Inflation with an SU(3) gauge field

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Doctoral Dissertation

Inflation with an SU(3) gauge field
（SU(3)ゲージ場を伴うインフレーション）

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Abstract

A coupling between the inflaton and the U(1) gauge field with a kinetic gauge function can give a stable anisotropic attractor solution, which is a counter-example for the cosmic no-hair conjecture. If the gauge field is of conventional SU(2) (with Pauli matrices as the generators), the nonlinear coupling will destabilize the anisotropic solution, which takes back the solution to no-hair(energy density of gauge field tends to 0). As there is SU(3) gauge field in the standard model of particle physics, it is natural to consider an SU(3) gauge field in inflation. From the viewpoint of group theory, that the differences of the structure constants in the SU(3) group may give different effective coupling between different gauge components and the inflaton, which may give different behavior compared to that of SU(2). Also, there are other two SU(2) subgroups in the SU(3) group. In each of the two SU(2) subgroups, there is one generator which is a combination of the Cartan generators. This difference may also give different behavior compared to the conventional SU(2).

Thus in this thesis, we study inflationary universes with an SU(3) gauge field coupled to an inflaton through a gauge kinetic function. In the general case, similar to that of SU(2), the nonlinear coupling between gauge components destabilizes the anisotropic solution. However, we found several features in inflation with an SU(3) gauge field, which do not appear in inflation with an conventional SU(2) gauge field. Firstly, in some special cases, anisotropy can generate transiently even from an isotropic initial condition. This is different from that of conventional SU(2) in which isotropy solution will keep on from isotropic condition. Secondly, we found for the other two SU(2) subgroups in the SU(3) group, in which there is one generator which is a combination of the Cartan generators, the gauge components corresponding to the Cartan generators can survive from the nonlinear coupling, which results in an anisotropic solution. It occurs due to flat directions in the potential of the gauge field. This can be generalized to Lie groups whose rank is higher than one. Thus, the conventional SU(2) gauge field has a specialty among general non-Abelian gauge fields in inflation.

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Contents

Chapter 1

Introduction

1.1 Overview of Big Bang Cosmology and Inflation

The modern experiment of cosmic microwave background radiation (CMBR), large scale structure, and type Ia supernovae data show that the universe is described by Einstein's general relativity and satisfies the so-called standard model-ΛCDM model, where Λ means a cosmological constant, CDM means cold dark matter. The standard model says that for the energy of our universe, 68% is from the cosmological constant, about 27% is matter that we still don't know the nature and only about 5% is the conventional matter we know. As we still don't know the nature of this cosmological constant and the 27% matter, we give them names dark energy and dark matter, respectively. The dark energy comes from the observation that our universe is accelerated expanding, so it needs a component that has negative pressure to account for the accelerated expansion. However, The stand model is not perfect, it suffers from the horizon problem and flatness problem.

The horizon problem is that according to the standard model of cosmology there should be many causally disconnected distinct patches in the CMB sky at recombination, while the experiment fact is that the CMB temperature is isotropic in a high degree of precision. The flatness problem is that why our universe is so flat even the curvature energy density is proportional to a^{-2}, while the matter and radiation energy density are proportional to a^{-3} and a^{-4} respectively.

A model called inflation can solve the horizon problem and flatness problem. Inflation is an accelerated expansion period at the early universe, when the comoving Hubble radius decrease, so that the large scale could have had a chance inside the comoving Hubble radius, then the CMB at recombination actually has a causal connection at the early universe. Besides, the quantum fluctuation in inflation supply the seed of the structure of the universe. No matter can survive in an expanding homogeneous universe in the presence of a positive cosmological constant except for the Bianchi type IX spacetime [1], which is the so-called cosmic no-hair theorem. Due to this cosmic no-hair theorem, it is believed that a hair such as a vector field never survives during inflation because an inflaton mimics the role of the cosmological constant. It is often called the cosmic no-hair conjecture.

Indeed, the CMB experiment from *WMAP* and *Planck* found the universe is consistent with ΛCDM model with spacial flatness and support the key predictions of isotropic slow-roll inflation. However, anomalies in the CMB temperature anisotropies on large angular scales are also found although the statistical significance is low.

Motivated by these anomalies in observation, many models have been proposed to explain the statistical anisotropy. On the other hand, there have been several attempts to seek a counterexample to the cosmic no-hair conjecture [2–5], although they suffer from instabilities in the models [6, 7].

1.2 Overview of Anisotropic Inflation

A healthy counterexample to the conjecture motivated by supergravity was found in [8], where a vector field is coupled to an inflaton through a gauge kinetic function. The point of the model is that the inflaton does not mimic a positive cosmological constant exactly, whose deviation is characterized by the slow-roll parameter. Then, an inflationary universe with a small anisotropy proportional to the slow-roll parameter can be realized [8–11].

Importantly, anisotropic inflation yields several observational signatures such as statistical anisotropy [12–25]. They have been tested by the observations of the cosmic microwave background [26, 27] and the large-scale structure of the Universe [28]. Implications from the observations [29, 30] and future perspectives are discussed [31, 32]. Considering phenomenological and observational importance, it is worth extending the anisotropic inflation model as far as possible [33–58] in order to explore the early universe. The original anisotropic inflation model [8] is endowed with a U(1) gauge field. In high-energy fundamental theories, we can expect the existence of multiple U(1) gauge fields in the early universe [36]. Interestingly, it was shown that multiple U(1) gauge fields tend to select a minimally anisotropic configuration dynamically [36]. A two-form field, which is also a gauge field, can give rise to a prolate-type anisotropy as opposed to an oblate-type anisotropy from a U(1) gauge field [21, 44, 50]. Moreover, an SU(2) gauge field coupled to an inflaton in the axially symmetric Bianchi type I spacetime was studied in [38]. It was shown that an SU(2) gauge field could result in both prolate- and oblate-type anisotropies. In general, nonlinear self-couplings in the kinetic term of an SU(2) gauge field cause the decay of the SU(2) gauge field after sufficient growth [39, 40]. That behavior would enrich the predictions for observations and support the cosmic no-hair conjecture.

1.3 Motivation of Research

In the standard model of particle physics, not only U(1) and SU(2) gauge fields but also an SU(3) gauge field plays an important role. In high-energy fundamental theories, there are many non-Abelian gauge fields including SU(3) and other Lie groups. From the viewpoint of group theory, the differences of the structure constant in the SU(3) group may result in a nontrivial phenomenon in the anisotropy of inflation. Therefore, it would be interesting to study the role of a SU(3) gauge field in the early universe in addition to the previous works for the cases of U(1) or SU(2) gauge fields [36, 38–40].

1.4 Outline of Thesis

This thesis is organized as follows:

- In Chap. 2, we review the standard cosmology and inflation theory. In §2.1 and §2.2 we review cosmology and its problem respectively. In §2.3 and §2.4, we review why inflation theory can solve the problem of cosmology. In §2.5, we review how inflation can provide the primordial fluctuations for the CMB anisotropy and large scale structure. §2.6 is the constraints from the *Plank* CMB anisotropy experiment.
- In Chap. 3, We review anisotropic inflation with U(1) gauge field(s). In §3.1 we review the mechanism of anisotropic inflation, see also [8, 34]. In §3.2 we review inflation with two U(1) gauge fields, see also [36]. In this case, the stable point is a state in which the two electric fields (the derivative of the gauge fields to time) are perpendicular. In §3.3 we review inflation with $N(N > 2)$ U(1) gauge fields, see also see also [36]. In this case, inflation will reach no-hair eventually.
- Chap. 4 is the main part of our research. In this chapter, we study inflation with an SU(3) gauge field. In §4.1, we first show that in the case of SU(2) subgroup, all the components decay. In §4.2 we show for the case of SU(2) $\otimes$ U(1) subgroup, the U(1) part can survive. In §4.3, we find the components corresponding to the

Cartan generators can survive from the non-linear self-coupling of the gauge field because of the existence of a flat direction. In §4.4 we study the general case of SU(3) gauge field, in which case all the components decay thus the universe becomes no-hair eventually.

- In Chap. 5 we extend the analysis of flat direction to inflation with an SU(N)($N > 2$) gauge field. We show that for each simple root with two non-zero components, there is a flat direction in some subspace of the corresponding SU(2) gauge field. The gauge components corresponding to the Cartan generators can survive from the non-linear self-coupling because of the existence of flat direction.
- Chap. 6 is the conclusion and discussion.
- Appendix A is the group theory which this thesis may use. This part is based on [59].
- Appendix B is the analysis of configurations of SU(3) gauge field in the axially symmetric Bianchi type I spacetime.

In this thesis, Chap. 4, Chap. 5, Chap. 6 and Appendix B is based on our published paper [60].

Chapter 2

Inflation Theory

2.1 Cosmology

2.1.1 Roberson-Walker Metric

At this section, we first review the standard cosmological model. In the background level, our universe is spatial homogeneous and isotropic at the large scale according to the observation. The background of the spacetime is governed by Einstein' general relativity. The spacetime metric can be decribed by the Robertson-Walker metric

$$ds^2 = -\mathrm{d}t^2 + a^2(t)\left[\frac{\mathrm{d}r^2}{1-\kappa r^2} + r^2 d\Omega^2\right], \tag{2.1}$$

where a is called scalar factor, it is the normalized size of the universe, and $a(t_0) = 1$ where t_0 is the time of today. κ is a value that depends on whether the space is open, flat or closed. Open, flat or closed universe correspond to $\kappa < 0$, $\kappa = 0$, $\kappa > 0$, respectively:

$$\begin{aligned}
&\kappa < 0 \rightarrow \text{open} \\
&\kappa = 0 \rightarrow \text{flat} \\
&\kappa > 0 \rightarrow \text{close.}
\end{aligned} \tag{2.2}$$

$\mathrm{d}\Omega^2$ is the metric of two-sphere

$$\mathrm{d}\Omega^2 = \mathrm{d}\theta^2 + \sin^2\theta \mathrm{d}\phi^2. \tag{2.3}$$

The Christoffel symbols in coordinate basis is given by

$$\Gamma^\lambda_{\mu\nu} = \frac{1}{2} g^{\lambda\sigma}\left(\partial_\mu g_{v\sigma} + \partial_\nu g_{\sigma\mu} - \partial_\sigma g_{\mu\nu}\right). \tag{2.4}$$

The Christoffel symbols for the Robertson-Walker metric are given by

$$\begin{array}{ll}
\Gamma^0_{11} = \frac{a\dot{a}}{1-\kappa r^2} & \Gamma^1_{11} = \frac{\kappa r}{1-\kappa r^2} \\
\Gamma^0_{22} = a\dot{a}r^2 & \Gamma^0_{33} = a\dot{a}r^2\sin^2\theta \\
\Gamma^1_{01} = \Gamma^2_{02} = \Gamma^3_{03} = \frac{\dot{a}}{a} & \\
\Gamma^1_{22} = -r\left(1-\kappa r^2\right) & \Gamma^1_{33} = -r\left(1-\kappa r^2\right)\sin^2\theta \\
\Gamma^2_{12} = \Gamma^3_{13} = \frac{1}{r} & \\
\Gamma^2_{33} = -\sin\theta\cos\theta & \Gamma^3_{23} = \cot\theta
\end{array} . \tag{2.5}$$

Note that the symmetry in coordinate basis $\Gamma^\alpha_{\beta\sigma} = \Gamma^\alpha_{\sigma\beta}$. The Riemann tensor is given by

$$R^\rho{}_{\sigma\mu\nu} = \partial_\mu \Gamma^\rho_{\nu\sigma} - \partial_\nu \Gamma^\rho_{\mu\sigma} + \Gamma^\rho_{\mu\lambda}\Gamma^\lambda_{\nu\sigma} - \Gamma^\rho_{\nu\lambda}\Gamma^\lambda_{\mu\sigma}. \tag{2.6}$$

The Ricci tesor is given by

$$R_{\alpha\beta} = R^\sigma{}_{\alpha\sigma\beta}. \tag{2.7}$$

and the nonzero components of Ricci tensor are

$$R_{00} = -3\frac{\ddot{a}}{a} \tag{2.8}$$

$$R_{11} = \frac{a\ddot{a} + 2\dot{a}^2 + 2\kappa}{1-\kappa r^2} \tag{2.9}$$

$$R_{22} = r^2\left(a\ddot{a} + 2\dot{a}^2 + 2\kappa\right) \tag{2.10}$$

$$R_{33} = r^2\left(a\ddot{a} + 2\dot{a}^2 + 2\kappa\right)\sin^2\theta. \tag{2.11}$$

The Ricci scalar is given by

$$R = R^\alpha{}_\alpha = 6\left[\frac{\ddot{a}}{a} + \left(\frac{\dot{a}}{a}\right)^2 + \frac{\kappa}{a^2}\right]. \tag{2.12}$$

2.1.2 The Friedmann Equation

We can now calculate the Einstein tensor using the above quantities. Now we need to deal with the energy-momentum tensor. We model matter and energy by a perfect fluid. A perfect fluid can be completely specified by two quantities, the rest-frame energy density ρ and an isotropic frame pressure p. We can choose a comoving coordinate so that the perfect fluid will be at rest and isotropic in this comoving coordinate. The four-velocity is then

$$U^\mu = (1, 0, 0, 0) \tag{2.13}$$

and the energy-momentum tensor

$$T_{\mu\nu} = (\rho + p)U_\mu U_\nu + p g_{\mu\nu} \tag{2.14}$$

becomes

$$T_{\mu\nu} = \begin{pmatrix} \rho & 0 & 0 & 0 \\ 0 & & & \\ 0 & & g_{ij}p & \\ 0 & & & \end{pmatrix}. \tag{2.15}$$

Raise one index give

$$T^\mu_\nu = \text{diag}(-\rho, p, p, p), \tag{2.16}$$

so the trace is given by

$$T = T^\mu_\mu = -\rho + 3p. \tag{2.17}$$

In general relativity, the conservation of energy equation is extended to

$$\nabla_\mu T^\mu_{\ \nu} = 0. \tag{2.18}$$

The zero component of the above equation gives

$$\begin{aligned} 0 &= \nabla_\mu T^\mu_0 \\ &= \partial_\mu T^\mu_0 + \Gamma^\mu_{\mu\lambda} T^\lambda_0 - \Gamma^\lambda_{\mu 0} T^\mu_\lambda \\ &= -\partial_0 \rho - 3\frac{\dot{a}}{a}(\rho + p). \end{aligned} \tag{2.19}$$

We can define an equation of state, a relationship between ρ and p by

$$p = w\rho. \tag{2.20}$$

The conservation of energy equation becomes to

$$\frac{\dot{\rho}}{\rho} = -3(1 + w)\frac{\dot{a}}{a}. \tag{2.21}$$

If w is a constant, we can integrate the above equation to obtain

$$\rho \propto a^{-3(1+w)}. \tag{2.22}$$

Thus if we know the w of a kind of energy, we know its evolution in terms of a. The conventional matter, which is any set of collisionless, nonrelativistic particles, have essentially zero pressure

$$p_{\rm M} = 0, \tag{2.23}$$

which means

$$w_{\rm M} = 0. \tag{2.24}$$

So we have

$$\rho_{\rm M} \propto a^{-3}. \tag{2.25}$$

The energy-momentum of the electromagnetic field is given by

$$T^{\mu\nu} = F^{\mu\lambda}F^{\nu}_{\lambda} - \frac{1}{4}g^{\mu\nu}F^{\lambda\sigma}F_{\lambda\sigma}. \tag{2.26}$$

The trace of this is given by

$$T^{\mu}_{\mu} = F^{\mu\lambda}F_{\mu\lambda} - \frac{1}{4}(4)F^{\lambda\sigma}F_{\lambda\sigma} = 0. \tag{2.27}$$

As the trace for any perfect fluid is given by $T = -\rho + 3p$, we know the equation of state for radiation is

$$p_{\rm R} = \frac{1}{3}\rho_{\rm R}. \tag{2.28}$$

Thus we have

$$\rho_{\rm R} \propto a^{-4}. \tag{2.29}$$

The vacuum energy has an equation of state

$$p_{\Lambda} = -\rho_{\Lambda}, \tag{2.30}$$

so we have

$$\rho_{\Lambda} \propto a^{0}. \tag{2.31}$$

Matter is also known as dust, any universes whose energy density is mostly due to matter are known as matter-dominated. Any universes whose energy density is mostly due to radiation are known as radiation-dominated. Any universes whose energy density is mostly due to vacuum are known as vacuum-dominated. De Sitter and anti-de Sitter are vacuum-dominated solutions.

Now let us do with Einstein's equation. Einstein's equation can be written in the form of

$$R_{\mu\nu} = 8\pi G \left(T_{\mu\nu} - \frac{1}{2} g_{\mu\nu} T \right). \tag{2.32}$$

The $\mu\nu = 00$ component is given by

$$-3\frac{\ddot{a}}{a} = 4\pi G(\rho + 3p). \tag{2.33}$$

The $\mu\nu = ij$ component is given by

$$\frac{\ddot{a}}{a} + 2\left(\frac{\dot{a}}{a}\right)^2 + 2\frac{\kappa}{a^2} = 4\pi G(\rho - p). \tag{2.34}$$

As the space is isotropic, there is only 1 independent equations from $\mu\nu = ij$ equations. The above two equations can be transformed to

$$\left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G}{3}\rho - \frac{\kappa}{a^2} \tag{2.35}$$

$$\frac{\ddot{a}}{a} = -\frac{4\pi G}{3}(\rho + 3p). \tag{2.36}$$

The two equations are known as the Friedmann equations. Metric (2.1) that obeys these equations defines Friedmann-Robertson-Walker (FRW) universe. The rate of expansion of universe is characterized by the Hubble parameter

$$H = \frac{\dot{a}}{a} \tag{2.37}$$

The current measurement of the Hubble parameter (Hubble constant) is 70$\pm$10 km/sec/Mpc. We often parameterize the Hubble constant as

$$H_0 = 100h \text{ km/sec/Mpc} \tag{2.38}$$

so that $h \approx 0.7$.

Let $\rho_i (i = m, r, \Lambda, \kappa)$ be the current energy density of matter, radiation, vacuum and curvature respectively, where ρ_κ is defined as

$$\rho_K := -\frac{3\kappa}{8\pi G}. \tag{2.39}$$

The first Friedmann equation can be rewritten as

$$H^2 = \frac{8\pi G}{3}\left(\rho_m a^{-3} + \rho_r a^{-4} + \rho_\Lambda + \rho_\kappa a^{-2}\right) \tag{2.40}$$

$$= \frac{8\pi G}{3}\rho_c\left(\frac{\rho_m}{\rho_c}a^{-3} + \frac{\rho_r}{\rho_c}a^{-4} + \frac{\rho_\Lambda}{\rho_c} + \frac{\rho_\kappa}{\rho_c}a^{-2}\right) \tag{2.41}$$

$$= H_0^2\left(\Omega_m a^{-3} + \Omega_r a^{-4} + \Omega_\Lambda + \Omega_\kappa a^{-2}\right), \tag{2.42}$$

where we have defined the critical energy density as

$$\rho_c := \frac{3}{8\pi G}H_0^2, \tag{2.43}$$

and the current density parameter for matter, radiation, vacuum and curvature respectively as

$$\Omega_i := \frac{\rho_i}{\rho_c} \quad \text{with} \quad i = m, r, \Lambda, \kappa. \tag{2.44}$$

2.1.3 Epoch of Matter-Radiation Equality

The epoch at which the energy density of matter equals that of radiation is called epoch of matter-radiation equality. Let a_{eq} as the scale factor of the epoch of matter-radiation equality, we have

$$\Omega_m a_{eq}^{-3} = \Omega_r a_{eq}^{-4}. \tag{2.45}$$

This give a_{eq} as

$$a_{eq} = \frac{\Omega_r}{\Omega_m}. \tag{2.46}$$

The observations of CMBR and the large scale structure give

$$\Omega_r = 4.15 \times 10^{-5} h^{-2}. \tag{2.47}$$

Thus

$$a_{eq} = \frac{4.15 \times 10^{-5}}{\Omega_m h^2}. \tag{2.48}$$

The redshift of equality is

$$1 + z_{eq} = \frac{1}{a_{eq}} = 2.4 \times 10^4 \Omega_m h^2. \tag{2.49}$$

As $\Omega_m \approx 0.3$, $h \approx 0.7$, we abtain

$$1 + z_{eq} = 3.6 \times 10^3. \tag{2.50}$$

So we know the redshift of equality is several times larger than the redshift of recombination, $z_r \approx 10^3$. Thus we know recombination happens in the matter-dominated era. Similarly, we can also know the universe evolve into a vacuum-dominated era at a very late time.

2.2 Why Inflation

2.2.1 Horizon Problem

The horizon problem begins from the question of why the cosmic microwave background (CMB) is isotropic to a so high degree of precision. According to the experiment on CMB, the CMB can be a seen very good black body with temperature $T = 2.7K$ and perturbation about 10^{-5}. However, according to the Big Bang cosmology, if we calculate the causally connected patches of CMB at recombination, we know the causally connected patch for one point is much smaller than the area of the CMB at recombination(or we can say the size of the horizon for a point of CMB at recombination is much smaller than the comoving distance between CMB at recoxa1mbination and the earth). That is, distinct patches of the CMB sky at recombination were causally disconnected. Thus the calculation using the naive Big Bang theory is contradictory to the experiment fact of CMB. To see this, we will do the calculation by imagining we are in a matter-dominated universe, for which

$$\rho \propto a^{-3}. \tag{2.51}$$

Substitute this to the Friedmann equation (Hamiltonian constraint)

$$H^2 \equiv (\frac{\dot{a}}{a})^2 = \frac{1}{3}\rho, \tag{2.52}$$

we obtain

$$a = (\frac{t}{t_0})^{2/3}, \tag{2.53}$$

where we have normalized the scale factor a to $a = 1$ for $t = t_0$, where t_0 is the time corresponding to $z = 0$. Note that we have used the reduced Plank mass as $M_{Pl} := 1$. From the above equation we know

$$H = \frac{2}{3}t^{-1} = H_0 a^{-2/3}. \tag{2.54}$$

The second equation is directly deduced from the Friedmann equation. We thus have

$$dt = H_0^{-1} a^{1/2} da. \tag{2.55}$$

On the other hand, the metric of a homogeneous isotropic spacetime can be given by

$$ds^2 = -dt^2 + a^2 dr^2. \tag{2.56}$$

So the comoving distance traveled by a photon in such spacetime between time t_1 and t_2 is given by

$$\Delta r = \int_{t_1}^{t_2} \frac{dt}{a}. \tag{2.57}$$

The comoving particle horizon for some time t (or the corresponding scale factor a) is just the comoving distance by integrating from the begining of Big Bang to the time time t:

$$d_h(a) = \int_0^t \frac{dt}{a} = H_0^{-1} \int_0^a a^{-1/2} da = 2H_0^{-1} a^{1/2}. \tag{2.58}$$

Recombination is the epoch at which charged electrons and protons first became bound to form electrically neutral hydrogen atoms and emit photons. Recombination occurred at redshift $z_{CMB} \approx 1100$ and thus the corresponding scale factor $a_{CMB} \approx 1/1100$. Thus when we look at the CMB we are observing the universe at a scale $a_{CMB} \approx 1/1100$, at the corresponding time when the recombination occurred. We can calculate the comoving distance between the point on the CMB (at recombination) and an observer on Earth

$$\begin{aligned} \Delta d &= 2H_0^{-1}(1 - \sqrt{a_{CMB}}) \\ &\approx 2H_0^{-1}. \end{aligned} \tag{2.59}$$

While the comoving horizon for such a point is

$$\begin{aligned} d_h(a_{CMB}) &= 2H_0^{-1}\sqrt{a_{CMB}} \\ &= 6 \times 10^{-2} H_0^{-1}. \end{aligned} \tag{2.60}$$

That is, actually, two widely-separated parts of the CMB have nonoverlapping horizons, widely-separated parts of the CMB were causally disconnected at recombination. A cartoon for the calculation above is also shown in Fig2.1. But our observation on CMB tells us that the CMB sky is at the same temperature at high precision. This implies that there is some problem in the conventional FRW cosmology and we must do some modifications.

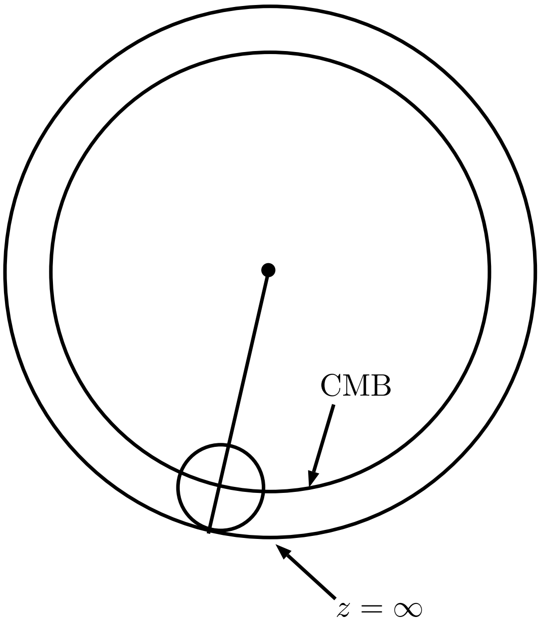

FIGURE 2.1: A cartoon for the horizon problem. $z = \infty$ corresponds to $a = 0$. The small circle centered at one point of CMB is the horizon of the point.

Similarly, for a radiation-dominated universe, we have $\rho \propto a^{-4}$, we can obtain

$$a = (\frac{t}{t_0})^{1/2} \tag{2.61}$$

$$H = \frac{1}{2t} = H_0 a^{-2} \tag{2.62}$$

$$\int_{t_1}^{t_2} \frac{dt}{a} = H_0^{-1}(a_2 - a_1). \tag{2.63}$$

The comoving distance between the point on the CMB (at recombination) and an observer on Earth is then

$$\begin{aligned} \Delta d &= H_0^{-1}(1 - a_{CMB}) \\ &\approx H_0^{-1}, \end{aligned} \tag{2.64}$$

while the comoving horizon distance for such a point is

$$\begin{aligned} d_h(a_{CMB}) &= H_0^{-1} a_{CMB} \\ &\approx \frac{1}{1200} H_0^{-1}. \end{aligned} \tag{2.65}$$

Thus in the radiation-dominated universe, there is still the horizon problem.

2.2.2 Flatness Problem

The flatness problem is that why our universe today is so flat. Consider the time when the vacuum energy is still small, the first Friedman equation is

$$H^2 = \frac{8\pi G}{3}\left(\rho_M + \rho_R - \frac{\kappa}{a^2}\right). \tag{2.66}$$

As $\rho_M \propto a^{-3}$, $\rho_R \propto a^{-4}$, this raises the question of why the energy density of curvature $\kappa a^{-2}/G$ isn't much larger than the energy density of matter and radiation $(\rho_M + \rho_R)$ given that a has increased by a factor of perhaps 10^{28}(see the next subsection) since the Plank epoch.

2.2.3 Solution to the Horizon Problem and Flatness Problem

The most well-known solution to the horizon problem and flatness problem is inflation theory. Inflation is an ear of acceleration($\ddot{a} > 0$) in the very early universe. The idea of inflation is that there is a period of the very early universe when the universe grows much more rapidly than the conventional cosmology model such that the true horizon is much larger than the naive estimate. To see this, let us rewrite the comoving horizon in a general form as

$$d_h(a) = \int_0^a \frac{da'}{a'}\frac{1}{a'H}. \tag{2.67}$$

The comoving horizon then is the logarithmic integral of the comoving Hubble radius $(aH)^{-1}$. The Hubble radius is the distance over which particles can travel in the course of one expansion time, $\frac{da}{a} = d\ln(a) = 1$, roughly the time in which the scale factor doubles. So the Hubble radius measure whether the particles are causally connected currently: if they are separated by a distance larger than the Hubble radius, then they can not currently communicate. While the comoving horizon says that if the particles are separated by comoving distances larger than d_h, they never could have communicated with one another. When the energy is smaller than the order of 10^2 GeV, the standard model of particle physics works very well. However, when the energy is larger than the order of this energy, although we have ideas, there is no experimental reason to prefer one theory over others. For the very early universe, The temperature is much larger than that today because $a \to 0$, we are actually assuming if we naively use the conventional cosmology in which the dominant component is radiation or matter. The inflation theory suggests a solution to the horizon problem: maybe in an epoch of the early universe, the universe is not dominated by radiation or matter, but by other components. So that the Hubble radius was much larger than the large scale at some initial time and decrease dramatically during this epoch. In that case, the integrated comoving horizon would get most of its contribution not from recent time but the primordial epoch. Then, the large scale of the CMB would causally connect and the isotropy of CMB would not be surprised.

If the comoving Hubble radius is to decrease, then aH must increase. Thus

$$\frac{d}{dt}\left[a\frac{da/dt}{a}\right] = \frac{d^2a}{dt^2} > 0. \tag{2.68}$$

That is, we need an epoch that the universe is accelerating. How many times does the universe need to expand during inflation? We first evaluate the Hubble radius at the end of inflation. To simplify the calculation, we ignore the relatively brief epoch of recent matter domination and assume that the universe has been radiation-dominated since the end of inflation. In radiation dominated universe, The Hubble parameter scale is a^{-2}, so we have

$$\frac{a_0 H_0}{a_e H_e} = \frac{a_e}{a_0}. \tag{2.69}$$

Where the "e" denote the "end" of inflation. If we denote the energy scale at the end of inflation as E_e, the temperature of radiation today as T_0, then we have

$$\frac{T_0}{E_e} = \frac{a_e}{a_0}. \tag{2.70}$$

If we know E_e, we can know a_0H_0/a_eH_e. For most inflationary model, they typically operate at enery scales of order 10^{15} GeV. Substitute this and the temperature of radiation today $T_0 = 3$K, we obtain

$$\frac{a_0 H_0}{a_e H_e} = \frac{a_e}{a_0} \approx \frac{10^{-4}\text{eV}}{10^{15}\text{GeV}} = 10^{-28}, \tag{2.71}$$

where we have use 1eV $= 11605$K and approximate as 1K $\sim 10^{-4}$eV. That is, the comoving Hubble radius at the end of inflation is 28 orders of magnitude smaller than it is today. For inflation to work, the comoving Hubble radius at the onset of inflation should be larger than large scale, i.e., larger than the comoving Hubble radius today. So during inflation, the comoving Hubble radius have to decrease at least 28 order of magnitude.

We can evaluate how much does the scale factor has changed during inflation by assuming a model that H does not change during inflation. In this case, the universe expands exponentially because the evolution of scale factor a satisfy

$$d\ln(a) = Hdt. \tag{2.72}$$

Actually, from observation, we know H does not change much during inflation. If H does not change during inflation, we have

$$\frac{a_i}{a_e} = 10^{-28}, \tag{2.73}$$

where a_i is the scale factor at the onset of inflation. Thus, if the universe expand (exponentially) for $\ln(10^{28}) \sim 64$ e-folds, the horizon problem can be solved.

For the flatness problem, consider the case where inflation is driven by constant vacuum energy. After a sufficiently long period during inflation, the term κ/a^2 dilutes to a very tiny number and its curvature density parameter close to 0 while the vacuum energy keeps unchanged. After inflation, the energy of vacuum energy is converted into matter and radiation, the density parameter will be sufficiently close to unity and the curvature density parameter will not have had a chance to noticeably change into the present era.

2.3 Condition for Inflation

Although we have point the the condition for inflation, we make it more clear in this section. The condition for inflation is

$$\frac{d}{dt}(aH)^{-1} < 0 \quad \Rightarrow \quad \varepsilon \equiv -\frac{\dot{H}}{H^2} < 1 \quad \Leftrightarrow \quad \frac{d^2 a}{dt^2} > 0 \quad \Leftrightarrow \quad \rho + 3p < 0. \tag{2.74}$$

Because of

$$\frac{d}{dt}(aH)^{-1} = -\frac{\dot{a}H + a\dot{H}}{(aH)^2} = -\frac{1}{a}(1-\varepsilon), \tag{2.75}$$

the decreasing comoving Hubble radius implies $\varepsilon < 1$. Actually, ε is the fractional change of the Hubble parameter per e-fold because

$$\varepsilon = -\frac{\dot{H}}{H^2} = -\frac{d\ln H}{dN}. \tag{2.76}$$

To solve the cosmological problems we want inflation to last for a sufficiently long time, So we require ε to change small in a Hubble time, it is measured by defining a second parameter

$$\eta \equiv \frac{\dot{\varepsilon}}{H\varepsilon} = \frac{d\ln\varepsilon}{dN}. \tag{2.77}$$

For $|\eta| < 1$. the fractional change of ε per e-fold is small and inflation persists.

Substitute the condition $\ddot{a} > 0$ to the second Friedmann equation

$$\dot{H} + H^2 \equiv \frac{\ddot{a}}{a} = -\frac{1}{6M_{\rm pl}^2}(\rho + 3p), \tag{2.78}$$

we obtain

$$\rho + 3p < 0, \tag{2.79}$$

i.e. inflation requires negative pressure and a violation of the strong energy condition. For strong energy condition, it requires

$$\left(T_{ab} - \frac{1}{2}g_{ab}T\right)t^a t^b \geqslant 0 \tag{2.80}$$

for any timelike t^a. For a perfect fluid and a pure time direction t^a, the strong energy condition can be rewritten as

$$\rho + 3p \geqslant 0. \tag{2.81}$$

Thus the condition for inflation violates the strong energy condition.

2.4 Physics of Inflation

In this section, we will take slow-roll inflation as an example as this model is simple and has a mechanism to end the inflation and successfully reheat the universe.

Consider a scalar field ϕ, the inflaton, minimally coupled to Einstein gravity

$$S = \int \mathrm{d}^4x\sqrt{-g}\left[\frac{M_{\rm pl}^2}{2}\mathcal{R} - \frac{1}{2}g^{\mu\nu}\partial_\mu, \phi\partial_\nu\phi - V(\phi)\right] \tag{2.82}$$

where $\mathcal{R}$ is the four-dimensional Ricci scalar derived from the metric $g_{\mu\nu}$ and $V(\phi)$ is an arbitraru function:

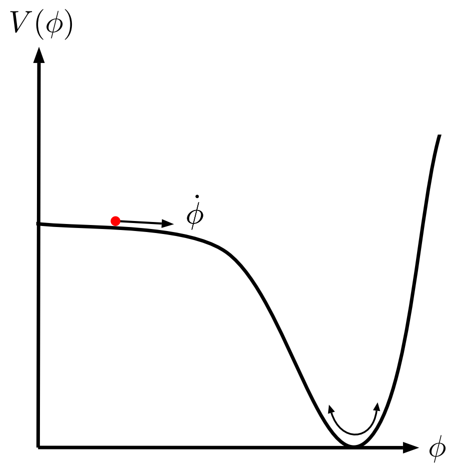

FIGURE 2.2

The equations of motion include the Hamiltonian constraint:

$$H^2 = \frac{1}{3M_{\rm pl}^2}\left[\frac{1}{2}\dot{\phi}^2 + V\right], \tag{2.83}$$

the continuity equation

$$\dot{H} = -\frac{1}{2}\frac{\dot{\phi}^2}{M_{\rm pl}^2}, \tag{2.84}$$

and the Klein-Gordon equation

$$\ddot{\phi} + 3H\dot{\phi} = -V'. \tag{2.85}$$

Note that there is only 2 independent equations in the above equations. Substitute the continuity equation into the definition of ε ,we obtain

$$\varepsilon = \frac{\frac{1}{2}\dot{\phi}^2}{M_{\rm pl}^2 H^2}. \tag{2.86}$$

Thus if the potential V dominates over the the kinetic energy $\frac{1}{2}\dot{\phi}^2$, inflation occurs. We can also check the negative pressure and violation of the strong energy condition. As V dominates over $\frac{1}{2}\dot{\phi}^2$, we have the energy density approximately

$$\rho = \frac{1}{2}\dot{\phi}^2 + V \approx V, \tag{2.87}$$

while the pressure p is approximate

$$p = \frac{1}{2}\dot{\phi}^2 - V \approx -V. \tag{2.88}$$

Thus the negative pressure and violation of the strong energy condition are satisfied.

We can calculate η from the expression of ε,

$$\begin{aligned}\eta &\equiv \frac{\dot{\varepsilon}}{H\varepsilon} \\ &= 2\varepsilon + 2\frac{\ddot{\phi}}{H\dot{\phi}} \\ &= 2(\varepsilon - \delta),\end{aligned} \tag{2.89}$$

where we have defined at the last equation

$$\delta \equiv -\frac{\ddot{\phi}}{H\dot{\phi}}. \tag{2.90}$$

Thus, if $\{\varepsilon, |\delta|\} \ll 1$, then both H and ε have small fractional changes per e-fold : $\{\varepsilon, |\eta|\} \ll 1$.

If there is a regime where $\{\varepsilon, |\eta|\} \ll 1$, then inflation exist. We can use these conditions to simplify the equations of motion. This is called *slow-roll approximation*. As the condition $\varepsilon = \frac{\frac{1}{2}\dot{\phi}^2}{M_{\rm pl}^2 H^2} \ll 1$ implies $\frac{1}{2}\dot{\phi}^2 \ll V$, we can simplify the Hamiltonian constraint by ignoring the kinetic term:

$$H^2 \approx \frac{V}{3M_{\rm pl}^2}. \tag{2.91}$$

On the other hand, by using the condition $|\delta| = \frac{|\ddot{\phi}|}{H|\dot{\phi}|} \ll 1$ we can simplify the Klein-Gordon equation to

$$3H\dot{\phi} \approx -V'. \tag{2.92}$$

Substitute $\dot{\phi}$ and H^2 in the above two equations to ε we obtain

$$\varepsilon = -\frac{\dot{H}}{H^2} = \frac{\frac{1}{2}\dot{\phi}^2}{M_{\rm pl}^2 H^2} \approx \frac{M_{\rm pl}^2}{2}\left(\frac{V'}{V}\right)^2 \equiv \epsilon_{\rm v}. \tag{2.93}$$

Take the time-derivative of (2.92) we obtain

$$3\dot{H}\dot{\phi} + 3H\ddot{\phi} = -V''\dot{\phi}. \tag{2.94}$$

Divide this by $-H^2\dot{\phi}$ and use $H^2 \approx V^2$ we obtain

$$-\frac{\ddot{\phi}}{H\dot{\phi}} - \frac{\dot{H}}{H^2} \approx M_{\rm pl}^2\frac{V''}{V} \equiv \eta_{\rm v} \tag{2.95}$$

or directly

$$\delta + \varepsilon \approx \eta_{\rm v}. \tag{2.96}$$

Here $\epsilon_{\rm v}$ and $\eta_{\rm v}$ are called potential slow-roll parameters. When they are small, slow-roll inflation occurs:

$$\epsilon_{\rm v} \equiv \frac{M_{\rm pl}^2}{2}\left(\frac{V'}{V}\right)^2 \ll 1 \tag{2.97}$$

$$|\eta_{\rm v}| \equiv M_{\rm pl}^2 \frac{|V''|}{V} \ll 1. \tag{2.98}$$

We can use the number of e-folds of accelerated expansion to measure the amount of inflation

$$N \equiv \int_{a_i}^{a_f} \mathrm{d}\ln a = \int_{t_i}^{t_f} H(t)\mathrm{d}t, \tag{2.99}$$

where the integrated regime is defined as that satisfies $\epsilon_{\rm v} < 1$. The calculation of N can be changed to the integration for ϕ because

$$H\ \mathrm{d}t = \frac{H}{\dot\phi}\mathrm{d}\phi \approx -\frac{3H}{V'} \cdot H\ \mathrm{d}\phi \approx \frac{1}{\sqrt{2\epsilon_{\rm v}}}\frac{|\mathrm{d}\phi|}{M_{\rm pl}}. \tag{2.100}$$

As to solve the horizon problem we need about 60 e-folding number, we have

$$N_{\rm cmb} = \int_{\phi_{\rm cmb}}^{\phi_f} \frac{1}{\sqrt{2\epsilon_{\rm v}}}\frac{|\mathrm{d}\phi|}{M_{\rm pl}} \approx 60. \tag{2.101}$$

Case of $m^2\phi^2$ inflation:
For potential

$$V(\phi) = \frac{1}{2}m^2\phi^2, \tag{2.102}$$

the slow-roll parameters are

$$\epsilon_{\rm v}(\phi) = \eta_{\rm v}(\phi) = 2\left(\frac{M_{\rm pl}}{\phi}\right)^2. \tag{2.103}$$

They need to be small than 1, thus

$$\phi > \sqrt{2}M_{\rm pl} \equiv \phi_f, \tag{2.104}$$

where we have assume ϕ is in the regime $\phi > 0$. Thus the e-folding number is

$$N(\phi) = \frac{\phi^2}{4M_{\rm pl}^2} - \frac{1}{2}. \tag{2.105}$$

Fluctuation of the large scale of CMB are created at

$$\phi_{\rm cmb} = 2\sqrt{N_{\rm cmb}}M_{\rm pl} \sim 15M_{\rm pl}. \tag{2.106}$$

2.5 Primordial Fluctuations

Inflation not only solves the horizon problem and flatness problem but also provides the seed of the cosmic structure generated at the late time through quantum fluctuations. In this chapter, we calculate the quantum perturbation during inflation. We will consider single-field slow-roll model of inflation,

$$S = \int \mathrm{d}^4x\sqrt{-g}\left[\frac{1}{2}\mathcal{R} - \frac{1}{2}g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi - V(\phi)\right]. \tag{2.107}$$

where we have set $M_{\rm Pl} = 1$.

2.5.1 Classical Perturbations

We first obtain the Classical equation of motion for the perturbations. We can write the inflaton and the metric in the perturbation form

$$g_{\mu\nu}(t, \mathbf{x}) \equiv \bar{g}_{\mu\nu}(t) + \delta g_{\mu\nu}(t, \mathbf{x}) \tag{2.108}$$

$$\phi(t, \mathbf{x}) \equiv \bar{\phi}(t) + \delta\phi(t, \mathbf{x}). \tag{2.109}$$

There are 10 degrees of freedom in the metric perturbations. The metric can be written as

$$ds^2 = -(1 + 2\Phi)dt^2 + w_i(dtdx^i + dx^i dt) + [(1 - 2\Psi)\delta_{ij} + 2s_{ij}]dx^i dx^j, \tag{2.110}$$

where the w_i can be decomposed to a transverse (divergence) part (2 Dof) and a longitudinal part (1 Dof), the traceless s_{ij} can be decomposed to a transverse part (2 Dof), a sonenloidal part (2 Dof) and a longitudinal part (1 Dof). Thus there are 4 scalar modes, 4 vector modes, and 2 tensor modes totally in the metric perturbations. In addition to $\delta\phi(t, \mathbf{x})$, we have 5 scalar perturbations. The tensor modes account for the Primordial gravitational waves.

In the 10 Degrees of freedom, there are 4 gauge Degrees of freedom because we can find a gauge vector ξ^μ such that the Riemann tensor $R^{\mu}_{\nu\rho\sigma}$ is unchanged under and thus the Ricci scalar is also unchanged under the transformation of

$$\delta g_{\mu\nu} \to \delta g^{(\epsilon)}_{\mu\nu} = \delta g_{\mu\nu} + 2\epsilon\partial_{(\mu}\xi_{\nu)}. \tag{2.111}$$

We can use the gauge invariance under scalar coordinate transformations

$$t \to t + \epsilon_0 \tag{2.112}$$

$$x_i \to x_i + \partial_i\epsilon \tag{2.113}$$

to remove two modes. The Einstein constraint equations remove two more modes, so that we are left with only 1 physical scalar mode. Using comoving gauge, defined by the vanishing of the momentum density, $\delta T_{0i} \equiv 0$. For slow-roll inflation, this becomes

$$\delta\phi = 0. \tag{2.114}$$

In this gauge, the perturbations of metric can be written as

$$\delta g_{ij} = a^2(1-2\zeta)\delta_{ij} + a^2 h_{ij}, \tag{2.115}$$

where h_{ij} is a transverse, traceless tensor and thus has only two degrees of freedom. ζ is referred to as teh comoving curvature perturbation because the three-curvature related to ζ by

$$R_{(3)} = \frac{4}{a^2}\nabla^2\zeta, \tag{2.116}$$

the second-order action for ζ is given by

$$S = \frac{1}{2}\int \mathrm{d}t\ \mathrm{d}^3\mathbf{x} a^3 \frac{\dot{\phi}^2}{H^2}\left[\dot{\zeta}^2 - \frac{1}{a^2}\left(\partial_i\zeta\right)^2\right]. \tag{2.117}$$

Define the canonically normalized Mukhanov variable

$$v \equiv z\zeta, \tag{2.118}$$

where

$$z^2 \equiv a^2\frac{\dot{\phi}^2}{H^2} = 2a^2\varepsilon M_{\mathrm{pl}}^2. \tag{2.119}$$

Switching to conformal time, the action can be rewritten as

$$S=\frac{1}{2}\int \mathrm{d}\tau \mathrm{d}^3\mathbf{x}\left[\left(v'\right)^2-\left(\partial_i v\right)^2+\frac{z''}{z}v^2\right]. \tag{2.120}$$

This is the action of an harmonic oscillator with a time-dependent mass

$$m_{\mathrm{eff}}^2(\tau)\equiv-\frac{z''}{z}=-\frac{H}{a\dot{\phi}}\frac{\partial^2}{\partial\tau^2}\frac{a\dot{\phi}}{H}. \tag{2.121}$$

From the action we can obtain the mukhanov-Sasaki equaiton

$$v_{\mathbf{k}}''+\underbrace{\left(k^2-\frac{z''}{z}\right)}_{\equiv\omega_k^2(\tau)}v_{\mathbf{k}}=0, \tag{2.122}$$

where v_{k} is the Fourier modes. From (2.119) we know $z''/z=a''/a$. In de Sitter space, $a=-(H\tau^{-1})$, $(aH)^{-1}=-\tau$, the comoving horizon can be caracterized as $-\tau$. We also have $a''/a=2/\tau^2$, so the effective frequency in de Sitter reduces to

$$\omega_k^2(\tau)=k^2-\frac{2}{\tau^2}\quad(\text{ de Sitter }). \tag{2.123}$$

Consider the subhorizon, $k^2\gg|z''/z|\quad(k\gg(aH)^{-1})$, we get

$$v_{\mathbf{k}}''+k^2v_{\mathbf{k}}=0\quad(\text{ subhorizon }). \tag{2.124}$$

This has oscillationg solutions: $v_{\mathbf{k}}\propto e^{\pm ik\tau}$. Thus the curvature perturbation $\zeta_{\mathbf{k}}(\propto z^{-1}v_{\mathbf{k}}\propto a^{-1}v_{\mathbf{k}}$ decay as a^{-1} in subhorizon scale. Consider the superhorizon, $k^2\ll|z''/z|\quad(k\ll(aH)^{-1})$, we have

$$\frac{v_{\mathrm{k}}''}{v_{\mathrm{k}}}=\frac{z''}{z}\approx\frac{2}{\tau^2}\quad(\text{ superhorizon }) \tag{2.125}$$

This has growing solution $v_{\mathrm{k}}\propto z\propto\tau^{-1}$ (and the decaying solution $v_{\mathrm{k}}\propto\tau^2$), thus the corvature perturbation ζ freezes on superhorizon scales: $\zeta_{\mathbf{k}}=z^{-1}v_{\mathbf{k}}\propto$ const.

2.5.2 Quantum Origin of Cosmological Perturbations

Vacuum in Minkowski Space

Consider the Mukhanov-Sasaki equation (with scalar k) in Minkowski space ($a\equiv1$) :

$$v_k''+k^2v_k=0. \tag{2.126}$$

We have the solution same as the subhorizon solution, that is

$$v_{\mathbf{k}} = A(k)e^{\pm ik\tau}, \tag{2.127}$$

where the amplitude $A(k)$ is still not determined. We can determine its expression by requiring the expectation value of the Hamiltonian in the vacuum state to be minimized. One can show that when

$$A(k) = \frac{1}{\sqrt{2k}}, \tag{2.128}$$

the expectation value of the Hamiltonian in the vacuum state in minimized and the vacuum is defined.

Zero-Point Fluctuation in de Sitter Spacetime

The exact solution for the Mukhanov-Sasaki Equation is

$$v_k(\tau) = \alpha\frac{e^{-ik\tau}}{\sqrt{2k}}\left(1 - \frac{i}{k\tau}\right) + \beta\frac{e^{ik\tau}}{\sqrt{2k}}\left(1 + \frac{i}{k\tau}\right). \tag{2.129}$$

The value of α and β can be determined as $\alpha = 1$ and $\beta = 0$ by using the initial condition

$$\lim_{\tau\to-\infty} v_k(\tau) = \frac{1}{\sqrt{2k}}e^{-ik\tau}. \tag{2.130}$$

The initial condition comes from that at $\tau \to \infty$, the Mukhanov-Sasaki Equation is just the same as that in Minkowski space. Thus the solution for the Mukhanov-Sasaki Equation in de Sitter Spacetime is

$$v_k(\tau) = \frac{e^{-ik\tau}}{\sqrt{2k}}\left(1 - \frac{i}{k\tau}\right). \tag{2.131}$$

The superhorizon solution is then

$$\lim_{k\tau\to 0} v_k(\tau) = \frac{1}{i\sqrt{2}}\cdot\frac{1}{k^{3/2}\tau}. \tag{2.132}$$

The most general general solution of Mukhanov-Sasaki equation (with vector $\mathbf{k}$)(2.122) can be written as

$$v_{\mathbf{k}} \equiv a^-_{\mathbf{k}}v_k(\tau) + a^+_{-\mathbf{k}}v^*_k(\tau). \tag{2.133}$$

The canonical quantization is done by promoting the field $v(\tau, \mathbf{x})$ and its cononically conjugate momentum $\pi \equiv v'$ to quantum operators $\hat{v}$ and $\hat{\pi}$, which satisfy the standard equal-time commutation relations

$$[\hat{v}(\tau,\mathbf{x}),\hat{\pi}(\tau,\mathbf{y})] = i\delta(\mathbf{x}-\mathbf{y}) \tag{2.134}$$

$$[\hat{v}(\tau,\mathbf{x}),\hat{v}(\tau,\mathbf{y})] = [\hat{\pi}(\tau,\mathbf{x}),\hat{\pi}(\tau,\mathbf{y})] = 0. \tag{2.135}$$

The constants $a^{\pm}_{\mathbf{k}}$ then become operators $\hat{a}^{\pm}_{\mathbf{k}}$, and satisfy

$$\left[\hat{a}^{-}_{\mathbf{k}},\hat{a}^{+}_{\mathbf{k}'}\right] = \delta\left(\mathbf{k}-\mathbf{k}'\right) \quad \text{and} \quad \left[\hat{a}^{-}_{\mathbf{k}},\hat{a}^{-}_{\mathbf{k}'}\right] = \left[\hat{a}^{+}_{\mathbf{k}},\hat{a}^{+}_{\mathbf{k}'}\right] = 0 \tag{2.136}$$

The effect of quantum zero-point fluctuations for the canonically-normalized field $v_{\mathbf{k}}$ is then

$$\begin{aligned}\langle \hat{v}_{\mathbf{k}}\hat{v}_{\mathbf{k}'}\rangle &= \langle 0\,|\hat{v}_{\mathbf{k}}\hat{v}_{\mathbf{k}'}|\,0\rangle \\ &= \left\langle 0\left|\left(a^{-}_{\mathbf{k}}v_k + a^{+}_{-\mathbf{k}}v^*_k\right)\left(a^{-}_{\mathbf{k}'}v_{k'} + a^{+}_{-\mathbf{k}'}v^*_{k'}\right)\right|0\right\rangle \\ &= v_k v^*_{k'}\left\langle 0\left|a^{-}_{\mathbf{k}}a^{+}_{-\mathbf{k}'}\right|0\right\rangle \\ &= v_k v^*_{k'}\left\langle 0\left|\left[a^{-}_{\mathbf{k}},a^{+}_{-\mathbf{k}'}\right]\right|0\right\rangle \\ &= |v_k|^2\,\delta\left(\mathbf{k}+\mathbf{k}'\right) \\ &\equiv P_v(k)\delta\left(\mathbf{k}+\mathbf{k}'\right)\end{aligned} \quad . \tag{2.137}$$

On superhorizon scales P_v is

$$P_v = \frac{1}{2k^3}\frac{1}{\tau^2} = \frac{1}{2k^3}(aH)^2. \tag{2.138}$$

The power spectrum of curvature perturbation is then

$$P_\zeta = \frac{1}{z^2}P_v. \tag{2.139}$$

2.5.3 Curvature Perturbation from Inflation

Actually, the curvature fluctuations $\zeta = z^{-1}v$ are ill-defined in perfect de Sitter since $z^2 = 2a^2\epsilon$ vanishes in de Sitter. In quasi-de Sitter space, ζ is well-defined. For the P_v in quasi-de Sitter, we still use the result of de Sitter (2.138). using $z^2 = 2a^2\epsilon$, we get the power spectrum of ζ for quasi-de Sitter space on superhorizon scales

$$P_\zeta = \frac{1}{z^2}P_v = \frac{1}{4k^3}\frac{H^2}{\varepsilon M^2_{\rm pl}} = \frac{1}{2k^3}\frac{H^4}{\dot{\phi}^2}. \tag{2.140}$$

As the power spectrum is for superhorizon scales, it need to satisfy $k(aH)^{-1} \to 0$. But as since ζ freezes at horizon crossing, the condition can be released to $k(aH)^{-1} = 1$.

The power spectrum (on superhorizon scale) can then be written in the form of

$$P_\zeta(k) = \frac{1}{4k^3}\frac{H^2}{\varepsilon M_{\rm pl}^2}\bigg|_{k=aH}. \tag{2.141}$$

We can define a dimensionless power spectrum

$$\Delta_s^2(k) \equiv \frac{k^3}{2\pi^2}P_\zeta(k) = \frac{1}{8\pi^2}\frac{H^2}{\varepsilon}\bigg|_{k=aH}. \tag{2.142}$$

Since the power spectrum is evaluated at the horizon scale $k = aH$, and H and possibly ϵ are now functions of time, thus $\Delta_s^2(k)$ will deviate slightly from the scale-invariant form $\Delta_s^2(k) \approx k^0$. We can define the scalar spectral index n_s:

$$n_s - 1 \equiv \frac{d\ln\Delta_{\rm S}^2}{d\ln k}. \tag{2.143}$$

We have

$$\frac{d\ln\Delta_{\rm S}^2}{d\ln k} = \frac{d\ln\Delta_{\rm S}^2}{dN}\times\frac{dN}{d\ln k} \tag{2.144}$$

$$= (2\frac{d\ln H}{dN} - \frac{d\ln\varepsilon}{dN})\times\frac{dN}{d\ln k}, \tag{2.145}$$

where

$$\frac{d\ln H}{dN} = H^{-1}\frac{dH}{dN} = H^{-1}\frac{\dot{H}}{\dot{N}} = -\epsilon \tag{2.146}$$

$$\frac{d\ln\varepsilon}{dN} = \epsilon^{-1}\frac{\dot{\epsilon}}{\dot{N}} = \eta \tag{2.147}$$

$$\frac{dN}{d\ln k} = 1 - \frac{d\ln H}{dN} = 1 + \varepsilon, \tag{2.148}$$

where the last equation (2.148) have used the horizon crossing condition $k = aH$, or

$$\ln k = N + \ln H. \tag{2.149}$$

Thus, to first order in the Hubble slow-roll parameters we find

$$n_{\rm s} - 1 = -2\varepsilon - \eta. \tag{2.150}$$

2.5.4 Gravitational Waves from Inflation

The tensor perturbations are transverse and traceless perturbations to the spatial metric $\delta g_{ij} = a^2 h_{ij}$. The tensor perturbation is gauge-invariant and doesn't backreact on the inflationary background. Expansion of the Einstein-Hilbert action gives the second-order action for tensor fluctuations

$$S = \frac{M_{\rm pl}^2}{8} \int d\tau d^3\mathbf{x} a^2 \left[\left(h'_{ij}\right)^2 - (\nabla h_{ij})^2 \right]. \tag{2.151}$$

As the tensor perturbation h_{ij} is transverse and traceless, we can express it in the form of

$$h_{ab} = \begin{pmatrix} h_+ & h_\times & 0 \\ h_\times & -h_+ & 0 \\ 0 & 0 & 0 \end{pmatrix}, \tag{2.152}$$

where we have assumed the gravitational wave propagate in the z direction ($\mathbf{k} = (0, 0, k)$). we can further introduce the two independent traceless, transverse polarization tensors

$$e_{ab}^+ = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad e_{ab}^\times = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \tag{2.153}$$

which satisfy

$$\epsilon_{ii}^\gamma = 0 \tag{2.154}$$

$$k^i \epsilon_{ij}^\gamma = 0 \tag{2.155}$$

$$\epsilon_{ij}^\gamma \epsilon_{ij}^{\gamma'} = 0, \tag{2.156}$$

for the traceless, transverse and independence properties respectively, with γ denoting $+$ and $\times$. Now we can express h_{ij} as

$$h_{ij} = \sum_\gamma h_\gamma \epsilon_{ij}^\gamma. \tag{2.157}$$

Define the cononically-normalized fields

$$v_\gamma \equiv \frac{a}{2} M_{\rm pl} h_\gamma, \tag{2.158}$$

then the action can be rewritten as

$$S = \sum_\gamma \frac{1}{2} \int d\tau d^3\mathbf{x} [(v'_\gamma - \frac{a'}{a} v_\gamma)^2 - (\partial_i v_\gamma)^2] \tag{2.159}$$

$$= \sum_\gamma \frac{1}{2} \int d\tau d^3\mathbf{x} [(v'^2_\gamma + \frac{a''}{a} v_\gamma^2 - (\partial_i v_\gamma)^2], \tag{2.160}$$

where the second form is obtained by integrating by parts and the total derivative term is dropped. This has the same form as the action of scalar perturbation (2.120) except that v is give by different definition. Thus the power spectrum of v_γ is the same as that of scalar (2.138)

$$P_v = \frac{1}{2k^3}(aH)^2. \tag{2.161}$$

Defining the tensor power spectrum P_t as the sum of the power spectra fro each polarization mode of h_{ij}, we have

$$P_t = 2 \cdot P_h = 2 \cdot \left(\frac{2}{aM_{\rm pl}}\right)^2 \cdot P_v = \frac{4}{k^3}\frac{H^2}{M_{\rm pl}^2}. \tag{2.162}$$

The dimensionless spectrum is then given by

$$\Delta_t^2(k) = \frac{k^3}{2\pi^2}P_t = \frac{2}{\pi^2}\frac{H^2}{M_{\rm pl}^2}\bigg|_{k=aH}. \tag{2.163}$$

One important quantity is tensor-to-scalar ratio

$$r \equiv \frac{\Delta_t^2}{\Delta_s^2} = 16\varepsilon = \frac{8}{M_{\rm pl}^2}\frac{\dot{\phi}^2}{H^2}. \tag{2.164}$$

From which we have

$$r = \frac{8}{M_{\rm pl}^2}\left(\frac{d\phi}{dN}\right)^2. \tag{2.165}$$

Thus the change of ϕ between CMB fluctuations exited the horizon at N_{cmb} and the end of inflation N_{end} is given by

$$\frac{\Delta\phi}{M_{\rm pl}} = \int_{N_{\rm end}}^{N_{\rm cmb}} \mathrm{d}N\sqrt{\frac{r}{8}}. \tag{2.166}$$

Consider the e-fold number is a few tens and r doesn't evolve much during inflation, we have the approximate relation called Lyth Bound,

$$\frac{\Delta\phi}{M_{\rm pl}} = \mathcal{O}(1) \times \left(\frac{r}{0.01}\right)^{1/2}. \tag{2.167}$$

If $r = 0.01$, then we have $\Delta\phi > M_{\rm pl}$, this case is called large-field inflation.

2.6 Constraints of Inflation Models from CMB Experiment

The evidence of inflation theory can be found in the experiment of cosmic microwave background (CMB) anisotropy measurements. The latest constraint on inflation models can be found in *Planck* data [61]. The CMB angular power spectra(see Fig 2.3) in the *Planck* data support the ΛCDM model with no spatial curvature. The predictions for (n_s,r) to first order in the slow-roll approximation for a few inflationary models are shown in Fig 2.4. For more detail, please read the original paper.

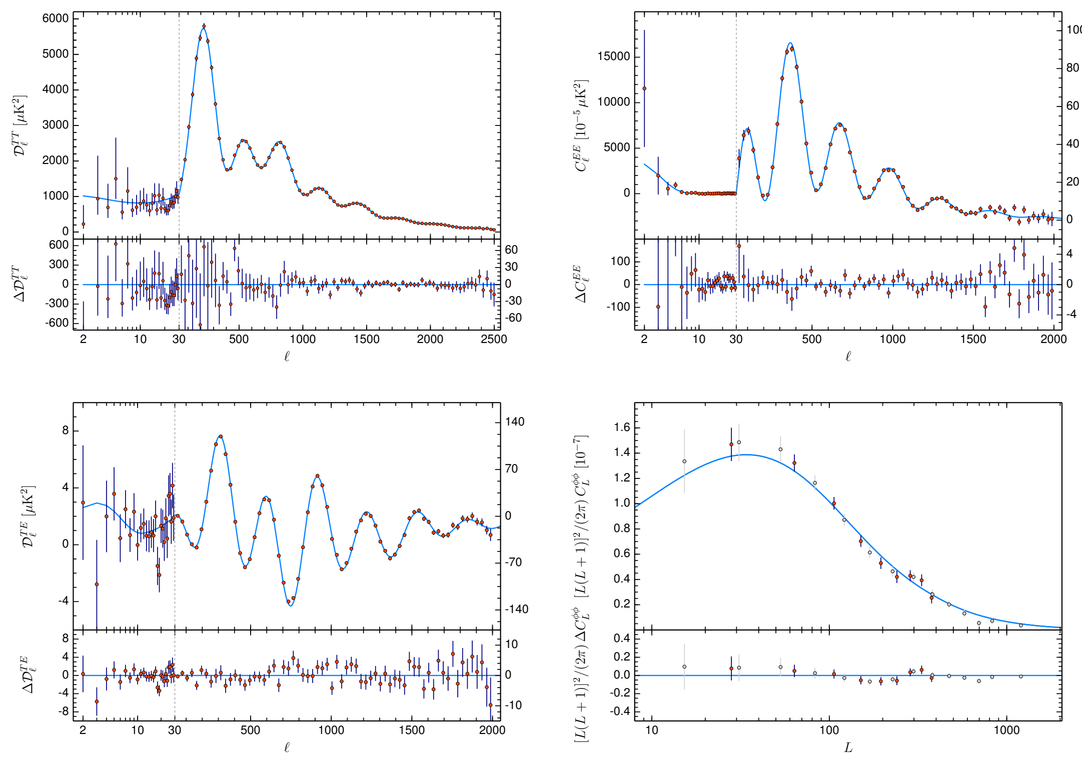

FIGURE 2.3: *Planck* 2018 CMB angular power spectra, compared with the base-ΛCDM best fit to the *Planck* TT,TE,EE+lowE+lensing data (blue curves). For each panel the residuals with respect to this baseline best fit are shown. Plotted are $\mathcal{D}_\ell = \ell(\ell+1)C_\ell/(2\pi)$ for TT and TE, C_ℓ for EE, and $L^2(L+1)^2C_L^{\phi\phi}/(2\pi)$ for lensing. This figure is taken from [61], for more details please see the paper.

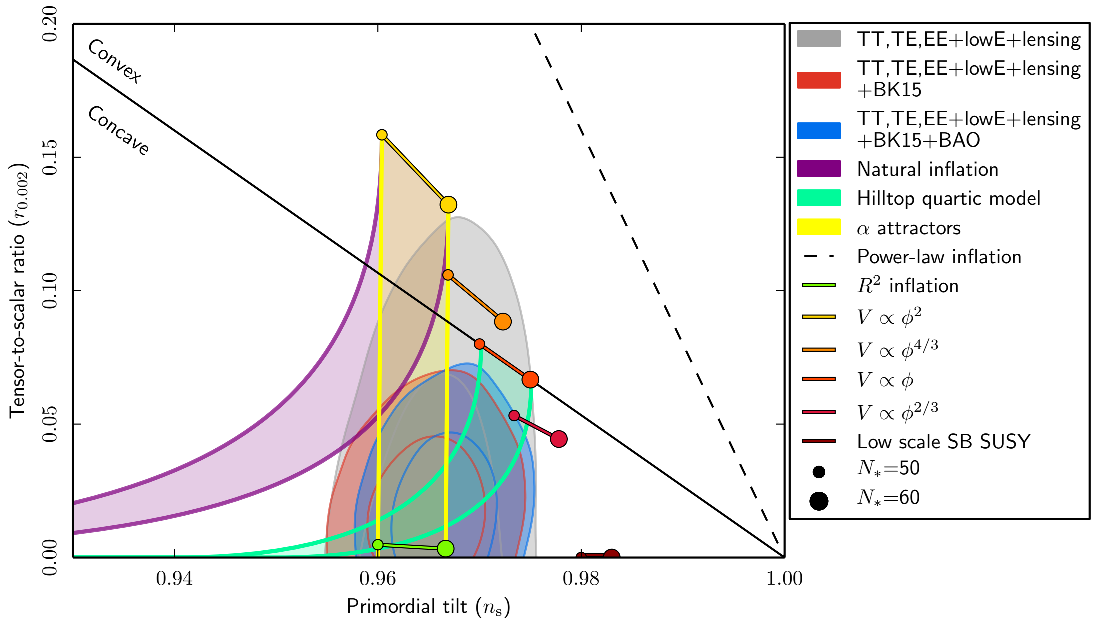

FIGURE 2.4: Marginalized joint 68 % and 95 % CL regions for n_s and r at $k = 0.002\,\mathrm{Mpc}^{-1}$ from *Planck* alone and in combination with BK15 or BK15+BAO data, compared to the theoretical predictions of selected inflationary models. Note that the marginalized joint 68 % and 95 % CL regions assume $dn_s/d\ln k = 0$. This figure is taken from [61], for more details please see the paper.

Chapter 3

Inflation with U(1) Gauge Field(s)

Although statistically isotropic primordial fluctuations are supported in the CMB experiments, anomalies are also shown at low statistical significance in the CMB temperature anisotropies[61, 62]. In precision cosmology, we need to study the fine structure of the fluctuation of CMB, e.g., Non-gaussianity and statistical anisotropy. Motivated by these anomalies in observation, many models have been proposed to explain the statistical anisotropy. In the case of statistical anisotropy, one needs to find some mechanism to achieve this. One direction is to find an anisotropic inflation model, so that the primordial fluctuation may have a chance to be statistically anisotropic.

However, it was proved that in the presence of a positive cosmological constant, the energy-momentum will not survive and any anisotropy would also decay to 0 in all Bianchi Type homogeneous universe except Bianchi type IX , which is the so-called No-hair Conjecture(Bianchi type IX universe also satisify No-hair theorem given an additional condition) [1]. Due to this cosmic no-hair theorem, it is believed that a hair such as a vector field never survives during inflation because an inflaton mimics the role of the cosmological constant. It is often called the cosmic no-hair conjecture.

There have been several attempts to seek a counterexample to the cosmic no-hair conjecture [2–5], although they suffer from instabilities in the models [6, 7]. However, a healthy counterexample to the conjecture motivated by supergravity was found in [8], where a vector field is coupled to an inflaton through a gauge kinetic function. The point of the model is that the inflaton does not mimic a positive cosmological constant exactly, whose deviation is characterized by the slow-roll parameter. Then, an inflationary universe with a small anisotropy proportional to the slow-roll parameter can be realized [8–11]. Interestingly, it was shown that multiple U(1) gauge fields tend to select a minimally anisotropic configuration dynamically [36]. In §3.1, we first review the mechanism of anisotropic inflation. In §3.2 we review inflation with two U(1) gauge fields. In this case, the stable point is a state in which the two electric fields (the time derivative of the gauge fields) are perpendicular. In §3.3 we review inflation with $N(N > 2)$ U(1) gauge fields. In this case, inflation will reach ho-hair eventually.

3.1 Anisotropic Inflation with One U(1) Gauge Field

3.1.1 Mechanism of Anisotropic Inflation

In supergravity, the bosonic sector for the supergravity action is given by

$$\begin{array}{c} S = \int d^4x\sqrt{-g}\left[\frac{1}{2}R - G_{\bar{i}j}\partial^\mu\bar{\phi}^{\bar{i}}\partial_\mu\phi^j - e^K\left(G^{\bar{i}}\bar{D}_i\bar{W}D_jW - 3\bar{W}W\right)\right. \\ \left. -\frac{1}{4}f_{ab}^2(\phi)F^{a\mu\nu}F^b_{\mu\nu} + \cdots \right. \end{array} \tag{3.1}$$

where $G_{\bar{i}j} = \partial K/\partial\bar{\phi}^{\bar{i}}\partial\phi^j, D_iW = \partial W/\partial\phi^i + \left(\partial K/\partial\phi^i\right)W, K(\phi,\bar{\phi})$ and $W(\phi)$ are the Kaler potential and the super potential, respectively. And there is a kinetic term for gauge fields with gauge kinetic functions f_{ab}, which we will concentrate on in this thesis.

Motivated by supergravity, Watanabe et al. give a counterexample to the no-hair conjecture in the paper [8]. It is achieved by coupling a U(1) gauge field with the inflaton with a nontrivial kinetic function. The action is given by

$$S = \int d^4x\sqrt{-g}\left[\frac{1}{2}R - \frac{1}{2}\left(\partial_\mu\phi\right)\left(\partial^\mu\phi\right) - V(\phi) - \frac{1}{4}f^2(\phi)F_{\mu\nu}F^{\mu\nu}\right], \tag{3.2}$$

where g is the determinant of the metric, R is the Ricci scalar, $V(\phi)$ is the inflaton potential. $f(\phi)$ is the coupling function of the inflaton field to the vector one, respectively. The field strength of the U(1) gauge field is given by

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu. \tag{3.3}$$

The gauge field can be taken to be the form of $A_\mu = (0, 0, 0, A_z(t))$ after choosing the gauge $A_0 = 0$ without loss generality. As the gauge field is set in the direction of x direction, the anisotropic metric is in the form of

$$ds^2 = -dt^2 + e^{2\alpha(t)}\left[e^{2\beta(t)}\left(dx^2 + dy^2\right) + e^{-4\beta(t)}dz^2\right] \tag{3.4}$$

in Bianchi Type I universe, where α denote the isotropic expansion and β denote the deviation from isotropy. Note that the matrix for the anisotropy defined as Diag$(2\beta, 2\beta, -4\beta)$ satisfies the property of traceless. The average Hubble parameter is the derivative of α

to time

$$H = \dot{\alpha}. \tag{3.5}$$

The anisotropy is defined by

$$\sigma \equiv \dot{\beta}/\dot{\alpha}. \tag{3.6}$$

The equation of motion of gauge field can be solved as

$$\dot{A}_z = f^{-2} e^{-\alpha - 4\beta} p_A, \tag{3.7}$$

where p_A is a constant related to the initial value of the gauge field. The equations of motion of this system then can be written as

$$\dot{\alpha}^2 = \dot{\beta}^2 + \frac{1}{3}\left[\frac{1}{2}\dot{\phi}^2 + V(\phi) + \frac{p_A^2}{2} f^{-2}(\phi) e^{-4\alpha-4\beta}\right] \tag{3.8}$$

$$\ddot{\alpha} = -3\dot{\alpha}^2 + V(\phi) + \frac{p_A^2}{6} f^{-2}(\phi) e^{-4\alpha-4\beta} \tag{3.9}$$

$$\ddot{\beta} = -3\dot{\alpha}\dot{\beta} + \frac{p_A^2}{3} f^{-2}(\phi) e^{-4\alpha-4\beta} \tag{3.10}$$

$$\ddot{\phi} = -3\dot{\alpha}\dot{\phi} - V'(\phi) + p_A^2 f^{-3}(\phi) f'(\phi) e^{-4\alpha-4\beta}. \tag{3.11}$$

The energy of the gauge field is given by

$$\rho_A := \frac{1}{2} f^2 g^{ij} A_i A_j = \frac{p_A^2}{2} f^{-2} e^{-4\alpha-4\beta}. \tag{3.12}$$

Thus in the critical case, $f(\phi) \propto e^{-2\alpha}$, the energy density of the gauge field remains almost constant during the slow-roll inflation. Using the slow-roll equations

$$\dot{\alpha}^2 = \frac{1}{3} V(\phi), \quad 3\dot{\alpha}\dot{\phi} = -V'(\phi), \tag{3.13}$$

α can be integrated as

$$\alpha = -\int V/V' d\phi. \tag{3.14}$$

Thus in the critical case $f(\phi)$ can be given by

$$f = e^{-2\alpha} = e^{2\int \frac{V}{V'} d\phi}. \tag{3.15}$$

For more general cases, we can take the form of f as

$$f = e^{-2\alpha} = e^{2c\int \frac{V}{V'} d\phi}, \tag{3.16}$$

where c is a parameter. The energy of the gauge field grows during inflation if $c > 1$. If the potential has the form of

$$V = \frac{1}{2}m^2\phi^2, \tag{3.17}$$

$f(\phi)$ become to

$$f(\phi) = e^{c\phi^2/2}. \tag{3.18}$$

Thus, the energy density of the gauge field can be written as

$$\rho_A = \frac{p_A^2}{2}e^{-c\phi^2-4\alpha-4\beta}. \tag{3.19}$$

As the energy density of the vector field should be subdominant during inflation, we can ignore β in the EOM of α, ϕ and the Hamiltionian constraint. The energy density of the gauge field, ignoring the β, can then be written as

$$\rho_A = \frac{p_A^2}{2}e^{-c\phi^2-4\alpha}. \tag{3.20}$$

Substitute $f(\phi) = e^{c\phi^2/2}$ into the Hamiltonian and EOM of ϕ, they become

$$\dot{\alpha}^2 = \frac{1}{3}\left[\frac{1}{2}\dot{\phi}^2 + \frac{1}{2}m^2\phi^2 + \frac{1}{2}e^{-c\phi^2-4\alpha}p_A^2\right] \tag{3.21}$$

$$\ddot{\phi} = -3\dot{\alpha}\dot{\phi} - m^2\phi + c\phi e^{-c\phi^2-4\alpha}p_A^2. \tag{3.22}$$

As $e^{-\phi^2/2}$ is the critical $f = e^{-2\alpha}$ to make the the energy density of gauge field keep nearly unchanged, thus we have $e^{-c\phi^2} \sim e^{4c\alpha}$ and $e^{-c\phi^2-4\alpha} \sim e^{4(c-1)\alpha}$ is then a growth factor. However, in (3.22), when this factor growth to m^2/c, that is , $c\phi^2 e^{-c\phi^2-4\alpha}p_A^2 \sim m^2\phi$, $\ddot{\phi}$ become positive and ϕ climb up, which in turn decrease the value of $c\phi^2 e^{-c\phi^2-4\alpha}$, thus $c\phi^2 e^{-c\phi^2-4\alpha}p_A^2$ will be attracted to $\sim m^2\phi$. In the view point of energy density, for $c > 1$, the energy density of gauge field obtain energy from the ϕ, while when the energy density of gauge field growth to large enough, it in turn climb up ϕ, which stop the process of energy transformation from ϕ to the gauge field. As a consequence, the energy of gauge field will be attracted to a value. Actually, we can calculate more precisely. The inflation dynamics after tracking is governed by the modified slow-roll equations

$$\dot{\alpha}^2 = \frac{1}{6}m^2\phi^2 \tag{3.23}$$

$$3\dot{\alpha}\dot{\phi} = -m^2\phi + c\phi p_A^2 e^{-c\phi^2-4\alpha}. \tag{3.24}$$

Using these two equations we can obtain

$$\phi\frac{d\phi}{d\alpha} = -2 + \frac{2cp_A^2}{m^2}e^{-c\phi^2-4\alpha}. \tag{3.25}$$

This can be integrated as

$$e^{-c\phi^2-4\alpha} = m^2(c-1)/c^2p_A^2\left[1 + De^{-4(c-1)\alpha}\right]^{-1}, \tag{3.26}$$

where D is a constant of integration. This solution rapidly converges to

$$e^{-c\phi^2-4\alpha} = \frac{m^2(c-1)}{c^2p_A^2}, \tag{3.27}$$

thus the energy density of gauge field become

$$\rho_A = \frac{m^2(c-1)}{2c^2} \tag{3.28}$$

in this phase. Substitute this to (3.24), we obtain

$$3\dot{\alpha}\dot{\phi} = -\frac{m^2}{c}\phi. \tag{3.29}$$

Fig (3.1) show the phase flow in $\phi - \dot{\phi}$ phase with $c = 2$, where indeed we see two slow-roll phases. As c=2, so the value of $\dot{\phi}$ in the second phase (anisotropic inflation phase) is about half of the first phase (isotropic inflation phase). The ratio of energy of

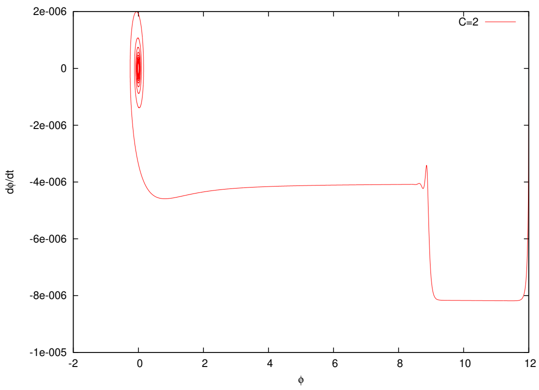

FIGURE 3.1: Phase flow for parameters ϕ for $c = 2, m = 10^{-5}$ and initial conditions $\phi_i = 12$, $\dot{\phi}_i = 0$. This figure is from [8].

gauge field to inflaton become

$$\mathcal{R} \equiv \frac{\rho_A}{\rho_\phi} = \frac{p_A^2 e^{-c\phi^2-4\alpha}}{m^2\phi^2} = \frac{c-1}{c^2\phi^2}. \tag{3.30}$$

For the anisotropy, in the second phase, assuming $\beta \ll c\phi^2, \ddot{\beta} \ll \dot{\alpha}\dot{\beta}$, the equation of β become to

$$3\dot{\alpha}\dot{\beta} = \frac{p_A^2}{3} e^{-c\phi^2 - 4\alpha}, \tag{3.31}$$

together with the Hamiltonian constaint in the second phase (3.23), we can directly obtain the anisotropy

$$\sigma := \frac{\dot{\beta}}{\dot{\alpha}} = \frac{p_A^2 e^{-c\phi^2 - 4\alpha}}{9\dot{\alpha}^2} = \frac{2}{3}\mathcal{R}(t). \tag{3.32}$$

The Fig. 3.2 shows the evolution of the anisotropy σ for various parameters under conditions $\sqrt{c}\phi_i = 17$. All the solutions show a quickly growth of the anisotropyin the first phase (isotropic inflation) but stop at the second phase (anisotropic inflation). Using (3.27), (3.29) and (3.31), one can obtain $\dot{\beta}^2 \ll \dot{\phi}^2$. Then combining the original

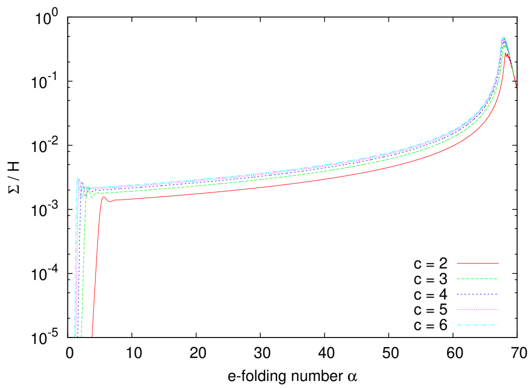

FIGURE 3.2: Evolutions of anisotropy $\sigma \equiv \Sigma/H$ for different values of c. This figure is from [8].

Hamiltonian constraint and the EOM of α, one obtain

$$\ddot{\alpha} = -\frac{1}{2}\dot{\phi}^2 - \frac{1}{3} e^{-c\phi^2 - 4\alpha} p_A^2, \tag{3.33}$$

after ignoring the $\dot{\beta}^2$ term. The slow-roll parameter is given by

$$\epsilon \equiv -\frac{\ddot{\alpha}}{\dot{\alpha}^2} = \frac{2}{c\phi^2}. \tag{3.34}$$

One can also get the relation between anisotropy and the slow-roll parameter

$$\sigma = \frac{1}{3}\frac{c-1}{c}\epsilon. \tag{3.35}$$

3.1.2 Anisotropic Power-law Inflation

In this section, we review anisotropic power-law inflation, where a U(1) gauge field are coupled to an inflaton ϕ through an exponential-type gauge kinetic function. Let us first consider the model with a single U(1) gauge field, which can be regarded as a subgroup of SU(3). In this case, the action is given by

$$S = \int \mathrm{d}^4x\sqrt{-g}\left[\frac{M_{\mathrm{Pl}}^2}{2}R - \frac{1}{2}\nabla_\mu\phi\nabla^\mu\phi - V(\phi) - \frac{1}{4}f^2(\phi)F_{\mu\nu}F^{\mu\nu}\right], \tag{3.36}$$

where g is the determinant of the metric, R is the Ricci scalar, $F_{\mu\nu} := \partial_\mu A_\nu - \partial_\nu A_\mu$ is the field strength of the U(1) gauge field A_μ, and M_{Pl} denotes the reduced Planck mass. We assume the potential $V(\phi)$ and the gauge kinetic function $f(\phi)$ respectively have the form

$$V(\phi) = V_0 \exp\left(\lambda\frac{\phi}{M_{\mathrm{Pl}}}\right), \qquad f(\phi) = f_0 \exp\left(\rho\frac{\phi}{M_{\mathrm{Pl}}}\right), \tag{3.37}$$

with V_0, f_0, λ, and ρ being positive constants. We introduce dimensionless quantities as

$$\hat{x}^\mu := M_{\mathrm{Pl}}x^\mu, \qquad \hat{V}_0 := \frac{V_0}{M_{\mathrm{Pl}}^4}, \qquad \hat{\phi} := \frac{\phi}{M_{\mathrm{Pl}}}, \qquad \hat{A}_\mu := \frac{A_\mu}{M_{\mathrm{Pl}}}. \tag{3.38}$$

Then, the model is characterized by the four parameters $\hat{V}_0$, f_0, λ, and ρ. In what follows, we omit hats from the dimensionless quantities for notational convenience.

The authors of [34] studied an exact solution of anisotropic power-law inflation in this model. They assumed a homogeneous spacetime and fields of the form,

$$\mathrm{d}s^2 = -\mathrm{d}t^2 + \mathrm{e}^{2\alpha(t)}\left[\mathrm{e}^{2\beta(t)}(\mathrm{d}x^2 + \mathrm{d}y^2) + \mathrm{e}^{-4\beta(t)}\mathrm{d}z^2\right], \qquad \phi = \phi(t), \qquad A_\mu \mathrm{d}x^\mu = A_3(t)\mathrm{d}z, \tag{3.39}$$

and showed that the following configuration solves the system of equations of motion (EOMs):

$$\alpha = \zeta \ln t, \qquad \beta = \eta \ln t, \qquad \phi = -\frac{2}{\lambda}\ln t + \phi_0, \qquad \dot{A}_3 = f^{-2}(\phi)\mathrm{e}^{-\alpha-4\beta}p_A. \tag{3.40}$$

Here, we have defined

$$\zeta := \frac{\lambda^2 + 8\lambda\rho + 12\rho^2 + 8}{6\lambda(\lambda + 2\rho)}, \qquad \eta := \frac{\lambda^2 + 2\lambda\rho - 4}{3\lambda(\lambda + 2\rho)}, \tag{3.41}$$

and the values of ϕ_0 and p_A are determined from

$$\begin{aligned} V_0 e^{\lambda\phi_0} &= \frac{\left(\lambda\rho + 2\rho^2 + 2\right)\left(-\lambda^2 + 4\lambda\rho + 12\rho^2 + 8\right)}{2\lambda^2(\lambda + 2\rho)^2} =: u, \\ p_A^2 f_0^{-2} e^{-2\rho\phi_0} &= \frac{\left(\lambda^2 + 2\lambda\rho - 4\right)\left(-\lambda^2 + 4\lambda\rho + 12\rho^2 + 8\right)}{2\lambda^2(\lambda + 2\rho)^2} =: w. \end{aligned} \tag{3.42}$$

Note that $V_0 e^{\lambda\phi_0}$ and $p_A^2 f_0^{-2} e^{-2\rho\phi_0}$ are intrinsically positive, and hence this type of solution exists only if $u > 0$ and $w > 0$.

For this solution, we obtain the Hubble parameter and the slow-roll parameter as

$$H := \dot{\alpha} = \frac{\zeta}{t}, \qquad \epsilon := -\frac{\dot{H}}{H^2} = \frac{1}{\zeta} = \frac{6\lambda(\lambda + 2\rho)}{\lambda^2 + 8\lambda\rho + 12\rho^2 + 8}, \tag{3.43}$$

where a dot denotes a derivative with respect to t. Hence, if we choose λ sufficiently small, then we have $\epsilon \ll 1$, i.e., an inflationary universe can be realized. For a small enough λ, the condition $u > 0$ is trivially satisfied, while we need $\lambda^2 + 2\lambda\rho > 4$ to guarantee $w > 0$. Also, the following parameter is useful to measure the anisotropy:

$$\sigma := \frac{\dot{\beta}}{H} = \frac{\eta}{\zeta} = \frac{2\left(\lambda^2 + 2\lambda\rho - 4\right)}{\lambda^2 + 8\lambda\rho + 12\rho^2 + 8}. \tag{3.44}$$

Finally, for later reference, let us compute the density parameter for the gauge field. The energy density of the gauge field is given by

$$\rho_g = \frac{1}{2} f^2 g^{33} (\dot{A}_3)^2, \tag{3.45}$$

and hence the density parameter is written as

$$\Omega_g := \frac{\rho_g}{3H^2} = \frac{w}{6\zeta^2} = \frac{3\left(\lambda^2 + 2\lambda\rho - 4\right)\left(-\lambda^2 + 4\lambda\rho + 12\rho^2 + 8\right)}{(\lambda^2 + 8\lambda\rho + 12\rho^2 + 8)^2}. \tag{3.46}$$

3.2 Power-law Inflation with Two U(1) Gauge Fields

Next, let us consider the model with two copies of U(1) gauge fields $A_\mu^{(1)}$ and $A_\mu^{(2)}$, which can also be embedded into SU(3). The action is written as

$$S = \int d^4x \sqrt{-g} \left[\frac{M_{\rm Pl}^2}{2} R - \frac{1}{2} \nabla_\mu \phi \nabla^\mu \phi - V(\phi) - \frac{1}{4} f^2(\phi) \sum_{n=1}^{2} F_{\mu\nu}^{(n)} F^{(n)\mu\nu} \right], \tag{3.47}$$

with $F_{\mu\nu}^{(n)} := \partial_\mu A_\nu^{(n)} - \partial_\nu A_\mu^{(n)}$ being the field strength of the nth gauge field. Also, the potential $V(\phi)$ and the gauge kinetic function $f(\phi)$ are of the same form as in (3.37).

From [36], we know there exists a stable fixed point with an orthogonal configuration of the two U(1) gauge fields, where the spacetime and fields are of the form,

$$\begin{aligned} \mathrm{d}s^2 &= -\mathrm{d}t^2 + \mathrm{e}^{2\alpha(t)}\left[\mathrm{e}^{2\beta(t)}(\mathrm{d}x^2+\mathrm{d}y^2)+\mathrm{e}^{-4\beta(t)}\mathrm{d}z^2\right], \\ \phi &= \phi(t), \qquad A^{(1)}_\mu \mathrm{d}x^\mu = A^{(1)}_1(t)\mathrm{d}x, \qquad A^{(2)}_\mu \mathrm{d}x^\mu = A^{(2)}_2(t)\mathrm{d}y, \end{aligned} \tag{3.48}$$

where

$$\alpha = \frac{(\lambda+2\rho)(\lambda+6\rho)+2}{3\lambda(\lambda+4\rho)}\ln t, \qquad \beta = -\frac{\lambda^2+2\lambda\rho-4}{3\lambda(\lambda+4\rho)}\ln t, \qquad \phi = -\frac{2}{\lambda}\ln t + \phi_0, \tag{3.49}$$

with ϕ_0 being constant. The anisotropy is given by

$$\sigma = \frac{\dot{\beta}}{H} = -\frac{\lambda^2+2\lambda\rho-4}{(\lambda+2\rho)(\lambda+6\rho)+2}, \tag{3.50}$$

and the total energy density of the gauge fields is given by

$$\rho_g = \frac{1}{2}\sum_{n=1}^{2} f^2 g^{ij}\dot{A}^{(n)}_i\dot{A}^{(n)}_j = \frac{18(\lambda^2+2\lambda\rho-4)(6\rho^2+2\lambda\rho+1)}{[(\lambda+2\rho)(\lambda+6\rho)+2]^2}H^2. \tag{3.51}$$

Here, $H=\dot{\alpha}$ is the Hubble parameter and each gauge field shares half of the total energy density. Hence, the density parameter for the gauge fields is written as

$$\Omega_g = \frac{\rho_g}{3H^2} = \frac{6(\lambda^2+2\lambda\rho-4)(6\rho^2+2\lambda\rho+1)}{[(\lambda+2\rho)(\lambda+6\rho)+2]^2}. \tag{3.52}$$

3.3 Power-law Inflation with Multi-U(1) Gauge Fields

In this section, let us consider the model with N copies of U(1) gauge fields $A^{(n)}_\mu$ $(n = 1,\cdots,N)$, with $N \geq 3$. These multiple U(1) cases cannot be embedded into SU(3). Nevertheless, it is useful to compare the generic behavior of the model with the SU(3) gauge field model. Now, the action is written as

$$S = \int \mathrm{d}^4x\sqrt{-g}\left[\frac{M_{\mathrm{Pl}}^2}{2}R - \frac{1}{2}\nabla_\mu\phi\nabla^\mu\phi - V(\phi) - \frac{1}{4}f^2(\phi)\sum_{n=1}^{N}F^{(n)}_{\mu\nu}F^{(n)\mu\nu}\right], \tag{3.53}$$

with $F^{(n)}_{\mu\nu} := \partial_\mu A^{(n)}_\nu - \partial_\nu A^{(n)}_\mu$ being the field strength of the nth gauge field. Also, the potential $V(\phi)$ and the gauge kinetic function $f(\phi)$ are of the same form as in (3.37). The authors of [36] studied a more general case where each gauge field is coupled to ϕ with a different coupling constant (i.e., different ρ for different n). However, we restrict ourselves to the model described by the action (3.53) for simplicity. Note also that we use dimensionless quantities similar to those in (3.38) in the following discussion.

As we did in the previous section, one can study power-law inflation in the present model. It was shown in [36] that there exists isotropic stable fixed points with nontrivial configuration of the gauge fields, where the spacetime and fields are of the form,

$$\mathrm{d}s^2 = -\mathrm{d}t^2 + \mathrm{e}^{2\alpha(t)}\left(\mathrm{d}x^2 + \mathrm{d}y^2 + \mathrm{d}z^2\right), \qquad \phi = \phi(t), \qquad A^{(n)}_\mu \mathrm{d}x^\mu = A^{(n)}_i(t)\mathrm{d}x^i. \tag{3.54}$$

where

$$\alpha = \frac{\lambda + 2\rho}{2\lambda}\ln t, \qquad \phi = -\frac{2}{\lambda}\ln t + \phi_0, \tag{3.55}$$

with ϕ_0 being constant, and the total energy density of the gauge fields is given by

$$\rho_g = \frac{1}{2}\sum_{n=1}^{N} f^2 g^{ij}\dot{A}^{(n)}_i\dot{A}^{(n)}_j = \frac{3(\lambda^2 + 2\lambda\rho - 4)}{(\lambda + 2\rho)^2}H^2. \tag{3.56}$$

Here, $H = \dot{\alpha}$ is the Hubble parameter. Hence, the density parameter for the gauge fields is written as

$$\Omega_g = \frac{\rho_g}{3H^2} = \frac{\lambda^2 + 2\lambda\rho - 4}{(\lambda + 2\rho)^2}. \tag{3.57}$$

Note that this value is the same for any $N \geq 3$. In the case of $N = 3$, each gauge field shares one-third of the total energy density.

Chapter 4

Inflation with an SU(3) Gauge Field

In this section, we study an inflationary universe with an SU(3) gauge field A^a_μ. The SU(3) gauge field is written in the form

$$\mathbf{A} = A^a_\mu T^a \mathrm{d}x^\mu, \tag{4.1}$$

where T^a's are the SU(3) generators defined by $T^a = \lambda^a/2$ with the Gell-Mann matrices λ^a. The generator matrices satisfy the normalization condition

$$\mathrm{Tr}\,(T^a T^b) = \frac{1}{2}\delta^{ab}, \tag{4.2}$$

and the commutation relation

$$\left[T^a, T^b\right] = i f^{abc} T^c. \tag{4.3}$$

Here, f^{abc} is the structure constant satisfying

$$f^{abc} = -2i\,\mathrm{Tr}\left(T^a\left[T^b, T^c\right]\right), \tag{4.4}$$

which is completely antisymmetric. The nonvanishing components of f^{abc} are

$$f^{123} = 1, \qquad f^{147} = f^{165} = f^{246} = f^{257} = f^{345} = f^{376} = \frac{1}{2}, \qquad f^{845} = f^{867} = \frac{\sqrt{3}}{2}. \tag{4.5}$$

The field strength of the gauge field is given by

$$F^a_{\mu\nu} = \nabla_\mu A^a_\nu - \nabla_\nu A^a_\mu + g_* f^{abc} A^b_\mu A^c_\nu, \tag{4.6}$$

where g_* is the gauge coupling constant.

Now, we are ready to write down the action. Similarly to the model with a U(1) gauge field studied in [34], we study a model with an SU(3) gauge field described by the following action:

$$S = \int \mathrm{d}^4x \sqrt{-g} \left[\frac{M_{\mathrm{Pl}}^2}{2} R - \frac{1}{2} \nabla_\mu \phi \nabla^\mu \phi - V(\phi) - \frac{1}{4} f^2(\phi) F^a_{\mu\nu} F^{a\mu\nu} \right]. \tag{4.7}$$

Here, the potential $V(\phi)$ and the gauge kinetic function $f(\phi)$ are of the same form as in (3.37), which we reproduce here for convenience:

$$V(\phi) = V_0 \exp\left(\lambda \frac{\phi}{M_{\mathrm{Pl}}} \right), \qquad f(\phi) = f_0 \exp\left(\rho \frac{\phi}{M_{\mathrm{Pl}}} \right), \tag{4.8}$$

with V_0, f_0, λ, and ρ being positive constants. As we did in §3.1, we introduce dimensionless quantities as follows:

$$\hat{x}^\mu := M_{\mathrm{Pl}} x^\mu, \qquad \hat{V}_0 := \frac{V_0}{M_{\mathrm{Pl}}^4}, \qquad \hat{f}_0 := \frac{f_0}{g_*}, \qquad \hat{\phi} := \frac{\phi}{M_{\mathrm{Pl}}}, \qquad \hat{A}^a_\mu := \frac{g_*}{M_{\mathrm{Pl}}} A^a_\mu. \tag{4.9}$$

Note that the gauge coupling constant g_* has been absorbed into the field redefinition. Moreover, one can set $\hat{f}_0 \to 1$ by shifting $\hat{\phi} \to \hat{\phi} - \rho^{-1} \ln \hat{f}_0$. Then, the model is characterized by the three parameters $\hat{V}_0$, λ, and ρ. In what follows, we suppress hats for notational convenience.

It is straightforward to obtain the EOMs. The Einstein equations read

$$R_{\mu\nu} = T_{\mu\nu} - \frac{1}{2} g_{\mu\nu} T^\rho{}_\rho, \tag{4.10}$$

where $R_{\mu\nu}$ is the Ricci tensor and $T_{\mu\nu} = T^\phi_{\mu\nu} + T^g_{\mu\nu}$ is the energy-momentum tensor. Here, $T^\phi_{\mu\nu}$ and $T^g_{\mu\nu}$ denote the contribution from the scalar

$$T^\phi_{\mu\nu} = \nabla_\mu \phi \nabla_\nu \phi - \frac{1}{2} g_{\mu\nu} \nabla_\rho \phi \nabla^\rho \phi - g_{\mu\nu} V, \tag{4.11}$$

and that from the gauge field

$$T^g_{\mu\nu} = f^2(\phi) \left(-\frac{1}{4} g_{\mu\nu} F^a_{\rho\sigma} F^{a\rho\sigma} + F^a_{\mu\sigma} F^{a\sigma}_\nu \right). \tag{4.12}$$

The EOM for the scalar is given by

$$-\nabla_\mu \nabla^\mu \phi + \frac{\mathrm{d}V}{\mathrm{d}\phi} = -\frac{1}{4} F^a_{\mu\nu} F^{a\mu\nu} \frac{\mathrm{d}f^2(\phi)}{\mathrm{d}\phi}. \tag{4.13}$$

The EOMs for the gauge fields are

$$\nabla_\mu F^{a\mu\nu} + f^{abc} A^b_\mu F^{c\mu\nu} = -\frac{\nabla_\mu f^2(\phi)}{f^2(\phi)} F^{a\mu\nu}. \tag{4.14}$$

We study a general Bianchi Type I universe having the metric of the form

$$g_{\mu\nu} \mathrm{d}x^\mu \mathrm{d}x^\nu = -\mathrm{d}t^2 + g_{ij}(t) \mathrm{d}x^i \mathrm{d}x^j, \tag{4.15}$$

accompanied by homogeneous scalar and gauge fields,

$$\phi = \phi(t), \qquad A^a_\mu \mathrm{d}x^\mu = A^a_i(t) \mathrm{d}x^i, \tag{4.16}$$

where we used the gauge symmetry to fix the time component of the gauge field. In this setup, the system of EOMs consists of the Hamiltonian constraint, 6 EOMs for g_{ij}, 8 Yang-Mills constraints, 24 EOMs for A^a_i, 1 EOM for ϕ. Namely, 9 constraint equations and 31 second-order ordinary differential equations in total. The Hamiltonian and the Yang-Mills constraints are used to provide a consistent set of initial data.

Let us introduce some useful notations. We define the Hubble parameter H by

$$H := \dot{\alpha}, \qquad \alpha := \frac{1}{6} \ln(\det(g_{ij})). \tag{4.17}$$

The e-folding number N is given by the change in the parameter α. Also, we define a matrix $(\mathrm{e}^{2\beta})_{ij}$ by

$$\left(\mathrm{e}^{2\beta}\right)_{ij} := \mathrm{e}^{-2\alpha} g_{ij}. \tag{4.18}$$

From (4.17), we know that the determinant of $(\mathrm{e}^{2\beta})_{ij}$ is unity, or equivalently, β_{ij} is traceless. In terms of this $(\mathrm{e}^{2\beta})_{ij}$, we define the anisotropy matrix by

$$\sigma_{ij} := \frac{1}{2H} \left(\mathrm{e}^{-2\beta} \frac{\mathrm{d}\mathrm{e}^{2\beta}}{\mathrm{d}t} \right)_{(ij)}, \tag{4.19}$$

where we have denoted the symmetrization of two indices by (ij). Also, we introduce the root-mean-square anisotropy as

$$\sigma := \sqrt{\frac{1}{6}\sum_{i,j=1}^{3}\sigma_{ij}\sigma_{ij}}\ . \tag{4.20}$$

Note that, if $\sigma = 0$, then all the components of σ_{ij} must vanish, which is nothing but the isotropic case.

To study the dynamics of the spacetime and the gauge field, it is useful to define the density parameters for the gauge field. In the present setup, the field strength takes the form

$$F^a_{0j} = \dot{A}^a_j, \qquad F^a_{ij} = f^{abc}A^b_iA^c_j, \tag{4.21}$$

which we call the electric part and the magnetic part, respectively. Then, the energy density of the gauge field is written as

$$\rho_g := T^g_{00} = f^2(\phi)\left(\frac{1}{2}g^{ij}\dot{A}^a_i\dot{A}^a_j + \frac{1}{4}f^{abc}f^{ade}A^b_iA^c_jA^{di}A^{ej}\right), \tag{4.22}$$

which can be separated into the contributions from the electric and the magnetic parts, i.e.,

$$\rho_E := \frac{f^2(\phi)}{2}g^{ij}\dot{A}^a_i\dot{A}^a_j, \qquad \rho_B := f^2(\phi)V_g. \tag{4.23}$$

Here, V_g is defined by

$$\begin{aligned} V_g \ :=\ & \frac{1}{4}f^{abc}f^{ade}A^b_iA^c_jA^{di}A^{ej} \\ =\ & \left(A^2_{[i}A^3_{j]} + \frac{1}{2}A^4_{[i}A^7_{j]} + \frac{1}{2}A^6_{[i}A^5_{j]}\right)^2 + \left(A^3_{[i}A^1_{j]} + \frac{1}{2}A^4_{[i}A^6_{j]} + \frac{1}{2}A^5_{[i}A^7_{j]}\right)^2 \\ & + \left(A^1_{[i}A^2_{j]} + \frac{1}{2}A^4_{[i}A^5_{j]} + \frac{1}{2}A^7_{[i}A^6_{j]}\right)^2 \\ & + \frac{1}{4}\left[A^1_{[i}A^7_{j]} + A^2_{[i}A^6_{j]} + \left(A^3_{[i} + \sqrt{3}A^8_{[i}\right)A^5_{j]}\right]^2 + \frac{1}{4}\left[A^1_{[i}A^6_{j]} + A^7_{[i}A^2_{j]} + \left(A^3_{[i} + \sqrt{3}A^8_{[i}\right)A^4_{j]}\right]^2 \\ & + \frac{1}{4}\left[A^5_{[i}A^1_{j]} + A^2_{[i}A^4_{j]} + \left(A^3_{[i} - \sqrt{3}A^8_{[i}\right)A^7_{j]}\right]^2 + \frac{1}{4}\left[A^1_{[i}A^4_{j]} + A^2_{[i}A^5_{j]} - \left(A^3_{[i} - \sqrt{3}A^8_{[i}\right)A^6_{j]}\right]^2 \\ & + \frac{3}{4}\left(A^4_{[i}A^5_{j]} + A^6_{[i}A^7_{j]}\right)^2, \end{aligned} \tag{4.24}$$

which amounts to the potential for the gauge field. Note that square brackets $[ij]$ denote antisymmetrization and we have denoted $(B_{ij})^2 := B_{ij}B^{ij}$ for an arbitrary quantity B_{ij} with spatial indices. Now, we can define the density parameter for each component,

$$\Omega_E := \frac{\rho_E}{3H^2}, \qquad \Omega_B := \frac{\rho_B}{3H^2}, \tag{4.25}$$

so that the total density parameter for the gauge field is given by $\Omega_T := \Omega_E + \Omega_B$. It should be noted that Ω_B measures the effect of nonlinear self-interactions. It is also useful to define the electric-part density parameter for each a, i.e.,

$$\Omega^a := \frac{f^2(\phi)}{6H^2} g^{ij} \dot{A}^a_i \dot{A}^a_j \qquad \text{(no sum over } a\text{)}, \tag{4.26}$$

so that $\sum_a \Omega^a = \Omega_E$.

We are now ready to investigate the evolution of inflationary universes with an SU(3) gauge field. It is useful to start with the simplest case and go step by step. The simplest subgroup of SU(3) is U(1). There are also U(1) ⊗ U(1) and SU(2) subgroups. These cases have been already studied. We shall start with the next simplest case SU(2) ⊗ U(1) and proceed step by step to explore the inflationary universe with an SU(3) gauge field. In numerical computations, we put the initial time to be $t = 1$ and set $A^a_i = 0$ so that the Yang-Mills constraints are satisfied. For a given set of parameters (V_0, λ, ρ), we fix initial values for ϕ, $\dot{\phi}$, and the velocity of the gauge field by use of the exact solutions mentioned in the previous section. We take $\lambda = 0.8$ and $\rho = 4$, for which the anisotropic inflation is realized in the U(1) model (see §3.1). In §4.2 and §4.3, as for the spatial part of the metric, we assume $g_{ij} = \delta_{ij}$ and $\dot{g}_{ij} = 2H_{\rm in}\delta_{ij}$ at the initial time, where the value of the constant $H_{\rm in}$ is determined from the Hamiltonian constraint.

4.1 SU(2) Subgroup

In this section, we consider the conventional SU(2) subgroup only, where the generators are the $\frac{1}{2}\times$ Pauli matrices $\{T^1, T^2, T^3\}$. The potential is given by

$$\begin{aligned} V_g &:= \frac{1}{4} f^{abc} f^{ade} A^b_i A^c_j A^{di} A^{ej} \\ &= \left(A^2_{[i} A^3_{j]}\right)^2 + \left(A^3_{[i} A^1_{j]}\right)^2 + \left(A^1_{[i} A^2_{j]}\right)^2 \end{aligned} \tag{4.27}$$

This SU(2) gauge field has an isotropic configuration

$$A^a_i = \psi \delta^a_i, \tag{4.28}$$

thus we can expect an isotropic initial condition of the gauge field will achieve a continued isotropic evolution. However, the self-coupling of the non-abelien gauge field makes the anisotropic inflation unstable. The Fig 4.1 shows that the energy of Electric field and magntic field both decay when the energy of magnetic field become large.

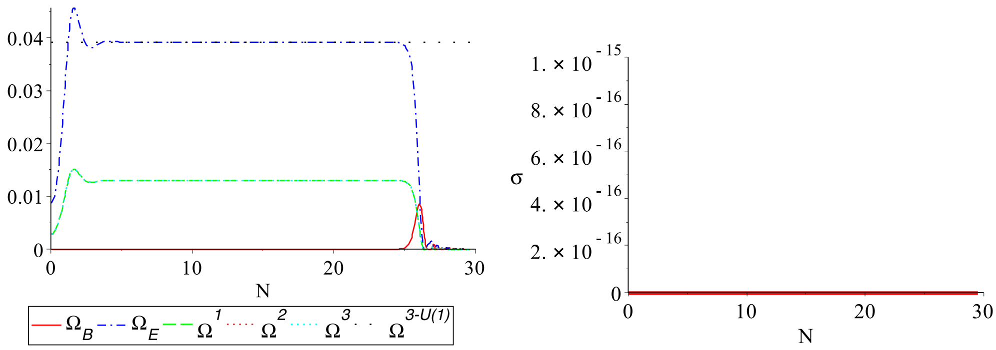

FIGURE 4.1: Evolution of the density parameters of gauge-field components (left) and anisotropy (right) against the number of e-folds for an initial condition with $\dot{A}^1_1 = \dot{A}^2_2 = \dot{A}^3_3$. In the left graph, the red solid, blue dash-dotted, green dashed, orange dotted, and cyan dotted curves respectively correspond to Ω_B, Ω_E, Ω^1, Ω^2, Ω^3. Because of the isotropic initial condition, the curves of Ω^1, Ω^2, Ω^3 overlap. The black space-dotted lines represent Ω_E for the isotropic three-U(1) case (3.57). In the right graph, the red solid curve corresponds to σ.

4.2 SU(2) ⊗ U(1) Subgroup

In this subsection, we investigate SU(2) ⊗ U(1) subgroup of the SU(3) gauge field. Namely, we consider the case where only A^1_1, A^2_2, A^3_3, and A^8_3 are nonvanishing. Moreover, we impose the axial symmetry along the z-direction, so that the spacetime and the gauge field have the following form:

$$\mathrm{d}s^2 = -\mathrm{d}t^2 + g_{11}\left(\mathrm{d}x^2 + \mathrm{d}y^2\right) + g_{33}\mathrm{d}z^2, \qquad A^1_1 = A^2_2. \tag{4.29}$$

As is shown in the Appendix, we can classify the gauge-field configurations which are consistent with the axial symmetry. Those classes of configurations can be treated similarly.

Now, the number of EOMs reduces to seven. All the other EOMs become trivial. Performing the transformation of variables

$$g_{11} = \exp(2\alpha + 2\beta), \qquad g_{33} = \exp(2\alpha - 4\beta), \tag{4.30}$$

we obtain the Hamiltonian constraint

$$\begin{aligned} &3\left(-\dot{\alpha}^2 + \dot{\beta}^2\right) + \frac{1}{2}\dot{\phi}^2 + V \\ &+ \frac{1}{2}\mathrm{e}^{-2\alpha} f^2 \left[2\mathrm{e}^{-2\beta}\dot{A}^2_{11} + \mathrm{e}^{4\beta}(\dot{A}^2_{33} + \dot{A}^2_{83}) + 2\mathrm{e}^{-2\alpha+2\beta}A^2_{11}A^2_{33} + \mathrm{e}^{-2\alpha-4\beta}A^4_{11}\right] = 0, \end{aligned} \tag{4.31}$$

the Einstein equations

$$2\ddot{\alpha}+3\left(\dot{\alpha}^2+\dot{\beta}^2\right)+\frac{1}{2}\dot{\phi}^2-V$$
$$+\frac{1}{6}\mathrm{e}^{-2\alpha}f^2\left[2\mathrm{e}^{-2\beta}\dot{A}_{11}^2+\mathrm{e}^{4\beta}(\dot{A}_{33}^2+\dot{A}_{83}^2)+2\mathrm{e}^{-2\alpha+2\beta}A_{11}^2A_{33}^2+\mathrm{e}^{-2\alpha-4\beta}A_{11}^4\right]=0, \quad (4.32)$$
$$\ddot{\beta}+3\dot{\alpha}\dot{\beta}-\frac{1}{3}\mathrm{e}^{-2\alpha}f^2\left[\mathrm{e}^{4\beta}(\dot{A}_{33}^2+\dot{A}_{83}^2)-\mathrm{e}^{-2\beta}\dot{A}_{11}^2-\mathrm{e}^{-2\alpha+2\beta}A_{11}^2A_{33}^2+\mathrm{e}^{-2\alpha-4\beta}A_{11}^4\right]=0, \quad (4.33)$$

the EOM for the inflaton

$$\ddot{\phi}+3\dot{\alpha}\dot{\phi}+V'-\mathrm{e}^{-2\alpha}ff'\left[\mathrm{e}^{4\beta}(\dot{A}_{33}^2+\dot{A}_{83}^2)+2\mathrm{e}^{-2\beta}\dot{A}_{11}^2-2\mathrm{e}^{-2\alpha+2\beta}A_{11}^2A_{33}^2-\mathrm{e}^{-2\alpha-4\beta}A_{11}^4\right]=0, \quad (4.34)$$

and the EOMs for the gauge field

$$\ddot{A}_{11}+2\frac{f'}{f}\dot{\phi}\dot{A}_{11}+(\dot{\alpha}-2\dot{\beta})\dot{A}_{11}+\mathrm{e}^{-2\alpha+4\beta}A_{33}^2A_{11}+\mathrm{e}^{-2\alpha-2\beta}A_{11}^3=0,$$
$$\ddot{A}_{33}+2\frac{f'}{f}\dot{\phi}\dot{A}_{33}+(\dot{\alpha}+4\dot{\beta})\dot{A}_{33}+2\mathrm{e}^{-2\alpha-2\beta}A_{11}^2A_{33}=0, \quad (4.35)$$
$$\ddot{A}_{83}+2\frac{f'}{f}\dot{\phi}\dot{A}_{83}+(\dot{\alpha}+4\dot{\beta})\dot{A}_{83}=0,$$

where $f':=\mathrm{d}f/\mathrm{d}\phi$ and we lowered the gauge index a for gauge-field components A_i^a for notational convenience. As the component A_3^8 is decoupled from the SU(2) sector, its EOM can be immediately integrated to yield

$$\dot{A}_{83}=f^{-2}\mathrm{e}^{-\alpha-4\beta}p_{83}, \quad (4.36)$$

where p_{83} is a constant.

In Fig. 4.2, we plot the evolution of density parameter (left panel) and the evolution of anisotropy (right panel) for an isotropic initial condition with $\dot{A}_1^1=\dot{A}_2^2=\sqrt{2}\dot{A}_3^3=\sqrt{2}\dot{A}_3^8$. For $N\lesssim 25$, the anisotropy of the universe is tiny due to the cancellation of electric fields between the z and $x(y)$ directions. In this period, the total electric density parameter Ω_E almost coincides with that of the isotropic three-U(1) fixed point [see (3.57)]. Notice that the magnetic energy density is negligible in this phase. After this stage, as the SU(2) sector grows, the density parameters associated with the SU(2) sector, Ω^1 and Ω^3, quickly decay, while the density parameter for the U(1) sector, Ω^8, quickly converges to the value for the anisotropic U(1) case [see (3.46)] in a few e-folds. During this transient phase, the magnetic density parameter Ω_B, which is proportional to the potential,

$$V_g=\left(A_{[i}^2A_{j]}^3\right)^2+\left(A_{[i}^3A_{j]}^1\right)^2+\left(A_{[i}^1A_{j]}^2\right)^2, \quad (4.37)$$

is important. Also, the anisotropy converges to the value for the one-U(1) case (3.44). Thus, the anisotropy is determined by the U(1) sector, i.e., A^8_3. Since A^8 has no coupling with the SU(2) sector, this state is stable.

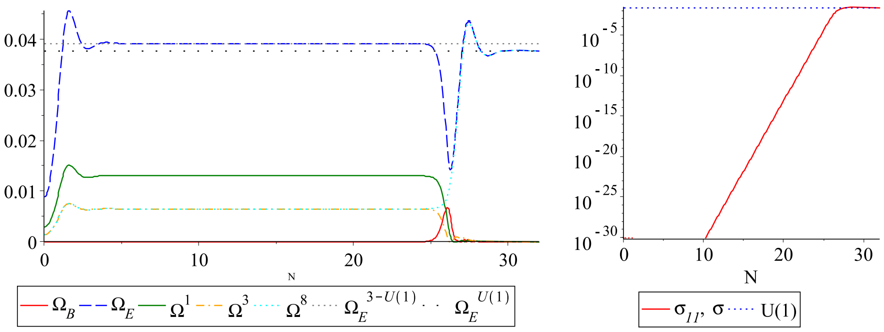

FIGURE 4.2: Evolution of the density parameters of gauge-field components (left) and anisotropy (right) against the number of e-folds for an initial condition with $\dot{A}^1_1 = \dot{A}^2_2 = \sqrt{2}\dot{A}^3_3 = \sqrt{2}\dot{A}^8_3$. In the left graph, the red solid, blue dashed, green solid, orange dash-dotted, and cyan dotted curves respectively correspond to Ω_B, Ω_E, Ω^1, Ω^3, and Ω^8. The gray dotted and black space-dotted lines represent Ω_E for the isotropic three-U(1) case (3.57) and Ω_E for the one-U(1) case (3.46), respectively. In the right graph, the red solid curve corresponds to $\sigma_{11}(=\sigma)$ and the blue dotted line represents the anisotropy for the one-U(1) case (3.44).

4.3 A Specific Example: Gauge-field Potential with a Flat Direction

In the previous subsection, we focused on the subgroup $\mathrm{SU}(2)\otimes\mathrm{U}(1)$, where the anisotropy remains due to the U(1) sector. In this subsection, we consider another specific case where the anisotropy can survive.

Let us study the case with nonvanishing $\{A^3, A^4, A^8\}$. We assume that only A^3_1, A^4_2, and A^8_3 have nontrivial initial velocities with $\dot{A}^3_1 = \dot{A}^4_2 = \dot{A}^8_3$, so that the spacetime is isotropic at the initial time. Note that the components A^3_3 and A^8_1 show up as the time evolves due to the nonlinear self-couplings. This results in off-diagonal components in the metric. Actually, in this setup, the metric takes the form,

$$\mathrm{d}s^2 = -\mathrm{d}t^2 + g_{11}\mathrm{d}x^2 + g_{22}\mathrm{d}y^2 + g_{33}\mathrm{d}z^2 + 2g_{13}\mathrm{d}x\mathrm{d}z. \tag{4.38}$$

The EOMs for the relevant components of the gauge field are given below:

$$\begin{aligned}
\ddot{A}_{31} &= -\frac{(\sqrt{3}A_{81}+A_{31})A_{42}^2}{4g_{22}} - 2\frac{f'}{f}\dot{\phi}\dot{A}_{31} + \left(\frac{[g_{33},g_{11}]}{2G_{13}} - \frac{\dot{g}_{22}}{2g_{22}}\right)\dot{A}_{31} + \frac{[g_{11},g_{13}]}{G_{13}}\dot{A}_{33}, \\
\ddot{A}_{33} &= -\frac{(\sqrt{3}A_{83}+A_{33})A_{42}^2}{4g_{22}} - 2\frac{f'}{f}\dot{\phi}\dot{A}_{33} + \left(\frac{[g_{11},g_{33}]}{2G_{13}} - \frac{\dot{g}_{22}}{2g_{22}}\right)\dot{A}_{33} + \frac{[g_{33},g_{13}]}{G_{13}}\dot{A}_{31}, \\
\ddot{A}_{81} &= -\frac{(\sqrt{3}A_{31}+3A_{81})A_{42}^2}{4g_{22}} - 2\frac{f'}{f}\dot{\phi}\dot{A}_{81} + \left(\frac{[g_{33},g_{11}]}{2G_{13}} - \frac{\dot{g}_{22}}{2g_{22}}\right)\dot{A}_{81} + \frac{[g_{11},g_{13}]}{G_{13}}\dot{A}_{83}, \\
\ddot{A}_{83} &= -\frac{(\sqrt{3}A_{33}+3A_{83})A_{42}^2}{4g_{22}} - 2\frac{f'}{f}\dot{\phi}\dot{A}_{83} + \left(\frac{[g_{11},g_{33}]}{2G_{13}} - \frac{\dot{g}_{22}}{2g_{22}}\right)\dot{A}_{83} + \frac{[g_{33},g_{13}]}{G_{13}}\dot{A}_{81}, \\
\ddot{A}_{42} &= -\frac{A_{42}}{4G_{13}}\left[g_{11}(A_{33}+\sqrt{3}A_{83})^2 + g_{33}(A_{31}+\sqrt{3}A_{81})^2 - 2g_{13}(A_{31}+\sqrt{3}A_{81})(A_{33}+\sqrt{3}A_{83})\right] \\
&\quad - 2\frac{f'}{f}\dot{\phi}\dot{A}_{42} + \left(\frac{\dot{g}_{22}}{2g_{22}} - \frac{g_{11}\dot{g}_{33}+g_{33}\dot{g}_{11}-2g_{13}\dot{g}_{13}}{2G_{13}}\right)\dot{A}_{42},
\end{aligned} \tag{4.39}$$

where we have defined $G_{13} := g_{11}g_{33} - g_{13}^2$ and $[f_1, f_2] := f_1\dot{f}_2 - f_2\dot{f}_1$ for any pair of functions f_1 and f_2 of t.

The evolution of the density parameters for the relevant gauge-field components and the evolution of the anisotropy is shown in Fig. 4.3. Similar to the case of SU(2) ⊗ U(1) subgroup in §4.2, the magnetic density parameter transiently grows but then quickly decays, implying that the nonlinear self-interactions of the gauge field are important in the transient phase. As a result, Ω^4 quickly decays after $N \sim 25$. However, Ω^3 and Ω^8 remain due to the existence of a flat direction in the potential of the gauge field (4.24). Actually, in the present case where only A^3, A^4, and A^8 are nonvanishing, the potential takes the following form:

$$V_g = \left[\left(f^{534}A^3_{[i} + f^{584}A^8_{[i}\right)A^4_{j]}\right]^2 = \frac{1}{4}\left[\left(A^3_{[i} + \sqrt{3}A^8_{[i}\right)A^4_{j]}\right]^2, \tag{4.40}$$

and thus there exists a flat direction defined by

$$A^3 + \sqrt{3}A^8 = 0. \tag{4.41}$$

Hence, we expect that A^3 and A^8 satisfy (4.41) after the potential becomes significant. This is indeed the case as we show in Fig. 4.4. The left panel shows the evolution of angles θ^3 and θ^8, defined by

$$\sin\theta^a = \frac{A_{a1}}{\sqrt{A_{a1}^2 + A_{a3}^2}}, \qquad \cos\theta^a = \frac{A_{a3}}{\sqrt{A_{a1}^2 + A_{a3}^2}}, \tag{4.42}$$

for $a = 3, 8$. We see that A^3 and A^8 are anti-parallel after a sufficiently long time. The right panel shows the evolution of the ratio of $(A^3/A^8)^2$, from which we see that

$(A^3/A^8)^2 \to 3$ at late times. Combining these results, we confirm the flat direction $A^3 + \sqrt{3}A^8 = 0$ really exist. Since the gauge field is trapped in the flat direction (4.41), the dynamics are similar to the one-U(1) case study in §3.1. Indeed, as shown in Fig. 4.3, the total electric density parameter Ω_E and the anisotropy σ approach to the values for the one-U(1) case. We note that, although we have only two nonvanishing components of the gauge field (i.e., A^3 and A^8) at late times, the final state here is different from the stable fixed point for the two-U(1) case study in §3.2, where the two U(1) gauge fields are orthogonal to each other. This is due to the existence of the flat direction in the potential of the gauge field mentioned above.

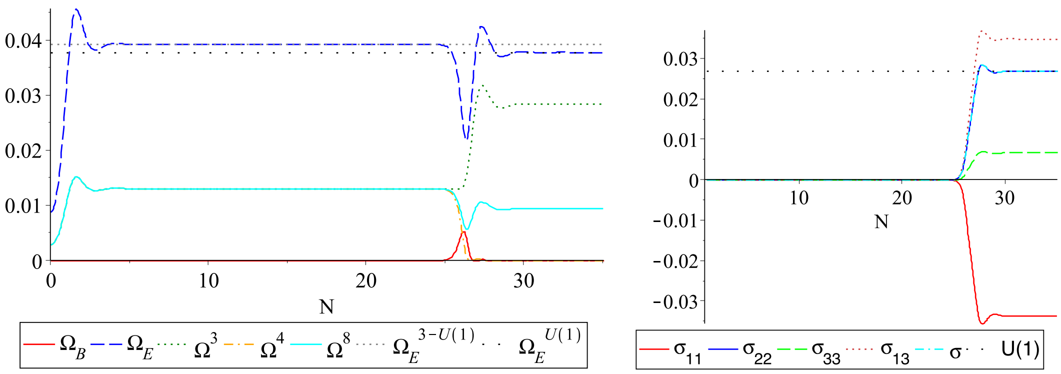

FIGURE 4.3: Evolution of the density parameters of gauge-field components (left) and anisotropy (right) against e-folding number for an initial condition with $\dot{A}^3_1 = \dot{A}^4_2 = \dot{A}^8_3$. In the left graph, the red solid, blue dashed, green dotted, orange dash-dotted, and cyan solid curves respectively correspond to Ω_B, Ω_E, Ω^3, Ω^4, and Ω^8. The gray dotted and black space-dotted lines represent Ω_E for the isotropic three-U(1) case (3.57) and Ω_E for the one-U(1) case (3.46), respectively. In the right graph, the red solid, blue solid, green dashed, orange dotted, and cyan dash-dotted curves respectively correspond to σ_{11}, σ_{22}, σ_{33}, σ_{13}, and the root-mean-square anisotropy σ. The black space-dotted line represents the anisotropy for the one-U(1) case (3.44). The curve for σ almost overlaps with that of σ_{22}.

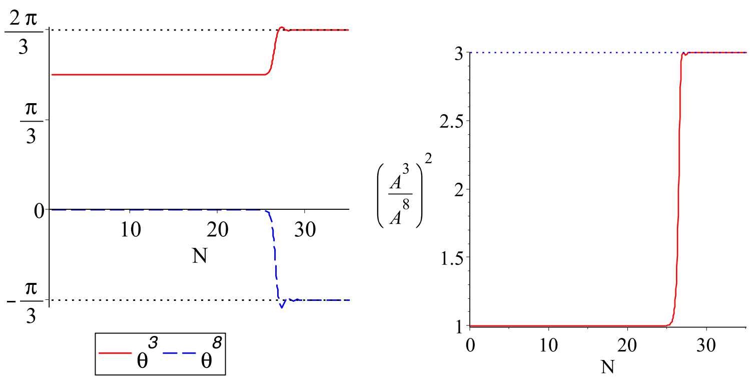

FIGURE 4.4: Evolution of the gauge-field components A^3 and A^8. In the left panel, the red solid and blue dashed curves represent θ^3 and θ^8, respectively. The right panel shows the evolution of $(A^3/A^8)^2$.

In fact, besides the conventional SU(2) subgroup whose generators are the Pauli matrices $\{T^1, T^2, T^3\}$, there are another 2 SU(2) subgroups whose generators are

$$\{T^4, T^5, T^x\} \quad \text{where} \quad T^x = \frac{1}{2}(T^3 + \sqrt{3}T^8) \tag{4.43}$$

$$\{T^6, T^7, T^x\} \quad \text{where} \quad T^y = \frac{1}{2}(-T^3 + \sqrt{3}T^8) \tag{4.44}$$

respectively in the SU(3) group. Actually, the flat direction exists in the SU(2) subgroup in which one of the generators is a linear combination of T^3 and T^8. Take the SU(2) subgroup whose generators are $\{T^4, T^5, T^x\}$ as an example. The potential of this SU(2) gauge field can be written as

We calculate the evolution of the energy density parameters and the anisotropy in Fig 4.5 and the value of A^3_3/A^8_3 in Fig 4.6 given an isotropic initial condition. They show that Ω^3 and Ω^8 remain and they source an anisotropy same as U(1) case. This is because the components A^4 and A^5 both decay and become negligible after the transient phase because of the non-linear coupling. While A^3 and A^8 survive and satisfy $A^3 + \sqrt{3}A^8 = 0$ because of the existence of flat direction.

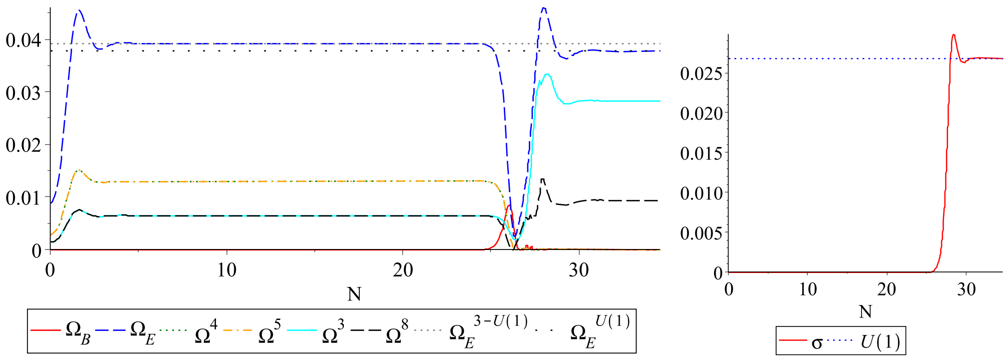

FIGURE 4.5: Evolution of the density parameters of gauge-field components (left) and anisotropy (right) against e-folding number for an initial condition with $\dot{A}^4_1 = \dot{A}^5_2 = \sqrt{2}\dot{A}^3_3 = \sqrt{2}\dot{A}^8_3$. In the left graph, the red solid, blue dashed, green dotted, orange dash-dotted, cyan solid and black dashed curves respectively correspond to Ω_B, Ω_E, Ω^4, Ω^5, Ω^3, and Ω^8. The gray dotted and black space-dotted lines represent Ω_E for the isotropic three-U(1) case (3.57) and Ω_E for the one-U(1) case (3.46), respectively. In the right graph, the red solid and blue dotted curves correspond to the root-mean-square anisotropy σ and the anisotropy for the one-U(1) case (3.44).

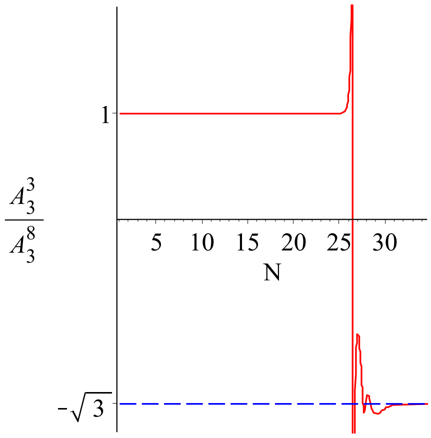

FIGURE 4.6: Evolution of (A_3^3/A_3^8) against e-folding number for an initial condition as the same as Fig 4.5.

Similarly for the SU(2) subgroup whose generators are $\{T^6, T^7, T^y\}$, the components A^3 and A^8 survive and satisfy

$$A^3 - \sqrt{3}A^8 = 0. \tag{4.45}$$

As we shall discuss in §5, the existence of such flat directions is a clear difference of SU(3) from the conventional SU(2) whose generators are $\{T^1, T^2, T^3\}$.

4.4 General Cases

So far, we have considered the special cases where the anisotropy survives. However, in general, the anisotropy decays once the nonlinearity of gauge fields becomes important. To see this, we study the situation where all the gauge-field components have initial velocities of the same order. In Fig. 4.7, we show the evolution of the density parameters for the gauge field and the spacetime anisotropy. The anisotropic expansion of spacetime lasts until $N \sim 25$, and then the nonlinear self-couplings of the gauge field become important and the anisotropy decays. Indeed, the magnetic density parameter Ω_B, which measures the effect of nonlinear self-couplings, is comparable to the total electric density parameter Ω_E at around $N \sim 25$.

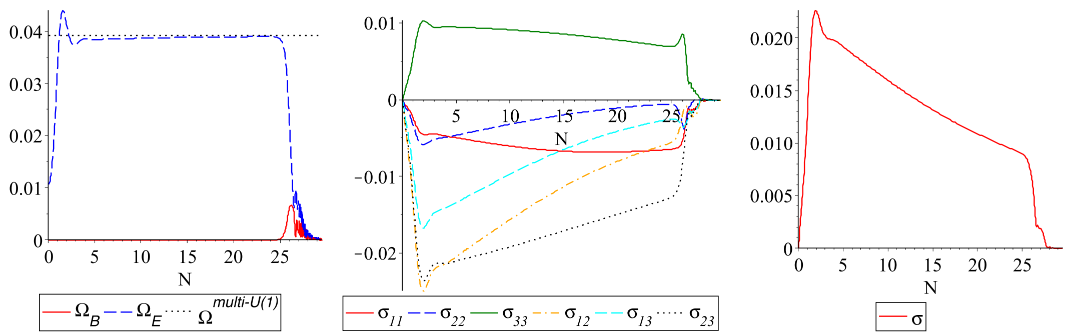

FIGURE 4.7: Evolution of the density parameters (left), σ_{ij} (middle), and σ (right) against e-folding number for an initial condition with all $\dot{A}_i^a$'s having the same order. In the left graph, the red solid and blue dashed curves correspond to the total electric and magnetic density parameters, respectively. The black dotted line represents the total electric density parameter for the case of isotropic multi-U(1) gauge fields (3.57). In the middle graph, the red solid, blue dashed, green solid, orange dash-dotted, cyan dashed, and black dotted curves correspond to σ_{11}, σ_{22}, σ_{33}, σ_{12}, σ_{13}, and σ_{23}, respectively.

To reiterate, the anisotropy decays at late times unless we fine-tune the initial condition as in §4.2 and §4.3. Thus, the cosmic no-hair conjecture generically holds. In a realistic universe, it is reasonable to expect that all the components of the SU(3) gauge field have initial values of the same order, and hence the expansion of the universe should become isotropic after a sufficiently long time after the onset of inflation.

Chapter 5

More on Inflation with Non-Abelian Gauge Fields

We have studied inflationary universes with an SU(3) gauge field. One can generalize the discussion to a non-Abelian SU(N) gauge field for arbitrary N. In this section, we will show that there are flat directions in the potential of an SU(N) gauge field if $N \geq 3$.

Let us consider an SU(N) gauge field with $N \geq 2$. The elements of the Cartan subalgebra are represented by

$$(H_m)_{ij} = \frac{1}{\sqrt{2m(m+1)}} \left(\sum_{k=1}^{m} \delta_{ik}\delta_{jk} - m\delta_{i,m+1}\delta_{j,m+1} \right), \qquad m = 1, 2, \cdots, N-1. \tag{5.1}$$

In particular, for $N = 3$, we have $H_1 = T^3$ and $H_2 = T^8$. The algebra is completely determined by the following simple roots [59]:

$$\begin{aligned}
\alpha^1 &= (1, 0, \cdots, 0), \\
\alpha^2 &= \left(-\frac{1}{2}, \frac{\sqrt{3}}{2}, 0, \cdots, 0 \right), \\
\alpha^3 &= \left(0, -\frac{1}{\sqrt{3}}, \sqrt{\frac{2}{3}}, 0, \cdots, 0 \right), \\
&\vdots \\
\alpha^m &= \left(0, \cdots, 0, -\sqrt{\frac{m-1}{2m}}, \sqrt{\frac{m+1}{2m}}, 0, \cdots, 0 \right), \\
&\vdots \\
\alpha^{N-1} &= \left(0, \cdots 0, -\sqrt{\frac{N-2}{2(N-1)}}, \sqrt{\frac{N}{2(N-1)}} \right),
\end{aligned} \tag{5.2}$$

where each root is an $(N-1)$-dimensional vector. From these, one can find flat directions in the potential of the gauge field. In group theory, for each pair of roots, there is an SU(2) subgroup of the SU(N) group, where

$$T := \alpha_i H_i \tag{5.3}$$

is a generator. In our case, for each simple root $\alpha_m (m > 1)$, there is an SU(2) subgroup of the SU(N) group, in which

$$T := \alpha_i H_i = -\sqrt{\frac{m-1}{2m}} H_{m-1} + \sqrt{\frac{m+1}{2m}} H_m \tag{5.4}$$

is a generator. Thus for inflation with this SU(2) gauge field there There will be two gauge components B^{m-1} and B^m that survive from the non-linear self-coupling after a transient phase and satisfy

$$B^{m-1} - \sqrt{\frac{m+1}{m-1}} B^m = 0., \tag{5.5}$$

where B^m is the gauge component corresponding to the Cartan generator H_m, i.e., $B^m = 2\,\mathrm{Tr}\,(\mathbf{B} H_m)$.

Chapter 6

Conclusion and Discussion

In this thesis, we first reviewed the standard cosmology, inflation theory, anisotropic inflation with U(1) gauge field(s). Then we studied inflationary universes in the presence of an SU(3) gauge field. We numerically solved the system of coupled EOMs to obtain the time evolution of the spacetime, the inflaton, and the gauge field. In general, even if we start from an isotropic spacetime, the anisotropy can be generated if the gauge field has an initial velocity.

There are special cases where the generated anisotropy does not decay and there remains a finite anisotropy. As an example, in §4.2, we studied the situation where the components of the SU(3) gauge field can be separated into the conventional SU(2) (the one with Pauli matrices as generators) and U(1) sectors. The energy density of the SU(2) sector decays due to the nonlinear self-interactions, but that of the U(1) sector remains, and hence the anisotropic expansion of spacetime lasts. The resultant anisotropy coincides with the one obtained in [34], where an exact solution of power-law anisotropic inflation with a U(1) gauge field was studied.

For another SU(2) group in which one generator is a linear combination of the Cartan generators in the SU(3) group, we found for inflation with such SU(2) group, the gauge components corresponding to the Cartan generators can survive against the non-linear coupling and source the anisotropy. This is because of the existence of flat direction inside the potential of this SU(2) gauge field.

Also, as we clarified in §5, such flat directions exist in general for non-Abelian gauge fields whose associated Lie group has a rank higher than one. It should be noted that there is no flat direction in the potential for the conventional SU(2) (the one with Pauli matrices as generators) gauge field. This gives rise to an interesting inflationary scenario with an SU(3) gauge field, which cannot be realized in the conventional SU(2) case.

On the contrary, in a realistic universe, it is reasonable to expect that all the components of the SU(3) gauge field have nonvanishing initial values of the same order of magnitude. We considered such a situation in §4.4. We found that the generated anisotropy eventually decays due to the nonlinear self-couplings of the gauge field (see also an analogous result in [63]). In this sense, the cosmic no-hair conjecture holds. However, the transient anisotropy should exist practically on the large scales and its effect would be imprinted on the cosmic microwave background and the large-scale structure.

There are several interesting directions for further developments. In our published paper, we have considered inflation with an SU(3) gauge field as a first step. It would be intriguing to study general non-Abelian gauge fields such as SU(N) in detail. For a root vector which has more than 2 non-zero components, inflation with this SU(2) gauge field will result in more than 2 gauge components that can survive from the non-linear self-coupling because of the existence of flat direction. This may result in more complicated anisotropic space but not just a U(1)-like axis-symmetric space. It is also interesting to investigate the Schwinger effect in the presence of non-Abelian gauge fields. Studying the Chern-Simons–type interaction for an SU(3) gauge field instead of the gauge kinetic function may also give interesting features. Another possible extension would be to study models with multiple scalar fields, where the field-space metric is not necessarily flat. Then, the nontrivial kinetic structure may change the dynamics [64]. Thus, it is worth studying the cosmic no-hair conjecture in a more general context. We leave these issues for future study.

Appendix A

Group Theory

A.1 Weights

A subset of commuting hermitian generators which is as large as possible is called a Cartan subalgebra. The generators of Cartan subalgebra are called Cattan generators, satisfying

$$H_i = H_i^\dagger, \quad \text{and} \quad [H_i, H_j] = 0 \tag{A.1}$$

for $i = 1$ to m. The Cartan generators form a linear space. Thus we can choose a basis which they satisfy

$$\mathrm{Tr}\,(\mathrm{H_i H_j}) = \mathrm{k_D}\delta_{\mathrm{ij}} \quad \text{for} \quad \mathrm{i, j = 1 \text{ to } m}, \tag{A.2}$$

where k_D is some constant that depends on the representation and on the normalization of the generators. The number of independent Cartan generators, m, is called the rank of the algebra.

As the Cartan generators commute to each other, they can be simultaneously diagonalized. After diagonalization of the Cartan generators, the states of the representation D can be written as $|\mu, x, D\rangle$ where

$$H_i|\mu, x, D\rangle = \mu_i|\mu, x, D\rangle \tag{A.3}$$

and x is any other label that is necessary to specify the state. The eigenvalues μ_i are called weights. They are real because they are eigenvalues of hermitian operators. The vector with m-component μ_i is called a weight vector. Note that μ_i and $|\mu, x, D\rangle$ are differ from different representation. In adjoint representation, $\mu_i = 0$, while in, for example, Gell-mann matrices defined representation of SU(3) group, $\mu_i \neq 0$. We will often use a vector notation in which

$$\alpha \cdot \mu \equiv \alpha_i \mu_i \quad \text{and} \quad \alpha^2 \equiv \alpha_i \alpha_i. \tag{A.4}$$

A.2 Adjoint Representation

The adjoint representation of an Lie algebra is that generated from the structure constants themselves. Consider a Lie group, the structure constants satisfy the Jacobi identity

$$f_{bcd} f_{ade} + f_{cad} f_{bde} + f_{abd} f_{cde} = 0. \tag{A.5}$$

Defining a set of matrices T_a

$$[T_a] \equiv -i f_{abc}, \tag{A.6}$$

then the above Jacobi identity can be rewritten as

$$[T_a, T_b] = i f_{abc} T_c. \tag{A.7}$$

That is, the structure constants themselves construct a representation of the algebra. This is called the adjoint representation. The dimension of the adjoint representation is the number of independent generators of the group. Because the rows and columns of the matrices defined by (A.6) are labeled by the same index that labels the generators, the states of the adjoint representation can correspond to the generators themselves. We can denote the state in the adjoint representation corresponding to an arbitrary generator X_a as

$$|X_a\rangle . \tag{A.8}$$

The linearity in the sate space also corresponds to the linearity in the algebra :

$$\alpha |X_a\rangle + \beta |X_b\rangle = |\alpha X_a + \beta X_b\rangle . \tag{A.9}$$

The scalar product on this space can be defined by

$$\langle X_a \mid X_b\rangle = \lambda^{-1} \operatorname{Tr}\left(X_a^{\dagger} X_b\right). \tag{A.10}$$

Using the definition of and the linearity of the state, we have

$$\begin{aligned} X_a |X_b\rangle &= |X_c\rangle \langle X_c |X_a| X_b\rangle \\ &= |X_c\rangle [T_a]_{cb} \\ &= -if_{acb} |X_c\rangle \\ &= if_{abc} |X_c\rangle \\ &= |if_{abc} X_c\rangle \\ &= |[X_a, X_b]\rangle \end{aligned} \tag{A.11}$$

A.3 Roots

The weights of the adjoint representation are called roots. As $[H_i, H_j] = 0$, the states in the adjoint representation corresponding to the Cartan generators have zero weight vectors

$$H_i |H_j\rangle = |[H_i, H_j]\rangle = 0. \tag{A.12}$$

The Cartan states are orthonormal, infact, useing (A.2) and (A.10), we have

$$\langle H_i \mid H_j\rangle = \lambda^{-1} \mathrm{Tr}\left(\mathrm{H_i H_j}\right) = \delta_{\mathrm{ij}}. \tag{A.13}$$

The other states of the adjoint representation that are not corresponding to Cartan generators, have non-zero weight vectors, α , with components α_i,

$$H_i |E_\alpha\rangle = \alpha_i |E_\alpha\rangle , \tag{A.14}$$

which means that the corresponding generators satisfy

$$[H_i, E_\alpha] = \alpha_i E_\alpha. \tag{A.15}$$

Take the adjoint of $[H_i, E_\alpha]^\dagger$ we have

$$[H_i, E_\alpha]^\dagger = (H_i E_\alpha - \alpha H_i)^\dagger \tag{A.16}$$

$$= E_\alpha^\dagger H_i^\dagger - H_i^\dagger E_\alpha^\dagger \tag{A.17}$$

$$= E_\alpha^\dagger H_i - H_i E_\alpha^\dagger \tag{A.18}$$

$$= \left[E_\alpha^\dagger, H_i\right]. \tag{A.19}$$

thus we have

$$\left[H_i, E_\alpha^\dagger\right] = -\alpha_i E_\alpha^\dagger, \tag{A.20}$$

so we have

$$E_\alpha^\dagger = E_{-\alpha}. \tag{A.21}$$

The weight α_i are called roots, and the weight vector α with components α_i is a root vector.

A.4 Raising/Lowering Operator and SU(2) Subgroups

The $E_{\pm\alpha}$ are raising and lowering operators for the weights, because the state $E_{\pm\alpha}|\mu, D\rangle$ has weight $\mu \pm \alpha-$:

$$H_i E_{\pm\alpha}|\mu, D\rangle = [H_i, E_{\pm\alpha}]\,|\mu, D\rangle + E_{\pm\alpha} H_i|\mu, D\rangle = (\mu \pm \alpha)_i E_{\pm\alpha}|\mu, D\rangle. \tag{A.22}$$

This equation is true for any presentation, but it is particularly important for the adjoint representation. In adjoint representation, consider $E_\alpha|E_{-\alpha}\rangle$ which has weight 0, thus it is a linear combination of states corresponding to Cartan generators:

$$E_\alpha|E_{-\alpha}\rangle = \beta_i|H_i\rangle = |\beta_i H_i\rangle \tag{A.23}$$

we can calculate β

$$\begin{aligned}\beta_i &= \langle H_i|E_\alpha|E_{-\alpha}\rangle \\ &= \langle H_i|\,[E_\alpha, E_{-\alpha}]\rangle \\ &= \lambda^{-1}\mathrm{Tr}(\mathrm{E}_{-\alpha}\,[\mathrm{H_i}, \mathrm{E}_\alpha]) \\ &= \lambda^{-1}\alpha_i\mathrm{Tr}(\mathrm{E}_{-\alpha}\mathrm{E}_\alpha) \\ &= \alpha_i.\end{aligned} \tag{A.24}$$

Because

$$[E_\alpha, E_{-\alpha}] = E_\alpha|E_{-\alpha}\rangle, \tag{A.25}$$

we have

$$[E_\alpha, E_{-\alpha}] = \alpha \cdot H. \tag{A.26}$$

The relations between $E_{\pm\alpha}$ and H_i are similar to that of $J^\pm$ and J^3 . Actually, for each nonzero pair of root vectors, $\pm\alpha$, there is an SU(2) subalgebra of the group, with generators

$$E^{\pm} \equiv |\alpha|^{-1}E_{\pm\alpha}$$
$$E_3 \equiv |\alpha|^{-2}\alpha \cdot H. \tag{A.27}$$

We can check

$$\begin{aligned}\left[E_3, E^{\pm}\right] &= |\alpha|^{-3}\left[\alpha \cdot H, E_{\pm\alpha}\right] \\ &= |\alpha|^{-3}\alpha \cdot (\pm\alpha)E_{\pm\alpha} \\ &= \pm|\alpha|^{-1}E_{\pm\alpha} \\ &= \pm E^{\pm}\end{aligned} \tag{A.28}$$

and

$$\begin{aligned}\left[E^{+}, E^{-}\right] &= |\alpha|^{-2}\left[E_{\alpha}, E_{-\alpha}\right] \\ &= |\alpha|^{-2}\alpha \cdot H \\ &= E_3,\end{aligned} \tag{A.29}$$

that is, $E^{\pm}$ and E_3 act as $J^{\pm}$ and J_3 respectively.

A.5 SU(3)

SU(3) is the group of 3 × 3 unitary matrices with determinant 1, where U stands for "unitary" and S stands for "special", which means determinant 1. SU(3) is generated by the 3 × 3 hermitian traceless matrices. The element of SU(3) is generated bu exponential the hermitian generators X_a

$$U(\alpha) = e^{i\alpha_a X_a}. \tag{A.30}$$

Let D be the diagonalized matrix of $\alpha_a X_a$, that is

$$V\alpha_a X_a V^{-1} = D, \tag{A.31}$$

then we have

$$\mathrm{det(U(\alpha))} = \det\left(\mathrm{e^{iD}}\right) = \prod_{\mathrm{j}} \mathrm{e^{i[D]_{jj}}} = \mathrm{e^{iTrD}} = \mathrm{e^{iTr\alpha_a X_a}}. \tag{A.32}$$

As $\alpha_a X_a$ is traceless, the determinant of U(α) is 1.

The standard basis of hermitian 3×3 matrices in the physics literature is the Gell-Mann matrices, which is a generalization of the Pauli matrices:

$$\begin{aligned}
\lambda^1 &= \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \lambda^2 = \begin{pmatrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \\
\lambda^3 &= \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \lambda^4 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \\
\lambda^5 &= \begin{pmatrix} 0 & 0 & -i \\ 0 & 0 & 0 \\ i & 0 & 0 \end{pmatrix}, \quad \lambda^6 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \\
\lambda^7 &= \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -i \\ 0 & i & 0 \end{pmatrix}, \quad \lambda^8 = \frac{1}{\sqrt{3}} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{pmatrix}.
\end{aligned} \tag{A.33}$$

The first 3 Gell-Mann matrices contain the Pauli matrices acting on a subspace:

$$\lambda_a = \begin{pmatrix} \sigma_a & 0 \\ 0 & 0 \end{pmatrix} \quad \text{for } a = 1 \text{ to } 3. \tag{A.34}$$

The SU(3) generators are conventionally defined by

$$T^a = \lambda^a/2 \tag{A.35}$$

and they satisfy the normalization condition

$$\mathrm{Tr}\,(T^a T^b) = \frac{1}{2}\delta^{ab} \tag{A.36}$$

and the commutation relation

$$\left[T^a, T^b\right] = i f^{abc} T^c, \tag{A.37}$$

where f^{abc} is the structure constant satisfying

$$f^{abc} = -2i\,\mathrm{Tr}\left(T^a\left[T^b, T^c\right]\right), \tag{A.38}$$

which is completely antisymmetric. The nonvanishing components of f^{abc} are

$$f^{123} = 1, \tag{A.39}$$

$$f^{147} = f^{165} = f^{246} = f^{257} = f^{345} = f^{376} = \frac{1}{2}, \tag{A.40}$$

$$f^{845} = f^{867} = \frac{\sqrt{3}}{2}. \tag{A.41}$$

It is convenient to put T_3 and T_8 in the Cartan subalgebra and take

$$H_1 = T_3 \quad H_2 = T_8. \tag{A.42}$$

A.5.1 Weights and Root of SU(3)

The eigenvectors and associated weight of this representation are

$$\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \to (1/2, \sqrt{3}/6)$$
$$\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \to (-1/2, \sqrt{3}/6) \tag{A.43}$$
$$\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \to (0, -\sqrt{3}/3).$$

If we plot the vector in a plane with H_1 and H_1 as the two coordinate directions, they form the vertices of an equilateral triangle.

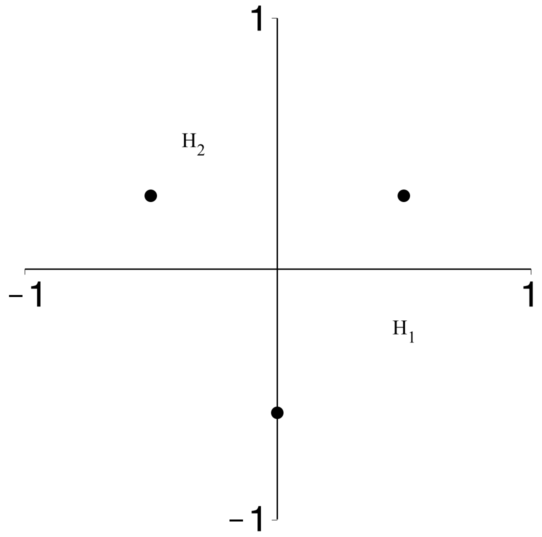

FIGURE A.1: Weights of SU(3).

The roots are the differences of weights (and thus the roots are pair, $\pm\alpha$), because the corresponding generators ($E_{\pm\alpha}$) must take us from one weight to another. The other generators associated to the roots are those that have only one off-diagonal entry:

$$\begin{aligned} \frac{1}{\sqrt{2}}\left(T_1 \pm iT_2\right) &= E_{\pm 1,0} \\ \frac{1}{\sqrt{2}}\left(T_4 \pm iT_5\right) &= E_{\pm 1/2, \pm\sqrt{3}/2} \\ \frac{1}{\sqrt{2}}\left(T_6 \pm iT_7\right) &= E_{\mp 1/2, \pm\sqrt{3}/2}, \end{aligned} \tag{A.44}$$

where the $\pm$ signs are correlated and the first (second) number in the Subscript of the generators are the roots corresponding to H_1 (H_2). The roots form a regular hexagon with H_1 and H_2 acting as the two coordinate directions.

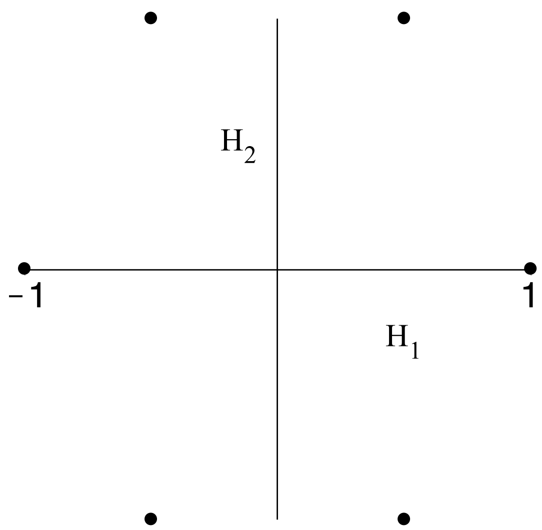

FIGURE A.2: Roost of SU(3).

A.6 SU(N)

We can general SU(3) to SU(N) by generalization the Gell-Mall matrices. As There are $N-1$ independent traceless diagonal real matrices, the SU(N) group is rank $N-1$. As a generalizaiton of the Cartan generators in SU(3), we can choose the $N-1$ Cartan generators as

$$[H_m]_{ij} = \frac{1}{\sqrt{2m(m+1)}} \left(\sum_{k=1}^{m} \delta_{ik}\delta_{jk} - m\delta_{i,m+1}\delta_{j,m+1} \right). \tag{A.45}$$

For example, the first 3 Cartan generators are

$$\begin{aligned} H_1 &= \tfrac{1}{2} \begin{pmatrix} 1 & 0 & \cdots \\ 0 & -1 & \cdots \\ \vdots & \vdots & \ddots \end{pmatrix} \\ H_2 &= \tfrac{1}{\sqrt{12}} \begin{pmatrix} 1 & 0 & 0 & \cdots \\ 0 & 1 & 0 & \cdots \\ 0 & 0 & -2 & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix} \\ H_3 &= \tfrac{1}{\sqrt{24}} \begin{pmatrix} 1 & 0 & 0 & 0 & \cdots \\ 0 & 1 & 0 & 0 & \cdots \\ 0 & 0 & 1 & 0 & \cdots \\ 0 & 0 & 0 & -3 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}. \end{aligned} \tag{A.46}$$

There are total N^2-1 independent traceless hermitian matrices that generate the N-dimensional defining representation of SU(N). The weights are $N-1$ dimensional vectors corresponding to $N-1$ Cartan generator,

$$\left[\nu^j\right]_m = [H_m]_{jj} = \frac{1}{\sqrt{2m(m+1)}} \left(\sum_{k=1}^{m} \delta_{jk} - m\delta_{j,m+1} \right). \tag{A.47}$$

We can explicitly the weights as

$$
\begin{aligned}
\nu^1 &= \left(\tfrac{1}{2}, \tfrac{1}{2\sqrt{3}}, \cdots, \tfrac{1}{\sqrt{2m(m+1)}}, \cdots, \tfrac{1}{\sqrt{2(N-1)N}}\right) \\
\nu^2 &= \left(-\tfrac{1}{2}, \tfrac{1}{2\sqrt{3}}, \cdots, \tfrac{1}{\sqrt{2m(m+1)}}, \cdots, \tfrac{1}{\sqrt{2(N-1)N}}\right) \\
\nu^3 &= \left(0, -\tfrac{1}{\sqrt{3}}, \cdots, \tfrac{1}{\sqrt{2m(m+1)}}, \cdots, \tfrac{1}{\sqrt{2(N-1)N}}\right) \\
&\cdots \\
\nu^{m+1} &= \left(0, 0, \cdots, -\tfrac{m}{\sqrt{2m(m+1)}}, \cdots, \tfrac{1}{\sqrt{2(N-1)N}}\right) \\
&\cdots \\
\nu^N &= \left(0, 0, \cdots, 0, \cdots, -\tfrac{N-1}{\sqrt{2(N-1)N}}\right).
\end{aligned}
\tag{A.48}
$$

For convenience, we can define the positive weight as one in which the LAST non-zero component is positive. With this definition, the weights satisfy

$$\nu^1 > \nu^2 \cdots > \nu^{N-1} > \nu^N. \tag{A.49}$$

As the Es change one weight to another, so the roots are differences of weights, $\nu^i - \nu^j$ for $i \neq j$. The positive roots are $\nu^i - \nu^j$ for $i < j$. The simple roots are

$$\alpha^i = \nu^i - \nu^{i+1} \text{ for } i = 1 \text{ to } N-1, \tag{A.50}$$

where each root is an $(N-1)$-dimensional vector. They can written explicitly as

$$
\begin{aligned}
\alpha^1 &= (1, 0, \cdots, 0), \\
\alpha^2 &= \left(-\frac{1}{2}, \frac{\sqrt{3}}{2}, 0, \cdots, 0\right), \\
\alpha^3 &= \left(0, -\frac{1}{\sqrt{3}}, \sqrt{\frac{2}{3}}, 0, \cdots, 0\right), \\
&\vdots \\
\alpha^m &= \left(0, \cdots, 0, -\sqrt{\frac{m-1}{2m}}, \sqrt{\frac{m+1}{2m}}, 0, \cdots, 0\right), \\
&\vdots \\
\alpha^{N-1} &= \left(0, \cdots 0, -\sqrt{\frac{N-2}{2(N-1)}}, \sqrt{\frac{N}{2(N-1)}}\right).
\end{aligned}
\tag{A.51}
$$

The roots all have length 1. They satisfy

$$\begin{aligned} \alpha^i \cdot \alpha^j &= \frac{1}{2}\left(\delta_{ij} - \delta_{i+1,j} - \delta_{i,j+1} + \delta_{i+1,j+1}\right) \\ &= \delta_{ij} - \frac{1}{2}\delta_{i,j\pm 1}. \end{aligned} \tag{A.52}$$

Appendix B

SU(3) Gauge Field in the Axially Symmetric Bianchi Type I Spacetime

In this appendix, we derive possible configurations of an SU(3) gauge field in the axially symmetric Bianchi type I spacetime. To this end, we extend the discussion for the case of an SU(2) gauge field [38, 65] to an SU(3) gauge field.

First of all, from the translation and the local SU(3) gauge invariance, one can write an SU(3) gauge field as

$$A^a = P^a(t)\mathrm{d}x + Q^a(t)\mathrm{d}y + R^a(t)\mathrm{d}z. \tag{A1}$$

In addition, we impose the axial symmetry along a particular direction, say, the z-direction on it. The rotational transformation along the z-direction, which is generated by a killing vector $\xi = x\partial_y - y\partial_x$, is given by

$$\mathcal{L}_\xi A^a = Q^a(t)\mathrm{d}x - P^a(t)\mathrm{d}y. \tag{A2}$$

In order to preserve the rotational symmetry, (A2) must be absorbed by the residual global SU(3) transformation:

$$\delta A^a = i[A, u]^a = f^{abc}u^b\left[P^c(t)\mathrm{d}x + Q^c(t)\mathrm{d}y + R^c(t)\mathrm{d}z\right], \tag{A3}$$

where u^a's are constant. Therefore, we require

$$\mathcal{L}_\xi A^a = \delta A^a. \tag{A4}$$

Configurations of A^a which satisfy this relation can be classified according to the direction and amplitude of u^a. A trivial case is $u^a = 0$, we have the condition $P^a(t) = Q^a(t) = 0$. Let us consider cases of $u^a = u^3, u^4, u^8$ as representative examples. We first consider the case of $u^a = u^3$. In this case, (A4) admits nontrivial configurations of A^a only if $u^3 = \pm 1$ or ± 2. For instance, $u^3 = 1$ yields

$$\begin{cases} P(t) = P^1(t)T^1 + P^2(t)T^2, \\ Q(t) = -P^2(t)T^1 + P^1(t)T^2, \\ R(t) = R^3(t)T^3 + R^8(t)T^8, \end{cases} \tag{A5}$$

and $u^3 = 2$ gives

$$\begin{cases} P(t) = P^4(t)T^4 + P^5(t)T^5 + P^6(t)T^6 + P^7(t)T^7, \\ Q(t) = -P^5(t)T^4 + P^4(t)T^5 + P^7(t)T^6 - P^6(t)T^7, \\ R(t) = R^3(t)T^3 + R^8(t)T^8. \end{cases} \tag{A6}$$

The case of (A5) includes the SU(2) ⊗ U(1) subgroup we studied in §4.2. Next, in the case of $u^a = u^4$, we have a solution of (A4) only if $u^4 = \pm 1$ or ± 2. For $u^4 = 1$, we have

$$\begin{cases} P(t) = P^5(t)T^5 + P^3(t)\left(T^3 + \sqrt{3}T^8\right), \\ Q(t) = -2P^3(t)T^5 + \frac{1}{2}P^5(t)\left(T^3 + \sqrt{3}T^8\right), \\ R(t) = R^4(t)T^4 + R^8(t)\left(-\sqrt{3}T^3 + T^8\right), \end{cases} \tag{A7}$$

while, for $u^4 = 2$, we obtain

$$\begin{cases} P(t) = P^1(t)T^1 + P^2(t)T^2 + P^6(t)T^6 + P^7(t)T^7, \\ Q(t) = P^7(t)T^1 + P^6(t)T^2 - P^2(t)T^6 - P^1(t)T^7, \\ R(t) = R^4(t)T^4 + R^8(t)\left(-\sqrt{3}T^3 + T^8\right). \end{cases} \tag{A8}$$

Finally, when $u^a = u^8$, the only possibility is $u^8 = \pm 2/\sqrt{3}$. For $u^8 = 2/\sqrt{3}$, the configuration of the gauge field which satisfies (A4) is

$$\begin{cases} P(t) = P^4(t)T^4 + P^5(t)T^5 + P^6(t)T^6 + P^7(t)T^7, \\ Q(t) = -P^5(t)T^4 + P^4(t)T^5 - P^7(t)T^6 + P^6(t)T^7, \\ R(t) = R^1(t)T^1 + R^2(t)T^2 + R^3(t)T^3 + R^8(t)T^8. \end{cases} \tag{A9}$$

In practice, it is necessary to impose the Yang-Mills constraint, i.e.,

$$\nabla_i F^{ai0} + f^{abc} A_i^b F^{ci0} = 0, \tag{A10}$$

which is nothing but the time component of the EOMs for the gauge field (4.14). This further constrains the gauge-field configuration. More explicitly, (A10) can be reduced as

$$f^{abc}\left[\left(P^b(t)\dot{P}^c(t)+Q^b(t)\dot{Q}^c(t)\right)g^{11}(t)+R^b(t)\dot{R}^c(t)g^{33}(t)\right]=0, \tag{A11}$$

in the axially symmetric Bianchi type I spacetime. For instance, for (A5), the above constraint yields $P^2/P^1 = \text{const}$. Likewise, one can obtain some relations among the functions $P^a(t)$, $Q^a(t)$, and $R^a(t)$ for other cases.

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