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(古典リー代数に対する高次カペリ元)

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Abstract

We construct the Capelli elements of lower degrees $C_k(u)$ $(k = 1, \dots, n)$ with a parameter u for the symplectic Lie algebras and orthogonal Lie algebras. They correspond to factorial Schur functions with parameter u attached to the column partitions (1^k). We also give explicit formulas for $C_k(u)$ arising from the expansion of $C_n(u)$ of the highest degree with respect to the parameter u. We use the Jacobi-Trudi formula for the factorial Schur functions $R_\lambda(x; u)$ to construct the higher Capelli elements $C_\lambda(u)$. They are expressed as determinants of matrices whose entries are Capelli elements of lower degrees.

Keywords: Classical Lie algebra, central element, higher Capelli element

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Introduction.

Let $\mathfrak{g}$ be a classical Lie algebra, i.e. one of the Lie algebras $\mathfrak{gl}_N$(general linear), $\mathfrak{o}_N$(orthogonal) and $\mathfrak{sp}_N$(symplectic). It is fundamental in representation theory that the center $\mathcal{ZU}(\mathfrak{g})$ of the universal enveloping algebra of $\mathfrak{g}$ is isomorphic to the ring $\mathbb{C}[x]^W$ of Weyl group invariant polynomials in certain variables $x = (x_1, \dots, x_n)$ through the Harish-Chandra isomorphism $\gamma : \mathcal{ZU}(\mathfrak{g}) \xrightarrow{\sim} \mathbb{C}[x]^W$. In this paper, we construct a class of central elements $C_\lambda(u) \in \mathcal{ZU}(\mathfrak{g})$ with a parameter u, which we call the *higher Capelli elements*. They are parametrized by partitions $\lambda = (\lambda_1, \dots, \lambda_n)$, and correspond by the Harish-Chandra isomorphism to the factorial Schur functions $R_\lambda(x; u) \in \mathbb{C}[x]^W$ with parameter u. (The definition of $R_\lambda(x; u)$ will be given below). In the $\mathfrak{gl}_N$ case, the higher Capelli elements $C_\lambda(0)$ with $u = 0$ have been constructed by Okounkov [9] and expressed in the form of quantum immanants.

The main point of this article is an explicit construction of the Capelli elements $C_k(u)$ *of lower degrees* $(k = 1, \dots, n)$ which correspond to factorial Schur functions attached to the column partitions (1^k). The higher Capelli element $C_\lambda(u)$ for an arbitrary partition λ is then obtained by applying the Jacobi-Trudi formula to the Capelli elements $C_k(u)$ of lower degrees. It is constructed as the determinant of a matrix whose entries are Capelli elements of lower degrees. This method of construction of the higher Capelli elements from the Capelli elements of lower degrees has already been discussed in our previous paper [5]. In the present paper, we mainly investigate explicit formulas for the Capelli elements $C_k(u)$ of lower degrees which arise from the expansion of the Capelli element $C_n(u)$ of the highest degree with respect to the parameter u.

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Following [7] and [10], we introduce the factorial Schur functions $R_\lambda^{(m)}(x;a)$ with a parameter a in m variables $x=(x_1,\dots,x_m)$. We first define the symbol $\langle z;a\rangle$ by

$$\langle z;a\rangle=\begin{cases} z-a & (\mathfrak{g}=\mathfrak{gl}_N) \\ z^2-a^2 & (\mathfrak{g}=\mathfrak{sp}_N,\mathfrak{o}_N)\end{cases}$$

depending on the choice of a classical Lie algebra, and the shifted factorials associated with $\langle z;a\rangle$ by

$$\langle z;a\rangle_k=\langle z;a\rangle\langle z;a+1\rangle\cdots\langle z;a+k-1\rangle \quad (k=0,1,2,\dots).$$

Then for each partition $\lambda=(\lambda_1,\dots,\lambda_m)$ with $l(\lambda)\le m$, we set

$$R_\lambda^{(m)}(x;a)=\frac{\det\left(\langle x_i;a\rangle_{\lambda_j+m-j}\right)_{1\le i,j\le m}}{\det\left(\langle x_i;a\rangle_{m-j}\right)_{1\le i,j\le m}}.$$

Note that, if $\lambda=(1^m)$ is the column partition of degree m, we have $R_{(1^m)}^{(m)}(x;a)=\langle x_1;a\rangle\cdots\langle x_m;a\rangle$. We remark that these functions $R_\lambda^{(m)}(x;a)$ form $\mathbb{C}$-basis of $\mathbb{C}[x_1,\dots,x_m]^{\mathfrak{S}_m}$ or $\mathbb{C}[x_1^2,\dots,x_m^2]^{\mathfrak{S}_m}$ according as $\mathfrak{g}=\mathfrak{gl}_N$ or $\mathfrak{g}=\mathfrak{sp}_N,\mathfrak{o}_N$. We simply write $R_\lambda^{(m)}(x;a)=R_\lambda(x;a)$ when the number of variables are obvious from the context.

In order to outline the idea of this paper, we now explain the case of $\mathfrak{gl}_N$ $(N=n)$ for comparison with the cases of $\mathfrak{sp}_N$ and $\mathfrak{o}_N$. It is a classical result due to Capelli [1] that the central elements $C_k(u)\in\mathcal{U}(\mathfrak{gl}_n)$ attached to column partitions $\left(1^k\right)$ $(k=1,\dots,n)$ are expressed as follows by column-determinants:

$$C_k(u)=\sum_{|I|=k}\det(\Pi_I-u)\in\mathcal{ZU}(\mathfrak{gl}_n),$$

$$\Pi_I=\begin{pmatrix} E_{i_1,i_1}+k-1 & E_{i_1,i_2} & \dots & E_{i_1,i_k} \\ E_{i_2,i_1} & E_{i_2,i_2}+k-2 & \dots & E_{i_2,i_k} \\ \vdots & \vdots & \ddots & \vdots \\ E_{i_k,i_1} & E_{i_k,i_2} & \dots & E_{i_k,i_k}\end{pmatrix},$$

where E_{ij} $(1\le i,j\le n)$ denote the elements of $\mathfrak{gl}_n$ corresponding to matrix units. According to Schur's lemma, each $C_k(u)$ acts on the irreducible representation (V_μ,π_μ) attached to a partition μ by scalar (eigenvalue). We can calculate the eigenvalue as follows:

$$\pi_\mu(C_k(u))=\sum_{1\le i_1<\dots<i_k\le n}\prod_{m=1}^{k}(\mu_{i_m}+k-m-u).$$

This eigenvalue corresponds to the factorial Schur function $R_{(1^k)}(x;u)\in\mathbb{C}[x]^{\mathfrak{S}_n}$ $(W=\mathfrak{S}_n)$ in $x=(x_1,\dots,x_n)$ under the identification of variables $x_i=\mu_i+n-i$ $(1\le i\le n)$. This means that $\gamma(C_k(u))=R_{(1^k)}(x;u)$. We remark that the Capelli elements $C_k(u)$ of lower degrees are obtained as the coefficients in the expansion

$$C_n(z)=\sum_{k=0}^{n}(-1)^k\langle z;u\rangle_k C_{n-k}(u)$$

of the Capelli element $C_n(z)$ of highest degree in terms of the shifted factorials. As we proved in [5], for an arbitrary partition λ, the higher Capelli element $C_\lambda(u)$ with Harish-Chandra image $\gamma(C_\lambda(u))=R_\lambda(x;u)$ is then constructed by means of the Jacobi-Trudi determinant

$$C_\lambda(u)=\det\left(C_{\lambda'_i-i+j}(u+j-1)\right)_{i,j=1}^{m}$$

of Capelli elements of lower degrees, where $\lambda' = (\lambda'_1, \ldots, \lambda'_m)$ denotes the conjugate partition of λ. It is known by Okounkov [9] that $C_\lambda(0)$ with parameter $u = 0$ is expressed as a quantum immanant

$$C_\lambda(0) = \frac{\chi_\lambda(1)}{p!} \sum_{I \in \{1,\ldots,n\}^p} \sum_{\sigma \in \mathfrak{S}_p} \langle \sigma \cdot \xi_T, \xi_T \rangle E_{i_1, i_{\sigma(1)}}(-c_T(1)) \cdots E_{i_p, i_{\sigma(p)}}(-c_T(p)),$$

where $|\lambda| = p$ (see [9] and [4] for details).

As for the symplectic Lie algebra $\mathfrak{sp}_{2n}$, Itoh [3] has constructed the Capelli element $C^{\mathfrak{sp}}(u) \in \mathcal{ZU}(\mathfrak{sp}_{2n})$ of the highest degree with a parameter u. This $C^{\mathfrak{sp}}(u)$ corresponds to $C_n(u)$ in the $\mathfrak{gl}_n$ case, and is expressed by a symmetrized determinant. To be more precise, $(-1)^n u^{-1} C^{\mathfrak{sp}}(u)$ corresponds to the factorial Schur function $R^{(n)}_{(1^n)}(x; u)$ by the Harish-Chandra isomorphism; in this case of $\mathfrak{sp}_{2n}$, $\mathbb{C}[x]^W = \mathbb{C}[x_1^2, \ldots, x_n^2]^{\mathfrak{S}_n}$. In Section 1, we deal with the Capelli elements of lower degrees for $\mathfrak{sp}_{2n}$. We give in Subsection 1.1 an alternative expression (Theorem 1.4) of Itoh's Capelli element $C^{\mathfrak{sp}}(u)$, putting the parameter u out of the symmetrized determinant. In Subsection 1.2, we expand the expression of Theorem 1.4 in terms of the shifted factorials $\langle u; a \rangle_k$ in order to construct the Capelli elements $C_k(a)$ of lower degrees. In this way, we obtain an explicit formula (Theorem 1.9) for the central elements $C_k(a)$ which correspond to the factorial Schur functions $R^{(n)}_{(1^k)}(x; a)$ by the Harish-Chandra isomorphism. Explicit formulas of Theorems 1.4 and 1.9 are the main results of this paper for the case of $\mathfrak{sp}_{2n}$. In Subsection 1.3, we construct the higher Capelli elements $C_\lambda(u)$ for an arbitrary partition λ by applying the Jacobi-Trudi formula for the factorial Schur functions to $C_k(a)$. This method of construction of the higher Capelli elements is already included in our previous paper [5]. It would be an interesting problem to find an expression of the higher Capelli elements $C_\lambda(u)$ in terms of quantum immanants.

In Sections 2 and 3, we treat the cases of $\mathfrak{o}_{2n}$ (type D) and $\mathfrak{o}_{2n+1}$ (type B) cases respectively. As for the orthogonal Lie algebra $\mathfrak{o}_N$, Wachi [11] has constructed the Capelli element $C^{\mathfrak{o}}(u) \in \mathcal{ZU}(\mathfrak{o})$ of the highest degree with a parameter u. Similarly to the case of $\mathfrak{sp}_{2n}$ (type C), we construct the Capelli elements of lower degrees by expanding Wachi's Capelli element $C^{\mathfrak{o}}(u)$ with respect to the parameter u. Explicit formulas for the Capelli elements of lower degrees in Theorems 2.2 and 3.1 are our main results for the case of $\mathfrak{o}_N$. In Section 4, we include the corresponding explicit formula for the Capelli elements of lower degrees of the case of $\mathfrak{gl}_n$ for comparison with those of the cases of $\mathfrak{sp}_{2n}$ and $\mathfrak{o}_N$.

1 Higher Capelli elements in the case of $\mathfrak{sp}_{2n}$

1.1 Putting the parameter u out of the symmetrized determinant of $C^{\mathfrak{sp}}(u)$

The symplectic Lie algebra $\mathfrak{sp}_{2n}$ is defined by

$$\mathfrak{sp}_{2n} = \{X \in \mathrm{Mat}_{2n}(\mathbb{C}) | {}^t X J + J X = 0\}$$

with the skew-symmetric matrix $J = \begin{pmatrix} 0 & -1_n \\ 1_n & 0 \end{pmatrix}$. Setting

$$F^{\mathfrak{sp}}_{ij} = E_{ij} - J^{-1} E_{ji} J \in \mathfrak{sp}_{2n} \quad (1 \leq i, j \leq 2n),$$

we introduce a $2n \times 2n$ matrix

$$F^{\mathfrak{sp}} = \left(F^{\mathfrak{sp}}_{ij}\right)_{1 \le i,j \le 2n}$$

with entries in the universal enveloping algebra $\mathcal{U}(\mathfrak{sp}_{2n})$. Itoh [3] defined a central element $C^{\mathfrak{sp}}(u) \in \mathcal{U}(\mathfrak{sp}_{2n})$ with parameter u which corresponds to $C_n(u)$ in the $\mathfrak{gl}_n$ case. This central element $C^{\mathfrak{sp}}(u)$ is expressed by the symmetrized determinant Det. For a general $k \times k$ matrix Z with non-commuting entries, the symmetrized determinant $\mathrm{Det}(Z; a_1, \ldots, a_k)$ with parameters $a_1, \ldots, a_k$ is defined by

$$\mathrm{Det}(Z; a_1, \ldots, a_k) = \frac{1}{k!} \sum_{\sigma, \tau \in \mathfrak{S}_k} \mathrm{sgn}(\sigma)\,\mathrm{sgn}(\tau) Z_{\sigma(1),\tau(1)}(a_1) \cdots Z_{\sigma(k),\tau(k)}(a_k)$$

where $Z_{i,j}(a) = Z_{i,j} + a\delta_{i,j}$ and $\delta_{i,j}$ denotes the Kronecker delta. When $a_1 = \cdots = a_k = 0$, we simply write $\mathrm{Det}(Z; 0, \ldots, 0) = \mathrm{Det}(Z)$. Following Itoh, we set

$$C^{\mathfrak{sp}}(u) = \mathrm{Det}(\widehat{F}^{\mathfrak{sp}} + u 1_{2n+1}; n, n-1, \ldots, -n)$$

with a $(2n+1) \times (2n+1)$ matrix

$$\widehat{F}^{\mathfrak{sp}} = \begin{pmatrix} & & & 0 \\ & F^{\mathfrak{sp}} & & \vdots \\ & & & \vdots \\ 0 & \ldots & \ldots & 0 \end{pmatrix}$$

whose entries of the last row and the last column are all zero. This central element $C^{\mathfrak{sp}}(u)$ is also called the *Capelli element*. From Subsection 1.1 to Subsection 1.3, we express $C^{\mathfrak{sp}}(u) = C(u)$ and $F^{\mathfrak{sp}} = F$ for simplicity. In this section, we rewrite $C(u)$ removing the hat of $\widehat{F}^{\mathfrak{sp}}$ and putting parameter u out of the symmetrized determinant. We introduce the polarization of the symmetrized determinant $\mathrm{Det}(A)$ by

$$P(A^{(1)}, \ldots, A^{(n)}) = \frac{1}{n!} \sum_{\sigma, \tau \in \mathfrak{S}_n} \mathrm{sgn}(\sigma)\,\mathrm{sgn}(\tau) A^{(1)}_{\sigma(1),\tau(1)} \cdots A^{(n)}_{\sigma(n),\tau(n)}$$

for $A^{(i)} \in \mathrm{Mat}_n$ $(i = 1, \ldots, n)$ so that $\mathrm{Det}(A; a_1, \ldots, a_n) = P(A + a_1 I, \ldots, A + a_n I)$.

Proposition 1.1. *Let A be a general $n \times n$ matrix with non-commuting entries. Then, for $a_i = b_i + c_i$ $(i = 1, \ldots, n)$, we have*

$$\mathrm{Det}(A; a_1, \ldots, a_n) = \sum_{r+s=n} \binom{n}{r}^{-1} \sum_{\substack{1 \le k_1 < \cdots < k_r \le n \\ 1 \le l_1 < \cdots < l_s \le n \\ K \bigsqcup L = \{1, \ldots, n\}}} c_{k_1} \cdots c_{k_r} \sum_{\substack{J \subseteq \{1, \ldots, n\} \\ |J| = s}} \mathrm{Det}(A_{J,J}; b_{l_1}, \ldots, b_{l_s}).$$

Here, $K = \{k_1, \ldots, k_r\}, L = \{l_1, \ldots, l_s\}$ and

$$A_{J,J} = \begin{pmatrix} A_{j_1,j_1} & \ldots & A_{j_1,j_s} \\ \vdots & \vdots & \vdots \\ A_{j_s,j_1} & \ldots & A_{j_s,j_s} \end{pmatrix}, \quad J = \{j_1, \ldots, j_s\}.$$

Proof. Since the matrix A and the identity matrix I commute with each other, we have

$$\begin{aligned}
\mathrm{Det}(A; a_1, \ldots, a_n) &= P(A + a_1 I, \ldots, A + a_n I) \\
&= P(A + b_1 I + c_1 I, \ldots, A + b_n I + c_n I) \\
&= \frac{1}{n!} \sum_{\sigma,\tau \in \mathfrak{S}_n} \mathrm{sgn}(\sigma)\,\mathrm{sgn}(\tau)\,(A + b_1 I + c_1 I)_{\sigma(1),\tau(1)} \cdots (A + b_n I + c_n I)_{\sigma(n),\tau(n)} \\
&= \frac{1}{n!} \sum_{\sigma,\tau} \sum_{\substack{1 \le k_1 < \cdots < k_r \le n \\ 1 \le l_1 < \cdots < l_s \le n \\ K \bigsqcup L = \{1,\ldots,n\}}} \mathrm{sgn}(\sigma)\,\mathrm{sgn}(\tau)\,(c_{k_1} I)_{\sigma(k_1),\tau(k_1)} \cdots (c_{k_r} I)_{\sigma(k_r),\tau(k_r)} \\
&\qquad \cdot (A + b_{l_1} I)_{\sigma(l_1),\tau(l_1)} \cdots (A + b_{l_s} I)_{\sigma(l_s),\tau(l_s)}\,.
\end{aligned}$$

Here, we put $\eta = \begin{pmatrix} 1 & \ldots & r & r+1 & \ldots & n \\ k_1 & \ldots & k_r & l_1 & \ldots & l_s \end{pmatrix}$, $\sigma' = \sigma\eta,\ \tau' = \tau\eta$. Then we have

$$\begin{aligned}
\mathrm{Det}(A; a_1, \ldots, a_n) &= \frac{1}{n!} \sum_{\sigma',\tau'} \sum_{\substack{1 \le k_1 < \cdots < k_r \le n \\ 1 \le l_1 < \cdots < l_s \le n \\ K \bigsqcup L = \{1,\ldots,n\}}} \mathrm{sgn}(\sigma'\eta^{-1})\,\mathrm{sgn}(\tau'\eta^{-1})(c_{k_1} I)_{\sigma'(1),\tau'(1)} \cdots (c_{k_r} I)_{\sigma'(r),\tau'(r)} \\
&\qquad \cdot (A + b_{l_1} I)_{\sigma'(r+1),\tau'(r+1)} \cdots (A + b_{l_s} I)_{\sigma'(n),\tau'(n)} \\
&= \frac{1}{n!} \sum_{\sigma,\tau} \sum_{\substack{1 \le k_1 < \cdots < k_r \le n \\ 1 \le l_1 < \cdots < l_s \le n \\ K \bigsqcup L = \{1,\ldots,n\}}} \mathrm{sgn}(\sigma)\,\mathrm{sgn}(\tau)(c_{k_1} I)_{\sigma(1),\tau(1)} \cdots (c_{k_r} I)_{\sigma(r),\tau(r)} \\
&\qquad \cdot (A + b_{l_1} I)_{\sigma(r+1),\tau(r+1)} \cdots (A + b_{l_s} I)_{\sigma(n),\tau(n)} \\
&= \frac{1}{n!} \sum_{\substack{\sigma,\tau \\ \sigma(1)=\tau(1)=m_1 \\ \vdots \\ \sigma(r)=\tau(r)=m_r}} \sum_{\substack{1 \le k_1 < \cdots < k_r \le n \\ 1 \le l_1 < \cdots < l_s \le n \\ K \bigsqcup L = \{1,\ldots,n\}}} \mathrm{sgn}(\sigma)\,\mathrm{sgn}(\tau) c_{k_1} \cdots c_{k_r} \\
&\qquad \cdot (A + b_{l_1} I)_{\sigma(r+1),\tau(r+1)} \cdots (A + b_{l_s} I)_{\sigma(n),\tau(n)}.
\end{aligned}$$

Taking the decomposition

$$\{i_1 < \cdots < i_r\} \bigsqcup \{j_1 < \cdots < j_s\} = \{1, \ldots, n\}$$

of the indexing set such that $\{m_1, \ldots, m_r\} = \{i_1, \ldots, i_r\}$, we express the pair σ, τ as

$$\sigma = \begin{pmatrix} 1 & \cdots & r & r+1 & \cdots & n \\ i_{\rho(1)} & \cdots & i_{\rho(r)} & j_{\pi(1)} & \cdots & j_{\pi(s)} \end{pmatrix}, \quad \tau = \begin{pmatrix} 1 & \cdots & r & r+1 & \cdots & n \\ i_{\rho(1)} & \cdots & i_{\rho(r)} & j_{\eta(1)} & \cdots & j_{\eta(s)} \end{pmatrix}$$

by permutations $\rho \in \mathfrak{S}_r$ and $\pi, \eta \in \mathfrak{S}_s$. Since

$$\mathrm{sgn}(\sigma) = \mathrm{sgn} \begin{pmatrix} 1 & \cdots & r & r+1 & \cdots & n \\ i_1 & \cdots & i_r & j_1 & \cdots & j_s \end{pmatrix} \mathrm{sgn}(\rho)\,\mathrm{sgn}(\pi),$$

$$\mathrm{sgn}(\tau) = \mathrm{sgn} \begin{pmatrix} 1 & \cdots & r & r+1 & \cdots & n \\ i_1 & \cdots & i_r & j_1 & \cdots & j_s \end{pmatrix} \mathrm{sgn}(\rho)\,\mathrm{sgn}(\eta),$$

we obtain

$$\begin{aligned}\mathrm{Det}(A;a_1,\dots,a_n) &= \frac{1}{n!}\sum_{\substack{\rho\in\mathfrak{S}_r\\ \pi,\eta\in\mathfrak{S}_s}}\ \sum_{\substack{1\le i_1<\cdots<i_r\le n\\ 1\le j_1<\cdots<j_s\le n\\ I\bigsqcup J=\{1,\dots,n\}}}\ \sum_{\substack{1\le k_1<\cdots<k_r\le n\\ 1\le l_1<\cdots<l_s\le n\\ K\bigsqcup L=\{1,\dots,n\}}} \mathrm{sgn}(\pi)\,\mathrm{sgn}(\eta)c_{k_1}\cdots c_{k_r}\\ &\qquad\cdot(A+b_{l_1}I)_{j_{\pi(1)},j_{\eta(1)}}\cdots(A+b_{l_s}I)_{j_{\pi(s)},j_{\eta(s)}}\\ &= \sum_{\substack{1\le k_1<\cdots<k_r\le n\\ 1\le l_1<\cdots<l_s\le n\\ K\bigsqcup L=\{1,\dots,n\}}} \frac{r!s!}{n!}c_{k_1}\cdots c_{k_r}\sum_{\substack{J\subseteq\{1,\dots,n\}\\ |J|=s}} P((A+b_{l_1}I)_{JJ},\dots,(A+b_{l_s}I)_{JJ})\\ &= \sum_{r+s=n}\binom{n}{r}^{-1}\sum_{\substack{1\le k_1<\cdots<k_r\le n\\ 1\le l_1<\cdots<l_s\le n\\ K\bigsqcup L=\{1,\dots,n\}}} c_{k_1}\cdots c_{k_r}\sum_{\substack{J\subseteq\{1,\dots,n\}\\ |J|=s}}\mathrm{Det}(A_{J,J};b_{l_1},\dots,b_{l_s}).\end{aligned}$$

□

Removing the hat of $\widehat{F}$ in $C(u)$, we express $C(u)$ as a linear combination of symmetrized determinants of F.

Proposition 1.2. *The Capelli element $C(u)$ of the highest degree can be expressed as follows:*

$$C(u) = \frac{1}{2n+1}\sum_{k=1}^{2n+1}(u+n-k+1)\,\mathrm{Det}\left(F;u+n,u+n-1,\dots,u+\widehat{n-k}+1,\dots,u-n\right).$$

Here, $u+\widehat{n-k}+1$ means removing $u+n-k+1$.

Proof.

$$\begin{aligned}C(u) &= \mathrm{Det}(\widehat{F}+uI;n,n-1,\dots,-n) = \mathrm{Det}(\widehat{F};u+n,u+n-1,\dots,u-n)\\ &= \frac{1}{(2n+1)!}\sum_{\sigma,\tau\in\mathfrak{S}_{2n+1}}\mathrm{sgn}(\sigma)\,\mathrm{sgn}(\tau)\widehat{F}_{\sigma(1)\tau(1)}(u+n)\cdots\widehat{F}_{\sigma(2n+1)\tau(2n+1)}(u-n)\\ &= \frac{1}{(2n+1)!}\sum_{k=1}^{2n+1}\sum_{\sigma(k)=\tau(k)=2n+1}\mathrm{sgn}(\sigma)\,\mathrm{sgn}(\tau)\widehat{F}_{\sigma(1)\tau(1)}(u+n)\cdots(u+n-k+1)\cdots\widehat{F}_{\sigma(2n+1)\tau(2n+1)}(u-n)\\ &= \frac{1}{(2n+1)!}(u+n-k+1)\\ &\qquad\sum_{k=1}^{2n+1}\sum_{\substack{\sigma',\tau'\in\mathfrak{S}_{2n+1}\\ \sigma'(2n+1)=\tau'(2n+1)=2n+1}}\mathrm{sgn}(\sigma')\,\mathrm{sgn}(\tau')\widehat{F}_{\sigma'(1)\tau'(1)}(u+n)\cdots\widehat{F}_{\sigma'(k-1)\tau'(k-1)}(u+n-k+2)\\ &\qquad\cdot\widehat{F}_{\sigma'(k)\tau'(k)}(u+n-k)\cdots\widehat{F}_{\sigma'(2n)\tau'(2n)}(u-n)\\ &= \frac{1}{2n+1}\sum_{k=1}^{2n+1}(u+n-k+1)\,\mathrm{Det}\left(F;u+n,u+n-1,\dots,u+\widehat{n-k}+1,\dots,u-n\right).\end{aligned}$$

□

In the following, for each $s=0,\dots,2n$ we denote by

$$\mathrm{Det}_s(F) = \sum_{|I|=s}\mathrm{Det}(F_{I,I})$$

the sum of all $s\times s$ symmetrized principal minor determinants of F. The next lemma follows from the property of $\mathfrak{sp}_{2n}$ only.

Lemma 1.3. *If s is odd,* $\operatorname{Det}_s(F)=0$.

Proof. Setting

$$i'=\begin{cases} n+i\ (1\le i\le n)\\ i-n\ (n+1\le i\le 2n)\end{cases}$$

and $\epsilon_{i,j}=\operatorname{sgn}(i)\operatorname{sgn}(j)$, we have $F_{i,j}=E_{i,j}-\epsilon_{i,j}E_{j',i'}$ and $F_{i,j}=-\epsilon_{i,j}F_{j',i'}$. Then, we obtain

$$\begin{aligned}\operatorname{Det}(F_{I,I})&=\frac{1}{s!}\sum_{\sigma,\tau\in\mathfrak{S}_s}\operatorname{sgn}(\sigma)\operatorname{sgn}(\tau)F_{i_{\sigma(1)},i_{\tau(1)}}\cdots F_{i_{\sigma(s)},i_{\tau(s)}}\\
&=\frac{1}{s!}\sum_{\sigma,\tau}\operatorname{sgn}(\sigma)\operatorname{sgn}(\tau)(-\epsilon_{i_{\sigma(1)},i_{\tau(1)}}F_{i_{\tau(1)}',i_{\sigma(1)}'})\cdots(-\epsilon_{i_{\sigma(s)},i_{\tau(s)}}F_{i_{\tau(s)}',i_{\sigma(s)}'})\\
&=\frac{1}{s!}\sum_{\sigma,\tau}\operatorname{sgn}(\sigma)\operatorname{sgn}(\tau)(-1)^sF_{i_{\tau(1)}',i_{\sigma(1)}'}\cdots F_{i_{\tau(s)}',i_{\sigma(s)}'}\\
&=(-1)^s\operatorname{Det}(F_{I',I'}),\end{aligned}$$

and hence

$$\operatorname{Det}_s(F)=(-1)^s\operatorname{Det}_s(F).$$

□

If we apply Proposition 1.1 and Lemma 1.3 to the central element $C(u)$, we can remove the hat of $\widehat{F}$ and put the parameter u out of the symmetrized determinant.

Theorem 1.4. *The Capelli element $C(u)$ of the highest degree can be expressed in terms of the minor determinants as follows:*

$$\begin{aligned}C(u)&=\sum_{r=0}^{n}\binom{2n+1}{2r+1}^{-1}e_{2r+1}(u+n,\dots,u-n)\operatorname{Det}_{2n-2r}(F)\\
&=\sum_{s=0}^{n}\binom{2n+1}{2s}^{-1}e_{2n+1-2s}(u+n,\dots,u-n)\operatorname{Det}_{2s}(F).\end{aligned}$$

Proof.

$$
\begin{aligned}
C(u) &= \mathrm{Det}(\widehat{F}; u+n, \ldots, u-n) \\
&= \sum_{r+s=2n+1} \binom{2n+1}{r}^{-1} \sum_{1\le k_1<\ldots<k_r\le 2n+1} (u+n-k_1+1)\cdots(u+n-k_r+1) \sum_{\substack{J\subseteq\{1,\ldots,2n+1\}\\ |J|=s}} \mathrm{Det}(\widehat{F}_{J,J}) \\
&= \sum_{r+s=2n+1} \binom{2n+1}{r}^{-1} \sum_{1\le k_1<\ldots<k_r\le 2n+1} (u+n-k_1+1)\cdots(u+n-k_r+1) \sum_{\substack{J\subseteq\{1,\ldots,2n\}\\ |J|=s}} \mathrm{Det}(\widehat{F}_{J,J}) \\
&= \sum_{r+s=2n+1} \binom{2n+1}{r}^{-1} \sum_{1\le k_1<\ldots<k_r\le 2n+1} (u+n-k_1+1)\cdots(u+n-k_r+1) \sum_{\substack{J\subseteq\{1,\ldots,2n\}\\ |J|=s}} \mathrm{Det}(F_{J,J}) \\
&= \sum_{r+s=2n+1} \binom{2n+1}{r}^{-1} e_r(u+n, \ldots, u-n)\, \mathrm{Det}_s(F) \\
&= \sum_{r=0}^{n} \binom{2n+1}{2r+1}^{-1} e_{2r+1}(u+n, \ldots, u-n)\, \mathrm{Det}_{2n-2r}(F) \\
&= \sum_{s=0}^{n} \binom{2n+1}{2s}^{-1} e_{2n+1-2s}(u+n, \ldots, u-n)\, \mathrm{Det}_{2s}(F).
\end{aligned}
$$

□

1.2 Construction of the Capelli elements of lower degrees

In this subsection, we construct the Capelli elements of lower degrees associated with Itoh's Capelli element $C(u) = C^{\mathfrak{sp}}(u)$ by a method similar to the $\mathfrak{gl}_n$ case. For the $\mathfrak{sp}_{2n}$ case, setting $\langle x; a\rangle = x^2 - a^2$, we define the shifted factorials $\langle x; a\rangle_k$ associated with $\langle x; a\rangle$ by

$$\langle x; a\rangle_k = \langle x; a\rangle\langle x; a+1\rangle \cdots \langle x; a+k-1\rangle \ (k = 0, 1, 2 \ldots).$$

We also introduce the central difference operator

$$\mathcal{D}_x = \frac{1}{2x}(T_x^{\frac{1}{2}} - T_x^{-\frac{1}{2}}),$$

where $T_x^c f(x) = f(x+c)$ for an arbitrary polynomial $f(x)$ in x. As we will see bellow (Theorem 1.7), the factorial Schur functions $R_{(1^{n-k})}(x; a)$ of lower degrees are obtained by expanding the factorial Schur function $R_{(1^n)}(x; u)$ of the highest degree in terms of the shifted factorials $\langle u; a\rangle_k$. This implies that the Capelli elements $C_{n-k}(u)$ of lower degrees are obtained from the Capelli element

$$C_n(u) = \frac{(-1)^n}{u} C(u)$$

of the highest degree as the expansion coefficients by $\langle u; a\rangle_k$. We use Propositions 1.5 and 1.6 below to expand $C_n(u)$ of Theorem 1.4 in terms of the shifted factorials $\langle u; a\rangle_k$. As a result we obtain an explicit formula for each Capelli element $C_k(u)$ of lower degree as a linear combination of symmetrized minor determinants $\mathrm{Det}_s(F)$ (Theorem 1.9).

Proposition 1.5. *Let $\varphi(x)$ be a polynomial in x and suppose that $\varphi(x)$ is even, i.e. $\varphi(-x) = \varphi(x)$. Then $\varphi(x)$ is expanded into the form*

$$\varphi(x) = \sum_{k \geq 0} \frac{c_k}{k!} \langle x; a \rangle_k \quad (\textit{finite sum}),$$

where the coefficients c_k are given by

$$c_k = \mathcal{D}_x^k \varphi(x)\big|_{x=a+\frac{k}{2}} \quad (k = 0, 1, 2, \ldots).$$

Proof. Note that

$$\begin{aligned}
(T_x^{\frac{1}{2}} - T_x^{-\frac{1}{2}})\langle x; a\rangle_k &= \langle x + \frac{1}{2}; a\rangle_k - \langle x - \frac{1}{2}; a\rangle_k \\
&= \{(x + \frac{1}{2} - a)(x + \frac{1}{2} + a + k - 1) - (x - \frac{1}{2} - a - k + 1)(x - \frac{1}{2} + a)\} \left\langle x; a + \frac{1}{2} \right\rangle_{k-1} \\
&= 2kx \left\langle x; a + \frac{1}{2} \right\rangle_{k-1}.
\end{aligned}$$

Therefore, we have

$$\begin{aligned}
\mathcal{D}_x^l \varphi(x) &= \sum_{k \geq 0} \frac{c_k}{(k-l)!} \left\langle x; a + \frac{l}{2} \right\rangle_{k-l} \\
&= \sum_{k \geq 0} \frac{c_{k+l}}{k!} \left\langle x; a + \frac{l}{2} \right\rangle_k,
\end{aligned}$$

and hence

$$\mathcal{D}_x^l \varphi(x)\big|_{x=a+\frac{l}{2}} = \sum_{k \geq 0} \frac{c_{k+l}}{k!} \left\langle a + \frac{l}{2}; a + \frac{l}{2} \right\rangle_k = c_l.$$

□

Noting that the difference operator $T_x^{\frac{1}{2}} - T_x^{-\frac{1}{2}}$ and the multiplication operator $\frac{1}{2x}$ are multiplied alternately in the kth power ${\mathcal{D}_x}^k$ of $\mathcal{D}_x$, we change the order of composition of operators.

Proposition 1.6. *For $k = 0, 1, 2, \ldots$, the kth power $\mathcal{D}_x^k$ of $\mathcal{D}_x$ can be expressed as follows:*

$$\begin{aligned}
\mathcal{D}_x^k &= \sum_{i=0}^{k} \binom{k}{i} \frac{1}{(2x-i)_{k-i}(-2x-k+i)_i} T^{\frac{k}{2}-i} \\
&= \sum_{i=0}^{k} \binom{k}{i} (-1)^i \frac{2x+k-2i}{(2x-i)_{k+1}} T_x^{\frac{k}{2}-i},
\end{aligned}$$

where $(x)_k = x(x+1)\cdots(x+k-1)$.

Proof. By the mathematical induction, we have

$$\begin{aligned}
\mathcal{D}_x^{k+1} &= \sum_{i=0}^{k} \binom{k}{i} \frac{1}{(2x-i)_{k-i}(-2x-k+i)_i} \cdot \frac{1}{2(x+\frac{k}{2}-i)} (T_x^{\frac{k+1}{2}-i} - T_x^{\frac{k-1}{2}-i}) \\
&= \sum_{i=0}^{k} \binom{k}{i} \frac{1}{(2x-i)_{k+1-i}(-2x-k+i)_i} (T_x^{\frac{k+1}{2}-i} - T_x^{\frac{k-1}{2}-i}) \\
&= \sum_{i=1}^{k} \left\{ \binom{k}{i} \frac{1}{(2x-i)_{k+1-i}(-2x-k+i)_i} - \binom{k}{i-1} \frac{1}{(2x-i+1)_{k+2-i}(-2x-k+i-1)_{i-1}} \right\} T_x^{\frac{k+1}{2}-i} \\
&\quad + \frac{1}{(2x)_{k+1}} T_x^{\frac{k+1}{2}} - \frac{1}{(2x-k)(-2x)_k} T_x^{-\frac{k+1}{2}}.
\end{aligned}$$

Here, we compute

$$\begin{aligned}
&\binom{k}{i} \frac{1}{(2x-i)_{k+1-i}(-2x-k+i)_i} - \binom{k}{i-1} \frac{1}{(2x-i+1)_{k+2-i}(-2x-k+i-1)_{i-1}} \\
&= \frac{\binom{k}{i}(2x-2i+k+1)(2x-2i+k+2)(-2x-k+i-1) - \binom{k}{i-1}(2x-i)(-2x-k+2i-2)(-2x-k+2i-1)}{(2x-i)_{k+3-i}(-2x-k+i-1)_{i+1}} \\
&= \frac{(2x-2i+k+1)(2x-2i+k+2)\binom{k+1}{i}\{\frac{k-i+1}{k+1}(-2x-k+i-1) - \frac{i}{k+1}(2x-i)\}}{(2x-i)_{k+3-i}(-2x-k+i-1)_{i+1}} \\
&= \binom{k+1}{i} \frac{1}{(2x-i)_{k+1-i}(-2x-(k+1)+i)_i}.
\end{aligned}$$

Thus, we obtain

$$\begin{aligned}
\mathcal{D}_x^{k+1} &= \sum_{i=1}^{k} \binom{k+1}{i} \frac{1}{(2x-i)_{k+1-i}(-2x-(k+1)+i)_i} T_x^{\frac{k+1}{2}-i} + \frac{1}{(2x)_{k+1}} T_x^{\frac{k+1}{2}} + \frac{1}{(-2x)_{k+1}} T_x^{-\frac{k+1}{2}} \\
&= \sum_{i=0}^{k+1} \binom{k+1}{i} \frac{1}{(2x-i)_{k+1-i}(-2x-(k+1)+i)_i} T_x^{\frac{k+1}{2}-i}.
\end{aligned}$$

□

The next theorem follows from the dual Cauchy formula of the factorial Schur functions (Theorem 2.2 of [5]).

Theorem 1.7. *For a set of variables $x = (x_1, \ldots, x_n)$, we have*

$$R_{(1^n)}^{(n)}(x; u) = \sum_{k=0}^{n} (-1)^k R_{(1^{n-k})}^{(n)}(x; a) \langle u; a \rangle_k.$$

Proof. Taking two matrices

$$X = \begin{pmatrix} x_{1,1} & \cdots & x_{1,m+n} \\ \vdots & & \vdots \\ x_{m,1} & \cdots & x_{m,m+n} \end{pmatrix},$$

$$Y = \begin{pmatrix} y_{1,1} & \cdots & y_{1,m+n} \\ \vdots & & \vdots \\ y_{n,1} & \cdots & y_{n,m+n} \end{pmatrix},$$

we set

$$Z = \begin{pmatrix} X \\ Y \end{pmatrix} = \begin{pmatrix} x_{1,1} & \cdots & x_{1,m+n} \\ \vdots & & \vdots \\ x_{m,1} & \cdots & x_{m,m+n} \\ y_{1,1} & \cdots & y_{1,m+n} \\ \vdots & & \vdots \\ y_{n,1} & \cdots & y_{n,m+n} \end{pmatrix}.$$

Then we have

$$\det Z = \sum_{\{j_1<\cdots<j_m\}\cup\{k_1<\cdots<k_n\}=\{1,\ldots,m+n\}} \operatorname{sgn}\begin{pmatrix} 1 & \cdots & m & m+1 & \cdots & m+n \\ j_1 & \cdots & j_m & k_1 & \cdots & k_n \end{pmatrix} \det X^{1,\ldots,m}_{j_1,\ldots,j_m} \det Y^{1,\ldots,n}_{k_1,\ldots,k_n},$$

where $X^{i_1,\ldots,i_m}_{j_1,\ldots,j_m} = \begin{pmatrix} X_{i_1,j_1} & \cdots & X_{i_m,j_m} \\ \vdots & \vdots & \vdots \\ X_{i_m,j_1} & \cdots & X_{i_m,j_m} \end{pmatrix}$. We put

$$\Psi_{mn}(X,Y) = \frac{\det Z}{\det X^{1,\ldots,m}_{1,\ldots,m} \det Y^{1,\ldots,n}_{1,\ldots,n}}$$

and $\mu_1 = j_m - m,\ \mu_2 = j_{m-1} - (m-1),\ \ldots,\ \mu_m = j_1 - 1$, so that

$$\Psi_{mn}(X,Y) = \sum_{\mu\subseteq n^m} (-1)^{|\mu|} S^{(m)}_{\mu}(X) S^{(n)}_{\mu^*}(Y).$$

Then, by the specialization

$$X = \begin{pmatrix} x_{1,1} & \cdots & x_{1,m+n} \\ \vdots & & \vdots \\ x_{m,1} & \cdots & x_{m,m+n} \end{pmatrix} = \left(\langle x_i; a\rangle_{j-1}\right)_{i,j},$$

$$Y = \begin{pmatrix} y_{1,1} & \cdots & y_{1,m+n} \\ \vdots & & \vdots \\ y_{n,1} & \cdots & y_{n,m+n} \end{pmatrix} = \left(\langle y_i; a\rangle_{j-1}\right)_{i,j},$$

we obtain

$$\begin{aligned} \Psi_{mn}(X,Y) &= \frac{\prod_{1\le i<j\le m+n}(z_j^2 - z_i^2)}{\prod_{1\le i<j\le m}(x_j^2 - x_i^2)\prod_{1\le i<j\le n}(y_j^2 - y_i^2)} \\ &= \prod_{i=1}^{m}\prod_{j=1}^{n}\langle y_j; x_i\rangle, \end{aligned}$$

where $(z_1,\ldots,z_{m+n}) = (x_1,\ldots,x_m,y_1,\ldots,y_n)$. This implies

$$\prod_{i=1}^{m}\prod_{j=1}^{n}\langle y_j; x_i\rangle = \sum_{\mu\subseteq n^m} (-1)^{|\mu|} R^{(m)}_{\mu}(x;a) R^{(n)}_{\mu^*}(y;a).$$

Putting $n = 1$, we obtain

$$\begin{aligned}\prod_{i=1}^{m}\langle y_1; x_i\rangle &= (-1)^m \prod_{i=1}^{m}\langle x_i; y_1\rangle \\ &= \sum_{r=0}^{m}(-1)^r R_{(1^r)}(x;a)R_{(m-r)}(y_1;a) \\ &= \sum_{r=0}^{m}(-1)^r R_{(1^r)}(x;a)\langle y_1;a\rangle_{m-r},\end{aligned}$$

and hence

$$\begin{aligned}\prod_{i=1}^{m}\langle x_i; y_1\rangle &= R^{(m)}_{(1^m)}(x;y_1) \\ &= \sum_{r=0}^{m}(-1)^{m-r} R_{(1^r)}(x;a)\langle y_1;a\rangle_{m-r} \\ &= \sum_{k=0}^{m}(-1)^k R^{(m)}_{(1^{m-k})}(x;a)\langle y_1;a\rangle_{k}.\end{aligned}$$

□

By Corollary 7.1 of [3], it is known that

$$\gamma\left(C(u)\right) = u(u^2 - x_1^2)\cdots(u^2 - x_n^2) = (-1)^n u R^{(n)}_{(1^n)}(x;u).$$

Hence $\frac{(-1)^n}{u}C(u)$ corresponds to $R^{(n)}_{(1^n)}(x;u)$ by the Harish-Chandra isomorphism. Thus, we put

$$C_n(u) = \frac{(-1)^n}{u}C(u) = \frac{(-1)^n}{u}\operatorname{Det}(\widehat{F}^{\mathfrak{sp}} + u1_{2n+1}; n, n-1, \ldots, -n).$$

The next theorem follows from the Theorem 1.7 by way of the Harish-Chandra isomorphism.

Theorem 1.8. *The Capelli element of the highest degree $C_n(u)$ can be expanded in terms of the shifted factorial as follows:*

$$C_n(u) = \sum_{k=0}^{n}(-1)^k\langle u;a\rangle_k C_{n-k}(a).$$

Here, $C_{n-k}(a)$ denote the Capelli elements of lower degrees corresponding to $R^{(n)}_{(1^{n-k})}(x;a)$ by the Harish-Chandra isomorphism.

From Theorem 1.4, we have

$$\begin{aligned}C_n(u) &= \frac{(-1)^n}{u}C(u) \\ &= \sum_{r=0}^{n}\binom{2n+1}{2r+1}^{-1}\frac{(-1)^n}{u}e_{2r+1}(u+n,\ldots,u-n)\operatorname{Det}_{2n-2r}(F).\end{aligned} \tag{1.1}$$

We now put

$$\varphi_r(u) = \frac{1}{u}e_{2r+1}(u+n,\ldots,u-n) \quad (r = 0, 1, \ldots, n).$$

Since $\varphi_r(u)$ is a polynomial in u and even, by Propositions 1.5 and 1.6, we can expand $\varphi_r(u)$ into the form

$$\varphi_r(u) = \sum_{k=0}^{r} \frac{c_{kr}}{k!} \langle u; a \rangle_k,$$

with the coefficients c_{kr} given by

$$c_{kr} = \sum_{i=0}^{k} \binom{k}{i} (-1)^i \frac{2}{(2a+k-i)_{k+1}} e_{2r+1}(a+k-i+n, a+k-i+n-1, \ldots, a+k-i-n).$$

Thus, comparing Theorem 1.8 with (1.1), we obtain the next theorem.

Theorem 1.9. *The Capelli elements $C_{n-k}(a)$ of lower degrees can be expanded in terms of the symmetrized minor determinants as follows:*

$$C_{n-k}(a) = (-1)^{n-k} \sum_{r=k}^{n} \binom{2n+1}{2r+1}^{-1} \frac{c_{kr}}{k!} \operatorname{Det}_{2(n-r)}(F),$$

where

$$c_{kr} = \sum_{i=0}^{k} \binom{k}{i} (-1)^i \frac{2}{(2a+k-i)_{k+1}} e_{2r+1}(a+k-i+n, a+k-i+n-1, \ldots, a+k-i-n).$$

Remark 1.10. The Capelli elements $C_l(1)$ of lower degrees with parameter $u = 1$ here equal to $(-1)^l C_{2l}^{\mathfrak{sp}}$ $(1 \leq l \leq n)$ in [3].

From these Capelli element of lower degrees, we can construct the higher Capelli elements by the Jacobi-Trudi formula.

1.3 Construction of the higher Capelli elements $C_\lambda(u)$

In the framework of [7], the general Schur functions $S_\lambda^{(m)}(X)$ have the Jacobi-Trudi formula

$$S_\lambda^{(m)}(X) = \det \left(e_{\lambda_i'-i+j}^{(m+j-1)}(X) \right)_{i,j=1}^{n},$$

where

$$e_r^{(m)}(X) = S_{(1^r)}^{(m)}(X).$$

In this Jacobi-Trudi formula, the dimension of the left-hand side $S_\lambda^{(m)}(X)$ is m, while the elementary symmetric functions of higher dimensions $m+j-1$ appear on the right-hand side. For relating the general Schur functions to the factorial Schur functions, we put $X = X(a) = \left(\langle x_i; a \rangle_{j-1}\right)_{i,j \geq 1}$ so that $R_\lambda^{(m)}(x; a) = S_\lambda^{(m)}(X(a))$. Then by Theorem 1.11 below we can change the dimensions of the right-hand side to m for adapting it to our setting of the Capelli elements of lower degrees. (Theorem 1.11 is included in [5] as Lemma 2.4 and Theorem 2.5.)

Theorem 1.11. *For each partition $\lambda \subseteq (n^m)$, we have the Jacobi-Trudi formula*

$$R_\lambda^{(m)}(x; a) = \det \left(e_{\lambda_i'-i+j}^{(m)}(x; a+j-1) \right)_{i,j=1}^{n}.$$

Proof. We set $R_\lambda^{(m)}(x;a) = R_\lambda^{(m)}(x_1,\dots,x_m;a)$ and $\langle x_i;a\rangle_j = x_i^{(j)}$. When $l(\lambda) \le m < n$,

$$
\begin{aligned}
&R_\lambda^{(n)}(x_1,\dots,x_m,a+n-m-1,a+n-m-2,\dots,a+1,a;a)\\
&= \frac{\det\left(\begin{array}{ccc|cccccc}
x_1^{(\lambda_1+n-1)} & \dots & x_1^{(\lambda_m+n-m)} & x_1^{(n-m-1)} & \dots & \dots & x_1^{(1)} & 1\\
\vdots & \dots & \vdots & \vdots & \dots & \dots & \dots & \vdots\\
x_m^{(\lambda_1+n-1)} & \dots & x_m^{(\lambda_m+n-m)} & x_m^{(n-m-1)} & \dots & \dots & x_m^{(1)} & 1\\
\hline
 & & & (a+n-m-1)^{(n-m-1)} & \dots & \dots & (a+n-m-1)^{(1)} & 1\\
 & 0 & & \vdots & \dots & \ddots & \dots & \vdots\\
 & & & 0 & \dots & 0 & (a+1)^{(1)} & 1\\
 & & & 0 & \dots & \dots & 0 & 1
\end{array}\right)}{\det\left(\begin{array}{ccc|cccccc}
x_1^{(n-1)} & \dots & x_1^{(n-m)} & x_1^{(n-m-1)} & \dots & \dots & x_1^{(1)} & 1\\
\vdots & \dots & \vdots & \vdots & \dots & \dots & \dots & \vdots\\
x_m^{(n-1)} & \dots & x_m^{(n-m)} & x_m^{(n-m-1)} & \dots & \dots & x_m^{(1)} & 1\\
\hline
 & & & (a+n-m-1)^{(n-m-1)} & \dots & \dots & (a+n-m-1)^{(1)} & 1\\
 & 0 & & \vdots & \dots & \ddots & \dots & \vdots\\
 & & & 0 & \dots & 0 & (a+1)^{(1)} & 1\\
 & & & 0 & \dots & \dots & 0 & 1
\end{array}\right)}\\
&= \frac{\det\begin{pmatrix} x_1^{(\lambda_1+n-1)} & \dots & x_1^{(\lambda_m+n-m)}\\ \vdots & \dots & \vdots\\ x_m^{(\lambda_1+n-1)} & \dots & x_m^{(\lambda_m+n-m)}\end{pmatrix}}{\det\begin{pmatrix} x_1^{(n-1)} & \dots & x_1^{(n-m)}\\ \vdots & \dots & \vdots\\ x_m^{(n-1)} & \dots & x_m^{(n-m)}\end{pmatrix}}\\
&= \frac{x_1^{(n-m)}\cdots x_m^{(n-m)}\det\begin{pmatrix} \langle x_1;a+n-m\rangle_{\lambda_1+m-1} & \dots & \langle x_1;a+n-m\rangle_{\lambda_m}\\ \vdots & \dots & \vdots\\ \langle x_m;a+n-m\rangle_{\lambda_1+m-1} & \dots & \langle x_m;a+n-m\rangle_{\lambda_m}\end{pmatrix}}{x_1^{(n-m)}\cdots x_m^{(n-m)}\det\begin{pmatrix} \langle x_1;a+n-m\rangle_{m-1} & \langle x_1;a+n-m\rangle_{m-2} & \dots & 1\\ \vdots & \dots & \dots & \vdots\\ \langle x_m;a+n-m\rangle_{m-1} & \langle x_m;a+n-m\rangle_{m-2} & \dots & 1\end{pmatrix}}\\
&= R_\lambda^{(m)}(x_1,\dots,x_m;a+n-m).
\end{aligned}
$$

When $m < l(\lambda)$, and there exists $j > 0$ such that $\lambda_{m+j} > 0$, the $(m+j,m+j)$-component of the numerator equals to $(a+n-m-j)^{(\lambda_{m+j}+n-m-j)} = 0$. Thus, $R_\lambda^{(n)}(x_1,\dots,x_m,a+n-m-1,a+n-m-2,\dots,a+1,a;a) = 0$. In the Jacobi-Trudi formula

$$S_\lambda^{(m)}(X) = \det\left(e_{\lambda'_i-i+j}^{(m+j-1)}(x)\right)_{i,j=1}^n .$$

of [7] for $X = X(a)$, the left-hand side does not depend on $x_{m+1},\dots,x_{m+j-1}$. Thus, we can put

$$x_{m+1} = a+j-2, x_{m+2} = a+j-3, \dots, x_{m+j-1} = a$$

in the right-hand side. Then, we have

$$R_\lambda^{(m)}(x;a) = \det\left(e_{\lambda'_i-i+j}^{(m+j-1)}(x;a)\right)_{i,j=1}^n$$

$$= \det\left(e^{(m)}_{\lambda'_i-i+j}(x;a+j-1)\right)^{n}_{i,j=1}.$$

□

Remark 1.12. Theorem 1.11 holds also in the $\mathfrak{gl}_n$ case if we put $\langle x;a\rangle = x-a$ and

$$\langle x;a\rangle_k = \langle x;a\rangle\langle x;a+1\rangle\cdots\langle x;a+k-1\rangle.$$

Since $C^{(n)}_k(u)$ corresponds to the factorial Schur function $e^{(n)}_k(x;u)$, from Theorem 1.11 we can construct the higher Capelli elements $C_\lambda(u)$ which corresponds to the factorial Schur function $R^{(n)}_\lambda(x;u)$.

Theorem 1.13 (Higher Capelli elements for $\mathfrak{sp}_{2n}$)**.** *For each partition $\lambda \subseteq (m^n)$, we define*

$$C_\lambda(u) = \det\left(C_{\lambda'_i-i+j}(u+j-1)\right)^{m}_{i,j=1}.$$

Then, $C_\lambda(u)$ corresponds to $R^{(n)}_\lambda(x;u)$ by the Harish-Chandra isomorphism.

2 Higher Capelli elements in the case of $\mathfrak{o}_{2n}$

The split realization of the orthogonal Lie algebra $\mathfrak{o}_m$ is defined by

$$\mathfrak{o}(S_m) = \{X \in \mathrm{Mat}_m(\mathbb{C}) | {}^tXS_m + S_mX = 0\}$$

with the symmetric matrix $S_m = (\delta_{i,m+1-j})_{1\le i,j\le m}$, where $\delta_{i,j}$ is the Kronecker delta. In this setting, we use the matrix

$$F^{\mathfrak{o}} = \left(F^{\mathfrak{o}}_{ij}\right)_{1\le i,j\le m} \in \mathrm{Mat_m}\left(\mathcal{U}(\mathfrak{o}(S_m))\right),$$
$$F^{\mathfrak{o}}_{ij} = E_{ij} - E_{m+1-j,m+1-i} \in \mathfrak{o}(S_m).$$

In this section we treat the case of $\mathfrak{o}_{2n}$, so we put $m = 2n$. Following Wachi [11], we define the central element $C^{\mathfrak{o}_{2n}}_n(u) \in \mathcal{U}(\mathfrak{o}((S_{2n}))$ of the highest degree which corresponds to $R^{(n)}_{(1^n)}(x;u)$ by Harish-Chandra isomorphism. Then, we construct the higher Capelli elements for the $\mathfrak{o}_{2n}$ case, by the same method as in the $\mathfrak{sp}_{2n}$ case. We define the Capelli element of $\mathcal{U}(\mathfrak{o}((S_{2n}))$ by the symmetrized determinant

$$C^{\mathfrak{o}_{2n}}_n(u) = (-1)^n\,\mathrm{Det}\left(uI - F^{\mathfrak{o}}; n-1, n-2, \dots, 0; 0, -1, \dots, -n+1\right),$$

where 0 appears twice in the diagonal shift in the nth and $(n+1)$th places. In [11], this $C^{\mathfrak{o}_{2n}}_n(u)$ is expressed as $(-1)^n C^{\mathrm{Det}}_{2n}(u)$. Although certain Capelli elements $C^{\mathrm{Det}}_d(u)$ $(d < 2n)$ of lower degrees are introduced as well in [11], their eigenvalues do not correspond to factorial Schur functions. Thus, we propose to construct the Capelli elements $C^{\mathfrak{o}_{2n}}_k(u)$ $(k < n)$ of lower degrees which correspond to the factorial Schur functions attached to the column partitions. In this section we express $F^{\mathfrak{o}} = F$ and $C^{\mathfrak{o}_{2n}}_k(u) = C_k(u)$ for simplicity. Firstly we confirm that $\mathrm{Det}_{2s+1}(F) = 0$.

Lemma 2.1. *If s is odd,* $\mathrm{Det}_s(F) = 0$.

Proof. Setting $i' = 2n+1-i$, we have $F_{i,j} = -F_{j',i'}$. Then,

$$\begin{aligned}
\operatorname{Det}(F_{I,I}) &= \frac{1}{s!}\sum_{\sigma,\tau\in\mathfrak{S}_s} \operatorname{sgn}(\sigma)(\tau) F_{i_{\sigma(1)},i_{\tau(1)}}\cdots F_{i_{\sigma(s)},i_{\tau(s)}} \\
&= \frac{1}{s!}\sum_{\sigma,\tau} \operatorname{sgn}(\sigma)\operatorname{sgn}(\tau)(-F_{i_{\tau(1)}',i_{\sigma(1)}'})\cdots(-F_{i_{\tau(s)}',i_{\sigma(s)}'}) \\
&= (-1)^s \operatorname{Det}(F_{I',I'}).
\end{aligned}$$

Thus, we obtain

$$\operatorname{Det}_s(F) = (-1)^s \operatorname{Det}_s(F).$$

□

Secondly we apply Proposition 1.1 to $C_n(u)$:

$$\begin{aligned}
C_n(u) &= \operatorname{Det}(-F; u+n-1, u+n-2, \dots, u; u, u-1, \dots, u-n+1) \\
&= \sum_{r+s=2n} \binom{2n}{r}^{-1} e_r(u+n-1, u+n-2, \dots, u; u, u-1, \dots, u-n+1)(-1)^s \operatorname{Det}_s(F) \\
&= \sum_{r=0}^{n} \binom{2n}{2r}^{-1} e_{2r}(u+n-1, u+n-2, \dots, u; u, u-1, \dots, u-n+1) \operatorname{Det}_{2n-2r}(F) \qquad (2.1) \\
&= \sum_{s=0}^{n} \binom{2n}{2s}^{-1} e_{2n-2s}(u+n-1, u+n-2, \dots, u; u, u-1, \dots, u-n+1) \operatorname{Det}_{2s}(F).
\end{aligned}$$

Thirdly for (2.1) we put

$$\psi_r(u) = e_{2r}(u+n-1, u+n-2, \dots, u; u, u-1, \dots, u-n+1).$$

Then $\psi_r(u)$ is a polynomial and an even function. Thus, by Propositions 1.5 and 1.6, we expand $\psi_r(u)$ into the form

$$\psi_r(u) = \sum_{k=0}^{r} \frac{c_{k,r}}{k!} \langle u; a\rangle_k,$$

where the coefficients $c_{k,r}$ are given by

$$c_{k,r} = \sum_{i=0}^{k} \binom{k}{i} (-1)^i \frac{2a+2k-2i}{(2a+k-i)_{k+1}} e_{2r}(a+k-i+n-1, a+k-i+n-2, \dots, a+k-i; a+k-i, \dots, a+k-i-n+1).$$

By Theorem 1.8,

$$\begin{aligned}
C_n(u) &= \sum_{r=0}^{n} \binom{2n}{2r}^{-1} (-1)^n \sum_{k=0}^{r} \frac{c_{k,r}}{k!} \langle u; a\rangle_k \operatorname{Det}_{2(n-r)}(F) \\
&= \sum_{k=0}^{n} (-1)^k \langle u; a\rangle_k C_{n-k}(a).
\end{aligned}$$

Then, comparing the coefficients of $\langle u; a\rangle_k$, we obtain explicit formulas for the Capelli elements of lower degrees.

Theorem 2.2. *The Capelli elements $C_{n-k}(a)$ of lower degrees can be expressed in terms of the minor symmetrized determinants as follows:*

$$C_{n-k}(a) = (-1)^{n-k} \sum_{r=k}^{n} \binom{2n}{2r}^{-1} \frac{c_{k,r}}{k!} \operatorname{Det}_{2(n-r)}(F),$$

where

$$c_{k,r} = \sum_{i=0}^{k} \binom{k}{i} (-1)^i \frac{2a+2k-2i}{(2a+k-i)_{k+1}} e_{2r}(a+k-i+n-1, a+k-i+n-2, \dots, a+k-i; a+k-i, \dots, a+k-i-n+1).$$

The higher Capelli elements for the $\mathfrak{o}_{2n}$ case have the same form as in Theorem 1.13.

Theorem 2.3 (Higher Capelli elements for $\mathfrak{o}_{2n}$)**.** *For each partition $\lambda \subseteq (m^n)$ we define*

$$C_\lambda(u) = \det \left(C_{\lambda'_i - i + j}(u + j - 1) \right)_{i,j=1}^{m}.$$

Then, $C_\lambda(u)$ corresponds to $R_\lambda^{(n)}(x;u)$ by the Harish-Chandra isomorphism.

3 Higher Capelli elements in the case of $\mathfrak{o}_{2n+1}$

We define the Capelli element of $\mathcal{U}(\mathfrak{o}((S_{2n+1}))$ by the symmetrized determinant

$$C_n^{\mathfrak{o}_{2n+1}}(u) = \frac{(-1)^n}{u} \operatorname{Det}(uI - F^{\mathfrak{o}}; n - \frac{1}{2}, n - \frac{3}{2}, \dots, \frac{1}{2}; 0; -\frac{1}{2}, -\frac{3}{2}, \dots, -n - \frac{1}{2}).$$

This Capelli element corresponds to $R_{(1^n)}^{(n)}(x;u)$ by Harish-Chandra isomorphism. In [11], this $C_n^{\mathfrak{o}_{2n+1}}(u)$ is expressed as $\frac{(-1)^n}{u} C_{2n+1}^{\mathrm{Det}}(u)$. In this section we express $F^{\mathfrak{o}} = F$ and $C_k^{\mathfrak{o}_{2n+1}}(u) = C_k(u)$ for simplicity. Firstly we apply Proposition 1.1 and Lemma 2.1 to $C_n(u)$:

$$\begin{aligned}
C_n(u) &= \frac{(-1)^m}{u} \operatorname{Det}\left(-F; u+n-\frac{1}{2}, u+n-\frac{3}{2}, \dots, u+\frac{1}{2}; u; u-\frac{1}{2}, u-\frac{3}{2}, \dots, u-n-\frac{1}{2}\right) \\
&= \frac{(-1)^n}{u} \sum_{r+s=2n+1} \binom{2n+1}{r}^{-1} e_r\left(u+n-\frac{1}{2}, \dots, u+\frac{1}{2}; u; u-\frac{1}{2}, \dots, u-n-\frac{1}{2}\right)(-1)^s \operatorname{Det}_s(F) \\
&= \frac{(-1)^n}{u} \sum_{r=0}^{n} \binom{2n+1}{2r+1}^{-1} e_{2r+1}\left(u+n-\frac{1}{2}, \dots, u+\frac{1}{2}; u; u-\frac{1}{2}, \dots, u-n-\frac{1}{2}\right) \operatorname{Det}_{2(n-r)}(F) \qquad (3.1) \\
&= \frac{(-1)^n}{u} \sum_{s=0}^{n} \binom{2n+1}{2s}^{-1} e_{2(m-s)+1}\left(u+n-\frac{1}{2}, \dots, u+\frac{1}{2}; u; u-\frac{1}{2}, \dots, u-n-\frac{1}{2}\right) \operatorname{Det}_{2s}(F).
\end{aligned}$$

Secondly for (3.1) we put

$$\xi_r(u) = \frac{1}{u} e_{2r+1}\left(u+n-\frac{1}{2}, \dots, u+\frac{1}{2}; u; u-\frac{1}{2}, \dots, u-n-\frac{1}{2}\right).$$

Then $\xi_r(u)$ is a polynomial and an even function. Thus, by Proposition 1.5 and 1.6, we expand $\xi_r(u)$ into the form

$$\xi_r(u) = \sum_{k=0}^{r} \frac{c_{k,r}}{k!} \langle u; a \rangle_k,$$

where the coefficients $c_{k,r}$ are given by

$$c_{k,r} = \sum_{i=0}^{k} \binom{k}{i} (-1)^i \frac{2}{(2a+k-i)_{k+1}} e_{2r+1} \left(a+k-i+n-\frac{1}{2}, \dots, a+k-i+\frac{1}{2}; a+k-i; a+k-i-\frac{1}{2}, \dots, a+k-i-n-\frac{1}{2} \right).$$

By Theorem 1.8,

$$\begin{aligned} C_n(u) &= \sum_{r=0}^{n} \binom{2n+1}{2r+1}^{-1} (-1)^n \sum_{k=0}^{r} \frac{c_{k,r}}{k!} \langle u; a \rangle_k \operatorname{Det}_{2(n-r)}(F) \\ &= \sum_{k=0}^{n} (-1)^k \langle u; a \rangle_k C_{n-k}(a). \end{aligned}$$

Then, comparing the coefficient of $\langle u; a \rangle_k$, we obtain explicit formulas for the Capelli elements of lower degrees.

Theorem 3.1. *The Capelli elements $C_{n-k}(a)$ of lower degrees can be expanded in terms of the symmetrized minor determinants as follows:*

$$C_{n-k}(a) = (-1)^{n-k} \sum_{r=k}^{n} \binom{2n+1}{2r+1}^{-1} \frac{c_{k,r}}{k!} \operatorname{Det}_{2(n-r)}(F),$$

where

$$c_{k,r} = \sum_{i=0}^{k} \binom{k}{i} (-1)^i \frac{2}{(2a+k-i)_{k+1}} e_{2r+1} \left(a+k-i+n-\frac{1}{2}, \dots, a+k-i+\frac{1}{2}; a+k-i; a+k-i-\frac{1}{2}, \dots, a+k-i-n-\frac{1}{2} \right).$$

The higher Capelli elements for the $\mathfrak{o}_{2n+1}$ case have the same form as in Theorems 1.13 and 2.3.

Theorem 3.2 (Higher Capelli elements for $\mathfrak{o}_{2n+1}$)**.** *For each partition $\lambda \subseteq (m^n)$, we define*

$$C_\lambda(u) = \det \left(C_{\lambda'_i - i + j}(u + j - 1) \right)_{i,j=1}^{m}.$$

Then, $C_\lambda(u)$ corresponds to $R_\lambda^{(n)}(x; u)$ by the Harish-Chandra isomorphism.

Remark 3.3. In [11], Wachi defined certain central elements $C_d(u)$ of lower degrees with a parameter u for $\mathfrak{o}_{2n}$ and $\mathfrak{o}_{2n+1}$ cases, but these are different from our Capelli elements, expect for the one of the highest degree.

4 Higher Capelli elements in the case of $\mathfrak{gl}_n$

In this section, we apply our method of construction of higher Capelli elements $C_\lambda(u)$ to the $\mathfrak{gl}_n$ case. In this case, the higher Capelli elements $C_\lambda(0)$ with $u = 0$ coincide with those constructed by Okounkov [9]. For a general $k \times k$ matrix Z with non-commuting entries, the column determinant $\det(Z)$ is defined by

$$\det(Z) = \sum_{\sigma \in \mathfrak{S}_k} \operatorname{sgn}(\sigma) Z_{\sigma(1),1} \cdots Z_{\sigma(k),k}.$$

Similarly to Proposition 1.1, we have

Proposition 4.1. *For $a_i = b_i + c_i$, we have*

$$\det\left(A + \mathrm{diag}(a_1, \ldots, a_n)\right) = \sum_{r+s=n} \sum_{\substack{1 \le k_1 < \cdots < k_r \le n \\ 1 \le l_1 < \cdots < l_s \le n \\ K \bigsqcup L = \{1, \ldots, n\}}} c_{k_1} \cdots c_{k_r} \det\left(A + \mathrm{diag}(b_1, \ldots, b_n)\right)_{L,L}.$$

Applying Proposition 4.1 to $C_n(u)$, we have

$$\begin{aligned} C_n(u) &= \det(E + \mathrm{diag}(n-u, n-1-u, \ldots, -u)) \\ &= \sum_{r+s=n} e_r(n-u, n-1-u, \ldots, -u)\mathrm{det}_s E \\ &= \sum_{r=0}^{n} e_r(n-u, n-1-u, \ldots, -u)\mathrm{det}_{n-r} E, \end{aligned}$$

where $\det_s E = \sum_{|I|=s} \det E_{I,I}$ denotes the sum of all $s \times s$ column principal minor determinants of E.

The relation between the factorial Schur function of the highest degree $R^n_{(1^n)}(x; u)$ and lower degrees $R^{(n)}_{(1^{n-k})}(x; a)$ is given in the same way as in Theorem 1.7.

Theorem 4.2. *For a set of variables $x = (x_1, \ldots, x_n)$, we have*

$$R^{(n)}_{(1^n)}(x; u) = \sum_{k=0}^{n} (-1)^k R^{(n)}_{(1^{n-k})}(x; a)\langle u; a\rangle_k.$$

Here, we set

$$\langle u; a\rangle = u - a \text{ and}$$

$$\langle u; a\rangle_k = \langle u; a\rangle\langle u; a+1\rangle \cdots \langle u; a+k-1\rangle.$$

We define the difference operator Δ_x by

$$\Delta_x \varphi(x) = \varphi(x+1) - \varphi(x)$$

for an arbitrary polynomial $\varphi(x)$, then expansion of $\varphi(x)$ by the shifted factorials $\langle x; a\rangle_k$ is determined as follows.

Proposition 4.3. *If we expand a polynomial $\varphi(x)$ in the form*

$$\varphi(x) = \sum_{k \ge 0} \frac{c_k}{k!} \langle x; a\rangle_k \quad (\textit{finite sum}),$$

then the coefficients c_k are given by

$$c_k = \Delta_x^k \varphi(x)\big|_{x=a} \qquad (k = 0, 1, 2, \ldots).$$

In this case, the powers of Δ_x are expanded simply by the binomial coefficients.

Proposition 4.4. *For $k = 0, 1, 2, \ldots$, the kth power Δ_x^k of Δ_x can be expressed as follows:*

$$\Delta_x^k = \sum_{i=0}^{k} \binom{k}{i} (-1)^{k-i} T_x^i.$$

By Theorem 4.2 and the Harish-Chandra isomorphism, we obtain

Theorem 4.5. *The Capelli element $C_n(u)$ of the highest degree can be expanded in terms of the shifted factorials as follows:*

$$C_n(u) = \sum_{k=0}^{n} (-1)^k C_{n-k}(a) \langle u; a \rangle_k.$$

Here, $C_{n-k}(a)$ denote the Capelli elements of lower degrees corresponding to $R^{(n)}_{(1^{n-k})}(x; a)$ by the Harish-Chandra isomorphism.

Thus we have

$$\begin{aligned} C_n(u) &= \sum_{r=0}^{n} e_r(n-u, n-1-u, \ldots, -u) \det\nolimits_{n-r} E \\ &= \sum_{k=0}^{n} (-1)^k C_{n-k}(a) \langle u; a \rangle_k. \end{aligned}$$

We put

$$\begin{aligned} \psi_r(u) &= e_r(n-u, n-1-u, \ldots, -u) \\ &= \sum_{k=0}^{r} \frac{c_{kr}}{k!} \langle u; a \rangle_k. \end{aligned}$$

Then, comparing the coefficient of $\langle u; a \rangle_k$, we have

$$C_{n-k}(a) = \sum_{r=k}^{n} (-1)^k \frac{c_{kr}}{k!} \det\nolimits_{n-r} E.$$

By Proposition 4.3, we can calculate these coefficients c_{kr} as follows:

$$\begin{aligned} c_{kr} &= \Delta_u^k \psi_r(u)\big|_{u=a} \\ &= \sum_{i=0}^{k} \binom{k}{i} (-1)^{k-i} e_r(n-a-i, n-1-a-i, \ldots, -a-i). \end{aligned}$$

In this way, we obtain explicit formulas for the Capelli elements $C_{n-k}(a)$ of lower degrees.

Theorem 4.6. *The Capelli elements $C_{n-k}(a)$ of lower degrees can be expanded in terms of the column minor determinants as follows:*

$$C_{n-k}(a) = \sum_{r=k}^{n} (-1)^k \frac{c_{kr}}{k!} \det\nolimits_{n-r} E,$$

where

$$c_{kr} = \sum_{i=0}^{k} \binom{k}{i} (-1)^{k-i} e_r(n-a-i, n-1-a-i, \ldots, -a-i).$$

The higher Capelli elements for the $\mathfrak{gl}_n$ case are obtained in the same way as in Theorems 1.13, 2.3, and 3.2.

Theorem 4.7 (Higher Capelli elements for $\mathfrak{gl}_n$)**.** *For each partition $\lambda \subseteq (m^n)$, we define*

$$C_\lambda(u) = \det \left(C_{\lambda'_i - i + j}(u + j - 1) \right)_{i,j=1}^{m}.$$

Then, $C_\lambda(u)$ corresponds to $R^{(n)}_\lambda(x; u)$ by the Harish-Chandra isomorphism.

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