations of parameters in the system. This can be understood as follows: From the properties of SUSY QM, the nonzero energy states with $(-1)^F = \pm 1$ should be always doubly degenerate and change together by continuous deformations. Therefore, even if nonzero energy states become zero energy by continuous deformations, the difference $\mathcal{N}_{E=0,+} - \mathcal{N}_{E=0,-}$ does not change and the Witten index is invariant as Figure 2.2.

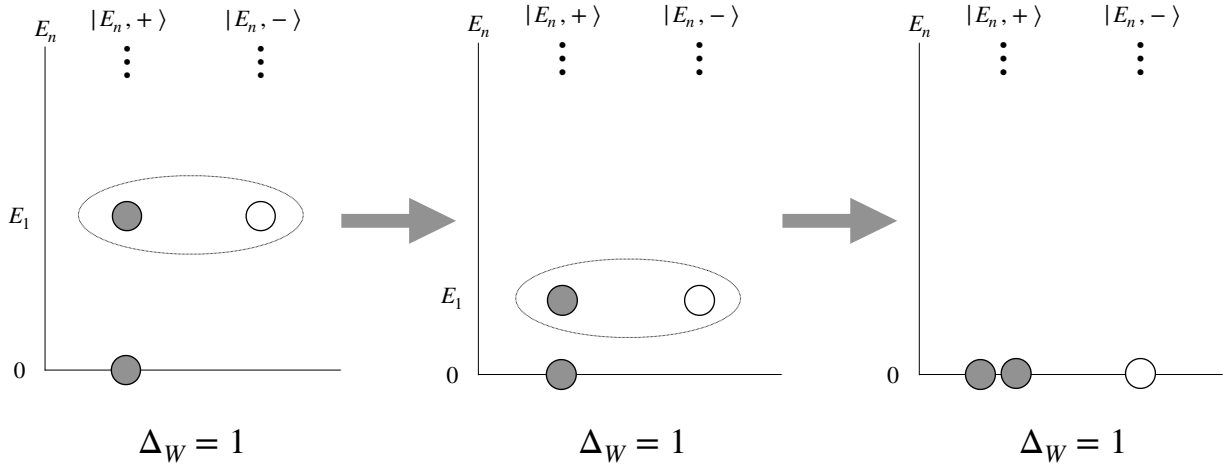


Figure 2.2: We illustrate the change of the energy spectrum by continuous deformations of parameters in the system. Nonzero energy states change together and the Witten index is invariant.

The Witten index characterizes the spontaneous supersymmetry breaking. We define the spontaneous supersymmetry breaking in quantum mechanics as nonexistence of zero energy states similarly to SUSY in quantum field theory, i.e.

$$Q|E_n, \pm\rangle \neq 0, \quad (2.1.18)$$

for the lowest energy states. Then we can find that

$$\Delta_W \neq 0 \implies \text{zero energy states exist} \iff \text{supersymmetry is restored}, \quad (2.1.19)$$

and the contraposition is given by

$$\text{spontaneous supersymmetry breaking} \implies \Delta_W = 0. \quad (2.1.20)$$

Therefore if supersymmetry is spontaneously broken, the Witten index should be zero.

2.1.3 Example: Witten model

In this section, we see the Witten model as an example of a model which realizes the $\mathcal{N} = 2$ SUSY QM in a one-dimensional quantum mechanics [46]. The Hamiltonian is given by 2×2 matrix

$$H = \begin{pmatrix} A^\dagger A & 0 \\ 0 & AA^\dagger \end{pmatrix} = \begin{pmatrix} H_+ & 0 \\ 0 & H_- \end{pmatrix} \quad (2.1.21)$$

$$= \begin{pmatrix} -\frac{\hbar^2}{2m} \frac{d^2}{dx^2} + \frac{1}{2m} (W'(x))^2 - \frac{\hbar}{2m} W''(x) & 0 \\ 0 & -\frac{\hbar^2}{2m} \frac{d^2}{dx^2} + \frac{1}{2m} (W'(x))^2 + \frac{\hbar}{2m} W''(x) \end{pmatrix}, \quad (2.1.22)$$

where $A, A^\dagger$ are defined by

$$A \equiv \frac{1}{\sqrt{2m}} \left(-i\hbar \frac{d}{dx} - iW'(x) \right), \quad A^\dagger \equiv \frac{1}{\sqrt{2m}} \left(-i\hbar \frac{d}{dx} + iW'(x) \right), \quad (2.1.23)$$

and m indicates a mass. $W(x)$ is a real function and called superpotential. $W'(x), W''(x)$ indicate first and second order derivatives for $W(x)$. Since the Hamiltonian is 2×2 matrix, a wave function is given by the form of 2-component vector

$$\Psi(x) = \begin{pmatrix} \psi_+(x) \\ \psi_-(x) \end{pmatrix}. \quad (2.1.24)$$

Then, we find that the following SUSY QM structure exists in this model: The supercharge Q , the Hamiltonian H and the fermion number operator $(-1)^F$ can be defined as

$$Q = \begin{pmatrix} 0 & A^\dagger \\ A & 0 \end{pmatrix}, \quad H = \begin{pmatrix} A^\dagger A & 0 \\ 0 & AA^\dagger \end{pmatrix}, \quad (-1)^F = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad (2.1.25)$$

and satisfy the SUSY relations (2.1.1), (2.1.2), (2.1.3). Here we require suitable boundary conditions for wave functions to satisfy the Hermiticity of Q , if boundaries exist. From the properties of SUSY QM, the eigenstates

$$\Psi_{E,+}(x) = \begin{pmatrix} \psi_{E,+}(x) \\ 0 \end{pmatrix}, \quad \Psi_{E,-}(x) = \begin{pmatrix} 0 \\ \psi_{E,-}(x) \end{pmatrix}, \quad (2.1.26)$$

$$H\Psi_{E,\pm}(x) = E\Psi_{E,\pm}(x), \quad (2.1.27)$$

$$(-1)^F \Psi_{E,\pm}(x) = \pm \Psi_{E,\pm}(x), \quad (2.1.28)$$

are doubly degenerate for nonzero energy states and related by the supercharge

$$Q\Psi_{E,\pm}(x) = \sqrt{E}\Psi_{E,\mp}(x). \quad (2.1.29)$$

If $W''(x) \neq 0$, The diagonal components of Hamiltonian H_+ and H_- are different with each other, i.e. $H_+ \neq H_-$. Therefore, when we consider the one-dimensional quantum-mechanical systems with the Hamiltonian H_+ and H_- respectively, these two systems are related by the SUSY in the Witten model although H_+ and H_- describe the different systems. Thus, if we obtain the nonzero energy eigenstates $\psi_{E,+}$ ($\psi_{E,-}$) for the system H_+ (H_-), we can also obtain the same energy eigenfunction $\psi_{E,-}$ ($\psi_{E,+}$) for the other system H_- (H_+) by the relation (2.1.29).

The following systems are examples which are related by SUSY in the Witten model:

• **Example 1 : harmonic oscillator**

We consider the superpotential

$$W(x) = \frac{1}{2}m\omega x^2, \quad (2.1.30)$$

where ω is a constant. Then H_+ and H_- are given as the Hamiltonian of the harmonic oscillator with constants $\mp \hbar\omega/2$

$$H_{\pm} = -\frac{\hbar^2}{2m} \frac{d^2}{dx^2} + \frac{m\omega^2}{2} x^2 \mp \frac{\hbar\omega}{2}. \quad (2.1.31)$$

The supercharge Q can be written by the creation-annihilation operator $a, a^\dagger$

$$Q = \begin{pmatrix} 0 & +i\sqrt{2\hbar\omega}a^\dagger \\ -i\sqrt{2\hbar\omega}a & 0 \end{pmatrix}, \quad (2.1.32)$$

$$a^\dagger = \sqrt{\frac{\hbar}{2m\omega}} \left(-\frac{d}{dx} + \frac{m\omega}{\hbar}x \right), \quad a = \sqrt{\frac{\hbar}{2m\omega}} \left(\frac{d}{dx} + \frac{m\omega}{\hbar}x \right), \quad (2.1.33)$$

and these two systems are related with each other due to the creation-annihilation operators which shift the energy by $\hbar\omega$.

• **Example 2 : free particle and Rosen-Morse potential**

We take $W'(x)$ as

$$W'(x) = \hbar \tanh(x). \quad (2.1.34)$$

In this case, H_+ and H_- are given as

$$H_+ = -\frac{\hbar^2}{2m} \frac{d^2}{dx^2} + \frac{\hbar^2}{2m} \left(1 - \frac{2}{\cosh^2(x)} \right), \quad (2.1.35)$$

$$H_- = -\frac{\hbar^2}{2m} \frac{d^2}{dx^2} + \frac{\hbar^2}{2m}. \quad (2.1.36)$$

H_- corresponds to the Hamiltonian of the free particle since the second term is merely the constant. On the other hand, H_+ has nontrivial potential which is called the Rosen-Morse potential. Nonzero energy eigenstates of these systems are related by the SUSY QM and the positive energy spectrum of H_+ has the same structure as the free particle system, in spite of the nontrivial potential.

2.2 Extended supersymmetry

In this section, we discuss the extensions of $\mathcal{N} = 2$ SUSY QM. One is the $\mathcal{N}$ -extended supersymmetry and the other is central extension.

2.2.1 $\mathcal{N}$ -extended supersymmetry

$\mathcal{N}$ -extended supersymmetry has $\mathcal{N}$ supercharges which satisfy the following algebra

$$\{Q_i, Q_j\} = 2H\delta_{ij}, \quad [H, Q_i] = 0 \quad (i, j = 1, \dots, \mathcal{N}), \quad (2.2.1)$$

where Q_i ($i = 1, \dots, \mathcal{N}$) are Hermitian. Due to the existence of many supercharges, there appear more degenerate nonzero energy eigenstates compared with the $\mathcal{N} = 2$ SUSY QM. We also find that zero energy eigenstates are not related by the supercharges Q and not necessarily degenerate as well as the $\mathcal{N} = 2$ SUSY QM.

Then, we study the nonzero energy eigenstates and show that they are $2^{\lfloor \mathcal{N}/2 \rfloor}$ -fold degenerate.¹ First, we introduce the following Hermitian operators

$$S_1 = -iQ_1Q_2, \quad \dots, \quad S_{\lfloor \mathcal{N}/2 \rfloor} = -iQ_{2\lfloor \mathcal{N}/2 \rfloor - 1}Q_{2\lfloor \mathcal{N}/2 \rfloor}. \quad (2.2.2)$$

These operators commute with each other and the Hamiltonian H . These operators also satisfy the relation

$$(S_p)^2 = H^2 \quad (p = 1, \dots, \lfloor \mathcal{N}/2 \rfloor). \quad (2.2.3)$$

This indicates the eigenvalues of S_p are given by $S_p = \pm E_n$ for the eigenstates. Thus, there exist simultaneous nonzero energy eigenstates and we can label them as

$$H|E_n, s_1, \dots, s_{\lfloor \mathcal{N}/2 \rfloor}\rangle = E_n|E_n, s_1, \dots, s_{\lfloor \mathcal{N}/2 \rfloor}\rangle, \quad (2.2.4)$$

$$S_p|E_n, s_1, \dots, s_{\lfloor \mathcal{N}/2 \rfloor}\rangle = s_p E_n|E_n, s_1, \dots, s_{\lfloor \mathcal{N}/2 \rfloor}\rangle \quad (p = 1, \dots, \lfloor \mathcal{N}/2 \rfloor), \quad (2.2.5)$$

where $s_p = \pm 1$ ($p = 1, \dots, \lfloor \mathcal{N}/2 \rfloor$).

Furthermore, S_p and the supercharges satisfy

$$Q_i S_p = \begin{cases} -S_p Q_i & (i = 2p - 1, 2p), \\ +S_p Q_i & (i \neq 2p - 1, 2p). \end{cases} \quad (2.2.6)$$

Therefore we find that the supercharges Q_{2p-1} and Q_{2p} flip the sign of the eigenvalues of S_p but do not change other eigenvalues

$$Q_i|E_n, s_1, \dots, s_p, \dots, s_{\lfloor \mathcal{N}/2 \rfloor}\rangle \propto \sqrt{E_n}|E_n, s_1, \dots, -s_p, \dots, s_{\lfloor \mathcal{N}/2 \rfloor}\rangle, \quad (2.2.7)$$

$$(p = 1, \dots, \lfloor \mathcal{N}/2 \rfloor, \quad i = 2p - 1, 2p).$$

From this relations, we can explicitly construct the supermultiplet from the $|E_n, +, \dots, +\rangle$ as

$$|E_n, s_1, \dots, s_{\lfloor \mathcal{N}/2 \rfloor}\rangle = \frac{1}{(\sqrt{E_n})^s} (Q_1)^{\frac{1-s_1}{2}} (Q_3)^{\frac{1-s_3}{2}} \dots (Q_{2\lfloor \mathcal{N}/2 \rfloor - 1})^{\frac{1-s_{\lfloor \mathcal{N}/2 \rfloor}}{2}} |E_n, +, \dots, +\rangle \quad (2.2.8)$$

¹The symbol $\lfloor \cdot \rfloor$ is the floor function and $[k]$ denotes the maximum integer less than or equal to k .

where $s = \frac{1}{2}(1-s_1) + \frac{1}{2}(1-s_3) + \cdots + \frac{1}{2}(1-s_{\lfloor N/2 \rfloor})$, and the supercharges relate $2^{\lfloor N/2 \rfloor}$ -fold degenerate states with each other.

In the case that N is odd, we can further introduce the operator

$$S_{\lfloor N/2 \rfloor + 1} \equiv S_1 S_2 \cdots S_{\lfloor N/2 \rfloor} Q_N. \quad (2.2.9)$$

This operator however commutes with all the other operators H, Q_i ($i = 1, \dots, N$), S_p ($p = 1, \dots, \lfloor N/2 \rfloor$) and satisfy

$$(S_{\lfloor N/2 \rfloor + 1})^2 = H^{N+1}. \quad (2.2.10)$$

Therefore, we obtain the two irreducible representations which are distinguished by the eigenvalues $S_{\lfloor N/2 \rfloor + 1} = \pm(E_n)^{(N+1)/2}$ and they are not related by the supercharges.

2.2.2 Central extension

The other is a central extension, which introduces central charges Z_{ij} in the algebra

$$\{Q_i, Q_j\} = 2H\delta_{ij} + 2Z_{ij}, \quad (2.2.11)$$

$$[H, Q_i] = [Q_i, Z_{jk}] = [H, Z_{jk}] = [Z_{ij}, Z_{kl}] = 0 \quad (i, j, k, l = 1, \dots, N). \quad (2.2.12)$$

Central charges are operators which commute with all the operators in the algebra, and they can make the size of supermultiplets small compared with the regular representation.

Since the Hamiltonian and the central charges commute with each other, we can consider simultaneous eigenstates of them. For these states, the algebra (2.2.11) is rewritten into

$$\{Q_i, Q_j\} = 2E_n\delta_{ij} + 2z_{ij}, \quad (2.2.13)$$

where E_n and z_{ij} denote the eigenvalues of H and Z_{ij} . From the Hermiticity of the supercharges and the symmetry of indices, z_{ij} becomes a real symmetric matrix ($(z)_{ij}^* = z_{ij}$, $z_{ij} = z_{ji}$) and it can be diagonalized by the orthogonal matrix U_{ij} . Therefore, by redefining the supercharges, we can obtain

$$\{Q'_i, Q'_j\} = 2(E_n + z'_i)\delta_{ij}, \quad (2.2.14)$$

where $z'_i\delta_{ij} = \sum_{k,l} U_{ik} Z_{kl} (U)_{lj}^T$ is the diagonalized matrix and the redefined supercharges are given by $Q'_i = \sum_k U_{ik} Q_k$.

Then we can find that $(Q'_i)^2 = 0$ for eigenstates with $z'_i = -E_n$ (if exists) and this implies $Q'_i = 0$ for such states since the supercharges Q'_i are Hermitian. Thus, the number of nontrivial supercharges effectively reduces for the states with $z'_i = -E_n$, and the size of supermultiplets becomes small compared with the regular representation. These states are called Bogomol'nyi–Prasad–Sommerfield (BPS) states.²

²In the case of supersymmetry in quantum field theory, central charges and BPS states are closely related to topological properties [77–79].

2.3 Summary

In this chapter, we have briefly reviewed the SUSY QM. In $\mathcal{N} = 2$ SUSY QM, we have found that the doubly degenerate states with nonzero energy is related by the supercharges and the Witten index, the difference of the number of zero energy states with $(-1)^F = +1$ and $(-1)^F = -1$ is given as the topological quantity. We have also considered the extension of the supersymmetry and shown that the $\mathcal{N}$ -extended supersymmetry leads to further degeneracy of nonzero energy eigenstates. Furthermore, if central charges appear in the supersymmetry algebra, the size of multiplets can become small compared with the regular representation.

In the following chapters, we will see that these supersymmetric structures exist in extra dimensional models and play important roles.

Chapter 3

Extra dimension and hidden supersymmetry

In this chapter, we consider KK decomposition of higher dimensional Dirac fields and show that the structure of the $\mathcal{N} = 2$ SUSY QM is hidden in the 4D mass spectrum and KK mode functions.

The KK decomposition divides a higher dimensional fields into 4D fields with 4D coordinates and KK mode functions which are given by the eigenfunctions of a Laplacian in extra dimensions. By this decomposition, we obtain infinite 4D fields whose masses are given by eigenvalues of extra dimensional Laplacian.

For 4D spinors, a 4D mass term are given as a pair of 4D left and right-handed spinors. Therefore, corresponding KK mode functions should be related with each other. In this relation, it is known that the $\mathcal{N} = 2$ SUSY QM is hidden [12, 13, 80–82]. KK mode functions for 4D left and right-handed fields form supermultiplets and are related by the supercharges.

This chapter is organized as follows: In the next section, we briefly review the higher dimensional spinors and gamma matrices. In Section 3.2, we consider the KK decomposition of higher dimensional Dirac fields with flat and curved extra dimensions respectively. Then, in Section 3.3, we show that the structure of $\mathcal{N} = 2$ SUSY QM is hidden in the KK mass spectrum.

3.1 Higher dimensional gamma matrices and spinors

First, we introduce higher dimensional gamma matrices and spinors.

3.1.1 Definitions of gamma matrices and spinors

We define the gamma matrices Γ^M in $4 + d$ -dimensional Minkowski space as the $2^{\lfloor d/2 \rfloor + 2} \times 2^{\lfloor d/2 \rfloor + 2}$ matrices which satisfy the $4 + d$ -dimensional Clifford algebra and the following Hermiticity condition

$$\{\Gamma^M, \Gamma^N\} = -2\eta^{MN} \mathbf{1}_{2^{\lfloor d/2 \rfloor + 2}} \quad (M, N = 0, 1, 2, 3, y_1, \dots, y_d), \quad (3.1.1)$$

$$(\Gamma^M)^\dagger = \begin{cases} \Gamma^M & (M = 0), \\ -\Gamma^M & (M \neq 0), \end{cases} \quad (3.1.2)$$

where η^{MN} is the $(4 + d)$ -dimensional Minkowski metric $\eta^{MN} = \eta^{MN} = \text{diag}(-1, +1, \dots, +1)$.

Then, let us consider the $4 + d$ -dimensional Lorentz transformation

$$x^M \longrightarrow x'^M = \Lambda^M_{\ N} x^N, \quad \Lambda^M_{\ M'} \Lambda^N_{\ N'} \eta^{M'N'} = \eta^{MN}. \quad (3.1.3)$$

A Dirac spinor Ψ in $4+d$ -dimensional Minkowski space is defined as the quantity which transforms as follows under the $4+d$ -dimensional Lorentz transformation:

$$\Psi(x) \rightarrow S\Psi(x') = e^{+\frac{i}{2}\epsilon_{MN}\Sigma^{MN}}\Psi(x'), \quad (3.1.4)$$

where ϵ_{MN} indicate the parameters of the Lorentz transformation and Σ^{MN} are the generators which satisfy the following $4+d$ -dimensional Lorentz algebra $so(1, 3+d)$

$$\Sigma^{MN} = \frac{i}{4} [\Gamma^M, \Gamma^N], \quad (3.1.5)$$

$$[\Sigma^{KL}, \Sigma^{MN}] = -i(\eta^{ML}\Sigma^{KN} - \eta^{MK}\Sigma^{LN} - \eta^{NL}\Sigma^{KM} + \eta^{NK}\Sigma^{LM}). \quad (3.1.6)$$

Therefore the Dirac spinor has $2^{\lfloor d/2 \rfloor + 2}$ components. The Dirac conjugate $\bar{\Psi}$ is defined by

$$\bar{\Psi} \equiv \Psi^\dagger \Gamma^0, \quad (3.1.7)$$

and transforms as follows

$$\bar{\Psi}(x) \longrightarrow \bar{\Psi}(x') S^{-1}. \quad (3.1.8)$$

It should be noted that the matrices S and Λ^M_N are related by

$$S^{-1} \Gamma^M S = \Lambda^M_N \Gamma^N. \quad (3.1.9)$$

In even dimensions, we can define the higher dimensional chiral matrix

$$\Gamma_{chi} \equiv i^{d/2+1} \Gamma^0 \dots \Gamma^3 \Gamma^{y_1} \dots \Gamma^{y_d}, \quad (3.1.10)$$

which satisfies

$$\{\Gamma_{chi}, \Gamma^M\} = 0 \quad (M = 0, 1, 2, 3, y_1, \dots, y_d), \quad (\Gamma_{chi})^2 = \mathbf{1}_{2^{\lfloor d/2 \rfloor + 2}}, \quad (\Gamma_{chi})^\dagger = \Gamma_{chi}. \quad (3.1.11)$$

Then, we can obtain the higher dimensional chiral spinors $\Psi_\pm$ with the eigenvalues $\Gamma_{chi} = \pm 1$

$$\Psi = \Psi_+ + \Psi_-, \quad \Psi_\pm = \frac{1 \pm \Gamma_{chi}}{2} \Psi, \quad \Gamma_{chi} \Psi_\pm = \pm \Psi_\pm. \quad (3.1.12)$$

Since the chiral matrix commutes with the Lorentz generators (3.1.5), the chiral spinors Ψ_+ and Ψ_- are not related by the higher dimensional Lorentz transformation and the Dirac spinors can be decomposed into two irreducible representations in even dimensions.

3.1.2 4D and internal chiral matrix

In the following sections, we will consider the decomposition of the fields with respect to the 4D Minkowski space and extra dimensions, and furthermore integrate out extra dimensions. Thus we focus on the 4D Lorentz group $SO(1, 3)$ which is the subgroup of the $4+d$ -dimensional Lorentz group $SO(1, 3+d)$.

Then it is convenient to introduce the 4D chiral matrix Γ^5

$$\Gamma^5 = i\Gamma^0 \Gamma^1 \Gamma^2 \Gamma^3. \quad (3.1.13)$$

This chiral matrix satisfies the relations

$$\{\Gamma^5, \Gamma^\mu\} = 0, \quad [\Gamma^5, \Gamma^y] = 0, \quad (\Gamma^5)^2 = \mathbf{1}_{2^{[d/2]+2}}, \quad (\Gamma^5)^\dagger = \Gamma^5, \quad (3.1.14)$$

$$(\mu = 0, 1, 2, 3, \quad y = y_1, \dots, y_d),$$

and commutes with the Lorentz generators in 4D and extra dimensions

$$[\Gamma^5, \Sigma^{\mu\nu}] = 0, \quad [\Gamma^5, \Sigma^{yy'}] = 0, \quad (\mu, \nu = 0, 1, 2, 3, \quad y, y' = y_1, \dots, y_d), \quad (3.1.15)$$

where μ, ν indicate the indices of the 4D coordinates and y, y' denote the indices of the extra dimensional ones. In addition, the generators of extra dimensional Lorentz group $\Sigma^{y_1 y_2}, \Sigma^{y_3 y_4}, \dots, \Sigma^{y_{2[d/2]-1} y_{2[d/2]}}$ satisfy

$$(\Sigma^{y_{2i-1} y_{2i}})^2 = \frac{1}{4}, \quad [\Sigma^{y_{2i-1} y_{2i}}, \Sigma^{\mu\nu}] = 0, \quad [\Sigma^{y_{2i-1} y_{2i}}, \Sigma^{y_{2j-1} y_{2j}}] = 0, \quad (3.1.16)$$

$$(i, j = 1, \dots, [d/2]).$$

Therefore, when we consider the KK decomposition, the higher dimensional spinor can be independently decomposed according to the eigenvalues $\Gamma^5 = \pm 1$, and also $\Sigma^{y_{2i-1} y_{2i}} = \pm 1/2$ ($i = 1, \dots, [d/2]$).

Furthermore, for the later convenience, we also define the internal chiral matrix Γ^{in} in the case of even d :

$$\Gamma^{in} \equiv i^{d/2} \Gamma^{y_1} \dots \Gamma^{y_d} \quad (d \text{ is even}). \quad (3.1.17)$$

This satisfies the following properties

$$[\Gamma^{in}, \Gamma^\mu] = 0, \quad \{\Gamma^{in}, \Gamma^y\} = 0, \quad (\Gamma^{in})^2 = \mathbf{1}_{2^{[d/2]+2}}, \quad (\Gamma^{in})^\dagger = \Gamma^{in}, \quad (3.1.18)$$

$$\Gamma_{chi} = \Gamma^5 \Gamma^{in}, \quad (\mu = 0, 1, 2, 3, \quad y = y_1, \dots, y_d). \quad (3.1.19)$$

3.1.3 Representation of gamma matrices

Hereafter, we use the following representation of the gamma matrices:

$$\Gamma^\mu = \mathbf{1}_{2^{[d/2]}} \otimes \gamma^\mu \quad (\mu = 0, 1, 2, 3), \quad (3.1.20)$$

$$\Gamma^y = \gamma^y \otimes \gamma^5 \quad (y = y_1, \dots, y_d), \quad (3.1.21)$$

where γ^μ represent the 4×4 4D gamma matrices

$$\{\gamma^\mu, \gamma^\nu\} = -2\eta^{\mu\nu} \mathbf{1}_4, \quad \eta^{\mu\nu} = \text{diag}(-1, +1, +1, +1), \quad (3.1.22)$$

$$(\gamma^\mu)^\dagger = \begin{cases} \gamma^\mu & (\mu = 0), \\ -\gamma^\mu & (\mu \neq 0), \end{cases} \quad (3.1.23)$$

and $\gamma^5 \equiv i\gamma^0\gamma^1\gamma^2\gamma^3$ is the 4×4 4D chiral matrix which satisfy

$$\{\gamma^5, \gamma^\mu\} = 0 \quad (\mu = 0, 1, 2, 3), \quad (\gamma^5)^2 = \mathbf{1}_4, \quad (\gamma^5)^\dagger = \gamma^5. \quad (3.1.24)$$

γ^y are the $2^{[d/2]} \times 2^{[d/2]}$ internal space gamma matrices and satisfy

$$\{\gamma^y, \gamma^{y'}\} = -2\delta^{yy'} \mathbf{1}_{2^{[d/2]}}, \quad (3.1.25)$$

$$(\gamma^y)^\dagger = -\gamma^y \quad (y, y' = y_1, \dots, y_d). \quad (3.1.26)$$

For the case of $d = 1$, we define γ^{y_1} as $-i$.

In this representation, the chiral matrices are written by

$$\Gamma_{chi} = \gamma^{in} \otimes \gamma^5, \quad (3.1.27)$$

$$\Gamma^5 = \mathbf{1}_{2^{[d/2]}} \otimes \gamma^5, \quad (3.1.28)$$

$$\Gamma^{in} = \gamma^{in} \otimes \mathbf{1}_4, \quad (3.1.29)$$

here we define the $2^{[d/2]} \times 2^{[d/2]}$ internal chiral matrix

$$\gamma^{in} \equiv i^{d/2} \gamma^{y_1} \dots \gamma^{y_d}, \quad (3.1.30)$$

for even d .

3.2 KK decomposition

In this section, we consider a higher dimensional Dirac field and the KK decomposition. Then, by integrating out extra dimensions, we see that infinite 4D fields with different masses appear in the effective 4D action.

In the following, Ω indicates the compact extra dimensional space in higher dimensions and $\partial\Omega$ is its boundaries. The coordinates of M^4 and Ω are represented by x^μ ($\mu = 0, 1, 2, 3$) and y_k ($k = 1, 2, \dots, d$), respectively.

3.2.1 5D Dirac field

First, we consider the case of the 5D Dirac action on $M^4 \times \Omega$

$$S = \int_{M^4} d^4x \int_{\Omega} dy \bar{\Psi}(x, y) (i\gamma^\mu \partial_\mu + i\gamma^y \partial_y + M) \Psi(x, y), \quad (3.2.1)$$

where M is a bulk mass and the suitable boundary conditions on $\partial\Omega$ are imposed for the 5D Dirac field Ψ . Because of the 5D space, Ψ is a 4-component spinor and $i\gamma^y$ can be give by the 4D chiral matrix $i\gamma^y = \gamma^5$.

Then, let us consider the KK decomposition. The 5D Dirac field Ψ is decomposed into 4-component 4D chiral fields $\psi_{R/L, \alpha}^{(n)}(x)$ ($\gamma^5 \psi_{R/L, \alpha}^{(n)} = \pm \psi_{R/L, \alpha}^{(n)}$) on M^4 and KK mode functions $f_\alpha^{(n)}(y)$, $g_\alpha^{(n)}(y)$ on Ω :

$$\Psi(x, y) = \sum_n \sum_\alpha \left\{ \psi_{R, \alpha}^{(n)}(x) f_\alpha^{(n)}(y) + \psi_{L, \alpha}^{(n)}(x) g_\alpha^{(n)}(y) \right\}, \quad (3.2.2)$$

where the index n indicates the n -th level of the KK modes and α denotes the index that distinguishes the degeneracy of the n -th KK modes (if exists). The mode functions $f_\alpha^{(n)}(y)$ ($g_\alpha^{(n)}(y)$) are assumed to form a complete set with respect to the internal space.

Substituting the expansion (3.2.2) into the action (3.2.1), we obtain

$$\begin{aligned} S = \sum_{n, m} \sum_{\alpha, \beta} \int d^4x & \left[\langle f_\alpha^{(n)} | f_\beta^{(m)} \rangle \bar{\psi}_{R, \alpha}^{(n)}(x) i\gamma^\mu \partial_\mu \psi_{R, \beta}^{(m)}(x) + \langle g_\alpha^{(n)} | g_\beta^{(m)} \rangle \bar{\psi}_{L, \alpha}^{(n)}(x) i\gamma^\mu \partial_\mu \psi_{L, \beta}^{(m)}(x) \right. \\ & \left. + \langle f_\alpha^{(n)} | \mathcal{A}^\dagger g_\beta^{(m)} \rangle \bar{\psi}_{R, \alpha}^{(n)}(x) \psi_{L, \beta}^{(m)}(x) + \langle g_\alpha^{(n)} | \mathcal{A} f_\beta^{(m)} \rangle \bar{\psi}_{L, \alpha}^{(n)}(x) \psi_{R, \beta}^{(m)}(x) \right], \end{aligned} \quad (3.2.3)$$

where we have defined the inner product and the operator $\mathcal{A}, \mathcal{A}^\dagger$ as

$$\langle X|Y\rangle = \int_{\Omega} dy X^\dagger(y) Y(y), \quad (3.2.4)$$

$$\mathcal{A} = \partial_y + M, \quad (3.2.5)$$

$$\mathcal{A}^\dagger = -\partial_y + M. \quad (3.2.6)$$

Then, by requiring that the KK mode functions satisfy the orthonormal relations

$$\begin{aligned} \langle f_\alpha^{(n)} | f_\beta^{(m)} \rangle &= \langle g_\alpha^{(n)} | g_\beta^{(m)} \rangle = \delta^{nm} \delta_{\alpha\beta}, \\ \langle f_\alpha^{(n)} | \mathcal{A}^\dagger g_\beta^{(m)} \rangle &= \langle g_\alpha^{(n)} | \mathcal{A} f_\beta^{(m)} \rangle = m_n \delta^{nm} \delta_{\alpha\beta}, \end{aligned} \quad (3.2.7)$$

we can obtain the following effective 4D action:

$$\begin{aligned} S = \int_{M^4} d^4x \Big\{ & \sum_{\alpha} \sum_n \bar{\psi}_\alpha^{(n)}(x) (i\gamma^\mu \partial_\mu + m_n) \psi_\alpha^{(n)}(x) \\ & + \sum_{\alpha} \bar{\psi}_{L,\alpha}^{(0)}(x) i\gamma^\mu \partial_\mu \psi_{L,\alpha}^{(0)}(x) + \sum_{\alpha} \bar{\psi}_{R,\alpha}^{(0)}(x) i\gamma^\mu \partial_\mu \psi_{R,\alpha}^{(0)}(x) \Big\}, \end{aligned} \quad (3.2.8)$$

where $\psi_\alpha^{(n)}(x) = \psi_{L,\alpha}^{(n)}(x) + \psi_{R,\alpha}^{(n)}(x)$ indicate 4D Dirac spinors with mass m_n and $\psi_{L/R,\alpha}^{(0)}(x)$ are massless 4D chiral spinors. The 4D mass spectrum can be described as Figure 3.1.

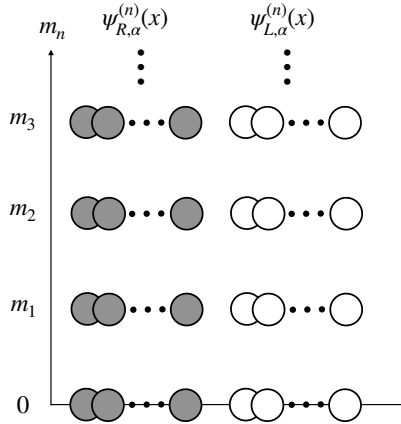


Figure 3.1: KK mass spectrum in extra dimensional models

Since we have assumed the completeness of the mode functions $f_\alpha^{(n)}(y)$ ($g_\alpha^{(n)}(y)$), the orthonormal conditions (3.2.7) lead to the relations

$$\mathcal{A} f_\alpha^{(n)}(y) = m_n g_\alpha^{(n)}, \quad (3.2.9)$$

$$\mathcal{A}^\dagger g_\alpha^{(n)}(y) = m_n f_\alpha^{(n)}, \quad (3.2.10)$$

and therefore

$$(-\partial_y^2 + M^2) f_\alpha^{(n)}(y) = m_n^2 f_\alpha^{(n)}, \quad (3.2.11)$$

$$(-\partial_y^2 + M^2) g_\alpha^{(n)}(y) = m_n^2 g_\alpha^{(n)}. \quad (3.2.12)$$

Then, we can find that the mode functions and the masses of 4D fields are given by the eigenfunctions and the square root of the eigenvalues of $-\partial_y^2 + M^2$.

In general, infinite 4D fields appear in the 4D effective action due to the eigenvalues of $-\partial_y^2 + M^2$. However, the nonzero mass eigenvalues are naively in inverse proportion to the size of extra dimensional space as we will see in examples. Therefore, the nonzero masses can become larger than our current energy scale when the size of the extra dimensional space is sufficiently small. Then we can consider that massless particles can appear and be identified as the ones of the SM, whereas those massive particles are not presently observed because of the low energy scale in the current experiment.

• **Example 1 : S^1 with periodic boundary condition**

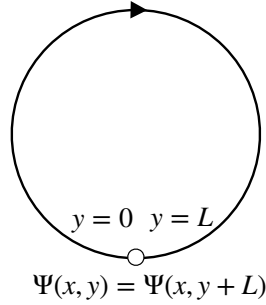


Figure 3.2: S^1 -extra dimension with the periodic boundary condition

As an example, let us consider the S^1 -extra dimension with the periodic boundary condition

$$\Psi(x, y) = \Psi(x, y + L), \quad (3.2.13)$$

or equivalently

$$f_\alpha^{(n)}(y) = f_\alpha^{(n)}(y + L), \quad g_\alpha^{(n)}(y) = g_\alpha^{(n)}(y + L), \quad (3.2.14)$$

where L denotes the period of S^1 . Here, we consider the case of the bulk mass $M = 0$. From the relation (3.2.9), (3.2.11) (3.2.12), the massive KK mode functions $f_\alpha^{(n)}(y)$ and $g_\alpha^{(n)}(y)$ can be given as

$$f_+^{(n)}(y) = \sqrt{\frac{4}{L}} \cos \frac{2\pi n}{L} y, \quad f_-^{(n)}(y) = \sqrt{\frac{4}{L}} \sin \frac{2\pi n}{L} y, \quad (3.2.15)$$

$$g_+^{(n)}(y) = \frac{1}{m_n} \mathcal{A} f_+^{(n)}(y), \quad g_-^{(n)}(y) = \frac{1}{m_n} \mathcal{A} f_-^{(n)}(y) \quad (n = 0, 1, 2, \dots), \quad (3.2.16)$$

where the mode functions satisfy the following equations.

$$-\partial_y^2 f_\alpha^{(n)}(y) = m_n^2 f_\alpha^{(n)}(y), \quad -\partial_y^2 g_\alpha^{(n)}(y) = m_n^2 g_\alpha^{(n)}(y), \quad (3.2.17)$$

$$m_n = \frac{2\pi n}{L} \quad (n = 1, 2, \dots, \quad \alpha = \pm). \quad (3.2.18)$$

In addition, the massless mode functions are given as the constants

$$f^{(0)}(y) = g^{(0)}(y) = \sqrt{\frac{2}{L}}. \quad (3.2.19)$$

Then, the 4D action is of the form

$$S = \int_{M^4} d^4x \left\{ \sum_{\alpha=\pm} \sum_{n=1}^{\infty} \bar{\psi}_{\alpha}^{(n)}(x) (i\gamma^{\mu} \partial_{\mu} + m_n) \psi_{\alpha}^{(n)}(x) + \bar{\psi}^{(0)}(x) i\gamma^{\mu} \partial_{\mu} \psi^{(0)}(x) \right\}, \quad (3.2.20)$$

where $\psi^{(0)}(x) = \psi_L^{(0)}(x) + \psi_R^{(0)}(x)$ indicates the massless Dirac spinor and the masses are $m_n = 2\pi n/L$ ($n = 1, 2, \dots$).¹ Thus we can find that masses are in proportion to $1/L$.

Although we obtained the massless field, it is not chiral but Dirac. This is because this model is symmetric for the 4D left and right-handed components of the higher dimensional field $\Psi_{R/L} = ((1 \pm \gamma^5)/2)\Psi$. Since the fermions in the SM are chiral, zero modes should be also chiral spinors from the viewpoint of the phenomenology. In the next example, we will consider the model which includes a 4D chiral spinor.

• **Example 2 : interval with Dirichlet boundary condition**

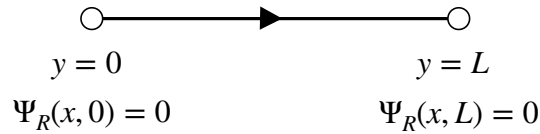


Figure 3.3: Interval with the Dirichlet boundary condition

Next, we consider the extra dimension as an interval with the Dirichlet boundary condition for $\Psi_R(x, y)$:

$$\Psi_R(x, 0) = \Psi_R(x, L) = 0, \quad \Psi_{R/L} = \frac{1 \pm \gamma^5}{2} \Psi, \quad (3.2.21)$$

or equivalently

$$f_{\alpha}^{(n)}(0) = f_{\alpha}^{(n)}(L) = 0, \quad (3.2.22)$$

where L denotes the length of the interval.

In this case, the massive KK mode functions are given as

$$f^{(n)}(y) = \sqrt{\frac{2}{L}} \sin \frac{\pi n}{L} y, \quad g^{(n)}(y) = \frac{1}{m_n} \mathcal{A} f^{(n)}(y), \quad (3.2.23)$$

¹If we consider the case of $M \neq 0$, the masses are given by $m_n = \sqrt{(2\pi n/L)^2 + M^2}$ ($n = 0, 1, 2, \dots$) and massless 4D fields dose not appear.

where the mode functions satisfy

$$(-\partial_y^2 + M^2)f^{(n)}(y) = m_n^2 f^{(n)}(y), \quad (3.2.24)$$

$$(-\partial_y^2 + M^2)g^{(n)}(y) = m_n^2 g^{(n)}(y), \quad (3.2.25)$$

$$m_n = \sqrt{\left(\frac{\pi n}{L}\right)^2 + M^2} \quad (n = 1, 2, \dots). \quad (3.2.26)$$

Furthermore, from the relations (3.2.9) and (3.2.10), zero modes should be of the form

$$f^{(0)}(y) = \sqrt{\frac{M}{e^{ML} - 1}} e^{My}, \quad g^{(0)}(y) = \sqrt{\frac{M}{e^{ML} - 1}} e^{-My}. \quad (3.2.27)$$

However, $f^{(0)}(y)$ cannot satisfy the boundary condition. Thus, the zero mode only exists for $g^{(0)}(y)$.

Then, the 4D action is given by

$$S = \int_{M^4} d^4x \left\{ \sum_{n=1}^{\infty} \bar{\psi}^{(n)}(x) (i\gamma^\mu \partial_\mu + m_n) \psi^{(n)}(x) + \bar{\psi}_L^{(0)}(x) i\gamma^\mu \partial_\mu \psi_L^{(0)}(x) \right\}, \quad (3.2.28)$$

with the masses $\sqrt{(\pi n/L)^2 + M^2}$ ($n = 1, 2, \dots$). Therefore we can find that this model has the left-handed chiral fermion due to the boundary condition, unlike the S^1 -extra dimension.

Although we have obtained the chiral fermion, the SM has three generations for the fermions and this model cannot explain them from the one higher dimensional Dirac field. In Chapter 4, we will discuss the generation of fermions in more detail, based on 1D extra dimensional models.

3.2.2 Higher dimensional Dirac field with flat extra dimensions

Second, we consider the case of the $(4 + d)$ -dimensional Dirac action with flat extra dimensions Ω

$$S = \int_{M^4} d^4x \int_{\Omega} d^d y \bar{\Psi}(x, y) (i\Gamma^\mu \partial_\mu + i\Gamma^{y_k} \partial_{y_k} + M \mathbf{1}_{2^{\lfloor d/2 \rfloor + 2}}) \Psi(x, y), \quad (3.2.29)$$

where a size of the gamma matrices is $2^{\lfloor d/2 \rfloor + 2} \times 2^{\lfloor d/2 \rfloor + 2}$ and we use the representation

$$\Gamma^\mu = \mathbf{1}_{2^{\lfloor d/2 \rfloor}} \otimes \gamma^\mu \quad (\mu = 0, 1, 2, 3), \quad (3.2.30)$$

$$\Gamma^y = \gamma^y \otimes \gamma^5 \quad (y = y_1, \dots, y_d), \quad (3.2.31)$$

according to (3.1.20) and (3.1.21). Since the Dirac field Ψ is $2^{\lfloor d/2 \rfloor + 2}$ components, the KK decomposition of the $(4 + d)$ -dimensional Dirac field $\Psi(x, y)$ is given by

$$\Psi(x, y) = \sum_n \sum_\alpha \left\{ f_\alpha^{(n)}(y) \otimes \psi_{L,\alpha}^{(n)}(x) + g_\alpha^{(n)}(y) \otimes \psi_{R,\alpha}^{(n)}(x) \right\}, \quad (3.2.32)$$

where $\psi_{R/L,\alpha}^{(n)}(x)$ represent the 4-component chiral spinors and the mode functions $f_\alpha^{(n)}(y)$, $g_\alpha^{(n)}(y)$ have $2^{\lfloor d/2 \rfloor}$ components unlike the 5D case.

The action with the expansion (3.2.32) is written as

$$S = \sum_{n,m} \sum_{\alpha,\beta} \int d^4x \left[\langle f_\alpha^{(n)} | f_\beta^{(m)} \rangle \bar{\psi}_{R,\alpha}^{(n)}(x) i\gamma^\mu \partial_\mu \psi_{R,\beta}^{(m)}(x) + \langle g_\alpha^{(n)} | g_\beta^{(m)} \rangle \bar{\psi}_{L,\alpha}^{(n)}(x) i\gamma^\mu \partial_\mu \psi_{L,\beta}^{(m)}(x) \right. \\ \left. + \langle f_\alpha^{(n)} | \mathcal{A}^\dagger g_\beta^{(m)} \rangle \bar{\psi}_{R,\alpha}^{(n)}(x) \psi_{L,\beta}^{(m)}(x) + \langle g_\alpha^{(n)} | \mathcal{A} f_\beta^{(m)} \rangle \bar{\psi}_{L,\alpha}^{(n)}(x) \psi_{R,\beta}^{(m)}(x) \right], \quad (3.2.33)$$

where the operator $\mathcal{A}, \mathcal{A}^\dagger$ are defined as follows:

$$\mathcal{A} = i\gamma^y \partial_y + M, \quad (3.2.34)$$

$$\mathcal{A}^\dagger = -i\gamma^y \partial_y + M. \quad (3.2.35)$$

Then, as well as the 5D case, we require that the KK mode functions satisfy the orthonormal relations

$$\begin{aligned} \langle f_\alpha^{(n)} | f_\beta^{(m)} \rangle &= \langle g_\alpha^{(n)} | g_\beta^{(m)} \rangle = \delta^{nm} \delta_{\alpha\beta}, \\ \langle f_\alpha^{(n)} | \mathcal{A}^\dagger g_\beta^{(m)} \rangle &= \langle g_\alpha^{(n)} | \mathcal{A} f_\beta^{(m)} \rangle = m_n \delta^{nm} \delta_{\alpha\beta}, \end{aligned} \quad (3.2.36)$$

and we can obtain the effective 4D action

$$\begin{aligned} S = \int_{M^4} d^4x \left\{ \sum_\alpha \sum_n \bar{\psi}_\alpha^{(n)}(x) (i\gamma^\mu \partial_\mu + m_n) \psi_\alpha^{(n)}(x) \right. \\ \left. + \sum_\alpha \bar{\psi}_{L,\alpha}^{(0)}(x) i\gamma^\mu \partial_\mu \psi_{L,\alpha}^{(0)}(x) + \sum_\alpha \bar{\psi}_{R,\alpha}^{(0)}(x) i\gamma^\mu \partial_\mu \psi_{R,\alpha}^{(0)}(x) \right\}. \end{aligned} \quad (3.2.37)$$

Since the KK mode functions have many components in higher dimensions, we can expect that there are many degenerate mode functions, unlike the 5D case. The degeneracy due to higher dimensional extra space will be discussed in Chapter 6.

3.2.3 Higher dimensional Dirac field with curved extra dimensions

Finally, we discuss the KK decomposition of a higher dimensional Dirac field with curved extra dimensions for a more general discussion. For the details of spinors in curved spacetime, see Appendix A.

In terms of the 4D Lorentz symmetry, we assume that the $(4 + d)$ -dimensional background metric G_{MN} ($M, N = 0, 1, 2, 3, y_1, \dots, y_d$) is of the form

$$ds^2 = G_{MN} dx^M dx^N = e^{2\sigma(y)} \eta_{\mu\nu} dx^\mu dx^\nu + g_{yy'}(y) dy^y dy^{y'}, \quad (3.2.38)$$

where $x^M = (x^\mu, y^y)$ is the $(4 + d)$ -dimensional coordinates. $\eta_{\mu\nu} = \text{diag}(-1, +1, +1, +1)$ denotes the 4D Minkowski metric, and $\sigma(y)$ and $g_{yy'}(y)$ depend on the extra dimensional coordinates.²

Then, for a general discussion, we study the $(4 + d)$ -dimensional Dirac action with a vector background field $A_N(y) = (0, A_y(y))$ and a scalar background field $W(y)$:

$$S = \int d^4x \int_\Omega d^d y \sqrt{-G} \bar{\Psi}(x, y) \left[i\Gamma^{\hat{N}} e_{\hat{N}}^N (\nabla_N + iqA_N(y)) + W(y) \right] \Psi(x, y), \quad (3.2.39)$$

where $G = \det G_{MN}$, q is the gauge coupling constant and $\hat{N} = \hat{0}, \hat{1}, \hat{2}, \hat{3}, \hat{y}_1, \dots, \hat{y}_d$ are the indices of the local Lorentz frame. We define the Dirac conjugate as $\bar{\Psi}(x, y) = \Psi^\dagger(x, y) \Gamma^{\hat{0}}$ and $\Gamma^{\hat{N}}$ indicate the gamma matrices with the local Lorentz indices

$$\{\Gamma^{\hat{M}}, \Gamma^{\hat{N}}\} = -2\eta^{\hat{M}\hat{N}} \mathbf{1}_{2[d/2]+2}, \quad (3.2.40)$$

$$(\Gamma^{\hat{M}})^\dagger = \begin{cases} \Gamma^{\hat{M}} & (\hat{M} = \hat{0}), \\ -\Gamma^{\hat{M}} & (\hat{M} \neq \hat{0}), \end{cases} \quad (3.2.41)$$

²In the case that $\sigma(y) = -k|y|$ and $g_{yy} = 1$ with the 1D extra dimension, this metric corresponds to the Randall-Sundrum warped metric [5, 15].

where $\eta_{\hat{M}\hat{N}} = \text{diag}(-1, +1, \dots, +1)$ is $(4+d)$ -dimensional Minkowski metric. $e_N^{\hat{N}}$ is the vielvein and satisfies

$$e_L^{\hat{M}} e_{\hat{M}}^N = \delta_L^N, \quad e_L^{\hat{M}} e_M^{\hat{N}} = \delta_L^{\hat{N}}, \quad G_{MN} = e_M^{\hat{M}} e_N^{\hat{N}} \eta_{\hat{M}\hat{N}}. \quad (3.2.42)$$

In this model, the nonzero components of the vielvein can be given by $e_\mu^{\hat{\mu}} = e^{\sigma(y)} \delta_\mu^{\hat{\mu}}$ ($\mu = 0, 1, 2, 3$, $\hat{\mu} = \hat{0}, \hat{1}, \hat{2}, \hat{3}$) and $e_y^{\hat{y}} (y = y_1 \cdots y_d, \hat{y} = \hat{y}_1, \dots, \hat{y}_d)$. ∇_N represents the covariant derivative whose behavior for the Dirac field $\Psi(x, y)$ is

$$\nabla_N \Psi(x, y) = \left(\partial_N + \frac{i}{2} \omega_{N\hat{K}\hat{L}} \Sigma^{\hat{K}\hat{L}} \right) \Psi(x, y), \quad (3.2.43)$$

where $\Sigma^{\hat{K}\hat{L}} = \frac{i}{4} [\Gamma^{\hat{K}}, \Gamma^{\hat{L}}]$ corresponds to the generator of $(4+d)$ -dimensional Lorentz transformation, and $\omega_{N\hat{K}\hat{L}}$ is the spin connection defined from the Christoffel symbol Γ_{NK}^L

$$\omega_{N\hat{K}\hat{L}} = -e_{\hat{L}}^K \partial_N e_K^{\hat{L}} + e_{\hat{L}}^K \Gamma_{NK}^L e_L^{\hat{K}}, \quad (3.2.44)$$

and whose nonzero components are $\omega_{\mu\hat{y}\hat{y}} = \frac{1}{2} (e^{-\sigma} \partial_y e^{2\sigma}) \eta_{\mu\hat{y}} e_{\hat{y}}^y$ and $\omega_{y\hat{y}'\hat{y}''}$.

We also adopt the following representation of the gamma matrices as well as the previous cases:

$$\Gamma^{\hat{\mu}} = \mathbf{1}_{2^{[d/2]}} \otimes \gamma^{\hat{\mu}}, \quad \Gamma^{\hat{y}} = \gamma^{\hat{y}} \otimes \gamma^5. \quad (3.2.45)$$

$\gamma^{\hat{\mu}}$ ($\hat{\mu} = \hat{0}, \hat{1}, \hat{2}, \hat{3}$) are the 4×4 4D gamma matrices and $\gamma^5 = i\gamma^{\hat{0}}\gamma^{\hat{1}}\gamma^{\hat{2}}\gamma^{\hat{3}}$ denotes the 4D chiral matrix. $\gamma^{\hat{y}}$ ($\hat{y} = \hat{y}_1, \dots, \hat{y}_d$) represent the $2^{[d/2]} \times 2^{[d/2]}$ d -dimensional internal gamma matrices which satisfy

$$\{\gamma^{\hat{y}}, \gamma^{\hat{y}'}\} = -2\delta^{\hat{y}\hat{y}'}, \quad (\gamma^{\hat{y}})^\dagger = -\gamma^{\hat{y}}. \quad (3.2.46)$$

From this representation, the Dirac operator can be rewritten into the form ³

$$i\Gamma^{\hat{N}} e_{\hat{N}}^N (\nabla_N + iqA_N) + W = \mathbf{1}_{2^{[d/2]}} \otimes e^{-\sigma} i\gamma^{\hat{\mu}} \partial_\mu + i\gamma^{\hat{y}} e_{\hat{y}}^y (\nabla_y + iqA_y + 2\partial_y \sigma) \otimes \gamma^5 + W \mathbf{1}_{2^{[d/2]}} \otimes \mathbf{1}_4, \quad (3.2.47)$$

where ∇_y in the right hand side of (3.2.47) means

$$\nabla_y = \partial_y + \frac{i}{2} \omega_{y\hat{y}'\hat{y}''} \sigma^{\hat{y}'\hat{y}''}, \quad \sigma^{\hat{y}\hat{y}'} = \frac{i}{4} [\gamma^{\hat{y}}, \gamma^{\hat{y}'}], \quad (3.2.48)$$

and this corresponds to the covariant derivative for spinors defined on the space Ω .

Next, we consider the KK decomposition of the higher dimensional Dirac field $\Psi(x, y)$. In the curved space time, we decompose the fields as follows:

$$\Psi(x, y) = \sum_n \sum_\alpha \left\{ e^{-2\sigma(y)} f_\alpha^{(n)}(y) \otimes \psi_{R,\alpha}^{(n)}(x) + e^{-2\sigma(y)} g_\alpha^{(n)}(y) \otimes \psi_{L,\alpha}^{(n)}(x) \right\}. \quad (3.2.49)$$

The mode functions $f_\alpha^{(n)}(y)$ ($g_\alpha^{(n)}(y)$) have $2^{[d/2]}$ components and are assumed to form a complete set. By substituting (3.2.49) and (3.2.47) into the action, we obtain

$$S = \sum_{n,m} \sum_{\alpha,\beta} \int d^4x \left[\langle f_\alpha^{(n)} | f_\beta^{(m)} \rangle \bar{\psi}_{R,\alpha}^{(n)}(x) i\gamma^\mu \partial_\mu \psi_{R,\beta}^{(m)}(x) + \langle g_\alpha^{(n)} | g_\beta^{(m)} \rangle \bar{\psi}_{L,\alpha}^{(n)}(x) i\gamma^\mu \partial_\mu \psi_{L,\beta}^{(m)}(x) \right. \\ \left. + \langle f_\alpha^{(n)} | \mathcal{A}^\dagger g_\beta^{(m)} \rangle \bar{\psi}_{R,\alpha}^{(n)}(x) \psi_{L,\beta}^{(m)}(x) + \langle g_\alpha^{(n)} | \mathcal{A} f_\beta^{(m)} \rangle \bar{\psi}_{L,\alpha}^{(n)}(x) \psi_{R,\beta}^{(m)}(x) \right], \quad (3.2.50)$$

³ γ^μ implies $\gamma^\mu = \gamma^{\hat{\mu}} \delta_\mu^{\hat{\mu}}$, where $\delta_\mu^{\hat{\mu}}$ appear from the vielvein.

where we have defined the inner product and the operator $\mathcal{A}, \mathcal{A}^\dagger$ in the case of curved spacetime as

$$\langle X|Y \rangle = \int_{\Omega} d^d y \sqrt{g} e^{-\sigma(y)} X^\dagger(y) Y(y), \quad g = \det g_{yy'}, \quad (3.2.51)$$

$$\mathcal{A} = e^\sigma \left[i\gamma^{\hat{y}} e_{\hat{y}}^y (\nabla_y + iqA_y) + W \right], \quad (3.2.52)$$

$$\mathcal{A}^\dagger = e^\sigma \left[-i\gamma^{\hat{y}} e_{\hat{y}}^y (\nabla_y + iqA_y) + W \right]. \quad (3.2.53)$$

Then, by requiring that the KK mode functions satisfy the orthonormal relations

$$\begin{aligned} \langle f_\alpha^{(n)} | f_\beta^{(m)} \rangle &= \langle g_\alpha^{(n)} | g_\beta^{(m)} \rangle = \delta^{nm} \delta_{\alpha\beta}, \\ \langle f_\alpha^{(n)} | \mathcal{A}^\dagger g_\beta^{(m)} \rangle &= \langle g_\alpha^{(n)} | \mathcal{A} f_\beta^{(m)} \rangle = m_n \delta^{nm} \delta_{\alpha\beta}, \end{aligned} \quad (3.2.54)$$

we can obtain the following action

$$\begin{aligned} S = \int d^4 x \Big\{ & \sum_\alpha \sum_n \bar{\psi}_\alpha^{(n)}(x) (i\gamma^\mu \partial_\mu + m_n) \psi_\alpha^{(n)}(x) \\ & + \sum_\alpha \bar{\psi}_{L,\alpha}^{(0)}(x) i\gamma^\mu \partial_\mu \psi_{L,\alpha}^{(0)}(x) + \sum_\alpha \bar{\psi}_{R,\alpha}^{(0)}(x) i\gamma^\mu \partial_\mu \psi_{R,\alpha}^{(0)}(x) \Big\}. \end{aligned} \quad (3.2.55)$$

The expression of the above effective 4D action coincides with the case of flat extra dimensions, although the effects of curved spaces and background fields appear as the mass spectrum through the definition of $\mathcal{A}, \mathcal{A}^\dagger$ and (3.2.54).

3.3 $\mathcal{N} = 2$ supersymmetric quantum mechanics in extra dimensional models

In the previous section, we have seen the KK decomposition of the extra dimensional models and found that the KK mode functions should satisfy the relations (3.2.7), (3.2.36) and (3.2.54) to obtain the 4D effective actions. Then, in this section, we reveal the structure of $\mathcal{N} = 2$ SUSY QM hidden in the KK mass spectrum and the mode functions. Furthermore, we also discuss the relation between the Hermiticity of the supercharge and boundary conditions for a higher dimensional Dirac field. Although we here focus on the curved extra dimensional models, the same discussions can be applied to the other cases.

3.3.1 Structure of $\mathcal{N} = 2$ supersymmetric quantum mechanics

Since we have assumed that the KK mode functions $f_\alpha^{(n)}$ and $g_\alpha^{(n)}$ form the complete set respectively, the orthonormal relations (3.2.54) lead to the following relations as well as the 5D case

$$\mathcal{A} f_\alpha^{(n)}(y) = m_n g_\alpha^{(n)}(y), \quad \mathcal{A}^\dagger g_\alpha^{(n)}(y) = m_n f_\alpha^{(n)}(y). \quad (3.3.1)$$

From the above relations, we can obtain

$$\begin{aligned} Q \begin{pmatrix} f_\alpha^{(n)}(y) \\ 0 \end{pmatrix} &= m_n \begin{pmatrix} 0 \\ g_\alpha^{(n)}(y) \end{pmatrix}, \quad H \begin{pmatrix} f_\alpha^{(n)}(y) \\ 0 \end{pmatrix} = m_n^2 \begin{pmatrix} f_\alpha^{(n)}(y) \\ 0 \end{pmatrix}, \quad (-1)^F \begin{pmatrix} f_\alpha^{(n)}(y) \\ 0 \end{pmatrix} = + \begin{pmatrix} f_\alpha^{(n)}(y) \\ 0 \end{pmatrix}, \\ Q \begin{pmatrix} 0 \\ g_\alpha^{(n)}(y) \end{pmatrix} &= m_n \begin{pmatrix} f_\alpha^{(n)}(y) \\ 0 \end{pmatrix}, \quad H \begin{pmatrix} 0 \\ g_\alpha^{(n)}(y) \end{pmatrix} = m_n^2 \begin{pmatrix} 0 \\ g_\alpha^{(n)}(y) \end{pmatrix}, \quad (-1)^F \begin{pmatrix} 0 \\ g_\alpha^{(n)}(y) \end{pmatrix} = - \begin{pmatrix} 0 \\ g_\alpha^{(n)}(y) \end{pmatrix}, \end{aligned} \quad (3.3.2)$$

where the supercharge Q , Hamiltonian H and “fermion” number operator $(-1)^F$ are defined by

$$Q = \begin{pmatrix} 0 & \mathcal{A}^\dagger \\ \mathcal{A} & 0 \end{pmatrix} = \begin{pmatrix} 0 & e^\sigma \left[-i\gamma^{\hat{y}} e_{\hat{y}^y} (\nabla_y + iqA_y) + W \right] \\ e^\sigma \left[i\gamma^{\hat{y}} e_{\hat{y}^y} (\nabla_y + iqA_y) + W \right] & 0 \end{pmatrix}, \quad (3.3.3)$$

$$H = Q^2 = e^{2\sigma} \left[-(\nabla_y + iqA_y)^2 + q\sigma^{\hat{y}\hat{y}'} e_{\hat{y}^y} e_{\hat{y}'^y} F_{yy'} + \frac{1}{4}R - i\gamma^{\hat{y}} e_{\hat{y}^y} (\partial_y W)(-1)^F + W^2 \right. \\ \left. - i\gamma^{\hat{y}} e_{\hat{y}^y} (\partial_y \sigma) \begin{pmatrix} \mathcal{A} & 0 \\ 0 & -\mathcal{A}^\dagger \end{pmatrix} \right], \quad (3.3.4)$$

$$(-1)^F = \begin{pmatrix} \mathbf{1}_{2[d/2]} & 0 \\ 0 & -\mathbf{1}_{2[d/2]} \end{pmatrix}. \quad (3.3.5)$$

$F_{yy'}$ is the field strength for A_y and R is the Ricci scalar defined on Ω .

Then, we can find that the relations (3.3.2) realize the $\mathcal{N} = 2$ SUSY QM, if the supercharge is Hermitian. In this model, the “bosonic” and “fermionic” states which form an $\mathcal{N} = 2$ supermultiplet correspond to the KK mode functions $(f_\alpha^{(n)}(y), 0)^T$ and $(0, g_\alpha^{(n)}(y))^T$. The 4D mass spectrum is given by the square root of eigenvalues of the Hamiltonian H . This supersymmetric structure results from the property that 4D left and right-handed chiral fields should be paired to construct a 4D mass term.

As well as Chapter 2, we can also consider the the Witten index Δ_W . In the extra dimensional model, this index is given as

$$\Delta_W = N_{f^{(0)}} - N_{g^{(0)}}, \quad (3.3.6)$$

where $N_{f^{(0)}}$ and $N_{g^{(0)}}$ denote the number of the massless mode functions $f_\alpha^{(0)}(y)$ and $g_\alpha^{(0)}(y)$. Δ_W is a topological quantity and invariant under the continuous deformations of parameters, *e.g.* shape of extra dimensions, boundary conditions and so on. Therefore, we can find that the difference between the numbers of the massless modes would be related to the topology of a extra dimensional space. From the viewpoint of the 4D effective theory, this index is identical to the number of massless chiral fermions (if $\Delta_W > 0$ (< 0), $|\Delta_W|$ right (left)-handed 4D chiral fermions exist), and then the model has $|\Delta_W|$ generations of chiral fermions. Thus, we should care about the topology of extra dimensions when we construct the phenomenological models.

3.3.2 Hermiticity of supercharge and boundary condition

Before closing this section, we comment about the Hermiticity of the supercharge and boundary conditions on $\partial\Omega$ for the higher dimensional Dirac field Ψ .

• Hermiticity of supercharge

The definition of the Hermiticity of the supercharge is that the supercharge satisfy

$$\langle \Phi' | Q\Phi \rangle = \langle Q\Phi' | \Phi \rangle, \quad (3.3.7)$$

or equivalently

$$\int_\Omega d^d y \sqrt{g} e^{-\sigma(y)} \Phi'^\dagger(y) (Q\Phi(y)) = \int_\Omega d^d y \sqrt{g} e^{-\sigma(y)} (Q\Phi'(y))^\dagger \Phi(y), \quad (3.3.8)$$

where Φ, Φ' are any $2^{[d/2]}$ -component functions given by the linear combination of the mode functions $(f_\alpha^{(n)}(y), 0)^T$ and $(0, g_\alpha^{(n)}(y))^T$. This condition can be rewritten into

$$\int_{\Omega} d^d y \sqrt{g} \nabla_y \left[(f_\alpha^{(n)}(y))^{\dagger} i \gamma^{\hat{y}} e_{\hat{y}}^y g_\beta^{(m)}(y) \right] = 0 \quad \text{for all } m, n, \alpha, \beta, \quad (3.3.9)$$

where ∇_y denotes the covariant derivative on Ω . Then, from the following relation for a vector V^M on Ω

$$\sqrt{g} \nabla_M V^M = \partial_M (\sqrt{g} V^M), \quad (3.3.10)$$

the Hermiticity condition for the supercharge is finally of the form

$$\int_{\partial\Omega} d^{d-1} y \sqrt{g} (f_\alpha^{(n)}(y))^{\dagger} i n_y(y) \gamma^{\hat{y}} e_{\hat{y}}^y g_\beta^{(m)}(y) = 0 \quad \text{for all } m, n, \alpha, \beta, \quad (3.3.11)$$

with an orthonormal vector $n_y(y)$ on $\partial\Omega$.

• Boundary condition

Next, we consider the boundary condition on $\partial\Omega$ for a higher dimensional field Ψ . By taking a variation of Ψ for the action, the variations of the action δS is given as

$$\begin{aligned} \delta S &= \int d^4 x \int_{\Omega} d^d y \sqrt{-G} \bar{\Psi}(x, y) \left[i \Gamma^{\hat{N}} e_{\hat{N}}^N (\nabla_N + i q A_N(y)) + W(y) \right] \delta \Psi(x, y) \\ &= \int d^4 x \int_{\Omega} d^d y \sqrt{g} \bar{\Psi} \left[i \Gamma^{\hat{N}} e_{\hat{N}}^N \left(-\overleftarrow{\nabla}_N + i q A_N(y) \right) + W(y) \right] \delta \Psi \\ &\quad + \int d^4 x \int_{\Omega} d^d y \left[\partial_N (\sqrt{g} \bar{\Psi} i \Gamma^{\hat{N}} e_{\hat{N}}^N \delta \Psi) \right]. \end{aligned} \quad (3.3.12)$$

Then, by the action principle $\delta S = 0$, we obtain the equation of motion for the Dirac conjugate $\bar{\Psi}$

$$\bar{\Psi}(x, y) \left[i \Gamma^{\hat{N}} e_{\hat{N}}^N \left(-\overleftarrow{\nabla}_N + i q A_N(y) \right) + W(y) \right] = 0, \quad (3.3.13)$$

and the boundary condition on $\partial\Omega$

$$\int_{\partial\Omega} d^{d-1} y \sqrt{g} \bar{\Psi}(x, y) i n_y(y) \Gamma^{\hat{y}} e_{\hat{y}}^y \delta \Psi(x, y) = 0. \quad (3.3.14)$$

The boundary condition (3.3.14) can be rewritten into the form with the use of 4D chiral decomposition of the Dirac field

$$\int_{\partial\Omega} d^{d-1} y \sqrt{g} \left\{ -\bar{\Psi}_R(x, y) [i n_y(y) \gamma^{\hat{y}} e_{\hat{y}}^y \otimes \mathbf{1}_4] \delta \Psi_L(x, y) + \bar{\Psi}_L(x, y) [i n_y(y) \gamma^{\hat{y}} e_{\hat{y}}^y \otimes \mathbf{1}_4] \delta \Psi_R(x, y) \right\} = 0, \quad (3.3.15)$$

where

$$\Psi_{R/L} = \frac{1 \pm \Gamma^5}{2} \Psi, \quad \delta \Psi_{R/L} = \frac{1 \pm \Gamma^5}{2} \delta \Psi. \quad (3.3.16)$$

Then we can obtain the following relation due to the independence of left and right-handed fields and also their variations:

$$\int_{\partial\Omega} d^{n-1}y \sqrt{-G} \bar{\Psi}_R(x, y) [i n_y(y) \gamma^{\hat{y}} e_{\hat{y}}^y \otimes \mathbf{1}_4] \delta \Psi_L(x, y) = 0. \quad (3.3.17)$$

Here we assume that the higher dimensional fields and variations are satisfy the same boundary condition and expand the variations as

$$\delta \Psi(x, y) = \sum_{n, \alpha} [e^{-2\sigma(y)} f_{\alpha}^{(n)}(y) \otimes \delta \psi_{R, \alpha}^{(n)}(x) + e^{-2\sigma(y)} g_{\alpha}^{(n)}(y) \otimes \delta \psi_{L, \alpha}^{(n)}(x)]. \quad (3.3.18)$$

$\delta \psi_{R/L, \alpha}^{(n)}(x)$ correspond to the variations of the 4D fields $\psi_{R/L, \alpha}^{(n)}(x)$. According to the independence of the 4D fields and the variations, the conditions (3.3.17) are reduced to

$$\int_{\partial\Omega} d^{n-1}y \sqrt{g} (f_{\alpha}^{(n)}(y))^{\dagger} i n_y(y) \gamma^{\hat{y}} e_{\hat{y}}^y g_{\beta}^{(m)}(y) = 0 \quad \text{for all } n, m, \alpha, \beta. \quad (3.3.19)$$

This boundary condition corresponds to the Hermiticity condition for the supercharge (3.3.11) and the supercharge Q is Hermitian as long as the action principle is required. Therefore, we can conclude that the $\mathcal{N} = 2$ SUSY QM is always realized in the 4D mass spectrum of the higher dimensional Dirac action.

3.4 Summary

In this chapter, we have considered the KK decomposition of a higher dimensional field and revealed the structure of $\mathcal{N} = 2$ SUSY QM. We have obtained that the effective 4D actions with an infinite number of 4D fields whose masses correspond to the eigenvalues of the SUSY Hamiltonian, and we can regard massless fields as the ones in the SM whereas massive fields as the new particles which are not observed so far due to their large masses from small extra dimensions. Since the 4D mass term consists of the 4D left and right-handed chiral spinors, we have shown that the corresponding KK mode functions should be related with each other. The structure of the $\mathcal{N} = 2$ SUSY QM exists in this relation.

As we have seen in Chapter 2, the Witten index Δ_W , the difference between the number of the massless mode functions $f_{\alpha}^{(0)}(y)$ and $g_{\alpha}^{(0)}(y)$, is given as a topological quantity. From the viewpoint of the 4D effective theory, this index corresponds to the number of chiral fermions (if $\Delta_W > 0$ (< 0), $|\Delta_W|$ right (left)-handed 4D chiral spinors exist). Therefore, we should construct extra dimensional models with the Witten index $|\Delta_W| = 3$, when we try to explain the three generations of the fermions in the SM. The structure of the fermion generation will be discussed in Chapter 4 from the viewpoint of general 1D extra dimensional models.

In addition to the 2-fold degeneracy from the $\mathcal{N} = 2$ SUSY QM, we will have more degeneracies in a 4D mass spectrum due to the degrees of freedom of extra dimensions. Thus we can expect that further hidden symmetries larger than the $\mathcal{N} = 2$ SUSY QM exist in the extra dimensional models. We will consider these structures in Chapter 6.

Chapter 4

5D fermion on quantum graph

In Chapter 3, we have given the overview of the KK decomposition of extra dimensional models and the hidden supersymmetry. Then, in this chapter, let us proceed to study the phenomenological structure of extra dimension models.

Here, we investigate a 5D Dirac fermion on a quantum graph that consists of a single vertex and N loops. We find that the model possesses a rich structure of boundary conditions for mode functions on the quantum graph and they can be classified into $(2N + 1)$ distinct categories. It is then shown that there appear degenerate four-dimensional chiral massless fermions in the 4D mass spectrum due to the topological properties in the boundary conditions as discussed in Chapter 3. We briefly discuss how our model could naturally solve the problems of the fermion generation, the fermion mass hierarchy and the origin of the flavor mixing and CP -violating phase.

4.1 Introduction

The discovery of the Higgs boson completes the SM. However, there still remain several mysteries and problems in the SM as discussed in 1. Here we focus on the generation problem, fermion mass hierarchy problem and origin of flavor mixing and CP violating phase.

Extra dimensional model is one of promising candidates which solve the above problems. The generation of the quarks and leptons will be explained if extra dimensional models give the degeneracy of 4D massless chiral fermions in 4D mass spectrum by the KK decomposition. Furthermore, the fermion mass hierarchy could be also resolved if the zero mode profiles are localized near some points on a extra space. This is because the mass matrix $m_{\alpha\beta}$ between the α -th generation and the β -th one of the fermions will be of the form

$$m_{\alpha\beta} = g_Y v \int dy (f_\alpha^{(0)}(y))^* g_\beta^{(0)}(y), \quad (4.1.1)$$

where g_Y is a Yukawa coupling constant¹, v is a VEV of the Higgs field, $\int dy$ denotes the integral over the extra dimensions, and $f_\alpha^{(0)}(y)$ ($g_\beta^{(0)}(y)$) is the profile of the α -th (β -th) generation of the 4D massless chiral fermions. Then, if $f_\alpha^{(0)}(y)$ and $g_\beta^{(0)}(y)$ are localized functions on the extra dimensions, we will get mass hierarchies, which sensitively depend on the positions of localization [14, 17, 19, 83–86]. The overlaps of the zero modes for different generations give off-diagonal components of

¹It should be emphasized that the Yukawa coupling g_Y has no index with respect to the generation. This is because we are considering extra dimensional models that could solve the generation problem, so that the models should be assumed to contain only one generation of the quarks and the lepton fields.

mass matrices and the flavor mixing. Furthermore, the mass matrix (4.1.1) can possess complex phases naturally and a CP -violating phase can appear if the zero mode functions $f_\alpha^{(0)}(y)$ and $g_\beta^{(0)}(y)$ are genuine complex functions [42].

In order to obtain extra dimensional models that possess the properties mentioned in the above paragraph, we investigate a 5D Dirac action on a quantum graph, which consists of edges and vertices. (For reviews of quantum graph, see [87, 88].) In this chapter, we take, as a quantum graph, a rose graph consisting of one vertex and N edges, where each edge forms a loop that begins and ends at the vertex (see fig. 4.1).

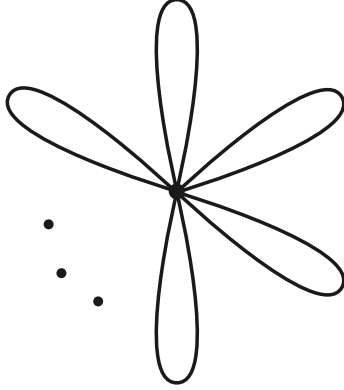


Figure 4.1: Rose graph with one vertex and N loops

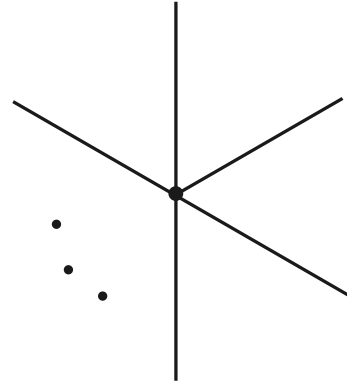


Figure 4.2: Star graph with one vertex and N edges



Figure 4.3: Interval with $(N - 1)$ point interactions

It is known that the rose graph with N loops can be regarded as a master quantum graph with N edges because the rose graph will reduce to so-called a star graph² (see fig. 4.2), an interval with $(N - 1)$ point interactions³ (see fig. 4.3) and so on, by appropriately tuning boundary conditions (or connection conditions) for mode functions at the vertex of the rose graph. Therefore, although ordinary 1D extra dimensions such as a circle or an interval could not solve the above problems as discussed in Section 3.2.1, the rose graph have a rich structure and will solve them simultaneously.

In this chapter, we show that allowed boundary conditions on the rose graph can be classified into $(2N + 1)$ distinct types, and clarify how many massless chiral fermions appear in the 4D mass

²There are many studies in terms of star graphs, see *e.g.* [89–92]. In particular, several works have been carried out from the viewpoint of extra dimensions on star graphs which are realized by flux compactifications in string theory, see [93–99].

³One-dimensional quantum mechanics with point interactions has attractive properties [100–102] and is applied to extra-dimension models to solve the generation problem and the fermion mass hierarchy [41, 43, 103].

spectrum for each type of boundary conditions. The results show that our model possesses the desired properties mentioned in the second paragraph in this section. We also confirm the relation between the topology of boundary condition and SUSY QM in this model.

This chapter is organized as follows: In the next section, we give a setup of our model. In Section 4.3, we classify allowed boundary conditions into $(2N + 1)$ -types of them. In Section 4.4, we examine zero mode solutions for each type of the boundary conditions. In Section 4.5, we discuss a topological nature of the zero modes and the relation to the hidden SUSY QM. Section 4.6 is devoted to summary and discussion.

4.2 5D Dirac action on quantum graph

Here, we consider the 5D Dirac action

$$S = \int d^4x \sum_{a=1}^N \int_{L_{a-1}}^{L_a} dy \bar{\Psi}(x, y) [i\gamma^\mu \partial_\mu + i\gamma^y \partial_y + M] \Psi(x, y), \quad (4.2.1)$$

where x^μ ($\mu = 0, 1, 2, 3$) are the coordinates of the 4D Minkowski spacetime and y is the coordinate of an extra dimension. $\Psi(x, y)$ represents a 4-component Dirac spinor on five dimensions and the Dirac conjugate $\bar{\Psi}$ is defined by $\bar{\Psi} = \Psi^\dagger \gamma^0$. γ^μ ($\mu = 0, 1, 2, 3$) are 4×4 gamma matrices and γ^y is taken to be $\gamma^y = -i\gamma^5$ ($\gamma^5 = i\gamma^0\gamma^1\gamma^2\gamma^3$). M denotes the bulk mass.

In this chapter, we consider the extra dimension of a rose graph with one vertex and N edges, each of which forms a loop with $L_{a-1} < y < L_a$ ($a = 1, 2, \dots, N$): see fig. 4.4.

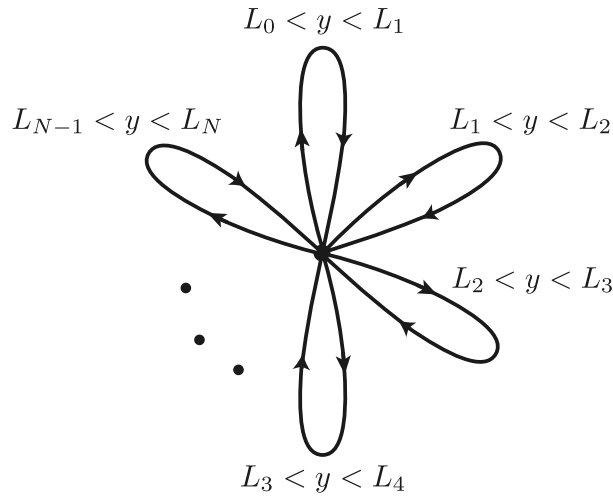


Figure 4.4: A rose graph consisting of one vertex and N loops.

The action principle $\delta S = 0$ leads to the 5D Dirac equation

$$[i\gamma^\mu \partial_\mu + i\gamma^y \partial_y + M] \Psi(x, y) = 0, \quad (4.2.2)$$

with the condition

$$\sum_{a=1}^N [\bar{\Psi}(x, y) \gamma^5 \delta \Psi(x, y)]_{y=L_{a-1}+\varepsilon}^{y=L_a-\varepsilon} = 0 \quad (4.2.3)$$

from (3.3.14), where ε is an infinitesimal positive constant. The condition (4.2.3) may be understood as the current conservation in the direction of the extra dimension at the center of the rose graph described in fig. 4.4. As we will see in the next section, Eq. (4.2.3) leads to boundary conditions that the field $\Psi(x, y)$ should obey at the boundaries $y = L_0 + \varepsilon, L_1 \pm \varepsilon, \dots, L_{N-1} \pm \varepsilon, L_N - \varepsilon$.

As well as Chapter 3, the 5D Dirac spinor $\Psi(x, y)$ will be expanded into the following form In terms of the 4D right-handed (left-handed) chiral spinors $\psi_{R,\alpha}^{(n)}(x)$ ($\psi_{L,\alpha}^{(n)}(x)$):

$$\Psi(x, y) = \sum_{\alpha} \sum_n \psi_{R,\alpha}^{(n)}(x) f_{\alpha}^{(n)}(y) + \sum_{\alpha} \sum_n \psi_{L,\alpha}^{(n)}(x) g_{\alpha}^{(n)}(y), \quad (4.2.4)$$

where the index n indicates the n -th level of the KK modes and α denotes the index that distinguishes the degeneracy of the n -th KK modes (if it exists). The mode functions $f_{\alpha}^{(n)}(y)$ and $g_{\alpha}^{(n)}(y)$ are assumed to form a complete set with respect to the extra dimensional space and satisfy the orthonormality relations

$$\sum_{a=1}^N \int_{L_{a-1}}^{L_a} dy (f_{\alpha}^{(m)}(y))^* f_{\beta}^{(n)}(y) = \delta^{mn} \delta_{\alpha\beta}, \quad (4.2.5)$$

$$\sum_{a=1}^N \int_{L_{a-1}}^{L_a} dy (g_{\alpha}^{(m)}(y))^* g_{\beta}^{(n)}(y) = \delta^{mn} \delta_{\alpha\beta}. \quad (4.2.6)$$

Substituting the expansion (4.2.4) into the action, we can rewrite the action S , in terms of the 4D spinors, as

$$\begin{aligned} S = \int d^4x \{ & \sum_{\alpha} \bar{\psi}_{R,\alpha}^{(0)}(x) i\gamma^{\mu} \partial_{\mu} \psi_{R,\alpha}^{(0)}(x) + \sum_{\beta} \bar{\psi}_{L,\beta}^{(0)}(x) i\gamma^{\mu} \partial_{\mu} \psi_{L,\beta}^{(0)}(x) \\ & + \sum_{\alpha} \sum_{n \neq 0} \bar{\psi}_{\alpha}^{(n)}(x) (i\gamma^{\mu} \partial_{\mu} + m_n) \psi_{\alpha}^{(n)}(x) \}, \end{aligned} \quad (4.2.7)$$

and we find the equations that the mode functions $f_{\alpha}^{(n)}(y)$ and $g_{\alpha}^{(n)}(y)$ should satisfy

$$(\partial_y + M) f_{\alpha}^{(n)}(y) = m_{\alpha} g_{\alpha}^{(n)}(y), \quad (4.2.8)$$

$$(-\partial_y + M) g_{\alpha}^{(n)}(y) = m_{\alpha} f_{\alpha}^{(n)}(y). \quad (4.2.9)$$

4.3 Classification of boundary conditions

In order to determine the mass eigenvalue m_n as well as the eigenfunctions $f_{\alpha}^{(n)}(y)$ and $g_{\alpha}^{(n)}(y)$, we need to specify some boundary conditions for the mode functions $f_{\alpha}^{(n)}(y)$ and $g_{\alpha}^{(n)}(y)$ at the boundaries $y = L_0 + \varepsilon, L_1 \pm \varepsilon, \dots, L_{N-1} \pm \varepsilon, L_N - \varepsilon$. In this section, we discuss allowed boundary conditions for $f_{\alpha}^{(n)}(y)$ and $g_{\alpha}^{(n)}(y)$ from Eq. (4.2.3) and classify them.

Substituting the expansion (4.2.4) of $\Psi(x, y)$ (and also $\delta\Psi(x, y)$) into Eq. (4.2.3) and using the fact that $\psi_{R/L,\alpha}^{(n)}(x)$ and $\delta\psi_{R/L,\alpha}^{(n)}(x)$ are independent fields, we find

$$\vec{F}_{\alpha}^{(n)\dagger} \vec{G}_{\beta}^{(m)} = \vec{G}_{\beta}^{(m)\dagger} \vec{F}_{\alpha}^{(n)} = 0 \quad \text{for } \forall n, m \text{ and } \forall \alpha, \beta, \quad (4.3.1)$$

4.3. CLASSIFICATION OF BOUNDARY CONDITIONS

where $\vec{F}_\alpha^{(n)}$ and $\vec{G}_\beta^{(m)}$ are $2N$ -component complex vectors defined by

$$\vec{F}_\alpha^{(n)} \equiv \begin{pmatrix} f_\alpha^{(n)}(L_0 + \varepsilon) \\ f_\alpha^{(n)}(L_1 - \varepsilon) \\ f_\alpha^{(n)}(L_1 + \varepsilon) \\ f_\alpha^{(n)}(L_2 - \varepsilon) \\ \vdots \\ f_\alpha^{(n)}(L_{a-1} + \varepsilon) \\ f_\alpha^{(n)}(L_a - \varepsilon) \\ \vdots \\ f_\alpha^{(n)}(L_{N-1} + \varepsilon) \\ f_\alpha^{(n)}(L_N - \varepsilon) \end{pmatrix}, \quad \vec{G}_\beta^{(m)} \equiv \begin{pmatrix} g_\beta^{(m)}(L_0 + \varepsilon) \\ -g_\beta^{(m)}(L_1 - \varepsilon) \\ g_\beta^{(m)}(L_1 + \varepsilon) \\ -g_\beta^{(m)}(L_2 - \varepsilon) \\ \vdots \\ g_\beta^{(m)}(L_{a-1} + \varepsilon) \\ -g_\beta^{(m)}(L_a - \varepsilon) \\ \vdots \\ g_\beta^{(m)}(L_{N-1} + \varepsilon) \\ -g_\beta^{(m)}(L_N - \varepsilon) \end{pmatrix}. \quad (4.3.2)$$

Here, we may call $\vec{F}_\alpha^{(n)}$ and $\vec{G}_\beta^{(m)}$ boundary vectors.

Let $\{\vec{F}_\alpha^{(n)}\}$ and $\{\vec{G}_\beta^{(m)}\}$ span the vectors space V and (W) , respectively. Then, Eq. (4.3.1) implies that V is orthogonal to W , and V will be regarded as the orthogonal complement space of W . Therefore Eq. (4.3.1) may be rewritten into the condition

$$\mathcal{P}_- \vec{F}_\alpha^{(n)} = \mathcal{P}_+ \vec{G}_\beta^{(m)} = 0 \quad \text{for } \forall n, m \text{ and } \forall i, j, \quad (4.3.3)$$

where $\mathcal{P}_\pm$ are projection matrices which maps the $2N$ -dimensional complex vector space into V (W), and they satisfy the relations $\mathcal{P}_+ + \mathcal{P}_- = \mathbf{1}_{2N}$, $\mathcal{P}_+ \mathcal{P}_- = \mathcal{P}_- \mathcal{P}_+ = 0$ and $(\mathcal{P}_\pm)^\dagger = \mathcal{P}_\pm$.

The projection matrices $\mathcal{P}_\pm$ can be written as

$$\mathcal{P}_\pm = \frac{1}{2}(\mathbf{1}_{2N} \pm U), \quad (4.3.4)$$

where U is a $2N \times 2N$ Hermitian unitary matrix with the property

$$U^2 = \mathbf{1}_{2N}, \quad U^\dagger = U^{-1} = U. \quad (4.3.5)$$

Thus, we find that the mode functions $f_\alpha^{(n)}(y)$ and $g_\beta^{(m)}(y)$ obey the following boundary conditions⁴

$$(\mathbf{1}_{2N} - U) \vec{F}_\alpha^{(n)} = \vec{0}, \quad (4.3.6)$$

$$(\mathbf{1}_{2N} + U) \vec{G}_\beta^{(m)} = \vec{0}, \quad (4.3.7)$$

for $\forall n, m$ and $\forall i, j$. Thus, we conclude that a Hermitian matrix U satisfying Eq. (4.3.5) specifies a 5D Dirac theory on a rose graph depicted in fig. 4.4.

Since $U^2 = \mathbf{1}_{2N}$, the eigenvalues of U are $+1$ or -1 , so that the matrix U is classified by the number of the eigenvalue $+1$ (or -1). We then call the boundary condition the type $(2N - k, k)$ if the number of the eigenvalue $+1$ (-1) of U is $2N - k$ (k) for $k = 0, 1, 2, \dots, 2N$, and the matrix U

⁴It will be worthwhile pointing out that for one-dimensional quantum mechanics on a quantum graph boundary conditions are generally imposed on not only $\psi(y)$ but also $\psi'(y) = \frac{d\psi(y)}{dy}$ [44, 104], though our model gives conditions only for the mode functions $f_\alpha^{(n)}(y)$ and $g_\alpha^{(n)}(y)$ but not $f'_\alpha^{(n)}(y)$ and $g'_\alpha^{(n)}(y)$.

in the type $(2N - k, k)$ can be expressed as

$$\text{type } (2N - k, k) : \quad U = V \left(\underbrace{\begin{pmatrix} +1 & & 0 \\ & \ddots & \\ 0 & & +1 \\ & & 0 \end{pmatrix}}_{2N-k} \underbrace{\begin{pmatrix} & & 0 \\ & -1 & \\ & & 0 \\ 0 & & -1 \end{pmatrix}}_k \right) V^\dagger, \quad (4.3.8)$$

for $V \in U(2N)$. Therefore the allowed boundary conditions on the rose graph are found to be classified into the $(2N + 1)$ -types of them. The form of (4.3.8) indicates that the parameter space of the type $(2N - k, k)$ boundary condition is found to be given by the coset space $U(2N)/(U(2N - k) \times U(k))$. It is worth noting that the parameter space of the type $(2N - k, k)$ boundary condition is distinct from that of the type $(2N - k', k')$ boundary condition for $k \neq k'$ and they are not continuously connected with each other.

4.4 Zero mode solutions and boundary conditions

In this chapter, we restrict our considerations to the zero mode solutions $f_\alpha^{(0)}(y)$ and $g_\alpha^{(0)}(y)$, which obey the equations (see. Eqs. (4.2.8) and (4.2.9))

$$(\partial_y + M)f_\alpha^{(0)}(y) = 0, \quad (4.4.1)$$

$$(-\partial_y + M)g_\beta^{(0)}(y) = 0. \quad (4.4.2)$$

Any solutions to (4.4.1) and (4.4.2) will be written into the form

$$f_\alpha^{(0)}(y) = \sum_{a=1}^N \theta(y - L_{a-1})\theta(L_a - y)A_{\alpha,a}C_a e^{-My}, \quad (4.4.3)$$

$$g_\beta^{(0)}(y) = \sum_{a=1}^N \theta(y - L_{a-1})\theta(L_a - y)B_{\beta,a}C'_a e^{My}, \quad (4.4.4)$$

for $\exists A_{\alpha,a}, B_{\beta,a} \in \mathbb{C}$ ($a = 1, 2, \dots, N$). The constants C_a and C'_a , which are independent of α and β , will be determined later for our convenience. The coefficients $A_{\alpha,a}$ and $B_{\beta,a}$ should be chosen to satisfy the boundary conditions (4.3.6) and (4.3.7). Here, $\theta(y)$ is the Heaviside step function defined as $\theta(y) = 1$ for $y > 0$ and $\theta(y) = 0$ for $y < 0$. Since we are considering functions on the rose graph described in fig. 4.4, we allow the mode functions $f_\alpha^{(n)}(y)$ and $g_\alpha^{(n)}(y)$ to be discontinuous at the boundaries $y = L_0, L_1, \dots, L_N$.

The criterion that there are K independent zero mode solutions $f_\alpha^{(0)}(y)$ ($\alpha = 1, 2, \dots, K$) is that the N -component complex vectors⁵ $\mathbf{A}_\alpha \equiv (A_{\alpha,1}, A_{\alpha,2}, \dots, A_{\alpha,N})^T$ ($\alpha = 1, 2, \dots, K$) associated with the solutions (4.4.3) are linearly independent. Similarly, K' solutions $g_\beta^{(0)}(y)$ ($\beta = 1, 2, \dots, K'$) can be independent if the N -component complex vectors $\mathbf{B}_\beta \equiv (B_{\beta,1}, B_{\beta,2}, \dots, B_{\beta,N})^T$ ($j = 1, 2, \dots, K'$)

⁵Throughout this chapter, we use the notation that $\vec{V}$ with the vector symbol “ $\rightarrow$ ” denotes a $2N$ -dimensional vector and the bold symbol $\mathbf{A}$ denotes an N -dimensional one.

4.4. ZERO MODE SOLUTIONS AND BOUNDARY CONDITIONS

associated with the solutions (4.4.4) are linearly independent. The above criterion will be used later.

In the following analysis, it will be convenient to introduce the $2N$ -dimensional vectors $\vec{\mathcal{F}}_a$ and $\vec{\mathcal{G}}_a$ ($a = 1, 2, \dots, N$) which are defined by

$$\vec{\mathcal{F}}_a \equiv C_a(\underbrace{0, \dots, 0}_{2(a-1)}, e^{-M(L_{a-1}+\varepsilon)}, e^{-M(L_a-\varepsilon)}, 0, \dots, 0)^T, \quad (4.4.5)$$

$$\vec{\mathcal{G}}_a \equiv C'_a(\underbrace{0, \dots, 0}_{2(a-1)}, e^{M(L_{a-1}+\varepsilon)}, -e^{M(L_a-\varepsilon)}, 0, \dots, 0)^T, \quad (4.4.6)$$

where C_a and C'_a ($a = 1, 2, \dots, N$) are the constants given in Eqs. (4.4.3) and (4.4.4), and are chosen to be

$$\vec{\mathcal{F}}_a^\dagger \vec{\mathcal{F}}_b = \vec{\mathcal{G}}_a^\dagger \vec{\mathcal{G}}_b = \delta_{ab} \quad (a, b = 1, 2, \dots, N). \quad (4.4.7)$$

The important observations are that $\vec{\mathcal{F}}_a$ and $\vec{\mathcal{G}}_b$ are orthogonal to each other, *i.e.*

$$\vec{\mathcal{F}}_a^\dagger \vec{\mathcal{G}}_b = \vec{\mathcal{G}}_a^\dagger \vec{\mathcal{F}}_b = 0 \quad (a, b = 1, 2, \dots, N). \quad (4.4.8)$$

Therefore, the set of $\{\vec{\mathcal{F}}_a, \vec{\mathcal{G}}_a \mid a = 1, 2, \dots, N\}$ can be regarded as an orthonormal basis of the $2N$ -dimensional complex vector space, so that any $2N$ -dimensional complex vector can be expressed as a linear combination of $\{\vec{\mathcal{F}}_a, \vec{\mathcal{G}}_a \mid a = 1, 2, \dots, N\}$. By means of $\{\vec{\mathcal{F}}_a, \vec{\mathcal{G}}_a \mid a = 1, 2, \dots, N\}$, the boundary vectors $\vec{F}_\alpha^{(0)}$ and $\vec{G}_\beta^{(0)}$ given in Eq. (4.3.2) associated with the zero mode solutions (4.4.3) and (4.4.4) can be expressed as follows:

$$\vec{F}_\alpha^{(0)} = \sum_{a=1}^N A_{\alpha,a} \vec{\mathcal{F}}_a, \quad (4.4.9)$$

$$\vec{G}_\beta^{(0)} = \sum_{a=1}^N B_{\beta,a} \vec{\mathcal{G}}_a. \quad (4.4.10)$$

In the following, we shall clarify how many zero mode solutions exist for each of the type $(2N - k, k)$ boundary conditions ($k = 0, 1, 2, \dots, 2N$). This is equivalent to find linearly independent boundary vectors $\vec{F}_\alpha^{(0)}$ and $\vec{G}_\beta^{(0)}$ which satisfy the boundary conditions (4.3.6) and (4.3.7) for a given U . To this end, it is convenient to use the fact that any $2N \times 2N$ unitary matrix V can be expressed as

$$V = (\vec{u}_1, \vec{u}_2, \dots, \vec{u}_{2N}), \quad (4.4.11)$$

where $\vec{u}_p$ ($p = 1, 2, \dots, 2N$) are $2N$ -dimensional orthonormal complex vectors satisfying $\vec{u}_p^\dagger \vec{u}_{p'} = \delta_{pp'}$ ($p, p' = 1, 2, \dots, 2N$). Then, the matrix U belonging to the type $(2N - k, k)$ boundary condition will be given by

$$U = \sum_{p=1}^{2N-k} \vec{u}_p \vec{u}_p^\dagger - \sum_{q=2N-k+1}^{2N} \vec{u}_q \vec{u}_q^\dagger. \quad (4.4.12)$$

It follows that the type $(2N - k, k)$ boundary conditions (4.3.6) and (4.3.7) reduce to

$$\vec{u}_q^\dagger \vec{F}_\alpha^{(0)} = 0 \quad \text{for } q = 2N - k + 1, 2N - k + 2, \dots, 2N, \quad (4.4.13)$$

$$\vec{u}_p^\dagger \vec{G}_\beta^{(0)} = 0 \quad \text{for } p = 1, 2, \dots, 2N - k. \quad (4.4.14)$$

4.4.1 Type $(2N - k, k)$ boundary condition with $0 \leq k \leq N$

Let us first consider the case of $0 \leq k \leq N$. Since the set of $\{\vec{\mathcal{F}}_a, \vec{\mathcal{G}}_a (a = 1, 2, \dots, N)\}$ forms a complete set of the $2N$ -dimensional complex vector space, the vectors $\{\vec{u}_q (q = 2N - k + 1, \dots, 2N)\}$ can be expressed as

$$\vec{u}_q = \sum_{a=1}^N u_{q,a}^{(\mathcal{F})} \vec{\mathcal{F}}_a + \sum_{a=1}^N u_{q,a}^{(\mathcal{G})} \vec{\mathcal{G}}_a \quad (4.4.15)$$

for $\exists u_{q,a}^{(\mathcal{F})}, u_{q,a}^{(\mathcal{G})} \in \mathbb{C} (q = 2N - k + 1, \dots, 2N)$. Inserting Eqs. (4.4.9) and (4.4.15) into Eq. (4.4.13) and using Eqs. (4.4.7) and (4.4.8), we have

$$\mathbf{u}_q^{(\mathcal{F})\dagger} \cdot \mathbf{A}_\alpha = 0, \quad (q = 2N - k + 1, \dots, 2N), \quad (4.4.16)$$

where $\mathbf{u}_q^{(\mathcal{F})} \equiv (u_{q,1}^{(\mathcal{F})}, u_{q,2}^{(\mathcal{F})}, \dots, u_{q,N}^{(\mathcal{F})})^T$.

If the maximal number of the linearly independent vectors for $\mathbf{u}_q^{(\mathcal{F})} (q = 2N - k + 1, \dots, 2N)$ is l with $0 \leq l \leq k$, it turns out that there exist $(N - l)$ linearly independent solutions of $\mathbf{A}^{(i)}$ ($i = 1, 2, \dots, N - l$) to Eq. (4.4.16). This immediately implies that there are $(N - l)$ linearly independent boundary vectors $\vec{F}_\alpha^{(0)}$ ($i = 1, 2, \dots, N - l$).

In order to obtain $\vec{G}_\beta^{(0)}$, we first notice that solutions of $\vec{G}_\beta^{(0)}$ to Eq. (4.4.14) could be written into the form

$$\vec{G}_\beta^{(0)} = \sum_{q=2N-k+1}^{2N} b_{\beta,q} \vec{u}_q \quad (4.4.17)$$

for $\exists b_{\beta,q} \in \mathbb{C} (q = 2N - k + 1, \dots, 2N)$. Furthermore, $\vec{G}_\beta^{(0)}$ should be given as a linear combination of the vectors $\vec{\mathcal{G}}_a$ from (4.4.10). Since we have assumed that the set of the vectors $\mathbf{u}_q^{(\mathcal{F})} (q = 2N - k + 1, \dots, 2N)$ in Eq. (4.4.15) consists of l linearly independent vectors, by appropriately choosing the coefficients $b_{\beta,q} \in \mathbb{C}$ and removing the components of $\vec{\mathcal{F}}_a$ in Eq. (4.4.17), we can obtain $(k - l)$ linearly independent solutions of the boundary vectors $\vec{G}_\beta^{(0)}$ ($j = 1, 2, \dots, k - l$).

Therefore, we conclude that for the type $(2N - k, k)$ boundary condition with $0 \leq k \leq N$, the number of the zero mode solutions $f_\alpha^{(0)}(y)$ and $g_\beta^{(0)}(y)$ are given as follows:

Type $(2N - k, k)$ boundary condition with $0 \leq k \leq N$		
l	$N_{f^{(0)}}$	$N_{g^{(0)}}$
0	N	k
1	$N - 1$	$k - 1$
$\vdots$	$\vdots$	$\vdots$
k	$N - k$	0

Table 4.1: The number of the zero mode solutions of $f_\alpha^{(0)}(y)$ and $g_\beta^{(0)}(y)$ for the type $(2N - k, k)$ boundary condition ($0 \leq k \leq N$). l denotes the maximal number of the linearly independent vectors $\mathbf{u}_q^{(\mathcal{F})} (q = 2N - k + 1, \dots, 2N)$ in Eq. (4.4.15). $N_{f^{(0)}} (N_{g^{(0)}})$ is the number of the zero mode solutions of $f_\alpha^{(0)}(y)$ ($g_\beta^{(0)}(y)$).

4.4.2 Type $(2N - k, k)$ boundary condition with $N \leq k \leq 2N$

Let us next consider the case of $N \leq k \leq 2N$. In this case, we will expand $\vec{u}_p$ ($p = 1, 2, \dots, 2N - k$) as

$$\vec{u}_p = \sum_{a=1}^N u_{p,a}^{(\mathcal{F})} \vec{\mathcal{F}}_a + \sum_{a=1}^N u_{p,a}^{(\mathcal{G})} \vec{\mathcal{G}}_a \quad (4.4.18)$$

for $\exists u_{p,a}^{(\mathcal{F})}, u_{p,a}^{(\mathcal{G})} \in \mathbb{C}$ ($p = 1, 2, \dots, 2N - k$). Inserting Eqs. (4.4.10) and (4.4.18) into Eq. (4.4.14) and using Eqs. (4.4.7) and (4.4.8), we have

$$\mathbf{u}_p^{(\mathcal{G})\dagger} \cdot \mathbf{B}_\beta = 0, \quad (p = 1, 2, \dots, 2N - k) \quad (4.4.19)$$

where $\mathbf{u}_p^{(\mathcal{G})} \equiv (u_{p,1}^{(\mathcal{G})}, u_{p,2}^{(\mathcal{G})}, \dots, u_{p,N}^{(\mathcal{G})})^T$. If the maximal number of the linearly independent vectors for $\mathbf{u}_p^{(\mathcal{G})}$ ($p = 1, 2, \dots, 2N - k$) is l with $0 \leq l \leq 2N - k$, it turns out that there exist $(N - l)$ linearly independent solutions of $\mathbf{B}_\beta$ ($j = 1, 2, \dots, N - l$) to Eq. (4.4.19). This implies that there $(N - l)$ linearly independent vectors $\vec{G}_\beta^{(0)}$ ($j = 1, 2, \dots, N - l$).

In order to obtain $\vec{F}_\alpha^{(0)}$, we first notice that solutions of $\vec{F}_\alpha^{(0)}$ to Eq. (4.4.13) could be written into the form

$$\vec{F}_\alpha^{(0)} = \sum_{p=1}^{2N-k} a_{\alpha,p} \vec{u}_p, \quad (4.4.20)$$

for $\exists a_p^{(i)} \in \mathbb{C}$ ($p = 1, 2, \dots, 2N - k$). Furthermore, $\vec{F}_\alpha^{(0)}$ should be given as a linear combination of the vectors $\vec{\mathcal{F}}_a$ from (4.4.9). Since we have assumed that the set of the vectors $\{\mathbf{u}_p^{(\mathcal{G})} (p = 1, 2, \dots, 2N - k)\}$ in Eq. (4.4.15) consists of l linearly independent vectors, by appropriately choosing the coefficients $a_{\alpha,p} \in \mathbb{C}$ and removing the components of $\vec{\mathcal{G}}_a$, we can obtain $(2N - k - l)$ linearly independent solutions of the boundary vectors $\vec{F}_\alpha^{(0)}$ ($i = 1, 2, \dots, 2N - k - l$), which are written into the form (4.4.9).

Therefore, we conclude that for the type $(2N - k, k)$ boundary condition with $N \leq k \leq 2N$, the number of the zero mode solutions $f_\alpha^{(0)}(y)$ and $g_\beta^{(0)}(y)$ are given as follows:

Type $(2N - k, k)$ boundary condition with $N \leq k \leq 2N$		
l	$N_{f^{(0)}}$	$N_{g^{(0)}}$
0	$2N - k$	N
1	$2N - k - 1$	$N - 1$
$\vdots$	$\vdots$	$\vdots$
$2N - k$	0	$-N + k$

Table 4.2: The number of the zero mode solutions of $f_\alpha^{(0)}(y)$ and $g_\beta^{(0)}(y)$ for the type $(2N - k, k)$ boundary condition ($N \leq k \leq 2N$). l denotes the maximal number of the linearly independent vectors $\mathbf{u}_p^{(\mathcal{G})}$ ($p = 1, 2, \dots, 2N - k$) in Eq. (4.4.18). $N_{f^{(0)}}$ ($N_{g^{(0)}}$) is the number of the zero mode solutions of $f_\alpha^{(0)}(y)$ ($g_\beta^{(0)}(y)$).

4.5 Zero mode solutions and Witten index

In the previous section, we have succeeded in finding the number $N_{f^{(0)}}$ ($N_{g^{(0)}}$) of the zero mode solutions $f_\alpha^{(0)}(y)$ ($g_\beta^{(0)}(y)$). Then, we have seen that $N_{f^{(0)}}$ and $N_{g^{(0)}}$ depend on l and k , where l is the maximal number of linearly independent vectors for $\{\mathbf{u}_q^{(\mathcal{F})} (q = 2N - k + 1, \dots, 2N)\}$ in Eq. (4.4.15) or $\{\mathbf{u}_p^{(\mathcal{G})} (p = 1, 2, \dots, 2N - k)\}$ in Eq. (4.4.18), and k is the number that specifies the type $(2N - k, k)$ boundary condition. It will be worthwhile noting that the difference $\Delta_W \equiv N_{f^{(0)}} - N_{g^{(0)}}$ is independent of l , *i.e.* $\Delta_W = N - k$ for the type $(2N - k, k)$ boundary condition with $0 \leq k \leq 2N$, though $N_{f^{(0)}}$ and $N_{g^{(0)}}$ depend on l (see Table 4.1 and Table 4.2). The fact that Δ_W is independent of l is not accidental, but it has a topological reason from the hidden supersymmetry.

In Chapter 3, we have shown that the supersymmetric structure is hidden in the 4D mass spectrum of extra dimensional models and the difference Δ_W corresponds to the Witten index which is topological index. Since the type $(2N - k, k)$ boundary condition is not continuously connected to other type $(2N - k', k')$ with $k \neq k'$, the Witten index Δ_W can depend on k but not on l because the boundary conditions with different l can continuously be deformed each other.

Thus, the three generation of fermion can be obtained from a model with $|\Delta_W| = 3$, *i.e.* the type $(N - 3, N + 3)$ or type $(N + 3, N - 3)$ boundary condition for the rose graph with N loops. The most simplest example is the case of $N = 3$ with the type $(6, 0)$ or type $(0, 6)$ boundary condition. For the type $(6, 0)$ (the type $(0, 6)$) boundary condition, the mode functions $g_\beta^{(n)}(y)$ ($f_\alpha^{(n)}(y)$) should satisfy the Dirichlet boundary condition at vertex and therefore the quantum graph in the extra dimension physically equivalent to the interval with point interactions as described Figure 4.5. We will discuss this case numerically in the next chapter.

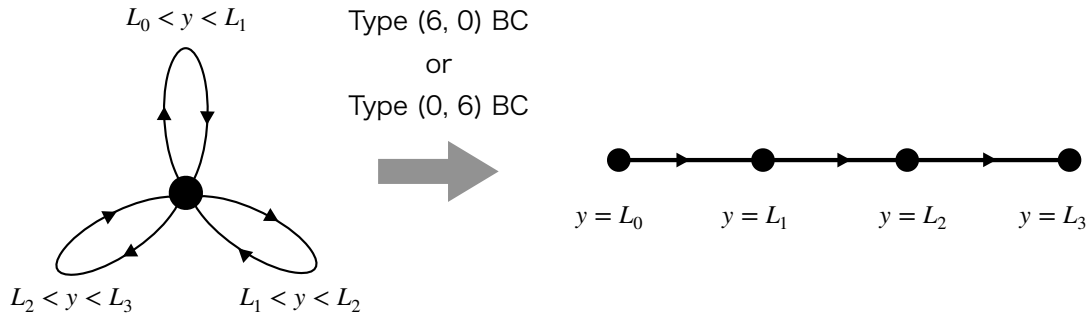


Figure 4.5: Interval with 2 point interactions

4.6 Summary and discussion

In this chapter, we have investigated the KK decomposition of the 5D Dirac fermion on the rose graph which consists of one vertex and N loops (see fig. 4.4). We have successfully classified the allowed boundary condition on the rose graph into the type $(2N - k, k)$ with $0 \leq k \leq 2N$ and clarified the number of the zero modes for each type of the boundary conditions.

Our results would become phenomenologically important in constructing models beyond the SM. The chiral fermions of the SM can be identified as the 4D chiral massless fermions $\psi_{R,\alpha}^{(0)}(x)$ and $\psi_{L,\beta}^{(0)}(x)$ associated with the zero mode solutions $f_\alpha^{(0)}(y)$ and $g_\beta^{(0)}(y)$. Thus, the three generations of chiral fermions could be obtained from a model with $|\Delta_W| = 3$, *i.e.* the type $(N - 3, N + 3)$ or

type $(N + 3, N - 3)$ boundary condition for the rose graph with N loops. Furthermore, the exponentially localized profiles of zero mode solutions $f_\alpha^{(0)}(y)$ and $g_\beta^{(0)}(y)$ and the overlap integrals of zero mode solutions can produce hierarchical masses for the quarks and leptons. Then the fermion mass hierarchy problem will be solved in our model. The overlaps of the mode functions for different generations also gives off-diagonal components of mass matrices and the flavor mixing. Furthermore, our model naturally provides a CP -violating phase in the CKM matrix. This is because the zero modes $f_\alpha^{(0)}(y)$ and $g_\beta^{(0)}(y)$ can become genuine complex functions in general, through the complex (Hermitian) matrix U , in the boundary conditions (4.3.6) and (4.3.7). Therefore, our model considered in this chapter is expected to shed a new light on the generation problem, the fermion mass hierarchy, the flavor mixing and the CP -violating phase of the SM.

It would be of great interest to note that the rose graph depicted in fig. 4.4 has non-trivial geometry and that the parameter space of the boundary conditions has been found to possess a rich structure. For instance, if we take the length of all edges to be equal ($L = L_a - L_{a-1}$ for $a = 1, 2, \dots, N$) and choose the boundary conditions appropriately, we will have higher degeneracy in the 4D mass spectrum and then extended supersymmetries might appear in the spectrum [101, 102]. Furthermore, we will expect non-abelian Berry phases in the space of the boundary conditions with the degeneracy of the spectrum [52, 53, 105]. In addition, although we have considered the 5D fermion, we should also investigate 5D scalar, gauge, gravity fields on the quantum graph and clarify allowed boundary conditions. The issues mentioned above will be reported in a future work.

Chapter 5

Dynamical generation of quark/lepton mass hierarchy in an extra dimension

In the previous chapter, we have clarified the structure of the boundary condition on the 1D extra dimensional model with the rose quantum graph, and shown that the $N-k$ ($= \Delta_W$) chiral zero modes appear for the type $(2N-k, k)$ boundary condition ($0 \leq k \leq 2N$) in the rose graph with N loops. The most simplest model which realizes $\Delta_W = 3$ is the case of $N = 3$ with the type $(6, 0)$ or type $(0, 6)$ boundary condition and this graph corresponds to the interval with point interactions. The extra dimensional models with point interactions were investigated in Refs. [41–43] and numerically reproduced the fermion generation structure by adjusting parameters “by hand”.

Then, in this chapter, we consider whether the observed quark/lepton mass hierarchy can be realized “dynamically” on an interval extra dimension with point interactions based on the idea in [103]. In our model, the positions of the point interactions play a crucial role to control the quark/lepton mass hierarchy and are determined by the minimization of the Casimir energy. By use of the exact extra-dimensional coordinate-dependent vacuum expectation value of a gauge singlet scalar, we find that there is a parameter set, where the positions of the point interactions are stabilized and fixed, which can reproduce the experimental values of the quark masses precisely enough, while the charged lepton part is less relevant. We also show that possible mixings among the charged leptons will improve the situation significantly.

5.1 Introduction

Even after the discovery of the Higgs boson, several issues associated with the Higgs boson still remain to be solved in the SM. One is why the observed quark/lepton mass hierarchy is so drastic. In the SM, quarks and leptons acquire their masses through the Yukawa interactions where they pick up the vacuum expectation value (VEV) of the Higgs field. Here, we should accept to impose $O(10^5)$ -order hierarchical values on Yukawa couplings, which look very unnatural.

Another unnatural request from the SM is associated with the matter content of the fermions. In both of the quark and the lepton sectors, there are three similar but different copies of quarks and leptons, which possess exactly the same quantum numbers except for their masses. The origin of such generation structure is beyond the scope of the SM, and is still unveiled.

Then, we address these problems by considering the extra dimensional model with the quantum graph given in the previous chapter.¹ Here we concentrate on the model of interval extra dimension

¹Another way is to introduce the magnetic flux or the magnetized orbifold in the extra dimensions [25–27, 30–40].

with point interactions, which belongs to the case of $N = 3$ loops with the type $(6, 0)$ or type $(0, 6)$ boundary condition in the rose graph as discussed in the section 4.5.

The point interactions can provide extra boundary conditions for each of 5D quarks and 5D leptons, and induce the degenerate chiral zero modes. This is interpreted as spontaneous generation of the three matter generations in the SM when we introduce two point interactions for each 5D fermion. Appealing properties in this direction are that a 5D fermion leads to three chiral zero modes, and they are localized to different segments each other, whose end points are identified by the two corresponding point interactions. A three-by-three mass matrix is realized as m_{ij} ($i, j = 1, 2, 3$) due to these properties. The generation dependence of the four-dimensional (4D) mass matrix m_{ij} is described through the following manner:

$$m_{ij} \propto \int_0^L dy \langle \Phi(y) \rangle (g_{\psi_{iL}}^{(0)}(y))^* f_{\psi_{jR}}^{(0)}(y), \quad (5.1.1)$$

where L denotes the length of the interval, $\int_0^L dy$ is an integration along the shown range of the extra-dimensional coordinate, $g_{\psi_{iL}}^{(0)}(y)$ ($f_{\psi_{jR}}^{(0)}(y)$) represents mode functions of i -th (j -th) generation left- (right-) handed chiral zero mode $\psi_{iL}(x)$ ($\psi_{jR}(x)$). $\langle \Phi(y) \rangle$ is the contribution from the classical configuration of the scalar which appears in the 5D Yukawa term and its VEV shows the dependence on y . If not only $g_{\psi_{iL}}^{(0)}(y)$ and $f_{\psi_{jR}}^{(0)}(y)$, but also $\langle \Phi(y) \rangle$ are localized functions, sizable mass hierarchy can appear through the overlap integrals in Eq. (5.1.1).

In this chapter, based on the above mechanism and Refs. [41–43], we perform a numerical research to appraise whether the scenario can reproduce the observed values of the quark/lepton hierarchical masses dynamically (see [111–114] for dynamical generations of the mass hierarchy). The positions of the point interactions, which play an important role to control the magnitude of the overlap integrals, seem to be free parameters. Nonetheless, they are not free but are determined dynamically through the vacuum configuration to minimize the Casimir energy (see [115, 116] and also [117–120] as related works). This dynamical determination of the positions of the point interactions reduces the number of the free parameters. Also, to accelerate hierarchical values in the overlap integral in Eq. (5.1.1), we employ a new type of extra-dimensional coordinate-dependent configuration in $\langle \Phi(y) \rangle$, which is also obtained dynamically by solving the equation of motion with the quartic interaction. It should be noted that in the research [103], the VEV of the scalar $\langle \Phi(y) \rangle$ is also obtained dynamically. However, the different functional type of the VEV was adopted (see subsection 5.3.2 in detail). Performing numerical searches brings us the conclusion that the hierarchical masses of the quarks are fully reproduced, while the charged leptons do not work well. Introduction of flavor mixings may improve our current results, especially for the charged leptons.

This chapter is organized as follows. In section 5.2, we briefly review the extra-dimensional model with point interactions on an interval. In section 5.3, we discuss the dynamics of the Casimir energy, the potential minimization of the scalar Φ and their implication for the mass hierarchy. In section 5.4, we investigate searches for sets of desirable input parameters and show our results. Section 5.5 is devoted to summary and discussion.

There are also several ways to address the SM problems in the context of four-dimensional gauge theories, by use of noncompact gauge symmetry [106–110].

5.2 The model with point interactions on an interval

5.2.1 Basic idea

In this subsection, we provide a brief review on the extra-dimensional model which sheds light on spontaneous generation of the three generations of the fermion matter content, the hierarchical structure of the observed quarks and charged leptons, and their mixing structures in the SM, simultaneously. This kind of scenario was firstly proposed in [41] on an interval to explore the quark sector, where objects so-called point interactions play a significant role. The point interactions describe possible singularities in one-dimensional quantum mechanics, by which profiles of particles along the extra spacial direction y are represented. This statement is rephrased that we can add “extra (or generalized) boundary conditions” for all kinds of 5D fields in the bulk space, in addition to the boundary points. By imposing the Dirichlet boundary conditions at the point interactions [41], three chiral zero modes are generated from a single 5D fermion. A key point to realize mass hierarchy is that we can introduce a bulk mass for each of such 5D fermions respectively, which makes the profiles of the three zero modes localized towards the boundary points as shown in Figure 5.1 and Figure 5.2.



Figure 5.1: A schematic figure of the interval extra dimension with the point interactions. The black dots at $y = L_1, L_2$ indicate the point interaction.

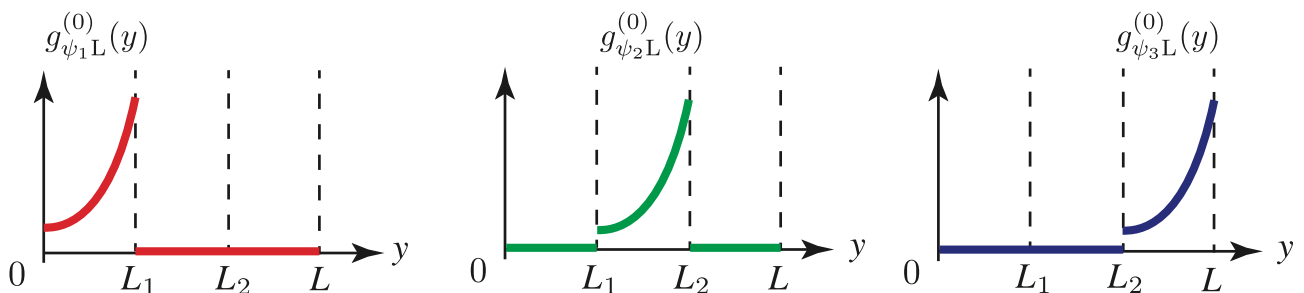


Figure 5.2: A schematic figure of the profiles of the three chiral zero modes. Each zero mode localizes towards the different boundary points but in a similar way.

Here, if the scalar appearing in the Yukawa sector of such 5D fermions possesses an extra-dimensional coordinate-dependent VEV, it is clear that sizable mass hierarchy is generated on an interval. It is noted that this scenario has been also developed in the S^1 geometry, where a complex phase for describing the CP -violating phase of the CKM matrix can emerge, originating from a twisted boundary condition [42, 43].

In the scenario, an $SU(3)_C \times SU(2)_L \times U(1)_Y$ gauge theory is unfolded on an interval, the relevant

part of the 5D action of which for our discussion reads,

$$S = S_{\text{quark}} + S_{\text{lepton}} + S_{\text{Higgs}} + S_{\text{singlet}} + S_{\text{Yukawa}}, \quad (5.2.1)$$

$$S_{\text{quark}} = \int d^4x \int_0^L dy \left[\bar{Q} (i\Gamma^M D_M^{(Q)} + M_Q) Q + \bar{\mathcal{U}} (i\Gamma^M D_M^{(\mathcal{U})} + M_{\mathcal{U}}) \mathcal{U} + \bar{\mathcal{D}} (i\Gamma^M D_M^{(\mathcal{D})} + M_{\mathcal{D}}) \mathcal{D} \right], \quad (5.2.2)$$

$$S_{\text{lepton}} = \int d^4x \int_0^L dy \left[\bar{L} (i\Gamma^M D_M^{(L)} + M_L) L + \bar{\mathcal{E}} (i\Gamma^M D_M^{(\mathcal{E})} + M_{\mathcal{E}}) \mathcal{E} \right], \quad (5.2.3)$$

$$S_{\text{Higgs}} = \int d^4x \int_0^L dy \left[H^\dagger (D^M D_M + M_H^2) H - \frac{\lambda_H}{2} (H^\dagger H)^2 \right], \quad (5.2.4)$$

$$S_{\text{singlet}} = \int d^4x \int_0^L dy \left[\Phi^\dagger (\partial^M \partial_M - M_\Phi^2) \Phi - \frac{\lambda_\Phi}{2} (\Phi^\dagger \Phi)^2 \right], \quad (5.2.5)$$

$$S_{\text{Yukawa}} = \int d^4x \int_0^L dy \left[\Phi (-\mathcal{Y}_u \bar{Q} (i\sigma_2 H^*) \mathcal{U} - \mathcal{Y}_d \bar{Q} H \mathcal{D} - \mathcal{Y}_e \bar{L} H \mathcal{E}) + (\text{h.c.}) \right], \quad (5.2.6)$$

which consist of a 5D quark doublet field Q and a 5D lepton doublet field L ; singlet fields for an up-type quark $\mathcal{U}$, a down-type quark $\mathcal{D}$ and a charged lepton $\mathcal{E}$; the Higgs doublet H and a gauge singlet complex scalar Φ ; the Pauli matrix σ_2 , where we skipped to show the apparent gauge sector of the current scenario and we usually suppress the 5D coordinates as variables of 5D fields. In this dissertation, we do not discuss the minuscule active neutrino mass generation so that the (5D) neutrino singlet field is not introduced (see the discussions on the S^1 geometry without/with Majorana-like mass terms [43]/ [121]). Various $D_M^{(\Psi)}$ ($\Psi = Q, \mathcal{U}, \mathcal{D}, L, \mathcal{E}$) symbols represent the corresponding covariant derivatives, M_Ψ ($\Psi = Q, \mathcal{U}, \mathcal{D}, L, \mathcal{E}$) are bulk masses for the 5D fermions, M_H^2 (M_Φ^2) and λ_H (λ_Φ) are squared mass scales and quartic couplings for the Higgs (the gauge-singlet scalar). $\mathcal{Y}_u, \mathcal{Y}_d$ and $\mathcal{Y}_e$ are overall Yukawa couplings which do not hold indices to discriminate matter generations since all the 5D fermions are assumed to be one-generation. The sign in front of M_H^2 is chosen to ignite the Higgs mechanism. Obedience to the conditions $\lambda_H > 0$ and $\lambda_\Phi > 0$ is compulsory to ensure the stability of the whole potential.²

Here we provide several comments.

- As we mentioned above, we introduce two point interactions for each of the 5D fermions to realize three generations at the chiral zero mode sector, while not for the scalars and the gauge bosons.
- We impose the discrete symmetry, $H \rightarrow -H$ and $\Phi \rightarrow -\Phi$, which prohibits the ‘ordinary’ Yukawa terms like $\bar{Q}(i\sigma_2 H^*)\mathcal{U}$ and $\Phi \bar{Q}Q$ and others likewise. We assume that H has the ordinary constant VEV $\langle H \rangle$ to break the SM gauge group suitably, and Φ acquires a y -dependent VEV $\langle \Phi \rangle$ to generate hierarchical structure in the Yukawa sector. Thus, the above ‘higher-dimensional’ Yukawa terms are suitable for our purpose. The detail of the y -dependent gauge-singlet scalar VEV profile will be provided in subsection 5.3.2.
- Under the current setup on an interval, no physical CP -violating phase can emerge since only overall complexities can be realized by the Yukawa couplings $\mathcal{Y}_u, \mathcal{Y}_d$ and $\mathcal{Y}_e$ and absorbed into field redefinitions. We conquer this obstacle by moving to the S^1 geometry easily [42, 43], while we do not discuss this point here.

²In general, the mixing term $(H^\dagger H)(\Phi^\dagger \Phi)$ can be written down. However, the existence of this term leads to gauge universality violation [41]. Then we set the coefficient as zero by hand.

5.2.2 Appearance of three-generation mass matrices

As discussed in [41], introduction of two suitable Dirichlet boundary conditions by point interactions for a 5D fermion Ψ in the bulk space, in addition to the two endpoints of the interval, as $\Psi_R \equiv \left(\frac{1+\gamma_5}{2}\right)\Psi = 0$ or $\Psi_L \equiv \left(\frac{1-\gamma_5}{2}\right)\Psi = 0$ leads to the realization of three localized left- or right-handed chiral zero modes via Ψ , respectively. Concretely, the following conditions are imposed for the 5D fermions with an infinitesimal positive constant ε :

$$Q_R = 0 \quad \text{at } y = 0, L_1^{(Q)} \pm \varepsilon, L_2^{(Q)} \pm \varepsilon, L, \quad (5.2.7)$$

$$\mathcal{U}_L = 0 \quad \text{at } y = 0, L_1^{(\mathcal{U})} \pm \varepsilon, L_2^{(\mathcal{U})} \pm \varepsilon, L, \quad (5.2.8)$$

$$\mathcal{D}_L = 0 \quad \text{at } y = 0, L_1^{(\mathcal{D})} \pm \varepsilon, L_2^{(\mathcal{D})} \pm \varepsilon, L, \quad (5.2.9)$$

$$L_R = 0 \quad \text{at } y = 0, L_1^{(L)} \pm \varepsilon, L_2^{(L)} \pm \varepsilon, L, \quad (5.2.10)$$

$$\mathcal{E}_L = 0 \quad \text{at } y = 0, L_1^{(\mathcal{E})} \pm \varepsilon, L_2^{(\mathcal{E})} \pm \varepsilon, L, \quad (5.2.11)$$

where $L_1^{(\Psi)}$ and $L_2^{(\Psi)}$ mean the positions of the first and second point interactions ($0 < L_1^{(\Psi)} < L_2^{(\Psi)} < L$) for the field Ψ ($\Psi = Q, \mathcal{U}, \mathcal{D}, L, \mathcal{E}$). Here we define the conventions $L_0^{(\Psi)} \equiv 0$ and $L_3^{(\Psi)} \equiv L$ for convenience soon later. The Kaluza–Klein (KK) expansions of the 5D fermions are represented as

$$Q(x, y) = \begin{pmatrix} U(x, y) \\ D(x, y) \end{pmatrix} = \sum_{i=1}^3 \begin{pmatrix} u_{iL}^{(0)}(x) \\ d_{iL}^{(0)}(x) \end{pmatrix} g_{q_{iL}}^{(0)}(y) + (\text{massive modes}), \quad (5.2.12)$$

$$\mathcal{U}(x, y) = \sum_{i=1}^3 u_{iR}^{(0)}(x) f_{u_{iR}}^{(0)}(y) + (\text{massive modes}), \quad (5.2.13)$$

$$\mathcal{D}(x, y) = \sum_{i=1}^3 d_{iR}^{(0)}(x) f_{d_{iR}}^{(0)}(y) + (\text{massive modes}), \quad (5.2.14)$$

$$L(x, y) = \begin{pmatrix} N(x, y) \\ E(x, y) \end{pmatrix} = \sum_{i=1}^3 \begin{pmatrix} \nu_{iL}^{(0)}(x) \\ e_{iL}^{(0)}(x) \end{pmatrix} g_{l_{iL}}^{(0)}(y) + (\text{massive modes}), \quad (5.2.15)$$

$$\mathcal{E}(x, y) = \sum_{i=1}^3 e_{iR}^{(0)}(x) f_{e_{iR}}^{(0)}(y) + (\text{massive modes}), \quad (5.2.16)$$

with the generation indices i ($= 1, 2, 3$). The concrete forms of the zero-mode profiles are easily derived as [41]

$$g_{q_{iL}}^{(0)}(y) = [\theta(y - L_{i-1}^{(Q)}) \theta(L_i^{(Q)} - y)] \cdot \mathcal{N}_i^{(Q)} e^{+M_Q(y - L_{i-1}^{(Q)})}, \quad (5.2.17)$$

$$f_{u_{iR}}^{(0)}(y) = [\theta(y - L_{i-1}^{(\mathcal{U})}) \theta(L_i^{(\mathcal{U})} - y)] \cdot \mathcal{N}_i^{(\mathcal{U})} e^{-M_{\mathcal{U}}(y - L_{i-1}^{(\mathcal{U})})}, \quad (5.2.18)$$

$$f_{d_{iR}}^{(0)}(y) = [\theta(y - L_{i-1}^{(\mathcal{D})}) \theta(L_i^{(\mathcal{D})} - y)] \cdot \mathcal{N}_i^{(\mathcal{D})} e^{-M_{\mathcal{D}}(y - L_{i-1}^{(\mathcal{D})})}, \quad (5.2.19)$$

$$g_{l_{iL}}^{(0)}(y) = [\theta(y - L_{i-1}^{(L)}) \theta(L_i^{(L)} - y)] \cdot \mathcal{N}_i^{(L)} e^{+M_L(y - L_{i-1}^{(L)})}, \quad (5.2.20)$$

$$f_{e_{iR}}^{(0)}(y) = [\theta(y - L_{i-1}^{(\mathcal{E})}) \theta(L_i^{(\mathcal{E})} - y)] \cdot \mathcal{N}_i^{(\mathcal{E})} e^{-M_{\mathcal{E}}(y - L_{i-1}^{(\mathcal{E})})}, \quad (i = 1, 2, 3) \quad (5.2.21)$$

where $\theta(y)$ denotes the Heaviside step function and the individual kinetic normalization constants are expressed

$$\begin{aligned} \mathcal{N}_i^{(Q)} &= \sqrt{\frac{2M_Q}{e^{2M_Q\Delta L_i^{(Q)}} - 1}}, & \mathcal{N}_i^{(\mathcal{U})} &= \sqrt{\frac{2M_{\mathcal{U}}}{1 - e^{-2M_{\mathcal{U}}\Delta L_i^{(\mathcal{U})}}}}, & \mathcal{N}_i^{(\mathcal{D})} &= \sqrt{\frac{2M_{\mathcal{D}}}{1 - e^{-2M_{\mathcal{D}}\Delta L_i^{(\mathcal{D})}}}}, \\ \mathcal{N}_i^{(L)} &= \sqrt{\frac{2M_L}{e^{2M_L\Delta L_i^{(L)}} - 1}}, & \mathcal{N}_i^{(\mathcal{E})} &= \sqrt{\frac{2M_{\mathcal{E}}}{1 - e^{-2M_{\mathcal{E}}\Delta L_i^{(\mathcal{E})}}}}, & (i = 1, 2, 3) \end{aligned} \quad (5.2.22)$$

with the lengths of segments

$$\Delta L_i^{(\Psi)} \equiv L_i^{(\Psi)} - L_{i-1}^{(\Psi)} \quad (\text{for } i = 1, 2, 3; \Psi = Q, \mathcal{U}, \mathcal{D}, L, \mathcal{E}). \quad (5.2.23)$$

After the integration along the y direction, we obtain three-by-three mass matrices, *e.g.*, for the up-type quarks as

$$S_{\text{up-type quark mass}} = - \int d^4x \sum_{i,j=1}^3 \bar{u}_{iL}^{(0)} M_{ij}^{(u)} u_{jR}^{(0)} + (\text{h.c.}). \quad (5.2.24)$$

It should be emphasized that the ordering of the positions of point interactions governs the form of the mass matrices. For example, when the following ordering is realized,

$$0 < L_1^{(\mathcal{U})} < L_1^{(Q)} < L_2^{(\mathcal{U})} < L_2^{(Q)} < L, \quad (5.2.25)$$

the corresponding concrete forms of the elements of the up-type quark mass matrix are taken

$$M^{(u)} = \begin{bmatrix} M_{11}^{(u)} & M_{12}^{(u)} & 0 \\ 0 & M_{22}^{(u)} & M_{23}^{(u)} \\ 0 & 0 & M_{33}^{(u)} \end{bmatrix}, \quad (5.2.26)$$

with

$$M_{11}^{(u)} = \frac{y_u v}{\sqrt{2}} \int_0^{L_1^{(\mathcal{U})}} dy \langle \Phi(y) \rangle g_{q_{1L}}^{(0)}(y) f_{u_{1R}}^{(0)}(y), \quad (5.2.27)$$

$$M_{22}^{(u)} = \frac{y_u v}{\sqrt{2}} \int_{L_1^{(Q)}}^{L_2^{(\mathcal{U})}} dy \langle \Phi(y) \rangle g_{q_{2L}}^{(0)}(y) f_{u_{2R}}^{(0)}(y), \quad (5.2.28)$$

$$M_{33}^{(u)} = \frac{y_u v}{\sqrt{2}} \int_{L_2^{(Q)}}^{L_3^{(\mathcal{U})}} dy \langle \Phi(y) \rangle g_{q_{3L}}^{(0)}(y) f_{u_{3R}}^{(0)}(y), \quad (5.2.29)$$

$$M_{12}^{(u)} = \frac{y_u v}{\sqrt{2}} \int_{L_1^{(\mathcal{U})}}^{L_1^{(Q)}} dy \langle \Phi(y) \rangle g_{q_{1L}}^{(0)}(y) f_{u_{2R}}^{(0)}(y), \quad (5.2.30)$$

$$M_{23}^{(u)} = \frac{y_u v}{\sqrt{2}} \int_{L_2^{(\mathcal{U})}}^{L_2^{(Q)}} dy \langle \Phi(y) \rangle g_{q_{2L}}^{(0)}(y) f_{u_{3R}}^{(0)}(y), \quad (5.2.31)$$

where we assumed the form for the Higgs doublet, $\langle H \rangle = (0, v/\sqrt{2})^T$, where v is the 5D Higgs VEV and can be treated as real without loss of generality as in the SM. If the ordering is different from Eq. (5.2.25), we can easily obtain the corresponding forms by looking at the profiles of fermion mode functions. See [41] (also [42, 43]) for more details.

5.3 Dynamics in the scenario and their implications

We first mention that, in [41], the observed quark mass hierarchy and the three mixing angles of the CKM matrix were successfully reproduced through the current strategy, under the following preconditions:

- All of the positions of the point interactions can be treated as individual free parameters.
- The form of the $\langle\Phi(y)\rangle$ is exponential as $\langle\Phi(y)\rangle \sim e^{M_{\Phi}y}$ (where the mass scale is not shown), whose form can be materialized by setting the parameters associated with Φ appropriately in the actual form of $\langle\Phi(y)\rangle$ (in Case (II) in subsection 5.3.2) [41].

These preconditions would be justified for phenomenological studies. However, the following criticisms will come in various points of view:

- From stability of the system:** the positions of the point interactions contribute to the background Casimir energy of this extra-dimensional system, which should be minimized (or extremized at least) to ensure the stability of the whole system. If one obeys this line, the positions of the point interactions cannot take arbitrary values.
- From the number of free parameters:** for quarks, in addition to the six parameters as the positions of point interactions for quark fields $\{L_i^{(Q)}, L_i^{(U)}, L_i^{(D)}; i = 1, 2\}$, the following parameters join to describe the quark profiles naively: one as the length of the system L , two as the overall Yukawa couplings $\{\mathcal{Y}_u, \mathcal{Y}_d\}$, three as the bulk masses $\{M_Q, M_U, M_D\}$, one as the 5D Higgs VEV v , and four to the gauge-singlet scalar VEV $\langle\Phi(y)\rangle$ (where details are provided in subsection 5.3.2); there are 15 free parameters in total (for the lepton sector, where apparently seven additional ones $\{L_1^{(L)}, L_2^{(L)}, L_1^{(E)}, L_2^{(E)}, \mathcal{Y}_e, M_L, M_E\}$ join when we take care of charged leptons).³ On the contrary, only the nine digits (six for mass eigenvalues, three for CKM mixing angles) are fitted. Such an unbalanced situation may lead to the criticism that the result of the previous fit is not surprising, looks trivial.
- From generality in the profile of $\langle\Phi(y)\rangle$:** As partially discussed in Section 4 of [41], the actual form of $\langle\Phi(y)\rangle$ is provided by the Jacobi's elliptic function. As we will see in subsection 5.3.2, there are two solutions for the VEV $\langle\Phi(y)\rangle$, which may give new insights into the geometric realization of the observed Yukawa texture in the 'generalized' setup.

Based on the result of [103] and the general information of $\langle\Phi(y)\rangle$, we can provide our answers for all of the criticisms, at least partially.

5.3.1 Criterion via stabilization of point interactions

The determination of the positions of the point interactions $L_i^{(\Psi)}$ by the minimization of the Casimir energy has been developed in [103]. The Casimir energy for the case of the interval-extra dimension without point interactions is discussed in Appendix B. For the case with point interactions, the Casimir energy is given by adding the contributions from each segment. Then, the Casimir energy E at one-loop order as the function of $L_i^{(\Psi)}$ takes to be the form of

$$E = 2 \sum_{\Psi=Q,L} E^{(\Psi)}[L_1^{(\Psi)}, L_2^{(\Psi)}] + \sum_{\Psi'=U,D,E} E^{(\Psi')}[L_1^{(\Psi')}, L_2^{(\Psi')}], \quad (5.3.1)$$

³As we will see in Section 5.4.1, this counting of parameters is naive a bit. The number of the degrees of freedom relevant for mass ratios is less than the digit.

where

$$E^{(\Psi)}[L_1^{(\Psi)}, L_2^{(\Psi)}] = \sum_{i=1}^3 \frac{|M^{(\Psi)}|^2}{8\pi^2(L_i^{(\Psi)} - L_{i-1}^{(\Psi)})^2} \sum_{w=1}^{\infty} \frac{e^{-2w|M^{(\Psi)}|(L_i^{(\Psi)} - L_{i-1}^{(\Psi)})}}{w^3} \times \left(1 + \frac{3}{2w|M^{(\Psi)}|(L_i^{(\Psi)} - L_{i-1}^{(\Psi)})} + \frac{3}{4w^2|M^{(\Psi)}|^2(L_i^{(\Psi)} - L_{i-1}^{(\Psi)})^2} \right). \quad (5.3.2)$$

The minimization of the above Casimir energy leads to the results in which the positions of the point interactions for the doublet fermions and the singlet fermions are the same as ⁴

$$\begin{aligned} L_1^{(\mathcal{Q})} &= L_1^{(\mathcal{U})} = L_1^{(\mathcal{D})} = L_1^{(L)} = L_1^{(\mathcal{E})}, \\ L_2^{(\mathcal{Q})} &= L_2^{(\mathcal{U})} = L_2^{(\mathcal{D})} = L_2^{(L)} = L_2^{(\mathcal{E})}, \end{aligned} \quad (5.3.3)$$

and all of the point interactions should be located at

$$L_1^{(\Psi)} = \frac{1}{3}L, \quad L_2^{(\Psi)} = \frac{2}{3}L, \quad (5.3.4)$$

to minimize the Casimir energy in spite of the type of the field Ψ . More concretely,

$$\begin{aligned} L_1^{(\mathcal{Q})} &= L_1^{(\mathcal{U})} = L_1^{(\mathcal{D})} = L_1^{(L)} = L_1^{(\mathcal{E})} = \frac{1}{3}L, \\ L_2^{(\mathcal{Q})} &= L_2^{(\mathcal{U})} = L_2^{(\mathcal{D})} = L_2^{(L)} = L_2^{(\mathcal{E})} = \frac{2}{3}L. \end{aligned} \quad (5.3.5)$$

It is noted that (5.3.5) implies no mixing terms in mass matrices. Taking account of this result in parameter fitting would definitely become a reasonable response (despite of being not complete) to the above criticisms (a) and (b), where the naive counting of the input parameters for quarks reduced to 11, from 17.

5.3.2 Criterion via the general solution of $\langle \Phi(y) \rangle$

To know the actual solution of Φ means to solve the following nonlinear equation derived through the variational principle from S_{singlet} in Eq. (5.2.5),

$$\frac{d^2\phi(y)}{dy^2} - M_\Phi^2\phi(y) - \lambda_\Phi\phi^3(y) = 0, \quad (5.3.6)$$

where we put the ansatz that the VEV of Φ depends on the extra dimensional coordinate y , *i.e.*

$$\langle \Phi \rangle = \phi(y), \quad (5.3.7)$$

where in general the following form comes $\langle \Phi \rangle = e^{i\theta(y)}\phi(y)$, but we have proven that without loss of generality we can set the y -dependent phase part $\theta(y)$ to be zero, where $\phi(y)$ is real. The above equation leads to

$$\frac{d}{dy} \left[\frac{1}{2} \left(\frac{d\phi(y)}{dy} \right)^2 - \frac{M_\Phi^2}{2} \phi^2(y) - \frac{\lambda_\Phi}{4} \phi^4(y) \right] = 0, \quad (5.3.8)$$

⁴In the case of the discussion about the total length L by the Casimir energy, the other fields contributions, *e.g.* gauge fields and the scalar fields, should be included to the minimization condition (see the discussion in [103]). On the other hand, in the case of the discussion about the positions of the point interactions, only the related fermion's contributions are important.

and can be deformed to

$$\frac{1}{2} \left(\frac{d\phi(y)}{dy} \right)^2 + U(\phi) = E_\Phi, \quad \left(U(\phi) \equiv -\frac{M_\Phi^2}{2} \phi^2(y) - \frac{\lambda_\Phi}{4} \phi^4(y) \right), \quad (5.3.9)$$

with an undetermined integration constant E_Φ as a parameter. Here, we should take account of the boundary conditions of $\phi(y)$ at $y = 0$ and $y = L$, where we impose the Robin boundary conditions, which are described with the length parameters $L_\pm$ ($-\infty \leq L_\pm \leq +\infty$) as below⁵

$$\begin{aligned} \phi(0) + L_+ \phi'(0) &= 0, \\ \phi(L) - L_- \phi'(L) &= 0, \end{aligned} \quad (5.3.10)$$

where the prime symbol represents the derivative of y (refer to [86]).

Through elliptic integrals, at least the following *two* solutions are possible in terms of Jacobi's elliptic functions:

(I) When $0 < E_\Phi < M_\Phi^4/(4\lambda_\Phi)$:

$$\phi(y) = \mu_- \frac{\text{sn}\left(\mu_+ \sqrt{\frac{\lambda_\Phi}{2}}(y - y_0), k\right)}{\text{cn}\left(\mu_+ \sqrt{\frac{\lambda_\Phi}{2}}(y - y_0), k\right)}, \quad (5.3.11)$$

where

$$\mu_\pm^2 \equiv \frac{M_\Phi^2}{\lambda_\Phi} \left(1 \pm \sqrt{1 - \frac{4\lambda_\Phi E_\Phi}{M_\Phi^4}} \right), \quad k^2 \equiv \frac{\mu_+^2 - \mu_-^2}{\mu_+^2}. \quad (5.3.12)$$

(II) When $E_\Phi < 0$:

$$\phi(y) = \frac{\nu}{\text{cn}\left(\frac{\mu}{k} \sqrt{\frac{\lambda_\Phi}{2}}(y - y_0), k\right)}, \quad (5.3.13)$$

where

$$\mu^2 \equiv \frac{M_\Phi^2}{\lambda_\Phi} \left(1 + \sqrt{1 - \frac{4\lambda_\Phi E_\Phi}{M_\Phi^4}} \right), \quad \nu^2 \equiv \frac{M_\Phi^2}{\lambda_\Phi} \left(\sqrt{1 - \frac{4\lambda_\Phi E_\Phi}{M_\Phi^4}} - 1 \right), \quad k^2 \equiv \frac{\mu^2}{\mu^2 + \nu^2}. \quad (5.3.14)$$

We point out that the variables $L_\pm$ in Eq. (5.3.10) to parametrize the Robin boundary conditions are automatically fixed by the above profiles. y_0 represents the degree of freedom of the translation along the y direction. The parameter k measures ellipticity defined in the range of $0 \leq k \leq 1$,⁶ where the two extremal cases $k = 0$ and 1 correspond to

$$\begin{aligned} \frac{\text{sn}(x, k)}{\text{cn}(x, k)} &= \text{sc}(x, k) \rightarrow \begin{cases} \tan x & \text{when } k \rightarrow 0, \\ \sinh x & \text{when } k \rightarrow 1, \end{cases} \\ \frac{1}{\text{cn}(x, k)} &\rightarrow \begin{cases} 1/\cos x & \text{when } k \rightarrow 0, \\ \cosh x & \text{when } k \rightarrow 1. \end{cases} \end{aligned} \quad (5.3.15)$$

⁵We note that the scalar field Φ is assumed not to feel point interactions.

⁶Actually, the limited region $1/\sqrt{2} \leq k \leq 1$ can be taken both in the cases of (I) and (II).

When $k \neq 1$, a very nontrivial feature of the two elliptic functions $\text{sc}(x, k)$ and $1/\text{cn}(x, k)$ rises up, where the functions get divergent periodically along the y -direction. The minimal periods between the nearest divergent points of $\phi(y)$ in Case (I) and Case (II) are

$$2 \left(\mu_+ \sqrt{\frac{\lambda_\Phi}{2}} \right)^{-1} K[k], \quad 2 \left(\frac{\mu}{k} \sqrt{\frac{\lambda_\Phi}{2}} \right)^{-1} K[k],$$

respectively with the complete elliptic integral of the first kind $K[k]$. A possible important difference between Case (I) and Case (II) is that $\text{sc}(x, k)$ increases monotonically from a divergent point x_d ($\text{sc}(x_d + \varepsilon, k) \sim -\infty$) to the next divergent point x'_d ($> x_d$) ($\text{sc}(x'_d - \varepsilon, k) \sim +\infty$), where the value of $\text{sc}(x, k)$ is negative in the first half part of the region $x_d < x < (x_d + x'_d)/2$. This property would be useful to realize gigantic hierarchy in the overlap integrals that express the components of the mass matrices (see Eqs.(5.2.27)–(5.2.31)) since negative contributions get to be possible. This motivates us to have a discussion for exploring the situation with the exact form of $\phi(y)$ in Case (I) in contrast with the prior case (II) researches. This will be the response to the criticism (c) shown above.

5.4 Numerical analysis on an interval

Following the discussions in the previous section, we perform a numerical parameter search in the current model on an interval. The differences in method between the analyses on this manuscript and in the prior researches [41–43] are summarized:

- The relative distances of the point interactions are fixed by the minimization of the Casimir energy [103] (see Eq. (5.3.5)), while previously the positions were treated as completely free parameters [41–43].
- Due to the minimization of the Casimir energy shown in Eq. (5.3.3), currently we cannot discuss the flavor mixing in a consistent way. Here, we take the strategy where at first we focus on only mass hierarchies, which may be justified as the first step to know how the geometry constrained by the Casimir energy works well to reproduce the observed magnitudes of the SM mass hierarchy, of course up to the inevitable distortion by mixings. This attitude will be reasonable especially for the quark sector since the observed mixing angles are small. On the other hand, we touch effects from mixings by introducing off-diagonal terms by hand phenomenologically for the charged leptons.
- The exact form of the y -dependent VEV of the Case (I) is adopted for the singlet Φ , while an approximate form of the Case (II) solution is used in the prior researches.

Now, the quark and lepton mass matrices are diagonal, where the elements of the matrices are formulated for ψ ($= u, d, e$) as

$$M_{ii}^{(\psi)} = \widetilde{\mathcal{Y}}_\psi \cdot \frac{v_H}{\sqrt{2}} \sqrt{\frac{\widetilde{M}_\Phi^2}{\lambda_\Phi}} (1 - X)^{1/2} \times \int_{\widetilde{L}_{i-1}}^{\widetilde{L}_i} d\widetilde{y} \, \text{sc} \left(\sqrt{\frac{\widetilde{M}_\Phi^2 (1 + X)}{2}} (\widetilde{y} - \widetilde{y}_0), \sqrt{\frac{2X}{1 + X}} \right) \widetilde{g}_{\psi_{iL}}^{(0)}(\widetilde{y}) \widetilde{f}_{\psi_{iR}}^{(0)}(\widetilde{y}) \quad (i = 1, 2, 3), \quad (5.4.1)$$

where

$$X \equiv \sqrt{1 - \frac{4\widetilde{E}_\Phi \widetilde{\lambda}_\Phi}{\widetilde{M}_\Phi^4}}, \quad (5.4.2)$$

and the formula is factorized as the product of the flavor independent and dependent parts, and here all of the parameters are scaled to the dimensionless ones by the total length of the system L , *e.g.*, $M_\Phi = \widetilde{M}_\Phi \cdot L^{-1}$ except for the 4D Higgs VEV $v_H (= 246 \text{ GeV})$ defined as $v_H \equiv v \sqrt{L}$ from the 5D Higgs VEV v . We adopted the notation that variables accompanying the tilde symbol are dimensionless, where the normalization part of the fermion wave functions are made dimensionless following the instruction. The uniformly located positions of the point interactions are represented as (*c.f.* Eq. (5.3.5))

$$\widetilde{L}_0 = 0, \quad \widetilde{L}_1 = \frac{1}{3}, \quad \widetilde{L}_2 = \frac{2}{3}, \quad \widetilde{L}_3 = 1. \quad (5.4.3)$$

We comment that the form in Eq. (5.4.1) does not depend on L itself since it describes phenomena of zero modes. The independent parameters for describing the deformation of the flavor dependent part via Φ are taken as X , $\widetilde{M}_\Phi$, and $\widetilde{y}_0$. We remind that the five bulk masses of the fermions $\{\widetilde{M}_Q, \widetilde{M}_U, \widetilde{M}_D, \widetilde{M}_L, \widetilde{M}_E\}$ and the three overall Yukawa couplings $\{\widetilde{\mathcal{Y}}_u, \widetilde{\mathcal{Y}}_d, \widetilde{\mathcal{Y}}_e\}$ also contribute to the flavor dependent part, where the 10 parameters in Eqs. (5.4.4) and (5.4.5) are relevant when we debate with the flavor dependence up to the overall scale (where one of the Yukawa couplings is included to the overall scale). The fit observables are the eight mass ratios of the nine SM fermion masses like $m_{\text{others}}/m_{\text{top}}$. We should mention that as was pointed out in the Ref. [42], more parameters than the experimental values of the quark/lepton masses do not mean that we can always reproduce the quark and lepton masses in the present scenario. The geometry of the extra dimension tightly restrict the form of the mass matrices to four-zero texture, see *e.g.*, Eq. (5.2.26). Therefore, it is quite nontrivial whether our model can reproduce the quark and lepton masses even though two extra parameters (out of 10 ones for fitting the eight observables) seem to remain.

5.4.1 Results for the numerical research

We found that, even in the current limited setup, when we take the following benchmark,⁷

$$\begin{aligned} \widetilde{M}_\Phi &= 4.3, & \widetilde{y}_0 &= 0.33, & X &= 0.74, \\ \widetilde{M}_Q &= 40, & \widetilde{M}_U &= 320, & \widetilde{M}_D &= 0.1, & \widetilde{M}_L &= 100, & \widetilde{M}_E &= 0.01, \end{aligned} \quad (5.4.4)$$

and take the degrees of freedom of the Yukawa couplings to adjust the third-generation fermions as

$$\frac{\widetilde{\mathcal{Y}}_d}{\widetilde{\mathcal{Y}}_u} \simeq 0.124, \quad \frac{\widetilde{\mathcal{Y}}_e}{\widetilde{\mathcal{Y}}_u} \simeq 0.0454, \quad (5.4.5)$$

⁷Here, the nearest divergent point of the Jacobi's elliptic function is located at $\widetilde{y} \simeq 1.00249$, which is outside the region where the interval geometry is elaborated.

the result comes

$$\begin{aligned}
 \frac{M_{11}^{(u)}}{m_{\text{up}}} &\simeq 0.746, & \frac{M_{22}^{(u)}}{m_{\text{charm}}} &\simeq 1.08, & \frac{M_{33}^{(u)}}{m_{\text{top}}} &= 1, \\
 \frac{M_{11}^{(d)}}{m_{\text{down}}} &\simeq -1.07, & \frac{M_{22}^{(d)}}{m_{\text{strange}}} &\simeq 0.966, & \frac{M_{33}^{(d)}}{m_{\text{bottom}}} &= 1, \\
 \frac{M_{11}^{(e)}}{m_{\text{electron}}} &\simeq -0.775, & \frac{M_{22}^{(e)}}{m_{\text{muon}}} &\simeq 0.253, & \frac{M_{33}^{(e)}}{m_{\text{tauon}}} &= 1,
 \end{aligned} \tag{5.4.6}$$

where we compare the obtained result with the derived pole masses at the leading order from the PDG digits [122].⁸ We reached a good fit for quarks as ratios,⁹ while the fit of the charged lepton is quite far from being precise. We do not pay attention to the overall flavor independent part since we may adjust this part by use of $\tilde{\mathcal{Y}}_u$ and $\tilde{\lambda}_\Phi$. The negativity for $M_{11}^{(d)}$, $M_{11}^{(e)}$ originates from the lower tail part of the sc function, where now the absolute values of them physically make sense.

5.4.2 An extra phenomenological study of the mixing effect in charged leptons

A possible origin of the deviated result in the charged leptons may be that no suitably-enough relevant degree of freedom remains for charged leptons after we fix the parameters to reproduce the observed mass hierarchy of quarks under the constraints in Eq. (5.4.3).

To look into possible improvements of the charged lepton part, we do a phenomenological trial for the charged lepton part, which seems to have a difficulty to regenerate the observed patterns of their mass eigenvalues. Now we focus on the speculation that the charged leptons can be mixed largely, because the Pontecorvo–Maki–Nakagawa–Sakata (PMNS) matrix which describes the mixings among the left-handed charged leptons and active neutrinos contains large mixing angles. Under such circumstances mass eigenvalues can alter remarkably by the introduction of sizable non-diagonal terms of mass matrix. The existence of non-diagonal elements might be realized when we evaluate the Casimir energy at two-loop order or introduce exotic fermions, which have no chiral zero modes and only affect to the Casimir energy. Therefore, to investigate such a case with phenomenologically introduced terms by hand would be fruitful to know further possibilities of the current scenario.

In what follows, we bring the following phenomenological form for a charged-lepton mass matrix into focus:

$$\mathcal{M}^{(e)} = \begin{bmatrix} M_{11}^{(e)} & m_{12}^{(e)} & 0 \\ 0 & M_{22}^{(e)} & m_{23}^{(e)} \\ m_{31}^{(e)} & 0 & M_{33}^{(e)} \end{bmatrix}, \tag{5.4.7}$$

⁸Concrete digits of the central values are as follows: $m_{\text{up}} = 2.5 \text{ MeV}$, $m_{\text{down}} = 5.2 \text{ MeV}$, $m_{\text{strange}} = 110 \text{ MeV}$, $m_{\text{charm}} = 1.448 \text{ GeV}$, $m_{\text{bottom}} = 4.557 \text{ GeV}$, $m_{\text{top}} = 173.0 \text{ GeV}$, $m_{\text{electron}} = 0.5109989461 \text{ MeV}$, $m_{\text{muon}} = 105.6583745 \text{ MeV}$, $m_{\text{tauon}} = 1.77686 \text{ GeV}$, respectively.

⁹On the other hand, these results are sensitive to the modulation of the input parameters in sub-leading magnitudes. For example, if we change $\tilde{M}_\Phi$ from 4.3 to 4.31, the accurate fit for quarks turns out to be disrupted as $M_{11}^{(u)}/m_{\text{up}} \simeq 0.416$, $M_{22}^{(u)}/m_{\text{charm}} \simeq 0.605$, $M_{11}^{(d)}/m_{\text{down}} \simeq -0.761$, $M_{22}^{(d)}/m_{\text{strange}} \simeq 0.687$, $M_{11}^{(e)}/m_{\text{electron}} \simeq -0.504$, $M_{22}^{(e)}/m_{\text{muon}} \simeq 0.165$, respectively, where the other ratios are still unities after suitable adjustments of $\mathcal{Y}_d/\mathcal{Y}_u$ and $\mathcal{Y}_e/\mathcal{Y}_u$. The position of the nearest divergent point of the elliptic function becomes $\tilde{y}_0 \simeq 1.00093$. This point is considered as an inevitable difficulty of the present scenario, where a suitable tuning of inputs is required.

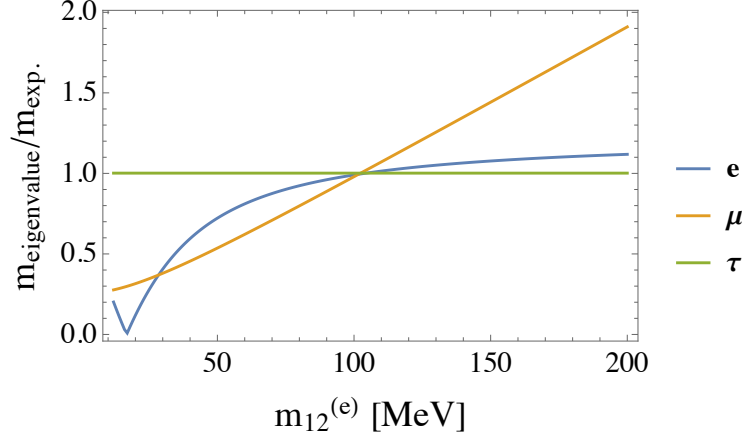


Figure 5.3: Shifts of mass eigenvalues of the phenomenological charged-lepton mass matrix $\mathcal{M}^{(e)}$ as functions of $m_{12}^{(e)}$, where $m_{23}^{(e)}$ and $m_{31}^{(e)}$ are pinned down as 100 MeV and 11.2 MeV, respectively. The three eigenvalues get to be very close to the measured digits when $m_{12}^{(e)} \simeq 104$ MeV simultaneously.

where we assume that the diagonal inputs take the same digits as in Eq. (5.4.6) without rounding, and $m_{12}^{(e)}$, $m_{23}^{(e)}$ and $m_{31}^{(e)}$ are phenomenological parameters.¹⁰ We found a desirable configuration to reproduce the measurements, where $m_{12}^{(e)}$, $m_{23}^{(e)}$ and $m_{31}^{(e)}$ are chosen as 104 MeV, 100 MeV and 11.2 MeV, respectively (see Figure 5.3). Here, we achieved the good accuracy: $(m_{\text{eigen}}/m_{\text{exp}})_{\text{electron}} \simeq 1.00052$, $(m_{\text{eigen}}/m_{\text{exp}})_{\text{muon}} \simeq 1.01612$ and $(m_{\text{eigen}}/m_{\text{exp}})_{\text{tauon}} \simeq 1.0016$. The current scope of this issue is just within a naive discussion founded on phenomenological assumptions so that it would be worth to investigate the full situation grounded on higher-loop calculations or modification of the matter contents.

5.5 Summary and discussion

In this chapter, we have performed the numerical research in our dynamical model to reproduce the quark/lepton mass hierarchy with the experimental values of the quark/lepton masses. Our model consists of 10 point interactions, positions of which were treated as free parameters in the prior researches [41–43]. Their positions have been determined by the minimization condition of the Casimir energy. Because of that, the parameters relevant for fixing the ratios of the fermion masses have been reduced to 10 from 20 in our model and only the remnant 10 parameters are tools for reemergence of the eight experimental values of the quark and lepton mass ratios. As was pointed out in the Ref. [42], more parameters than the experimental values of the quark/lepton masses do not mean that we can always reproduce the quark and lepton masses in our model. The geometry of the extra dimension tightly restricts the form of the mass matrices to four-zero texture, see Eq. (5.2.26). Therefore, it is quite nontrivial whether our model can reproduce the quark and lepton masses.

To obtain the desired mass hierarchy, we introduced the extra-dimension coordinate-dependent VEV of the gauge singlet scalar $\langle \Phi(y) \rangle$, which is also obtained dynamically by minimizing the potential of the scalar Φ (refer to Eq. (5.2.5)). As a result of our numerical analysis, we found that there is a parameter set, which can reproduce the experimental values of the quark well, while some

¹⁰It is pointed out that no $1 \leftrightarrow 3$ mixing term can be realized on an interval. But it is easy to activate such mixing term after we switch the geometry to S^1 (see [42]).

difficulties remain for the charged leptons. On the other hand, we pointed out that the mixing terms among the charged leptons will resolve remaining discrepancies, even though it is not justified fully in the theoretical point of view within the current scope.

A definite next step is to contemplate how to realize non-diagonal terms in the quarks and the charged leptons, which are necessary ingredients to generate the observed textures of the mixings represented by the CKM and PMNS matrices. The calculation of the Casimir energy was achieved at one-loop order, and consequently the positions of the point interactions are determined as to divide the interval extra dimension equally among three, *i.e.* the mass matrices of quarks and leptons automatically become diagonal ones. If off-diagonal components appear after introducing a suitable mechanism, the eigenvalues of the mass matrices deviate from the diagonal components of the mass matrices and the experimental values might be recovered with flavor mixings, as is discussed in subsection 5.4.2.

One possible idea to generate off-diagonal components in the mass matrices is to introduce exotic fermions. Boundary conditions for the exotic fermions should be chosen to produce no massless chiral zero modes, and then they will contribute only to the Casimir energy at low energies. A possible boundary condition for the exotic fermion $\Psi^{\text{ex}}(x, y)$ is given by

$$\Psi_L^{\text{ex}}(x, y) = 0 \quad \text{at} \quad y = 0, L_1 + \varepsilon, \quad \Psi_R^{\text{ex}}(x, y) = 0 \quad \text{at} \quad y = L_1 - \varepsilon, L. \quad (5.5.1)$$

It is noted that the exotic fermion feels the point interaction only at $y = L_1$ but not the one at $y = L_2$. The boundary condition (5.5.1) can prohibit the presence of chiral zero modes [86], and will force the point interaction at $y = L_1$ to move from $L_1 = L/3$ to $L_1 > L/3$. This is because the exotic fermion contributes to the position $y = L_1$ of the point interaction as to divide the interval extra dimension equally among two. After combining the contributions of the exotic fermion with the (5D) SM fermions, the minimization condition of the Casimir energy is modified and the off-diagonal component M_{12} (and probably other off-diagonal components) can appear due to $L_1 > L/3$. It should be mentioned that there is a way to obtain $L_1 < L/3$ by extending the mechanism. We can introduce extra three point interactions only for the exotic fermion and impose the boundary conditions at $y = L'_2, L'_3, L'_4$ in addition to $y = L_1$. In this case, the exotic fermion may contribute to the position $y = L_1$ of the point interaction as to divide the interval extra dimension equally among four, in which $L_1 < L/3$ can be realized from the modified minimization condition of the Casimir energy. The key ingredient is that this mechanism accomplishes the appearance of off-diagonal components dynamically and the contribution of the exotic fermion might be the same order as the (5D) SM fermions, so that large off-diagonal components could emerge. If bulk masses of some exotic fermions are chosen to be much larger than those of the (5D) SM fermions, their effects on the Casimir energy are expected to be limited and will give small contribution to off-diagonal components of the mass matrices. Therefore we can expect that both of the small/large mixing in the quark/lepton sector might be obtained dynamically by use of this mechanism.

Another idea is to take account of higher-order effects of the Casimir energy. Since the flavor structures of the quarks and leptons are nontrivial, the effects of the matter contents at two-loop order might produce corrections to the minimization conditions of the Casimir energy. That may lead to off-diagonal components of the mass matrices through the modulations of the positions of the point interactions. Origin of modulations may also be the (zero-mode)-(KK-mode) mixing which can make small mass perturbations. The issues stated above remain to be pursued as the future work.

Chapter 6

Extended supersymmetry in extra dimensional models

In Chapter 3, it has been shown that the structure of the $\mathcal{N} = 2$ SUSY QM is hidden in the 4D mass spectrum of the higher dimensional Dirac action with extra dimensions, and the KK mode functions for the 4D right-handed and left-handed spinors form the supermultiplets. Since a higher dimensional Dirac spinor can be decomposed into 4D Dirac spinors with many “flavors” in the Kaluza–Klein (KK) decomposition, we expect that some symmetries larger than the $\mathcal{N} = 2$ SUSY QM will be hidden in the 4D mass spectrum of higher dimensional Dirac actions.

Then, in this chapter, we discuss the $\mathcal{N}$ -extended SUSY QM with central charges which are obtained from symmetries in the higher dimensional Dirac action with extra dimensions. We show that central charges appear from those symmetries and the supermultiplets in this SUSY algebra become the BPS states. As an example, we will confirm that this $\mathcal{N}$ -extended SUSY QM is realized in the S^2 -extra dimension with the magnetic monopole background and explains the degeneracy of the 4D mass spectrum.

This chapter is organized as follows: In Section 6.1, we construct the $\mathcal{N}$ -extended SUSY QM with central charges from the symmetries in extra dimensions and discuss the representation of this SUSY algebra. Then, in Section 6.2, we confirm that this $\mathcal{N}$ -extended SUSY QM can be realized in the model of S^2 -extra dimension with the magnetic monopole background, and the KK mode functions correspond to the representation given in Section 6.1. The Section 6.3 is devoted to summary and discussion.

6.1 Structure of extended supersymmetry

In Chapter 3, we have described the $\mathcal{N} = 2$ SUSY QM hidden in the two-fold degeneracy of $f_\alpha^{(n)}$ and $g_\alpha^{(n)}(y)$. However, we can expect that further hidden structures exist in the 4D mass spectrum and these would lead to the extra degeneracy due to the index α .

In this section, we show that the $\mathcal{N}$ -extended SUSY QM with central charges can be constructed from symmetries in the extra dimensions. This SUSY QM can explain the extra degeneracy in the 4D mass spectrum. Then, we clarify the representation of this algebra for the nonzero energy states and it will turn out that the eigenstates become BPS states. This section is devoted to the general discussion, and a concrete example will be given in the next section.

6.1.1 $\mathcal{N}$ -extended SUSY algebra with central charges

In Chapter 3, we have obtained the structure of $\mathcal{N} = 2$ SUSY QM

$$\begin{aligned} Q \begin{pmatrix} f_\alpha^{(n)}(y) \\ 0 \end{pmatrix} &= m_n \begin{pmatrix} 0 \\ g_\alpha^{(n)}(y) \end{pmatrix}, \quad H \begin{pmatrix} f_\alpha^{(n)}(y) \\ 0 \end{pmatrix} = m_n^2 \begin{pmatrix} f_\alpha^{(n)}(y) \\ 0 \end{pmatrix}, \quad (-1)^F \begin{pmatrix} f_\alpha^{(n)}(y) \\ 0 \end{pmatrix} = + \begin{pmatrix} f_\alpha^{(n)}(y) \\ 0 \end{pmatrix}, \\ Q \begin{pmatrix} 0 \\ g_\alpha^{(n)}(y) \end{pmatrix} &= m_n \begin{pmatrix} f_\alpha^{(n)}(y) \\ 0 \end{pmatrix}, \quad H \begin{pmatrix} 0 \\ g_\alpha^{(n)}(y) \end{pmatrix} = m_n^2 \begin{pmatrix} 0 \\ g_\alpha^{(n)}(y) \end{pmatrix}, \quad (-1)^F \begin{pmatrix} 0 \\ g_\alpha^{(n)}(y) \end{pmatrix} = - \begin{pmatrix} 0 \\ g_\alpha^{(n)}(y) \end{pmatrix}, \end{aligned} \quad (6.1.1)$$

where the supercharge Q , the Hamiltonian H and the “fermion” number operator $(-1)^F$ are defined by

$$Q = \begin{pmatrix} 0 & \mathcal{A}^\dagger \\ \mathcal{A} & 0 \end{pmatrix} = \begin{pmatrix} 0 & e^\sigma [-i\gamma^{\hat{y}} e_{\hat{y}^y} (\nabla_y + iqA_y) + W] \\ e^\sigma [i\gamma^{\hat{y}} e_{\hat{y}^y} (\nabla_y + iqA_y) + W] & 0 \end{pmatrix}, \quad (6.1.2)$$

$$\begin{aligned} H = Q^2 = e^{2\sigma} &\left[-(\nabla_y + iqA_y)^2 + q\sigma^{\hat{y}\hat{y}'} e_{\hat{y}^y} e_{\hat{y}'^y} F_{yy'} + \frac{1}{4}R - i\gamma^{\hat{y}} e_{\hat{y}^y} (\partial_y W)(-1)^F + W^2 \right. \\ &\left. - i\gamma^{\hat{y}} e_{\hat{y}^y} (\partial_y \sigma) \begin{pmatrix} \mathcal{A} & 0 \\ 0 & -\mathcal{A}^\dagger \end{pmatrix} \right], \end{aligned} \quad (6.1.3)$$

$$(-1)^F = \begin{pmatrix} \mathbf{1}_{2[d/2]} & 0 \\ 0 & -\mathbf{1}_{2[d/2]} \end{pmatrix}. \quad (6.1.4)$$

$F_{yy'}$ is the field strength for the back ground gauge potential A_y and R is the Ricci scalar defined on extra space Ω .

Then, we discuss a new realization of $\mathcal{N}$ -extended SUSY QM from symmetries. First, we consider sets of operators $\{\hat{a}_i (i = 1, 2, \dots, N_a)\}$, $\{\hat{b}_i (i = 1, 2, \dots, N_b)\}$, $\dots$, $\{\hat{\alpha}_i (i = 1, 2, \dots, N_\alpha)\}$, $\{\hat{\beta}_i (i = 1, 2, \dots, N_\beta)\}$, $\dots$, which are Hermitian and consistent with an imposed boundary condition for the mode functions $f_\alpha^{(n)}$,¹ and commute with $\mathcal{A}^\dagger \mathcal{A}$

$$[\hat{a}_i, \mathcal{A}^\dagger \mathcal{A}] = [\hat{b}_i, \mathcal{A}^\dagger \mathcal{A}] = \dots = [\hat{\alpha}_i, \mathcal{A}^\dagger \mathcal{A}] = [\hat{\beta}_i, \mathcal{A}^\dagger \mathcal{A}] = \dots = 0. \quad (6.1.5)$$

Therefore, these operators do not change the mass eigenvalues and will be related to the symmetries in the extra dimensions. Furthermore, we require that these operators commute with the ones in the same sets

$$[\hat{a}_i, \hat{a}_j] = [\hat{b}_i, \hat{b}_j] = \dots = 0, \quad [\hat{\alpha}_i, \hat{\alpha}_j] = [\hat{\beta}_i, \hat{\beta}_j] = \dots = 0, \quad (6.1.6)$$

and anticommute with the ones in the different sets for Roman and Greek letters

$$\{\hat{a}_i, \hat{b}_j\} = \{\hat{a}_i, \hat{c}_j\} = \dots = \{\hat{b}_i, \hat{c}_j\} = \{\hat{b}_i, \hat{d}_j\} = \dots = 0, \quad (6.1.7)$$

$$\{\hat{\alpha}_i, \hat{\beta}_j\} = \{\hat{\alpha}_i, \hat{\gamma}_j\} = \dots = \{\hat{\beta}_i, \hat{\gamma}_j\} = \{\hat{\beta}_i, \hat{\delta}_j\} = \dots = 0, \quad (6.1.8)$$

and the operators with the Roman letters and the ones with the Greek letters commute with each other

$$[\hat{a}_i, \hat{\alpha}_j] = [\hat{a}_i, \hat{\beta}_j] = \dots = [\hat{b}_i, \hat{\alpha}_j] = [\hat{b}_i, \hat{\beta}_j] = \dots = 0. \quad (6.1.9)$$

¹More precisely, we require that the functions $\hat{a}_i f_\alpha^{(n)} (i = 1, 2, \dots, N_a)$, $\hat{b}_i f_\alpha^{(n)} (i = 1, 2, \dots, N_b)$, $\dots$ also satisfy the imposed boundary condition.

Then, we define the following extended supercharges

$$Q_i^{(a)} = \begin{pmatrix} & i\hat{a}_i\mathcal{A}^\dagger \\ -i\mathcal{A}\hat{a}_i & \end{pmatrix}, \quad Q_i^{(b)} = \begin{pmatrix} & i\hat{b}_i\mathcal{A}^\dagger \\ -i\mathcal{A}\hat{b}_i & \end{pmatrix}, \quad \dots, \quad (6.1.10)$$

$$Q_i^{(\alpha)} = \begin{pmatrix} & \hat{\alpha}_i\mathcal{A}^\dagger \\ \mathcal{A}\hat{\alpha}_i & \end{pmatrix}, \quad Q_i^{(\beta)} = \begin{pmatrix} & \hat{\beta}_i\mathcal{A}^\dagger \\ \mathcal{A}\hat{\beta}_i & \end{pmatrix}, \quad \dots, \quad (6.1.11)$$

and obtain $\mathcal{N} = (N_a + N_b + \dots + N_\alpha + N_\beta + \dots)$ SUSY algebra with the central charges

$$\{Q_i^{(A)}, Q_j^{(B)}\} = 2H\delta_{ij}\delta^{AB} + 2Z_{ij}^{(A)}\delta^{AB}, \quad (6.1.12)$$

$$[Q_i^{(A)}, Z_{jk}^{(B)}] = [H, Z_{jk}^{(A)}] = [Z_{ij}^{(A)}, Z_{kl}^{(B)}] = [Q_i^{(A)}, H] = 0 \quad (A, B = a, b, \dots, \alpha, \beta, \dots), \quad (6.1.13)$$

where H denotes the Hamiltonian given by (3.3.4) and the central charges $Z_{ij}^{(A)}$ are defined as

$$Z_{ij}^{(a)} = -H\delta_{ij} + \begin{pmatrix} \hat{a}_i\hat{a}_j\mathcal{A}^\dagger\mathcal{A} & 0 \\ 0 & \mathcal{A}\hat{a}_i\hat{a}_j\mathcal{A}^\dagger \end{pmatrix}, \quad Z_{ij}^{(b)} = -H\delta_{ij} + \begin{pmatrix} \hat{b}_i\hat{b}_j\mathcal{A}^\dagger\mathcal{A} & 0 \\ 0 & \mathcal{A}\hat{b}_i\hat{b}_j\mathcal{A}^\dagger \end{pmatrix}, \quad \dots, \quad (6.1.14)$$

$$Z_{ij}^{(\alpha)} = -H\delta_{ij} + \begin{pmatrix} \hat{\alpha}_i\hat{\alpha}_j\mathcal{A}^\dagger\mathcal{A} & 0 \\ 0 & \mathcal{A}\hat{\alpha}_i\hat{\alpha}_j\mathcal{A}^\dagger \end{pmatrix}, \quad Z_{ij}^{(\beta)} = -H\delta_{ij} + \begin{pmatrix} \hat{\beta}_i\hat{\beta}_j\mathcal{A}^\dagger\mathcal{A} & 0 \\ 0 & \mathcal{A}\hat{\beta}_i\hat{\beta}_j\mathcal{A}^\dagger \end{pmatrix}, \quad \dots. \quad (6.1.15)$$

Therefore, we can consider that the central charges in this SUSY algebra result from the symmetries in the extra dimensions.

Especially, the supercharges $Q_i^{(A)}$ ($i = 1, \dots, N_A$) for each index A commute with each other

$$[Q_i^{(A)}, Q_j^{(A)}] = 0 \quad (i, j = 1, \dots, N_A). \quad (6.1.16)$$

This property is important for the discussion of the BPS states in the next subsection.

6.1.2 Representation of SUSY algebra

Then, let us clarify the representation of this algebra for the nonzero energy states as well as Chapter 2. Since the Hamiltonian and the central charges commute with each other, we first look at the simultaneous eigenstates of them:

$$H\Phi_{s,z}^{(n)} = m_n^2\Phi_{s,z}^{(n)}, \quad Z_{ij}^{(A)}\Phi_{s,z}^{(n)} = z_{ij}^{(A)}m_n^2\Phi_{s,z}^{(n)}, \quad (6.1.17)$$

where n and z indicate the labels of their eigenvalues m_n and $z_{ij}^{(A)}$,² and s denotes the extra index to further classify the eigenstates in the following discussions. For these states, the algebra (6.1.12) is rewritten into

$$\{Q_i^{(A)}, Q_j^{(B)}\} = 2m_n^2\delta_{ij}\delta^{AB} + 2z_{ij}^{(A)}m_n^2\delta^{AB}. \quad (6.1.18)$$

Since $z_{ij}^{(A)}$ is the real symmetric matrix, it can be diagonalized by the orthogonal matrix $U_{ij}^{(A)}$. Then, by redefining the supercharges, we can obtain

$$\{Q_i'^{(A)}, Q_j'^{(B)}\} = 2m_n^2\delta_{ij}\delta^{AB} + 2z_i'^{(A)}m_n^2\delta^{AB}, \quad (6.1.19)$$

²In general, $z_{ij}^{(A)}$ depend on the index n although we do not explicitly label $z_{ij}^{(A)}$ with it.

where $z_i^{(A)} \delta_{ij} = \sum_{k,l} U_{ik}^{(A)} z_{kl}^{(A)} (U^{(A)})_{lj}^T$ is the diagonalized matrix and $Q_i^{(A)} = \sum_k U_{ik}^{(A)} Q_k^{(A)}$, $Q_j^{(B)} = \sum_k U_{jk}^{(B)} Q_k^{(B)}$ indicate the redefined supercharges. Then we can find that the square of the supercharge $(Q_i^{(A)})^2$ equals 0 for eigenstates with $z_i^{(A)} = -1$ (if exist). This relation leads to $Q_i^{(A)} = 0$ for such states since the supercharge $Q_i^{(A)}$ is Hermitian. Thus, for states with $z_i^{(A)} = -1$, the number of nontrivial supercharges effectively reduces and the size of supermultiplets become small compared with the regular representation. These states are called BPS states.

It should be noted that, in this algebra, at most one supercharge among $Q_i^{(A)}$ ($i = 1, \dots, N_A$) can become nontrivial and the others must equal 0 for any eigenstates. This is because the supercharges satisfy the relation

$$Q_i^{(A)} Q_j^{(A)} = 0 \quad (i \neq j) \quad (6.1.20)$$

due to (6.1.19) and the commutativity of $Q_i^{(A)}$ and $Q_j^{(A)}$. Therefore, the number of nontrivial supercharges for the eigenstates are maximally given by the number of the sets of the operators related to symmetries, and the eigenstates necessarily become the BPS states if the sets have more than one operator.

Next, to construct the supermultiplets in this SUSY algebra, we introduce the following operators by means of the nontrivial redefined supercharges for the eigenstates:

$$S_{ij}^{(AB)} = -i Q_i^{(A)} Q_j^{(B)}, \quad S_{kl}^{(CD)} = -i Q_k^{(C)} Q_l^{(D)}, \quad \dots, \quad (6.1.21)$$

where A, B, C , and D are different with each other according to the above discussion. These operators commute with each other, and therefore, we can further classify the eigenfunctions by these operators. Since these operators satisfy

$$(S_{ij}^{(AB)})^2 = m_n^4 (1 + z_i^{(A)}) (1 + z_j^{(B)}) \quad (6.1.22)$$

for the eigenstates, we can parametrize their eigenvalues as ³

$$S_{ij}^{(AB)} \Phi_{s_{ij}^{(AB)} s_{kl}^{(CD)} \dots, z}^{(n)} = s_{ij}^{(AB)} \sqrt{(1 + z_i^{(A)}) (1 + z_j^{(B)})} m_n^2 \Phi_{s_{ij}^{(AB)} s_{kl}^{(CD)} \dots, z}^{(n)}, \quad (6.1.23)$$

where $s_{ij}^{(AB)} = \pm$. Here, we have described the index s as $s_{ij}^{(AB)} s_{kl}^{(CD)} \dots$.

From the relation

$$S_{ij}^{(AB)} Q_m^{(E)} = \begin{cases} -Q_m^{(E)} S_{ij}^{(AB)} & (E = A, m = i \text{ or } E = B, m = j), \\ +Q_m^{(E)} S_{ij}^{(AB)} & (\text{the others}), \end{cases} \quad (6.1.24)$$

we can see that $Q_i^{(A)}$ and $Q_j^{(B)}$ flip the sign of the eigenvalue of $S_{ij}^{(AB)}$ but do not change other eigenvalues. Thus, if there are K nontrivial supercharges, this relation implies that a supermultiplet composes of the $2^{\lfloor K/2 \rfloor}$ -fold degenerate states $\{\Phi_{s_{ij}^{(AB)} s_{kl}^{(CD)} \dots, z}^{(n)} \text{ with } s_{ij}^{(AB)} = \pm, s_{kl}^{(CD)} = \pm, \dots\}$ and we can explicitly construct the supermultiplet from $\Phi_{++\dots, z}^{(n)}$ as

$$\begin{aligned} \Phi_{s_{ij}^{(AB)} s_{kl}^{(CD)} \dots, z}^{(n)} &= \left(\frac{1}{m_n \sqrt{1 + z_i^{(A)}}} \right)^{(1-s_{ij}^{(AB)})/2} \left(\frac{1}{m_n \sqrt{1 + z_k^{(C)}}} \right)^{(1-s_{kl}^{(CD)})/2} \dots \\ &\quad \times \left[(Q_i^{(A)})^{(1-s_{ij}^{(AB)})/2} (Q_k^{(C)})^{(1-s_{kl}^{(CD)})/2} \dots \right] \Phi_{++\dots, z}^{(n)}. \end{aligned} \quad (6.1.25)$$

³ $z_i^{(A)} \geq -1$ because $(Q_i^{(A)})^2 = m_n^2 + z_i^{(A)} m_n^2 \geq 0$.

6.2 S^2 with magnetic monopole

In this section, we will confirm that the $\mathcal{N}$ -extended SUSY QM given in the previous section can be realized in higher dimensional Dirac action with curved extra dimensions. As an example, we examine the S^2 -extra dimension with the Wu-Yang magnetic monopole background.

6.2.1 Spin-weighted spherical harmonics

As is well known, the mode functions on S^2 can be expressed by the spin-weighted spherical harmonics [123–125]. Thus, we briefly review the Newman-Penrose δ (eth) formalism and this function. For more detail, see Appendix D.

First, we consider the rotation of the orthogonal basis $\mathbf{e}_\theta, \mathbf{e}_\phi$ defined in tangent space on S^2

$$\mathbf{e}_\theta = (\cos \theta \cos \phi, \cos \theta \sin \phi, -\sin \theta), \quad (6.2.1)$$

$$\mathbf{e}_\phi = (-\sin \phi, \cos \phi, 0), \quad (6.2.2)$$

$$\begin{cases} \mathbf{e}_\theta \rightarrow \mathbf{e}'_\theta = \mathbf{e}_\theta \cos \alpha - \mathbf{e}_\phi \sin \alpha, \\ \mathbf{e}_\phi \rightarrow \mathbf{e}'_\phi = \mathbf{e}_\theta \sin \alpha + \mathbf{e}_\phi \cos \alpha, \end{cases} \quad (6.2.3)$$

We call that a quantity η has spin weight s , if η transforms as follows under the above transformation:

$$\eta \rightarrow e^{is\alpha} \eta. \quad (6.2.4)$$

Furthermore, we introduce the operators δ (eth) and $\bar{\delta}$ (eth bar) which act as follows for η with the spin weight s :

$$\delta \eta = - \left[\partial_\theta + \frac{i}{\sin \theta} \partial_\phi - s \cot \theta \right] \eta = -(\sin \theta)^s \left[\partial_\theta + \frac{i}{\sin \theta} \partial_\phi \right] (\sin \theta)^{-s} \eta, \quad (6.2.5)$$

$$\bar{\delta} \eta = - \left[\partial_\theta - \frac{i}{\sin \theta} \partial_\phi + s \cot \theta \right] \eta = -(\sin \theta)^{-s} \left[\partial_\theta - \frac{i}{\sin \theta} \partial_\phi \right] (\sin \theta)^s \eta. \quad (6.2.6)$$

We can show that $\delta \eta$ has spin weight $s + 1$ and $\bar{\delta} \eta$ has spin weight $s - 1$. Therefore, δ and $\bar{\delta}$ correspond to the spin weight raising and lowering operators, respectively.

From (6.2.5) and (6.2.6), we obtain

$$\begin{aligned} \bar{\delta} \delta \eta &= \left[\frac{1}{\sin \theta} \partial_\theta \sin \theta \partial_\theta + \frac{1}{\sin^2 \theta} \partial_\phi^2 + 2is \frac{\cos \theta}{\sin^2 \theta} \partial_\phi - \frac{s^2}{\sin^2 \theta} + s(s+1) \right] \eta, \\ \delta \bar{\delta} \eta &= \left[\frac{1}{\sin \theta} \partial_\theta \sin \theta \partial_\theta + \frac{1}{\sin^2 \theta} \partial_\phi^2 + 2is \frac{\cos \theta}{\sin^2 \theta} \partial_\phi - \frac{s^2}{\sin^2 \theta} + s(s-1) \right] \eta. \end{aligned} \quad (6.2.7)$$

The spin-weighted spherical harmonics ${}_s Y_{jm}$ with the spin weight s is given as the eigenfunction of $\bar{\delta} \delta$ and $\delta \bar{\delta}$:

$$\bar{\delta} \delta {}_s Y_{jm} = -\{j(j+1) - s(s+1)\} {}_s Y_{jm}, \quad \delta \bar{\delta} {}_s Y_{jm} = -\{j(j+1) - s(s-1)\} {}_s Y_{jm}, \quad (6.2.8)$$

or equivalently

$$\left[-\frac{1}{\sin \theta} \partial_\theta \sin \theta \partial_\theta - \frac{1}{\sin^2 \theta} \partial_\phi^2 - 2is \frac{\cos \theta}{\sin^2 \theta} \partial_\phi + \frac{s^2}{\sin^2 \theta} \right] {}_s Y_{jm} = j(j+1) {}_s Y_{jm}, \quad (6.2.9)$$

where the spin weight is given by $s = 0, \pm 1/2, \pm 1, \pm 3/2, \dots$. The index $j (= |s|, |s| + 1, |s| + 2, \dots)$ denotes the main total angular momentum quantum number and the index $m (= -j, -j + 1, \dots, j - 1, j)$ indicates the secondary total angular momentum quantum number. Since the spin-weighted spherical harmonics ${}_s Y_{jm}$ form a complete set for the fixed spin weight s , any function on S^2 with the spin weight s can be decomposed into ${}_s Y_{jm}$.

The explicit form of the normalized spin-weighted spherical harmonics is written as

$${}_s Y_{jm}(\theta, \phi) = e^{im\pi} \sqrt{\frac{2j+1}{4\pi} (j+m)!(j-m)!(j+s)!(j-s)!} \times \sum_{k=\max\{0, -m-s\}}^{\min\{j-s, j-m\}} \frac{(-1)^k \left(\sin \frac{\theta}{2}\right)^{m+s+2k} \left(\cos \frac{\theta}{2}\right)^{2j-m-s-2k}}{k!(j-m-k)!(j-s-k)!(m+s+k)!} e^{im\phi}, \quad (6.2.10)$$

which satisfies the orthonormal relation

$$\int d\Omega ({}_s Y_{jm}(\theta, \phi))^* {}_s Y_{j'm'}(\theta, \phi) = \delta_{jj'} \delta_{mm'}. \quad (6.2.11)$$

Here, we have chosen the phase of this function in such a way that the function satisfies

$$\eth {}_s Y_{jm} = \sqrt{j(j+1) - s(s+1)} {}_{s+1} Y_{jm}, \quad \bar{\eth} {}_s Y_{jm} = -\sqrt{j(j+1) - s(s-1)} {}_{s-1} Y_{jm}. \quad (6.2.12)$$

In the case of $s = 0$, the spin-weighted spherical harmonics corresponds to the spherical harmonic Y_{jm} .

As well as the ordinary spherical harmonics, this function corresponds to the representation of $\text{su}(2)$ algebra

$$L^2 {}_s Y_{jm} = j(j+1) {}_s Y_{jm}, \quad L_z {}_s Y_{jm} = m {}_s Y_{jm}, \quad L_{\pm} {}_s Y_{jm} = \sqrt{(j \mp m)(j+1 \pm m)} {}_s Y_{jm \pm 1}, \quad (6.2.13)$$

where L^2, L_z and $L_{\pm}$ are the angular momentum operators for quantities with spin weight s

$$L_z = -i\partial_{\phi}, \quad (6.2.14)$$

$$L_{\pm} = e^{\pm i\phi} \left(\pm \partial_{\theta} + i \cot \theta \partial_{\phi} - \frac{s}{\sin \theta} \right), \quad (6.2.15)$$

$$L^2 = \frac{1}{2}(L_+ L_- + L_- L_+) + L_z^2 = -\frac{1}{\sin \theta} \partial_{\theta} \sin \theta \partial_{\theta} - \frac{1}{\sin^2 \theta} \partial_{\phi}^2 - 2is \frac{\cos \theta}{\sin^2 \theta} \partial_{\phi} + \frac{s^2}{\sin^2 \theta}, \quad (6.2.16)$$

and satisfy

$$[L_z, L_{\pm}] = \pm L_{\pm}, \quad [L_+, L_-] = 2L_z, \quad [L^2, L_z] = [L^2, L_{\pm}] = 0. \quad (6.2.17)$$

Furthermore, this function satisfies the following properties:

$${}_s Y_{jm}(\pi - \theta, \phi) = (-1)^{j-m} {}_{-s} Y_{jm}(\theta, \phi), \quad (6.2.18)$$

$${}_s Y_{jm}(\theta, -\phi) = (-1)^{m-s} {}_{-s} Y_{j-m}(\theta, \phi). \quad (6.2.19)$$

6.2.2 KK mode functions and mass spectrum of S^2 -extra dimension with magnetic monopole

Then, we discuss the KK mode functions and the mass spectrum of S^2 -extra dimension with a magnetic monopole. We consider the space $M^4 \times S^2$ with the radius a

$$ds^2 = \eta_{\mu\nu} dx^{\mu} dx^{\nu} + a^2 (d\theta^2 + \sin^2 \theta d\phi^2), \quad (6.2.20)$$

and we choose the basis of the vielbein as

$$e_K^{\hat{K}} = \text{diag}(1, 1, 1, 1, a, a \sin \theta) \quad (K = 0, 1, 2, 3, \theta, \phi, \quad \hat{K} = \hat{0}, \hat{1}, \hat{2}, \hat{3}, \hat{\theta}, \hat{\phi}). \quad (6.2.21)$$

Furthermore, we introduce the Wu-Yang magnetic monopole background

$$A^{N/S} = -\frac{n}{2q}(\cos \theta \mp 1)d\phi \quad (n = 0, \pm 1, \pm 2, \dots), \quad (6.2.22)$$

where q is the gauge coupling constant.⁴ As is well known, the magnetic charge of the monopole is quantized. See Appendix C.

The gauge fields A^N and A^S are defined on the north patch ($0 \leq \theta < \pi$, $0 \leq \phi < 2\pi$) and the south patch ($0 < \theta \leq \pi$, $0 \leq \phi < 2\pi$) on S^2 , respectively.

Then, we consider the 6D Dirac action with the monopole background and the bulk mass M :

$$S = \int d^4x \int_{S^2} d\theta d\phi a^2 \sin \theta \bar{\Psi}(x, y) \left[i\Gamma^{\hat{K}} e_{\hat{K}}^K (\nabla_K + iqA_K(y)) + M \right] \Psi(x, y), \quad (6.2.23)$$

where we require that the Dirac fields on the north patch $\Psi^N(x, \theta, \phi)$ and on the south patch $\Psi^S(x, \theta, \phi)$ are related by gauge transformation

$$\Psi^S = e^{-in\phi} \Psi^N. \quad (6.2.24)$$

Here, we define the internal chiral matrix γ^{in} which satisfies

$$\gamma^{in} = i\gamma^{\hat{\theta}}\gamma^{\hat{\phi}}, \quad (\gamma^{in})^2 = \mathbf{1}_{2[d/2]}, \quad (\gamma^{in})^\dagger = \gamma^{in}, \quad \{\gamma^{in}, \gamma^{\hat{\phi}}\} = 0, \quad (6.2.25)$$

and we decomposition $\Psi^{N/S}(x, \theta, \phi)$ as

$$\Psi^{N/S}(x, \theta, \phi) = \sum_k \sum_{\alpha=\pm} \left[f_\alpha^{(k)N/S}(\theta, \phi) \otimes \psi_{R,\alpha}^{(k)}(x) + g_\alpha^{(k)N/S}(\theta, \phi) \otimes \psi_{L,\alpha}^{(k)}(x) \right], \quad (6.2.26)$$

where the index $\alpha = \pm$ indicate the eigenvalues of $\gamma^{in} = \pm$ for $f_\alpha^{(k)N/S}$ and massless $g_\alpha^{(k)N/S}$ (if exist).⁵

In this model, the operator $\mathcal{A}$ and $\mathcal{A}^\dagger$ in the supercharge (3.3.3) can be written into the form

$$\mathcal{A} = e^{\mp i\frac{n}{2}\phi} \left[-i\gamma^{\hat{\theta}} \frac{1}{a} (\bar{\delta}_{s_+} P_+ + \bar{\delta}_{s_-} P_-) + M \right] e^{\pm i\frac{n}{2}\phi}, \quad (6.2.27)$$

$$\mathcal{A}^\dagger = e^{\mp i\frac{n}{2}\phi} \left[i\gamma^{\hat{\theta}} \frac{1}{a} (\bar{\delta}_{s_+} P_+ + \bar{\delta}_{s_-} P_-) + M \right] e^{\pm i\frac{n}{2}\phi}, \quad (6.2.28)$$

where the upper and lower signs in the exponential denote the ones for the north and south patches, respectively. (Here after, we use the same notation for the signs in the exponential if appear.) The operators $\bar{\delta}_{s_+}$ and $\bar{\delta}_{s_-}$ represent the spin weight raising and lowering operators for the spin weight

⁴In the case of $n = 0$, the Einstein equation leads to $a \rightarrow \infty$. However, in the case of $n \neq 0$, the radius a is stabilized and given by $a^2 = n^2 \kappa^2 / 8q^2$ where κ is the 6D gravitational coupling constant [126]. Here, we concentrate on the structure of the mass spectrum of this model and we will not take into account the stability of a .

⁵The massive mode functions $g_\alpha^{(k)N/S} \propto \mathcal{A} f_\alpha^{(k)N/S}$ are not the eigenfunction with $\gamma^{in} = \alpha$ because A and γ^{in} do not commute with each other.

$s_{\pm} = -(n \pm 1)/2$, and $P_{\pm} = (1 \pm \gamma^{in})/2$ are the projection matrices for γ^{in} . Then, $\mathcal{A}^{\dagger}\mathcal{A}$ and $\mathcal{A}\mathcal{A}^{\dagger}$ are given by

$$\mathcal{A}^{\dagger}\mathcal{A} = \mathcal{A}\mathcal{A}^{\dagger} = e^{\mp i\frac{n}{2}\phi} \left[-\frac{1}{a^2} (\bar{\delta}_{s_{+}+1} \delta_{s_{+}} P_{+} + \bar{\delta}_{s_{-}-1} \bar{\delta}_{s_{-}} P_{-}) + M^2 \right] e^{\pm i\frac{n}{2}\phi}. \quad (6.2.29)$$

Thus, we can find that the mode functions and the mass eigenvalues are obtained as follows from (3.3.2) and (6.2.8):

$$f_{\alpha}^{(jm)N/S}(\theta, \phi) = \frac{s_{\alpha} Y_{jm}(\theta, \phi) e^{\mp i\frac{n}{2}\phi}}{a} \mathbf{e}_{\alpha}, \quad \mathbf{g}_{\alpha}^{(jm)N/S}(\theta, \phi) = \frac{1}{m_j} \mathcal{A} f_{\alpha}^{(jm)N/S}(\theta, \phi), \quad (\alpha = \pm), \quad (6.2.30)$$

$$m_j = \sqrt{\frac{j(j+1)}{a^2} - \frac{n^2-1}{4a^2} + M^2}, \quad (6.2.31)$$

where $s_{\pm}$, j and m are given by

$$s_{\pm} = -\frac{n \pm 1}{2}, \quad j = \begin{cases} |s_{+}|, |s_{+}|+1, |s_{+}|+2, \dots & (\text{for } \alpha = +) \\ |s_{-}|, |s_{-}|+1, |s_{-}|+2, \dots & (\text{for } \alpha = -) \end{cases}, \quad m = -j, -j+1, \dots, j-1, j, \quad (6.2.32)$$

and we define $\mathbf{e}_{\pm}$ as the 2-component orthonormal vectors which satisfy

$$\gamma^{in} \mathbf{e}_{\pm} = \pm \mathbf{e}_{\pm}, \quad \gamma^{\hat{\theta}} \mathbf{e}_{\pm} = i \mathbf{e}_{\mp}, \quad \gamma^{\hat{\phi}} \mathbf{e}_{\pm} = \mp \mathbf{e}_{\mp}. \quad (6.2.33)$$

Then, the degeneracy of j -th KK level is $2j+1$ for each mode function.

We can see that the mode functions $f_{\pm}^{(jm)N/S}$ and $\mathbf{g}_{\pm}^{(jm)N/S}$ with $j = |s_{\pm}|$ have the lowest mass eigenvalue $\sqrt{1/a^2 + M^2}$ in the case without the monopole ($n = 0$). On the other hand, in the case with the monopole ($n \neq 0$), the lowest mass eigenvalue decrease to M and the corresponding mode functions are given by $f_{-}^{(|s_{-}|m)N/S}$ and $\mathbf{g}_{-}^{(|s_{-}|m)N/S}$ for $n > 0$ and $f_{+}^{(s_{+}m)N/S}$ and $\mathbf{g}_{+}^{(s_{+}m)N/S}$ for $n < 0$. Thus, there are no massless modes unless the bulk mass M equals 0 and the magnetic monopole exists. If we consider the model with $M = 0$ and $n > 0$ ($n < 0$), $|n|$ massless modes exist for the mode functions $f_{\alpha}^{(jm)N/S}$ and $\mathbf{g}_{\alpha}^{(jm)N/S}$ with $\alpha = -$ ($\alpha = +$) respectively, and $|n|$ massless 4D Dirac fields appear. It should be noted that if we consider the action with a 6D chiral field with the eigenvalue $\Gamma_{chi} = -\Gamma^{\hat{0}}\Gamma^{\hat{1}}\Gamma^{\hat{2}}\Gamma^{\hat{3}}\Gamma^{\hat{\theta}}\Gamma^{\hat{\phi}} = + (-)$, there appear n 4D left (right)-handed fields in the case of $n > 0$ and $|n|$ 4D right (left)-handed fields in the case of $n < 0$.

$f_{+}^{(jm)N/S}$ and $\mathbf{g}_{+}^{(jm)N/S}$			$f_{-}^{(jm)N/S}$ and $\mathbf{g}_{-}^{(jm)N/S}$		
j	mass	degeneracy	j	mass	degeneracy
$j = \frac{1}{2}$	$\sqrt{\frac{1}{a^2} + M^2}$	2	$j = \frac{1}{2}$	$\sqrt{\frac{1}{a^2} + M^2}$	2
$\vdots$			$\vdots$		
$j = \frac{1}{2} + l$	$\sqrt{\frac{(l+1)^2}{a^2} + M^2}$	$2(l+1)$	$j = \frac{1}{2} + l$	$\sqrt{\frac{(l+1)^2}{a^2} + M^2}$	$2(l+1)$
$\vdots$			$\vdots$		

Table 6.1: mass spectrum for $n = 0$

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$f_+^{(jm)N/S}$ and $g_+^{(jm)N/S}$			$f_-^{(jm)N/S}$ and $g_-^{(jm)N/S}$		
j	mass	degeneracy	j	mass	degeneracy
			$j = \frac{n-1}{2}$	M	n
$j = \frac{n+1}{2}$	$\sqrt{\frac{(1+n)}{a^2}} + M^2$	$n + 2$	$j = \frac{n+1}{2}$	$\sqrt{\frac{(1+n)}{a^2}} + M^2$	$n + 2$
$\vdots$			$\vdots$		
$j = \frac{n+1}{2} + l - 1$	$\sqrt{\frac{l(l+n)}{a^2}} + M^2$	$n + 2l$	$j = \frac{n-1}{2} + l$	$\sqrt{\frac{l(l+n)}{a^2}} + M^2$	$n + 2l$
$\vdots$			$\vdots$		

Table 6.2: mass spectrum for $n > 1$

$f_+^{(jm)N/S}$ and $g_+^{(jm)N/S}$			$f_-^{(jm)N/S}$ and $g_-^{(jm)N/S}$		
j	mass	degeneracy	j	mass	degeneracy
$j = -\frac{n+1}{2}$	M	$ n $			
$j = -\frac{n-1}{2}$	$\sqrt{\frac{(1+ n)}{a^2}} + M^2$	$ n + 2$	$j = -\frac{n-1}{2}$	$\sqrt{\frac{(1+ n)}{a^2}} + M^2$	$ n + 2$
$\vdots$			$\vdots$		
$j = -\frac{n+1}{2} + l$	$\sqrt{\frac{l(l+ n)}{a^2}} + M^2$	$ n + 2l$	$j = -\frac{n-1}{2} + l - 1$	$\sqrt{\frac{l(l+ n)}{a^2}} + M^2$	$ n + 2l$
$\vdots$			$\vdots$		

Table 6.3: mass spectrum for $n < 1$

6.2.3 $\mathcal{N}$ -extended SUSY QM in S^2 -extra dimension with magnetic monopole

Let us construct the $\mathcal{N}$ -extended SUSY QM in S^2 -extra dimension with the magnetic monopole. For this purpose, we first summarize the well defined operators for the KK mode functions $f_\alpha^{(jm)N/S}$ and their properties in this model:

- **properties of $\gamma^{in}\gamma^{\hat{\delta}}\delta_{in}$**

We write the operator δ_{in} as

$$\delta_{in} = e^{\mp i\frac{\eta}{2}\phi}(\delta_{s+}P_+ + \bar{\delta}_{s-}P_-)e^{\pm i\frac{\eta}{2}\phi}, \quad (6.2.34)$$

which appears in the $\mathcal{A}$ and $\mathcal{A}^\dagger$. Then, the operators $i\gamma^{in}\gamma^{\hat{\delta}}\delta_{in}$ ($\hat{y} = \hat{\theta}, \hat{\phi}$) satisfy

$$[i\gamma^{in}\gamma^{\hat{\delta}}\delta_{in}, \mathcal{A}^\dagger \mathcal{A}] = 0, \quad \{i\gamma^{in}\gamma^{\hat{\theta}}\delta_{in}, i\gamma^{in}\gamma^{\hat{\phi}}\delta_{in}\} = 0. \quad (6.2.35)$$

These operators relate $f_{\alpha}^{(jm)N/S}$ with $f_{-\alpha}^{(jm)N/S}$ since the operator δ_{in} raises the spin weight for the functions with $\gamma^{in} = +$ and lowers the spin weight for the functions with $\gamma^{in} = -$. Furthermore, these are Hermitian for $f_{\alpha}^{(jm)N/S}$ in the sense that

$$\langle i\gamma^{in}\gamma^{\hat{y}}\delta_{in}f_{\alpha'}^{(j'm')N/S} | f_{\alpha}^{(jm)N/S} \rangle = \langle f_{\alpha'}^{(j'm')N/S} | i\gamma^{in}\gamma^{\hat{y}}\delta_{in}f_{\alpha}^{(jm)N/S} \rangle. \quad (6.2.36)$$

We can use these operators to construct the $\mathcal{N}$ -extended SUSY QM.

• properties of angular momentum operator

From the rotational symmetries of S^2 -extra dimension, the angular momentum operator L_z given in (6.2.14) with gauge transformation

$$L_z^{(n)} = e^{\mp i\frac{n}{2}\phi} L_z e^{\pm i\frac{n}{2}\phi}, \quad (6.2.37)$$

are Hermitian and well defined for the mode functions. Furthermore this satisfy

$$[L_z^{(n)}, \mathcal{A}^{\dagger}\mathcal{A}] = [L_z^{(n)}, \delta_{in}] = 0. \quad (6.2.38)$$

This is also useful for the construction of the $\mathcal{N}$ -extended SUSY QM.

• properties of reflection operators

In the model without the magnetic monopole ($n = 0$), we can consider the reflection operators with gamma matrices $\gamma^{in}\gamma^{\hat{\theta}}R_{\theta}$ and $\gamma^{in}\gamma^{\hat{\phi}}R_{\phi}$ where R_{θ} and R_{ϕ} represent the reflections for the coordinates θ and ϕ

$$R_{\theta} : \theta \rightarrow \pi - \theta, \quad R_{\phi} : \phi \rightarrow -\phi. \quad (6.2.39)$$

These operators are well defined for the mode functions $f_{\alpha}^{(jm)}$ and commute with $\mathcal{A}^{\dagger}\mathcal{A}$ because of the reflection symmetries in S^2 -extra dimension.

In the case of S^2 with monopole ($n \neq 0$), the above operators are ill defined and do not commute with $\mathcal{A}^{\dagger}\mathcal{A}$ by the monopole background. However, the operator with the combination of the above reflections and the gauge transformation

$$\mathcal{R} = e^{\mp i\frac{n}{2}\phi} R_{\theta} R_{\phi} e^{\pm i\frac{n}{2}\phi} \quad (6.2.40)$$

is Hermitian and well defined for the mode functions and satisfies

$$[\mathcal{R}, \mathcal{A}^{\dagger}\mathcal{A}] = \{\mathcal{R}, \delta_{in}\} = \{\mathcal{R}, L_z^{(n)}\} = 0, \quad \mathcal{R}^2 = \mathbf{1}. \quad (6.2.41)$$

We find that this operator connects the mode functions $f_{\alpha}^{(jm)N/S}$ to $f_{\alpha}^{(j-m)N/S}$.

Then, we can obtain various kinds of $\mathcal{N}$ -extended SUSY QM from the above operators. As examples, we consider the following two $\mathcal{N}$ -extended SUSY QM:

• $\mathcal{N} = 6$ extended SUSY QM with central charges from the angular momentum operator

First, we consider the $\mathcal{N} = 6$ extended SUSY QM with central charges which consist of the angular momentum operator $L_z^{(n)}$:

$$\{Q_k, Q_l\} = 2H\delta_{kl} + 2Z_{kl}, \quad (6.2.42)$$

$$[Q_{k'}, Z_{kl}] = [H, Z_{kl}] = [Z_{kl}, Z_{k'l'}] = [Q_{k'}, H] = 0 \quad (k, l, k', l' = 1, \dots, 6), \quad (6.2.43)$$

where the supercharges Q_k and the nonzero components of central charges Z_{kl} are given by

$$Q_1 = \begin{pmatrix} 0 & -\gamma^{in}\gamma^{\hat{\theta}}\delta_{in}\mathcal{A}^\dagger \\ \mathcal{A}\gamma^{in}\gamma^{\hat{\theta}}\delta_{in} & 0 \end{pmatrix}, \quad Q_2 = \begin{pmatrix} 0 & -\gamma^{in}\gamma^{\hat{\phi}}\delta_{in}\mathcal{A}^\dagger \\ \mathcal{A}\gamma^{in}\gamma^{\hat{\phi}}\delta_{in} & 0 \end{pmatrix}, \quad (6.2.44)$$

$$Q_3 = \begin{pmatrix} 0 & i\gamma^{in}L_z^{(n)}\mathcal{A}^\dagger \\ -i\mathcal{A}\gamma^{in}L_z^{(n)} & 0 \end{pmatrix}, \quad Q_4 = \begin{pmatrix} 0 & i\gamma^{in}\mathcal{A}^\dagger \\ -i\mathcal{A}\gamma^{in} & 0 \end{pmatrix}, \quad (6.2.45)$$

$$Q_5 = \begin{pmatrix} 0 & L_z^{(n)}\mathcal{A}^\dagger \\ \mathcal{A}L_z^{(n)} & 0 \end{pmatrix}, \quad Q_6 = Q, \quad (6.2.46)$$

$$Z_{11} = Z_{22} = H(-1 + a^2H - a^2M^2), \quad (6.2.47)$$

$$Z_{34} = Z_{56} = \begin{pmatrix} L_z^{(n)}\mathcal{A}^\dagger\mathcal{A} & 0 \\ 0 & \mathcal{A}L_z^{(n)}\mathcal{A}^\dagger \end{pmatrix}, \quad Z_{33} = Z_{55} = -H + \begin{pmatrix} (L_z^{(n)})^2\mathcal{A}^\dagger\mathcal{A} & 0 \\ 0 & \mathcal{A}(L_z^{(n)})^2\mathcal{A}^\dagger \end{pmatrix}, \quad (6.2.48)$$

In this SUSY QM, Q_3 and Q_4 commute with each other and also Q_5 and Q_6 .

According to the discussion given in the Section 6.1.2, we redefine the supercharges for the eigenfunctions with the eigenvalues $H = m_j^2$, $Z_{34} = m m_j^2$ and $Z_{33} = (-1 + m^2)m_j^2$ (where m indicates the angular momentum number):

$$Q'_1 = Q_1, \quad Q'_2 = Q_2, \quad (6.2.49)$$

$$Q'_3 = \frac{1}{\sqrt{1+m^2}}(mQ_3 + Q_4), \quad Q'_4 = \frac{1}{\sqrt{1+m^2}}(Q_3 - mQ_4), \quad (6.2.50)$$

$$Q'_5 = \frac{1}{\sqrt{1+m^2}}(mQ_5 + Q_6), \quad Q'_6 = \frac{1}{\sqrt{1+m^2}}(Q_5 - mQ_6), \quad (6.2.51)$$

The SUSY algebra (6.2.42) is written into the following form for the redefined supercharges

$$\{Q'_k, Q'_l\} = 2m_j^2\delta_{kl} + 2z'_k m_j^2\delta_{kl}, \quad (6.2.52)$$

where z'_k ($k = 1, \dots, 6$) is given by

$$z'_1 = z'_2 = (-1 + a^2m_j^2 - a^2M^2), \quad z'_3 = z'_5 = m^2, \quad z'_4 = z'_6 = -1. \quad (6.2.53)$$

From this relation, we find that $Q'_4 = Q'_6 = 0$ for any eigenstates. Furthermore the supercharges Q'_1 and Q'_2 are equal to 0 for the states with $m_j = M$ (or equivalently $H = M^2$), that is the case of $j = |s_-|$ for $n > 0$ and $j = s_+$ for $n < 0$. Thus, the BPS states appear as discussed in Section 6.1.

Then we can construct the eigenfunctions $\Phi_{s_{12}s_{35},z'}^{(jm)}(y)$ with $H = m_j^2 \neq M^2$ as

$$\Phi_{++,z'}^{(jm)N/S} = \begin{pmatrix} f_+^{(jm)N/S} \\ 0 \end{pmatrix}, \quad \Phi_{-+,z'}^{(jm)N/S} = \begin{pmatrix} 0 \\ -i\mathbf{g}_-^{(jm)N/S} \end{pmatrix}, \quad \Phi_{+-,z'}^{(jm)N/S} = \begin{pmatrix} 0 \\ -i\mathbf{g}_+^{(jm)N/S} \end{pmatrix}, \quad \Phi_{--,z'}^{(jm)N/S} = \begin{pmatrix} f_-^{(jm)N/S} \\ 0 \end{pmatrix}, \quad (6.2.54)$$

and also the eigenfunctions $\Phi_{s_{35},z'}^{(jm)}(y)$ with $H = M^2$ as

$$\begin{cases} \Phi_{+,z'}^{(|s_-|m)N/S} = \begin{pmatrix} 0 \\ -i\mathbf{g}_-^{(|s_-|m)N/S} \end{pmatrix}, & \Phi_{-,z'}^{(|s_-|m)N/S} = \begin{pmatrix} -f_-^{(|s_-|m)N/S} \\ 0 \end{pmatrix} & \text{for } n > 0 \text{ and } j = |s_-|, \\ \Phi_{+,z'}^{(s_+m)N/S} = \begin{pmatrix} f_+^{(s_+m)N/S} \\ 0 \end{pmatrix}, & \Phi_{-,z'}^{(s_+m)N/S} = \begin{pmatrix} 0 \\ -i\mathbf{g}_+^{(s_+m)N/S} \end{pmatrix} & \text{for } n < 0 \text{ and } j = s_+. \end{cases} \quad (6.2.55)$$

Here z' indicates $\{z'_i (i = 1, \dots, 6)\}$ given in (6.2.53) and $s_{12} = \pm$, $s_{35} = \pm$ denote the signs of the eigenvalues of $S_{12} = -iQ_1Q_2$, $S_{35} = -iQ_3Q_5$.

Then we can show that the above eigenfunctions form the supermultiplets and satisfy the same relation as (6.1.25). In this extended SUSY QM, the four (two) -fold degenerate eigenfunctions with $H \neq M^2$ ($H = M^2$) are related by the supercharges Q'_k with $k = 1, 2, 3, 5$ ($k = 3, 5$). Furthermore, the extra $(2j+1)$ -fold degeneracy exists in the eigenfunctions. The origin of this degeneracy comes from the angular momentum number m , which corresponds to the eigenvalues of central charges.

• **$\mathcal{N} = 6$ extended SUSY QM with central charges from the angular momentum operator and the reflection operator**

Next, we construct the $\mathcal{N} = 6$ extended SUSY QM with central charges from the angular momentum operator and the reflection operator:

$$\begin{aligned} \{Q_k, Q_l\} &= 2H\delta_{kl} + 2Z_{kl}, \\ [Q_{k'}, Z_{kl}] &= [H, Z_{kl}] = [Z_{kl}, Z_{k'l'}] = [Q_{k'}, H] = 0 \quad (k, l, k', l' = 1, \dots, 6), \end{aligned} \quad (6.2.56)$$

where the supercharges Q_k and the nonzero components of the central charges are taken to be the form of

$$Q_1 = \begin{pmatrix} 0 & -\gamma^{in}\gamma^{\hat{\theta}}\delta_{in}\mathcal{A}^\dagger \\ \mathcal{A}\gamma^{in}\gamma^{\hat{\theta}}\delta_{in} & 0 \end{pmatrix}, \quad Q_2 = \begin{pmatrix} 0 & -\gamma^{in}\gamma^{\hat{\phi}}\delta_{in}\mathcal{A}^\dagger \\ \mathcal{A}\gamma^{in}\gamma^{\hat{\phi}}\delta_{in} & 0 \end{pmatrix}, \quad (6.2.57)$$

$$Q_3 = \begin{pmatrix} 0 & i\gamma^{in}L_z^{(n)}\mathcal{A}^\dagger \\ -i\mathcal{A}\gamma^{in}L_z^{(n)} & 0 \end{pmatrix}, \quad Q_4 = \begin{pmatrix} 0 & -L_z^{(n)}\mathcal{R}\mathcal{A}^\dagger \\ \mathcal{A}L_z^{(n)}\mathcal{R} & 0 \end{pmatrix}, \quad (6.2.58)$$

$$Q_5 = \begin{pmatrix} 0 & i\mathcal{R}\mathcal{A}^\dagger \\ -i\mathcal{A}\mathcal{R} & 0 \end{pmatrix}, \quad Q_6 = Q, \quad (6.2.59)$$

$$Z_{11} = Z_{22} = H(-1 + a^2H - a^2M^2), \quad Z_{33} = Z_{44} = -H + \begin{pmatrix} (L_z^{(n)})^2\mathcal{A}^\dagger\mathcal{A} & 0 \\ 0 & \mathcal{A}(L_z^{(n)})^2\mathcal{A}^\dagger \end{pmatrix}, \quad (6.2.60)$$

As well as the previous extended SUSY QM, the supercharges Q_1 and Q_2 equal 0 for the states with $H = M^2$. In addition, the eigenstates with $Z_{33} = -m_j^2 (L_z^{(n)} = 0)$ become BPS states and Q_3 and Q_4 equal 0 for them.

Here, we redefine the basis of the KK mode functions to obtain the representation of this SUSY QM

$$\begin{aligned} f_{\alpha,r}^{(jm)N/S}(\theta, \phi) &= \frac{1}{\sqrt{2}} \left(f_{\alpha}^{(jm)N/S}(\theta, \phi) + r(-1)^{j+s_{\alpha}} f_{\alpha}^{(j-m)N/S}(\theta, \phi) \right), \\ g_{\alpha,r}^{(jm)N/S}(\theta, \phi) &= \frac{1}{\sqrt{2}} \left(g_{\alpha}^{(jm)N/S}(\theta, \phi) + r(-1)^{j+s_{\alpha}} g_{\alpha}^{(j-m)N/S}(\theta, \phi) \right), \end{aligned} \quad (6.2.61)$$

where $r = \pm$ and $m \geq 0$. The mode functions $f_{\alpha,r}^{(jm)N/S}$ with $r = \pm$ correspond to the parity even and odd functions for $\mathcal{R}$, respectively

$$\mathcal{R}f_{\alpha,\pm}^{(jm)N/S} = \pm f_{\alpha,\pm}^{(jm)N/S}. \quad (6.2.62)$$

Then, we can obtain the eigenfunctions $\Phi_{s_{12}s_{34}s_{56},z}^{(jm)}(y)$ for non-BPS states ($H = m_j^2 \neq M^2$, $Z_{33} =$

$(-1 + m^2)m_j^2 \neq -m_j^2$) as follows:

$$\begin{aligned} \Phi_{+++,\mathbf{z}}^{(jm)} &= \begin{pmatrix} f_{+,+}^{(jm)N/S} \\ 0 \end{pmatrix}, & \Phi_{-++,\mathbf{z}}^{(jm)} &= \begin{pmatrix} 0 \\ -i\mathbf{g}_{-,-}^{(jm)N/S} \end{pmatrix}, & \Phi_{+-+,\mathbf{z}}^{(jm)} &= \begin{pmatrix} 0 \\ -i\mathbf{g}_{+,-}^{(jm)N/S} \end{pmatrix}, & \Phi_{++-, \mathbf{z}}^{(jm)} &= \begin{pmatrix} 0 \\ -i\mathbf{g}_{+,+}^{(jm)N/S} \end{pmatrix}, \\ \Phi_{--+,\mathbf{z}}^{(jm)} &= \begin{pmatrix} f_{-,-}^{(jm)N/S} \\ 0 \end{pmatrix}, & \Phi_{-+-,\mathbf{z}}^{(jm)} &= \begin{pmatrix} f_{-,-}^{(jm)N/S} \\ 0 \end{pmatrix}, & \Phi_{+-,-,\mathbf{z}}^{(jm)} &= \begin{pmatrix} f_{+,-}^{(jm)N/S} \\ 0 \end{pmatrix}, & \Phi_{---,\mathbf{z}}^{(jm)} &= \begin{pmatrix} 0 \\ -i\mathbf{g}_{-,-}^{(jm)N/S} \end{pmatrix}, \end{aligned} \quad (6.2.63)$$

where $\mathbf{z}$ indicates z_k ($k = 1, 2, 3, 5$) which are the coefficient of m_j^2 in the eigenvalues of Z_{kk}

$$z_1 = z_2 = -1 + a^2 m_j^2 - a^2 M^2, \quad z_3 = z_5 = -1 + m^2, \quad (6.2.64)$$

and the dependence of $\mathbf{z}$ in the right hand side of (6.2.63) is described by j and m . $s_{12} = \pm$, $s_{34} = \pm$ and $s_{56} = \pm$ denote the signs of the eigenvalues of $S_{12} = -iQ_1Q_2$, $S_{34} = -iQ_3Q_4$, $S_{56} = -iQ_5Q_6$.

We can show that the above eigenfunctions form the supermultiplets and satisfy the same relation as (6.1.25). The supermultiplets for the BPS states can be constructed by use of the mode functions $f_{\alpha,r}^{(jm)N/S}$ and $\mathbf{g}_{\alpha,r}^{(jm)N/S}$ in the same way.

In this extended SUSY QM, the eight-fold degenerate non-BPS states is related by the six supercharges Q_k ($k = 1, \dots, 6$). Therefore, we see that the additional two-fold degeneracy can be further explained by the supercharges compared with the previous extended SUSY QM (although the total degeneracy including the BPS states with $m = 0$ is not changed). This additional degeneracy corresponds to the parity even and odd for the reflection. Since the eigenfunctions with $m = 0$ are equivalent for the parity even from the definition (6.2.61), they should become the BPS states.

Although we considered two examples, there are more extended SUSY QM constructed from the symmetries. For instance, we can obtain extended SUSY QM by using the reflection operators $\gamma^{in}\gamma^{\hat{\theta}}R_{\theta}$ and $\gamma^{in}\gamma^{\hat{\phi}}R_{\phi}$ in the case without the magnetic monopole, since those operators correspond to the symmetries in such model.

6.3 Summary and discussion

In this chapter, we have constructed the new realization of the $\mathcal{N}$ -extended SUSY QM with central charges hidden in the higher dimensional Dirac action with extra dimensions. This extended SUSY QM results from symmetries in extra dimensions, and the supercharges and the central charges are obtained by use of them. We have also investigated the representation of the SUSY algebra and shown that the supermultiplets would become the BPS states. In addition, we considered the model of S^2 -extra dimension with the magnetic monopole as the concrete example. Then we have confirmed that the KK mode functions properly correspond to the representations in the two types of $\mathcal{N}$ -extended SUSY QM which are obtained from the rotational and reflection symmetries in the extra dimensions. Although we have discussed the extended SUSY QM in S^2 -extra dimension as the example, the same structures are also investigated for extra dimensions of the hyperrectangle and the torus in Refs. [80, 81].

The characteristic property of our extended SUSY QM is that certain supercharges commute with each other. Recently, a new generalization of supersymmetry is proposed, that is $\mathbb{Z}_2^n$ -graded supersymmetry whose supercharges have n degrees [127–129]. For their supercharges with different degrees, the algebra may not close in anticommutator but commutator. Therefore, it is interesting to investigate the relation between this supersymmetry and our SUSY QM.

Our analysis is not perfect. As we have seen in Section 6.2, there appear various extended SUSY QM in a model, according to sets of symmetries. Thus, we should clarify what kinds of extended SUSY QM can be obtained from a model. It is also important to reveal that what symmetries exist by the choice of boundary conditions, extra dimensional spaces and background fields since they will affect the structures of the extended QM SUSY. We can also expected that there is hidden extended SUSY in the extra dimensional model with the quantum graph and we should investigate its structure. Furthermore, it is known that central charges are closely related to topological properties [77–79]. Although the central charges given in Section 6.2 might be related to the topology of S^2 and the magnetic monopole, the detailed structures are not unclear. Thus, we should investigate more models with nontrivial topology. We can also expect the possibilities that the further structures are hidden in the 4D mass spectrum.

Chapter 7

Summary and discussion

In this dissertation, we have studied the hidden supersymmetries and fermion generation structure in extra dimensional models.

The SM is phenomenologically established model which successfully explains the nature up to $O(100)$ GeV. However, there remain many unanswered mysteries, *e.g.* the fermion generation problem, the fermion mass hierarchy, the origin of the flavor mixing and CP -violating phase and so on. One of the promising candidates which explain these problems are extra dimensional models. In extra dimensional models, we decompose higher dimensional fields into 4D fields on the 4D Minkowski space and KK mode functions on an extra space. Then, by integrating out the extra dimensions, we obtain effective 4D actions with the infinite 4D fields. The particles of the SM correspond to zero modes for the Laplacian in terms of a extra space, whereas we regard massive modes as new particles and they are not observed so far since their masses are much larger than current experimental energy scale. In extra dimensional models, a geometry and boundary conditions in a extra space are very important, since they affect the realization of the SM and what new particles appear in the obtained 4D effective theories. Then, in this dissertation, we studied the theoretical structure in extra dimensional models with respect to a geometry and boundary conditions of extra dimensions.

Here we have focused on the hidden SUSY QM in the 4D mass spectrum of a higher dimensional Dirac field and the fermion generation structure in extra dimensional models.

In Chapter 2, we have briefly reviewed the SUSY QM. The supercharge relates doubly degenerate states with a nonzero energy and the Witten index, which is defined by the difference between the number of zero energy states with $(-1)^F = +1$ and $(-1)^F = -1$, is given as the topological quantity. We have also considered the extension of the supersymmetry. The $\mathcal{N}$ -extended supersymmetry which includes $\mathcal{N}$ supercharges leads to more degeneracies of nonzero energy eigenstates. Furthermore, if central charges appear in the supersymmetry algebra, the size of multiplets can become small compared with the regular representation.

Then, in Chapter 3, we have considered the KK decomposition of a higher dimensional Dirac field and shown that the structure of $\mathcal{N} = 2$ SUSY QM is always hidden in the 4D mass spectrum of extra dimensional models. This structure is due to the fact that a 4D mass term should consist of 4D left and right-handed chiral fields, and the supercharge relates corresponding massive KK mode functions. Furthermore, we have found that the number of 4D massless chiral fermions corresponds to the Witten index in the $\mathcal{N} = 2$ SUSY QM, and therefore it is related to the topology in extra dimensions.

However, further degeneracy generally exists due to the degrees of freedom of an extra space and this cannot be explained by the $\mathcal{N} = 2$ SUSY QM. This leads to many KK particles with same

masses in an effective 4D theory. We have focused on this additional degeneracy and revealed the hidden structure of $\mathcal{N}$ -extended SUSY QM with central charges in Chapter 6. This extended SUSY results from symmetries in extra dimensions and degenerate KK mode functions correspond to BPS states. We confirmed that the model of S^2 -extra dimension with the magnetic monopole properly realizes this extend SUSY QM from the rotation and reflection symmetries.

Furthermore, from the viewpoint of general 1D extra dimensional models, we have studied the 5D fermion on the rose quantum graph with N loops in Chapter 4. We have successfully classified the allowed boundary condition into the type $(2N-k, k)$ boundary condition with $0 \leq k \leq 2N$. It is also shown that there exist $|N-k|$ chiral zero modes for each type of the boundary conditions and the Witten index $\Delta_W = N - k$ is related to the topological properties that the different type boundary conditions cannot be continually deformed to each other. Thus, the extra dimensional model of the rose graph with $|\Delta_W| = 3$, *i.e.* the type $(N-3, N+3)$ or type $(N+3, N-3)$ boundary condition leads to the three generation of chiral fermions. Furthermore, since the massless mode functions $f_\alpha^{(0)}(y)$ and $g_\beta^{(0)}(y)$ are exponentially localized at some boundaries, their overlap integrals become hierarchical and will resolve the fermion mass hierarchy problem in the SM. The overlaps of the mode functions for different generations also lead to the off-diagonal components of the mass matrices and flavor mixing. Moreover, the zero mode solutions $f_\alpha^{(0)}(y)$ and $g_\beta^{(0)}(y)$ could become genuine complex functions due to the boundary conditions. Then they can produce the CP -violating phase in the CKM matrix. Therefore, the extra dimensional model with the quantum graph is expected to shed a new light on the fermion generation structure.

Moreover, in Chapter 5, we have numerically investigate the fermion mass hierarchy in the model of the interval extra dimension with point interactions which corresponds to the models of the quantum graph with the type $(6, 0)$ or type $(0, 6)$ boundary condition. The positions of point interactions and the gauge singlet scalar $\langle \Phi(y) \rangle$ are obtained dynamically by minimizing the Casimir energy and the scalar potential. Then we have shown that the experimental values of the masses for the quarks can be reproduced, although some difficulties remain for the charged leptons. We also noted that the mixing terms among the charged leptons will resolve remaining discrepancies.

There are many tasks to be addressed in the future. There will exist higher degeneracy in the 4D mass spectrum of the extra dimensional model with the quantum graph. Then, the $\mathcal{N}$ -extended SUSY will be also hidden in the quantum graph and we should reveal this structure. Moreover, since our discussions are not complete, further degeneracies will also exist in extra dimensional models in addition to the ones from the $\mathcal{N}$ -extended SUSY. We can expect that additional symmetries are hidden behind them.

In Chapter 5, we have focused on the model with point interactions and we consider whether the fermion generation structure can be realized numerically and dynamically by the Casimir energy. Then, we should also consider the cases with the other types of the boundary conditions on the quantum graph. Although the boundary conditions were adjusted by hand, we might also dynamically determine them by the Casimir energy. Moreover, we have only investigated the 5D fermion. Thus, we should study the structure of 5D scalar, gauge, gravity fields on the quantum graph. Especially, parameter spaces of boundary conditions for gauge, gravity fields are not unclear. Furthermore, although we considered the flat quantum graph, we may realize the quantum graph with a warped metric as a generalization of the Randall-Sundrum warped geometry.

It should be noted that several works have been carried out from the viewpoint of extra dimensions on star graphs and the rose graph with simple boundary conditions. These models addressed the low scale SUSY breaking and also dark matter [93], gauge hierarchy and corrections for Newton's law of gravitation from the viewpoint of ADD model [94, 99], doublet-triplet splitting in GUT [95], baryon asymmetry [96], flavor structure [97], stabilizing lengths of edges in the star

graph [98]. Our quantum graph model corresponds to the generalization of them and has larger parameter space. Therefore, by applying these ideas to our model, we would obtain a realistic model which explains not the fermion generation structure but also other mysteries in the SM simultaneously. It is also known that star graphs are realized as low energy effective theories from flux compactifications in string theory. Then, we are interested in clarifying whether the general quantum graph model is realized as an effective theory from string theories as well as the case of the star graph. We can also expect that the 1D quantum graphs can be generalized to higher dimensional ones. This would lead to more fascinating structures of geometries and boundary conditions in extra dimensions.

These issues discussed above will be addressed in further works.

Appendix A

Spinors on curved spacetime

In this chapter, we consider spinors on curved spacetime.

A.1 Summary of curved spacetime

First, we summarize properties of curved spacetime.

A.1.1 Metric

We consider an n -dimensional curved spacetime

$$-ds^2 = g_{MN}(x) dx^M dx^N \quad (M, N = 0, 1, 2, \dots, n-1). \quad (\text{A.1.1})$$

The metric $g_{MN}(x)$ is a symmetric matrix and its inverse matrix is defined by g^{MN} :

$$g_{MN} = g_{NM}, \quad (\text{A.1.2})$$

$$g_{ML} g^{LN} = \delta_M^N. \quad (\text{A.1.3})$$

We take one of the eigenvalues of g_{MN} as negative. This metric is called Lorentz metric.¹

Indices of tensors are raised and lowered by the metric

$$V_M = g_{MN} V^N, \quad V^M = g^{MN} V_N. \quad (\text{A.1.4})$$

A.1.2 General coordinate transformation

Under the general coordinate transformation $x^M \rightarrow x'^M$, dx^M and $\partial/\partial x^M$ transform as

$$dx^M \rightarrow dx'^M = \frac{\partial x'^M}{\partial x^K} dx^K, \quad (\text{A.1.5})$$

$$\frac{\partial}{\partial x^M} \rightarrow \frac{\partial}{\partial x'^M} = \frac{\partial x^K}{\partial x'^M} \frac{\partial}{\partial x^K}. \quad (\text{A.1.6})$$

Therefore, the metric transforms as

$$g_{MN}(x) \rightarrow g'_{MN}(x') = \frac{\partial x^K}{\partial x'^M} \frac{\partial x^L}{\partial x'^N} g_{KL}(x) \quad (\text{A.1.7})$$

¹In the case of flat space, this corresponds to $\eta_{MN} = \text{diag}(-1, +1, \dots, +1)$.

from the requirement of the invariance of ds^2 .

Transformation properties for General tensors $V^{M_1 \dots M_i}_{N_1 \dots N_j}(x)$ are given by

$$V^{M_1 \dots M_i}_{N_1 \dots N_j}(x) \rightarrow V'^{M_1 \dots M_i}_{N_1 \dots N_j}(x') = \frac{\partial x'^{M_1}}{\partial x^{K_1}} \dots \frac{\partial x'^{M_i}}{\partial x^{K_i}} \frac{\partial x^{L_1}}{\partial x'^{N_1}} \dots \frac{\partial x^{L_j}}{\partial x'^{N_j}} V^{K_1 \dots K_i}_{L_1 \dots L_j}(x). \quad (\text{A.1.8})$$

A.1.3 Volume element

The ordinary volume element $d^n x = dx^0 \dots dx^{n-1}$ is not invariant under the General coordinate transformation since the Jacobian $|\det(\partial x' / \partial x)|$ appears as follows

$$d^n x \rightarrow \left| \det \left(\frac{\partial x'^M}{\partial x^K} \right) \right| d^n x. \quad (\text{A.1.9})$$

Instead of this, we can obtain the following invariant volume element

$$\sqrt{-g} d^n x, \quad (\text{A.1.10})$$

since the determinant $g = \det g_{MN}$ transforms as

$$g \rightarrow \det \left(\frac{\partial x'^M}{\partial x'^K} \right)^2 g, \quad (\text{A.1.11})$$

from (A.1.7). The minus sign in the root is required because of the negative eigenvalue of the metric.

A.1.4 Parallel transport

When we consider the difference of vectors on different points, we should introduce the parallel transport for vectors on curved space.

We define the parallel transport for a vector field V^N from the point Q with the coordinate $x^M + \epsilon t^M$ to the infinitesimally apart point P with the coordinate x^M as follows:

$$V^N(Q \rightarrow P) = V^N(Q) + \Gamma_{ML}^N(Q) \epsilon t^M V^L(Q), \quad (\text{A.1.12})$$

where Γ_{ML}^N is called affine connection coefficient or connection, and is introduced to take into account the variation of the vector due to the curve of the space.

Then the difference between $V^N(Q \rightarrow P)$ and $V^N(P)$ is given by

$$V^N(Q \rightarrow P) - V^N(P) = V^N(Q) - V^N(P) + \epsilon \Gamma_{ML}^N(Q) t^M V^L(Q). \quad (\text{A.1.13})$$

A.1.5 Covariant derivative

Dividing both sides of (A.1.13) by ϵ and taking the limit of $\epsilon \rightarrow 0$, we obtain

$$t^M \nabla_M V^N(x) = t^M \left(\partial_M V^N(x) + \Gamma_{ML}^N(x) V^L(x) \right). \quad (\text{A.1.14})$$

Here we defined the covariant derivative ∇_M for vectors as

$$\nabla_M V^N = \partial_M V^N + \Gamma_{ML}^N V^L, \quad (\text{A.1.15})$$

$$\nabla_M V_N = \partial_M V_N - \Gamma_{MN}^L V_L. \quad (\text{A.1.16})$$

We require that $\nabla_M V^N$ and $\nabla_M V_N$ transform as tensors, or equivalently the difference of vectors are also a vector. Then, the connection Γ_{ML}^N is not a tensor and transforms as follows

$$\Gamma_{ML}^N \rightarrow \Gamma'_{ML}^N = \frac{\partial x'^N}{\partial x^O} \frac{\partial x^P}{\partial x'^M} \frac{\partial x^Q}{\partial x'^L} \Gamma_{PQ}^O + \frac{\partial x'^N}{\partial x^O} \frac{\partial^2 x^O}{\partial x'^M \partial x'^L}. \quad (\text{A.1.17})$$

The covariant derivative for a tensor $V^{M_1 \dots M_i}_{N_1 \dots N_j}$ is given by

$$\begin{aligned} \nabla_L V^{M_1 \dots M_i}_{N_1 \dots N_j} &= \partial_L V^{M_1 \dots M_i}_{N_1 \dots N_j} + \Gamma_{LK}^{M_1} V^{K \dots M_i}_{N_1 \dots N_j} + \dots + \Gamma_{LK}^{M_i} V^{M_1 \dots K}_{N_1 \dots N_j} \\ &\quad - \Gamma_{LM_1}^K V^{M_1 \dots M_i}_{K \dots N_j} - \dots - \Gamma_{LM_j}^K V^{M_1 \dots M_i}_{N_1 \dots K}. \end{aligned} \quad (\text{A.1.18})$$

For an arbitrary curve with the tangent vector t^M , we consider parallel vectors X^M and Y^M which satisfy $t^L \nabla_L X^M = t^L \nabla_L Y^M = 0$. If we require that the inner product of these vectors is invariant under parallel transport, we can obtain the metricity condition: the covariant derivative for the metric should be zero

$$\begin{aligned} 0 &= t^L \nabla_L (X^M Y_M) = t^L \nabla_L (g_{MN} X^M Y^N) \\ &= t^L (\nabla_L g_{MN}) X^M Y^N \quad (\because t^L \nabla_L X^N = t^L \nabla_L Y^N = 0) \\ \implies \nabla_L g_{MN} &= 0. \end{aligned} \quad (\text{A.1.19})$$

A.1.6 Torsion tensor

When the metricity condition (A.1.19) is satisfied, the connection can be written in to the form

$$\Gamma_{MN}^L = \frac{1}{2} (\Gamma_{MN}^L + \Gamma_{NM}^L) + \frac{1}{2} T_{MN}^L \quad (\text{A.1.20})$$

$$= \frac{1}{2} g^{LK} (\partial_M g_{NK} + \partial_N g_{KM} - \partial_K g_{MN}) + \frac{1}{2} g^{LK} (T_{KMN} - T_{MKN} - T_{NMK}). \quad (\text{A.1.21})$$

where T_{MN}^L indicates a torsion tensor which is defined by the antisymmetric part of the connection

$$T_{MN}^L = \Gamma_{MN}^L - \Gamma_{NM}^L. \quad (\text{A.1.22})$$

We can find that the torsion tensor transform as a tensor from (A.1.17).

The first term of (A.1.21) corresponds to Christoffel symbols and the second term is called contorsion tensor K_{MN}^L

$$K_{MN}^L = \frac{1}{2} g^{LK} (T_{KMN} - T_{MKN} - T_{NMK}). \quad (\text{A.1.23})$$

From (A.1.21), we can show that

$$\sqrt{-g} \nabla_M V^M = \partial_M (\sqrt{-g} V^M) + \sqrt{-g} T_{LM}^L V^M. \quad (\text{A.1.24})$$

$$\nabla_L g_{MN} = 0,$$

$$\begin{aligned} \Rightarrow & \begin{cases} \partial_L g_{MN} - \Gamma_{LM}^K g_{KN} - \Gamma_{LN}^K g_{MK} = 0, \\ \partial_M g_{NL} - \Gamma_{MN}^K g_{KL} - \Gamma_{ML}^K g_{NK} = 0 \quad (\because \nabla_M g_{NL} = 0), \\ \partial_N g_{LM} - \Gamma_{NL}^K g_{KM} - \Gamma_{NM}^K g_{LK} = 0 \quad (\because \nabla_N g_{LM} = 0), \end{cases} \\ \Rightarrow & \partial_M g_{NL} + \partial_N g_{LM} - \partial_L g_{MN} = (\Gamma_{MN}^K + \Gamma_{NM}^K) g_{KL} + (\Gamma_{ML}^K - \Gamma_{LM}^K) g_{NK} + (\Gamma_{NL}^K - \Gamma_{LN}^K) g_{KM}, \\ \Rightarrow & \partial_M g_{NL} + \partial_N g_{LM} - \partial_L g_{MN} = (\Gamma_{MN}^K + \Gamma_{NM}^K) g_{KL} + T_{NML} + T_{MNL}, \\ \Rightarrow & \frac{1}{2}(\Gamma_{MN}^L + \Gamma_{NM}^L) = \frac{1}{2}g^{LK}(\partial_M g_{NK} + \partial_N g_{KM} - \partial_K g_{MN}) - \frac{1}{2}g^{LK}(T_{MNK} + T_{NML}). \end{aligned}$$

derivation of (A.1.24)

$$\begin{aligned} \nabla_M V^M &= g^{MN} \nabla_M V_N = g^{MN} (\partial_M V_N - \Gamma_{MN}^L V_L) \\ &= g^{MN} \partial_M V_N - \frac{1}{2} g^{MN} g^{LK} (\partial_M g_{NK} + \partial_N g_{KM} - \partial_K g_{MN}) V_L - \frac{1}{2} g^{MN} g^{LK} (T_{MNK} + T_{NMK} - T_{KMN}) V_L \\ &= \underline{g^{MN} \partial_M V_N + (\partial_M g^{ML}) V_L} + \frac{1}{2} \underline{g^{LK} g^{MN} \partial_K g_{MN} V_L + T^M{}_M{}^L V_L} \quad (\because g^{MN} \partial_M g_{NK} = -(\partial_M g^{MN}) g_{NK}) \\ &= \underline{\frac{1}{\sqrt{-g}} \sqrt{-g} \partial_M V^M} + \frac{1}{\sqrt{-g}} (\partial_K \sqrt{-g}) V^K + T^M{}_M{}^L V_L \quad (\because \partial_K g = g g_{MN} \partial_K g^{MN}) \\ &= \frac{1}{\sqrt{-g}} \partial_M (\sqrt{-g} V^M) + T^M{}_M{}^L V_L. \end{aligned}$$

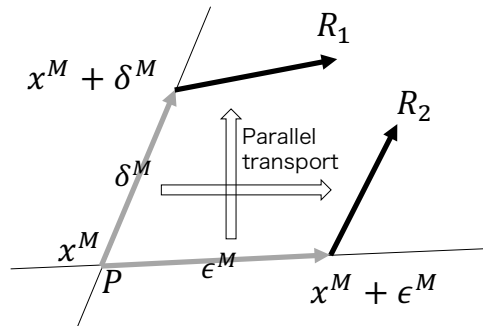


Figure A.1: The parallel transport of ϵ^M along with δ^M and the parallel transport of δ^M along with ϵ^M

Let us consider the meaning of the torsion tensor. We introduce the infinitesimal vectors ϵ^M and δ^M on a point P with a coordinate x^M , and consider the point R_1 and R_2 which are obtained by the parallel transport of ϵ^M along with δ^M and the parallel transport of δ^M along with ϵ^M .

Since ϵ^M and δ^M are infinitesimal, the vectors PR_1 and PR_2 are given by

$$PR_1 = \delta^M + \underbrace{\epsilon^M - \Gamma_{LK}^M \delta^L \epsilon^K}_{\text{parallel transport of } \epsilon^M}, \quad (\text{A.1.25})$$

$$PR_2 = \epsilon^M + \underbrace{\delta^M - \Gamma_{KL}^M \epsilon^K \delta^L}_{\text{parallel transport of } \delta^M}, \quad (\text{A.1.26})$$

and therefore their difference is

$$PR_1 - PR_2 = (\Gamma_{LK}^M - \Gamma_{KL}^M) \delta^L \epsilon^K = T_{LK}^M \delta^L \epsilon^K. \quad (\text{A.1.27})$$

This implies that the torsion tensor represents the non-closure of the parallelogram which consists of the small displacement vectors.

In the theory of general relativity, we assume that the metricity condition is satisfied and furthermore the torsion tensor vanishes. Then the connection is uniquely determined by the metric. The theory which treats the torsion tensor as a new degree of freedom is called Einstein-Cartan theory. However, it is known that the Einstein-Cartan theory is equivalent to the general relativity if matters dose not exist. Hereafter we focus on the case of ordinary general relativity which have no torsion.

A.1.7 Local Lorentz frame and vielbein

Vectors and tensors on a curved space considered in the previous section correspond to generalizations for ones on the Minkowski space. Although we also want to introduce spinors on a curved space, we are not sure that the spinor representation of the Lorentz group can be extended to a curved space and how we can define gamma matrices.

Therefore, to define spinors, we consider a local Minkowski space as a tangent space on each point of curved spacetime. Specifically, we obtain the Minkowski metric by forcing the coordinate dependence in the metric $g_{MN}(x)$ to the coordinate basis dx^M as follows:

$$g_{MN}(x) dx^M dx^N = \eta_{\hat{M}\hat{N}} e^{\hat{M}}(x) e^{\hat{N}}(x), \quad (\text{A.1.28})$$

$$\eta_{\hat{M}\hat{N}} = \text{diag}(-1, +1, \dots, +1), \quad (\hat{M}, \hat{N} = 0, \dots, n-1), \quad (\text{A.1.29})$$

where $e^{\hat{M}}(x)$ is called the non-coordinate basis and $\hat{M}, \hat{N}$ indicate the indices of the local Lorentz frame but not general coordinates. The metric $g_{MN}(x)$, $\eta_{\hat{M}\hat{N}}$ and the basis dx^M , $e^{\hat{M}}(x)$ are related by

$$g_{MN}(x) = \eta_{\hat{M}\hat{N}} e_M^{\hat{M}}(x) e_N^{\hat{N}}(x), \quad \eta_{\hat{M}\hat{N}} = e_{\hat{M}}^M e_{\hat{N}}^N g_{MN}, \quad (\text{A.1.30})$$

$$e^{\hat{M}}(x) = e_M^{\hat{M}}(x) dx^M, \quad dx^M = e_{\hat{M}}^M(x) e^{\hat{M}}(x), \quad (\text{A.1.31})$$

$$e_M^{\hat{M}} e_{\hat{M}}^N = \delta_M^N, \quad e_{\hat{M}}^M e_M^{\hat{N}} = \delta_{\hat{M}}^{\hat{N}}, \quad (\text{A.1.32})$$

This matrix $e_M^{\hat{M}}$ is called the zweibein in two-dimensions, the dreibein in three-dimensions, the vierbein in four-dimensions, and the vielbein in general dimensions.

The vielbein $e_M^{\hat{M}}$ has n^2 degrees of freedom, so the difference between ones of $e_M^{\hat{M}}$ and g_{MN} are given by $n(n-1)/2$. These degrees of freedom correspond to the local Lorentz transformations

on the local Minkowski space on each point. Under the local Lorentz transformations, the vielbein transforms as

$$e_M^{\hat{M}}(x) \rightarrow (\Lambda(x))^{\hat{M}}_{\hat{L}} e_M^{\hat{L}}(x), \quad e_{\hat{M}}^M(x) \rightarrow e_{\hat{L}}^M(x) (\Lambda^{-1}(x))^{\hat{L}}_{\hat{M}}, \quad (\text{A.1.33})$$

where $(\Lambda(x))^{\hat{M}}_{\hat{L}}$ is a matrix which satisfies

$$(\Lambda(x))^{\hat{M}}_{\hat{L}} (\Lambda(x))^{\hat{N}}_{\hat{K}} \eta_{\hat{M}\hat{N}} = \eta_{\hat{L}\hat{K}}. \quad (\text{A.1.34})$$

Furthermore, under the general coordinate transformations,

$$e_M^{\hat{M}}(x) \rightarrow \frac{\partial x'^L}{\partial x'^M} e_L^{\hat{M}}(x), \quad e_{\hat{M}}^M(x) \rightarrow \frac{\partial x'^M}{\partial x'^L} e_{\hat{M}}^L(x). \quad (\text{A.1.35})$$

By means of the vielbeins we can relate the quantity with indices of general coordinates and local Lorentz frame

$$V^M = e_{\hat{M}}^M V^{\hat{M}}, \quad V_M = e_M^{\hat{M}} V_{\hat{M}}, \quad (\text{A.1.36})$$

$$V^{\hat{M}} = e_M^{\hat{M}} V^M, \quad V_{\hat{M}} = e_{\hat{M}}^M V_M. \quad (\text{A.1.37})$$

A.1.8 Covariant derivative in local Lorentz frame

As well as the general coordinate transformation, $\partial_M V^{\hat{M}}$ dose not behave as a vector under the local Lorentz transformation

$$\partial_M V^{\hat{M}}(x) \rightarrow \partial_M \left((\Lambda(x))^{\hat{M}}_{\hat{K}} V^{\hat{K}}(x) \right) = (\Lambda(x))^{\hat{M}}_{\hat{L}} \partial_M V^{\hat{L}}(x) + \left(\partial_M (\Lambda(x))^{\hat{M}}_{\hat{L}} \right) V^{\hat{L}}(x). \quad (\text{A.1.38})$$

Therefore, we define a covariant derivative for local Lorentz vectors as

$$\nabla_M V^{\hat{M}}(x) = \partial_M V^{\hat{M}}(x) + \omega_M^{\hat{M}}_{\hat{N}}(x) V^{\hat{N}}(x), \quad (\text{A.1.39})$$

where we introduce the new quantity $\omega_M^{\hat{M}}_{\hat{N}}(x)$ which is called spin connection and transforms as

$$\omega_M^{\hat{M}}_{\hat{N}} \rightarrow \Lambda^{\hat{M}}_{\hat{K}} \omega_M^{\hat{K}}_{\hat{L}} (\Lambda^{-1})^{\hat{L}}_{\hat{N}} - \left[\partial_M \Lambda^{\hat{M}}_{\hat{L}} \right] (\Lambda^{-1})^{\hat{L}}_{\hat{N}}. \quad (\text{A.1.40})$$

Then we find that $\nabla_M V^{\hat{M}}(x)$ is a vector under the local Lorentz transformation.

From the condition that the covariant derivative for $X^{\hat{M}} Y_{\hat{M}}$ corresponds to ordinary derivative, we can obtain

$$\nabla_M V_{\hat{M}} = \partial_M V_{\hat{M}} - \omega_M^{\hat{N}}_{\hat{M}} V_{\hat{N}}. \quad (\text{A.1.41})$$

derivation of (A.1.41)

$$\begin{aligned} \partial_M (X^{\hat{M}} Y_{\hat{M}}) &= \nabla_M (X^{\hat{M}} Y_{\hat{M}}) = (\nabla_M X^{\hat{M}}) Y_{\hat{M}} + X^{\hat{M}} \nabla_M Y_{\hat{M}} \\ &= (\partial_M X^{\hat{M}}) Y_{\hat{M}} + \omega_M^{\hat{M}}_{\hat{N}} X^{\hat{N}} Y_{\hat{M}} + X^{\hat{M}} \nabla_M Y_{\hat{M}}, \\ \implies X^{\hat{M}} \nabla_M Y_{\hat{M}} &= X^{\hat{M}} \partial_M Y_{\hat{M}} - \underbrace{\omega_M^{\hat{M}}_{\hat{N}} X^{\hat{N}} Y_{\hat{M}}}_{\text{replacement of } \hat{M} \text{ and } \hat{N}}, \\ \implies X^{\hat{M}} \nabla_M Y_{\hat{M}} &= X^{\hat{M}} \partial_M Y_{\hat{M}} - \omega_M^{\hat{N}}_{\hat{M}} X^{\hat{M}} Y_{\hat{N}}, \\ \implies &(\text{A.1.41}). \end{aligned}$$

We also define the covariant derivative for a quantity with the indices of general and local Lorentz frame as

$$\nabla_M V^{N\hat{N}} = \partial_M V^{N\hat{N}} + \Gamma_{ML}^N V^{L\hat{N}} + \omega_M^{\hat{N}\hat{L}} V^{N\hat{L}}. \quad (\text{A.1.42})$$

A.1.9 Spin connection

So far, the spin connection is not uniquely determined. We consider the parallel transport for a vector $V^{\hat{M}}$ from the point Q with the coordinate $x^M + \epsilon t^M$ to the infinitesimally apart point P with the coordinate x^M

$$V^{\hat{M}}(Q \rightarrow P) = V^{\hat{M}}(Q) + \omega_M^{\hat{M}\hat{N}}(Q) \epsilon t^M V^{\hat{N}}(Q), \quad (\text{A.1.43})$$

and require that the local Lorentz vector $V^{\hat{M}}(Q \rightarrow P)$ can be written by the parallel transported vector $V^M(Q \rightarrow P)$

$$V^{\hat{M}}(Q \rightarrow P) = e_M^{\hat{M}}(P) V^M(Q \rightarrow P), \quad (\text{A.1.44})$$

or equivalently

$$V^{\hat{M}}(Q) + \omega_M^{\hat{M}\hat{N}}(Q) \epsilon t^M V^{\hat{N}}(Q) = e_M^{\hat{M}}(P) \left(V^M(Q) + \Gamma_{NL}^M(Q) \epsilon t^N V^L(Q) \right). \quad (\text{A.1.45})$$

In the first order of ϵ ,

$$\epsilon t^M V^N(x) \omega_M^{\hat{M}\hat{N}}(x) e_N^{\hat{N}}(x) = \epsilon t^N \left(\partial_N e_M^{\hat{M}}(x) \right) V^M(x) + e_M^{\hat{M}}(x) \Gamma_{NL}^M(x) \epsilon t^N V^L(x), \quad (\text{A.1.46})$$

$$\implies \epsilon t^N V^M(x) \left(\omega_N^{\hat{M}\hat{N}}(x) e_M^{\hat{N}}(x) - \partial_N e_M^{\hat{M}}(x) - e_L^{\hat{M}}(x) \Gamma_{NM}^L(x) \right) = 0. \quad (\text{A.1.47})$$

Then we obtain the condition that the covariant derivative for the vielbein is zero,

$$\nabla_L e_M^{\hat{M}} = \partial_L e_M^{\hat{M}} - \Gamma_{LM}^K e_K^{\hat{M}} + \omega_L^{\hat{M}\hat{K}} e_M^{\hat{K}} = 0. \quad (\text{A.1.48})$$

This leads to the relation between the Christoffel connection and the spin connection

$$\Gamma_{LM}^K = e_{\hat{M}}^K \partial_L e_M^{\hat{M}} + e_{\hat{M}}^K \omega_L^{\hat{M}\hat{K}} e_M^{\hat{K}}, \quad (\text{A.1.49})$$

$$\omega_L^{\hat{M}\hat{K}} = -e_{\hat{K}}^M \partial_L e_M^{\hat{M}} + e_{\hat{K}}^M \Gamma_{LM}^K e_K^{\hat{M}}. \quad (\text{A.1.50})$$

Furthermore, we find that the spin connection $\omega_{L\hat{M}\hat{K}} (= \eta_{\hat{M}\hat{N}} \omega_L^{\hat{N}\hat{K}})$ is antisymmetric for $\hat{M}$ and $\hat{K}$.

$$\omega_{L\hat{M}\hat{K}} = -\omega_{L\hat{K}\hat{M}}. \quad (\text{A.1.51})$$

derivation of (A.1.51)

$$\begin{aligned} 0 &= \nabla_L g_{KM} = \nabla_L (e_K^{\hat{K}} e_M^{\hat{M}} \eta_{\hat{K}\hat{M}}) = e_K^{\hat{K}} e_M^{\hat{M}} \nabla_L \eta_{\hat{K}\hat{M}} \\ &= e_K^{\hat{K}} e_M^{\hat{M}} (-\omega_L^{\hat{O}\hat{K}} \eta_{\hat{O}\hat{M}} - \omega_L^{\hat{O}\hat{M}} \eta_{\hat{K}\hat{O}}) \\ &= -e_K^{\hat{K}} e_M^{\hat{M}} (\omega_{L\hat{M}\hat{K}} + \omega_{L\hat{K}\hat{M}}), \\ &\implies \omega_{L\hat{M}\hat{K}} = -\omega_{L\hat{K}\hat{M}}. \end{aligned}$$

From this antisymmetry, we can also write the spin connection as

$$\omega_{LMK} = e_{\hat{M}M} e_K^{\hat{K}} \omega_L^{\hat{M}}{}_{\hat{K}} \quad (\text{A.1.52})$$

$$= \frac{1}{2} \left[-e_{\hat{O}L} (\partial_M e_K^{\hat{O}} - \partial_K e_M^{\hat{O}}) + e_{\hat{O}M} (\partial_K e_L^{\hat{O}} - \partial_L e_K^{\hat{O}}) + e_{\hat{O}K} (\partial_L e_M^{\hat{O}} - \partial_M e_L^{\hat{O}}) \right], \quad (\text{A.1.53})$$

where

$$e_{\hat{M}M} = \eta_{\hat{M}\hat{L}} e_M^{\hat{L}} = g_{ML} e_{\hat{M}}^L. \quad (\text{A.1.54})$$

derivation of (A.1.53)

From (A.1.49),

$$\begin{aligned} & \begin{cases} \Gamma_{KLM} - \Gamma_{KML} = 0 = e_{\hat{O}K} (\partial_L e_M^{\hat{O}} - \partial_M e_L^{\hat{O}}) + \omega_{LKM} - \omega_{MKL}, \\ \Gamma_{LMK} - \Gamma_{LKM} = 0 = e_{\hat{O}L} (\partial_M e_K^{\hat{O}} - \partial_K e_M^{\hat{O}}) + \omega_{MLK} - \omega_{KLM}, \\ \Gamma_{MKL} - \Gamma_{MLK} = 0 = e_{\hat{O}M} (\partial_K e_L^{\hat{O}} - \partial_L e_K^{\hat{O}}) + \omega_{KML} - \omega_{LMK}. \end{cases} \\ \implies & + e_{\hat{O}K} (\partial_L e_M^{\hat{O}} - \partial_M e_L^{\hat{O}}) - e_{\hat{O}L} (\partial_M e_K^{\hat{O}} - \partial_K e_M^{\hat{O}}) + e_{\hat{O}M} (\partial_K e_L^{\hat{O}} - \partial_L e_K^{\hat{O}}) \\ & = -\frac{(\omega_{LKM} - \omega_{MKL})}{K \leftrightarrow M} + \omega_{MLK} - \omega_{KLM} - \frac{(\omega_{KML} - \omega_{LMK})}{M \leftrightarrow L} \\ & = \frac{\omega_{LMK} - \omega_{MLK}}{K \leftrightarrow M} + \omega_{MLK} - \omega_{KLM} + \frac{\omega_{KLM}}{M \leftrightarrow L} + \omega_{LMK} \quad (\because \omega_{LMK} = -\omega_{LKM}) \\ & = 2\omega_{LMK} \implies (\text{A.1.53}). \end{aligned}$$

A.1.10 Curvature

In general relativity, the curvature tensor $R^K{}_{LMN}$ is defined as

$$[\nabla_M, \nabla_N] V^K = R^K{}_{LMN} V^L, \quad (\text{A.1.55})$$

$$R^K{}_{LMN} = \partial_M \Gamma_{NL}^K - \partial_N \Gamma_{ML}^K + \Gamma_{MO}^K \Gamma_{NL}^O - \Gamma_{NO}^K \Gamma_{ML}^O. \quad (\text{A.1.56})$$

The curvature tensor can be also rewritten by the spin connection

$$R^K{}_{LMN} = e_{\hat{K}}^K e_L^{\hat{L}} R^{\hat{K}}{}_{\hat{L}MN}, \quad (\text{A.1.57})$$

$$R^{\hat{K}}{}_{\hat{L}MN} = \partial_M \omega_N^{\hat{K}}{}_{\hat{L}} - \partial_N \omega_M^{\hat{K}}{}_{\hat{L}} + \omega_M^{\hat{K}}{}_{\hat{O}} \omega_N^{\hat{O}}{}_{\hat{L}} - \omega_N^{\hat{K}}{}_{\hat{O}} \omega_M^{\hat{O}}{}_{\hat{L}}, \quad (\text{A.1.58})$$

$$[D_M, D_N] V^{\hat{K}} = R^{\hat{K}}{}_{\hat{L}MN} V^{\hat{L}}, \quad (\text{A.1.59})$$

where we introduce the covariant derivative D_M for only the local Lorentz indices. (For example, $D_M V^{N\hat{N}} = \partial_M V^{N\hat{N}} + \omega_M^{\hat{N}}{}_{\hat{L}} V^{N\hat{L}}$.)

R_{KLMN} is antisymmetric for the indices K and L , and also M and N

$$R_{KLMN} = -R_{LKMN}, \quad R_{KLMN} = -R_{KLNM}. \quad (\text{A.1.60})$$

Furthermore, R_{KLMN} satisfies the Bianchi identity

$$R^K{}_{LMN} + R^K{}_{MNL} + R^K{}_{NLM} = 0. \quad (\text{A.1.61})$$

$$\begin{aligned}
 & -\partial_N(\partial_M e_L^{\hat{P}} + \omega_M^{\hat{P}}{}_{\hat{Q}} e_L^{\hat{Q}}) + \omega_N^{\hat{P}}{}_{\hat{Q}}(\partial_M e_L^{\hat{Q}} + \omega_M^{\hat{Q}}{}_{\hat{O}} e_L^{\hat{O}}) \\
 & = e_{\hat{P}}^K e_L^{\hat{O}} \left[\partial_M \omega_N^{\hat{P}}{}_{\hat{O}} - \partial_N \omega_M^{\hat{P}}{}_{\hat{O}} + \omega_M^{\hat{P}}{}_{\hat{Q}} \omega_N^{\hat{Q}}{}_{\hat{O}} - \omega_N^{\hat{P}}{}_{\hat{Q}} \omega_M^{\hat{Q}}{}_{\hat{O}} \right] \\
 & = e_{\hat{P}}^K e_L^{\hat{O}} R^{\hat{P}}{}_{\hat{O}MN},
 \end{aligned}$$

$$(*) \implies [D_M, D_N] V^{\hat{P}} = R^{\hat{P}}{}_{\hat{O}MN} V^{\hat{O}}.$$

derivation of (A.1.60)

$$\begin{aligned}
 R_{\hat{K}\hat{L}MN} &= \partial_M \omega_{N\hat{K}\hat{L}} - \partial_N \omega_{M\hat{K}\hat{L}} + \omega_{M\hat{K}\hat{O}} \omega_N^{\hat{O}}{}_{\hat{L}} - \omega_{N\hat{K}\hat{O}} \omega_M^{\hat{O}}{}_{\hat{L}}, \\
 &\implies R_{\hat{K}\hat{L}MN} = -R_{\hat{L}\hat{K}MN} \implies R_{KLMN} = -R_{LKMN}.
 \end{aligned}$$

derivation of (A.1.61)

$$\begin{aligned}
 (*) \implies R^{\hat{P}}{}_{LMN} &= (D_M D_N - D_N D_M) e_L^{\hat{P}}, \\
 &\implies R^{\hat{P}}{}_{LMN} + R^{\hat{P}}{}_{MNL} + R^{\hat{P}}{}_{NLM} \\
 &= (D_M D_N - D_N D_M) e_L^{\hat{P}} + (D_N D_L - D_L D_N) e_M^{\hat{P}} + (D_L D_M - D_M D_L) e_N^{\hat{P}} \\
 &= D_M (D_N e_L^{\hat{P}} - D_L e_N^{\hat{P}}) + D_N (D_L e_M^{\hat{P}} - D_M e_L^{\hat{P}}) + D_L (D_M e_N^{\hat{P}} - D_N e_M^{\hat{P}}) \\
 &= D_M T^{\hat{P}}{}_{NL} + D_N T^{\hat{P}}{}_{LM} + D_L T^{\hat{P}}{}_{MN} \quad (\because \text{(A.1.49)}) \\
 &= 0 \quad (\because \text{we require that } T^{\hat{P}}{}_{NL} = 0).
 \end{aligned}$$

derivation of (A.1.62)

$$R_{KLMN} + \underline{R_{KMNL}} + \underline{R_{KNLM}} = 0 \quad (1)$$

$$\underline{R_{LMNK}} + \underline{R_{LNKM}} + R_{LKMN} = 0 \quad (2)$$

$$R_{MNKL} + \underline{R_{MKLN}} + \underline{R_{MLNK}} = 0 \quad (3)$$

$$\underline{R_{NKLM}} + \underline{R_{NLMK}} + R_{NMKL} = 0 \quad (4)$$

$$\begin{aligned}
 & (1) - (2) - (3) + (4) \\
 & \implies 2R_{KLMN} - 2R_{MNKL} = 0, \\
 & \implies \text{(A.1.62)}
 \end{aligned}$$

A.2 Spinors on curved space time

A.2.1 Definition of spinors

First, in the local Minkowski space, we define the Gamma matrices $\Gamma^{\hat{M}}$ which satisfy

$$\{\Gamma^{\hat{M}}, \Gamma^{\hat{N}}\} = -2\eta^{\hat{M}\hat{N}}, \quad (\Gamma^{\hat{M}})^\dagger = \Gamma^{\hat{0}}\Gamma^{\hat{M}}\Gamma^{\hat{0}}. \quad (\text{A.2.1})$$

Next, we consider the local Lorentz transformation

$$\Lambda^{\hat{M}}_{\hat{N}}(x) = \left(e^{+\frac{i}{2}\epsilon_{\hat{K}\hat{L}}(x)J^{\hat{K}\hat{L}}} \right)_{\hat{N}}^{\hat{M}}, \quad (\text{A.2.2})$$

$$(J^{\hat{K}\hat{L}})^{\hat{M}}_{\hat{N}} = -i(\eta^{\hat{K}\hat{M}}\delta^{\hat{L}}_{\hat{N}} - \eta^{\hat{L}\hat{M}}\delta^{\hat{K}}_{\hat{N}}), \quad (\text{A.2.3})$$

where the $\epsilon_{\hat{K}\hat{L}}(x)$ is antisymmetric parameter and $(J^{\hat{K}\hat{L}})^{\hat{M}}_{\hat{N}}$ denotes the generator of the local Lorentz transformation which satisfies

$$[J^{\hat{K}\hat{L}}, J^{\hat{M}\hat{N}}] = -i(\eta^{\hat{M}\hat{L}}J^{\hat{K}\hat{N}} - \eta^{\hat{M}\hat{K}}J^{\hat{L}\hat{N}} - \eta^{\hat{N}\hat{L}}J^{\hat{K}\hat{M}} + \eta^{\hat{N}\hat{K}}J^{\hat{L}\hat{M}}). \quad (\text{A.2.4})$$

In the case that $\epsilon_{\hat{K}\hat{L}}(x)$ is infinitesimal,

$$\Lambda^{\hat{M}}_{\hat{N}}(x) = \delta^{\hat{M}}_{\hat{N}} + \epsilon^{\hat{M}}_{\hat{N}}(x). \quad (\text{A.2.5})$$

Then, we define the spinor on the local Minkowski space as a quantity which transforms as follows under the local Lorentz transformation:

$$\Psi(x) \rightarrow S(x)\Psi(x) = e^{+\frac{i}{2}\epsilon_{\hat{M}\hat{N}}(x)\Sigma^{\hat{M}\hat{N}}}\Psi(x), \quad (\text{A.2.6})$$

where $\Sigma^{\hat{M}\hat{N}}$ indicates the generator for the spinor

$$\Sigma^{\hat{M}\hat{N}} = \frac{i}{4}[\Gamma^{\hat{M}}, \Gamma^{\hat{N}}], \quad (\text{A.2.7})$$

$$[\Sigma^{\hat{K}\hat{L}}, \Sigma^{\hat{M}\hat{N}}] = -i(\eta^{\hat{M}\hat{L}}\Sigma^{\hat{K}\hat{N}} - \eta^{\hat{M}\hat{K}}\Sigma^{\hat{L}\hat{N}} - \eta^{\hat{N}\hat{L}}\Sigma^{\hat{K}\hat{M}} + \eta^{\hat{N}\hat{K}}\Sigma^{\hat{L}\hat{M}}). \quad (\text{A.2.8})$$

The Dirac conjugate $\bar{\Psi}(x) \equiv (\Psi(x))^\dagger \Gamma^{\hat{0}}$ transforms as

$$\bar{\Psi}(x) \rightarrow \bar{\Psi}(x)S^{-1}(x) = \bar{\Psi}(x)e^{-\frac{i}{2}\epsilon_{\hat{M}\hat{N}}(x)\Sigma^{\hat{M}\hat{N}}}. \quad (\text{A.2.9})$$

From the relation

$$[\Sigma^{\hat{K}\hat{L}}, \Gamma^{\hat{M}}] = -i(\eta^{\hat{M}\hat{L}}\Gamma^{\hat{K}} - \eta^{\hat{M}\hat{K}}\Gamma^{\hat{L}}), \quad (\text{A.2.10})$$

we can show that

$$S^{-1}(x)\Gamma^{\hat{M}}S(X) = \Lambda^{\hat{M}}_{\hat{N}}(x)\Gamma^{\hat{N}}. \quad (\text{A.2.11})$$

If $\epsilon_{\hat{M}\hat{N}}$ is infinitesimal,

$$(\text{lhs of (A.2.11)}) = e^{-\frac{i}{2}\epsilon_{\hat{K}\hat{L}}(x)\Sigma^{\hat{K}\hat{L}}}\Gamma^{\hat{M}}e^{+\frac{i}{2}\epsilon_{\hat{K}\hat{L}}(x)\Sigma^{\hat{K}\hat{L}}} = \Gamma^{\hat{M}} - \frac{i}{2}\epsilon_{\hat{K}\hat{L}}(x)[\Sigma^{\hat{K}\hat{L}}, \Gamma^{\hat{M}}], \quad (\text{A.2.12})$$

$$(\text{rhs of (A.2.11)}) = (\delta^{\hat{M}}_{\hat{N}} + \epsilon^{\hat{M}}_{\hat{N}}(x))\Gamma^{\hat{N}} = \Gamma^{\hat{M}} - \frac{i}{2}\epsilon_{\hat{K}\hat{L}}(x)(-i)(\eta^{\hat{M}\hat{L}}\Gamma^{\hat{K}} - \eta^{\hat{M}\hat{K}}\Gamma^{\hat{L}}), \quad (\text{A.2.13})$$

and (A.2.11) is satisfied. The finite case can be also shown by the repetition of the infinitesimal transformation.

A.2.2 Correspondence to the gauge transformation

The local Lorentz transformation can be viewed as the gauge transformation as follows:

Gauge transformation

- Local gauge transformation for a field $\Phi(x)$ with the gauge group G

$$\Phi(x) \rightarrow U(x)\Phi(x), \quad U(x) = e^{iq \sum_a \epsilon^a(x) T^a} \in G, \quad T^a : \text{generators}. \quad (\text{A.2.14})$$

- Covariant derivative with the gauge field $A_M(x)$

$$D_M \Phi(x) = (\partial_M + iq A_M(x)) \Phi(x), \quad A_M(x) = \sum_a A_M^a(x) T^a. \quad (\text{A.2.15})$$

- Transformation properties of $A_M(x)$

$$A_M(x) \rightarrow U(x) A_M(x) U^{-1}(x) - (\partial_M U(x)) U^{-1}(x). \quad (\text{A.2.16})$$

Local Lorentz transformation

- Local Lorentz transformation for a vector $V^{\hat{M}}(x)$

$$V^{\hat{M}}(x) \rightarrow \Lambda^{\hat{M}}_{\hat{N}}(x) V^{\hat{N}}(x), \quad \Lambda^{\hat{M}}_{\hat{N}}(x) = \left(e^{+\frac{i}{2} \epsilon_{\hat{K}\hat{L}}(x) J^{\hat{K}\hat{L}}} \right)^{\hat{M}}_{\hat{N}}, \quad J^{\hat{K}\hat{L}} : \text{generators}, \quad (\text{A.2.17})$$

- Covariant derivative with the spin connection

$$\begin{aligned} \frac{i}{2} \omega_{M\hat{K}\hat{L}} (J^{\hat{K}\hat{L}})^{\hat{M}}_{\hat{N}} &= \frac{1}{2} \omega_{M\hat{K}\hat{L}} \left(\eta^{\hat{K}\hat{M}} \delta^{\hat{L}}_{\hat{N}} - \eta^{\hat{L}\hat{M}} \delta^{\hat{K}}_{\hat{N}} \right) \\ &= \frac{1}{2} \left(\omega_{M\hat{K}\hat{L}} \eta^{\hat{K}\hat{M}} \delta^{\hat{L}}_{\hat{N}} + \omega_{M\hat{L}\hat{K}} \eta^{\hat{L}\hat{M}} \delta^{\hat{K}}_{\hat{N}} \right) = \omega_M^{\hat{M}}_{\hat{N}}, \end{aligned} \quad (\text{A.2.18})$$

$$\Rightarrow \nabla_M V^{\hat{M}}(x) = \partial_M V^{\hat{M}}(x) + \omega_M^{\hat{M}}_{\hat{N}}(x) V^{\hat{N}}(x) = \left(\delta^{\hat{M}}_{\hat{N}} \partial_M + \frac{i}{2} \omega_{M\hat{K}\hat{L}}(x) (J^{\hat{K}\hat{L}})^{\hat{M}}_{\hat{N}} \right) V^{\hat{N}}(x). \quad (\text{A.2.19})$$

- Transformation properties of $\omega_M^{\hat{M}}_{\hat{N}} = \frac{i}{2} \omega_{M\hat{K}\hat{L}} (J^{\hat{K}\hat{L}})^{\hat{M}}_{\hat{N}}$

$$\omega_M^{\hat{M}}_{\hat{N}} \rightarrow \Lambda^{\hat{M}}_{\hat{K}} \omega_M^{\hat{K}}_{\hat{L}} (\Lambda^{-1})^{\hat{L}}_{\hat{N}} - \left(\partial_M \Lambda^{\hat{M}}_{\hat{L}} \right) (\Lambda^{-1})^{\hat{L}}_{\hat{N}}. \quad (\text{A.2.20})$$

From the above, we can find that there exists a correspondence

$$A^a(x) \Longleftrightarrow \omega_{M\hat{K}\hat{L}}(x), \quad (\text{A.2.21})$$

$$T^a \Longleftrightarrow (J^{\hat{K}\hat{L}})^{\hat{M}}_{\hat{N}}, \quad (\text{A.2.22})$$

and we can regard the local Lorentz transformation as the gauge transformation.

A.2.3 Covariant derivative for spinor

$\partial_M \Psi$ dose not transform as a spinor under the local Lorentz transformation

$$\partial_M \Psi(x) \rightarrow \partial_M (S(x) \Psi(x)) = S(x) \partial_M \Psi(x) + (\partial_M S(x)) \Psi(x). \quad (\text{A.2.23})$$

Therefore we define the covariant derivative for the spinor

$$\nabla_M \Psi(x) = (\partial_M + \Omega_M(x)) \Psi(x), \quad (\text{A.2.24})$$

where we introduce the quantity $\Omega_M(x)$ whose local Lorentz transformation property is given by

$$\Omega_M \rightarrow S \Omega_M S^{-1} - (\partial_M S) S^{-1}. \quad (\text{A.2.25})$$

Then we find that $\nabla_M \Psi$ transforms as a spinor.

Ω_M can be written as

$$\Omega_M = \frac{i}{2} \omega_{M\hat{K}\hat{L}} \Sigma^{\hat{K}\hat{L}}, \quad (\text{A.2.26})$$

by replacing the generators of vector representation $(J^{\hat{K}\hat{L}})^{\hat{M}}_{\hat{N}}$ to spinor representation $\Sigma^{\hat{K}\hat{L}}$ in (A.2.19). We can show that this Ω_M satisfies the transformation property (A.2.25).

Since $\bar{\Psi}\Psi$ is Lorentz scalar, we require $\nabla_M(\bar{\Psi}\Psi) = \partial_M(\bar{\Psi}\Psi)$ and obtain

$$\nabla_M \bar{\Psi} = \bar{\Psi} (\overleftarrow{\partial}_M - \Omega_M). \quad (\text{A.2.27})$$

Furthermore, the covariant derivative for the gamma matrices $\Gamma^N = e_{\hat{N}}^N \Gamma^{\hat{N}}$ is zero, and therefore Γ^N are constant vectors on a curved spacetime

$$\nabla_M \Gamma^N = 0. \quad (\text{A.2.28})$$

derivation of transformation property of $\nabla_M \Psi$

Let us show the relation

$$\Gamma_{\hat{O}} S (\partial_M S^{-1}) \Gamma^{\hat{O}} = (n-4) (\partial_M S) S^{-1}. \quad (\text{A.2.29})$$

First, we consider the infinitesimal case. Then

$$\begin{aligned} S &= 1 + \frac{i}{2} \epsilon_{\hat{M}\hat{N}}(x) \Sigma^{\hat{M}\hat{N}}, & S^{-1} &= 1 - \frac{i}{2} \epsilon_{\hat{M}\hat{N}}(x) \Sigma^{\hat{M}\hat{N}}, \\ \partial_M S &= \left(\frac{i}{2} \partial_M \epsilon_{\hat{K}\hat{L}}(x) \right) \Sigma^{\hat{K}\hat{L}} S, & \partial_M S^{-1} &= - \left(\frac{i}{2} \partial_M \epsilon_{\hat{K}\hat{L}}(x) \right) \Sigma^{\hat{K}\hat{L}} S^{-1}, \end{aligned}$$

and we can show the relation as follows

$$\begin{aligned} \Gamma_{\hat{O}} S (\partial_M S^{-1}) \Gamma^{\hat{O}} &= -\Gamma_{\hat{O}} \left(\frac{i}{2} \partial_M \epsilon_{\hat{K}\hat{L}}(x) \Sigma^{\hat{K}\hat{L}} \right) \Gamma^{\hat{O}} \\ &= - \underbrace{\left(\frac{i}{2} \partial_M \epsilon_{\hat{K}\hat{L}}(x) \Sigma^{\hat{K}\hat{L}} \right) \Gamma_{\hat{O}} \Gamma^{\hat{O}}}_{\Downarrow} + \left(\frac{i}{2} \partial_M \epsilon_{\hat{K}\hat{L}}(x) \right) [\Sigma^{\hat{K}\hat{L}}, \Gamma^{\hat{O}}] \Gamma_{\hat{O}} \end{aligned}$$

$$\begin{aligned}
 &= -(\partial_M S) S^{-1} \Gamma_{\hat{O}} \Gamma^{\hat{O}} + \left(\frac{i}{2} \partial_M \epsilon_{\hat{K}\hat{L}}(x) \right) (-i) [\eta^{\hat{O}\hat{L}} \Gamma^{\hat{K}} - \eta^{\hat{O}\hat{K}} \Gamma^{\hat{L}}] \Gamma_{\hat{O}} \\
 &= -(\partial_M S) \Gamma_{\hat{O}} \Gamma^{\hat{O}} S^{-1} - 4 \left(\frac{i}{2} \partial_M \epsilon_{\hat{K}\hat{L}}(x) \right) \Sigma^{\hat{K}\hat{L}} \\
 &= +n(\partial_M S) S^{-1} - 4(\partial_M S) S^{-1} \quad (\because \Gamma_{\hat{O}} \Gamma^{\hat{O}} = -n) \\
 &= (n-4)(\partial_M S) S^{-1}.
 \end{aligned}$$

Furthermore, we also obtain

$$\begin{aligned}
 \Gamma_{\hat{O}} S^k \{ \partial_M (S^{-1})^k \} \Gamma^{\hat{O}} &= \Gamma_{\hat{O}} S^k \partial_M \{ (S^{-1})^{k-1} S^{-1} \} \Gamma^{\hat{O}} \\
 &= \Gamma_{\hat{O}} S^k \{ \partial_M (S^{-1})^{k-1} \} S^{-1} \Gamma^{\hat{O}} + \Gamma_{\hat{O}} S^k (S^{-1})^{k-1} \{ \partial_M S^{-1} \} \Gamma^{\hat{O}} \\
 &= \Gamma_{\hat{O}} S S^{k-1} \{ \partial_M (S^{-1})^{k-1} \} \Gamma^{\hat{O}} + \Gamma_{\hat{O}} S \{ \partial_M S^{-1} \} \Gamma^{\hat{O}} \\
 &= (\Lambda^{-1})^{\hat{K}} \underline{\Gamma_{\hat{O}} S \Gamma_{\hat{K}} S^{k-1} \{ \partial_M (S^{-1})^{k-1} \}} (\Lambda)^{\hat{O}} \Gamma_{\hat{L}} S^{-1} + (n-4)(\partial_M S) S^{-1} \\
 &\quad (\because S^{-1} \Gamma_{\hat{O}} S = (\Lambda^{-1})^{\hat{K}} \Gamma_{\hat{K}}, \quad S^{-1} \Gamma^{\hat{O}} S = (\Lambda)^{\hat{O}} \Gamma^{\hat{L}}) \\
 &= S \Gamma_{\hat{K}} S^{k-1} \{ \partial_M (S^{-1})^{k-1} \} \Gamma^{\hat{K}} S^{-1} + (n-4)(\partial_M S) S^{-1} \\
 &= S^2 \Gamma_{\hat{K}} S^{k-2} \{ \partial_M (S^{-1})^{k-2} \} \Gamma^{\hat{K}} (S^{-1})^2 + (n-4) S (\partial_M S) (S^{-1})^2 \\
 &\quad + (n-4)(\partial_M S) S^{-1} \\
 &= \dots \\
 &= (n-4) \left[S^{k-1} (\partial_M S) (S^{-1})^k + S^{k-2} (\partial_M S) (S^{-1})^{k-1} + \dots + (\partial_M S) S^{-1} \right] \\
 &= (n-4)(\partial_M S^k) (S^{-1})^k.
 \end{aligned}$$

Therefore, we can also show the finite case by the repetition of the infinitesimal one.

From the relation (A.2.29), we can confirm the transformation property of Ω_M :

$$\begin{aligned}
 \Omega_M &\rightarrow \Omega'_M = \frac{i}{2} \omega'_M{}^{\hat{K}}{}_{\hat{L}} \Sigma_{\hat{K}}{}^{\hat{L}} \\
 &= \frac{i}{2} \left[\underbrace{\Lambda^{\hat{K}}{}_{\hat{O}} \omega_M{}^{\hat{O}}{}_{\hat{P}} (\Lambda^{-1})^{\hat{P}}{}_{\hat{L}}}_{\Downarrow} - \underbrace{(\partial_M \Lambda^{\hat{K}}{}_{\hat{O}}) (\Lambda^{-1})^{\hat{O}}{}_{\hat{L}}}_{\Downarrow} \right] \Sigma_{\hat{K}}{}^{\hat{L}} \\
 &= \frac{i}{2} \left[\omega_M{}^{\hat{O}}{}_{\hat{P}} S \Sigma_{\hat{O}}{}^{\hat{P}} S^{-1} - \frac{i}{4} [\partial_M (S \Gamma_{\hat{O}} S^{-1}), S \Gamma^{\hat{O}} S^{-1}] \right] \\
 &\quad (\because S \Gamma^{\hat{M}} S^{-1} = (\Lambda^{-1})^{\hat{M}}{}_{\hat{N}} \Gamma^{\hat{N}}, \quad S \Gamma_{\hat{M}} S^{-1} = \Lambda^{\hat{N}}{}_{\hat{M}} \Gamma_{\hat{N}}) \\
 &= S \Omega_M S^{-1} + \frac{1}{4} \partial_M (S \Gamma_{\hat{O}} S^{-1}) S \Gamma^{\hat{O}} S^{-1} \\
 &= S \Omega_M S^{-1} + \frac{1}{4} (\partial_M S) \Gamma_{\hat{O}} \Gamma^{\hat{O}} S^{-1} + \frac{1}{4} S \Gamma_{\hat{O}} (\partial_M S^{-1}) S \Gamma^{\hat{O}} S^{-1} \\
 &= \underbrace{S \Omega_M S^{-1} - \frac{n}{4} (\partial_M S) S^{-1} + \frac{1}{4} S \Gamma_{\hat{O}} S^{-1} S (\partial_M S^{-1}) S \Gamma^{\hat{O}} S^{-1}}_{=\dots} \quad (\because \Gamma_{\hat{O}} \Gamma^{\hat{O}} = -n) \\
 &= \dots + \frac{1}{4} \Lambda^{\hat{K}}{}_{\hat{O}} \Gamma_{\hat{K}} S (\partial_M S^{-1}) (\Lambda^{-1})^{\hat{O}}{}_{\hat{L}} \Gamma^{\hat{L}}
 \end{aligned}$$

$$\begin{aligned}
&= \cdots + \frac{1}{4} \Gamma_{\hat{K}} S (\partial_M S^{-1}) \Gamma^{\hat{K}} \\
&= \cdots + \frac{1}{4} (n-4) (\partial_M S) S^{-1} \quad (\because \text{(A.2.29)}) \\
&= S \Omega_M S^{-1} - (\partial_M S) S^{-1}
\end{aligned}$$

derivation of (A.2.28)

$$\begin{aligned}
\nabla_M \Gamma^N &= \partial_M \Gamma^N + \frac{i}{2} \omega_{M\hat{K}\hat{L}} [\Sigma^{\hat{K}\hat{L}}, \Gamma^N] + \Gamma_{ML}^N \Gamma^L \\
&= (\partial_M e_{\hat{N}}^N) \Gamma^{\hat{N}} + \frac{i}{2} \omega_{M\hat{K}\hat{L}} e_{\hat{N}}^N (-i) [\eta^{\hat{N}\hat{L}} \Gamma^{\hat{K}} - \eta^{\hat{K}\hat{N}} \Gamma^{\hat{L}}] + \Gamma_{ML}^N e_{\hat{L}}^L \Gamma^{\hat{L}} \\
&= (\partial_M e_{\hat{N}}^N) \Gamma^{\hat{N}} + \frac{1}{2} \omega_{M\hat{K}\hat{L}} [e^{\hat{L}N} \Gamma^{\hat{K}} - e^{\hat{K}N} \Gamma^{\hat{L}}] + \Gamma_{ML}^N e_{\hat{L}}^L \Gamma^{\hat{L}} \\
&= [\partial_M e_{\hat{K}}^N - \omega_M^{\hat{L}}{}_{\hat{K}} e_{\hat{L}}^N + \Gamma_{ML}^N e_{\hat{K}}^L] \Gamma^{\hat{K}} \\
&= (\nabla_M e_{\hat{K}}^N) \Gamma^{\hat{K}} = 0.
\end{aligned}$$

A.2.4 Action for spinor on curved spacetime

In the Minkowski space, the action for the spinor with a gauge interaction is given by

$$S = \int d^n x \bar{\Psi}(x) \left[i \Gamma^N (\partial_N + i q A_N(x)) - M \right] \Psi(x), \quad (\text{A.2.30})$$

where $A_N(x)$ indicates the gauge field, q is the gauge coupling constant and M denotes the mass.

Then, in the curved spacetime, the action for the spinor, which is invariant under the general coordinate and local Lorentz transformation, can be given by the replacement of $d^n x \rightarrow d^n x \sqrt{-g}$, $\Gamma^N \rightarrow \Gamma^{\hat{N}} e_{\hat{N}}^N$, $\partial_N \rightarrow \nabla_N$:

$$S = \int d^n x \sqrt{-g} \bar{\Psi}(x) \left[i \Gamma^{\hat{N}} e_{\hat{N}}^N (\nabla_N + i q A_N(x)) - M \right] \Psi(x). \quad (\text{A.2.31})$$

A.2.5 Dirac equation and Klein-Gordon equation on curved spacetime

By taking the variation for $\bar{\Psi}(x)$ and requiring the action principle, we obtain the Dirac equation on the curved spacetime

$$\left[i \Gamma^{\hat{N}} e_{\hat{N}}^N (\nabla_N + i q A_N(x)) - M \right] \Psi(x) = 0. \quad (\text{A.2.32})$$

Multiplying the Dirac equation by $-i \Gamma^{\hat{N}} e_{\hat{N}}^N (\nabla_N + i q A_N(x)) - M$ on the left, we also obtain the Klein-Gordon equation for spinors on the curved spacetime

$$\left[-(\nabla^M + i q A^M)(\nabla_M + i q A_M) + \frac{1}{4} R + q e_{\hat{M}}^M e_{\hat{N}}^N \Sigma^{\hat{M}\hat{N}} F_{MN} + M^2 \right] \Psi(x) = 0, \quad (\text{A.2.33})$$

where $F_{MN} = \partial_M A_N - \partial_N A_M + i q [A_M, A_N]$ is a field strength for the gauge field and R denotes the Ricci scalar. The derivation is given in Section A.2.7.

A.2.6 Curvature for spinors

As well as the curvature tensor for vectors, we consider the curvature for spinors $\mathcal{R}_{MN}$

$$\begin{aligned} [\nabla_M, \nabla_N]\Psi &= \mathcal{R}_{MN}\Psi, \\ \mathcal{R}_{MN} &= \frac{i}{2}R_{\hat{K}\hat{L}MN}\Sigma^{\hat{K}\hat{L}} = -\frac{1}{4}R_{\hat{K}\hat{L}MN}\Gamma^{\hat{K}}\Gamma^{\hat{L}}. \end{aligned} \quad (\text{A.2.34})$$

$\mathcal{R}_{MN}$ is related to the Ricci tensor and Ricci scalar as follows

$$\mathcal{R}_{MN}\Gamma^M = -\frac{1}{2}R_{MN}\Gamma^M, \quad (\text{A.2.35})$$

$$\mathcal{R}_{MN}\Gamma^M\Gamma^N = \frac{1}{2}R. \quad (\text{A.2.36})$$

derivation of (A.2.34)

$$\begin{aligned} [\nabla_M, \nabla_N]\Psi &= \nabla_M\nabla_N\Psi - \nabla_N\nabla_M\Psi \\ &= \left[\partial_M\nabla_N - \Gamma_{MN}^L\nabla_L + \Omega_M\nabla_N - (M \leftrightarrow N) \right] \Psi \\ &= \left[\partial_M\partial_N + \partial_M\Omega_N + \Omega_N\partial_M - \Gamma_{MN}^L\nabla_L + \Omega_M\partial_N + \Omega_M\Omega_N - (M \leftrightarrow N) \right] \Psi \\ &= \left[\partial_M\Omega_N - \partial_N\Omega_M + \Omega_M\Omega_N - \Omega_N\Omega_M - \underbrace{T_{MN}^L}_{=0}\nabla_L \right] \Psi \\ &= \left[\partial_M\Omega_N - \partial_N\Omega_M - \frac{1}{4}\omega_{M\hat{K}\hat{L}}\omega_{N\hat{O}\hat{P}}[\Sigma^{\hat{K}\hat{L}}, \Sigma^{\hat{O}\hat{P}}] \right] \Psi \\ &= \left[\partial_M\Omega_N - \partial_N\Omega_M - \frac{1}{4}\omega_{M\hat{K}\hat{L}}\omega_{N\hat{O}\hat{P}}(-i)\left(\eta^{\hat{O}\hat{L}}\Sigma^{\hat{K}\hat{P}} - \eta^{\hat{O}\hat{K}}\Sigma^{\hat{L}\hat{P}} + \eta^{\hat{P}\hat{L}}\Sigma^{\hat{O}\hat{K}} - \eta^{\hat{P}\hat{K}}\Sigma^{\hat{O}\hat{L}}\right) \right] \Psi \\ &= \left[\partial_M\Omega_N - \partial_N\Omega_M + \frac{i}{2}\left(\omega_{M\hat{K}\hat{L}}\omega_{N\hat{P}}\Sigma^{\hat{K}\hat{P}} - \omega_{N\hat{O}\hat{P}}\omega_{M\hat{L}}\Sigma^{\hat{O}\hat{L}}\right) \right] \Psi \\ &= \left[\partial_M\omega_{N\hat{K}\hat{L}} - \partial_N\omega_{M\hat{K}\hat{L}} + \omega_{M\hat{K}\hat{O}}\omega_{N\hat{L}}^{\hat{O}} - \omega_{N\hat{K}\hat{P}}\omega_{M\hat{L}}^{\hat{P}} \right] \frac{i}{2}\Sigma^{\hat{K}\hat{L}}\Psi + \dots \\ &= \frac{i}{2}R_{\hat{K}\hat{L}MN}\Sigma^{\hat{K}\hat{L}}\Psi = (\text{A.2.34}). \end{aligned}$$

derivation of (A.2.35)

$$\begin{aligned} R_{LN}\Gamma^L &= R_{KLMN}\Gamma^L g^{KM} \\ &= -\frac{1}{2}R_{KLMN}\Gamma^L\left(\Gamma^K\Gamma^M + \Gamma^M\Gamma^K\right) = -2\mathcal{R}_{MN}\Gamma^M - \frac{1}{2}R_{KLMN}\Gamma^L\Gamma^M\Gamma^K \quad (\because (\text{A.2.34})) \\ &= -2\mathcal{R}_{MN}\Gamma^M + \frac{1}{2}\left[R_{KMNL} + \underline{R_{KNLM}}\right]\Gamma^L\Gamma^M\Gamma^K \quad (\because (\text{A.1.61})) \\ &= -2\mathcal{R}_{MN}\Gamma^M + \frac{1}{2}\left[R_{KMNL} + \underline{R_{LMKN}}\right]\Gamma^L\Gamma^M\Gamma^K \end{aligned}$$

($\because$ (A.1.62))

$$\begin{aligned}
 &= -4\mathcal{R}_{MN}\Gamma^M + \frac{1}{2}R_{KMNL}\Gamma^L\Gamma^M\Gamma^K \\
 &= -4\mathcal{R}_{MN}\Gamma^M + \frac{1}{2}R_{KMNL}\left[\Gamma^M\Gamma^K\Gamma^L - 2g^{LM}\Gamma^K + 2g^{LK}\Gamma^M\right] \\
 &= -4\mathcal{R}_{MN}\Gamma^M + 2\mathcal{R}_{NL}\Gamma^L - R_{KN}\Gamma^K - R_{MN}\Gamma^M \\
 &= -6\mathcal{R}_{MN}\Gamma^M - 2R_{MN}\Gamma^M \\
 &\implies \mathcal{R}_{MN}\Gamma^M = -\frac{1}{2}R_{MN}\Gamma^M.
 \end{aligned}
 \tag{\because (A.2.34)}$$

derivation of (A.2.36)

$$\begin{aligned}
 \mathcal{R}_{MN}\Gamma^M\Gamma^N &= -\frac{1}{2}R_{MN}\Gamma^M\Gamma^N \\
 &= -\frac{1}{2}(R_{MN} + R_{NM})\Gamma^M\Gamma^N - \frac{1}{2}(R_{MN} - R_{NM})\Gamma^M\Gamma^N \\
 &= \frac{1}{2}(R_{MN} + R_{NM})g^{MN} - \frac{1}{2}(R_{MN} - R_{NM})\Gamma^M\Gamma^N \\
 &= \frac{1}{2}R \quad (\because (A.1.62)).
 \end{aligned}$$

A.2.7 Derivation of Klein-Gordon equation for spinors on curved spacetime

We can show that

$$\begin{aligned}
 i\Sigma^{MN}[\nabla_M, \nabla_N]\Psi &= i\Sigma^{MN}\mathcal{R}_{MN}\Psi \quad (\because (A.2.34)) \\
 &= -\frac{1}{2}\Gamma^M\Gamma^N\mathcal{R}_{MN}\Psi = -\frac{1}{4}R\Psi. \quad (\because (A.2.36))
 \end{aligned}$$

Then we obtain

$$\begin{aligned}
 0 &= \left[-i\Gamma^N(\nabla_N + iqA_N) - M\right]\left[i\Gamma^M(\nabla_M + iqA_M) - M\right]\Psi \\
 &= \left[\Gamma^M\Gamma^N(\nabla_M + iqA_M)(\nabla_N + iqA_N) + M^2\right]\Psi \\
 &= \left[\underbrace{\frac{1}{2}\{\Gamma^M, \Gamma^N\}}_{=-2g^{MN}} + \frac{1}{2}[\Gamma^M, \Gamma^N]\right](\nabla_M + iqA_M)(\nabla_N + iqA_N) + \dots\right]\Psi \\
 &= \left[-(\nabla^M + iqA^M)(\nabla_M + iqA_M) - i\Sigma^{MN}[\nabla^M + iqA^M, \nabla_N + iqA_N] + \dots\right]\Psi \\
 &= \left[-(\nabla^M + iqA^M)(\nabla_M + iqA_M) - i\Sigma^{MN}\left\{[\nabla_M, \nabla_N] + iq\left(\frac{[\nabla_M, A_N] + [A_M, \nabla_N] + iq[A_M, A_N]}{=\partial_MA_N - \partial_NA_M + iq[A_M, A_N]=F_{MN}}\right)\right\} + \dots\right]\Psi \\
 &= \left[-(\nabla^M + iqA^M)(\nabla_M + iqA_M) + \frac{1}{4}R + q\Sigma^{MN}F_{MN} + M^2\right]\Psi.
 \end{aligned}$$

Appendix B

Casimir energy

B.1 Casimir energy for Dirac field on an interval-extra dimension

In this section, we discuss the Casimir energy for the 5D Dirac field on an interval-extra dimension with the Dirichlet boundary condition as discussed in Example 2 of Section 3.2 and see how it depends on the length of the interval.

The effective 4D action obtained from the KK decomposition of the 5D Dirac action is given as

$$S = \int_{M^4} d^4x \left\{ \sum_{n=1}^{\infty} \bar{\psi}^{(n)}(x)(i\gamma^\mu + m_n)\psi^{(n)}(x) + (\eta^{(0)}(x))^\dagger i\bar{\sigma}^\mu \partial_\mu \eta^{(0)}(x) \right\}, \quad (\text{B.1.1})$$

with the masses $\sqrt{(\pi n/L)^2 + M^2}$ ($n = 1, 2, \dots$). Here L is the length of the interval, M denotes the bulk mass and we use the chiral representation of gamma matrices

$$\gamma^\mu = \begin{pmatrix} 0 & \bar{\sigma}^\mu \\ \sigma^\mu & 0 \end{pmatrix}, \quad \sigma^\mu = (\mathbf{1}_2, \sigma^i), \quad \bar{\sigma}^\mu = (\mathbf{1}_2, -\sigma^i), \quad \psi_L^{(0)} = \begin{pmatrix} 0 \\ \eta^{(0)}(x) \end{pmatrix}. \quad (\text{B.1.2})$$

The partition function can be written as

$$Z_F[M, L] = \int \mathcal{D}\psi \mathcal{D}\bar{\psi} e^{iS} \quad (\text{B.1.3})$$

$$= \int \mathcal{D}\psi \mathcal{D}\bar{\psi} \exp \left\{ i \int_{M^4} d^4x \left\{ \sum_{n=1}^{\infty} \bar{\psi}^{(n)}(x)(i\gamma^\mu \partial_\mu + m_n)\psi^{(n)}(x) + \bar{\psi}_L^{(0)}(x)i\gamma^\mu \partial_\mu \psi_L^{(0)}(x) \right\} \right\} \quad (\text{B.1.4})$$

$$\propto \det(i\bar{\sigma}^\mu \partial_\mu) \prod_n \det(i\gamma^\mu \partial_\mu + m_n). \quad (\text{B.1.5})$$

The determinants satisfy the relations

- $\det(i\bar{\sigma}^\mu \partial_\mu) = \det(i\bar{\sigma}^\mu \partial_\mu)^T \quad (\because \det A = \det A^T)$

$$= \det(i\sigma^0 \partial_0 - i\sigma^1 \partial_1 + i\sigma^2 \partial_2 - i\sigma^3 \partial_3)$$

$$= \det(\{\sigma^2\}^2 \{i\sigma^0 \partial_0 - i\sigma^1 \partial_1 + i\sigma^2 \partial_2 - i\sigma^3 \partial_3\})$$

$$= \det(\sigma^2 \{i\partial_0 - i\sigma^1 \partial_1 + i\bar{\sigma}^2 \partial_2 - i\sigma^3 \partial_3\} \underset{\leftarrow}{\sigma^2}) \quad (\because \det AB = \det BA)$$

$$\begin{aligned}
 &= \det(i\sigma^0\partial_0 + i\sigma^1\partial_1 + i\sigma^2\partial_2 + i\sigma^3\partial_3) \\
 &= \det(i\sigma^\mu\partial_\mu) \\
 &= \det(-i\sigma^\mu\partial_\mu) \quad (\because \det(-\mathbf{1}_{2k}) = \det(\mathbf{1}_{2k})), \tag{B.1.6}
 \end{aligned}$$

$$\begin{aligned}
 \Rightarrow \det(i\bar{\sigma}^\mu\partial_\mu) &= [\det(i\bar{\sigma}^\mu\partial_\mu)\det(-i\sigma^\nu\partial_\nu)]^{\frac{1}{2}} = [\det(\bar{\sigma}^\mu\sigma^\nu\partial_\mu\partial_\nu)]^{\frac{1}{2}} = \left[\det\left(\frac{1}{2}(\bar{\sigma}^\mu\sigma^\nu + \bar{\sigma}^\nu\sigma^\mu)\partial_\mu\partial_\nu\right) \right]^{\frac{1}{2}} \\
 &= [\det(-\partial^\mu\partial_\mu \cdot \mathbf{1}_2)]^{\frac{1}{2}} \\
 &= \det(-\partial^\mu\partial_\mu) \\
 &= \exp[\text{Tr} \ln(-\partial^\mu\partial_\mu)] \quad (\because \ln \det A = \text{Tr} \ln A), \tag{B.1.7}
 \end{aligned}$$

$$\begin{aligned}
 \bullet \det(i\gamma^\mu\partial_\mu + m_n) &= \det((\gamma^5)^2\{i\gamma^\mu\partial_\mu + m_n\}) \\
 &= \det(\gamma^5\{i\gamma^\mu\partial_\mu + m_n\}\gamma^5) \quad (\because \det AB = \det BA) \\
 &= \det(-i\gamma^\mu\partial_\mu + m_n), \tag{B.1.8}
 \end{aligned}$$

$$\begin{aligned}
 \Rightarrow \det(i\gamma^\mu\partial_\mu) &= [\det(i\gamma^\mu\partial_\mu + m_n)\det(-i\gamma^\mu\partial_\mu + m_n)]^{\frac{1}{2}} = [\det(\gamma^\mu\gamma^\nu\partial_\mu\partial_\nu + m_n^2)]^{\frac{1}{2}} \\
 &= [\det(\{-\partial^\mu\partial_\mu + m_n^2\}\mathbf{1}_4)]^{\frac{1}{2}} \\
 &= [\det(-\partial^\mu\partial_\mu + m_n^2)]^2 \\
 &= \exp[2\text{Tr} \ln(-\partial^\mu\partial_\mu + m_n^2)] \quad (\because \ln \det A = \text{Tr} \ln A). \tag{B.1.9}
 \end{aligned}$$

Then the partition function is of the following form by using the momentum representation:

$$Z_F \propto \exp \left[\text{Tr} \ln(-\partial^\mu\partial_\mu) + \sum_{n=1}^{\infty} 2\text{Tr} \ln(-\partial^\mu\partial_\mu + m_n^2) \right] \tag{B.1.10}$$

$$\begin{aligned}
 &= \exp \left[\int \frac{d^4p}{(2\pi)^4} \langle p | \ln(\hat{p}^\mu \hat{p}_\mu) + \sum_{n=1}^{\infty} 2 \ln(\hat{p}^\mu \hat{p}_\mu + m_n^2) | p \rangle \right] \\
 &= \exp \left[\int \frac{d^4p}{(2\pi)^4} \left\{ \ln(p^\mu p_\mu) + \sum_{n=1}^{\infty} 2 \ln(p^\mu p_\mu + m_n^2) \right\} \right] \\
 &= \exp \left[\int d^4x \int \frac{d^4p}{(2\pi)^4} \left\{ \ln(p^\mu p_\mu) + \sum_{n=1}^{\infty} 2 \ln(p^\mu p_\mu + m_n^2) \right\} \right]. \tag{B.1.11}
 \end{aligned}$$

After moving to the Euclidean space

$$x^0 \rightarrow -ix_E^4, \quad x^i \rightarrow x_E^i, \quad d^4x \rightarrow -id^4x_E, \tag{B.1.12}$$

$$p^0 \rightarrow ip_E^4, \quad p^i \rightarrow p_E^i, \quad d^4p \rightarrow id^4p_E, \tag{B.1.13}$$

we obtain the Casimir energy $E_F[M, L]$ for the Dirac fields

$$Z_F^{\text{Euclid}}[M, L] \propto \exp \left[-E_F[M, L] \int d^4x_E \right], \tag{B.1.14}$$

where

$$E_F[M, L] = - \int \frac{d^4 p_E}{(2\pi)^4} \left\{ \ln p_E^2 + \sum_{n=1}^{\infty} 2 \ln \left(p_E^2 + \left(\frac{n\pi}{L} \right)^2 + M^2 \right) \right\} \quad (\text{B.1.15})$$

$$= - \int \frac{d^4 p_E}{(2\pi)^4} \left\{ \ln p_E^2 - \ln (p_E^2 + M^2) + \sum_{n=-\infty}^{\infty} \ln \left(p_E^2 + \left(\frac{n\pi}{L} \right)^2 + M^2 \right) \right\}. \quad (\text{B.1.16})$$

Hereafter, we are interested in the L -dependence of the Casimir energy. Thus, we decompose the Casimir energy into the L -independent part $E_{F1}[M]$ and the L -dependent part $E_{F2}[M, L]$

$$E_F[M, L] = E_{F1}[M] + E_{F2}[M, L], \quad (\text{B.1.17})$$

$$E_{F1}[M] = - \int \frac{d^4 p_E}{(2\pi)^4} \left\{ \ln p_E^2 - \ln (p_E^2 + M^2) \right\}, \quad (\text{B.1.18})$$

$$E_{F2}[M, L] = - \int \frac{d^4 p_E}{(2\pi)^4} \sum_{n=-\infty}^{\infty} \ln \left(p_E^2 + \left(\frac{n\pi}{L} \right)^2 + M^2 \right). \quad (\text{B.1.19})$$

and focus on the $E_{F2}[M, L]$. By using the formulas

$$-\ln A = \frac{d}{ds} A^{-s} \Big|_{s=0}, \quad (\text{B.1.20})$$

$$A^{-s} = \frac{1}{\Gamma(s)} \int_0^{\infty} dt t^{s-1} e^{-At}, \quad \Gamma(s) = \int_0^{\infty} dt t^{s-1} e^{-t}, \quad (\text{B.1.21})$$

$$\frac{d}{ds} \frac{t^s}{\Gamma(s)} \Big|_{s=0} = 1, \quad (\text{B.1.22})$$

we can rewrite $E_{F2}[M, L]$ as

$$\begin{aligned} E_{F2}[M, L] &= \int \frac{d^4 p_E}{(2\pi)^4} \sum_{n=-\infty}^{\infty} \frac{d}{ds} \left(p_E^2 + \left(\frac{n\pi}{L} \right)^2 + M^2 \right)^{-s} \Big|_{s=0} \quad (\because (\text{B.1.20})) \\ &= \int \frac{d^4 p_E}{(2\pi)^4} \sum_{n=-\infty}^{\infty} \frac{d}{ds} \frac{1}{\Gamma(s)} \int_0^{\infty} dt t^{s-1} e^{-\left(p_E^2 + \left(\frac{n\pi}{L} \right)^2 + M^2 \right)t} \Big|_{s=0} \quad (\because (\text{B.1.21})) \\ &= \frac{1}{16\pi^4} \sum_{n=-\infty}^{\infty} \frac{d}{ds} \frac{1}{\Gamma(s)} \int_0^{\infty} dt \left(\sqrt{\frac{\pi}{t}} \right)^4 t^{s-1} e^{-\left(\left(\frac{n\pi}{L} \right)^2 + M^2 \right)t} \Big|_{s=0} \quad (\because \text{integration of } p_E) \\ &= \frac{1}{16\pi^2} \sum_{n=-\infty}^{\infty} \int_0^{\infty} dt t^{-3} e^{-\left(\left(\frac{n\pi}{L} \right)^2 + M^2 \right)t}. \quad (\because (\text{B.1.22})) \end{aligned} \quad (\text{B.1.23})$$

Furthermore, the Poisson summation formula

$$\sum_{n=-\infty}^{\infty} e^{-\left(\frac{n\pi}{L} \right)^2} = \sum_{w=-\infty}^{\infty} \frac{L}{\sqrt{\pi t}} e^{-\frac{w^2 L^2}{t}} \quad (\text{B.1.24})$$

leads to

$$E_{F2}[M, L] = \frac{L}{16\pi^{\frac{5}{2}}} \sum_{w=-\infty}^{\infty} \int_0^{\infty} dt t^{-\frac{7}{2}} e^{-\frac{w^2 L^2}{t}} e^{-M^2 t}. \quad (\text{B.1.25})$$

We can find that the $w \neq 0$ terms correspond to the winding modes around the extra dimension and give finite contributions for the Casimir energy without UV divergences. On the other hand the $w = 0$ term corresponds to unwinding mode and the energy density $E_{F2}[M, L]/L$ has a UV divergence at $t = 0$. This divergence is equivalent to the ones in the case of $L \rightarrow \infty$ and therefore we regularize it as

$$\frac{1}{L}E_F[M, L]_{\text{reg}} \equiv \frac{1}{L}E_{F2}[M, L] - \frac{1}{L}E_{F2}[M, L] \Big|_{L \rightarrow \infty} \quad (\text{B.1.26})$$

$$= \frac{1}{8\pi^{\frac{5}{2}}} \sum_{w=1}^{\infty} \int_0^{\infty} dt t^{-\frac{7}{2}} e^{-\frac{w^2 L^2}{t}} e^{-M^2 t}. \quad (\text{B.1.27})$$

This $E_F[M, L]_{\text{reg}}$ corresponds to the contribution of the compactification of the extra dimension.

By means of the modified Bessel function

$$2 \left(\frac{A}{B} \right)^{\frac{\nu}{2}} K_{\nu}(2 \sqrt{AB}) = \int_0^{\infty} dt t^{-\nu-1} e^{-At - \frac{B}{t}}, \quad (\text{B.1.28})$$

$E_F[M, L]_{\text{reg}}$ is of the form

$$\begin{aligned} E_F[M, L]_{\text{reg}} &= \frac{L}{8\pi^{\frac{5}{2}}} \sum_{w=1}^{\infty} 2 \left(\frac{M^2}{w^2 L^2} \right)^{\frac{5}{4}} K_{\frac{5}{2}}(2w|M|L) \\ &= \frac{L}{4\pi^{\frac{5}{2}}} \sum_{w=1}^{\infty} \left(\frac{|M|}{wL} \right)^{\frac{5}{2}} K_{\frac{5}{2}}(2w|M|L). \end{aligned} \quad (\text{B.1.29})$$

Moreover, the modified Bessel function $K_{\frac{D}{2}}(z)$ with odd D is given by

$$K_{\frac{D}{2}}(z) = \sqrt{\frac{\pi}{2z}} e^{-z} \sum_{k=0}^{\frac{D-1}{2}} \frac{\left(\frac{D-1}{2} + k \right)!}{k! \left(\frac{D-1}{2} - k \right)! (2z)^k}. \quad (\text{B.1.30})$$

Therefore, we finally obtain the L -dependent part of the Casimir energy as

$$E_F[M, L]_{\text{reg}} = \frac{|M|^2}{8\pi^2 L^2} \sum_{w=1}^{\infty} \frac{e^{-2w|M|L}}{w^3} \left(1 + \frac{3}{2w|M|L} + \frac{3}{4w^2|M|^2 L^2} \right). \quad (\text{B.1.31})$$

The schematic figures of $E_F[M, L]_{\text{reg}}$ and its derivative $\frac{d}{dL}E_F[M, L]_{\text{reg}}$ are described in Figure B.1.

Then, we can find the Casimir energy for Dirac fields dose not have a minimum. However, it is known that a minimum can appear if we consider the contribution of bosons such as scalar fields and gauge fields as discussed in [103], and the size of the interval L becomes stable by considering the minimization of the Casimir energy.

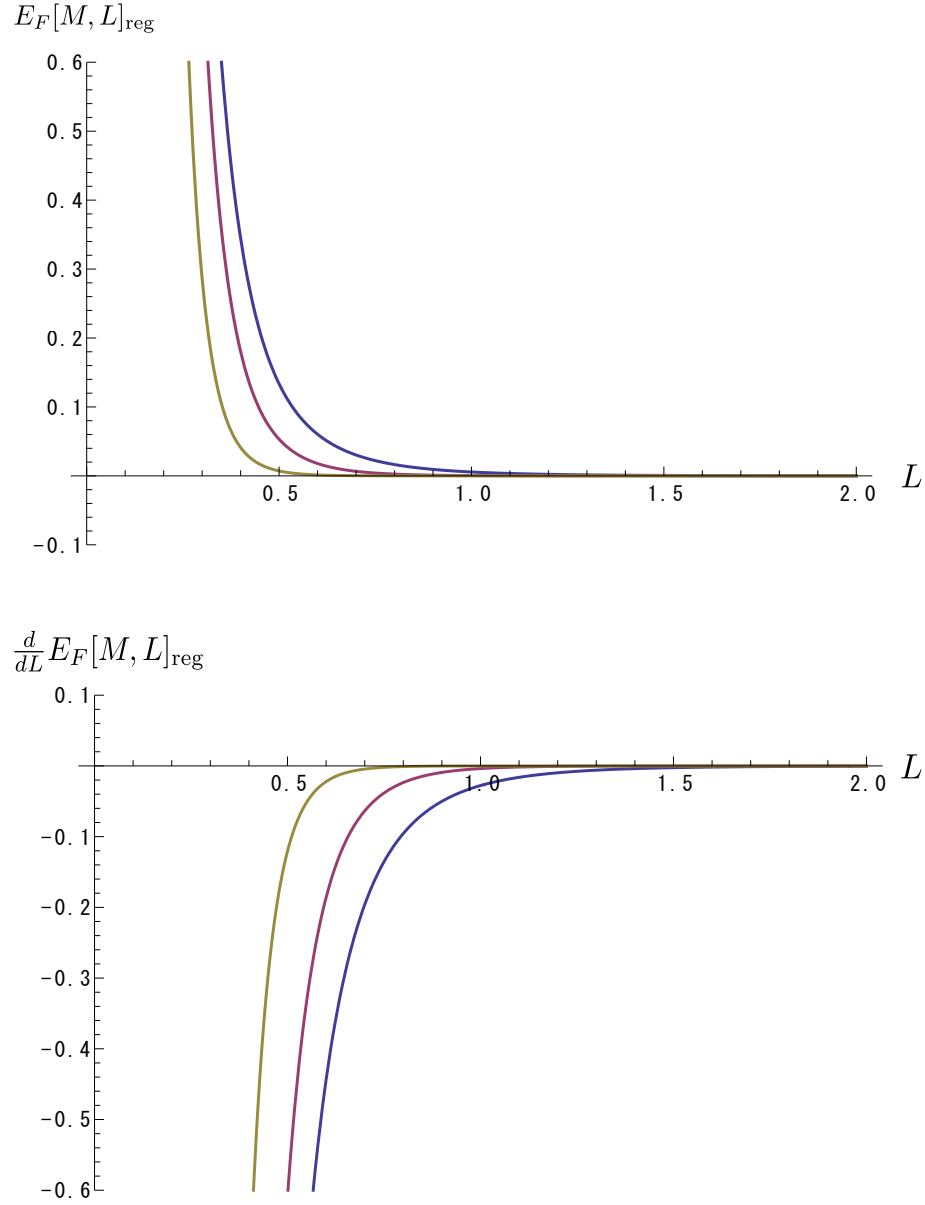


Figure B.1: The figures of $E_F[M, L]_{\text{reg}}$ and $\frac{d}{dL} E_F[M, L]_{\text{reg}}$ as the functions of L are described. The blue, magenta, yellow lines are the functions with $M = 1$, $M = 3$, $M = 6$, respectively. Here, M and L are treated as the dimensionless parameters by multiplying the fundamental scale in a model.

Appendix C

Magnetic monopole

C.1 Dirac monopole and Wu–Yang monopole

The magnetic monopole with the magnetic charge g at $\mathbf{r} = (x, y, z) = 0$ are given by

$$\nabla \cdot \mathbf{B} = 4\pi g \delta^3(\mathbf{r}), \quad (\text{C.1.1})$$

$$\mathbf{B} = g \frac{\mathbf{r}}{r}. \quad (\text{C.1.2})$$

Then the magnetic flux is

$$\Phi = \int_S \mathbf{B} \cdot d\mathbf{S} = 4\pi g. \quad (\text{C.1.3})$$

Since $\nabla \cdot \mathbf{B} \neq 0$, we cannot express $\mathbf{B}$ as $\mathbf{B} = \nabla \times \mathbf{A}$ with the vector potential $\mathbf{A}$ in the entire space. However, using vector potential $A^N(A^S)$, we can express the monopole except for the negative (positive) z axis:

$$A_x^{N/S} = \frac{-gy}{r(z \pm r)}, \quad A_y^{N/S} = \frac{gx}{r(z \pm r)}, \quad A_z^{N/S} = 0, \quad (\text{C.1.4})$$

$$\mathbf{B} = \nabla \times \mathbf{A}^{N/S} = g \frac{\mathbf{r}}{r} \pm 4\pi g \theta(\mp z) \delta(x) \delta(y) \mathbf{e}_z. \quad (\text{C.1.5})$$

This configuration is known as the Dirac monopole and the line singularity is called the Dirac string [130]. For this $\mathbf{B}$, we can show that

$$\nabla \cdot \mathbf{B} = 0, \quad (\text{C.1.6})$$

due to the cancellation of the Dirac string and (C.1.1).

In the polar coordinate

$$d\mathbf{r} = dr \frac{\partial \mathbf{r}}{\partial r} + d\theta \frac{\partial \mathbf{r}}{\partial \theta} + d\phi \frac{\partial \mathbf{r}}{\partial \phi} = dr \mathbf{e}_r + r d\theta \mathbf{e}_\theta + r \sin \theta d\phi \mathbf{e}_\phi, \quad (\text{C.1.7})$$

$$\mathbf{e}_r = \frac{\partial \mathbf{r}}{\partial r} = \sin \theta \cos \phi \mathbf{e}_x + \sin \theta \sin \phi \mathbf{e}_y + \cos \theta \mathbf{e}_z, \quad (\text{C.1.8})$$

$$\mathbf{e}_\theta = \frac{1}{r} \frac{\partial \mathbf{r}}{\partial \theta} = \cos \theta \cos \phi \mathbf{e}_x + \cos \theta \sin \phi \mathbf{e}_y - \sin \theta \mathbf{e}_z, \quad (\text{C.1.9})$$

$$\mathbf{e}_\phi = \frac{1}{r \sin \theta} \frac{\partial \mathbf{r}}{\partial \phi} = -\sin \phi \mathbf{e}_x + \cos \phi \mathbf{e}_y, \quad (\text{C.1.10})$$

$$\mathbf{e}_r \times \mathbf{e}_\theta = \mathbf{e}_\phi, \quad \mathbf{e}_\theta \times \mathbf{e}_\phi = \mathbf{e}_r, \quad \mathbf{e}_\phi \times \mathbf{e}_r = \mathbf{e}_\theta, \quad (\text{C.1.11})$$

the vector potential is given by

$$\begin{aligned} \mathbf{A}^{N/S} &= -\frac{g \sin \theta \sin \phi}{r(\cos \theta \pm 1)} \mathbf{e}_x + \frac{g \sin \theta \cos \phi}{r(\cos \theta \pm 1)} \mathbf{e}_y = \frac{g(\cos \theta \mp 1) \sin \phi}{r \sin \theta} \mathbf{e}_x - \frac{g(\cos \theta \mp 1) \cos \phi}{r \sin \theta} \mathbf{e}_y \\ &= -\frac{g(\cos \theta \mp 1)}{r \sin \theta} \mathbf{e}_\phi. \end{aligned} \quad (\text{C.1.12})$$

Although we introduced the Dirac string, we can avoid its singularity by using two vector potentials [131]. This monopole is called the Wu–Yang monopole. We consider the sphere and the potentials in the north and south patch are given by $\mathbf{A}^N$ and $\mathbf{A}^S$, respectively. On the boundary of them, both potentials are well defined and related by the gauge transformation

$$\mathbf{A}^N - \mathbf{A}^S = \frac{2g}{r \sin \theta} \mathbf{e}_\phi = \nabla(2g\phi), \quad (\text{C.1.13})$$

$$\nabla = \mathbf{e}_r \frac{d}{dr} + \mathbf{e}_\theta \frac{1}{r} \frac{d}{d\theta} + \mathbf{e}_\phi \frac{1}{r \sin \theta} \frac{d}{d\phi}. \quad (\text{C.1.14})$$

The magnetic flux corresponds to (C.1.3):

$$\begin{aligned} \Phi &= \int_{S^N} d\mathbf{S} \cdot \mathbf{B} + \int_{S^S} d\mathbf{S} \cdot \mathbf{B} = \int_C d\mathbf{s} \cdot \mathbf{A}^N - \int_C d\mathbf{s} \cdot \mathbf{A}^S = \int_C d\mathbf{s} \cdot \nabla(2g\phi) \\ &= 4\pi g. \end{aligned} \quad (\text{C.1.15})$$

In the quantum mechanics, the wavefunctions ψ^N and ψ^S defined on the north and south patch are related by the gauge transformation

$$\psi^S = e^{-i\frac{q}{\hbar c} \cdot 2g\phi} \psi^N, \quad (\text{C.1.16})$$

and we require that the wavefunction is single valued when we go round the sphere from $\phi = 0$ to $\phi = 2\pi$. To be satisfied the above condition, the product of the electric charge and magnetic charge should be quantized

$$\frac{2qg}{\hbar c} = n \quad (n = 0, \pm 1, \pm 2, \dots). \quad (\text{C.1.17})$$

This is well known as the Dirac quantization condition.

derivation of (C.1.1)

$$\begin{aligned} \nabla \frac{1}{r} &= -\frac{\mathbf{r}}{r^2}, \\ \nabla \cdot \nabla \frac{1}{r} &= \Delta \frac{1}{r} = \frac{1}{r} \frac{d}{dr} \left(r \frac{1}{r} \right) = 0 \quad (\text{at } r \neq 0), \\ \int_V dV \Delta \frac{1}{r} &= \int_S d\mathbf{S} \cdot \nabla \frac{1}{r} = \int_S d\mathbf{S} \cdot \frac{-\mathbf{r}}{r^2} = -4\pi, \\ \implies \Delta \frac{1}{r} &= -\nabla \cdot \frac{\mathbf{r}}{r^2} = -4\pi \delta^3(\mathbf{r}) \implies (\text{C.1.1}). \end{aligned}$$

derivation of (C.1.5)

$$\circ \frac{\mathbf{e}_\phi}{r \sin \theta} = -\frac{\sin \phi}{r \sin \theta} \mathbf{e}_x + \frac{\cos \phi}{r \sin \theta} \mathbf{e}_y = -\frac{y}{x^2 + y^2} \mathbf{e}_x + \frac{x^2}{x^2 + y^2} \mathbf{e}_y = \frac{1}{2} \mathbf{e}_z \times \nabla \log(x^2 + y^2),$$

$$\begin{aligned} \circ \nabla \times \frac{\mathbf{e}_\phi}{r \sin \theta} &= \frac{1}{2} \nabla \times (\mathbf{e}_z \times \nabla) \log(x^2 + y^2) \\ &= \frac{1}{2} \left(\mathbf{e}_z \nabla^2 - \underbrace{(\mathbf{e}_z \cdot \nabla) \nabla}_{=\partial_z} \right) \log(x^2 + y^2) \quad (\because a \times (b \times c) = (a \cdot c)b - (a \cdot b)c) \\ &= \frac{1}{2} \mathbf{e}_z \nabla^2 \log(x^2 + y^2) = 2\pi \mathbf{e}_z \delta(x) \delta(y) \quad (\because (\partial_x^2 + \partial_y^2) \log(\sqrt{x^2 + y^2}) = 2\pi \delta(x) \delta(y)), \end{aligned}$$

$$\circ \mathbf{A}^{N/S} = -\frac{g(\cos \theta \mp 1)}{r \sin \theta} \mathbf{e}_\phi,$$

$$\begin{aligned} \Rightarrow \nabla \times \mathbf{A}^{N/S} &= -\nabla \times \left(\frac{g(\cos \theta \mp 1)}{r \sin \theta} \mathbf{e}_\phi \right) = -g(\nabla \cos \theta) \times \frac{\mathbf{e}_\phi}{r \sin \theta} - g(\cos \theta \mp 1) \nabla \times \frac{\mathbf{e}_\phi}{r \sin \theta} \\ &= -g \frac{-\sin \theta}{r} \mathbf{e}_\theta \times \frac{\mathbf{e}_\phi}{r \sin \theta} - g(\cos \theta \mp 1) \nabla \times \frac{\mathbf{e}_\phi}{r \sin \theta} \\ &= g \frac{1}{r^2} \mathbf{e}_r - g(\cos \theta \mp 1) \nabla \times \frac{\mathbf{e}_\phi}{r \sin \theta} \\ &= g \frac{1}{r^2} \mathbf{e}_r - 2\pi g \mathbf{e}_z \underbrace{(\cos \theta \mp 1) \delta(x) \delta(y)}_{=\mp 2\theta(\mp z)} \\ &= g \frac{\mathbf{r}}{r} \pm 4\pi g \mathbf{e}_z \theta(\mp z) \delta(x) \delta(y). \end{aligned}$$

derivation of $(\partial_x^2 + \partial_y^2) \log(\sqrt{x^2 + y^2}) = 2\pi \delta(x) \delta(y)$

For $r = \sqrt{x^2 + y^2} \neq 0$,

$$(\partial_x^2 + \partial_y^2) \log r = \left[\frac{1}{r} \partial_r (r \partial_r) + \frac{1}{r^2} \partial_\theta^2 \right] \log r = \frac{1}{r} \partial_r (1) = 0,$$

$$\begin{aligned} \int dx dy \nabla^2 \log(r) &= \int dx dy \left[\partial_x \frac{x}{r^2} + \partial_y \frac{y}{r^2} \right] \\ &= \oint dy \frac{x}{r^2} - dx \frac{y}{r^2} \\ &= \oint (\sin \theta dr + r \cos \theta d\theta) \frac{\cos \theta}{r} - (\cos \theta dr - r \sin \theta d\theta) \frac{\sin \theta}{r} \\ &= \oint d\theta = 2\pi. \end{aligned}$$

$$\Rightarrow (\partial_x^2 + \partial_y^2) \log(\sqrt{x^2 + y^2}) = 2\pi \delta(x) \delta(y). \quad (\text{C.1.18})$$

Appendix D

Spin-weighted spherical harmonics

In this chapter, with reference to [124], we review the spin-weighted spherical harmonics and the Newman-Penrose δ (eth) formalism [123].

D.1 Spherical harmonics

First, we consider the three dimensional laplace equation

$$\nabla^2 f = \left[\frac{1}{r} \partial_r^2 r - \frac{1}{r^2} L^2 \right] f = 0, \quad (\text{D.1.1})$$

where $L = -i\mathbf{r} \times \nabla$ is the angular momentum operator. The solutions are given by the form

$$f(r, \theta, \phi) = r^l g(\theta, \phi), \quad (\text{D.1.2})$$

on the condition that $Lg(\theta, \phi) = l(l+1)g(\theta, \phi)$. The function $g(\theta, \phi)$ corresponds to the spherical harmonics of order l .

For the following discussions, we expand f by the polynomial in the Cartesian coordinates $(x, y, z) = (x^1, x^2, x^3)$ with l indices

$$f = d_{ij\dots k} \underbrace{x^i x^j \dots x^k}_l, \quad (\text{D.1.3})$$

where $d_{ij\dots k}$ is a constant and totally symmetric. Since f satisfy the Laplace equation $\nabla^2 f = 0$, $d_{ij\dots k}$ should satisfies

$$d_{iij\dots k} = 0. \quad (\text{D.1.4})$$

Then, when we rewrite f as

$$f = r^l d_{ij\dots k} N^i N^j \dots N^k, \quad (\text{D.1.5})$$

$$N^i = x^i / r, \quad (\text{D.1.6})$$

the sperical harmonics of order l is given by

$$g = d_{ij\dots k} N^i N^j \dots N^k. \quad (\text{D.1.7})$$

N_i corresponds to the point of S^2 and can be expressed as follows:

$$N_i = \hat{o}_A \sigma_i^A{}_B o^B, \quad (D.1.8)$$

where $\sigma_i^A{}_B$ is the pauli matrix and $\hat{o}_A, o^B$ are the 2 component spinors which transform by SU(2) matrix under the rotation of the point of S^2 .

$$o = \begin{pmatrix} o^1 \\ o^2 \end{pmatrix} = \begin{pmatrix} e^{-i\frac{\phi}{2}} \cos \frac{\theta}{2} \\ e^{i\frac{\phi}{2}} \sin \frac{\theta}{2} \end{pmatrix}, \quad \hat{o} = o^\dagger = (\hat{o}_1, \hat{o}_2) = \left(e^{i\frac{\phi}{2}} \cos \frac{\theta}{2}, e^{-i\frac{\phi}{2}} \sin \frac{\theta}{2} \right), \quad (D.1.9)$$

$$o^A \hat{o}_A = 1. \quad (D.1.10)$$

Then we introduce the Levi-Civita symbol

$$\epsilon_{AB} = \begin{cases} +1 & (A=1, B=2), \\ -1 & (A=2, B=1), \\ 0 & (\text{the others}), \end{cases} \quad \epsilon^{AB} = \begin{cases} -1 & (A=1, B=2), \\ +1 & (A=2, B=1), \\ 0 & (\text{the others}), \end{cases} \quad (D.1.11)$$

and use this in order for raising and lowering the index A, B

$$\hat{o}^A = \epsilon^{AB} \hat{o}_B, \quad \begin{pmatrix} \hat{o}^1 \\ \hat{o}^2 \end{pmatrix} = \begin{pmatrix} -e^{-i\frac{\phi}{2}} \sin \frac{\theta}{2} \\ e^{i\frac{\phi}{2}} \cos \frac{\theta}{2} \end{pmatrix}, \quad (D.1.12)$$

$$\sigma_{iAB} = \epsilon_{AC} \sigma_i^C{}_B, \quad \sigma_{1AB} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad \sigma_{2AB} = \begin{pmatrix} -i & 0 \\ 0 & -i \end{pmatrix}, \quad \sigma_{3AB} = \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}. \quad (D.1.13)$$

Since arbitrary spinors ξ^A, η^A satisfy

$$\xi^A \eta_A = \epsilon^{AB} \xi_B \eta_A = -\epsilon^{BA} \xi_B \eta_A = -\xi_B \eta^B, \quad (D.1.14)$$

(D.1.8) can be written as

$$N_i = -\sigma_{iAB} \hat{o}^A o^B. \quad (D.1.15)$$

Therefore the spherical harmonics is given by

$$g = d_{ij\dots k} N^i N^j \dots N^k \quad (D.1.16)$$

$$= d_{AB\dots CDE\dots F} \underbrace{o^A o^B \dots o^C}_{l} \underbrace{\hat{o}^D \hat{o}^E \dots \hat{o}^F}_{l}, \quad (D.1.17)$$

where we define $d_{AB\dots CDE\dots F}$ as

$$d_{AB\dots CD} = (-1)^l d_{ij\dots k} \sigma_{iAB} \dots \sigma_{kCD}. \quad (D.1.18)$$

$d_{ABCD\dots}$ satisfy

$$d_{ABCD\dots} = d_{BACD\dots} = d_{CDAB\dots} = d_{ACBD\dots}, \quad (D.1.19)$$

and thus $d_{ABCD\dots}$ is totally symmetric.

Because o^1 and $\hat{o}^1$ (o^2 and $\hat{o}^2$) have the factor $e^{-i\phi}$ ($e^{+i\phi}$), the spherical harmonics with n_1 indices of o^1 or $\hat{o}^1$ and n_2 indices of o^2 or $\hat{o}^2$ is the eigenfunction of $L_z = -i\frac{\partial}{\partial \phi}$ with the eigenvalue $m \equiv \frac{1}{2}(n_1 - n_2)$ ($m = -l, -l+1, \dots, l-1, l$). For example, spherical harmonics of order 1 is given by

$$d_{AB} o^A \hat{o}^B = d_{11} o^1 \hat{o}^1 + d_{12} (o^1 \hat{o}^2 + o^2 \hat{o}^1) + d_{22} o^2 \hat{o}^2$$

$$\begin{aligned}
 &= d_{11} \left(-\frac{1}{2} e^{-i\phi} \sin \theta \right) + d_{12} \cos \theta + d_{22} \left(\frac{1}{2} e^{i\phi} \sin \theta \right) \\
 &= d_{11} \left(-\sqrt{\frac{2\pi}{3}} Y_{1,-1} \right) + d_{12} \left(\sqrt{\frac{4\pi}{3}} Y_{1,0} \right) + d_{22} \left(-\sqrt{\frac{2\pi}{3}} Y_{1,1} \right). \tag{D.1.20}
 \end{aligned}$$

derivation of (D.1.4)

$$\begin{aligned}
 \nabla^2 f &= \partial_m \partial_m d_{ijk\dots n} x^i x^j x^k \dots x^n = \partial_m d_{ijk\dots n} \left[\delta_{mi} x^j \dots x^n + x_i \delta_{mj} x^k \dots x^n + \dots + x^i x^j \dots \delta_{mn} \right] \\
 &= l \partial_m d_{ij\dots n} \delta_{im} x^j x^k \dots x^n \quad (\because d_{ij\dots n} \text{ is totally symmetric}) \\
 &= l(l-1) d_{ijk\dots n} \delta_{mi} \delta_{mj} x^k \dots x^n \\
 &= l(l-1) d_{mmk\dots n} x^k \dots x^n = 0 \implies d_{mmk\dots n} = 0.
 \end{aligned}$$

derivation of (D.1.19)

$$\begin{aligned}
 \circ \sigma_{iAB} &= \sigma_{iBA} \implies d_{ABCD\dots} = d_{BACD\dots} \\
 \circ d_{ij\dots k} &= d_{ji\dots k} \implies d_{ABCD\dots} = d_{CDAB\dots} \\
 \circ d_{ABCD\dots} - d_{ACBD\dots} &= (-1)^l d_{ij\dots} (\sigma_{iAB} \sigma_{jCD} \dots - \sigma_{iAC} \sigma_{jBD} \dots) \\
 &= (-1)^l d_{ij\dots} \epsilon_{BC} \sigma_{iAE} \sigma_j^E \dots \quad (\because M_{AB} - M_{BA} = \epsilon_{AB} M_C^C) \\
 &= (-1)^l d_{ij\dots} \epsilon_{BC} \frac{1}{2} (\sigma_{iAE} \sigma_j^E \dots + \sigma_{jAE} \sigma_i^E \dots) \quad (\because M_{AB} - M_{BA} = \epsilon_{AB} M_C^C) \\
 &= (-1)^l d_{ij\dots} \epsilon_{BC} \delta_{ij} \epsilon_{AD} \dots \quad (\because \sigma_{iAE} \sigma_j^E \dots + \sigma_{jAE} \sigma_i^E \dots = 2\delta_{ij} \epsilon_{AD}) \\
 &= (-1)^l d_{ii\dots} \epsilon_{BC} \epsilon_{AD} \\
 &= 0.,
 \end{aligned}$$

D.2 Spin-weighted spherical harmonics

First we consider the transformation

$$o^A \rightarrow e^{i\frac{\alpha}{2}} o^A, \quad \hat{o}^A \rightarrow e^{-i\frac{\alpha}{2}} \hat{o}^A. \tag{D.2.1}$$

This transformation corresponds to the rotation for the vectors which form an orthogonal basis of the tangent plane of sphere at point N_i

$$\begin{cases} \mathbf{e}_r &= (N_1, N_2, N_3) = (\sin \theta \cos \phi, \sin \theta \sin \phi, \cos \theta), \\ \mathbf{e}_\theta + i\mathbf{e}_\phi &= (\sigma_{1AB} o^A o^B, \sigma_{2AB} o^A o^B, \sigma_{3AB} o^A o^B) = (\cos \theta \cos \phi - i \sin \phi, \cos \theta \sin \phi + i \cos \phi, -\sin \theta), \\ \mathbf{e}_\theta - i\mathbf{e}_\phi &= (\sigma_{1AB} \hat{o}^A \hat{o}^B, \sigma_{2AB} \hat{o}^A \hat{o}^B, \sigma_{3AB} \hat{o}^A \hat{o}^B) = (\cos \theta \cos \phi + i \sin \phi, \cos \theta \sin \phi - i \cos \phi, -\sin \theta), \end{cases} \tag{D.2.2}$$

$$\mathbf{e}_\theta \pm i\mathbf{e}_\phi \rightarrow e^{\pm i\alpha}(\mathbf{e}_\theta \pm i\mathbf{e}_\phi) \implies \begin{cases} \mathbf{e}_\theta \rightarrow \mathbf{e}_\theta \cos \alpha - \mathbf{e}_\phi \sin \alpha \\ \mathbf{e}_\phi \rightarrow \mathbf{e}_\theta \sin \alpha + \mathbf{e}_\phi \cos \alpha \end{cases}. \quad (\text{D.2.3})$$

We call that a quantity η has spin weight s , if η transforms as follows under the transformation $o^A \rightarrow e^{i\frac{\alpha}{2}} o^A$:

$$\eta \rightarrow e^{is\alpha} \eta. \quad (\text{D.2.4})$$

Thus, by definition, o^A and $\hat{o}^A$ have spin weight $1/2$ and $-1/2$ respectively.

In the previous section, we have considered the spherical harmonics (D.1.17), and we can find that spherical harmonics has spin weight 0. Here, we generalize it to the spin-weighted spherical harmonics with spin weight s :

$$d_{AB\dots CDE\dots F} \underbrace{o^A o^B \dots o^C}_{j+s} \underbrace{\hat{o}^D \hat{o}^E \dots \hat{o}^F}_{j-s}, \quad (\text{D.2.5})$$

where $j+s$, $j-s$ are the non-negative integer and furthermore, j and s are integer or half-integer which satisfy

$$j \geq |s|. \quad (\text{D.2.6})$$

Next, we consider the normalization of spin-weighted spherical harmonics. We can obtain the following integral

$$\begin{aligned} \int_{S^2} d\Omega \underbrace{o^A o^B \dots o^C}_n \underbrace{\hat{o}_P \hat{o}_Q \dots \hat{o}_R}_n &= \frac{4\pi}{n+1} \delta_P^{(A} \delta_Q^B \dots \delta_R^{C)} \\ &= \frac{4\pi}{n+1} \frac{1}{n!} \underbrace{(\delta_P^A \delta_Q^B \dots \delta_R^C + \delta_P^B \delta_Q^A \dots \delta_R^C + \dots)}_{\text{all displacements of } A, B, \dots, C}. \end{aligned} \quad (\text{D.2.7})$$

derivation of (D.2.7)

From the symmetry of the indices and transformation properties of rotation, the integral should be given by the form

$$\int_{S^2} d\Omega \underbrace{o^A o^B \dots o^C}_n \underbrace{\hat{o}_P \hat{o}_Q \dots \hat{o}_R}_n = \lambda_n \delta_P^{(A} \delta_Q^B \dots \delta_R^{C)}.$$

Then, we obtain following integral by contracting the pair of indices A and P :

$$\begin{aligned} \int_{S^2} d\Omega \underbrace{o^A o^B \dots o^C}_{n-1} \underbrace{\hat{o}_P \hat{o}_Q \dots \hat{o}_R}_{n'-1} &= \frac{4\pi}{n+1} \frac{1}{n!} \underbrace{(\delta_A^A \delta_Q^B \dots \delta_R^C + \delta_A^B \delta_Q^A \dots \delta_R^C + \dots)}_{\text{all displacements of } A, B, \dots, C} \\ &= \lambda_n \frac{1}{n} \underbrace{(2\delta_Q^{(B} \dots \delta_R^{C)} + \delta_Q^{(B} \dots \delta_R^{C)} + \dots)}_{n-1 \text{ terms}} \\ &= \lambda_n \frac{n+1}{n} \delta_Q^{(B} \dots \delta_R^{C)}. \end{aligned}$$

Therefore we can find that

$$\begin{aligned} (n+1)\lambda_n &= n\lambda_{n-1} = \dots = 2\lambda_1 = \lambda_0 \\ &= 4\pi, \\ \Rightarrow \lambda_n &= \frac{4\pi}{n+1}. \end{aligned}$$

Then, the inner product of two spin weighted spherical harmonics with spin weight s is given by

$$\begin{aligned} &\langle d_{A\dots BC\dots D} \underbrace{o^A \dots o^B \hat{o}^C \dots \hat{o}^D}_{j+s} | h_{P\dots QR\dots S} \underbrace{o^P \dots o^Q \hat{o}^R \dots \hat{o}^S}_{j'+s} \rangle \\ &= (-1)^{(j-s)} \hat{d}^{A\dots BC\dots D} h_{P\dots QR\dots S} \int_{S^2} d\Omega \hat{o}_A \dots \hat{o}_B o_C \dots o_D o^P \dots o^Q \hat{o}^R \dots \hat{o}^S \\ &\quad (\because (o^A)^\dagger = \hat{o}_A, (\hat{o}^A)^\dagger = -o_A, \hat{d}^{A\dots BC\dots D} \equiv (d_{A\dots BC\dots D})^\dagger) \\ &= (-1)^{j'-s} \hat{d}^{A\dots B}{}_{C\dots D} h_{P\dots Q}{}^{R\dots S} \int_{S^2} d\Omega \underbrace{o^C \dots o^D}_{j-s} \underbrace{o^P \dots o^Q}_{j'+s} \underbrace{\hat{o}_A \dots \hat{o}_B}_{j+s} \underbrace{\hat{o}_R \dots \hat{o}_S}_{j'-s} \\ &= (-1)^{j'-s} \frac{4\pi}{j+j'+1} \hat{d}^{A\dots B}{}_{C\dots D} h_{P\dots Q}{}^{R\dots S} \delta_A^{(C} \dots \delta_B^D \delta_R^P \dots \delta_S^{Q)}. \end{aligned} \quad (\text{D.2.8})$$

If $j \neq j'$, there are contractions of the indices in $\hat{d}^{A\dots B}{}_{C\dots D}$ or $h_{P\dots Q}{}^{R\dots S}$ and (D.2.8) is equal to 0. This means that two spherical harmonics with different angular momentum are orthogonal with each other.

In the case of $j = j'$, the inner product is obtained as

$$\begin{aligned} &\langle d_{A\dots BC\dots D} \underbrace{o^A \dots o^B \hat{o}^C \dots \hat{o}^D}_{j+s} | h_{P\dots QR\dots S} \underbrace{o^P \dots o^Q \hat{o}^R \dots \hat{o}^S}_{j-s} \rangle \\ &= (-1)^{j-s} \frac{4\pi}{2j+1} \frac{(j+s)!(j-s)!}{2j!} \hat{d}^{A\dots B}{}_{C\dots D} h_{A\dots B}{}^{C\dots D} = (-1)^{2j} \frac{4\pi}{2j+1} \frac{(j+s)!(j-s)!}{2j!} \hat{d}_{A\dots BC\dots D} h^{A\dots BC\dots D} \\ &= (-1)^{2j} \frac{4\pi}{2j+1} \frac{(j+s)!(j-s)!}{(2j)!} \sum_{m=-j}^j (-1)^{j+m} \frac{(2j)!}{(j+m)!(j-m)!} \hat{d}_{1\dots 1 \ 2\dots 2}{}_{j+m \ j-m} h_{2\dots 2 \ 1\dots 1}{}_{j+m \ j-m} \\ &= \frac{4\pi}{2j+1} \sum_{m=-j}^j \frac{(j+s)!(j-s)!}{(j+m)!(j-m)!} (d_{1\dots 1 \ 2\dots 2}{}_{j-m \ j+m})^\dagger h_{2\dots 2 \ 1\dots 1}{}_{j+m \ j-m}. \end{aligned} \quad (\text{D.2.9})$$

Therefore the normalized spin-weighted spherical harmonics ${}_s Y_{jm}(\theta, \phi)$ is given by

$$\begin{aligned} d_{A\dots BC\dots D} \underbrace{o^A \dots o^B \hat{o}^C \dots \hat{o}^D}_{j+s} &= \sum_{m=-j}^j \left(e^{-im\pi} \sqrt{\frac{4\pi}{2j+1} \frac{(j+s)!(j-s)!}{(j+m)!(j-m)!}} \right) d_{1\dots 1 \ 2\dots 2}{}_{j-m \ j+m} \\ &\quad \times \frac{(2j)!}{(j+m)!(j-m)!} \left(e^{im\pi} \sqrt{\frac{2j+1}{4\pi} \frac{(j+m)!(j-m)!}{(j+s)!(j-s)!}} \right) \underbrace{o^{(1} \dots o^2 \hat{o}^1 \dots \hat{o}^2)}_{j+s} \end{aligned}$$

$$= \sum_{m=-j}^j e^{-im\pi} \sqrt{\frac{4\pi}{2j+1} \frac{(j+s)!(j-s)!}{(j+m)!(j-m)!}} d_{\frac{1}{j-m} \dots \frac{1}{2} \dots \frac{2}{j+m}} s Y_{jm}(\theta, \phi), \quad (\text{D.2.10})$$

$$\begin{aligned} {}_s Y_{jm}(\theta, \phi) &= \left(e^{im\pi} \sqrt{\frac{2j+1}{4\pi} \frac{(j+m)!(j-m)!}{(j+s)!(j-s)!}} \right) \frac{(2j)!}{(j+m)!(j-m)!} \\ &\quad \times \sum_{k=\max\{0, -m-s\}}^{\min\{j-s, j-m\}} \frac{(j+s)!(j-s)!}{(2j)!} {}_{j-m} C_k {}_{j+m} C_{j-s-k} (\hat{o}^1)^k (\hat{o}^2)^{j-s-k} (o^1)^{j-m-k} (o^2)^{m+s+k} \\ &= e^{im\pi} \sqrt{\frac{2j+1}{4\pi} (j+m)!(j-m)!(j+s)!(j-s)!} \sum_{k=\max\{0, -m-s\}}^{\min\{j-s, j-m\}} \frac{(-1)^k \left(\sin \frac{\theta}{2}\right)^{m+s+2k} \left(\cos \frac{\theta}{2}\right)^{2j-m-s-2k}}{k!(j-m-k)!(j-s-k)!(m+s+k)!} e^{im\phi}. \end{aligned} \quad (\text{D.2.11})$$

D.3 δ and $\bar{\delta}$

We introduce the operators δ (eth) and $\bar{\delta}$ (eth bar) which raise and lower a spin weight. For η with spin weight s , δ and $\bar{\delta}$ act as follows:

$$\delta\eta = -\left[\partial_\theta + \frac{i}{\sin\theta}\partial_\phi - s\cot\theta\right]\eta = -(\sin\theta)^s \underbrace{\left[\partial_\theta + \frac{i}{\sin\theta}\partial_\phi\right]}_{=(\mathbf{e}_\theta + i\mathbf{e}_\phi) \cdot \mathbf{L}} (\sin\theta)^{-s}\eta, \quad (\text{D.3.1})$$

$$\bar{\delta}\eta = -\left[\partial_\theta - \frac{i}{\sin\theta}\partial_\phi + s\cot\theta\right]\eta = -(\sin\theta)^{-s} \underbrace{\left[\partial_\theta - \frac{i}{\sin\theta}\partial_\phi\right]}_{=(\mathbf{e}_\theta - i\mathbf{e}_\phi) \cdot \mathbf{L}} (\sin\theta)^s\eta, \quad (\text{D.3.2})$$

$$\left(\mathbf{L} = -i\mathbf{r} \times \nabla = -i\left(\mathbf{e}_\theta\partial_\theta - \mathbf{e}_\phi\frac{1}{\sin\theta}\partial_\phi\right)\right).$$

We can find that $\delta\eta$ has spin weight $s + 1$ and $\bar{\delta}\eta$ has spin weight $s - 1$, because $\mathbf{e}_\theta \pm i\mathbf{e}_\phi$ has spin weight ± 1 and the physical vector $\mathbf{L}$ doesn't transform under the rotation of the basis of tangent space.

From (D.3.1) and (D.3.2), we obtain

$$\begin{aligned} \bar{\delta}\delta\eta &= \left[\frac{1}{\sin\theta}\partial_\theta\sin\theta\partial_\theta + \frac{1}{\sin^2\theta}\partial_\phi^2 + 2is\frac{\cos\theta}{\sin^2\theta}\partial_\phi - \frac{s^2}{\sin^2\theta} + s(s+1)\right]\eta, \\ \delta\bar{\delta}\eta &= \left[\frac{1}{\sin\theta}\partial_\theta\sin\theta\partial_\theta + \frac{1}{\sin^2\theta}\partial_\phi^2 + 2is\frac{\cos\theta}{\sin^2\theta}\partial_\phi - \frac{s^2}{\sin^2\theta} + s(s-1)\right]\eta, \end{aligned} \quad (\text{D.3.3})$$

$$[\bar{\delta}, \delta]\eta = 2s\eta, \quad (\text{D.3.4})$$

and furthermore, for η with spin weight s and ξ with spin weight $s - 1$,

$$\langle\eta|\delta\xi\rangle = -\langle\bar{\delta}\eta|\xi\rangle. \quad (\text{D.3.5})$$

Especially, since the action of δ and $\bar{\delta}$ for two component spinors o^A and $\hat{o}^A$ is

$$\begin{aligned} \delta o^A &= 0, & \bar{\delta} o^A &= -\hat{o}^A, \\ \delta \hat{o}^A &= o^A, & \bar{\delta} \hat{o}^A &= 0, \end{aligned} \quad (\text{D.3.6})$$

the spin-weighted spherical harmonics with different spin weight are related by

$$\delta d_{AB\dots CDE\dots F} \underbrace{o^A o^B \dots o^C}_{j+s} \underbrace{\hat{o}^D \hat{o}^E \dots \hat{o}^F}_{j-s} = (j-s) d_{AB\dots CDE\dots F} \underbrace{o^A o^B \dots o^C}_{j+s+1} \underbrace{\hat{o}^D \hat{o}^E \dots \hat{o}^F}_{j-s-1}, \quad (\text{D.3.7})$$

$$\bar{\delta} d_{AB\dots CDE\dots F} \underbrace{o^A o^B \dots o^C}_{j+s} \underbrace{\hat{o}^D \hat{o}^E \dots \hat{o}^F}_{j-s} = -(j+s) d_{AB\dots CDE\dots F} \underbrace{o^A o^B \dots o^C}_{j+s-1} \underbrace{\hat{o}^D \hat{o}^E \dots \hat{o}^F}_{j-s+1}, \quad (\text{D.3.8})$$

and therefore

$$\delta {}_s Y_{jm} = \sqrt{j(j+1) - s(s+1)} {}_{s+1} Y_{jm}, \quad \bar{\delta} {}_s Y_{jm} = -\sqrt{j(j+1) - s(s-1)} {}_{s-1} Y_{jm}. \quad (\text{D.3.9})$$

Then the spin-weighted spherical harmonics satisfies the following equation from (D.3.3):

$$\bar{\delta}\delta {}_s Y_{jm} = -\{j(j+1) - s(s+1)\} {}_s Y_{jm}, \quad \delta\bar{\delta} {}_s Y_{jm} = -\{j(j+1) - s(s-1)\} {}_s Y_{jm}, \quad (\text{D.3.10})$$

or equivalently

$$\left[-\frac{1}{\sin \theta} \partial_\theta \sin \theta \partial_\theta - \frac{1}{\sin^2 \theta} \partial_\phi^2 - 2is \frac{\cos \theta}{\sin^2 \theta} \partial_\phi + \frac{s^2}{\sin^2 \theta} \right] {}_s Y_{jm} = j(j+1) {}_s Y_{jm}. \quad (\text{D.3.11})$$

derivation of (D.3.3)

$$\begin{aligned} \begin{pmatrix} \bar{\partial} \partial \eta \\ \partial \bar{\partial} \eta \end{pmatrix} &= \begin{bmatrix} \partial_\theta \mp \frac{i}{\sin \theta} \partial_\phi \pm (s+1) \cot \theta \\ \partial_\theta \pm \frac{i}{\sin \theta} \partial_\phi \mp s \cot \theta \end{bmatrix} \begin{bmatrix} \partial_\theta \pm \frac{i}{\sin \theta} \partial_\phi \mp s \cot \theta \\ \partial_\theta \mp \frac{i}{\sin \theta} \partial_\phi \pm s \cot \theta \end{bmatrix} \eta \\ &= \begin{bmatrix} \partial_\theta \mp \frac{i}{\sin \theta} \partial_\phi \pm s \cot \theta \\ \partial_\theta \pm \frac{i}{\sin \theta} \partial_\phi \mp s \cot \theta \end{bmatrix} \begin{bmatrix} \partial_\theta \pm \frac{i}{\sin \theta} \partial_\phi \mp s \cot \theta \\ \partial_\theta \mp \frac{i}{\sin \theta} \partial_\phi \pm s \cot \theta \end{bmatrix} \eta + \cot \theta \begin{bmatrix} \partial_\theta \pm \frac{i}{\sin \theta} \partial_\phi \mp s \cot \theta \\ \partial_\theta \mp \frac{i}{\sin \theta} \partial_\phi \pm s \cot \theta \end{bmatrix} \eta \\ &= \left\{ \partial_\theta^2 \pm \left(\partial_\theta \frac{i}{\sin \theta} \right) \partial_\phi \mp s(\partial_\theta \cot \theta) - \left(\frac{i}{\sin \theta} \partial_\phi \mp s \cot \theta \right)^2 + \cot \theta \left[\partial_\theta \pm \frac{i}{\sin \theta} \partial_\phi \mp s \cot \theta \right] \right\} \eta \\ &= \left\{ \partial_\theta^2 \mp i \frac{\cos \theta}{\sin^2 \theta} \partial_\phi \mp s \left(-1 - \frac{\cos^2 \theta}{\sin^2 \theta} \right) + \frac{1}{\sin^2 \theta} \partial_\phi^2 + 2is \frac{\cos \theta}{\sin^2 \theta} \partial_\phi \right. \\ &\quad \left. - s^2 \cot^2 \theta + \cot \theta \partial_\theta \pm i \frac{\cos \theta}{\sin^2 \theta} \partial_\phi \mp s \cot^2 \theta \right\} \eta \\ &= \left\{ \partial_\theta^2 + \cot \theta \partial_\theta + \frac{1}{\sin^2 \theta} \partial_\phi^2 + 2is \frac{\cos \theta}{\sin^2 \theta} \partial_\phi - s^2 \left(\frac{1}{\sin^2 \theta} - 1 \right) \pm s \right\} \eta \\ &= \left\{ \frac{1}{\sin \theta} \partial_\theta \sin \theta \partial_\theta + \frac{1}{\sin^2 \theta} \partial_\phi^2 + 2is \frac{\cos \theta}{\sin^2 \theta} \partial_\phi - \frac{s^2}{\sin^2 \theta} + s(s \pm 1) \right\} \eta. \end{aligned}$$

derivation of (D.3.5)

$$\begin{aligned} \langle \eta | \bar{\partial} \xi \rangle &= \int_{S^2} \sin \theta d\theta d\phi \eta^\dagger \bar{\partial} \xi = - \int_{S^2} d\theta d\phi \eta^\dagger (\sin \theta)^s \left[\vec{\partial}_\theta + \frac{i}{\sin \theta} \vec{\partial}_\phi \right] (\sin \theta)^{-s+1} \xi \\ &= \int_{S^2} d\theta d\phi \eta^\dagger (\sin \theta)^s \left[\overleftarrow{\partial}_\theta + \frac{i}{\sin \theta} \overleftarrow{\partial}_\phi \right] (\sin \theta)^{-s+1} \xi \\ &= \int_{S^2} d\theta d\phi \left(\left[\partial_\theta - \frac{i}{\sin \theta} \partial_\phi \right] (\sin \theta)^s \eta \right)^\dagger (\sin \theta)^{-s+1} \xi \\ &= \int_{S^2} \sin \theta d\theta d\phi \left((\sin \theta)^{-s} \left[\partial_\theta - \frac{i}{\sin \theta} \partial_\phi \right] (\sin \theta)^s \eta \right)^\dagger \xi \\ &= - \langle \bar{\partial} \eta | \xi \rangle. \end{aligned}$$

derivation of (D.3.6)

$$\begin{aligned}
\circ \begin{cases} \delta o^1 \\ \bar{\delta} o^1 \end{cases} &= - \left[\partial_\theta \pm \frac{i}{\sin \theta} \partial_\phi \mp \frac{1}{2} \cot \theta \right] e^{-i\frac{\phi}{2}} \cos \frac{\theta}{2} = -e^{-i\frac{\phi}{2}} \left[-\frac{1}{2} \sin \frac{\theta}{2} \pm \frac{1}{2} \frac{\cos \frac{\theta}{2}}{\sin \theta} \mp \frac{1}{2} \cot \theta \cos \frac{\theta}{2} \right] \\
&= -\frac{1}{2} e^{-i\frac{\phi}{2}} \left[-\sin \frac{\theta}{2} \pm \frac{1}{2} \frac{1}{\sin \frac{\theta}{2}} \mp \frac{1}{2} \frac{1 - 2 \sin^2 \frac{\theta}{2}}{\sin \frac{\theta}{2}} \right] = -\frac{1}{2} e^{-i\frac{\phi}{2}} \left[-\sin \frac{\theta}{2} \pm \sin \frac{\theta}{2} \right] = \begin{cases} 0 \\ -\delta^1 \end{cases} . \\
\circ \begin{cases} \delta o^2 \\ \bar{\delta} o^2 \end{cases} &= - \left[\partial_\theta \pm \frac{i}{\sin \theta} \partial_\phi \mp \frac{1}{2} \cot \theta \right] e^{i\frac{\phi}{2}} \sin \frac{\theta}{2} = -e^{i\frac{\phi}{2}} \left[\frac{1}{2} \cos \frac{\theta}{2} \mp \frac{1}{2} \frac{\sin \frac{\theta}{2}}{\sin \theta} \mp \frac{1}{2} \cot \theta \sin \frac{\theta}{2} \right] \\
&= -\frac{1}{2} e^{i\frac{\phi}{2}} \left[\cos \frac{\theta}{2} \mp \frac{1}{2} \frac{1}{\cos \frac{\theta}{2}} \mp \frac{1}{2} \frac{2 \cos^2 \frac{\theta}{2} - 1}{\cos \frac{\theta}{2}} \right] = -\frac{1}{2} e^{i\frac{\phi}{2}} \left[\cos \frac{\theta}{2} \mp \cos \frac{\theta}{2} \right] = \begin{cases} 0 \\ -\delta^2 \end{cases} . \\
\circ \begin{cases} \delta \hat{o}^1 \\ \bar{\delta} \hat{o}^1 \end{cases} &= \left[\partial_\theta \pm \frac{i}{\sin \theta} \partial_\phi \pm \frac{1}{2} \cot \theta \right] e^{-i\frac{\phi}{2}} \sin \frac{\theta}{2} = e^{-i\frac{\phi}{2}} \left[\frac{1}{2} \cos \frac{\theta}{2} \pm \frac{1}{2} \frac{\sin \frac{\theta}{2}}{\sin \theta} \pm \frac{1}{2} \cot \theta \sin \frac{\theta}{2} \right] \\
&= \frac{1}{2} e^{-i\frac{\phi}{2}} \left[\cos \frac{\theta}{2} \pm \cos \frac{\theta}{2} \right] = \begin{cases} o^1 \\ 0 \end{cases} . \\
\circ \begin{cases} \delta \hat{o}^2 \\ \bar{\delta} \hat{o}^2 \end{cases} &= - \left[\partial_\theta \pm \frac{i}{\sin \theta} \partial_\phi \pm \frac{1}{2} \cot \theta \right] e^{i\frac{\phi}{2}} \cos \frac{\theta}{2} = -e^{i\frac{\phi}{2}} \left[-\frac{1}{2} \sin \frac{\theta}{2} \mp \frac{1}{2} \frac{\cos \frac{\theta}{2}}{\sin \theta} \pm \frac{1}{2} \cot \theta \cos \frac{\theta}{2} \right] \\
&= -\frac{1}{2} e^{i\frac{\phi}{2}} \left[-\sin \frac{\theta}{2} \mp \sin \frac{\theta}{2} \right] = \begin{cases} o^2 \\ 0 \end{cases} .
\end{aligned}$$

derivation of (D.3.9)

$$\begin{aligned}
\delta d_{AB \dots CDE \dots F} \underbrace{o^A o^B \dots o^C}_{j+s} \underbrace{\hat{o}^D \hat{o}^E \dots \hat{o}^F}_{j-s} &= (j-s) d_{AB \dots CDE \dots F} \underbrace{o^A o^B \dots o^C}_{j+s+1} \underbrace{\hat{o}^D \hat{o}^E \dots \hat{o}^F}_{j-s-1}, \\
\bar{\delta} d_{AB \dots CDE \dots F} \underbrace{o^A o^B \dots o^C}_{j+s} \underbrace{\hat{o}^D \hat{o}^E \dots \hat{o}^F}_{j-s} &= -(j+s) d_{AB \dots CDE \dots F} \underbrace{o^A o^B \dots \hat{o}^C}_{j+s-1} \underbrace{\hat{o}^D \hat{o}^E \dots \hat{o}^F}_{j-s+1}, \\
\Rightarrow \sum_{m=-j}^j e^{-im\pi} \sqrt{\frac{4\pi}{2j+1} \frac{(j+s)!(j-s)!}{(j+m)!(j-m)!}} d_{\substack{1 \dots 1 \\ j-m}} \substack{2 \dots 2 \\ j+m} \times \begin{cases} \delta_s Y_{jm} \\ \bar{\delta}_s Y_{jm} \end{cases} & (\because \text{(D.2.10)}) \\
&= \pm(j \mp s) \sum_{m=-j}^j e^{-im\pi} \sqrt{\frac{4\pi}{2j+1} \frac{(j+s \pm 1)!(j-s \mp 1)!}{(j+m)!(j-m)!}} d_{\substack{1 \dots 1 \\ j-m}} \substack{2 \dots 2 \\ j+m} s \pm 1 Y_{jm} \\
&= \pm \sqrt{(j \mp s)(j \pm s + 1)} \sum_{m=-j}^j e^{-im\pi} \sqrt{\frac{4\pi}{2j+1} \frac{(j+s)!(j-s)!}{(j+m)!(j-m)!}} d_{\substack{1 \dots 1 \\ j-m}} \substack{2 \dots 2 \\ j+m} s \pm 1 Y_{jm} .
\end{aligned}$$

$\delta, \bar{\delta}$ and $-i\partial_\phi$ commute with each other, and thus $\delta_s Y_{jm}, \bar{\delta}_s Y_{jm}$ and $s \pm 1 Y_{jm}$ have the same eigenvalue m . Then we obtain

$$\begin{aligned}
\delta_s Y_{jm} &= \sqrt{(j-s)(j+s+1)}_{s+1} Y_{jm} = \sqrt{j(j+1) - s(s+1)}_{s+1} Y_{jm}, \\
\bar{\delta}_s Y_{jm} &= -\sqrt{(j+s)(j-s+1)}_{s+1} Y_{jm} = -\sqrt{j(j+1) - s(s-1)}_{s-1} Y_{jm}.
\end{aligned}$$

D.4 Properties

- complex conjugate

$$({}_sY_{jm}(\theta, \phi))^* = {}_sY_{jm}(\theta, 2\pi - \phi) = (-1)^{-m-s} {}_sY_{j-m}(\theta, \phi). \quad (\text{D.4.1})$$

- reflection for xz -plane

$${}_sY_{jm}(\theta, -\phi) = e^{2im\pi} ({}_sY_{jm}(\theta, \phi))^* = (-1)^{m-s} {}_sY_{j-m}(\theta, \phi). \quad (\text{D.4.2})$$

- reflection for xy -plane

$${}_sY_{jm}(\pi - \theta, \phi) = (-1)^{j-m} {}_sY_{jm}(\theta, \phi). \quad (\text{D.4.3})$$

- translation for ϕ

$${}_sY_{jm}(\theta, \phi + \pi) = e^{im\pi} {}_sY_{jm}(\theta, \phi). \quad (\text{D.4.4})$$

- reflection for origin

$${}_sY_{jm}(\pi - \theta, \phi + \pi) = e^{ij\pi} {}_sY_{jm}(\theta, \phi). \quad (\text{D.4.5})$$

derivation of (D.4.1)

$$\begin{aligned} ({}_sY_{jm}(\theta, \phi))^* &= e^{-im\pi} \sqrt{\frac{2j+1}{4\pi}} (j+m)!(j-m)!(j+s)!(j-s)! \\ &\quad \times \sum_{k=\max\{0, -m-s\}}^{\min\{j-s, j-m\}} \frac{(-1)^k \left(\sin \frac{\theta}{2}\right)^{m+s+2k} \left(\cos \frac{\theta}{2}\right)^{2j-m-s-2k}}{k!(j-m-k)!(j-s-k)!(m+s+k)!} e^{-im\phi} \\ &= \dots \sum_{k'=\max\{0, m+s\}}^{\min\{j+s, j+m\}} \frac{(-1)^{k'-m-s} \left(\sin \frac{\theta}{2}\right)^{-m-s+2k'} \left(\cos \frac{\theta}{2}\right)^{2j+m+s-2k'}}{(k'-m-s)!(j+s-k')!(j+m-k')!k'!} e^{-im\phi} \quad (k \rightarrow k' = k + m + s) \\ &= (-1)^{-m-s} {}_sY_{j-m}(\theta, \phi). \end{aligned}$$

derivation of (D.4.3)

$$\begin{aligned} {}_sY_{jm}(\pi - \theta, \phi) &= \dots \sum_{k=\max\{0, -m-s\}}^{\min\{j-s, j-m\}} \frac{(-1)^k \left(\cos \frac{\theta}{2}\right)^{m+s+2k} \left(\sin \frac{\theta}{2}\right)^{2j-m-s-2k}}{k!(j-m-k)!(j-s-k)!(m+s+k)!} e^{im\phi} \\ &= \dots \sum_{k'=\max\{0, -m+s\}}^{\min\{j+s, j-m\}} \frac{(-1)^{j-m-k'} \left(\cos \frac{\theta}{2}\right)^{2j-m+s-2k'} \left(\sin \frac{\theta}{2}\right)^{m-s+2k'}}{(j-m-k')!k'!(m-s+k')!(j+s-k')!} e^{im\phi} \quad (k \rightarrow k' = -k + j - m) \\ &= (-1)^{j-m} {}_sY_{jm}(\theta, \phi). \end{aligned}$$

D.5 Completeness

In this section, we show the completeness of the spin-weighted spherical harmonics which means any function f with spin weight s can be decomposed by ${}_s Y_{jm}$

$$f = \sum_{j=|s|}^{\infty} \sum_{m=-j}^j c_{jm} {}_s Y_{jm}. \quad (\text{D.5.1})$$

Here, we consider the case of $s > 0$. Then, the function with spin weight 0 can be obtained as

$$f \underbrace{\hat{\partial}^{A'} \hat{\partial}^{B'} \dots \hat{\partial}^{C'}}_{2s}. \quad (\text{D.5.2})$$

Therefore, the above function can be decomposed as follows due to the completeness of the usual spherical harmonics:

$$f \underbrace{\hat{\partial}^{A'} \hat{\partial}^{B'} \dots \hat{\partial}^{C'}}_{2s} = \sum_{j=0}^{\infty} b^{(j)A'B'\dots C'} d_{AB\dots CDE\dots F}^{(j)} \underbrace{o^{(A} o^B \dots o^C}_{j} \underbrace{\hat{\partial}^D \hat{\partial}^E \dots \hat{\partial}^F}_{j}, \quad (\text{D.5.3})$$

where $b^{(j)A'B'\dots C'}$, $d_{AB\dots CDE\dots F}^{(j)}$ are totally symmetric constants. Then, contracting with $o^{A'} o^{B'} \dots o^{C'}$, we obtain

$$f = \sum_{j=0}^{\infty} b_{A'B'\dots C'}^{(j)} d_{AB\dots CDE\dots F}^{(j)} \underbrace{o^{A'} o^{B'} \dots o^{C'}}_{2s} \underbrace{o^{(A} o^B \dots o^C}_{j} \underbrace{\hat{\partial}^D \hat{\partial}^E \dots \hat{\partial}^F}_{j}. \quad (\text{D.5.4})$$

Here, we can show that

$$\begin{aligned} & o^{C'} \underbrace{o^{(A} o^B \dots o^C}_{j} \underbrace{\hat{\partial}^D \hat{\partial}^E \dots \hat{\partial}^F}_{j'} \\ &= \frac{1}{j+j'+1} \left[\underbrace{o^{C'} o^{(A} o^B \dots o^C \hat{\partial}^D \hat{\partial}^E \dots \hat{\partial}^F}_{j+j'+1} + o^A o^{(C'} o^B \dots o^C \hat{\partial}^D \hat{\partial}^E \dots \hat{\partial}^F} + \dots + o^F o^{(A} o^B \dots o^C \hat{\partial}^D \hat{\partial}^E \dots \hat{\partial}^{C')} \right. \\ & \quad \left. + (j+j') o^{C'} o^{(A} o^B \dots o^C \hat{\partial}^D \hat{\partial}^E \dots \hat{\partial}^F} - o^A o^{(C'} o^B \dots o^C \hat{\partial}^D \hat{\partial}^E \dots \hat{\partial}^F} - \dots - o^F o^{(A} o^B \dots o^C \hat{\partial}^D \hat{\partial}^E \dots \hat{\partial}^{C')} \right] \\ &= \underbrace{o^{(C'} o^A o^B \dots o^C \hat{\partial}^D \hat{\partial}^E \dots \hat{\partial}^F}_{j+j'+1} \\ & \quad + \frac{1}{j+j'+1} \left[(j+j') o^{C'} o^{(A} o^B \dots o^C \hat{\partial}^D \hat{\partial}^E \dots \hat{\partial}^F} - o^A o^{(C'} o^B \dots o^C \hat{\partial}^D \hat{\partial}^E \dots \hat{\partial}^F} - \dots \right. \\ & \quad \left. - o^F o^{(A} o^B \dots o^C \hat{\partial}^D \hat{\partial}^E \dots \hat{\partial}^{C')} \right] \\ &= o^{(C'} o^A o^B \dots o^C \hat{\partial}^D \hat{\partial}^E \dots \hat{\partial}^F} \\ & \quad + \frac{1}{j+j'+1} \left[o^{C'} o^{(A} o^B \dots o^C \hat{\partial}^D \hat{\partial}^E \dots \hat{\partial}^F} - o^A o^{(C'} o^B \dots o^C \hat{\partial}^D \hat{\partial}^E \dots \hat{\partial}^F} \right. \\ & \quad + \dots \\ & \quad \left. + o^{C'} o^{(A} o^B \dots o^C \hat{\partial}^D \hat{\partial}^E \dots \hat{\partial}^F} - o^F o^{(A} o^B \dots o^C \hat{\partial}^D \hat{\partial}^E \dots \hat{\partial}^{C')} \right] \end{aligned}$$

$$= \underbrace{o^{(C'} o^A o^B \dots o^C \hat{o}^D \hat{o}^E \dots \hat{o}^F)}_{j+1} \quad (D.5.5)$$

$$- \frac{j'}{(j+j'+1)(j+j')} \left[\epsilon^{C'A} \underbrace{o^{(B} \dots o^C \hat{o}^D \hat{o}^E \dots \hat{o}^F)}_j + \dots + \epsilon^{C'F} \underbrace{o^{(A} o^B \dots o^C \hat{o}^D \hat{o}^E \dots \hat{o}^G)}_{j'-1} \right]. \quad (D.5.6)$$

$$\left(\begin{aligned} \because \underbrace{o^{(C'} o^A o^B \dots o^C \hat{o}^D \hat{o}^E \dots \hat{o}^F)}_j &= \frac{j}{j+j'} o^{C'} o^A \underbrace{o^{(B} \dots o^C \hat{o}^D \hat{o}^E \dots \hat{o}^F)}_{j-1} + \frac{j'}{j+j'} o^{C'} \hat{o}^A \underbrace{o^{(B} \dots o^C \hat{o}^D \hat{o}^E \dots \hat{o}^F)}_{j'-1} \\ o^{C'} o^A - o^A o^{C'} &= 0, \quad o^{C'} \hat{o}^A - o^A \hat{o}^{C'} = -\epsilon^{C'A} \end{aligned} \right)$$

Therefore, from above relation, f can be rewritten into the form

$$\begin{aligned} f &= \sum_{j=0}^{\infty} \sum_{k=0}^{2s} B_{AB\dots CDE\dots F}^{(jk)} \underbrace{o^{(A} o^B \dots o^C \hat{o}^D \hat{o}^E \dots \hat{o}^F)}_{j+2s-k} \underbrace{\phantom{o^{(A} o^B \dots o^C \hat{o}^D \hat{o}^E \dots \hat{o}^F)}}_{j-k} \\ &= \sum_{j'=s}^{\infty} \sum_{k=0}^{2s} B_{AB\dots CDE\dots F}^{(j'k)} \underbrace{o^{(A} o^B \dots o^C \hat{o}^D \hat{o}^E \dots \hat{o}^F)}_{j'+s-k} \underbrace{\phantom{o^{(A} o^B \dots o^C \hat{o}^D \hat{o}^E \dots \hat{o}^F)}}_{j'-s-k} \quad (j \rightarrow j' = j + s) \end{aligned} \quad (D.5.7)$$

where $B_{AB\dots CDE\dots F}^{(j')}$ is constant and totally symmetric. Then, from the definition (D.2.5), we find that f can be decomposed by the spin-weighted spherical harmonics with spin weight s .

D.6 su(2) algebra

The spin-weighted spherical harmonics corresponds to the representation of su(2) algebra:

$$[L_z, L_{\pm}] = \pm L_{\pm}, \quad [L_+, L_-] = 2L_z, \quad [L^2, L_z] = [L^2, L_{\pm}] = 0. \quad (D.6.1)$$

$$L^2 {}_s Y_{jm} = j(j+1) {}_s Y_{jm}, \quad (D.6.2)$$

$$L_z {}_s Y_{jm} = m {}_s Y_{jm}, \quad (D.6.3)$$

$$L_{\pm} {}_s Y_{jm} = \sqrt{(j \mp m)(j+1 \pm m)} {}_s Y_{jm \pm 1}, \quad (D.6.4)$$

where L^2, L_z and $L_{\pm}$ are angular momentum operators for quantities with spin weight s

$$L_x = i \left(\sin \phi \partial_{\theta} + \frac{\cos \theta \cos \phi}{\sin \theta} \partial_{\phi} + i s \frac{\cos \phi}{\sin \theta} \right), \quad (D.6.5)$$

$$L_y = -i \left(\cos \phi \partial_{\theta} - \frac{\cos \theta \sin \phi}{\sin \theta} \partial_{\phi} - i s \frac{\sin \phi}{\sin \theta} \right), \quad (D.6.6)$$

$$L_z = -i \partial_{\phi}, \quad (D.6.7)$$

$$L_{\pm} = \frac{1}{2} (L_x \pm i L_y) = e^{\pm i \phi} \left(\pm \partial_{\theta} + i \cot \theta \partial_{\phi} - \frac{s}{\sin \theta} \right), \quad (D.6.8)$$

$$\begin{aligned} L^2 &= L_x^2 + L_y^2 + L_z^2 = \frac{1}{2} (L_+ L_- + L_- L_+) + L_z^2 \\ &= -\frac{1}{\sin \theta} \partial_{\theta} \sin \theta \partial_{\theta} - \frac{1}{\sin^2 \theta} \partial_{\phi}^2 - 2i s \frac{\cos \theta}{\sin^2 \theta} \partial_{\phi} + \frac{s^2}{\sin^2 \theta}. \end{aligned} \quad (D.6.9)$$

These operators commute with δ and $\bar{\delta}$

$$[\delta, L_z] = [\bar{\delta}, L_z] = [\delta, L_{\pm}] = [\bar{\delta}, L_{\pm}] = 0. \quad (\text{D.6.10})$$

derivation of (D.6.9)

$$\begin{aligned} \circ L_{\pm} L_{\mp} &= e^{\pm i\phi} \left(\pm \partial_{\theta} + i \cot \theta \partial_{\phi} - \frac{s}{\sin \theta} \right) e^{\mp i\phi} \left(\mp \partial_{\theta} + i \cot \theta \partial_{\phi} - \frac{s}{\sin \theta} \right) \\ &= \left(\pm \partial_{\theta} + i \cot \theta \partial_{\phi} \pm \cot \theta - \frac{s}{\sin \theta} \right) \left(\mp \partial_{\theta} + i \cot \theta \partial_{\phi} - \frac{s}{\sin \theta} \right) \\ &= -\partial_{\theta}^2 \pm i(\partial_{\theta} \cot \theta) \partial_{\phi} \mp \left(\partial_{\theta} \frac{s}{\sin \theta} \right) + \left(i \cot \theta \partial_{\phi} - \frac{s}{\sin \theta} \right)^2 \pm \cot \theta \left(\mp \partial_{\theta} + i \cot \theta \partial_{\phi} - \frac{s}{\sin \theta} \right) \\ &= -\partial_{\theta}^2 \mp i \left(1 + \frac{\cos^2 \theta}{\sin^2 \theta} \right) \partial_{\phi} \pm s \frac{\cos \theta}{\sin^2 \theta} - \cot^2 \theta \partial_{\phi}^2 - 2is \frac{\cos \theta}{\sin^2 \theta} + \frac{s^2}{\sin^2 \theta} \\ &\quad - \cot \theta \partial_{\theta} \pm \cot^2 \theta \partial_{\phi} \mp s \frac{\cos \theta}{\sin^2 \theta} \\ &= -\frac{1}{\sin \theta} \partial_{\theta} \sin \theta \partial_{\theta} \mp i \partial_{\phi} - \frac{1}{\sin^2 \theta} \partial_{\phi}^2 + \partial_{\phi}^2 - 2is \frac{\cos \theta}{\sin^2 \theta} \partial_{\phi} + \frac{s^2}{\sin^2 \theta} \\ &= L^2 - L_z^2 \pm L_z, \\ \implies L^2 &= \frac{1}{2}(L_+ L_- + L_- L_+) + L_z^2. \end{aligned}$$

derivation of (D.6.1)

$$\begin{aligned} \circ [L_z, L_{\pm}] &= \left[-i \partial_{\phi} e^{\pm i\phi} \left(\pm \partial_{\theta} + i \cot \theta - \frac{s}{\sin \theta} \right) \right] = \pm e^{\pm i\phi} \left(\pm \partial_{\theta} + i \cot \theta - \frac{s}{\sin \theta} \right) \\ &= \pm L_{\pm}. \end{aligned}$$

$$\circ [L_+, L_-] = L_+ L_- - L_- L_+ = 2L_z \quad (\because L_{\pm} L_{\mp} = L^2 - L_z^2 \pm L_z).$$

derivation of (D.6.4)

$$\begin{aligned} L_{\pm} &= e^{\pm i\phi} \left(\pm \partial_{\theta} + i \frac{\cos \frac{\theta}{2}}{\sin \frac{\theta}{2}} \partial_{\phi} - \frac{1}{2 \sin \frac{\theta}{2} \cos \frac{\theta}{2}} (i \partial_{\phi} + s) \right), \\ \implies \\ \bullet L_{\pm} o^1 &= e^{\pm i\phi} \left(\pm \partial_{\theta} + i \frac{\cos \frac{\theta}{2}}{\sin \frac{\theta}{2}} \partial_{\phi} - \frac{1}{2 \sin \frac{\theta}{2} \cos \frac{\theta}{2}} \left(i \partial_{\phi} + \frac{1}{2} \right) \right) e^{-i\frac{\phi}{2}} \cos \frac{\theta}{2} \\ &= e^{\pm i\phi} e^{-i\frac{\phi}{2}} \left(\mp \frac{1}{2} \sin \frac{\theta}{2} + \frac{1}{2 \sin \frac{\theta}{2}} - \frac{1}{2} \sin \frac{\theta}{2} - \frac{1}{2 \sin \frac{\theta}{2}} \right) \end{aligned}$$

$$= \begin{cases} -e^{i\frac{\phi}{2}} \sin \frac{\theta}{2} \\ 0 \end{cases} = \begin{cases} -o^2 \\ 0 \end{cases},$$

$$\begin{aligned} \bullet L_{\pm} o^2 &= e^{\pm i\phi} \left(\pm \partial_{\theta} + i \frac{\cos \frac{\theta}{2}}{\sin \frac{\theta}{2}} \partial_{\phi} - \frac{1}{2 \sin \frac{\theta}{2} \cos \frac{\theta}{2}} \left(i \partial_{\phi} + \frac{1}{2} \right) \right) e^{i\frac{\phi}{2}} \sin \frac{\theta}{2} \\ &= e^{\pm i\phi} e^{i\frac{\phi}{2}} \left(\pm \frac{1}{2} \cos \frac{\theta}{2} - \frac{1}{2} \cos \frac{\theta}{2} \right) \\ &= \begin{cases} 0 \\ -e^{-i\frac{\phi}{2}} \cos \frac{\theta}{2} \end{cases} = \begin{cases} 0 \\ o^1 \end{cases}, \end{aligned}$$

$$\begin{aligned} \bullet L_{\pm} \hat{o}^1 &= -e^{\pm i\phi} \left(\pm \partial_{\theta} + i \frac{\cos \frac{\theta}{2}}{\sin \frac{\theta}{2}} \partial_{\phi} - \frac{1}{2 \sin \frac{\theta}{2} \cos \frac{\theta}{2}} \left(i \partial_{\phi} - \frac{1}{2} \right) \right) e^{-i\frac{\phi}{2}} \sin \frac{\theta}{2} \\ &= -e^{\pm i\phi} e^{-i\frac{\phi}{2}} \left(\pm \frac{1}{2} \cos \frac{\theta}{2} + \frac{1}{2} \cos \frac{\theta}{2} \right) \\ &= \begin{cases} -e^{i\frac{\phi}{2}} \cos \frac{\theta}{2} \\ 0 \end{cases} = \begin{cases} -\hat{o}^2 \\ 0 \end{cases}, \end{aligned}$$

$$\begin{aligned} \bullet L_{\pm} \hat{o}^2 &= e^{\pm i\phi} \left(\pm \partial_{\theta} + i \frac{\cos \frac{\theta}{2}}{\sin \frac{\theta}{2}} \partial_{\phi} - \frac{1}{2 \sin \frac{\theta}{2} \cos \frac{\theta}{2}} \left(i \partial_{\phi} - \frac{1}{2} \right) \right) e^{i\frac{\phi}{2}} \cos \frac{\theta}{2} \\ &= e^{\pm i\phi} e^{i\frac{\phi}{2}} \left(\mp \frac{1}{2} \sin \frac{\theta}{2} - \frac{1}{2 \sin \frac{\theta}{2}} + \frac{1}{2} \sin \frac{\theta}{2} + \frac{1}{2 \sin \frac{\theta}{2}} \right) \\ &= \begin{cases} 0 \\ e^{-i\frac{\phi}{2}} \sin \frac{\theta}{2} \end{cases} = \begin{cases} 0 \\ -\hat{o}^1 \end{cases}. \end{aligned}$$

$$\begin{aligned} \bullet L_{+s} Y_{jm} &= L_{+} e^{im\pi} \sqrt{\frac{2j+1}{4\pi} \frac{(j+s)!(j-s)!}{(j+m)!(j-m)!}} \\ &\quad \times \sum_{k=\max\{0, -m-s\}}^{\min\{j-s, j-m\}} j_{-m} C_{k \ j+m} C_{j-s-k} (\hat{o}^1)^k (\hat{o}^2)^{j-s-k} (o^1)^{j-m-k} (o^2)^{m+s+k} \quad (\because \text{(D.2.11)}) \\ &= \cdots (-) \sum_{k=\max\{0, -m-s\}}^{\min\{j-s, j-m\}} j_{-m} C_{k \ j+m} C_{j-s-k} \left[k (\hat{o}^1)^{k-1} (\hat{o}^2)^{j-s-k+1} (o^1)^{j-m-k} (o^2)^{m+s+k} \right. \\ &\quad \left. + (j-m-k) (\hat{o}^1)^k (\hat{o}^2)^{j-s-k} (o^1)^{j-m-k-1} (o^2)^{m+s+k+1} \right] \\ &= \cdots \left[- \sum_{k'=\max\{0, -m-s-1\}}^{\min\{j-s-1, j-m-1\}} \frac{j_{-m} C_{k'+1 \ j+m} C_{j-s-k'-1} (k' + 1)}{\times (\hat{o}^1)^{k'} (\hat{o}^2)^{j-s-k'} (o^1)^{j-m-k'-1} (o^2)^{m+s+k'+1}} \right] \quad (k \rightarrow k' = k - 1) \end{aligned}$$

$$\begin{aligned}
 & - \sum_{k=\max\{0, -m-s\}}^{\min\{j-s, j-m\}} \underbrace{j-m C_k}_{j-m} \underbrace{C_{j+m} C_{j-s-k}}_{j+m} (j-m-k) \\
 & \quad \times (\hat{o}^1)^k (\hat{o}^2)^{j-s-k} (o^1)^{j-m-k-1} (o^2)^{m+s+k+1} \Big] \\
 & = \cdots \left[- \sum_{k'=\max\{0, -m-s-1\}}^{\min\{j-s-1, j-m-1\}} \underbrace{j-m-1 C_{k'}}_{j-m-1} \underbrace{C_{j+m+1} C_{j-s-k'}}_{j+m+1} \frac{(j-m)(j-s-k')}{j+m+1} \right. \\
 & \quad \times (\hat{o}^1)^{k'} (\hat{o}^2)^{j-s-k'} (o^1)^{j-m-k'-1} (o^2)^{m+s+k'+1} \\
 & \quad - \sum_{k=\max\{0, -m-s\}}^{\min\{j-s, j-m\}} \underbrace{j-m-1 C_k}_{j-m-1} \underbrace{C_{j+m+1} C_{j-s-k}}_{j+m+1} \frac{(j-m)(m+s+k+1)}{j+m+1} \\
 & \quad \times (\hat{o}^1)^k (\hat{o}^2)^{j-s-k} (o^1)^{j-m-k-1} (o^2)^{m+s+k+1} \Big] \\
 & = \cdots - \sum_{k=\max\{0, -m-s-1\}}^{\min\{j-s, j-m-1\}} j-m-1 C_k C_{j+m+1} C_{j-s-k} \left[\frac{(j-m)(j-s-k)}{j+m+1} (\hat{o}^1)^k (\hat{o}^2)^{j-s-k} (o^1)^{j-m-k-1} (o^2)^{m+s+k+1} \right. \\
 & \quad \left. + \frac{(j-m)(m+s+k+1)}{j+m+1} (\hat{o}^1)^k (\hat{o}^2)^{j-s-k} (o^1)^{j-m-k-1} (o^2)^{m+s+k+1} \right] \\
 & = \cdots - (j-m) \sum_{k=\max\{0, -m-s-1\}}^{\min\{j-s, j-m-1\}} j-m-1 C_k C_{j+m+1} C_{j-s-k} \left[(\hat{o}^1)^k (\hat{o}^2)^{j-s-k} (o^1)^{j-m-k-1} (o^2)^{m+s+k+1} \right] \\
 & = \sqrt{(j-m)(j+m+1)} {}_s Y_{jm+1}.
 \end{aligned}$$

derivation of (D.6.10)

$$\begin{aligned}
 [\check{D}, L_{\pm}] &= \check{D} L_{\pm} - L_{\pm} \check{D} \\
 &= - \left(\partial_{\theta} + \frac{i}{\sin \theta} \partial_{\phi} - s \cot \theta \right) e^{\pm i \phi} \left(\pm \partial_{\theta} + i \cot \theta \partial_{\phi} - \frac{s}{\sin \theta} \right) \\
 & \quad + e^{\pm i \phi} \left(\pm \partial_{\theta} + i \cot \theta \partial_{\phi} - \frac{s+1}{\sin \theta} \right) \left(\partial_{\theta} + \frac{i}{\sin \theta} \partial_{\phi} - s \cot \theta \right) \\
 &= -e^{\pm i \phi} \left(\partial_{\theta} + \frac{i}{\sin \theta} \partial_{\phi} \mp \frac{1}{\sin \theta} - s \cot \theta \right) \left(\pm \partial_{\theta} + i \cot \theta \partial_{\phi} - \frac{s}{\sin \theta} \right) \\
 & \quad + e^{\pm i \phi} \left(\pm \partial_{\theta} + i \cot \theta \partial_{\phi} - \frac{s+1}{\sin \theta} \right) \left(\partial_{\theta} + \frac{i}{\sin \theta} \partial_{\phi} - s \cot \theta \right) \\
 &= -e^{\pm i \phi} \left[\left(\partial_{\theta} + \frac{i}{\sin \theta} \partial_{\phi} - s \cot \theta \right) \left(\pm \partial_{\theta} + i \cot \theta \partial_{\phi} - \frac{s}{\sin \theta} \right) \right. \\
 & \quad - \left(\pm \partial_{\theta} + i \cot \theta \partial_{\phi} - \frac{s}{\sin \theta} \right) \left(\partial_{\theta} + \frac{i}{\sin \theta} \partial_{\phi} - s \cot \theta \right) \\
 & \quad \left. \mp i \frac{\cot \theta}{\sin \theta} \partial_{\phi} \pm \frac{s}{\sin^2 \theta} + \frac{i}{\sin^2 \theta} \partial_{\phi} - s \frac{\cot \theta}{\sin \theta} \right]
 \end{aligned}$$

$$\begin{aligned}
 = -e^{\pm i\phi} & \left[\cancel{i(\partial_\theta \cot \theta) \partial_\phi} - \cancel{\partial_\theta \frac{s}{\sin \theta}} \mp \underbrace{\left(\partial_\theta \frac{i}{\sin \theta} \right)}_{\text{}} \partial_\phi \pm \underbrace{s \partial_\theta \cot \theta}_{\text{}} \mp \underbrace{i \frac{\cot \theta}{\sin \theta} \partial_\phi}_{\text{}} \right. \\
 & \left. \pm \underbrace{\frac{s}{\sin^2 \theta}}_{\text{}} + \cancel{\frac{i}{\sin^2 \theta} \partial_\phi} - \cancel{s \frac{\cot \theta}{\sin \theta}} \right] = 0.
 \end{aligned}$$

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