Analysis of Consensus State and Design of Triggering Functions for Event-triggered Control of Multi-agent Systems

程, 子洋

(Degree)
博士（工学）
(Date of Degree)
2022-03-25
(Date of Publication)
2023-03-01
(Resource Type)
doctoral thesis
(Report Number)
甲第8362号
(URL)
https://hdl.handle.net/20.500.14094/D1008362

博 士 論 文

Analysis of Consensus State and Design of Triggering Functions for Event-triggered Control of Multi-agent Systems

マルチエージェントシステムのイベントトリガ制御における合意状態解析及びトリガー関数設計

2022 年 1 月

神戸大学システム情報学研究科

システム科学専攻

（指導教員：羅　志偉教授）

程 子洋

CHENG ZIYANG

Abstract

During the last decades, increasing attention has been focused on multi-agent system (MAS) due to their widespread existence, e.g., manufacturing, process control, network management. The MAS consists of multiple agents, each agent is an autonomous component that perceives its environment and cooperates with other agents in pursuit of its tasks. According to the location of decision maker, MAS can be divided into centralized, decentralized, and distributed systems. In a centralized system, all agents transmit their information to a central network owner, and receive the control command from it. Obviously, it is vulnerable and inefficient. Decentralized systems divide the agents into small groups, use multiple central owners to collect the information and make decisions for the agents in each group. The central owners form a communication network. It is more robust and has better performance compared with centralized systems. Nevertheless, it requires higher equipment cost and operating cost for more central agents. The distributed systems eliminate centralization, each agent have equal access to data and make decisions by its own. It has high fault-tolerance and scalability and is widely applied in many aspects of industries. However, it needs the highest equipment cost and operating cost among the three kinds of systems. Because an independent set of devices is demanded by each agent to fulfill various physical and cyber processes, e.g., sampling, storage, computation, communication, actuation. The computation and communication resource of each agent is limited and more efficient deployment is required. Hence, the idea of event-triggering is introduced. The desired operation is executed only when the prescribed event is triggered. It not only converts the continuous operations into discrete actions, but also offers an efficient organization on the working rhythm.

There are various problems in multi-agent cooperation of distributed system, e.g., leader-following, flocking, formation, distributed optimization, consensus control. Among them, consensus control plays a fundamental role and is regarded as

the basis of distributed problems. Generally, a group of agents is called consensus when they agree on a certain physical or virtual state. The consensus state refers to the state of agents when they reach consensus. Obviously, there must exist some changes when introduce event-triggered control protocol into a continuous system.

In this research, we investigate the consensus state of event-triggered MAS. The research results are summarized into the following three cases:

1. Observe the consensus state of general linear MAS with event-triggered control. (Chapter 3)

2. Design a sampled-state based triggering function for finite-time consensus of multi-agent system, which only requires sampled state. (Chapter 4)

3. Design a time-based triggering function for finite-time consensus of multi-agent system. (Chapter 5)

At first, we study the consensus state of general linear MAS with event-triggered control. By now, previous research studied the consensus state of simple linear event-triggered MAS with undirected topology. The simple linear system here means the dynamic of each agent is linear without state feedback. Based on this method, we analyze the consensus state of a general linear event-triggered system with directed graph. It is clarified that, the consensus state is a constant vector or a periodic varying vector, and such difference depends on the system matrices and the topology. In addition, these parameters and the trigger threshold affect the value of consensus state.

Secondly, we design a state-based triggering function for finite-time consensus of multi-agent system. In practical MAS, it is of particular interest to achieve cooperation in a finite time to meet specific requirements. In previous researches, although the update of control signal is determined by the evet-triggered control law, the communication between each neighboring agents still requires continuous operation, which requires a waste of communication resource. To solve this problem, we design the triggering function based on the sampled state. The sampled state refers to the state sampled and transmitted by the agents when event is triggered. With this condition, the communication is converted into event-triggered operations. Additionally, we analyze the conditions to ensure positive minimum inter-event time.

It should be mentioned that the existence of positive minimum inter-event time is a sufficient condition for the avoidance of Zeno behavior. It is much more applicable to the practical implementation. Simulation results show that although sampled-state based triggering function has greater estimation error in consensus time, it saves more computation and communication resource compared with exact-state based case, where the exact-state means the continuous state of agent.

Finally, we design a time-based triggering function for finite-time consensus of multi-agent systems. By now, most of triggering thresholds in the field of finite-time consensus problems are designed based on the state of agents. Different from such state-based triggering functions, here the triggering threshold is only time-dependent, the state information received from neighbors are only required to update the controller, which is easier to achieve a desired balance between the convergence time and the number of triggering times. Computer simulation shows that, (1) although the time-based event-triggered control protocol has larger error in the estimation of settling time in certain conditions, it has fewer number of triggering times and less settling time compared with exact-state based case, (2) the time-based case has smaller error in the estimation of settling time in certain conditions, fewer number of triggering times and less settling time compared with sampled-state based cases. Therefore, the time-based approach requires least computation and communication resource to achieve consensus.

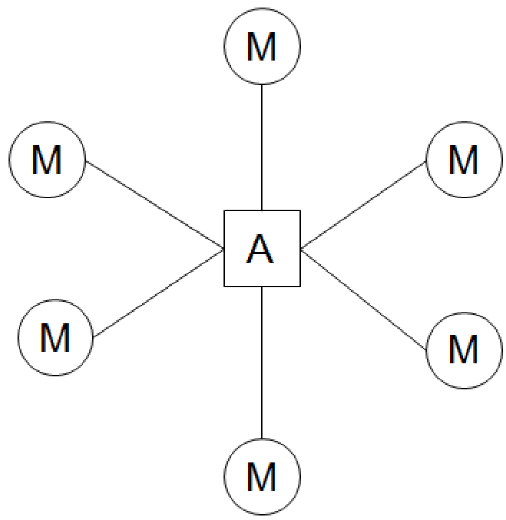
Contents

Chapter 1 Introduction

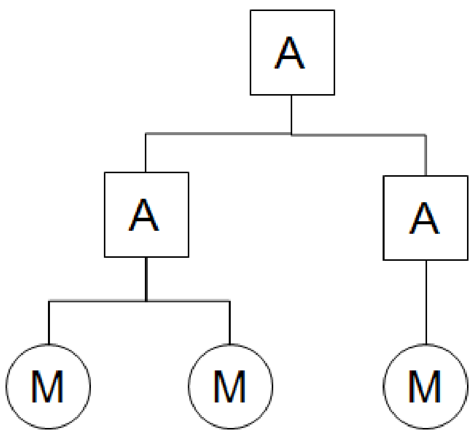
1.1 Background

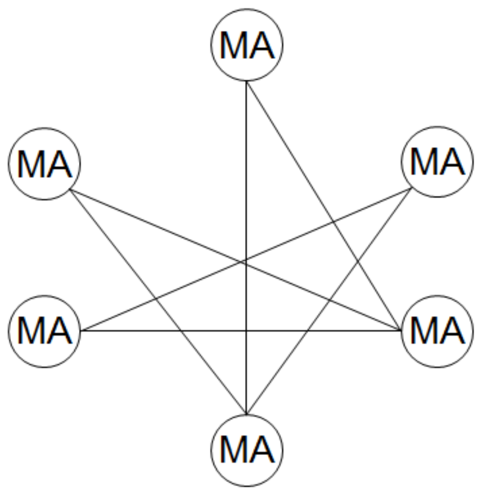
With the development of information technology, the application of multi-agent system (MAS) has expanded to every corner in our daily life, such as Internet of Things (IoT), sensor network, unmanned aerial vehicles, and electronic power grids [1-4]. Although these systems may have various functions and applications, the operations of these networks share some common characteristics, such as consisting of agents with local information and processing capacity, interactions based on the network, the goal is to realize certain local or global cooperation. This paper focus on the consensus problem for distributed MAS with event-triggered control. Its background is divided into three parts: 1. the type of MAS is distributed; 2. the control method is event-triggered; 3. the purpose is to solve consensus problem. They are introduced as follows.

According to the network layout, the control of MAS can be categorized into three kinds: centralized control decentralized control, and distributed control. The general structures of them are described in Fig. 1.1, and the corresponding examples are given as follows. A micro grid (MG) is an intelligent small-scale power supply network designed to provide power for small communities [5]. It contains renewable energy source, distributed power generation and energy storage systems, which make it flexible and efficient. Because of the random, intermittent, unpredictable nature of renewable energy source, it requires the exact information of every agent and arranges the power flow of transmission and distribution lines to ensure a continuous power supply. In this condition, a centralized control strategy is necessary. Since the physical limitation on the structure of large-scale power system makes information transfer among subsystems unfeasible [6], each distributed generator works freely according to the measured local signals. Subsequently, all generators are at a separate control level, i.e., it is a decentralized system. In the production systems, it is quite difficult for a single central system to deal with the flexible conditions, e.g., complexity of data management, uncertainty of demand and available resource. It requires industry to change from static optimization to dynamic agility and respond to environmental changes more actively. Hence, distributed manufacturing control systems are proposed to fulfill the gap. The global control process is divided into several activities of subsystems. Each entity has its own decision process, such as information collection, problem formulation and solving, actuation [7]. The structure of the access to information can be expressed as a topology. For better understanding, the topologies of centralized, decentralized, and distributed systems are presented in Fig. 1.2 [53].



(a) Centralized control





(b) Decentralized control (c) Distributed control

Fig. 1.1 Control architectures that consist of monitors (M) and analysis units (A)

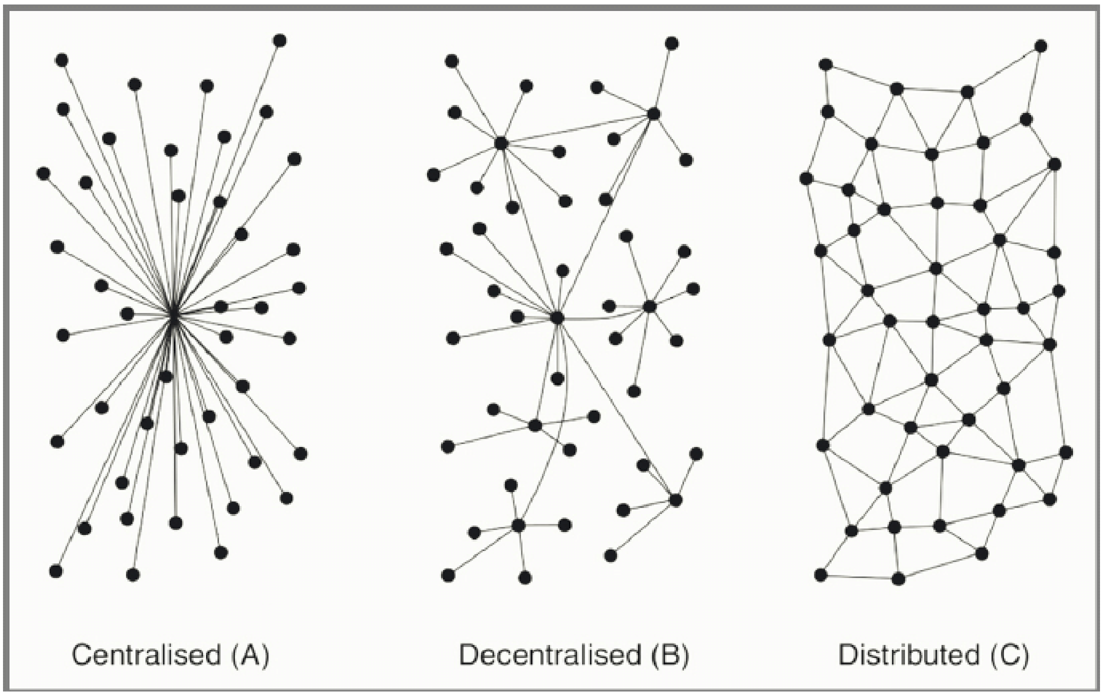


Fig. 1.2 The topologies of centralized, decentralized, and distributed systems

Traditionally, the digital controller takes periodic operations to manipulate continuous-time systems [8]. However, it is impossible to ensure an exact operating frequency in actual systems. Therefore, the discrete system with aperiodic control has become a long-term research field [9]. There exist various methods to solve this problem and analyze the stability of system [10]. For example, aperiodic control can be regard as a specific case of periodic control with time-delay. Similarly, the system can be modeled as a hybrid system with impulsive dynamics. The main idea of these methods is to formulate a linear time-invariant system with aperiodic sampling and control as a linear time-varying system or linear parameter-varying system. It is also common in the field of robust control, which analyzes the influence of aperiodic control and sampling on the output [11]. Obviously, the above examples regard the aperiodic operation as a kind of disturbance with respect to the ideal case of periodic operation. And the solution to these problems is usually setting the lower bound of operation frequency to guarantee the robustness of system. Although it is true that better performance can be acquired eventually by increasing the frequency, this method seems to be quite limited in many applications. However, the event-triggering method treats the aperiodic operation as an opportunity and the solution idea of this problem changes into finding appropriate time of operation. The main motivation for researching event-triggering is its application in IoT and other large-scale networks [12]. Since IoT devices are designed to support services and applications that interact with the physical world, they consist of various sensors and actuators aside from the traditional network devices, the functions of which contain processing, storage, communication, etc. Hence, IoT devices have following characteristics. First, these devices are usually battery-powered and portable, so energy is a scarce resource. Second, high computational complexity is demanded to serve a wide range of functions that it can be integrated seamlessly in arbitrary position of IoT. Third, wireless congestion should be taken into consideration because of the wide use of wireless communication. As a result, the operation of computation and communication is a precious resource and should be scheduled properly.

The idea of event-triggered control is proposed for the efficient deployment of computation and communication. It updates the control action only when the performance of dynamical control system meets a certain condition. This condition is determined by the restrictions of system, e.g., stability and hardware parameters. The controller only requires the state of event instants to update the output. Therefore, the computation of control signal is discrete instead of continuous, which saves computation resource. If the judgement of triggering condition also only requires the state of event instants, then the operation of communication is also discrete, and the communication resource is reduced. The event-triggering methods can address the instants of different operations precisely to maintain the desired attributes, which is more flexible and scalable. As a result, event-triggered methods have received more and more attention.

The consensus problem is one of the most fundamental problems of distributed MAS [13], which mean the processes in the network begins with an arbitrary initial value of a particular type and eventually output the same value of that same type. It has practical implications for many distributed applications in which some type of agreement is required [54]. For example, processes in a communication system may need to agree on whether a message has been received. Processes in a control system may need to agree on whether a particular other process is realizable. Processes may try to agree on an estimate of an airplane's altitude based on the readings of multiple altimeters. Then, the consensus implies that there is asynchronous algorithm that reaches the needed agreement.

1.2 Relevant Researches

The idea of event-triggered control was proposed by Åarzén [14]. Different from working in a certain frequency, the operation of controller only occurs at event-based instants which are determined by a triggering function. The function evaluates the state error between last event instant and current time, the event triggers when the error meets a prescribed threshold. The research in event-triggered mechanisms began to mature and has a place in the field of systems and control by Heemels [15]. The event-triggered strategy was applied in MAS by Dimarogonas [16]. Both centralized and distributed control protocols were researched for MAS consensus problem. It should be mentioned that event-triggered control might exhibit Zeno behavior, that is, the event is triggered infinite times in a finite amount of time. In this condition, the time interval between two consecutive triggering operations becomes infinite small, which is unacceptable in practical application. A non-Zeno behavior strategy of simple linear MAS with undirected topology was introduced by Seyboth [19]. The simple linear system means the dynamics of each agent is linear without feedback. It not only avoids Zeno behavior, but also ensures a positive minimum inter-event time. Inspired by this idea, the event-triggered consensus problem was solved by Yang [35] for a general linear system with directed topology. However, this paper only focused on the controller and triggering function for consensus problem. The consensus state of agents and the influence of triggering function on it were not researched.

In practical application, the operation time plays a significant role. To observe the factors affecting the consensus convergence rate, Olfati-Saber [21] found that the second smallest eigenvalue of Laplacian matrix of the network topology has relation to the consensus time. Kim [22] tried to optimize the network topology to maximize the second smallest eigenvalue. However, this method only improves the convergence rate to achieve asymptotic consensus in an infinite settling time. It is difficult to identify whether the convergence rate meets the actual needs. Therefore, Haimo [23] proposed a continuous feedback controller to realize finite-time stability of system. The finite-time stabilization of dynamical systems not only enables the state of system converge to the equilibrium point in prescribed time, but also gives rise to a high-precision performance. In addition, Bhat [24] investigated the settling time function of finite-time stable system and introduced the relations between the Lyapunov function and the settling time function. Xiao [25] extended this method into the multi-agent system and developed a finite-time formation control framework. Zhang [26] combined the finite-time consensus problem of MAS with event-triggered control, and proposed a nonlinear distributed control protocol under fixed and switching topologies. Hu [27] not only designed a distributed event-triggered control protocol for finite-time consensus, but also showed the relations between the settling time and the triggering condition. However, it only took use of event-triggering method in control protocol, the communication between agents remained continuous operation.

It should be mentioned that the above triggering thresholds for finite-time consensus problems are all depend on the state of agents. However, there are various kinds of triggering functions in other consensus problems. Borgers [17] analyzed the event-triggering mechanisms and divided them into three classes: relative, absolute, and mixed. Relative triggering means the triggering threshold is designed based on the state of agents. Absolute triggering means the triggering threshold depends on a positive constant. Mixed triggering not only considers the state, but also introduces a positive constant into the triggering threshold. A more general classification of the event-triggering mechanisms was proposed by Nowzari [18]: state-dependent, time-dependent, and dynamic-dependent. The triggering threshold of state-dependent triggering function only contains local information, i.e., it consists of the agent's and its neighbors' state. The time-dependent triggering function was first introduced by Seyboth [19], whose threshold is a time variable. The dynamic-dependent triggering function was proposed to avoid Zeno behavior [20], whose triggering function depends on a user-defined variable with its own dynamic. Liu [28] addressed the finite-time consensus problems with dynamic triggering function for second-order MAS with uncertain disturbance. However, there were few researches on time-dependent triggering functions for finite-time consensus problems. Different from the state-based method, it is much easier to achieve a desired balance between the convergence time and the number of triggering times, where the number of triggering times is positive related to the consumption of computation and communication resource.

1.3 Research Objective

As mentioned before, computation and communication become precious resources in MAS and require efficient management by event-triggering methods. Hence, this research focuses on the consensus problem of event-triggered control for MAS.

At first, we focus on the consensus state of linear event-triggered MAS. Different from continuous-time MAS, the consensus state must have some changes when event-triggered controller is introduced into the system. Hence, it is necessary to investigate the influence of triggering function on the consensus state.

Then, the condition of finite-time stability is taken into consideration. We concentrate on the communication load of event-triggered MAS for finite-time consensus problems. In existing researches, the state-dependent triggering function requires the exact state of neighbors, i.e., the agents keep continuous communication between each other. Therefore, a sampled-state triggering function based should be taken into consideration to convert the continuous communication into event-triggered operation. The sampled state refers to the state sampled and transmitted by the agents when event is triggered.

Finally, we set our sights on the design of time-based triggering threshold for finite-time consensus problems. Different from the case of state-based triggering function, the agents only require the state received from neighbors to update the controller and the triggering threshold is a time-dependent variable, which is much easier to achieve a desired balance between the convergence time and the number of triggering times. Additionally, a comparison should be given between time-based and state-based cases in settling time and resource consumption.

1.4 Main Structure of Paper

The structure of this thesis is introduced as follows.

In Chapter 2, the definitions of symbols are described at first. Then, background of graph theory is introduced, including the definition of Laplacian matrix and the related properties. In addition, the conditions and theorem of finite-time consensus are presented. The necessary lemmas for the proof of following chapters are introduced. Finally, the relation between Zeno behavior and minimum inter-event time is analyzed.

In Chapter 3, we start from introducing existing research on consensus state of simple linear event-triggered MAS to observe the process of analysis. Then, this method is applied into general linear MAS to study the consensus state and the influence of triggering function on it. Finally, simulations are carried out to check the results.

In Chapter 4, an existing state-dependent triggering function for finite-time consensus of MAS is introduced at first, which requires continuous communication. Then, a triggering function based on sampled state is developed, whose communication is event-triggered, and the corresponding criterion is established. In addition, the estimation of settling time and the existence of positive minimum inter-event time is analyzed. Finally, a comparison between the existing control protocol and our proposed control protocol is described in simulations.

In Chapter 5, an event-triggered controller with time-dependent triggering function for finite-time consensus of MAS is designed. The relation between the settling time and the triggering condition is analyzed. In addition, the condition of existing positive minimum inter-event time is introduced. Finally, the comparison between state-dependent and time-dependent triggering function is described in simulations.

In Chapter 6, the conclusion and our future work are presented.

Chapter 2 Related Work

In this chapter, we prepare the contents discussed in the later chapters. First, the definition of symbols is presented. The network structure between agents is expressed by Laplacian matrix, so the definition and the properties of Laplacian matrix are introduced. Then, the condition and theory of finite-time stability are provided. The necessary lemmas for the design of triggering function for finite-time consensus problem are reported. Finally, the relation between Zeno behavior and minimum inter-event time is analyzed.

2.1 Notation

$\mathbb{R}$ means the set of real numbers. $\mathbb{R}_+$ is the set of positive real numbers. $\mathbb{Z}_{\geq 0}$ is the set of nonnegative integer numbers. $\mathbb{R}^n$ is the n dimensional vector with real numbers as elements. $\mathbb{R}^{m\times n}$ is the $m \times n$ dimensional matrix with real numbers as elements. 1_N and 0_N denote the $N \times 1$ column vector of all ones and zeros respectively.

$|\cdot|$ denotes the absolute value. $\|\cdot\|$ means the Euclidean norm. $\|\cdot\|_F$ is the Frobenius norm. A^T means the transpose of matrix A. A^{-1} is the inverse matrix of A. $\otimes$ denotes their Kronecker product. $\mathrm{Re}(a)$ and $\mathrm{Im}(a)$ refer to the real part and the imaginary part of complex number a respectively.

The diagonal matrix with elements $A_i,\ i = 1, \cdots, n$ is defined as

$$\mathrm{diag}(A_1, \cdots, A_n) \equiv \begin{bmatrix} A_1 & & 0 \\ & \ddots & \\ 0 & & A_n \end{bmatrix}.$$

The distance of vector $x \in \mathbb{R}^n$ on set $\mathcal{N} \subset \mathbb{R}^n$ is defined as

$$\mathrm{dist}(x, \mathcal{N}) \equiv \inf_{y\in\mathcal{N}} \|x - y\|^2.$$

$[x]^{[\mu]} = \mathrm{sign}(x)|x|^{\mu}$ and $\mathrm{sign}(x)$ denotes the sign of variable x.

2.2 Graph Theory

The communication network structure between robots is represented as a topology with each agent as the vertices. The topology and corresponding characteristics are defined as follows.

Definition 2.1. Graph G contains N vertices is defined as $\{V, E, A\}$. $V = \{v_1, v_2, \cdots, v_N\}$ is the set of vertices. $E \subseteq V \times V$ denotes the set of edges based on non-sequential pairs of two vertices. $A \in \mathbb{R}^{N\times N}$ refers to the adjacency matrix.

Definition 2.2. The adjacency matrix is associated with the information of edges, and is defined as

$$A = \begin{bmatrix} a_{11} & \cdots & a_{1N} \\ \vdots & \ddots & \vdots \\ a_{N1} & \cdots & a_{NN} \end{bmatrix}.$$

where $a_{ii} = 0,\ a_{ij} \geq 0,\ i \in \{1, \cdots, N\},\ j \in \{1, \cdots, N\}$.

$a_{ij} > 0$ means the node v_j can receive information from node v_i, or v_i can broadcast information to v_j. And the value of a_{ij} refers to the weight of edge. Then, $a_{ij} = 0$ means there is no communication between the node v_i and the node v_j.

Definition 2.3. The Laplacian matrix $L \in \mathbb{R}^{N\times N}$ is defined as

$$l_{ij} = \begin{cases} \sum_{k=1,k\neq i}^{N} a_{ik} & j = i \\ -a_{ij} & j \neq i \end{cases},$$

where l_{ij} refers to the ijth element of matrix L, $i \in \{1, \cdots, N\}$, $j \in \{1, \cdots, N\}$.

$\lambda \in \mathbb{R}^n$ refers in particular to the eigenvalues of Laplacian matrix $L \in \mathbb{R}^{n\times n}$ which will be introduced latter. The elements of λ are arranged as follows,

$$\mathrm{Re}(\lambda_1) \leq \mathrm{Re}(\lambda_2) \leq \cdots \leq \mathrm{Re}(\lambda_n).$$

Definition 2.4. (Directed Spanning Tree) The topology contains a node such that there always exists a directed path from this node to every other node.

In this paper, the directed topology means there is at least one directed spanning tree in the topology.

For a directed topology G with N vertices, it has the following properties,

$$L 1_N = 0_N, \tag{2.1}$$

$$\lambda_1 = 0, \tag{2.2}$$

$$\lambda_2 > 0. \tag{2.3}$$

Undirected topology can be viewed as a topology consists of two directed topologies with opposite directions, i.e., the pair of connected vertices can communicate with each other. In this paper, the undirected topology means there exists at least one pair of directed spanning trees with opposite direction in the topology.

For an undirected topology G with N vertices, it has the following properties besides the ones of directed topology,

$$L = L^T, \tag{2.4}$$

$$1_N L = 0_N, \tag{2.5}$$

$$\lambda_2^2 x^T x \leq \lambda_2 x^T L x \leq x^T L^T L x \leq \lambda_N x^T L x \leq \lambda_N^2 x^T x. \tag{2.6}$$

Lemma 2.5 [29]. Suppose L is the Laplacian matrix of an undirected topology G with N vertices. Then, for all $t \geq 0$ and $v \in \mathbb{R}^N$ with $1_N^T v = 0$, it holds that

$$\|e^{-Lt} v\| \leq e^{-\lambda_2 t} \|v\|. \tag{2.7}$$

Proof. See Appendix.

2.3 Theory of Finite-time Stability

Different from classical stability, the finite-time stability has following two important features [30]. First, it deals with systems whose operation is limited to a fixed finite time interval. Second, it requires system variables to have prescribed bounds. When it is applied in the event-triggered system, some points should be taken into consideration. Because the operation of controller in event-triggered MAS is discrete, the state of system is continuous but not Lipschitz continuous. In this section, an existing theory of finite-time stability for non-Lipschitz continuous systems is introduced.

Consider the system of differential equations

$$\dot{y}(t) = g\big(y(t)\big), \tag{2.8}$$

where $g: \mathcal{D} \to \mathbb{R}^n$ is continuous on an open neighborhood $\mathcal{D} \subseteq \mathbb{R}^n$ of the origin and $g(0) = 0$.

A continuously differentiable function $y: I \to \mathcal{D}$ is said to be a solution of (2.8) on the interval $I \subset \mathbb{R}$ if y satisfies (2.8) for all $t \in I$. The continuity of f implies that, for every $x \in \mathcal{D}$, there exist $\tau_0 < 0 < \tau_1$ and a solution $y(\cdot)$ of (2.8) defined on (τ_0, τ_1) such that $y(0) = x$.

Remark 2.6. The continuity of f implies that it is applicable to non-Lipschitz continuous system, which satisfies the condition of event-triggered control.

Definition 2.7. The origin is said to be a finite-time-stable equilibrium of (2.8) if there exists an open neighborhood $\mathcal{N} \subseteq \mathcal{D}$ of the origin and a function $T: \mathcal{N} \backslash \{0\} \to (0, \infty)$, called the settling-time function, such that the following statements hold:

(i) Finite-time convergence: For every $x \in \mathcal{N} \backslash \{0\}$, ψ^x is defined on $\left[0, T(x)\right)$, $\psi^x \in \mathcal{N} \backslash \{0\}$ for all $t \in \left[0, T(x)\right)$, and $\lim_{t \to T(x)} \psi^x(t) = 0$,

(ii) $\psi^x(t) = 0$ for all $t > T$.

where $\psi^x = \mathrm{dist}(x(t), \mathcal{N}): [0, \tau_x) \to \mathcal{D}$ is the unique solution of (2.8).

Theorem 2.8 [24]. Suppose $V: \mathcal{D} \to \mathbb{R}_+$ is a continuous and positive definite function, the Dini derivative of V satisfies

$$\dot{V}(x) + c\big(V(x)\big)^{\mu} \leq 0, \tag{2.9}$$

where $c > 0$, $\mu \in (0,1)$, $x \in \mathcal{V} \backslash \{0\}$, $\mathcal{V} \subseteq \mathcal{D}$.

Then the system (2.8) is finite time stable with the settling time

$$T(x) \leq \frac{1}{c(1-\mu)} V(x)^{1-\mu}, \tag{2.10}$$

$x \in \mathcal{N}$, and T is continuous on $\mathcal{N}$. If in addition $\mathcal{D} = \mathbb{R}^N$, V is proper, and $\dot{V}$ takes negative values on $\mathbb{R}^N \backslash \{0\}$, then the origin is a globally finite-time-stable equilibrium of (2.8)

Proof. See Appendix.

2.4 Preliminaries

During the process of analyzing and designing triggering function for finite-time consensus of MAS, many mathematical inequalities are required. Here we present some necessary lemmas which will be utilized in the following chapters. In addition, an inference of finite-time stability is proposed based on the lemmas.

Lemma 2.9 [27]. For any $y, z \in \mathbb{R}$ and $0 < \mu \leq 1$,

$$|y + z|^{\mu} \leq |y|^{\mu} + |z|^{\mu}. \tag{2.11}$$

Proof. See Appendix.

Lemma 2.10 (Young's inequality) [31]. If $a \geq 0$, $b \geq 0$ and if $p > 1$, $q > 1$ such that $\frac{1}{p} + \frac{1}{q} = 1$, then

$$ab \leq \frac{a^p}{p} + \frac{b^q}{q}. \tag{2.12}$$

Proof. See Appendix.

Lemma 2.11 [27]. For any $\xi_1, \cdots, \xi_N \in \mathbb{R}_+$, $0 < \mu < 1$,

$$(\sum_{r=1}^{N} \xi_r)^{\mu} \leq \sum_{r=1}^{N} \xi_r^{\mu} \leq N^{1-\mu}(\sum_{r=1}^{N} \xi_r)^{\mu}. \tag{2.13}$$

Proof. See Appendix.

Lemma 2.12 [27]. For $\tilde{y}$, $\tilde{z} \in \mathbb{R}$, $0 < \mu < 1$, $|\tilde{y}| \geq |\tilde{z}|$,

$$-\tilde{y}[\tilde{y} + \tilde{z}]^{[\mu]} \leq -|\tilde{y}|^{\mu+1} + |\tilde{y}||\tilde{z}|^{\mu}. \tag{2.14}$$

Proof. See Appendix.

Lemma 2.13. For any $\tilde{y}$, $\tilde{z} \in \mathbb{R}$, $\mu > 0$, $|\tilde{y}| < |\tilde{z}|$,

$$-|\tilde{y} + \tilde{z}|^{\mu+1} \leq -|\tilde{y}|^{\mu+1} + |\tilde{z}|^{\mu+1}. \tag{2.15}$$

Proof. Consider $|\tilde{y}| < |\tilde{z}|$, then $|\tilde{y}|^{\mu+1} < |\tilde{z}|^{\mu+1}$.

Thus, $-|\tilde{y} + \tilde{z}|^{\mu+1} < 0 < -|\tilde{y}|^{\mu+1} + |\tilde{z}|^{\mu+1}$.

The inequality is verified. □

Theorem 2.14. If there exists a continuous function $V = V_1 + V_2$, V, V_1, $V_2 : \mathcal{D} \to \mathbb{R}_+$ such that 1) $V_1(x) = 0 \Rightarrow x \in \mathcal{N}$; 2) any solution $x(t)$ of system (2.8) satisfies the inequality $\dot{V}(t) \leq -\alpha_1 V_1{}^{\mu}\big(x(t)\big) - \alpha_2 V_2{}^{\mu}(t)$, for $\alpha_1 > 0$, $\alpha_2 > 0$, $\mu \in (0,1)$. Then the set $\mathcal{N} \subset \mathbb{R}^N$ is globally finite-time attractive for the system (2.8) and $T \leq \frac{V^{1-\mu}(x_0)}{\min\{\alpha_1, \alpha_2\}(1-\mu)}$, $x_0 = x(0)$.

Proof. According to **Lemma 2.11**,

$$\alpha_1 {V_1}^{\mu}\big(x(t)\big) + \alpha_2 {V_2}^{\mu}(t) \geq \min\{\alpha_1, \alpha_2\}\left[V_1\big(x(t)\big) + V_2(t)\right]^{\mu},$$

$$\dot{V}(t) \leq -\min\{\alpha_1, \alpha_2\} V^{\mu}(t).$$

Obviously, $V(x) = 0$ when $V(t) = 0$. Hence, the set $\mathcal{N} \subset \mathbb{R}^N$ is globally finite-time attractive for any $x(t)$. Therefore,

$$T \leq \frac{V^{1-\mu}(x_0)}{\min\{\alpha_1,\alpha_2\}(1-\mu)}.$$

The proof of this theorem is finished. □

2.5 Zeno Behavior and Minimum Inter-event time

According to the definition, the Zeno behavior occurs when infinite number of events is triggered in a finite time interval. However, the avoidance of Zeno behavior does not mean there exists a minimum inter-event time. Some examples are provided in [18] to introduce the relation between them.

(i) Zeno behavior:

Suppose the sequence of inter-event time satisfies

$$t_{k+1} - t_k = \frac{1}{(k+1)^2},$$

where $k \in \mathbb{Z}_{\geq 0}$, $t_0 = 0$, t_k is the kth event instant.

Hence,

$$\lim_{k\to\infty} t_k = \lim_{k\to\infty} \sum_{n=1}^{k} \frac{1}{n^2} = \frac{\pi^2}{6}.$$

This means the event is triggered infinite times before $T = \frac{\pi^2}{6}$.

(ii) Non Zeno behavior without a positive minimum inter-event time:

Suppose the sequence of inter-event time satisfies

$$t_{k+1} - t_k = \frac{1}{k+1},$$

where $k \in \mathbb{Z}_{\geq 0}$, $t_0 = 0$, t_k is the kth event instant.

Then,

$$\lim_{k\to\infty} t_k = \lim_{k\to\infty} \sum_{n=1}^{k} \frac{1}{n} = \infty.$$

In this condition, the existence of Zeno behavior is ruled out. However, the inter-event time $\lim_{k\to\infty} t_{k+1} - t_k = 0$, i.e., there does not exist a positive minimum inter-event time.

(iii) Positive minimum inter-event time:

Suppose the sequence of inter-event time satisfies

$$t_{k+1} - t_k = \frac{1}{k+1} + c,$$

where $k \in \mathbb{Z}_{\geq 0}$, $t_0 = 0$, t_k is the kth event instant, $c > 0$ is a constant.

Thus,

$$\lim_{k\to\infty} t_k = \lim_{k\to\infty} ck + \sum_{n=1}^{k} \frac{1}{n} = \infty.$$

It not only avoids Zeno behavior, but also ensures the existence of a positive minimum inter-event time $\lim_{k\to\infty} t_{k+1} - t_k > c > 0$.

Hence, the avoidance of Zeno behavior is weaker than the existence of positive minimum inter-event time. In physical implementation, ensuring the existence of positive minimum inter-event time is more appropriate than simply ruling out Zeno behavior.

Chapter 3 Consensus State of Event-triggered Control

3.1 Introduction

In this chapter, we focus on the consensus state of time-based event-triggered control for general linear MAS, where general linear MAS means the dynamic of agent is linear with state feedback. As mentioned in introduction, state-based event-triggered control for MAS has been well developed for various systems, e.g., general linear dynamics with distributed control [32], second-order dynamics with decentralized control [33] and linear dynamics for leader-following consensus with switching topologies [34]. By contrast, the research of time-based event-triggered control is a developing field and first introduced by Seyboth [19]. He introduced a non-Zeno behavior control protocol and corresponding triggering function to solve the consensus problem of simple linear MAS with undirected topology, where the simple linear system means the dynamic of each agent is linear without state feedback. What's more, the consensus state of MAS was investigated. Yang extended this research to a general linear system with directed topology [35]. However, it only introduced the design of controller and triggering function. The consensus state and the influence of triggering function on it were still unclear, which is the gap we want to fill in this section. The process of research is given as follows. The research method of consensus state for simple linear event-triggered MAS is observed at first, which is presented in [19]. Then, such method is applied into the conditions of general linear system [35] with directed topology to study the consensus state and the influence of triggering function on it. Finally, the result is checked by simulation.

3.2 Simple Linear System

We start from introducing the research method of the consensus state for the simple linear event-triggered MAS with undirected topology, which is given in [19].

Denote the state of agent i with dynamics

$$\dot{x}_i(t) = u_i(t) , \tag{3.1}$$

where $x_i(t) \in \mathbb{R}$ is the state and $u_i(t) \in \mathbb{R}$ is the control input.

The event-based controller is given as following:

$$u_i(t) = -\sum_{j=1}^{N} a_{ij}\left(x_i\left(t_{k_i}^i\right) - x_j\left(t_{k_j}^j\right)\right), \tag{3.2}$$

where $t_{k_i}^i$ is the most recent event instant of agent i, a_{ij} is the ijth element in adjacency matrix, N is the number of agents.

For each agent i and $t > 0$, by defining the state error

$$e_i(t) = x_i\left(t_{k_i}^i\right) - x_i(t). \tag{3.3}$$

The triggering function for agent i is

$$f_i(t) = \|e_i(t)\| - (c_0 + c_1 e^{-\alpha t}), \tag{3.4}$$

where $c_0 \geq 0,\ c_1 \geq 0,\ c_0 + c_1 > 0,\ \alpha > 0$ is a constant to be determined.

In event-triggered system, the event instants of each agent is determined by the triggering function, i.e., $t_{k_i+1}^i = \inf\left\{t{:}\, t > t_{k_i}^i, f_i\left(t, e_i(t)\right) > 0\right\}$.

Define the vector $x(t) = [x_1(t), \cdots, x_N(t)]^T$, and $e(t) = [e_1(t), \cdots, e_N(t)]^T$, the overall system with agent dynamics of (1) can be expressed as

$$\dot{x}(t) = -L\left(x(t) + e(t)\right).$$

Define the disagreement vector

$$\delta(t) = x(t) - \bar{x}(0),$$

where $\bar{x}(0) = \frac{1}{N} 1_N^T x(0) 1_N$ refers to the average of initial state.

Theorem 3.1[19]. Consider the MAS (3.1) with a connected and undirected graph G. Suppose the controller (3.2) and triggering function (3.4) with $0 < \alpha < \lambda_2$, then the system does not exhibit Zeno behavior. Moreover, the state $x(t)$ converges to a region centered at $\bar{x}(0)$ with radius $\frac{\|L\|\sqrt{N}c_0}{\lambda_2}$.

Proof. See Appendix.

Remark 3.2. From the proof of **Theorem 3.1**, the estimation of consensus state consists of two steps. Firstly, only consider the exact state $x(t)$ to acquire the center region of consensus state. In this case, $\lim_{t\to\infty} e^{-Lt}x(0) = \bar{x}(0)$ and the proof is given in [12]. Secondly, check the influence of triggering function by taking Euclidean norm of state error to acquire the bounds of distance between the center region and exact consensus state. Inspired by this process, we apply it to analyze a more general case.

3.3 General Linear System

Then, the condition is converted into general linear MAS with directed topology. The controller and triggering function were designed by [35], we extend this research to consensus state and the influence of triggering function on it.

The dynamics of the ith agent is given as

$$\dot{x}_i(t) = Ax_i(t) + Bu_i(t), \tag{3.5}$$

where $x_i(t) \in \mathbb{R}^n$ is the state, $u_i(t) \in \mathbb{R}^p$ is the control input, $A \in \mathbb{R}^{n\times n}$ and $B \in \mathbb{R}^{n\times p}$ are the system matrices.

According to [35], the consensus controller is proposed as

$$u_i(t) = -cK \sum_{j=1}^{N} a_{ij} \left(e^{A\left(t-t_{k_i}^i\right)} x_i\left(t_{k_i}^i\right) - e^{A\left(t-t_{k_j}^j\right)} x_j\left(t_{k_j}^j\right) \right), \tag{3.6}$$

where $c > 0$ is the coupling gain, $K \in \mathbb{R}^{p\times n}$ is the feedback gain to be determined, $t_{k_i}^i$ is the most recent event instant of agent i.

The state error defined as

$$e_i(t) = e^{A\left(t-t_{k_i}^i\right)} x_i\left(t_{k_i}^i\right) - x_i(t),$$

and the triggering function is designed as

$$f_i(t) = \|e_i(t)\| - c_1 e^{-\alpha t}, \tag{3.7}$$

where $c_1 > 0$, $\alpha > 0$ is a constant to be determined.

The overall system can be written as

$$\dot{x}(t) = (I_N \otimes A - cL \otimes BK)x(t) - (cL \otimes BK)e(t),$$

where $x(t) = [x_1^T(t), \cdots, x_N^T(t)]^T, e(t) = [e_1^T(t), \cdots, e_N^T(t)]^T$.

Then the solution is

$$x(t) = e^{\Psi t}x(0) - \int_0^t e^{\Psi(t-s)}(cL \otimes BK)e(s)ds,$$

where $\Psi = I_N \otimes A - \mathrm{c}L \otimes BK$.

Lemma 3.3 [35]**.** Consider the MAS (3.5) with connected and directed topology. Suppose the triggering function (3.7) with $0 < \alpha < -\max_i \mathrm{Re}\big(\lambda_i(\Pi)\big)$. Then, with controller (3.6), the MAS reaches consensus for any initial state if and only if all matrices $A - c\lambda_i BK$ are Hurwitz. Moreover, the existence of Zeno behavior can be ruled out.

$\Pi \triangleq I_{N-1} \otimes A + c\Delta \otimes BK \in \mathbb{C}^{(N-1)n\times(N-1)n}$, Δ is part of Jordan normal form of Laplacian matrix $J(L) = [0, 0_{N-1}^T; 0_{N-1}, \Delta]$, $\lambda_i \neq 0$ is the ith smallest eigenvalue of Laplacian matrix L.

Proof. See Appendix.

Theorem 3.4. Consider the MAS (3.5) with controller (3.6) and triggering function (3.7), the consensus state $\lim_{t\to\infty} x(t)$ depends on the value of $\lambda_m(\Psi)$, where $\lambda_m(\Psi)$ is the eigenvalue of $J_m \in \mathbb{R}^{n_m\times n_m}$ corresponding to the condition: $\lim_{t\to\infty} \max_{i=1,\cdots,s} \|e^{J_i t}\|_F = \lim_{t\to\infty} \|e^{J_m t}\|_F$, J_i is a block of $J_\Psi = \mathrm{diag}(J_1, \cdots, J_i, \cdots, J_s)$, J_Ψ is Jordan normal form of Ψ, s is the number of different eigenvalues of Ψ.

If $\mathrm{Re}\big(\lambda_m(\Psi)\big) < 0$, then $\lim_{t\to\infty} x(t) = 0_{Nn}$.

If $\mathrm{Re}\big(\lambda_m(\Psi)\big) = 0$, $n_m = 1$, then

(i) If $\mathrm{Im}\big(\lambda_m(\Psi)\big) = 0$, the consensus state is a constant vector and locates in a region centered at $rw^T x(0)$ with radius $\frac{c_1\sqrt{N}}{\alpha}\|P_\Psi^{-1}\|\|P_\Psi\|\|(cL \otimes BK)\|$.

(ii) If $\mathrm{Im}\big(\lambda_m(\Psi)\big) \neq 0$, the consensus state is a periodic varying vector and locates in a region centered at $e^{\lambda_m(\Psi)t}$ with radius $\frac{c_1\sqrt{N}}{\alpha}\|P_\Psi^{-1}\|\|P_\Psi\|\|(cL \otimes BK)\|$.

Proof. Consider the triggering function (3.7), it implies that $\lim_{t\to\infty} \|e_i(t)\| \leq \lim_{t\to\infty} c_1 e^{-\alpha t} = 0$, i.e., $\lim_{t\to\infty} e(t) = 0_{Nn}$. Therefore, the change of consensus state

$\lim_{t\to\infty} x(t)$ is parallel to $\lim_{t\to\infty} e^{\Psi t}x(0)$. In another word, $\lim_{t\to\infty} e^{\Psi t}x(0)$ describes the center region of consensus state and $\lim_{t\to\infty} \int_0^t e^{\Psi(t-s)}(cL\otimes BK)e(s)ds$ gives the distance between the center region and exact consensus state.

Here the influence of triggering function is analyzed at first. Similarly, we take the Euclidean norm on the both sides to acquire the bounds of distance,

$$\|x(t)-e^{\Psi t}x(0)\| \le \int_0^t \left\|e^{\Psi(t-s)}\right\| \|cL\otimes BK\| \|e(s)\| ds.$$

The triggering function (3.7) means $t_{k_i+1}^i = \inf\{t: t > t_{k_i}^i,\ \|e_i(t)\| - c_1 e^{-\alpha t} > 0\}$, which implies $\|e_i(t)\| \le c_1 e^{-\alpha t}$,

$$\|x(t)-e^{\Psi t}x(0)\| \le \int_0^t \left\|e^{\Psi(t-s)}\right\| \|cL\otimes BK\| \sqrt{N} c_1 e^{-\alpha s} ds.$$

Then it comes to analyzing $e^{\Psi t}$. It should be mentioned that, **Theorem 3.1** cannot be utilized here because there is no guarantee that Ψ is diagonalizable. As a result, a more general method is applied here.

Assume Ψ has s distinct eigenvalues $\{\lambda_1(\Psi),\cdots,\lambda_s(\Psi)\}$, there exists a nonsingular matrix $P_\Psi \in \mathbb{R}^{Nn\times Nn}$ that $P_\Psi^{-1}\Psi P_\Psi = J_\Psi = \mathrm{diag}(J_1,\cdots,J_s)$, J_Ψ is the Jordan normal form of the matrix Ψ and it consists of s Jordan blocks $J_i \in \mathbb{R}^{n_i\times n_i}$, n_i means there are n_i eigenvalues equal to $\lambda_i(\Psi)$,

$$J_i = \begin{bmatrix} \lambda_i(\Psi) & 1 & \cdots & 0 \\ \vdots & \ddots & \ddots & \vdots \\ 0 & \cdots & \lambda_i(\Psi) & 1 \\ 0 & \cdots & 0 & \lambda_i(\Psi) \end{bmatrix}.$$

The natural exponential function of each Jordan block is

$$e^{J_i t} = e^{\lambda_i(\Psi)t} \begin{bmatrix} 1 & t & \cdots & \frac{t^{n_i-1}}{(n_i-1)!} \\ \vdots & \ddots & \ddots & \vdots \\ 0 & \cdots & 1 & t \\ 0 & \cdots & 0 & 1 \end{bmatrix}.$$

It follows from the property,

$$\lim_{t\to\infty}\|e^{Jt}\| \le \lim_{t\to\infty}\max_{i=1,\cdots,s}\|e^{J_i t}\|_F = \lim_{t\to\infty} e^{\lambda_m(\Psi)t}\left(\sum_{j=1}^{n_m}\left|\frac{(n_m-j+1)t^{j-1}}{(j-1)!}\right|^2\right)^{\frac{1}{2}},$$

where $\lambda_m(\Psi)$ is the eigenvalue of J_m corresponding to the condition: $\lim_{t\to\infty}\max_{i=1,\cdots,s}\|e^{J_i t}\|_F = \lim_{t\to\infty}\|e^{J_m t}\|_F$.

Consider **Lemma 2.9**,

$$\lim_{t\to\infty}\|e^{\Psi t}\| \le \lim_{t\to\infty}\|P_\Psi^{-1}\|\|P_\Psi\|e^{\lambda_m(\Psi)t}\sum_{j=1}^{n_m}\frac{(n_m-j+1)t^{j-1}}{(j-1)!}. \tag{3.8}$$

To ensure the estimation of distance is limited, which means $\lim_{t\to\infty}\|e^{\Psi t}\|$ is upper bounded, it is necessary that $\mathrm{Re}\big(\lambda_m(\Psi)\big) < 0$ or $\mathrm{Re}\big(\lambda_m(\Psi)\big) = 0,\ n_m = 1$. In detail, in the case when

(1) $\mathrm{Re}\big(\lambda_m(\Psi)\big) < 0$

The distance is given as follows

$$\lim_{t\to\infty}\|x(t) - e^{\Psi t}x(0)\| \le \lim_{t\to\infty}\int_0^t \left\|e^{\Psi(t-s)}\right\|\|cL\otimes BK\|\sqrt{N}c_1 e^{-\alpha s}ds$$

$$\le \lim_{t\to\infty}\sqrt{N}\|P_\Psi^{-1}\|\|P_\Psi\|\|cL\otimes BK\|c_1 * \int_0^t e^{\lambda_m(\Psi)(t-s)-\alpha s}\sum_{j=1}^{n_m}\frac{(n_m-j+1)s^{j-1}}{(j-1)!}ds$$

$$= 0.$$

Then consider the center of consensus region. According to (3.8),

$$\lim_{t\to\infty}\|e^{\Psi t}\| \le \lim_{t\to\infty}\|P_\Psi^{-1}\|\|P_\Psi\|e^{\lambda_m(\Psi)t}\sum_{j=1}^{n_m}\frac{(n_m-j+1)t^{j-1}}{(j-1)!} = 0.$$

Thus, $\lim_{t\to\infty}e^{\Psi t}x(0) = 0_{Nn}$.

The consensus state is $\lim_{t\to\infty}x(t) = 0_{Nn}$.

(2) $\mathrm{Re}\big(\lambda_m(\Psi)\big) = 0,\ n_m = 1$

$$\|e^{\Psi t}\| \le \|P_\Psi^{-1}\|\|P_\Psi\|.$$

The distance between $x(t)$ and $e^{\Psi t}x(0)$ is given as follows.

$$\|x(t) - e^{\Psi t}x(0)\| \le \int_0^t\left\|e^{\Psi(t-s)}\right\|\|cL\otimes BK\|\|e(s)\|ds$$

$$\leq \sqrt{N}\|cL \otimes BK\| \int_0^t \|P_\Psi^{-1}\|\|P_\Psi\|c_1 e^{-\alpha s} ds$$

$$\leq \sqrt{N}\|P_\Psi^{-1}\|\|P_\Psi\|\|cL \otimes BK\| \left(\frac{c_1}{\alpha} e^{-\alpha t} + \frac{c_1}{\alpha}\right).$$

Thus,

$$\lim_{t\to\infty}\|x(t) - e^{\Psi t}x(0)\| \leq \lim_{t\to\infty}\sqrt{N}\|P_\Psi^{-1}\|\|P_\Psi\|\|cL \otimes BK\| \left(\frac{c_1}{\alpha} e^{-\alpha t} + \frac{c_1}{\alpha}\right)$$

$$= \frac{c_1\sqrt{N}}{\alpha}\|P_\Psi^{-1}\|\|P_\Psi\|\|(cL \otimes BK)\|.$$

The distance between $x(t)$ and $e^{\Psi t}x(0)$ is a certain value influenced by the topology, the system matrix as well as α and c_1 in the triggering function.

Then, the center of consensus region $\lim_{t\to\infty} e^{\Psi t}x(0)$ is analyzed in the next step.

According to **Lemma 3.3**, if there exists multiple J_m such that $\|e^{J_m t}\|_F = \cdots = \|e^{J_p t}\|_F$, $p \geq m$, and $\lambda_m(\Psi) \nless 0$, then the total size of the corresponding Jordan blocks meets the restriction $\sum_{i=m}^{p} n_i \leq n$, where n refers to the size of vector x_i.

(i) If $\mathrm{Im}\big(\lambda_m(\Psi)\big) = 0$,

In this condition, the existence of multiple solutions means $\lambda_m = \cdots = \lambda_p = 0$, then

$$\lim_{t\to\infty} e^{\Psi t}x(0) = rw^T x(0),$$

where r, $w \in \mathbb{R}^{Nn\times n_p}$ are right and left eigenvectors of Ψ coresponding to $\lambda_m(\Psi) \cdots \lambda_p(\Psi)$, $w^T r = I_{n_p\times n_p}$, $n_p = \sum_{i=m}^{p} n_i \leq n$.

However,

$$\Psi = I_N \otimes A - \mathrm{c}L \otimes BK$$

The consensus state locates in a region centered at $rw^T x(0)$ with radius $\frac{c_1\sqrt{N}}{\alpha}\|P_\Psi^{-1}\|\|P_\Psi\|\|(cL \otimes BK)\|$.

(ii) If $\mathrm{Im}\big(\lambda_m(\Psi)\big) \neq 0$,

Consider the Euler's formula, $e^{\lambda_m(\Psi)t}$ is a sine wave. However, there might exist other $\lambda_i(\Psi)$ with similar property, suppose

$$\mathrm{Re}\big(\lambda_m(\Psi)\big) = \cdots = \mathrm{Re}\left(\lambda_q(\Psi)\right) = 0,$$

where $\mathrm{Im}\big(\lambda_m(\Psi)\big) \neq \cdots \neq \mathrm{Im}\left(\lambda_q(\Psi)\right)$, $n_m = \cdots = n_q = 1$.

Because $\sum_{i=m}^{p} n_i \leq n$, the center of consensus region contains $q - m + 1$ different sine waves $e^{\lambda_m(\Psi)t}$ and the radius is $\frac{c_1\sqrt{N}}{\alpha}\|P_\Psi^{-1}\|\|P_\Psi\|\|(cL \otimes BK)\|$. □

3.4 Simulation

The results of above theorems are illustrated by simulations.

3.4.1 Simple Linear System

Consider a group of 6 agents with the network in Fig. 3.1, and the weight of each edge $a_{ij} = 1$ for $i, j = 1, \cdots, 6$. The triggering function is defined as (3.4), where $c_0 = 0.4$, $c_1 = 1$ and $\alpha = 0.5$. Set initial state $x(0) = [2.5, 1.6, -1, -1.2, 0.3, 2.2]^T$, then the estimated radius is $\frac{\|L\|\sqrt{N}c_0}{\lambda_2} = 1.96$ and the average of initial state is $\bar{x}(0) = 0$. The system state is shown in Fig. 3.2, the state changes around 0 within distance 1.96 which is same as **Theorem 3.1**.

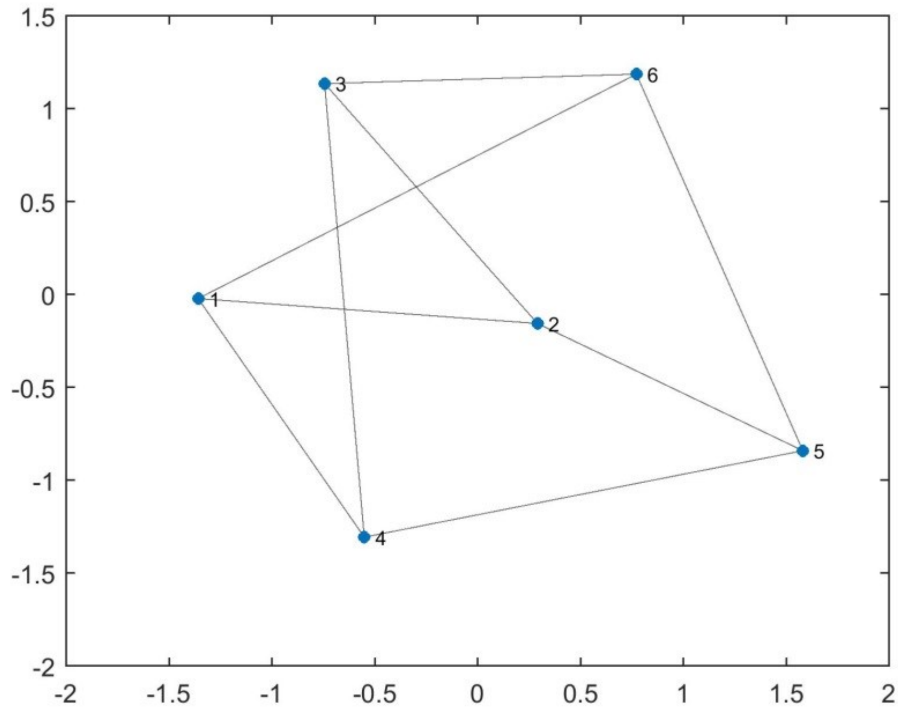


Fig. 3.1 Topology of simple system

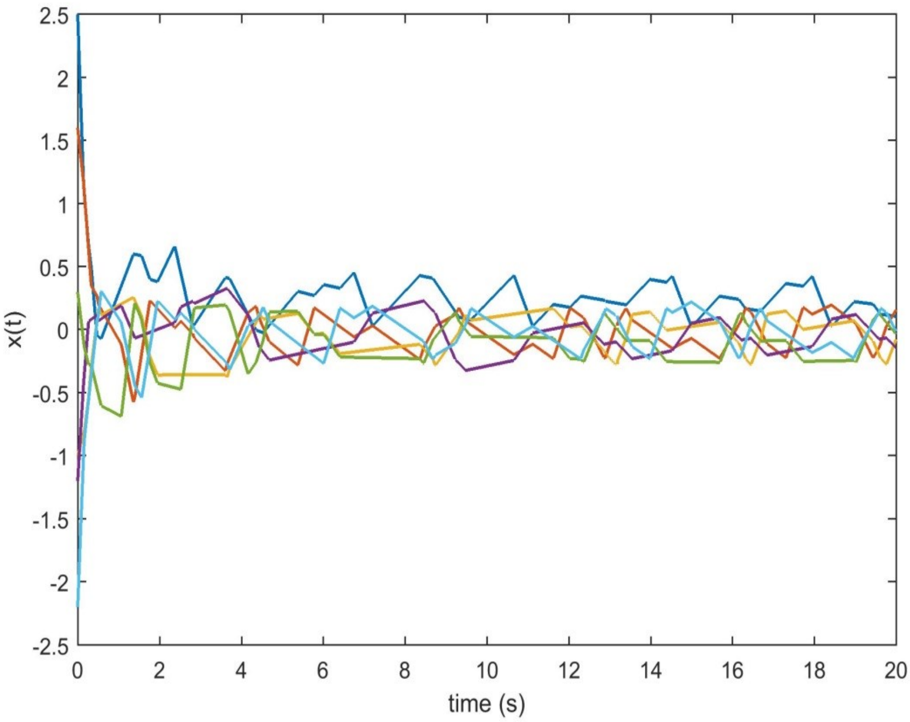


Fig. 3.2 State of simple system

3.4.2 General Linear System

The topology of general linear system is given in Fig. 3.3 and the weight of each edge $a_{ij} = 1$ for $i, j = 1, \cdots, 6$. The parameter of controller in (3.6) is given as $c = 1.1$ and $K = \begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix}$. The triggering function is defined as (3.7), where $c_1 = 0.6$ and $\alpha = 0.2$. Set initial state, $x_1(0) = [-0.3, 2.1]^{\mathrm{T}}$, $x_2(0) = [-1, 0]^T$, $x_3(0) = [1, 0.8]^T$, $x_4(0) = [0.4, -2]^{\mathrm{T}}$, $x_5(0) = [-1.6, 1.5]^T$ and $x_6(0) = [-2, -1.2]^T$. Then, $\lambda_m(\Psi)$ has the following conditions for various system matrix A and B.

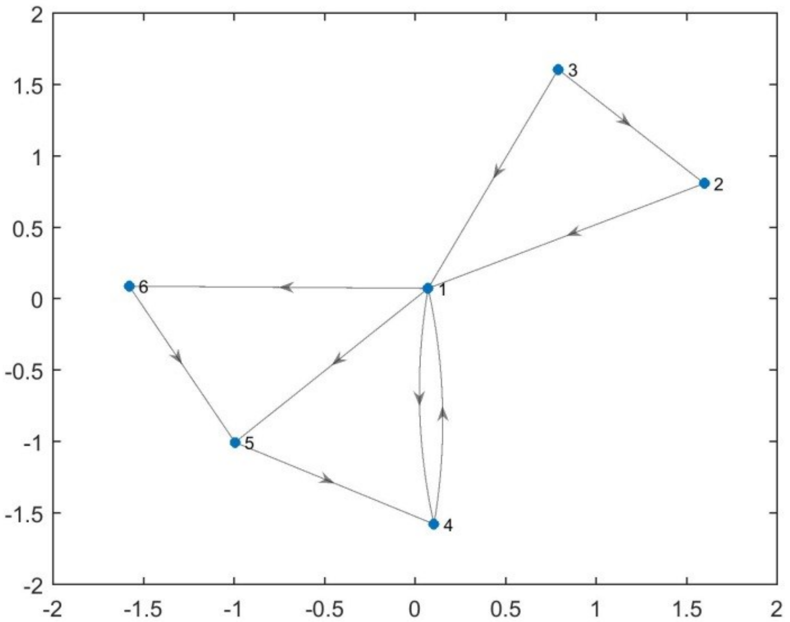


Fig. 3.3 Topology of general linear system

(1) $\mathrm{Re}\big(\lambda_m(\Psi)\big) < 0$

Set the system matrix $A = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$. In this condition, $\lambda_m(\Psi) = -1 < 0$, the state converges to 0_{Nn} which is shown in Fig. 3.4.

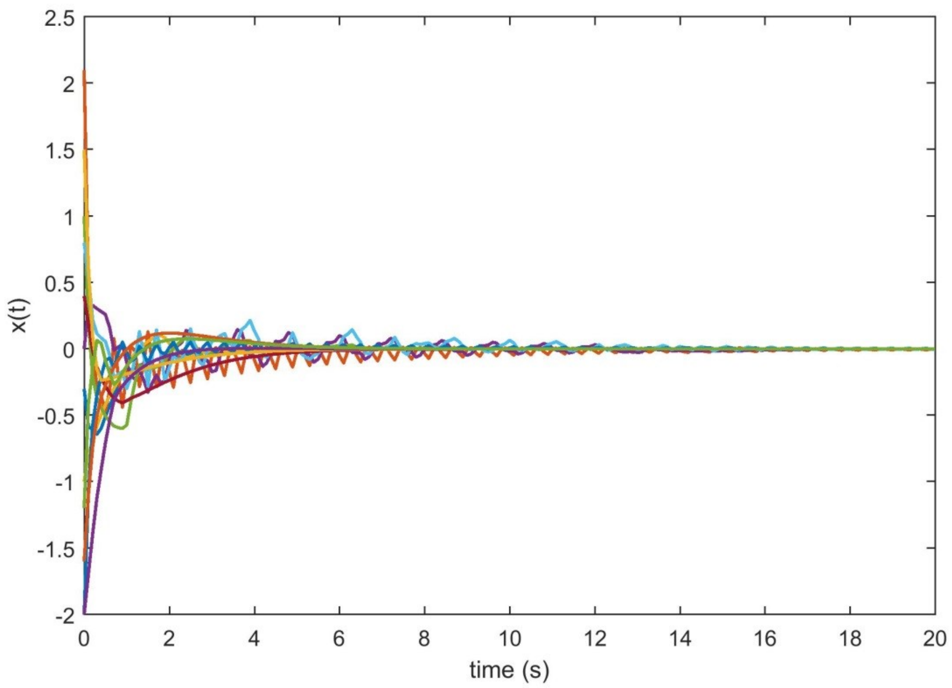


Fig. 3.4 State of general linear system ($Re\left(\lambda_m(\Psi)\right) < 0$)

(2) $\mathrm{Re}\left(\lambda_m(\Psi)\right) = 0$

(i) if $\mathrm{Im}\left(\lambda_m(\Psi)\right) = 0$

Set the system matrix $A = \begin{bmatrix} -1 & 0 \\ -1 & 0 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$. In this condition, $\lambda_m(\Psi) = 0$, the estimated distance $\frac{c_1\sqrt{N}}{\alpha}\|P_\Psi^{-1}\|\|P_\Psi\|\|(cL \otimes BK)\| = 94.69$, the state converges to a certain position in Fig. 3.5. And the exact distance $\|x(t) - e^{\Psi t}x(0)\|$ changes in a range smaller than the estimated value 94.69 in Fig. 3.6.

(ii) if $\mathrm{Im}\left(\lambda_m(\Psi)\right) \neq 0$

Set the system matrix $A = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$. In this condition, $\lambda_m(\Psi) = \pm i$, the estimated distance $\frac{c_1\sqrt{N}}{\alpha}\|P_\Psi^{-1}\|\|P_\Psi\|\|(cL \otimes BK)\| = 94.69$, the state converges to a periodic route in Fig. 3.7. And the exact distance $\|x(t) - e^{\Psi t}x(0)\|$ changes in a range smaller than the estimated value 94.69 in Fig. 3.8.

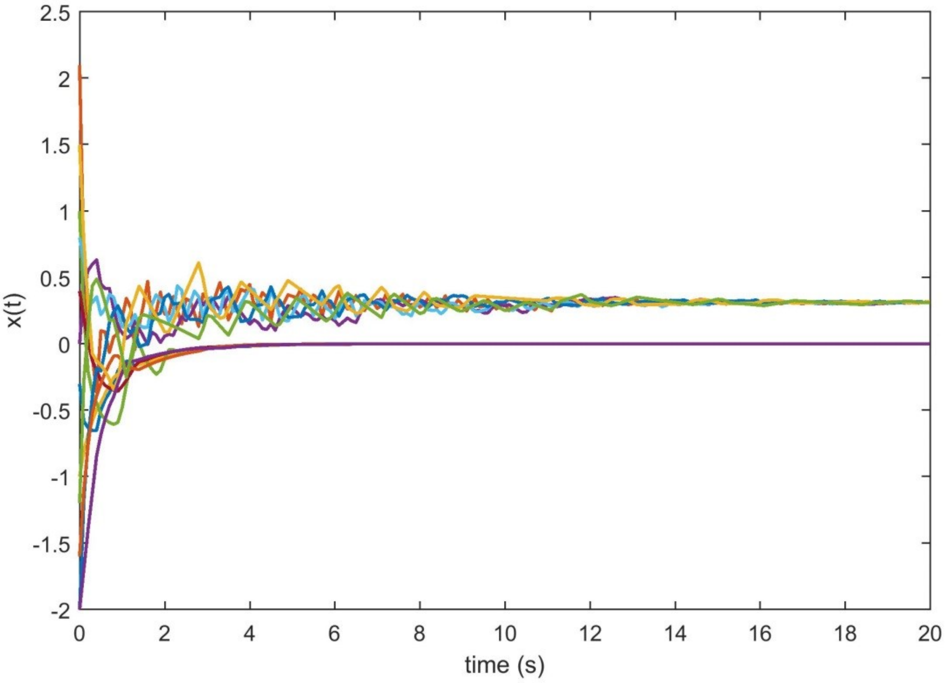


Fig. 3.5 State of general linear system ($\lambda_m(\Psi) = 0$)

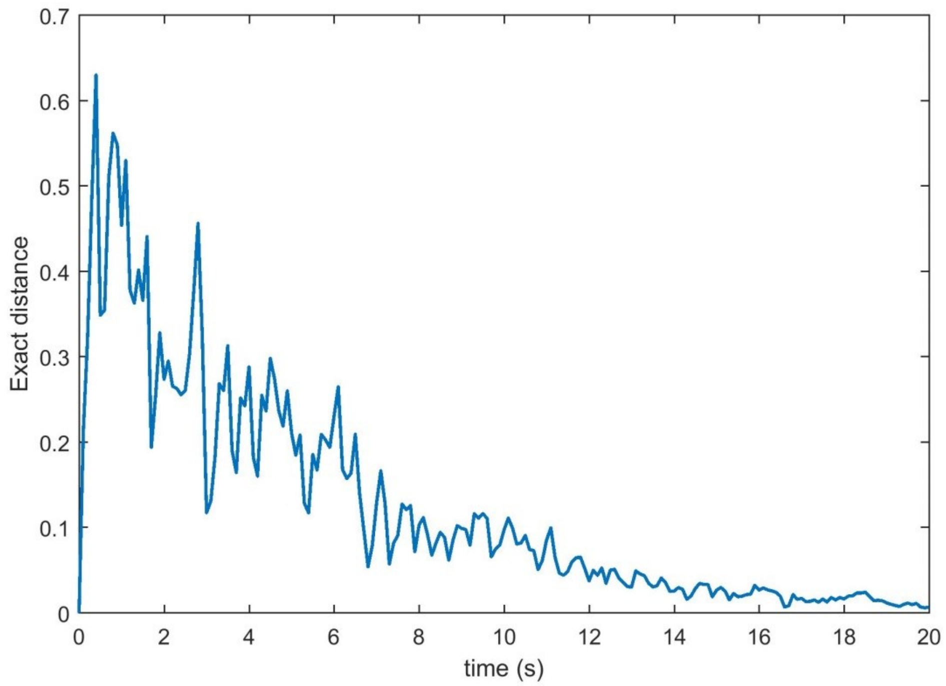


Fig. 3.6 Exact distance of general linear system ($\lambda_m(\Psi) = 0$)

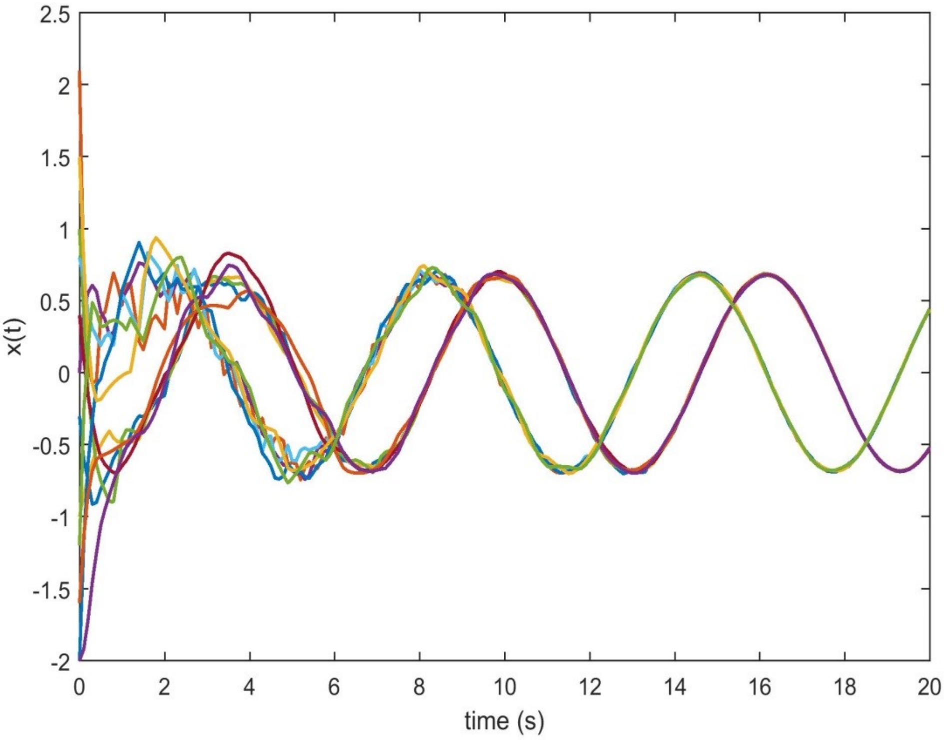


Fig. 3.7 State of general linear system ($\lambda_m(\Psi) = \pm i$)

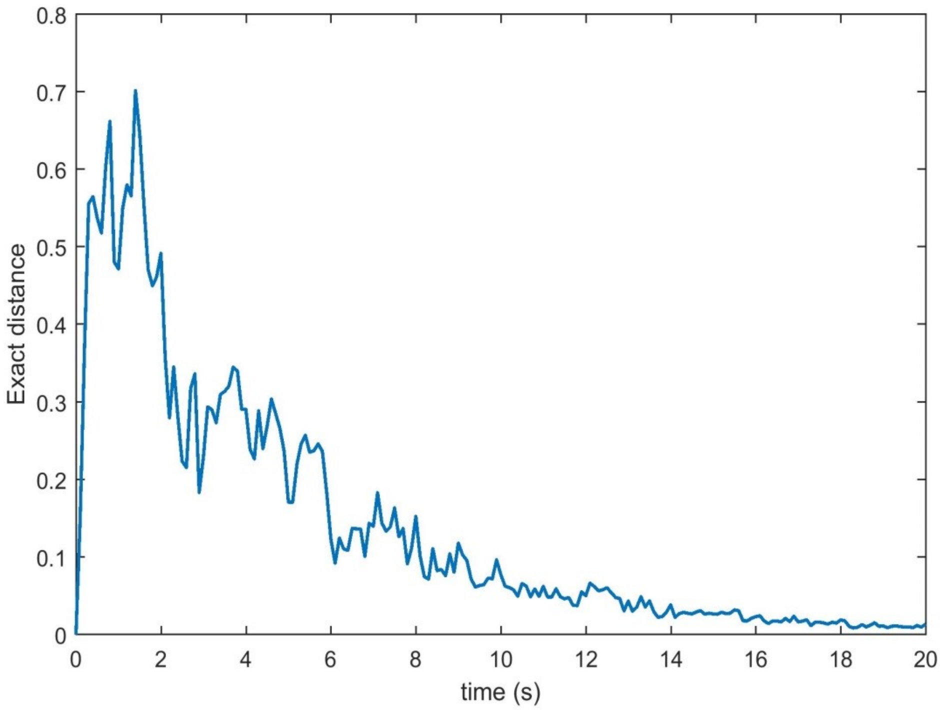


Fig. 3.8 Exact distance of general linear system ($\lambda_m(\Psi) = \pm i$)

3.5 Conclusion

This chapter studied the consensus state of event-triggered MAS with time-based triggering function. An existing research on consensus state of simple linear system with undirected topology is introduced at first to observe the analysis method. During the process, such method includes two steps: firstly, research the consensus state of system in continuous-time condition; secondly, take the influence of triggering function into consideration. Then, this method is utilized into a general linear system. The consensus state might be a constant vector or a periodic varying vector. Such difference depends on the system matrices and the topology. These parameters determine the center region of consensus state. In addition, the triggering threshold affects distance between the center region and exact consensus state. The results are verified by simulations. Because too many norms are utilized, the estimation error is quite large. Hence, a more precise estimation of consensus state is our future work.

Chapter 4 State-based Event-triggered Control for Finite-time Consensus

4.1 Introduction

In this chapter, we focus on applying the event-triggered paradigm into communication in the field of finite-time consensus. As mentioned in Chapter 1, Hu [27] proposed a state-based event-triggered control protocol for finite-time consensus in MAS. However, the triggering function of each agent in this paper contains the exact state of the neighbors, where the exact state means the continuous state of agent. That is to say, the triggering threshold requires real-time information from its neighbors and the agents need to communicate with each other all the time, which endows a tremendous burden in the communication load. And this problem exists widely in many other researches [37-39]. The research process of this problem is given as follows. We start from introducing the exact-state based event-triggered control strategy for finite-time consensus of MAS as a background. Then, we use sampled state to substitute the exact state in the triggering function and acquire the restrictions of triggering threshold based on the theorem of finite-time consensus. The sampled state refers to the state sampled and transmitted by the agents when the event is triggered. Hence, the communication is event-triggered in this condition. The relation between the estimated settling time and triggering function is also acquired. What's more, the problems of minimum inter-event time and Zeno behavior are taken into account. Finally, a comparison between aperiodic and continuous event detection is described in simulations.

4.2 Previous Research

Consider a group of N agents with an undirected and connected communication topology. The dynamics of the ith agent are given by

$$\dot{x}_i(t) = u_i(t), \tag{4.1}$$

where $x_i(t) \in \mathbb{R}$ is the state, $u_i(t) \in \mathbb{R}$ is the control input.

The corresponding controller is given as

$$u_i(t) = -\alpha \left[\sum_{j=1}^{N} a_{ij} \left(x_i\left(t_{k_i}^i\right) - x_j\left(t_{k_j}^j\right)\right)\right]^{[\mu]}, \tag{4.2}$$

where $t_{k_i}^i$ is the most recent event instant of agent i, a_{ij} is the ijth element in adjacency matrix and refers to the weight of communication between agent i and agent j, $\alpha > 0$ is the control gain and $0 < \mu < 1$.

For agent i and $t > 0$, the state error is defined as

$$e_i(t) = x_i\left(t_{k_i}^i\right) - x_i(t),\ t \in \left[t_{k_i}^i, t_{k_i+1}^i\right). \tag{4.3}$$

The triggering function is defined as

$$f_i(t) = |e_i(t)| - c_i \left|\sum_{j=1}^{N} a_{ij} \left(x_i(t) - x_j(t)\right)\right|. \tag{4.4}$$

In event-triggered system, the event instant of each agent is determined by the triggering function, i.e., $t_{k_i}^i = \inf\left\{t : t > t_{k_i}^i, f_i(t) \geq 0\right\}$.

Consider the Lyapunov function

$$V(t) = \frac{1}{2} x^T(t) L x(t). \tag{4.5}$$

Theorem 4.1[27]. Suppose the topology is undirected and connected. Then, with the controller (4.2) and the threshold of triggering function (4.4) satisfies

$$0 < c_{\max} = \max_i(c_i) < \sqrt{\frac{\lambda_2}{\lambda_N^3 N^{\frac{1-\mu}{1+\mu}}}},$$

(i) The MAS (4.1) achieves consensus within the estimated time

$$T_s = \frac{1}{1-\mu} \frac{V(0)^{\frac{1-\mu}{2}}}{\hat{\alpha} 2^{\frac{\mu-1}{2}} \lambda_2^{\frac{\mu+1}{2}}},$$

(ii) The MAS does not exhibit Zeno behavior on time interval $[0, T_s)$,

where $\hat{\alpha} = \alpha \frac{\mu}{1+\mu} \left[1 - N^{\frac{1-\mu}{2}} \left(\frac{\lambda_N^3 c_{\max}^2}{\lambda_2}\right)^{\frac{\mu+1}{2}}\right] > 0,\ \alpha > 0,\ 0 < \mu < 1.$

Proof. See Appendix.

Remark 4.2. From the triggering function (4.4), the estimation of settling time is based on the exact sate, which means each agent requires continuous monitoring its neighbors. By contrast, the controller only works when the sampled state is updated. It only reduces the computation load and the communication load keeps the same as continuous control.

4.3 Sampled-State based Control

In order to reduce the communication load, the triggering function is designed based on sampled state

$$f_i(t) = |e_i(t)| - c_i \left|\sum_{j=1}^{N} a_{ij} \left(x_i(t_{k_i}^i) - x_j\left(t_{k_j}^j\right)\right)\right|. \tag{4.6}$$

In this case, both triggering function and controller only require the sampled state of neighbors. Hence, the communication is also event-triggered.

The Lyapunov function is defined same as (4.5). Then we analyze the condition of finite-time consensus and the influence on the estimated settling time.

Theorem 4.3. Suppose that the graph G is undirected and connected. The controller (4.2) is triggered by the function (4.6) with the threshold $c_i,\ i = 1,\cdots,N$ satisfies

$$\frac{\lambda_2}{1+\lambda_N^2 c_{\max}^2} > \frac{4\lambda_N^3 c_{\max}^2 N^{\frac{1-\mu}{\mu+1}}}{1-2\lambda_N^2 c_{\max}^2},\ c_{\max} = \max_i\{c_i\},\ c_i > 0.$$

Then the following statements hold:

(i) The MAS (4.1) reaches finite-time consensus within the settling time

$$T_s = \frac{2V(0)^{\frac{1-\mu}{2}}}{(1-\mu)\beta},$$

where $\beta = \frac{\alpha}{1+\mu}\left[\left(\frac{\lambda_2}{1+\lambda_N^2 c_{\max}^2}\right)^{\frac{\mu+1}{2}} - \left(\frac{4\lambda_N^3 c_{\max}^2 N^{\frac{1-\mu}{\mu+1}}}{1-2\lambda_N^2 c_{\max}^2}\right)^{\frac{\mu+1}{2}}\right] > 0,\ 0 < \mu < 1,\ \alpha > 0,\ N$ is the number of agents.

(ii) The positive minimum inter-event time exists if and only if the triggering threshold of $|e_i(t)|$ is lower bounded.

Proof. (i) The proof is divided into following three steps.

At first, because the error is related to the sampled state $x_i\left(t_{k_i}^i\right)$ according to the triggering function (4.6), we establish an inequality of $\dot{V}(t)$ such that $\dot{V}(t)$ is no larger than a function expressed by $x_i\left(t_{k_i}^i\right)$.

Secondly, find the relation between $V(t)$ and $x_i\left(t_{k_i}^i\right)$, i.e., $x_i\left(t_{k_i}^i\right)$ locates between the functions expressed by $V(t)$.

Finally, take the result of second step into the inequality of $\dot{V}(t)$, and acquire the consensus condition from the inequality of $\dot{V}(t)$ and $V(t)$.

For the first step, the derivative of $V(t)$ satisfies

$$\begin{aligned}
\dot{V}(t) &= \sum_{i=1}^{N}\left\{\left[\sum_{j=1}^{N} a_{ij}\left(x_i(t)-x_j(t)\right)\right] u_i(t)\right\} \\
&= -\alpha \sum_{i=1}^{N}\left\{\left[\sum_{j=1}^{N} a_{ij}\left(x_i(t)-x_j(t)\right)\right]\left[\sum_{j=1}^{N} a_{ij}\left(x_i\left(t_{k_i}^i\right)-x_j\left(t_{k_j}^j\right)\right)\right]^{[\mu]}\right\} \\
&= -\alpha \sum_{i=1}^{N}\left[\sum_{j=1}^{N} a_{ij}\left(x_i\left(t_{k_i}^i\right)-x_j\left(t_{k_j}^j\right)-e_i(t)+e_j(t)\right)\right] \\
&\quad * \left[\sum_{j=1}^{N} a_{ij}\left(x_i\left(t_{k_i}^i\right)-x_j\left(t_{k_j}^j\right)\right)\right]^{[\mu]} \\
&= -\alpha \sum_{i=1}^{N}\left\{\left[\hat{X}_i-E_i\right]\left[\hat{X}_i\right]^{[\mu]}\right\},
\end{aligned}$$

where $E_i=\sum_{j=1}^{N} a_{ij}\left(e_i(t)-e_j(t)\right)$ refers to the i th element of $Le(t)$ and $e(t)=\left[e_1(t),\cdots,e_N(t)\right]^T$, $\hat{X}_i=\sum_{j=1}^{N} a_{ij}\left(x_i\left(t_{k_i}^i\right)-x_j\left(t_{k_j}^j\right)\right)$ refers to the i th element of $L\hat{x}(t)$ and $\hat{x}(t)=\left[\hat{x}_1(t),\cdots,\hat{x}_N(t)\right]^T=\left[x_1\left(t_{k_1}^1\right),\cdots,x_N\left(t_{k_N}^N\right)\right]^T$.

According to **Lemma 2.10**,

$$\begin{aligned}
\dot{V}(t) &\leq -\alpha \sum_{i=1}^{N}\left|\hat{X}_i\right|^{\mu+1}+\alpha \sum_{i=1}^{N}\left[\frac{\mu}{1+\mu}\left|\hat{X}_i\right|^{\mu+1}+\frac{1}{1+\mu}\left|E_i\right|^{\mu+1}\right] \\
&= -\frac{\alpha}{1+\mu} \sum_{i=1}^{N}\left|\hat{X}_i\right|^{\mu+1}+\frac{\alpha}{1+\mu} \sum_{i=1}^{N}\left|E_i\right|^{\mu+1}.
\end{aligned} \tag{4.7}$$

Using the **Lemma 2.11**,

$$\sum_{i=1}^{N}\left|E_i\right|^{\mu+1}=\sum_{i=1}^{N}\left|E_i^2\right|^{\frac{\mu+1}{2}} \leq N^{\frac{1-\mu}{2}}\left(\sum_{i=1}^{N} E_i^2\right)^{\frac{\mu+1}{2}}, \tag{4.8}$$

$$\sum_{i=1}^{N}\left|\hat{X}_i\right|^{\mu+1}=\sum_{i=1}^{N}\left|\hat{X}_i^2\right|^{\frac{\mu+1}{2}} \geq \left(\sum_{i=1}^{N} \hat{X}_i^2\right)^{\frac{\mu+1}{2}}. \tag{4.9}$$

Take (4.8) and (4.9) into (4.7),

$$\dot{V}(t) \le -\frac{\alpha}{1+\mu}\left(\sum_{i=1}^{N}\hat{X}_i^2\right)^{\frac{\mu+1}{2}} + \frac{\alpha}{1+\mu}N^{\frac{1-\mu}{2}}\left(\sum_{i=1}^{N}E_i^2\right)^{\frac{\mu+1}{2}}. \tag{4.10}$$

The graph G is undirected and connected, recalling the properties (2.6),

$$\sum_{i=1}^{N}\hat{X}_i^2 = \hat{x}^T(t)L^TL\hat{x}(t) \ge \lambda_2\hat{x}^T(t)L\hat{x}(t), \tag{4.11}$$

$$\sum_{i=1}^{N}E_i^2 = e^T(t)L^TLe(t) \le \lambda_N^2 e^T(t)e(t). \tag{4.12}$$

Take (4.11) and (4.12) into (4.10),

$$\dot{V}(t) \le -\frac{\alpha}{1+\mu}\left(\lambda_2\hat{x}^T(t)L\hat{x}(t)\right)^{\frac{\mu+1}{2}} + \frac{\alpha}{1+\mu}N^{\frac{1-\mu}{2}}\left(\lambda_N^2 e^T(t)e(t)\right)^{\frac{\mu+1}{2}}. \tag{4.13}$$

The triggering function (4.6) and the property (2.6) imply that,

$$\begin{aligned} e^T(t)e(t) &= \sum_{i=1}^{N} e_i^2(t) \le c_{\max}^2\sum_{i=1}^{N}\hat{X}_i^2 = c_{\max}^2\hat{x}^T(t)L^TL\hat{x}(t) \\ &\le \lambda_N c_{\max}^2\hat{x}^T(t)L\hat{x}(t). \end{aligned} \tag{4.14}$$

Therefore, inequality of $\dot{V}(t)$ and $\hat{x}_i(t)$ is acquired by taking (4.14) into (4.13)

$$\dot{V}(t) \le -\frac{\alpha}{1+\mu}\left(\lambda_2\hat{x}^T(t)L\hat{x}(t)\right)^{\frac{\mu+1}{2}} + \frac{\alpha}{1+\mu}N^{\frac{1-\mu}{2}}\left(\lambda_N^3 c_{\max}^2\hat{x}^T(t)L\hat{x}(t)\right)^{\frac{\mu+1}{2}}. \tag{4.15}$$

In the second step, the Lyapunov function has the following property according to the definition of error (4.3),

$$\begin{aligned} V(t) &= \frac{1}{4}\sum_{i=1}^{N}\sum_{j=1}^{N} a_{ij}\left(x_i(t) - x_j(t)\right)^2 \\ &= \frac{1}{4}\sum_{i=1}^{N}\sum_{j=1}^{N} a_{ij}\left[\left(x_i(t) - x_j(t)\right) + \left(e_i(t) - e_j(t)\right)\right]^2 \\ &\le \frac{1}{2}\sum_{i=1}^{N}\sum_{j=1}^{N} a_{ij}\left[\left(\hat{x}_i(t) - \hat{x}_j(t)\right)^2 + \left(e_i(t) - e_j(t)\right)^2\right]. \end{aligned} \tag{4.16}$$

Consider the property of Laplacian matrix (2.6), (4.16) can be written as

$$V(t) \le \hat{x}^T(t)L\hat{x}(t) + \lambda_N^2 c_{\max}^2\hat{x}^T(t)L\hat{x}(t). \tag{4.17}$$

Similarly, consider the definition of error (4.3) and the property (2.6),

$$\frac{1}{2}\hat{x}^T(t)L\hat{x}(t) = \frac{1}{2}\left(x(t) + e(t)\right)^T L\left(x(t) + e(t)\right)$$

$$= \frac{1}{2}\left(x^T(t)Lx(t) + x^T(t)Le(t) + e^T(t)Lx(t) + e^T(t)Le(t)\right)$$
$$\leq x^T(t)Lx(t) + e^T(t)Le(t)$$
$$\leq x^T(t)Lx(t) + \lambda_N^2 c_{\max}^2 \hat{x}^T(t)L\hat{x}(t). \tag{4.18}$$

Because the Lyapunov function is defined as (4.5), (4.18) can be converted as

$$\frac{1}{2}\hat{x}^T(t)L\hat{x}(t) \leq 2V(t) + \lambda_N^2 c_{\max}^2 \hat{x}^T(t)L\hat{x}(t). \tag{4.19}$$

Another inequality of $V(t)$ and $\hat{x}(t)$ is acquired,

$$V(t) \geq \left(\frac{1}{4} - \frac{1}{2}\lambda_N^2 c_{\max}^2\right)\hat{x}^T(t)L\hat{x}(t). \tag{4.20}$$

According to (4.17) and (4.20), the relation between $V(t)$ and $x_i(t_{k_i}^i)$ is

$$\frac{1}{1+\lambda_N^2 c_{\max}^2} V(t) \leq \hat{x}^T(t)L\hat{x}(t) \leq \frac{4}{1-2\lambda_N^2 c_{\max}^2} V(t). \tag{4.21}$$

Thirdly, take (4.21) into (4.15),

$$\dot{V}(t) \leq -\frac{\alpha}{1+\mu}\left(\frac{\lambda_2}{1+\lambda_N^2 c_{\max}^2} V(t)\right)^{\frac{\mu+1}{2}} + \frac{\alpha}{1+\mu} N^{\frac{1-\mu}{2}}\left(\frac{4\lambda_N^3 c_{\max}^2}{1-2\lambda_N^2 c_{\max}^2} V(t)\right)^{\frac{\mu+1}{2}}$$
$$= -\beta\left(V(t)\right)^{\frac{\mu+1}{2}}.$$

Since $0 < \frac{\mu+1}{2} < 1$, the system will reach consensus in finite time $T_s = \frac{2V(0)^{\frac{1-\mu}{2}}}{(1-\mu)\beta}$ follow from **Theorem 2.8**.

(ii) This part shows the conditions to ensure the existence of positive minimum inter-event time. In another word, to prove that the time interval between two consecutive triggering operations, i.e., $\tau_{k_i}^i = t_{k_i+1}^i - t_{k_i}^i$, is lower bounded.

(Sufficiency) Assume the last event instant of agent i is $t_{k_i}^i$, then $x_i(t_{k_i}^i)$ remains unchanged before t, $t_{k_i}^i \leq t \leq t_{k_i+1}^i$. Let $\tilde{\tau}_{k_i}^i = t - t_{k_i}^i$, it follows that

$$|e_i(t)| = \left|\int_{t_{k_i}^i}^t \dot{e}_i(s)ds\right| = \left|\int_{t_{k_i}^i}^t \alpha\left[\hat{X}_i\right]^{[\mu]}ds\right| \leq \alpha\theta\left|\hat{X}_i\right|^{\mu}\tilde{\tau}_{k_i}^i,$$

where $x_j\left(t_{k_j}^j\right)$ might update during the time interval $\left[t_{k_i}^i, t\right]$, and there exists a finite constant $\theta > 0$ such that $\theta\left|\hat{X}_i\right|^\mu \geq \max\left\{\left|\hat{X}_i\right|^\mu : t_{k_i}^i \leq t \leq t_{k_i+1}^i\right\}$.

From the triggering function (4.6), the next event of agent i will not be triggered before $\alpha\theta\left|\hat{X}_i\right|^\mu \tilde{\tau}_{k_i}^i = c_i\left|\hat{X}_i\right|$. Hence the lower bound of $\tau_{k_i}^i$ is given by

$$\tau_{k_i}^i \geq \tilde{\tau}_{k_i}^i = \frac{c_i}{\alpha\theta}\left|\hat{X}_i\right|^{1-\mu} \geq \frac{c_i^\mu}{\alpha\theta}|e_i(t)|^{1-\mu}.$$

Therefore, minimum inter-event time exists when set lower bound of the triggering threshold of $|e_i(t)|$.

(Necessity) Note that $\lim_{t\to T_s}|e_i(t)| = 0$. It requires $\tau_{k_i}^i \geq \tilde{\tau}_{k_i}^i = \kappa$ to ensure the existence of positive minimum inter-event time, where κ is a certain constant. In this condition, $|e_i(t)| - f_i(t)$ and $\left|\int_{t_{k_i}^i}^t \dot{e}_i(s)ds\right|$ have the same exponent of base $x_i\left(t_{k_i}^i\right)$, where $\left|\int_{t_{k_i}^i}^t \dot{e}_i(s)ds\right| \sim \left|\hat{X}_i\right|^\mu$ in previous proof. Without loss of generality, suppose $f_i(t) = |e_i(t)| - \left|\hat{X}_i\right|^\mu$. Then, the Lyapunov function has following properties, $\dot{V}(t) \sim \left|\hat{X}_i\right|^{\mu+1} + \left|\hat{X}_i\right|^{\mu(\mu+1)}$ and $V(t) \sim \left|\hat{X}_i\right|^2 + \left|\hat{X}_i\right|^{\mu(\mu+1)}$. Hence, it does not satisfy the condition of **Theorem 2.8**, and the MAS will not reach consensus within a predictable time. □

Remark 4.4. To the best of our knowledge, it's quite common to take use of $e_i(t)$ or $X_i(t)$ to express the lower bound of the time interval $\tau_{k_i}^i$ in the existing papers of finite-time consensus problem for event-triggered MAS, e.g., the lower bound $\tilde{\tau}_{k_i}^i$ of $t_{k_i+1}^i - t_{k_i}^i$ has relation with $e_i(t)$ in paper [27]. However, such expression does not equal to the exclusion of Zeno behavior. Suppose $\tau_{k_i}^i = t_{k_i+1}^i - t_{k_i}^i \geq \tilde{\tau}_{k_i}^i$, $\lim_{t_{k_i}^i \to T_s} k_i = n_i$, the lower bound is expressed by $\tilde{\tau}_{k_i}^i = \beta_i\left|\hat{X}_i\left(t_{k_i}^i\right)\right|^{1-\mu}$, where $\beta_i > 0$ is a certain constant. Because the system is finite-time consensus, there exists a positive constant ζ_i such that $\left|\hat{X}_i\left(t_{k_i+1}^i\right)\right| \leq \left|\hat{X}_i\left(t_{k_i}^i\right)\right| - \tilde{\tau}_{k_i}^i\zeta_i\left|\hat{X}_i\left(t_{k_i}^i\right)\right|^\mu$. In this condition, $\frac{\left|\hat{x}_i\left(t^i_{k_i+1}\right)\right|-\left|\hat{x}_i\left(t^i_{k_i}\right)\right|}{\left|\hat{x}_i\left(t^i_{k_i}\right)\right|} \leq -\frac{\tilde{\tau}^i_{k_i}\zeta_i\left|\hat{x}_i\left(t^i_{k_i}\right)\right|^{\mu}}{\left|\hat{x}_i\left(t^i_{k_i}\right)\right|} = -\frac{\beta_i\zeta_i\left|\hat{x}_i\left(t^i_{k_i}\right)\right|}{\left|\hat{x}_i\left(t^i_{k_i}\right)\right|} = -\beta_i\zeta_i$, i.e., $\lim_{t^i_{k_i}\to T_s} \frac{\tilde{\tau}^i_{k_i+1}}{\tilde{\tau}^i_{k_i}} = \lim_{t^i_{k_i}\to T_s} \left(\frac{\left|\hat{x}_i\left(t^i_{k_i+1}\right)\right|}{\left|\hat{x}_i\left(t^i_{k_i}\right)\right|}\right)^{\gamma} = \lim_{t^i_{k_i}\to T_s} (1-\beta_i\zeta_i)^{\gamma} < 1$, the series $\tilde{\tau}^i_{k_i}$ is convergent. However, the sum of series $\tau^i_{k_i}$ is a finite value $\sum_{k_i=1}^{n_i} \tau^i_{k_i} \leq \mathrm{T_s}$, which means the series $\tau^i_{k_i}$ is also convergent. Therefore, it cannot prove that $n_i < \infty$, i.e., the Zeno behavior cannot be excluded. The case of $e_i(t)$ can be verified similarly.

Remark 4.5. As introduced in Chapter 2.5, the existence of positive minimum inter-event time is a sufficient condition for the avoidance of Zeno behavior. What's more, it is much more applicable in practical operations. As a result, we only consider the positive minimum inter-event time here.

4.4 Simulation

In this section, we investigate the relation between triggering times and settling time at first, and then give a comparison between exact-state based and sampled-state based event-triggered control by simulation.

4.4.1 Relation between Parameters

According to the definition of triggering function (4.6), c_i increase will lead to triggering threshold $c_i\left|\sum_{j=1}^{N} a_{ij}\left(x_i\left(t^i_{k_i}\right) - x_j\left(t^j_{k_j}\right)\right)\right|$ increase, and then triggering times will decrease. So, c_i is inverse proportional to triggering times. On the other hand, β is inverse proportional to settling time $T_s = \frac{2V(0)^{\frac{1-\mu}{2}}}{(1-\mu)\beta}$. Because it is quite difficult to estimate the relation between c_i and β according to the equation $\beta = \frac{\alpha}{1+\mu}\left[\left(\frac{\lambda_2}{1+\lambda_N^2 c_{\max}^2}\right)^{\frac{\mu+1}{2}} - \left(\frac{4\lambda_N^3 c_{\max}^2 N^{\frac{1-\mu}{\mu+1}}}{1-2\lambda_N^2 c_{\max}^2}\right)^{\frac{\mu+1}{2}}\right]$, several examples are taken. Set $\mu = \alpha = 0.8$, suppose $c_i = c_{\max}$, $i = 1, \dots, N$, choose ring Fig. 4.1, star Fig. 4.3 and fully connected topology Fig. 4.5.

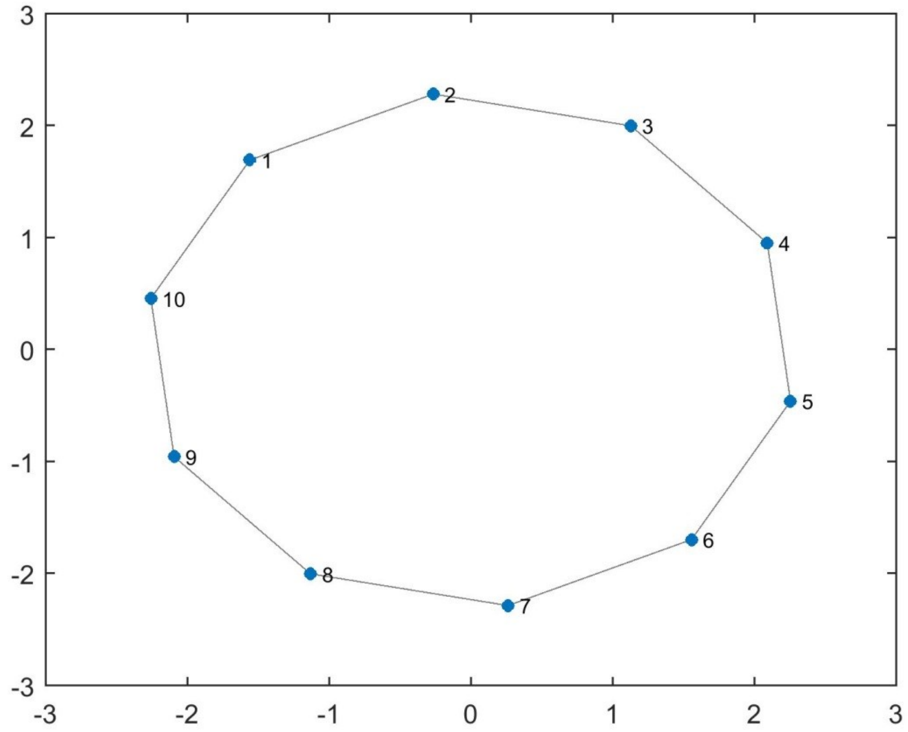


Fig. 4.1 Ring topology

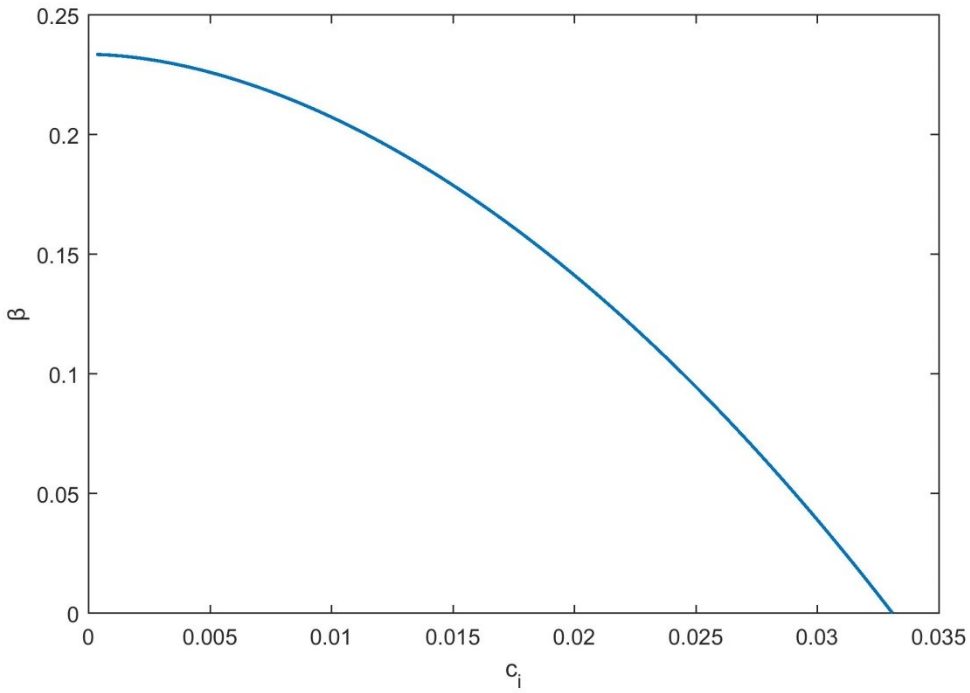


Fig. 4.2 Relation between c_i and β for ring topology

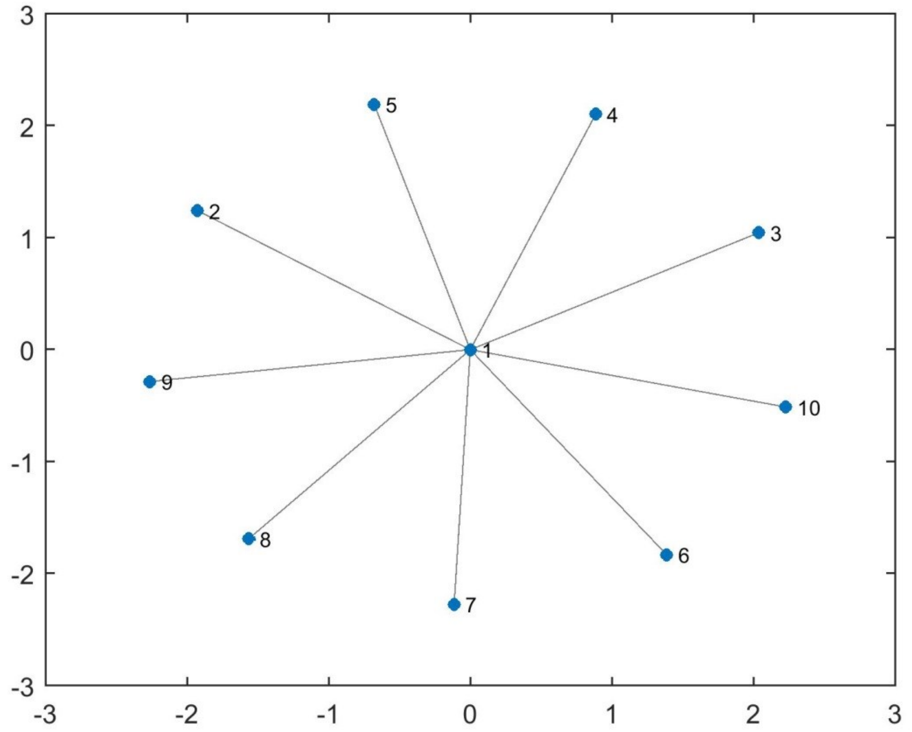


Fig. 4.3 Star topology

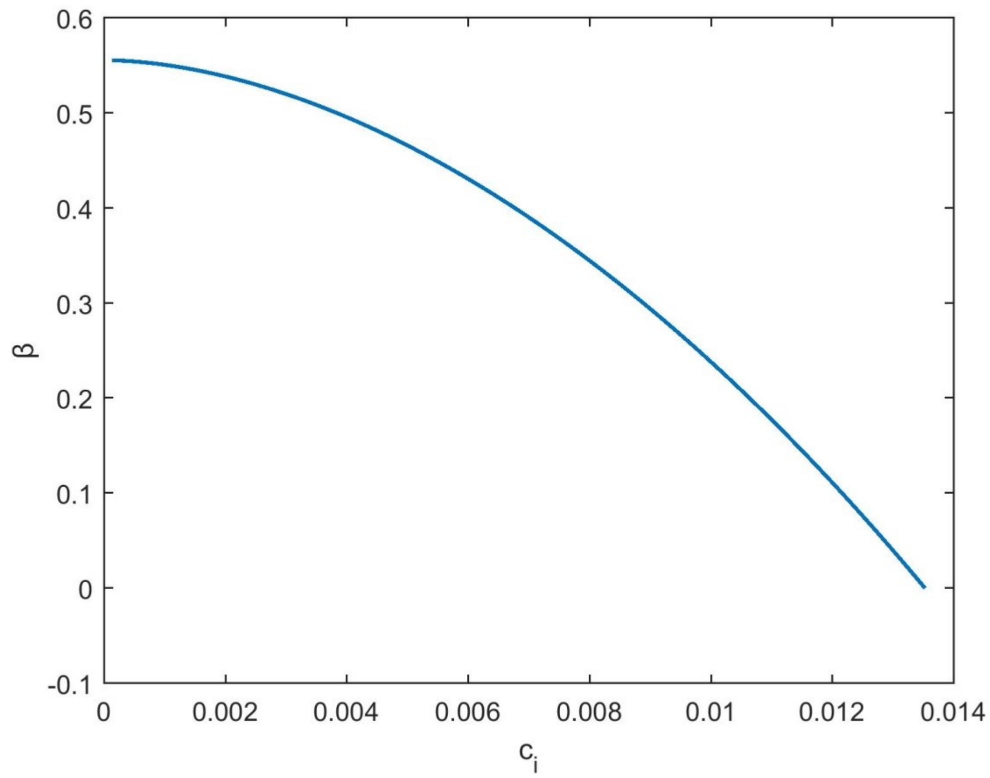


Fig. 4.4 Relation between c_i and β for star topology

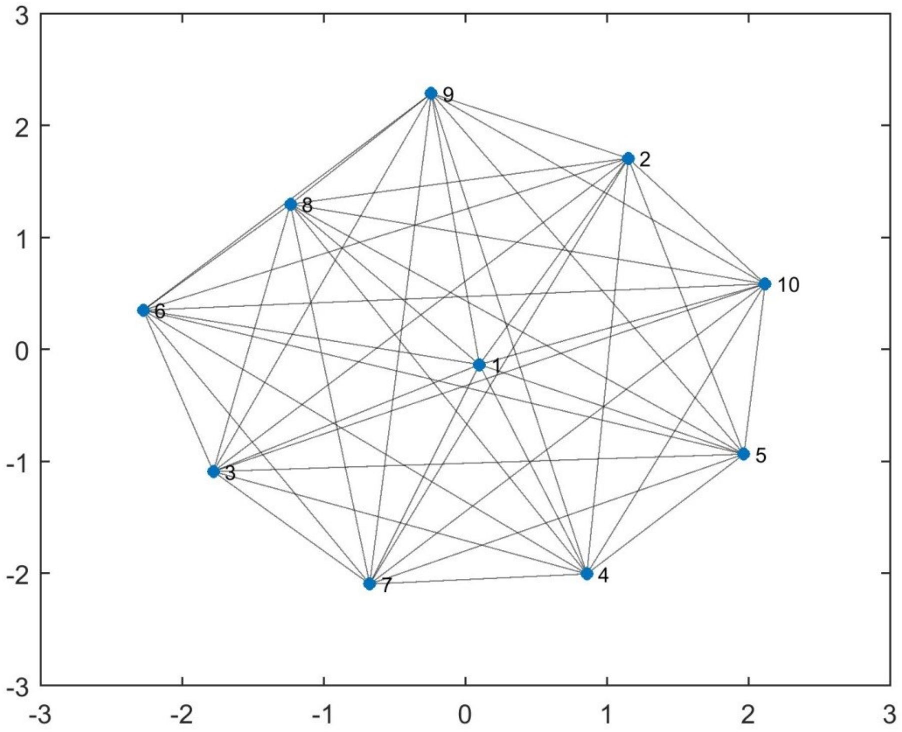


Fig. 4.5 Fully connected topology

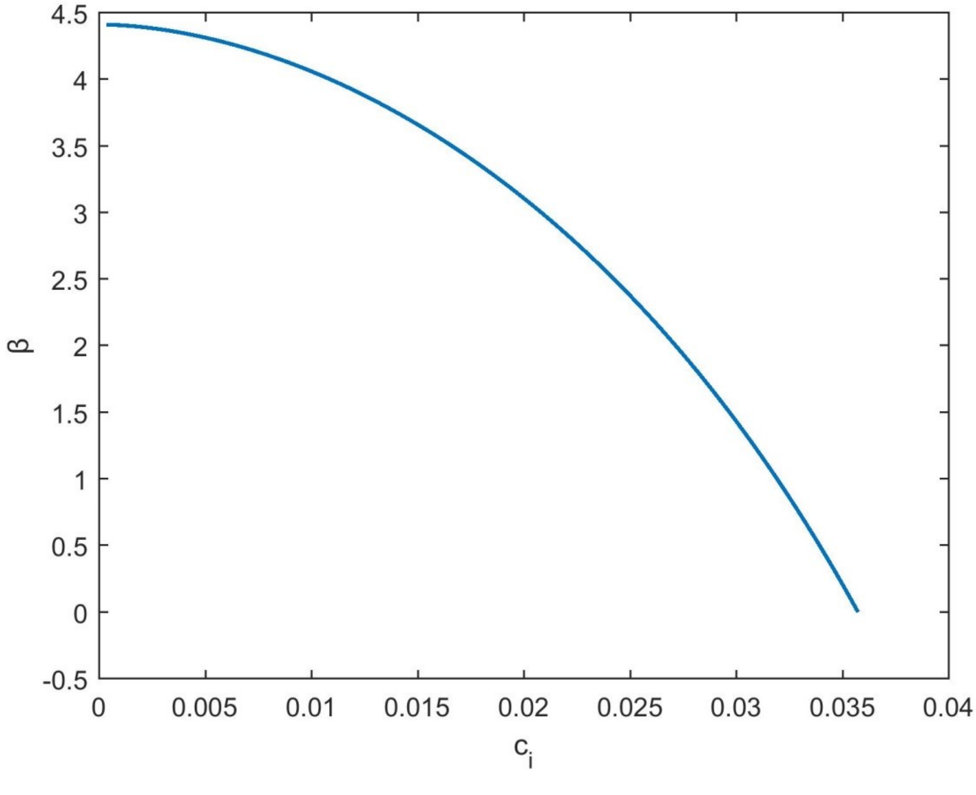


Fig. 4.6 Relation between c_i and β for fully connected topology

According to the result of Fig. 4.2, Fig. 4.4, Fig. 4.6, c_i is inverse proportional to β. As a result, to reduce the number of triggering times, we should increase c_i, it will reduce β, and increase the settling time. In real application, the tradeoff between the number of triggering times and the settling time should be taken into consideration. Obviously, the exact-state based case also has the same property according to **Theorem 4.1**.

4.4.2 Comparison between Different Methods

Consider the MAS consists of 6 agents with the network given in Fig. 4.7, and the weight of each edge $a_{ij} = 1$ for $i, j = 1, \cdots, 6$. The second smallest and largest eigenvalues of the corresponding Laplacian matrix L are $\lambda_2 = 3$ and $\lambda_N = 6$ respectively. The dynamics and controller of each agent are defined as (4.1) and (4.2), where $\alpha = 1$ and $\mu = 0.8$. Set initial state $x(0) = [-2.5, 5.6, 8, -7.2, 1.3, -5.2]^T$. Then the upper bound of the threshold c_i in triggering function (4.6) is $0 < c_{\max} < 0.042$. To ensure the existence of positive minimum inter-event time, set the lower bound of the triggering threshold $e_i(t)$ is 10^{-3}.

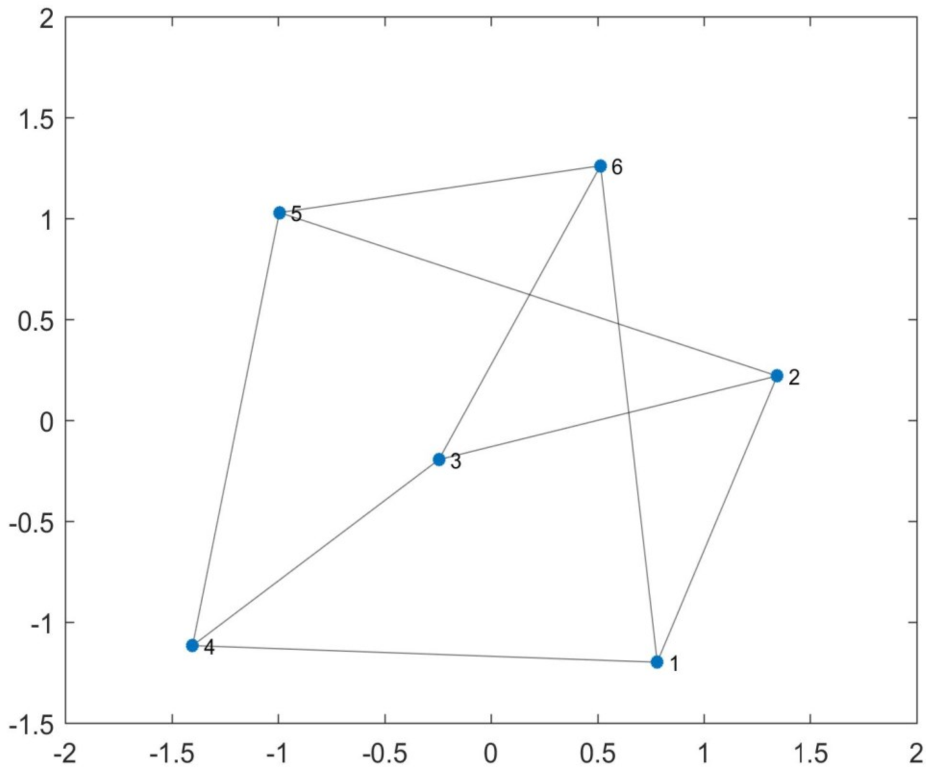


Fig. 4.7 The communication network

Here we take two sets of event thresholds c_i. In each case, results of our sampled-state based triggering function and the exact-state based triggering function in [27] are given as a comparison.

(i) $c_1 = \cdots = c_6 = 0.01$

In the sampled-state based case, the estimated settling time is $T_s = 12.58$ s. The sampled state $x_i\left(t_{k_i}^i\right)$ and event instants are shown in Fig. 4.8 and Fig. 4.9, the real settling time is $T = 2.09$ s which denotes $\inf\{t: V(t) \leq 10^{-3}\}$. The total number of triggering times is 892, which refers to the number of updating times of the controller and triggering function.

In continuous event detection, the estimated settling time is $T_s = 8.10$ s. The sampled state $x_i\left(t_{k_i}^i\right)$ and event instants are shown in Fig. 4.10 and Fig. 4.11, the real settling time is $T = 2.09$ s. The number of triggering times is 905, which refers to the number of updating times of the controller.

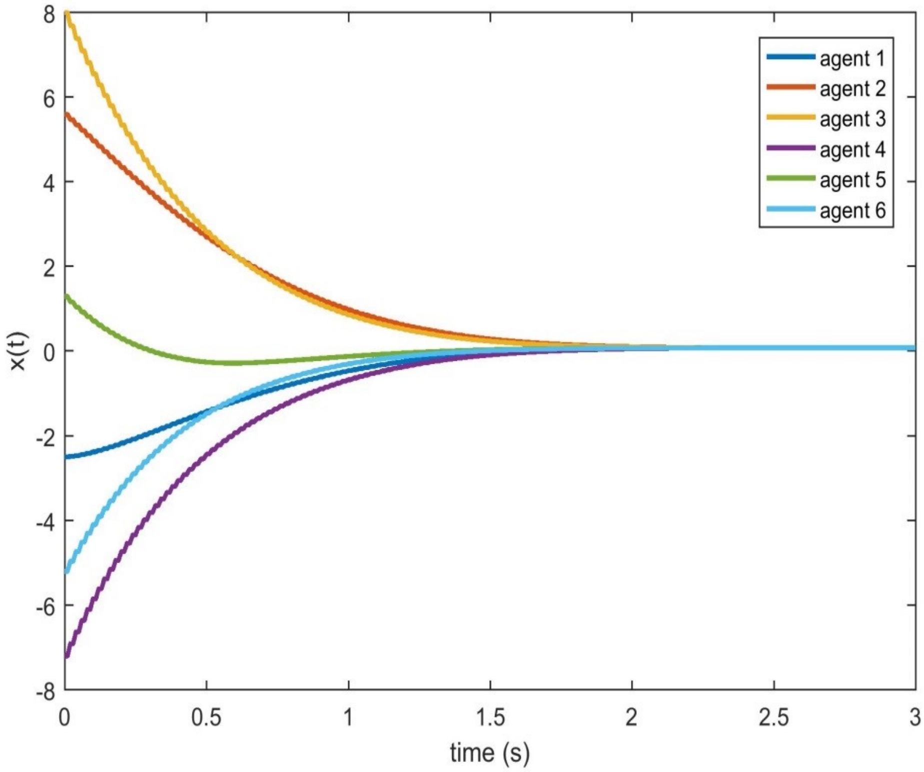


Fig. 4.8 The sampled state of the sampled-state based case (i)

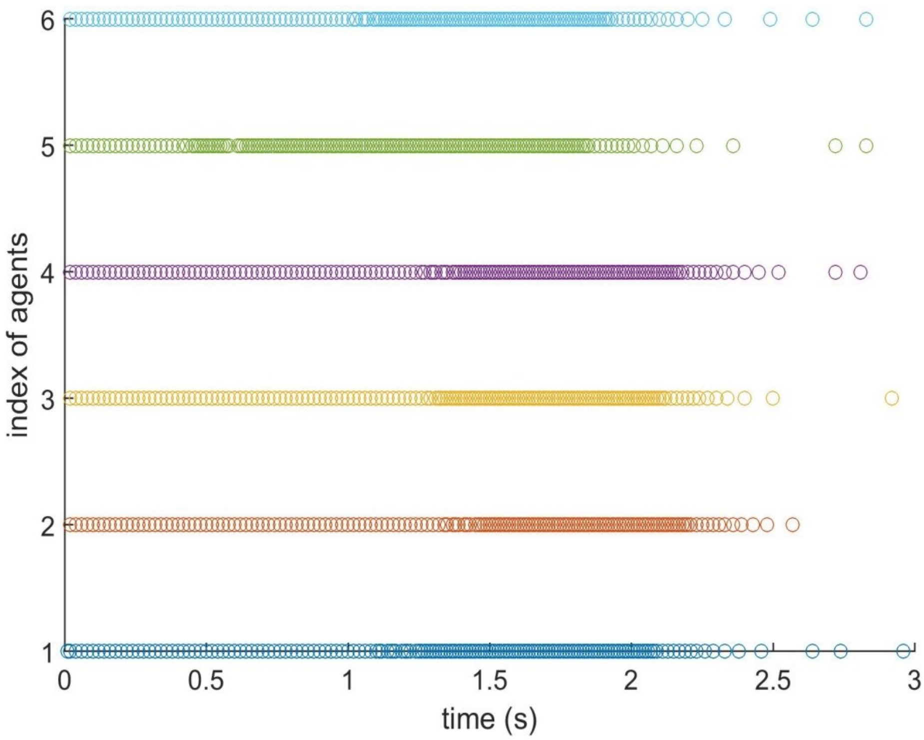


Fig. 4.9 Event instants of the sampled-state based case (i)

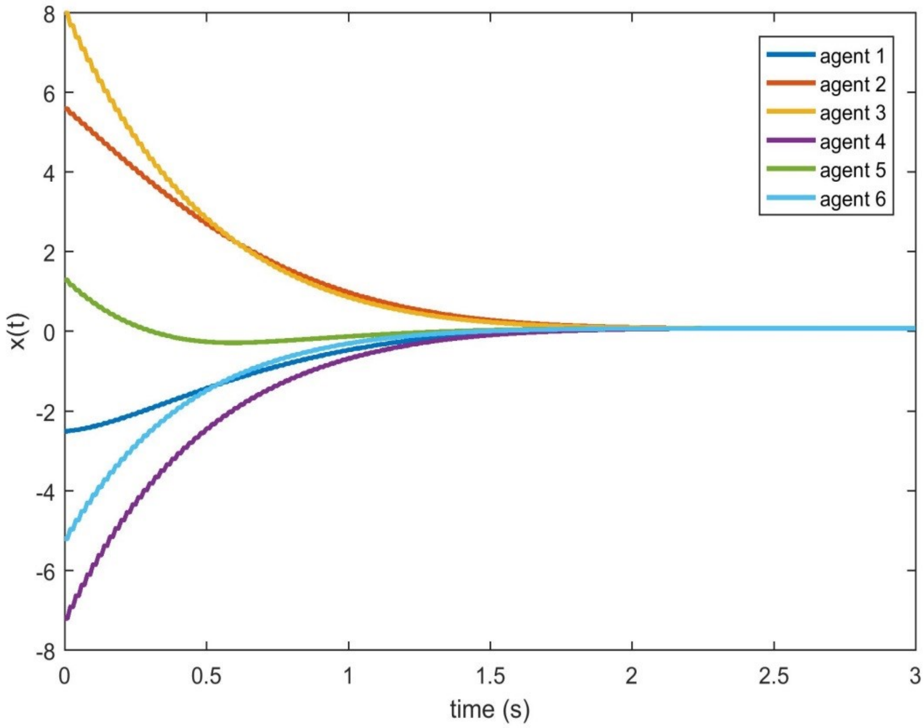


Fig. 4.10 The sampled state of the exact-state based case (i)

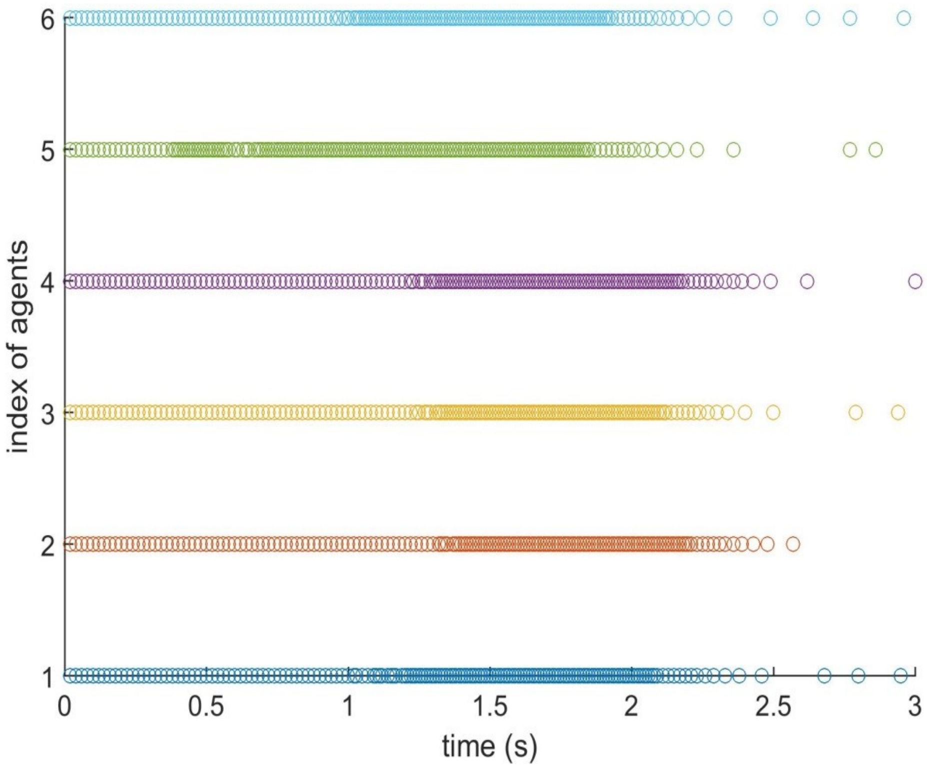


Fig. 4.11 Event instants of the exact-state based case (i)

(ii) $c_1 = \cdots = c_6 = 0.04$

In the sampled-state based case, the estimated settling time is $T_s = 41.70$ s. The sampled state $x_i(t_{k_i}^i)$ and event instants are shown in Fig. 4.12 and Fig. 4.13, the real settling time is $T = 2.01$ s. The total number of triggering times is 274, which refers to the number of updating times of the controller and trigger function.

In continuous event detection, the estimated settling time is $T_s = 9.63$ s. The sampled state $x_i(t_{k_i}^i)$ and event instants are shown in Fig. 4.14 and Fig. 4.15, the real settling time is $T = 2.02$ s. The total number of triggering times is 307, which refers to the number of updating times of the controller.

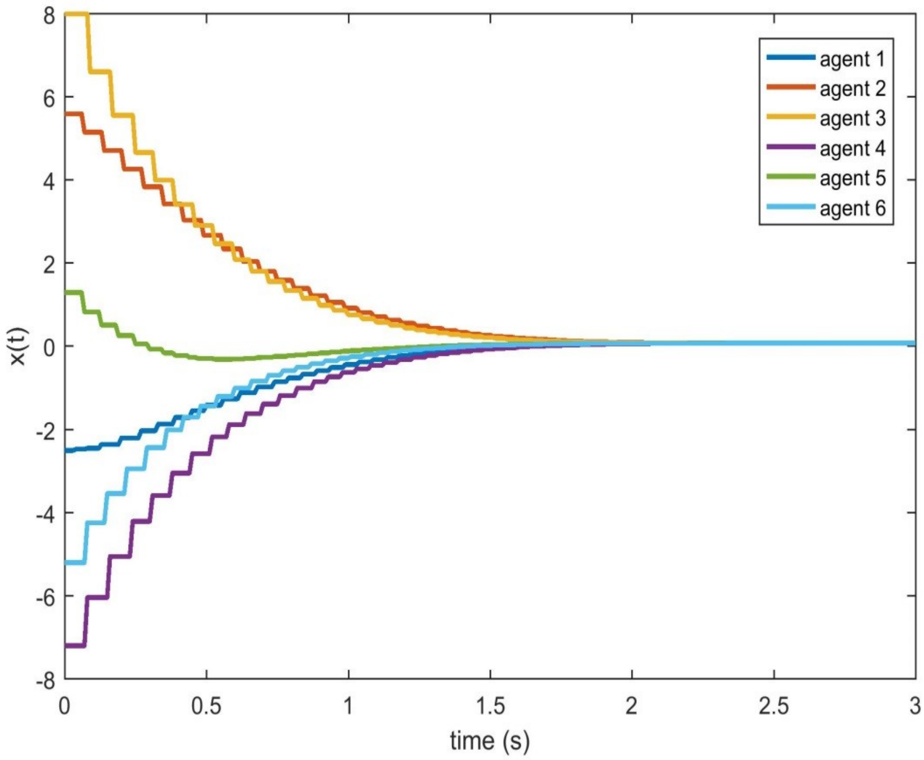


Fig. 4.12 The sampled state of the sampled-state based case (ii)

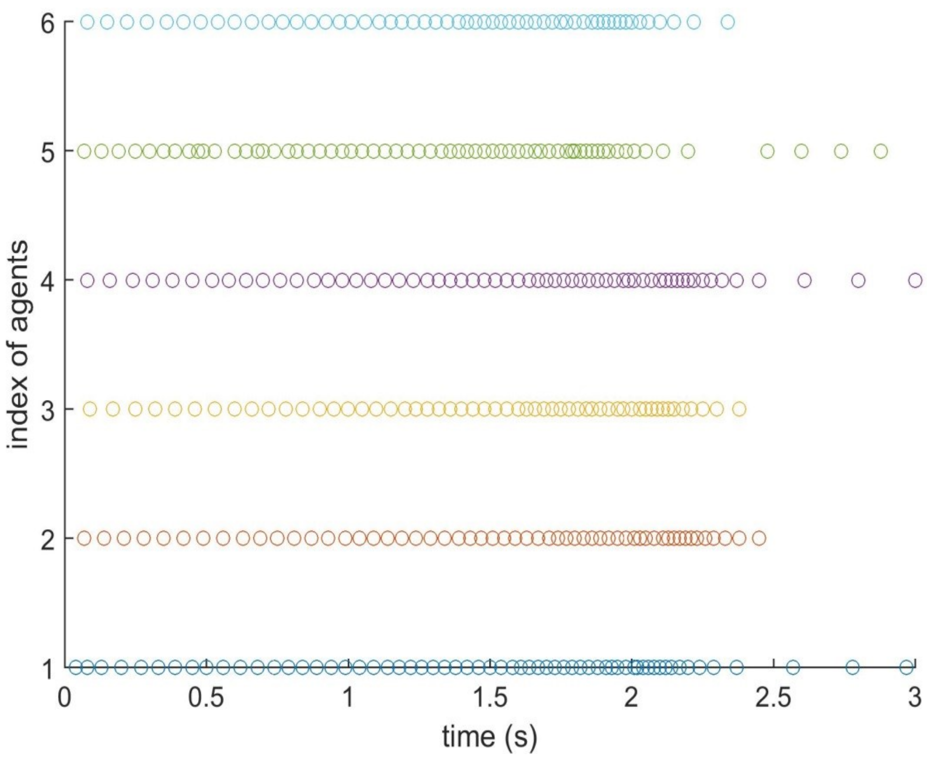


Fig. 4.13 Event instants of the sampled-state based case (ii)

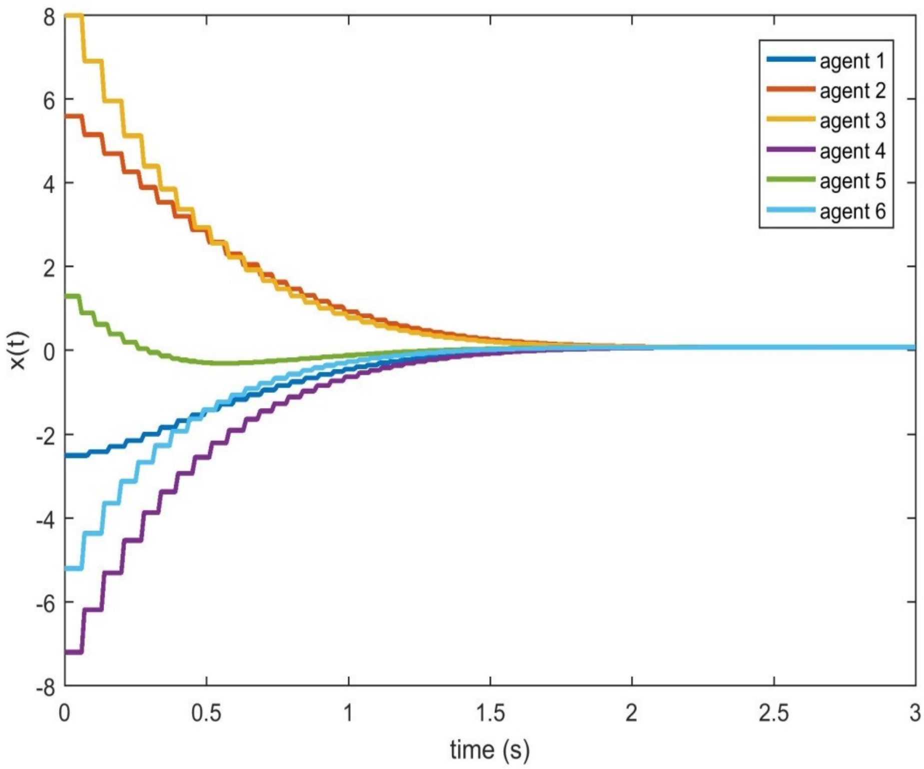


Fig. 4.14 The sampled state of the exact-state based case (ii)

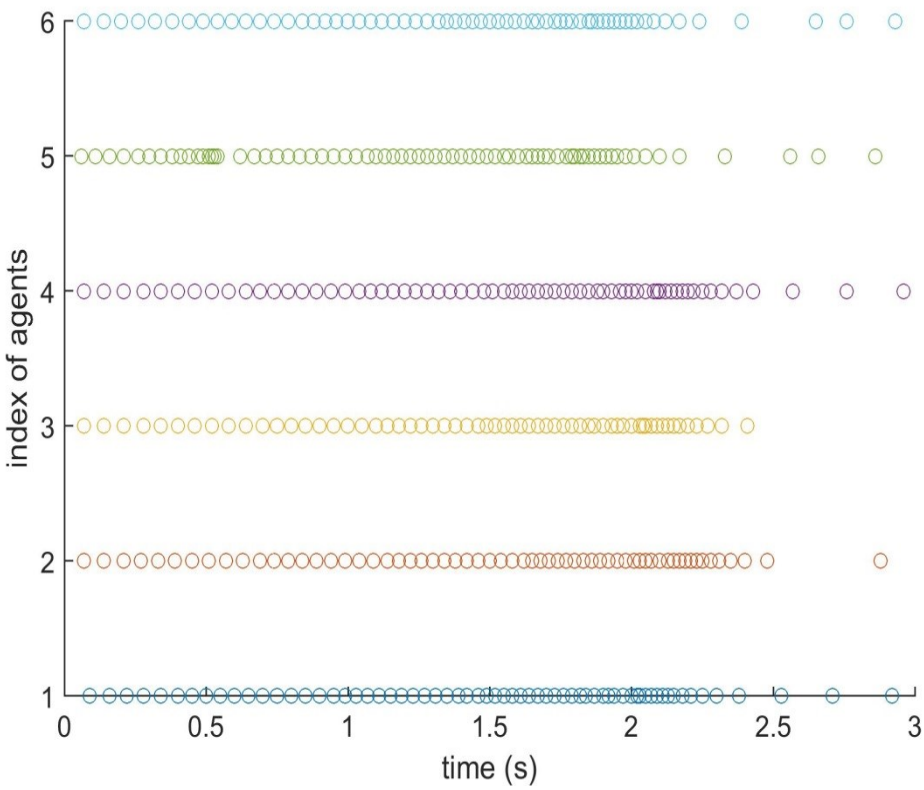


Fig. 4.15 Event instants of the exact-state based case (ii)

To have a better comparison, the change of $V(t)$ in both triggering function is shown in Fig. 4.16. The above examples show that the sampled-state based case has larger estimation error and similar dynamic performance compared with continuous event detection in [40]. However, it shows less the total trigger number in this condition. To verify the property, various undirected topologies are taken into consideration. The result of simulation is shown in Fig. 4.17, λ_2 in the x-axis refers to the second smallest eigenvalue of Laplacian matrix and it only represents different topologies. Fig. 4.17 reveals that the sampled-state based case reduces the total trigger number by 5.1%–16.5%, which means it requires less computational cost.

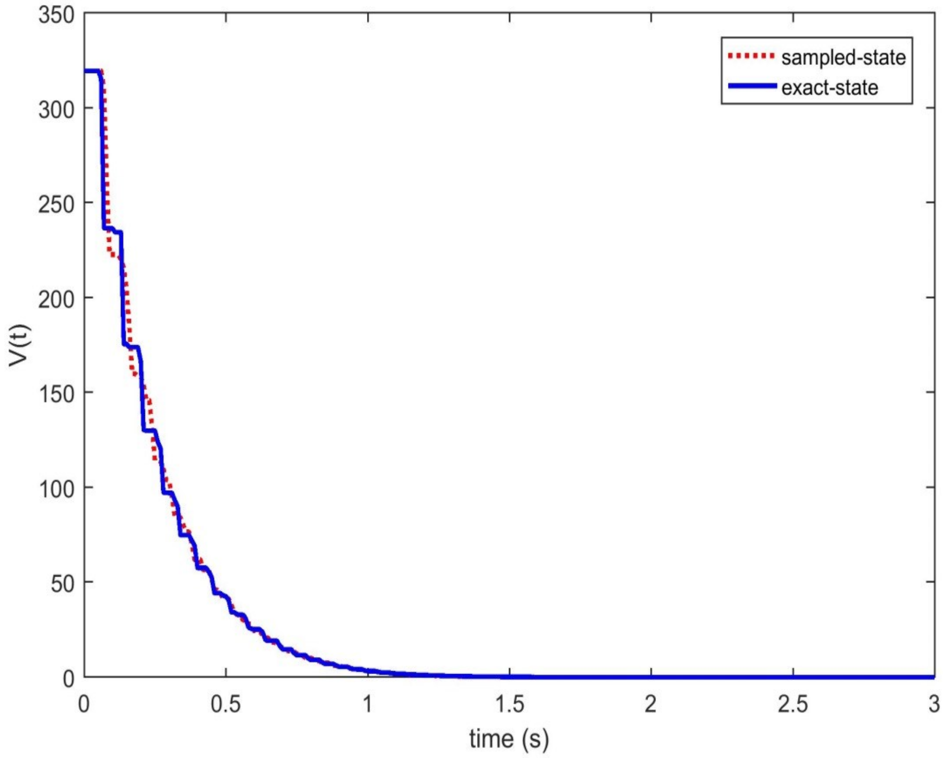


Fig. 4.16 Lyapunov function: the red line represents the sampled-state based case, and the blue line represents the exact-state based case

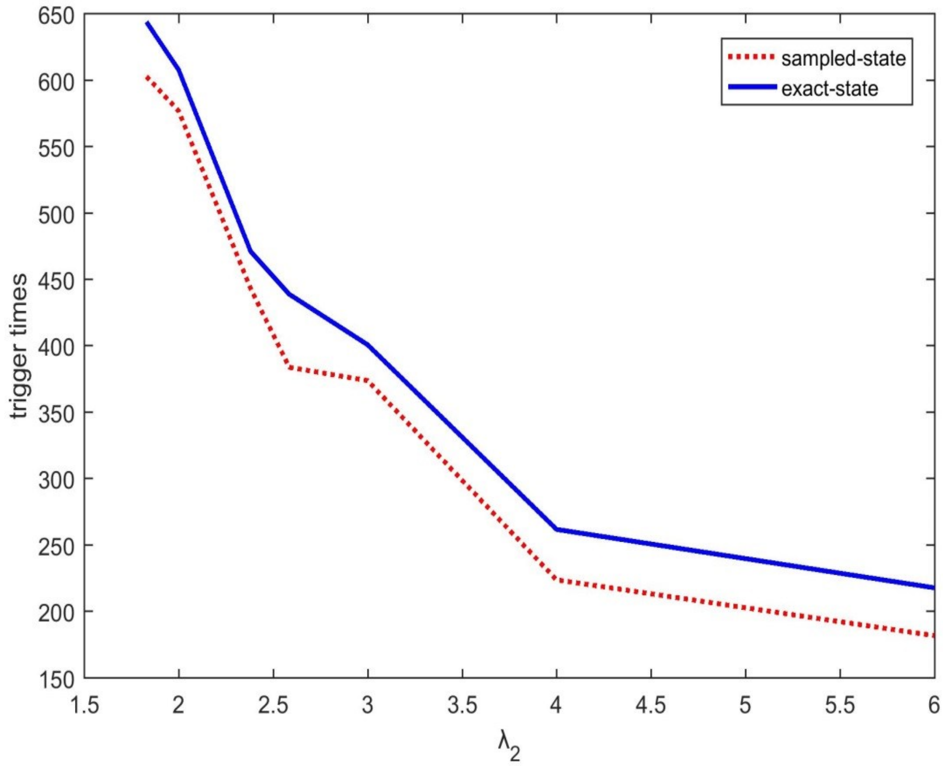


Fig. 4.17 The comparison of total trigger number

4.5 Conclusion

The application of the sampled-state based triggering function for finite-time consensus problem of MAS is studied in this section. Compared with the exact-state case, it not only saves communication load by converting continuous communication into event-triggered operation but also reduces the cost of computation by cutting down the number of triggering times. The existence of positive minimum inter-event time is guaranteed by setting a lower bound of state error. In this condition, Zeno behavior also can be avoided. What's more, there exists a tradeoff between triggering times and settling time in both exact-state and sampled-state based methods. The above results are verified by simulations.

Chapter 5 Time-based Event-triggered Control for Finite-time Consensus

5.1 Introduction

In this chapter, we focus on designing a time-based triggering function for finite-time consensus of MAS. Different from the case of state-based triggering function, the agents only require the state received from neighbors to update the controller and the triggering function doesn't need any communication information, which is much easier to achieve a desired balance between the convergence time and the number of triggering times. The relation between the settling time and the triggering condition is introduced. In addition, the existence of positive minimum inter-event time is analyzed. Finally, a comparison between state-based and time-based triggering function is described in simulations.

5.2 Time-based Triggering Function

Consider a group of N agents with an undirected and connected communication topology. The dynamics of the ith agent are given by

$$\dot{x}_i(t) = u_i(t), \tag{5.1}$$

where $x_i(t) \in \mathbb{R}$ is the state, $u_i(t) \in \mathbb{R}$ is the control input.

The corresponding controller is given as

$$u_i(t) = -\alpha \left[\sum_{j=1}^{N} a_{ij} \left(x_i(t_{k_i}^i) - x_j\left(t_{k_j}^j\right)\right)\right]^{[\mu]}, \tag{5.2}$$

where $t_{k_i}^i$ is the most recent triggering instant of agent i, a_{ij} is the ijth element in adjacency matrix and refers to the weight of communication between agent i and agent j, $\alpha > 0$ is the control gain, $0 < \mu < 1$.

For each agent i and $t > 0$, the state error is defined as

$$e_i(t) = x_i(t_{k_i}^i) - x_i(t),\ t \in \left[t_{k_i}^i, t_{k_i+1}^i\right).$$

Different from the state-dependent triggering function, we introduce a variable with its dynamic only relates to time.

The triggering function is defined as

$$f_i(t) = |e_i(t)|^2 - \omega(t). \tag{5.3}$$

where $\omega(t) \geq 0$ is a time-based threshold to be determined.

The triggering function implies that $|e_i(t)|^2 \leq \omega(t)$.

Remark 5.1. From the definition of triggering threshold, the agent i does not need the information from its neighbors. Therefore, each agent only requires the sampled state of its neighbors to update the control input. In this condition, the communication is event-triggered.

Consider the following Lyapunov function

$$V(t) = V_1\big(x(t)\big) + V_2\big(\omega(t)\big), \tag{5.4}$$

where $V_1\big(x(t)\big) = \frac{1}{2}x^T(t)Lx(t)$, $V_2\big(\omega(t)\big) = \beta_1\omega(t)$, $x(t) = [x_1(t), \cdots, x_N(t)]^T$, $\beta_1 > 0$.

Theorem 5.2. Suppose that the graph G is undirected and connected. The controller (5.2) is triggered by the function (5.3) with the threshold

$$\omega(t) = \left[\omega_0^{\frac{1-\mu}{2}} - \frac{1-\mu}{2}\beta_1^{\frac{\mu-1}{2}}\beta_2 t\right]^{\frac{2}{1-\mu}}, \tag{5.5}$$

where $\omega_0 = \omega(0)$, $\beta_2 > \frac{2\alpha}{1+\mu}N\lambda_N^{\mu+1}\beta_1^{-\frac{\mu+1}{2}}$, and when $t \geq \frac{2\beta_1^{\frac{1-\mu}{2}}\left(\omega_0^{\frac{1-\mu}{2}} - \varepsilon^{\frac{1-\mu}{2}}\right)}{\beta_2(1-\mu)}$, $\omega(t) = \varepsilon$, $\varepsilon > 0$ is the lower bound of $\omega(t)$.

Then the following statements hold:

(i) The MAS (5.1) achieves consensus within the estimated settling time

$$T_s = \frac{2V^{\frac{1-\mu}{2}}(x_0)}{\min\{\alpha_1, \alpha_4\}(1-\mu)},$$

where $\alpha_1 = \frac{\alpha\mu}{1+\mu}(2\lambda_2)^{\frac{\mu+1}{2}}$, $\alpha_4 = \beta_2 - \frac{2\alpha}{1+\mu}N\lambda_N^{\mu+1}\beta_1^{-\frac{\mu+1}{2}}$.

(ii) The positive minimum inter-event time exists if and only if $\omega(t)$ is lower bounded.

Proof. (i) The proof is divided into following three steps.

At first, the error is only related to a time-dependent variable according to the triggering function (5.3). Therefore, $\dot{V}_1(x(t))$ can be divided into two parts: the first part is a function of $x(t)$, the second part is a function of $\omega(t)$. The first part is corresponding to $-\alpha_1 {V_1}^{\mu}(x(t))$ in the condition of finite-time stability in **Theorem 2.14**.

Secondly, express $\dot{V}_2(t)$ by the function of $\omega(t)$, the combination of this function and the second part of $\dot{V}_1(x(t))$ in previous step can be expressed as a function of $V_2(t)$.

Finally, the result of second step is corresponding to $-\alpha_2 {V_2}^{\mu}(t)$ in the condition of finite-time stability in **Theorem 2.14**. Hence, the restriction of $\omega(t)$ and the estimated settling time can be acquired.

Here it contains two conditions which depend the value of $|X_i(t)|$ and $|E_i(t)|$, where $X_i(t) = \sum_{j=1}^{N} a_{ij}\left(x_i(t) - x_j(t)\right)$ refers to the ith element of $Lx(t)$, $E_i(t) = \sum_{j=1}^{N} a_{ij}\left(e_i(t) - e_j(t)\right)$ refers to the ith element of $Le(t)$ and $e(t) = [e_1(t), \cdots, e_N(t)]^T$.

1) If $|E_i(t)| \le |X_i(t)|$

Firstly, analyze $\dot{V}_1(x(t))$

$$
\begin{aligned}
\dot{V}_1(x(t)) &= -\alpha \sum_{i=1}^{N} \left[\sum_{j=1}^{N} a_{ij}\left(x_i(t) - x_j(t)\right)\right] \\
&\quad * \left[\sum_{j=1}^{N} a_{ij}\left(x_i(t_{k_i}^{i}) - x_j\left(t_{k_j}^{j}\right)\right)\right]^{[\mu]} \\
&= -\alpha \sum_{i=1}^{N} \left[\sum_{j=1}^{N} a_{ij}\left(x_i(t) - x_j(t)\right)\right] \\
&\quad * \left[\sum_{j=1}^{N} a_{ij}\left(x_i(t) - x_j(t) + e_i(t) - e_j(t)\right)\right]^{[\mu]} \\
&= -\alpha \sum_{i=1}^{N} \{X_i(t)[X_i(t) + E_i(t)]^{[\mu]}\}.
\end{aligned}
\tag{5.6}
$$

According to **Lemma 2.12**, (5.6) has following property,

$$
\dot{V}_1(x(t)) \le -\alpha \sum_{i=1}^{N} |X_i(t)|^{\mu+1} + \alpha \sum_{i=1}^{N} |X_i(t)||E_i(t)|^{\mu}. \tag{5.7}
$$

Use **Lemma 2.10** to divide (5.7) into two parts: the first part is a function of $x(t)$, the second part is a function of $e(t)$,

$$\dot{V}_1(x(t)) \le -\frac{\alpha\mu}{1+\mu}\sum_{i=1}^{N}|X_i(t)|^{\mu+1} + \frac{\alpha\mu}{1+\mu}\sum_{i=1}^{N}|E_i(t)|^{\mu+1}. \tag{5.8}$$

Apply **Lemma 2.11**, (5.8) can be written as

$$\dot{V}_1(x(t)) \le -\frac{\alpha\mu}{1+\mu}\{\sum_{i=1}^{N}|X_i(t)|^2\}^{\frac{\mu+1}{2}} + \frac{\alpha\mu}{1+\mu}N^{\frac{1-\mu}{2}}\left\{\sum_{i=1}^{N}[E_i(t)]^2\right\}^{\frac{\mu+1}{2}}. \tag{5.9}$$

Recalling the property of Laplacian matrix (2.6) and triggering function (5.3), the two parts in (5.9) can be converted into the function of $V_1(t)$ and the function of $V_2(t)$ separately,

$$\begin{aligned}\dot{V}_1(x(t)) &\le -\frac{\alpha\mu}{1+\mu}[x^T(t)L^TLx(t)]^{\frac{\mu+1}{2}} + \frac{\alpha\mu}{1+\mu}N^{\frac{1-\mu}{2}}[e^T(t)L^TLe(t)]^{\frac{\mu+1}{2}} \\ &\le -\frac{\alpha\mu}{1+\mu}[2\lambda_2V_1(t)]^{\frac{\mu+1}{2}} + \frac{\alpha\mu}{1+\mu}N^{\frac{1-\mu}{2}}[N\lambda_N^2\omega(t)]^{\frac{\mu+1}{2}} \\ &\le -\frac{\alpha\mu}{1+\mu}[2\lambda_2V_1(t)]^{\frac{\mu+1}{2}} + \frac{\alpha\mu}{1+\mu}N\lambda_N^{\mu+1}\beta_1^{-\frac{\mu+1}{2}}[V_2(t)]^{\frac{\mu+1}{2}}.\end{aligned} \tag{5.10}$$

Secondly, according to the triggering threshold (5.5), $\dot{V}_2(x(t))$ is written as,

$$\dot{V}_2(t) = \beta_1\dot{\omega}(t) = -\beta_2[V_2(t)]^{\frac{\mu+1}{2}}. \tag{5.11}$$

Then take (5.10) and (5.11) into $\dot{V}(t)$,

$$\dot{V}(t) \le -\alpha_1(V_1(t))^{\frac{\mu+1}{2}} - \alpha_2(V_2(t))^{\frac{\mu+1}{2}}, \tag{5.12}$$

where $\alpha_1 = \frac{\alpha\mu}{1+\mu}(2\lambda_2)^{\frac{\mu+1}{2}}$, $\alpha_2 = \beta_2 - \frac{\alpha\mu}{1+\mu}N\lambda_N^{\mu+1}\beta_1^{-\frac{\mu+1}{2}}$.

According to **Theorem 2.14**, the system reaches finite-time consensus within $T_1 = \frac{2V^{\frac{1-\mu}{2}}(x_0)}{\min\{\alpha_1,\alpha_2\}(1-\mu)}$.

2) If $|E_i(t)| > |X_i(t)|$

Firstly, analyze $\dot{V}_1(x(t))$

$$\dot{V}_1(x(t)) = -\alpha\sum_{i=1}^{N}\left[\sum_{j=1}^{N}a_{ij}\left(x_i(t) - x_j(t)\right)\right]$$

$$
\begin{aligned}
&* \left[\sum_{j=1}^{N} a_{ij}\left(x_i\left(t_{k_i}^{i}\right) - x_j\left(t_{k_j}^{j}\right)\right)\right]^{[\mu]} \\
&= -\alpha \sum_{i=1}^{N} \left[\sum_{j=1}^{N} a_{ij}\left(x_i\left(t_{k_i}^{i}\right) - x_j\left(t_{k_j}^{j}\right) - e_i(t) + e_j(t)\right)\right] \\
&* \left[\sum_{j=1}^{N} a_{ij}\left(x_i\left(t_{k_i}^{i}\right) - x_j\left(t_{k_j}^{j}\right)\right)\right]^{[\mu]} \\
&= -\alpha \sum_{i=1}^{N} \left\{\left[\hat{X}_i(t) - E_i(t)\right]\left[\hat{X}_i(t)\right]^{[\mu]}\right\}, \qquad (5.13)
\end{aligned}
$$

where $\hat{X}_i(t) = \sum_{j=1}^{N} a_{ij}\left(\hat{x}_i(t) - \hat{x}_j(t)\right) = \sum_{j=1}^{N} a_{ij}\left(x_i\left(t_{k_i}^{i}\right) - x_j\left(t_{k_j}^{j}\right)\right)$.

According to **Lemma 2.10**, (5.13) has following property,

$$
\dot{V}_1\big(x(t)\big) \le -\frac{\alpha}{1+\mu}\sum_{i=1}^{N}\left|\hat{X}_i(t)\right|^{\mu+1} + \frac{\alpha}{1+\mu}\sum_{i=1}^{N}|E_i(t)|^{\mu+1}. \qquad (5.14)
$$

Apply **Lemma 2.13** to convert $\hat{X}_i(t)$ into $X_i(t)$ in (5.14),

$$
\begin{aligned}
\dot{V}_1\big(x(t)\big) &\le -\frac{\alpha}{1+\mu}\sum_{i=1}^{N}|X_i(t) + E_i(t)|^{\mu+1} + \frac{\alpha}{1+\mu}\sum_{i=1}^{N}|E_i(t)|^{\mu+1} \\
&\le -\frac{\alpha}{1+\mu}\sum_{i=1}^{N}|X_i(t)|^{\mu+1} + \frac{2\alpha}{1+\mu}\sum_{i=1}^{N}|E_i(t)|^{\mu+1}. \qquad (5.15)
\end{aligned}
$$

Use **Lemma 2.11**, (5.15) can be written as

$$
\dot{V}_1\big(x(t)\big) \le -\frac{\alpha}{1+\mu}\left\{\sum_{i=1}^{N}|X_i(t)|^2\right\}^{\frac{\mu+1}{2}} + \frac{2\alpha}{1+\mu}N^{\frac{1-\mu}{2}}\left\{\sum_{i=1}^{N}[E_i(t)]^2\right\}^{\frac{\mu+1}{2}}. \qquad (5.16)
$$

Similarly, the two parts in (5.16) can be converted into the function of $V_1(t)$ and the function of $V_2(t)$ separately according to the property of Laplacian matrix (2.6) and triggering function (5.3),

$$
\begin{aligned}
\dot{V}_1\big(x(t)\big) &\le -\frac{\alpha}{1+\mu}\left[x^T(t)L^TLx(t)\right]^{\frac{\mu+1}{2}} + \frac{2\alpha}{1+\mu}N^{\frac{1-\mu}{2}}\left[e^T(t)L^TLe(t)\right]^{\frac{\mu+1}{2}} \\
&\le -\frac{\alpha}{1+\mu}\left[2\lambda_2 V_1(t)\right]^{\frac{\mu+1}{2}} + \frac{2\alpha}{1+\mu}N^{\frac{1-\mu}{2}}\left[N\lambda_N^2\omega(t)\right]^{\frac{\mu+1}{2}} \\
&\le -\frac{\alpha}{1+\mu}\left[2\lambda_2 V_1(t)\right]^{\frac{\mu+1}{2}} + \frac{2\alpha}{1+\mu}N\lambda_N^{\mu+1}\beta_1^{-\frac{\mu+1}{2}}\left[V_2(t)\right]^{\frac{\mu+1}{2}}. \qquad (5.17)
\end{aligned}
$$

Secondly, according to the triggering threshold (5.5), $\dot{V}_2\big(x(t)\big)$ can be written as,

$$
\dot{V}_2(t) = \beta_1\dot{\omega}(t) = -\beta_2[V_2]^{\frac{\mu+1}{2}}. \qquad (5.18)
$$

Then take (5.17) and (5.18) into $\dot{V}(t)$,

$$\dot{V}(t) \leq -\alpha_3(V_1)^{\frac{\mu+1}{2}} - \alpha_4(V_2)^{\frac{\mu+1}{2}}, \tag{5.19}$$

where $\alpha_3 = \frac{\alpha}{1+\mu}(2\lambda_2)^{\frac{\mu+1}{2}}$, $\alpha_4 = \beta_2 - \frac{2\alpha}{1+\mu}N\lambda_N^{\mu+1}\beta_1^{-\frac{\mu+1}{2}}$.

According to **Theorem 2.14**, the system reaches finite-time consensus within $T_2 = \frac{2V^{\frac{1-\mu}{2}}(x_0)}{\min\{\alpha_3,\alpha_4\}(1-\mu)}$.

Obviously, $\min\{\alpha_1, \alpha_2, \alpha_3, \alpha_4\} = \min\{\alpha_1, \alpha_4\}$. As a result, the estimated settling time is not greater than $T_s = \frac{2V^{\frac{1-\mu}{2}}(x_0)}{\min\{\alpha_1,\alpha_4\}(1-\mu)}$ in any conditions.

(ii) (Sufficiency) Assume agent i is triggered at $t_{k_i}^i$, then $x_i\left(t_{k_i}^i\right)$ remains unchanged before $t_{k_i+1}^i$.

$$|e_i(t)| = \left|\int_{t_{k_i}^i}^{t} \dot{e}_i(s)ds\right| = \left|\int_{t_{k_i}^i}^{t} u_i(s)ds\right| \leq \int_{t_{k_i}^i}^{t} \alpha|X_i(s) + E_i(s)|^{\mu}ds$$
$$\leq \int_{t_{k_i}^i}^{t} \alpha(\|X(s)\|^{\mu} + \|E(s)\|^{\mu})ds,$$

where $t_{k_i}^i < t \leq t_{k_i+1}^i$, $X(t) = [X_1(t), \cdots, X_N(t)]^T$, $E(t) = [E_1(t), \cdots, E_N(t)]^T$.

Suppose the lower bound of $\omega(t)$ is ω_d. Because the system is finite-time consensus, $\|X(t)\| \leq \|X(0)\|$ and $\|E(t)\| \leq \omega_0$. There must exist positive constants α_x and α_e such that $\|X(s)\| \leq \alpha_x\omega_d$ and $\|E(s)\| \leq \alpha_e\omega_d$. So τ_i is not smaller than the solution of the equation $\alpha[(\alpha_x\omega_d)^{\mu} + (\alpha_e\omega_d)^{\mu}]\tilde{\tau} = \omega_d$. Therefore, τ_i has a positive lower bound $\tilde{\tau}$, and Zeno behavior does not occur.

(Necessity) The triggering function (5.3) implies that

$$|e_i(t)| \leq \omega^{\frac{1}{2}}(t) = \omega^{\frac{1}{2}}\left(t_{k_i}^i\right) + \int_{t_{k_i}^i}^{t} \frac{1}{2}\omega^{-\frac{1}{2}}(s)\dot{\omega}(s)ds.$$

The above inequality can be written as

$$\left|\int_{t_{k_i}^i}^{t} -\dot{x}_i(s)\,ds\right| - \int_{t_{k_i}^i}^{t} \frac{1}{2}\omega^{-\frac{1}{2}}(s)\dot{\omega}(s)ds \leq \omega^{\frac{1}{2}}\left(t_{k_i}^i\right).$$

The lower bound of τ_i exists only when there exists a positive constant

$\varphi = \frac{\omega^{\frac{1}{2}}\left(t_{k_i}^i\right)}{\left|\dot{x}_i\left(t_{k_i}^i\right)\right| - \frac{1}{2}\omega\left(t_{k_i}^i\right)^{-\frac{1}{2}}\dot{\omega}\left(t_{k_i}^i\right)}$. Take the threshold (5.5) into it and the result is given as follows, $\varphi = \frac{\omega^{\frac{1}{2}}\left(t_{k_i}^i\right)}{\left|\dot{x}_i\left(t_{k_i}^i\right)\right| + \frac{1}{2}\beta_1^{\frac{\mu-1}{2}}\beta_2\omega^{\frac{\mu}{2}}\left(t_{k_i}^i\right)}$.

Therefore, as $t_{k_i}^i \to \frac{2\omega_0^{\frac{1-\mu}{2}}\beta_1^{\frac{1-\mu}{2}}}{(1-\mu)\beta_2}$, $\varphi \to 0_+$. In other words, the positive minimum inter-event time does not exist. □

Remark 5.3. According to the definition of the lower bound of triggering threshold, the estimated settling time T_s is greater than the time $T = \{t: \omega(t) = \varepsilon\}$, i.e., $T_s > \frac{2\beta_1\left(\omega_0^{\frac{1-\mu}{2}} - \varepsilon^{\frac{1-\mu}{2}}\right)}{\beta_2(1-\mu)}$. Thus, the period of event-triggered control is divided into two parts: 1. triggering threshold is a time-dependent value, 2. triggering threshold is a fixed value ε. Although the time interval of time-based triggered control depends on the parameters of topology, it can be adjusted by β_1.

Remark 5.4. Here we suppose $\dot{V}_2(t) = \beta_1\dot{\omega}(t) = -\beta_2[V_2(t)]^{\frac{\mu+1}{2}}$ for the sake of calculating the restriction of $\omega(t)$, which has the same exponent of $V_2(t)$ with the second part of $\dot{V}_1(x(t)) \le -\frac{\alpha}{1+\mu}[2\lambda_2 V_1(t)]^{\frac{\mu+1}{2}} + \frac{2\alpha}{1+\mu}N\lambda_N^{\mu+1}\beta_1^{-\frac{\mu+1}{2}}[V_2(t)]^{\frac{\mu+1}{2}}$. However, there might exist more general condition of $\dot{\omega}(t)$. It still requires further research.

5.3 Simulation

In this section, we give a comparison between state-based and time-based event-triggered control by simulation.

Consider the MAS consists of 6 agents with the network given in Fig. 5.1, and the weight of each edge $a_{ij} = 1$ for $i,j = 1,\cdots,6$. The second smallest and largest eigenvalues of the corresponding Laplacian matrix L are $\lambda_2 = 1$ and $\lambda_N = 4$ respectively. The dynamics and controller of each agent are defined as (5.1) and (5.2), where $\alpha = 1$ and $\mu = 0.8$. Set initial state $x(0) = [-1.5, 0.6, 2, -2.2, 1.3, -0.2]^T$. Both state-based and time-based triggering thresholds share the same lower bound $\varepsilon = 10^{-4}$, i.e., the upper bound of $\left|x_i\left(t_{k_i}^i\right) - x_i(t)\right|$ is no smaller than 10^{-2}. In this condition, the real settling time is $T = \inf\left\{t: \frac{1}{2} x^T(t) L x(t) \leq 1.2 \times 10^{-3}\right\}$ according to the property of Laplacian matrix in (2.6).

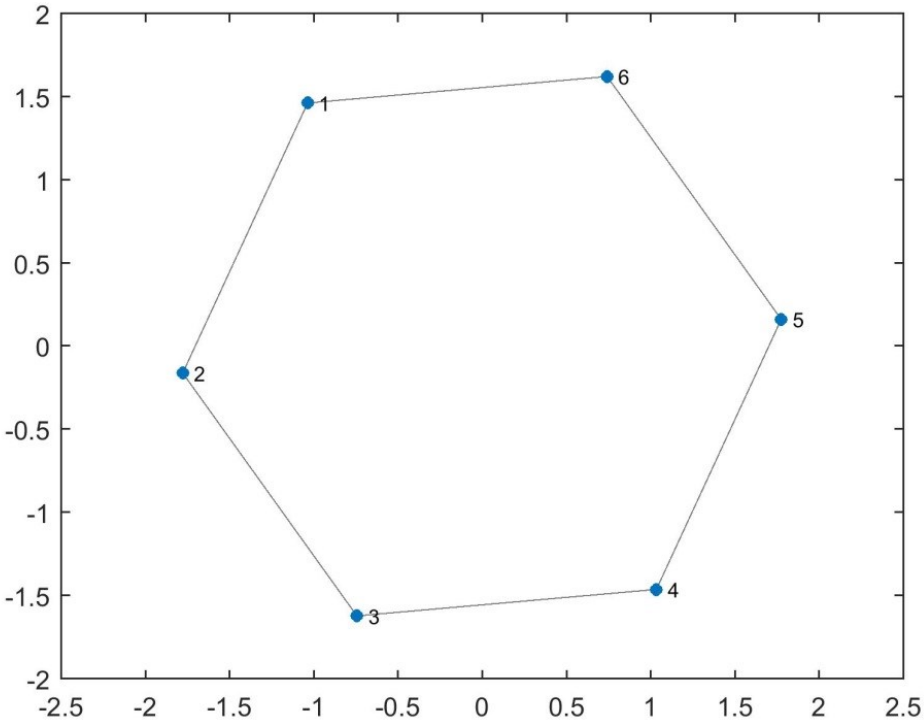


Fig. 5.1 The communication network

5.4.1 State-based Triggering Function

Based on the research in chapter 4, this section is divided into the exact-state based and sampled-state based.

(i) Exact-state based case

Consider **Theorem 2.8**, the upper bound of the threshold c_i in triggering function (4.4) is $0 < c_{\max} < 0.11$, set $c_1 = \cdots = c_6 = 0.03$. Then, the estimated settling time is $T_s = 17.92$ s. The sampled state $x_i\left(t_{k_i}^i\right)$ and event instants are shown in Fig. 5.2 and Fig. 5.3, the real settling time is $T = 2.03$ s. The total number of triggering times is 213, which refers to the updating times of the controller and triggering function.

(ii) Sampled-state based case

Consider **Theorem 4.3**, the upper bound of the threshold c_i in triggering function (4.6) is $0 < c_{\max} < 0.053$, set $c_1 = \cdots = c_6 = 0.03$. Then, the estimated settling time is $T_s = 36.84$ s. The sampled state $x_i\left(t_{k_i}^i\right)$ and event instants are shown in Fig. 5.4 and Fig. 5.5, the real settling time is $T = 2.03$ s. The total number of triggering times is 207, which refers to the updating times of the controller and triggering function.

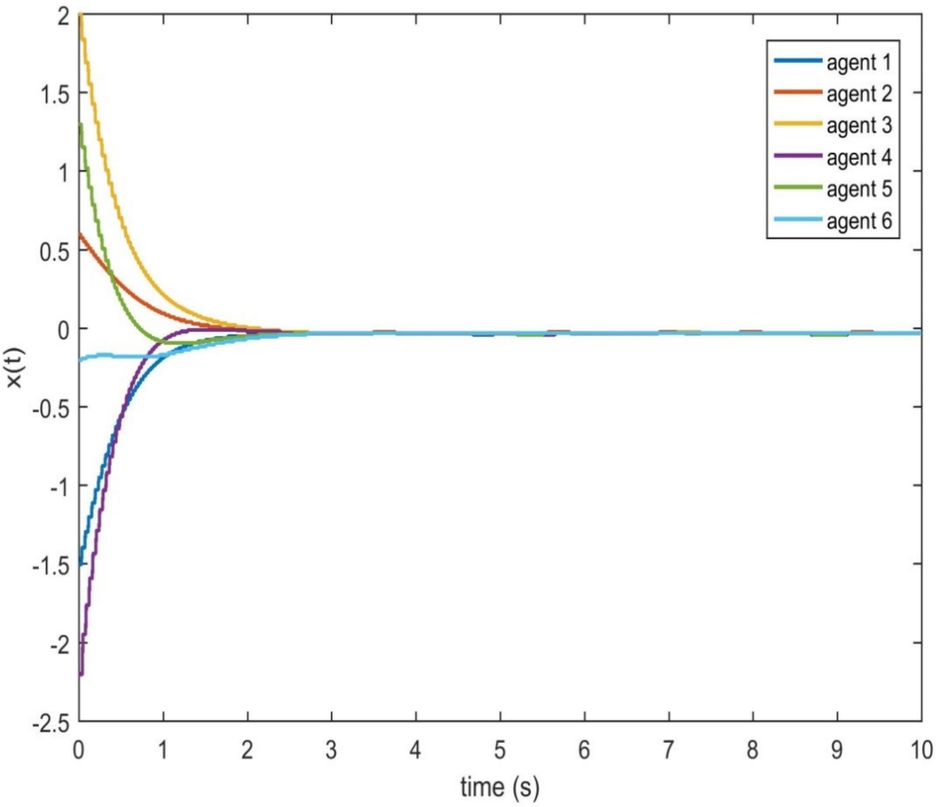


Fig. 5.2 The sampled state of exact-state based case

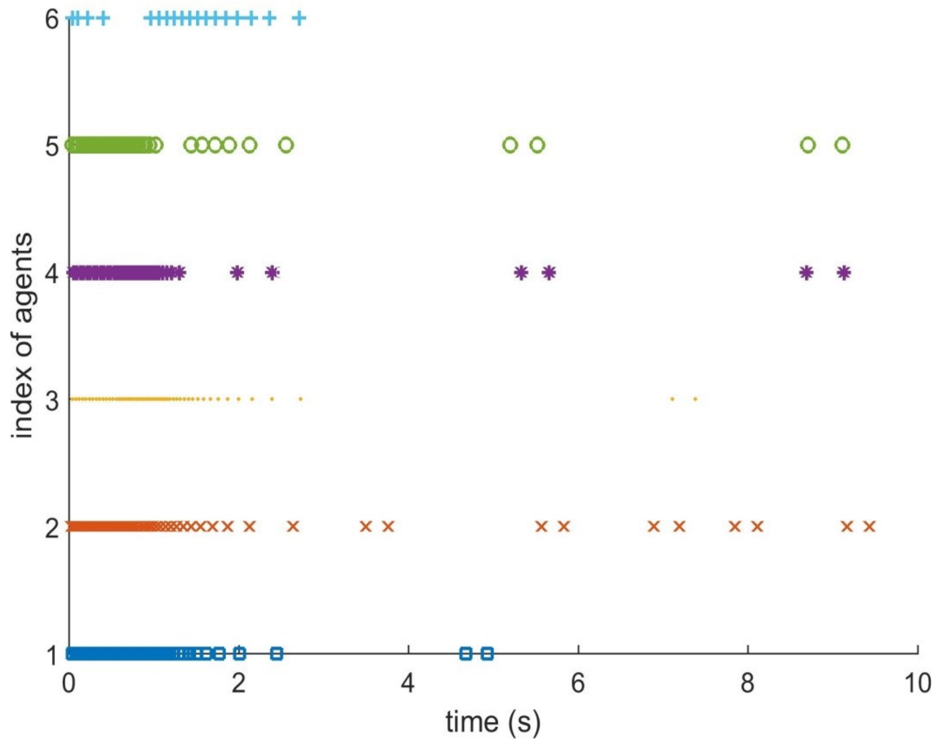


Fig. 5.3 Event instants of exact-state based case

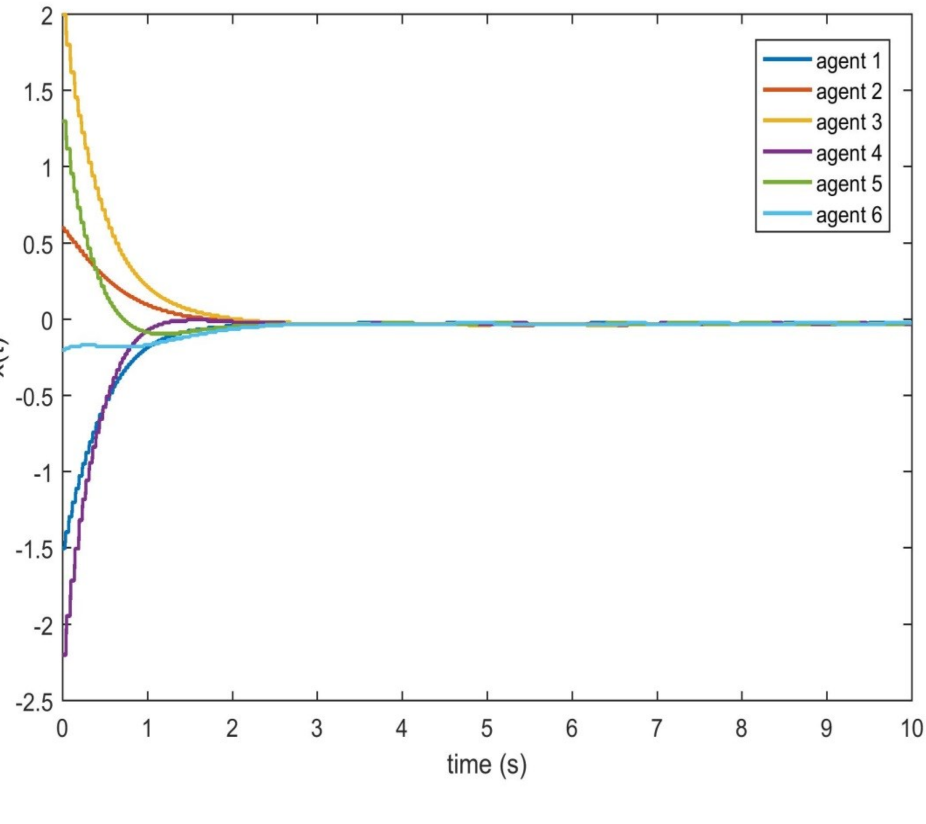


Fig. 5.4 The sampled state of sampled-state based case

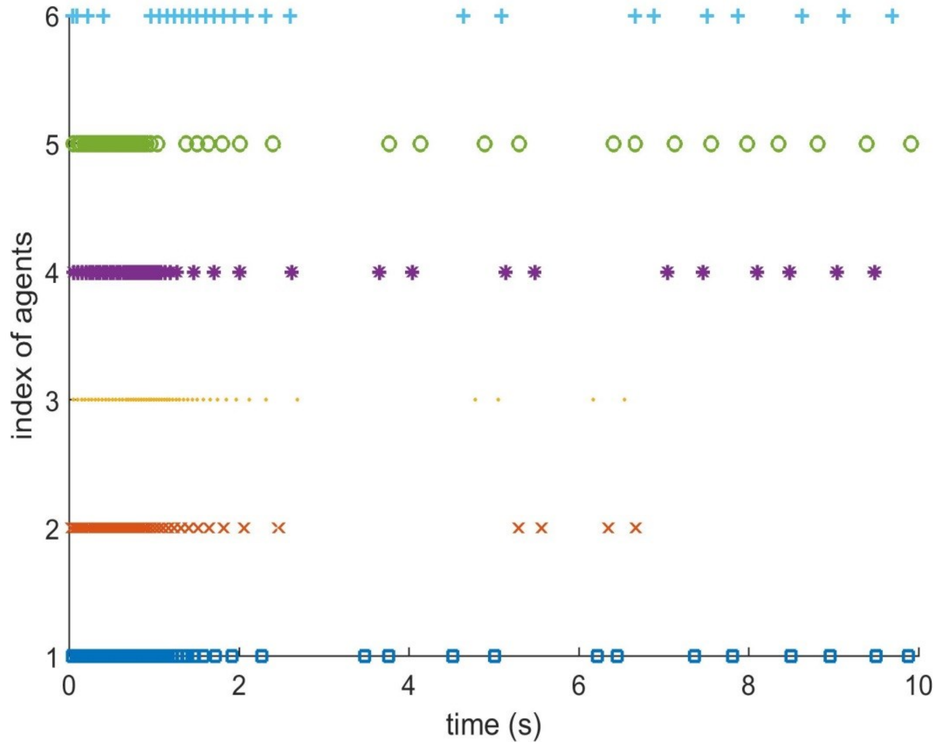


Fig. 5.5 Event instants of sampled-state based case

5.4.2 Time-based Triggering Function

In **Theorem 5.2**, set $\beta_1 = 60$, the lower bound of the parameter β_2 in triggering threshold (5.5) is $\beta_2 \geq 2.03$, set $\omega_0 = 1$ and $\beta_2 = 4.5$. Then, the estimated settling time is $T_s = 28.97$ s. The sampled state $x_i\left(t_{k_i}^i\right)$ and event instants are shown in Fig. 5.6 and Fig. 5.7, the real settling time is $T = 1.93$ s. The total number of triggering times is 77, which refers to the updating times of the controller.

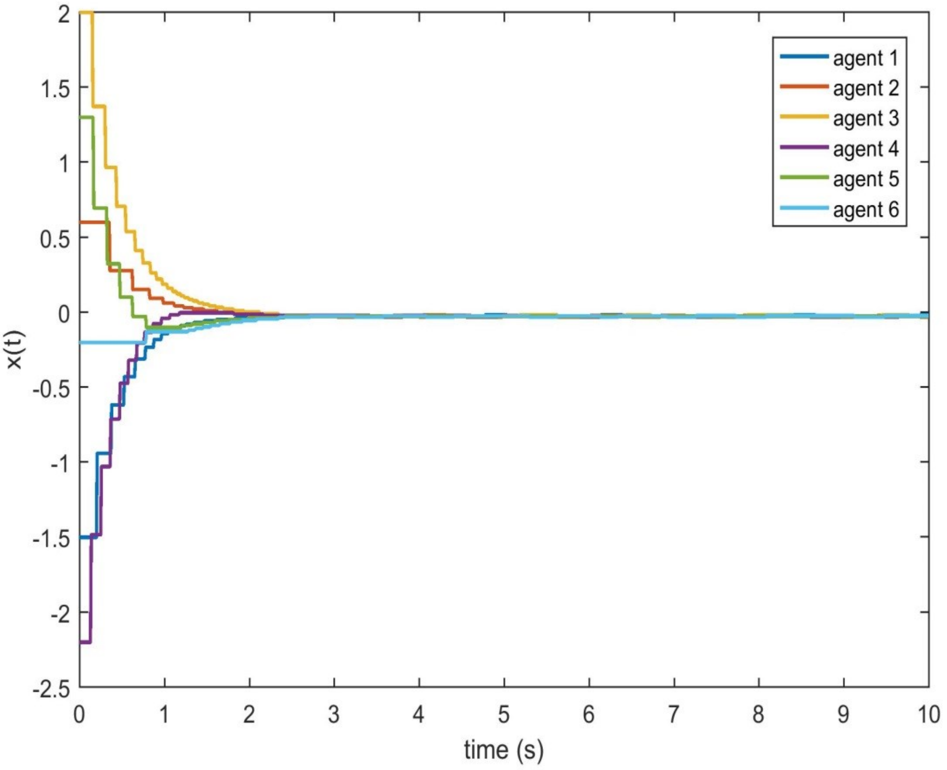


Fig. 5.6 The sampled state of time-based case

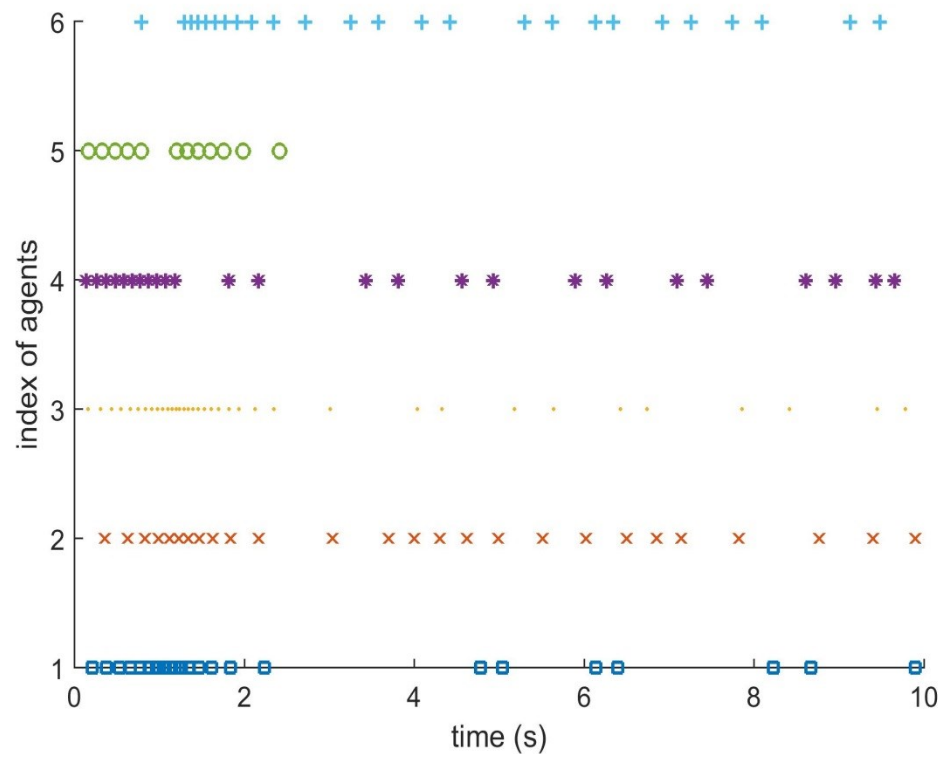


Fig. 5.7 Event instants of time-based case

In this example, the time-based triggering function has medium estimation error and the least real settling time compared with exact-state and sampled-state based case. What's more, it has the least triggering times, i.e., it requires the least computation and communication resource.

5.4.3 Comparison between Different Cases

However, the above properties of time-based method are based on a simple case. More general comparisons should be taken to observe whether these properties still hold for other cases. At first, different c_i is selected to observe the estimated and real settling time, and the number of triggering times, other parameters stay unchanged. The results are shown in Fig. 5.8 – Fig. 5.10. In Fig. 5.8, estimated settling time for different c_i shows that the time-based method only has medium estimation error for certain c_i. In Fig. 5.9, the time-based method cuts down the real settling time by 2.5%–5.4% compared with state-based methods. In Fig. 5.10, the time-based method only requires 24.8%–68.8% triggering times of exact-state based method, and 25.6%–47.8% triggering times of sampled-state based method.

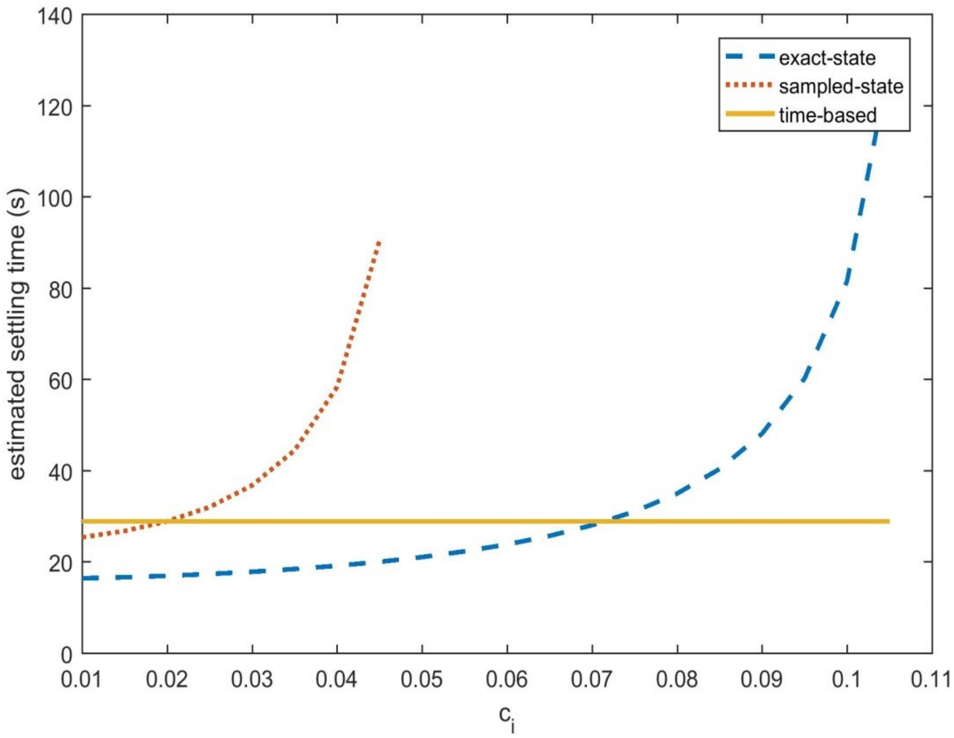


Fig. 5.8 Estimated settling time for different c_i

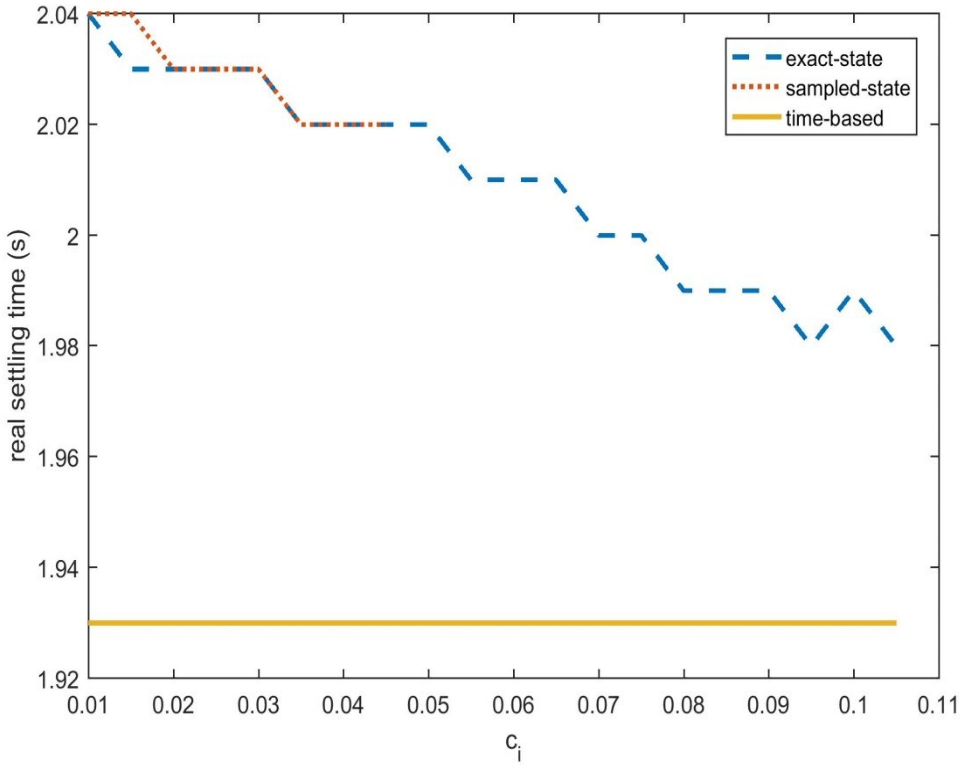


Fig. 5.9 Real settling time for different c_i

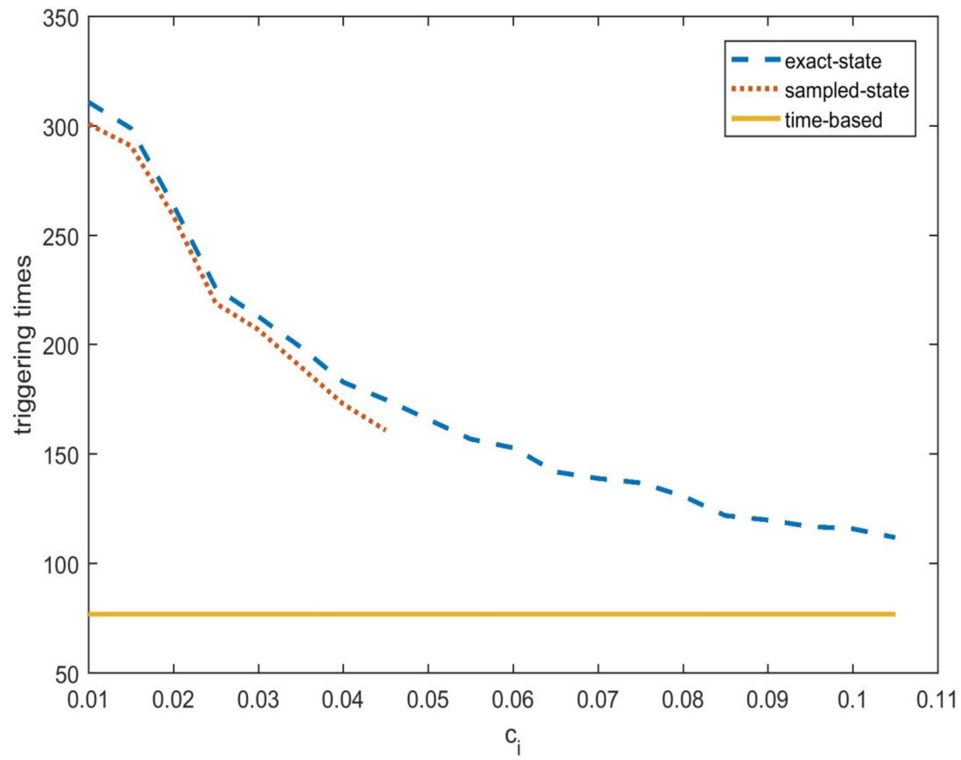


Fig. 5.10 Number of triggering times for different c_i

Then, different topologies are chosen to observe the estimated and real settling time, and the number of triggering times, other parameters stay unchanged ($c_1 = \cdots = c_6 = 0.03$). The results are shown in Fig. 5.11 – Fig. 5.13, where λ_N is the largest eigenvalue of Laplacian matrix and it only represents different topologies. In Fig. 5.11, the time-based method has medium estimation error for certain topologies. In Fig. 5.12, the time-based method cuts down the real settling time by 4.9%–10.1% compared with state-based methods. In Fig. 5.13, the time-based method only requires 18.6%–36.2% triggering times of exact-state based method, and 19.7%–37.2% triggering times of sampled-state based method.

Although the time-based method has the best performance in real settling time and triggering times, it is quite difficult to adjust the parameters β_1 and β_2 in triggering threshold (5.5). Because the change of time-based triggering threshold (5.5) is independent from the system state, there might be different optimal dynamics of triggering threshold for different initial state and system dynamics.

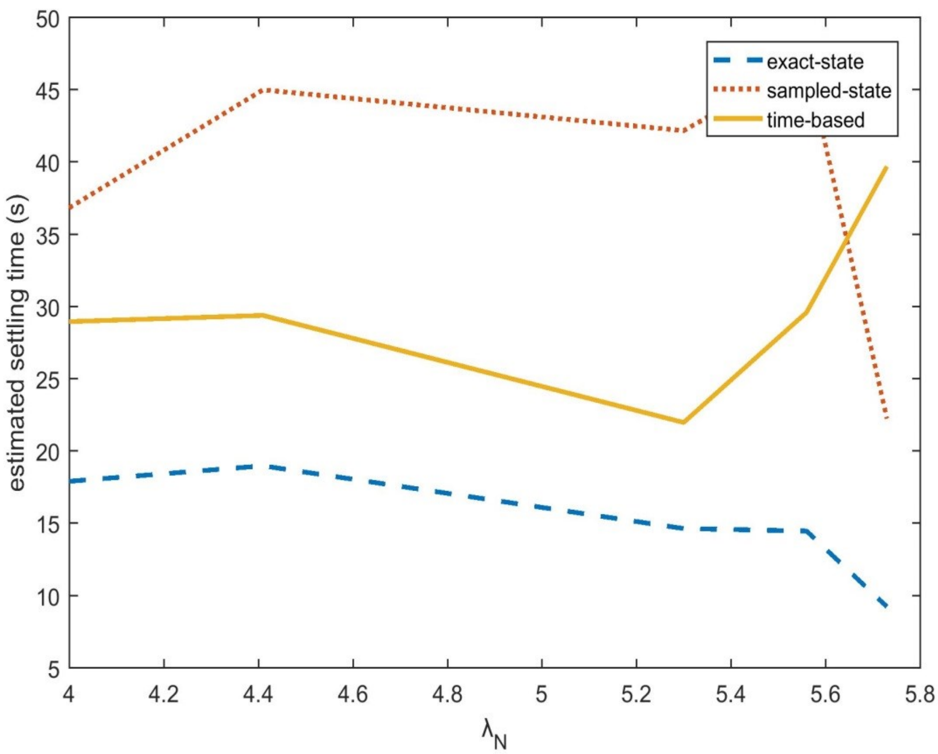


Fig. 5.11 Estimated settling time for different topologies

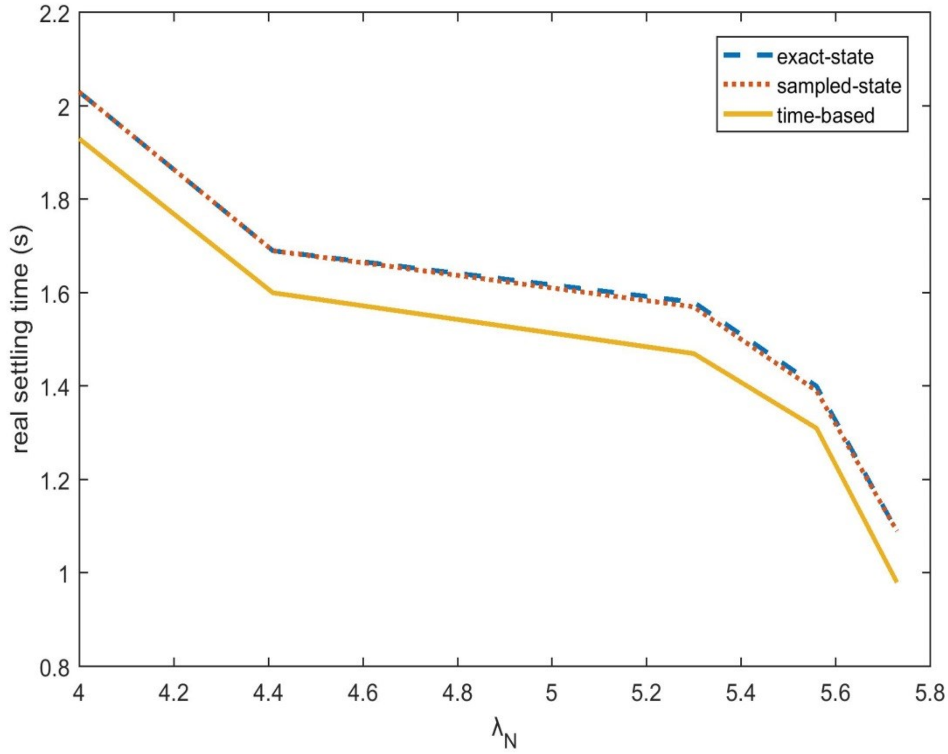


Fig. 5.12 Real settling time for different topologies

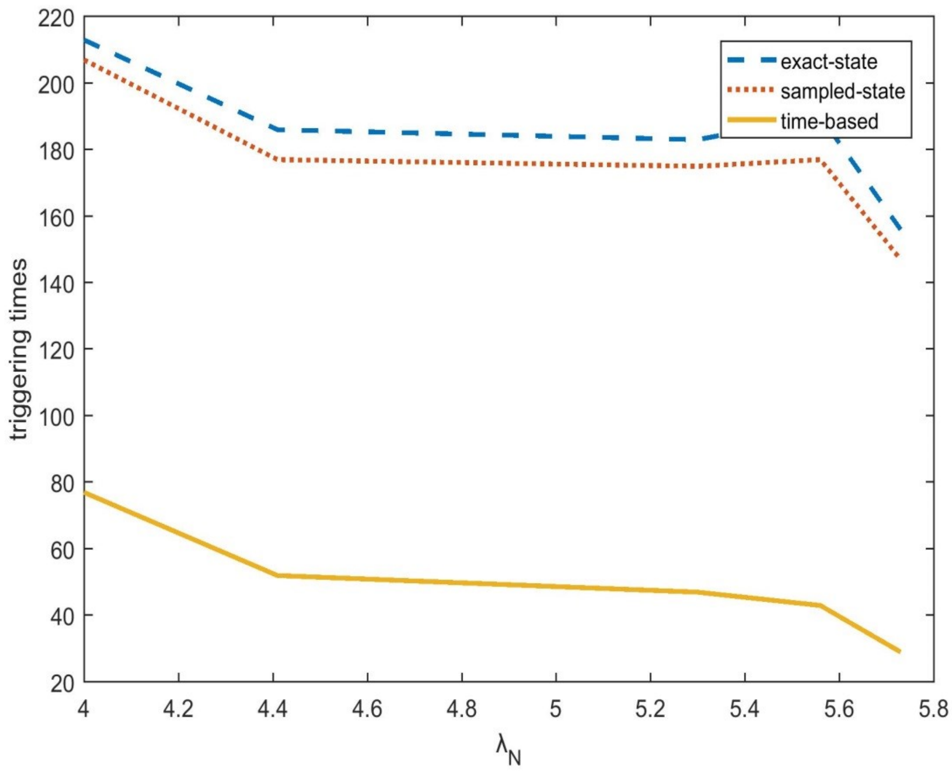


Fig. 5.13 Number of triggering times for different topologies

5.4 Conclusion

In this chapter, a time-based triggering function is designed for finite-time consensus of MAS. As same as the sampled-state based case, the existence of positive minimum inter-event time and non-Zeno executions are ensured by arranging a lower bound of triggering threshold, and the communication is also event-driven. According to the result of simulation, (1) although the time-based event-triggered control protocol has larger error in the estimation of settling time in certain conditions, it has fewer number of triggering times and less real settling time compared with exact-state based case, (2) the time-based case has smaller error in the estimation of settling time only in certain conditions, fewer number of triggering times and less real settling time compared with sampled-state based cases. Therefore, the time-based approach requires the least computation and communication resource to achieve consensus.

Chapter 6 Conclusion and Future Work

6.1 Summary

In this paper, our contributions are divided into the following three aspects: 1. research the consensus state of general linear MAS with event-triggered control and the influence of triggering function on it; 2. design a sampled-state based triggering function for finite-time consensus of MAS to realize event-triggered communication; 3. design a time-dependent triggering function for finite-time consensus of MAS to further reduce the cost of computation and communication.

At first, the consensus state of general linear MAS with event-triggered control is studied. It is found that the consensus state is a constant vector or a periodic varying vector. Such difference depends on the system matrices and the topology. These parameters determine the center region of consensus state. In addition, the triggering threshold affects distance between the center region and consensus state.

Then, the condition of finite-time stability is taken into consideration. In existing researches, event-triggered control for finite-time consensus of multi-agent systems still needs continuous communication to update the triggering function. Therefore, a triggering function based on sampled state is introduced, which only updates at each event instant. The system has positive minimum inter-event time if and only if the threshold of state error is lower bounded. In this condition, Zeno behavior can be ruled out. Simulation results show that although the sampled-state based triggering function has greater estimation error in convergence time, it saves computation resource for fewer triggering times compared with exact-state case. What's more, large amount of communication resource is saved by converting continuous communication into event-triggered.

In consensus problems, time-based event-driven control is much easier to achieve a balance between the convergence time and the number of triggering times compared with state-based method. However, it has not been researched in the field of finite-time consensus. Hence, this gap is filled in this paper. In time-based event-driven control, each agent only needs to broadcast its state to the neighbors at each event instant without continuous monitoring. The convergence time can be estimated according to the initial condition and topology. Furthermore, the system has positive minimum inter-event time if and only if the triggering threshold is lower bounded. Simulation results show that, (1) although the time-based event-triggered control protocol has larger error in the estimation of settling time in certain conditions, it has fewer number of triggering times and less real settling time compared with exact-state based case, (2) the time-based case has smaller error in the estimation of settling time in certain conditions, fewer number of triggering times and less real settling time compared with sampled-state based cases. Therefore, the time-based method requires the least computation and communication resource to achieve consensus.

6.2 Future Work

During the process of designing an event-triggered coordination strategy, the following aspects should be taken into account. Firstly, select appropriate parameters and variables that the triggering functions depend on. Secondly, how to monitor the events defined by the triggering function. Finally, what the action should be taken by the agent when the event is triggered. To sum up, the future prospect of this research is categorized as following points: dynamics, topology, trigger dependence, event detection and trigger response.

The triggering functions introduced in Chapter 4 and Chapter 5 are all designed for simple dynamics. Hence, it can be extended to more general cases, such as second-order MASs [41] and high-order MASs with external disturbances [42]. However, the triggering function still requires exact or high frequency periodic sampled state of neighboring agents in these two papers.

Compared with the consensus problem of MAS with fixed topology, the case with switching topologies is more challenging due to the time-varying matrices in the state equations of agents. Because of the flexibility and scalability, it has become a research hot in event-triggered control. There are abundant literatures utilizing event-triggered control method to solve consensus problem of MAS with switching topologies [43-45]. However, there are few researches on event-triggered control for finite-time consensus problem of MAS with switching topologies. The second smallest and the largest eigenvalues of topology are required to design the triggering function in finite-time consensus problem. The parameter of topology is global information and requires the knowledge of the whole system. Therefore, it is a complicate challenge to be conquered.

Aside from state-based triggering threshold and time-based triggering threshold, an internal dynamic variable with its own dynamics is also can be designed as triggering threshold, which is called dynamic event-triggered coordination [46-48]. However, most of event-triggered studies for finite-time consensus are state-based. Therefore, many other kinds of event-triggered control still need development in the field of finite-time consensus.

In the sampled-state based event-triggered control strategy, each agent still needs to monitor its own state and check the triggering condition continuously. In order to relax the continuous self-monitoring and computation, self-triggered scheme is proposed [49]. The next event instant is preset at the previous event instant, and no monitoring is required between the consecutive trigger events. To the best of our knowledge, the self-triggered control for finite-time consensus of MAS has not been researched.

Finally, the agents might take different actions in response to the event. In this paper, we only consider the condition that the agents send their information to the neighbors and update the controller when the event is triggered. There still exist many other operations, e.g. only update the controller [50], update the controller and acquire the state from neighbors [15], update the controller and exchange the information [51]. Additionally, the gossiping protocol can be introduced into event-triggered control, which is called edge-based event-triggered coordination [52]. These ideas are also valuable to event-triggered control for finite-time consensus of MAS.

Appendix

In this section, the proof of lemmas and theorems quoted from other papers is presented.

Lemma 2.5 [29]. Suppose L is the Laplacian matrix of an undirected topology G with N vertices. Then, for all $t \geq 0$ and $v \in \mathbb{R}^N$ with $1_N^T v = 0$, it holds that

$$\|e^{-Lt}v\| \leq e^{-\lambda_2 t}\|v\|. \tag{2.7}$$

Proof. Since G is undirected, $L = L^T$ according to the property (2.4) of Laplacian matrix. It is diagonalizable with an orthogonal matrix $T = [v_1, v_2, \dots, v_N]$, v_i is the eigenvector corresponding to eigenvalue λ_i, $i \in V$. Consequently, it holds that $e^{-Lt} = T\mathrm{diag}\left(1, e^{-\lambda_2 t}, \dots, e^{-\lambda_N t}\right)T^T$. With $v_1 = \left(\frac{1}{\sqrt{N}}\right) 1_N$

$$e^{-Lt}v = \frac{1}{N} 1_N 1_N^T v + T\mathrm{diag}\left(0, e^{-\lambda_2 t}, \dots, e^{-\lambda_N t}\right)T^T v.$$

By assumption, $1_N^T v = 0$, and consequently

$$\begin{aligned}\|e^{-Lt}v\| &= \left\|T\mathrm{diag}\left(0, e^{-\lambda_2 t}, \dots, e^{-\lambda_N t}\right)T^T v\right\| \\ &\leq \|T\|\left\|\mathrm{diag}\left(0, e^{-\lambda_2 t}, \dots, e^{-\lambda_N t}\right)\right\|\|T^T\|\|v\| \\ &= e^{-\lambda_2 t}\|v\|. \ \square\end{aligned}$$

Theorem 2.8 [24]. Suppose $V: \mathcal{D} \to \mathbb{R}_+$ is a continuous and positive definite function, the Dini derivative of V satisfies

$$\dot{V}(x) + c\left(V(x)\right)^{\mu} \leq 0, \tag{2.9}$$

where $c > 0$, $\mu \in (0,1)$, $x \in \mathcal{V}\backslash\{0\}$, $\mathcal{V} \subseteq \mathcal{D}$.

Then the system (2.8) is finite time stable with the settling time

$$T(x) \leq \frac{1}{c(1-\mu)} V(x)^{1-\mu}, \tag{2.10}$$

$x \in \mathcal{N}$, and T is continuous on $\mathcal{N}$. If in addition $\mathcal{D} = \mathbb{R}^N$, V is proper, and $\dot{V}$ takes negative values on $\mathbb{R}^N\backslash\{0\}$, then the origin is a globally finite-time-stable equilibrium of (2.8)

Proof. Since V is positive definite and $\dot{V}$ takes negative values on $\mathcal{V}\backslash\{0\}$, it follows that $y \equiv 0$ is the unique solution of (2.8) on $\mathbb{R}_+$ satisfying $y(0) = 0$. Thus, every initial condition in $\mathcal{D}$ has a unique solution in forward time. Moreover, $\dot{V}(x) = 0$ and thus (2.9) holds on $\mathcal{V}$.

Let $U \subseteq \mathcal{V}$ be a bounded open set such that $0 \in U$ and $\overline{U} \subset \mathcal{D}$. Then bd U is compact and $0 \notin$ bd U, where $\overline{U}$ and bd U denote the closure and the boundary of the set U. The continuous function V attains a minimum on bd U and by positive definiteness, $\min_{x\in \mathrm{bd}\, U} V(x) > 0$. Let $0 < \beta < \min_{x\in \mathrm{bd}\, U} V(x)$ and $\mathcal{N} = \{x \in U: V(x) < \beta\}$. $\mathcal{N}$ is nonempty since $0 \in \mathcal{N}$, open since V is continuous, and bounded since U is bounded.

Now, consider $x \in \mathcal{N}$ and let c and μ be as in the theorem statement above. By uniqueness, $\psi^x: [0, \tau_x) \to \mathcal{D}$ is the unique right maximally defined solution of (2.8) for the initial condition x. For every $t \in [0, \tau_x)$ such that $\psi^x \in U$, $\dot{V}\big(\psi^x(t)\big) = \big(D^+(V \circ \psi^x)\big)(t)$ yield

$$\big(D^+(V \circ \psi^x)\big)(t) \leq -c\big(V \circ \psi^x(t)\big)^{\mu}, \tag{7.1}$$

where $V: \mathcal{D} \to \mathbb{R}_+$, $\psi^x: [0, \tau_x) \to \mathcal{D}$, $V \circ \psi^x: [0, \tau_x) \to \mathbb{R}_+$, $\psi^x(t) = \psi(t, x)$ is the unique solution of $\dot{x}(t) = g\big(x(t)\big)$ on $[0, \tau_x)$.

Thus $y = \psi^x$ satisfies $\big(D^+(V \circ y)\big)(t) \leq -c\big(V \circ y(t)\big)^{\mu}$. Now ψ^x satisfies that $\psi^x: [0, \tau_x) \to \mathcal{D}$ is a right maximally defined solution of (2.8) such that $\psi^x \in \mathcal{N}$ for all $t \in [0, \tau_x)$, where $\overline{\mathcal{N}} \subset \mathcal{D}$ is compact, then $\tau_x = \infty$. Thus, $\psi: \mathbb{R}_+ \times \mathcal{N} \to \mathcal{N}$ is a continuous global semi flow satisfying $\psi(0, x) = x$ and $\psi\big(t, \psi(h, x)\big) = \psi(t + h, x)$.

Next, applying the comparison lemma to the differential inequality (7.1) and the scalar differential equation $\dot{y}(t) = -k\mathrm{sign}\big(y(t)\big)|y(t)|^{\mu}$ yields

$$V\big(\psi(t, x)\big) \leq \varepsilon\big(t, V(x)\big),\ t \in \mathbb{R}_+,\ x \in \mathcal{N}, \tag{7.2}$$

Where $k > 0$, ε is given by the direct integration of $\dot{y}(t) = -k\mathrm{sign}\big(y(t)\big)|y(t)|^{\mu}$

as $\varepsilon(t,x)$ with $k=c$, where

$$\varepsilon(t,x)=\operatorname{sign}(x)[|x|^{1-\mu}-k(1-\alpha)t]^{\frac{1}{1-\mu}},\ t<\frac{1}{k(1-\mu)}|x|^{1-\mu},\ x\neq 0,$$

$$\varepsilon(t,x)=0, \qquad t\geq\frac{1}{k(1-\mu)}|x|^{1-\mu},\ x\neq 0,$$

$$\varepsilon(t,x)=0, \qquad t\geq 0,\ x=0.$$

From (7.2), $\varepsilon(t,x)$ and the positive-definiteness of V, we conclude that

$$\psi(t,x)=0,\ t\geq\frac{1}{c(1-\mu)}|x|^{1-\mu},\ x\in\mathcal{N}. \tag{7.3}$$

Since $\psi(0,x)=x$ and ψ is continuous, $\inf\{t\in\mathbb{R}_+:\psi(t,x)=0\}>0$ for $x\in\mathcal{N}\backslash\{0\}$. Also, it follows from (7.3) that $\inf\{t\in\mathbb{R}_+:\psi(t,x)=0\}<\infty$ for $x\in\mathcal{N}$. Define $T(x)=\inf\{t\in\mathbb{R}_+:\psi(t,x)=0\}$. It is a simple matter to verify that T and $\mathcal{N}$ satisfy **Definition 2.7**. (i) and thus T is the settling-time function on $\mathcal{N}$. Lyapunov stability follows by noting from (2.9) that $\dot{V}$ takes negative values on $\mathcal{V}\backslash\{0\}$. Equation (2.10) follows from (7.3) and $T(x)=\inf\{t\in\mathbb{R}_+:\psi(t,x)=0\}$. Equation (2.10) implies that T is continuous at the origin and hence continuous on $\mathcal{N}$.

If $\mathcal{D}=\mathbb{R}^n$ and V is proper, then global finite-time-stability is proven in the same way that global asymptotic stability is proven using radially unbounded Lyapunov functions. □

Lemma 2.9 [27]. For any $y,z\in\mathbb{R}$ and $0<\mu\leq 1$,

$$|y+z|^{\mu}\leq|y|^{\mu}+|z|^{\mu}. \tag{2.11}$$

Proof. If $y=z$, then $|y+z|^{\mu}=2^{\mu}|y|^{\mu}\leq 2|y|^{\mu}=|y|^{\mu}+|z|^{\mu}$.

Else, suppose $|y|>|z|$, assume $z=\theta y$ with $-1<\theta<1$. Then,

$$|y+z|^{\mu}=(1+\theta)^{\mu}|y|^{\mu}\leq(1+|\theta|)^{\mu}|y|^{\mu}=|y|^{\mu}+|z|^{\mu}. \ \square$$

Lemma 2.10 (Young's inequality) [31]. If $a\geq 0$, $b\geq 0$ and if $p>1$, $q>1$ such that $\frac{1}{p}+\frac{1}{q}=1$, then

$$ab\leq\frac{a^p}{p}+\frac{b^q}{q}. \tag{2.12}$$

Proof. The claim is certainly true if $a=0$ or $b=0$. So assume that $a>0$ and

$b > 0$. Considering Jensen's inequality, suppose $f(x)$ is a convex function, then

$$f(tx_1 + (1-t)x_2) \le tf(x_1) + (1-t)f(x_2),$$

where $t \in [0,1]$.

Set $f(x) = \ln(x)$, $x > 0$, it is a concave function obviously. Put $t = \frac{1}{p}$ and $1 - t = \frac{1}{q}$,

$$\ln(ta^p + (1-t)b^q) \ge t\ln(a^p) + (1-t)\ln(b^q) = \ln(a) + \ln(b) = \ln(ab)$$

with the equality holding if and only if $a^p = b^q$. □

Lemma 2.11 [27]. For any $\xi_1, \cdots, \xi_N \in \mathbb{R}_+$, $0 < \mu < 1$,

$$(\textstyle\sum_{r=1}^N \xi_r)^\mu \le \sum_{r=1}^N \xi_r^\mu \le N^{1-\mu}(\sum_{r=1}^N \xi_r)^\mu. \tag{2.13}$$

Proof. Define vector $\xi = [\xi_1, \cdots, \xi_N]^{\mathrm{T}}$ and the norm $\|\xi\|_s = (\sum_{i=1}^N \xi_i^s)^{\frac{1}{s}}$. Using the norm equivalence property

$$\|\xi\|_s \le \|\xi\|_r \le N^{\frac{1}{r}-\frac{1}{s}}\|\xi\|_s,\ 0 < r < s,$$

the inequalities can be verified. □

Lemma 2.12 [27]. For $\tilde{y},\ \tilde{z} \in \mathbb{R}$, $0 < \mu < 1$, $|\tilde{y}| \ge |\tilde{z}|$,

$$-\tilde{y}[\tilde{y} + \tilde{z}]^{[\mu]} \le -|\tilde{y}|^{\mu+1} + |\tilde{y}||\tilde{z}|^\mu. \tag{2.14}$$

Proof. Consider $|\tilde{y}| = |\tilde{z}|$. It is obvious that the inequality holds in the case of $\tilde{y} = \tilde{z}$ since the left-side is no greater than zero while the right-side equals zero, and particularly the equality holds when $\tilde{y} = \tilde{z} = 0$. Moreover, the inequality with $\tilde{y} = -\tilde{z}$ is also true since both sides of (2.14) equal zero.

Consider $|\tilde{y}| > |\tilde{z}|$ ($\tilde{y} \ne 0$). Assume that there exists a real number $\theta: -1 < \theta < 1$, such that $\tilde{z} = \theta\tilde{y}$. With $1 + \theta > 0$, the left side of (2.14) can be written as

$$-\tilde{y}[\tilde{y} + \tilde{z}]^{[\mu]} = -\tilde{y}[\tilde{y} + \theta\tilde{y}]^{[\mu]} = -(1+\theta)^\mu\tilde{y}[\tilde{y}]^{[\mu]} - (1+\theta)^\mu|\tilde{y}|^{1+\mu}.$$

On the other hand, the right-hand side of (2.14) satisfies

$$-\tilde{y}[\tilde{y}]^{[\mu]} + |\tilde{y}||\tilde{z}|^\mu = -\tilde{y}[\tilde{y}]^{[\mu]} + |\theta|^\mu|\tilde{y}|^{1+\mu} = (-1 + |\theta|^\mu)|\tilde{y}|^{1+\mu}.$$

Note that $0 < \mu < 1$. For $0 \le \theta < 1$, one has

$$(1+\theta)^{\mu}+|\theta|^{\mu}-1 \geq 1^{\mu}+\theta^{\mu}-1=\theta^{\mu} \geq 0.$$

For $-1<\mu<0$, one gets

$$(1+\theta)^{\mu}+|\theta|^{\mu}-1 \geq (1+\theta)-\theta-1=0.$$

Thus, for $-1<\theta<1$, it follows that

$$(-1+|\theta|^{\mu})|\tilde{y}|^{1+\mu} \geq -(1+\theta)^{\mu}|\tilde{y}|^{1+\mu},$$

which yields the desired result. □

Theorem 3.1[19]. Consider the MAS (3.1) with a connected and undirected graph G. Suppose the controller (3.2) and triggering function (3.4) with $0<\alpha<\lambda_2$, then the system does not exhibit Zeno behavior. Moreover, the state $x(t)$ converges to a region centered at $\bar{x}(0)$ with radius $\frac{\|L\|\sqrt{N}c_0}{\lambda_2}$.

Proof. Since $L\frac{1}{N}1_N^T x(0)1_N=0$, therefore $L\delta(t)=Lx(t)$, so we have

$$\dot{\delta}(t)=-L\big(x(t)+e(t)\big)=-L\big(\delta(t)+e(t)\big).$$

Its solution is

$$\delta(t)=e^{-Lt}\delta(0)-\int_0^t e^{-L(t-s)}Le(s)ds.$$

Then the norm of $\delta(t)$ becomes

$$\|\delta(t)\| \leq \|e^{-Lt}\delta(0)\|+\int_0^t \left\|e^{-L(t-s)}Le(s)\right\|ds.$$

Consider **Lemma 2.5**,

$$\|\delta(t)\| \leq e^{-\lambda_2 t}\|\delta(0)\|+\|L\|\int_0^t e^{-\lambda_2(t-s)}\|e(s)\|ds.$$

According to the triggering function (3.4), we have $\|e_i(t)\| \leq (c_0+c_1e^{-\alpha t})$, then

$$\|\delta(t)\| \leq e^{-\lambda_2 t}\left(\|\delta(0)\|-\|L\|\sqrt{N}\left(\frac{c_0}{\lambda_2}+\frac{c_1}{\lambda_2-\alpha}\right)\right)+e^{-\alpha t}\frac{\|L\|\sqrt{N}c_1}{\lambda_2-\alpha}+\frac{\|L\|\sqrt{N}c_0}{\lambda_2}.$$

When time approaches to infinity,

$$\lim_{t\to\infty}\|\delta(t)\| = \lim_{t\to\infty}\left\|x(t) - \frac{1}{N}1_N^T x(0)1_N\right\| \le \frac{\|L\|\sqrt{N}c_0}{\lambda_2}.$$

As a result, the consensus state equals a vector located a region centered at the average of initial state with radius $\frac{\|L\|\sqrt{N}c_0}{\lambda_2}$. □

Lemma 3.3 [35]**.** Consider the MAS (3.5) with connected and directed topology. Suppose the triggering function (3.7) with $0 < \alpha < -\max_i \mathrm{Re}(\lambda_i(\Pi))$, where $\Pi \triangleq I_{N-1} \otimes A + c\Delta \otimes BK \in \mathbb{C}^{(N-1)n\times(N-1)n}$, $J(L) = [0, 0_{N-1}^T; 0_{N-1}, \Delta]$. Then, with controller (3.6), the MAS reaches consensus for any initial state if and only if all matrices $A - c\lambda_i BK$ are Hurwitz, where $\lambda_i \ne 0$ is the ith smallest eigenvalue of Laplacian matrix L. Moreover, the existence of Zeno behavior can be ruled out.

Proof. (Sufficiency) Without loss of generality, we assume that the latest triggered event occurred at agent i and denote $t^* = t_{k_i}^i$ as the latest event instant. That is, there is no event triggered from t^* to t for all agents. With the stack vectors $x(t) = [x_1^T(t), \dots, x_N^T(t)]^T$, $x(t^*) = [x_1^T(t^*), \dots, x_N^T(t^*)]^T$, and $e(t) = [e_1^T(t), \dots, e_N^T(t)]^T$, the closed-loop system of (3.5) using (3.6) can be written as

$$\begin{aligned}\dot{x}(t) &= (I_N \otimes A)x(t) + (cL \otimes BK)e^{(I_N\otimes A)(t-t^*)}x(t^*)\\ &= (I_N \otimes A + cL \otimes BK)x(t) + (cL \otimes BK)e(t). \qquad (7.4)\end{aligned}$$

Since G is a directed graph, there exist matrices $T = [1_N, Y]$ and $T^{-1} = [r, W^T]^T$, where $Y \in \mathbb{C}^{N\times(N-1)}$ and $W \in \mathbb{C}^{(N-1)\times N}$, such that $T^{-1}LT = J(L) = [0, 0_{N-1}^T; 0_{N-1}, \Delta]$, where $J(L)$ is the Jordan canonical form of the matrix L and $\Delta \in \mathbb{C}^{(N-1)\times(N-1)}$ is a block diagonal matrix. Define the disagreement vector $\delta(t) = x(t) - (1_N r^T \otimes I_n)x(t)$ and a new vector

$$\varepsilon(t) = (T^{-1} \otimes I_n)\delta(t) = [\varepsilon_1^T(t), \varepsilon_{2-N}^T(t)]^T, \qquad (7.5)$$

where $\varepsilon_1(t) \in \mathbb{C}^n$ and $\varepsilon_{2-N}(t) \in \mathbb{C}^{(N-1)n}$. The consensus problem for general linear agents can be converted to the stability problem of $\delta(t)$ or $\varepsilon(t)$ under the event-triggered consensus controller (3.6). Also it follows that $\varepsilon_1(t) \equiv 0_n$ and the vector $\varepsilon_{2-N}(t)$

$$\dot{\varepsilon}_{2-N}(t) = \Pi\varepsilon_{2-N}(t) + (c\Delta W \otimes BK)e_{2-N}(t). \qquad (7.6)$$

where $\Pi \triangleq I_{N-1} \otimes A + c\Delta \otimes BK \in \mathbb{C}^{(N-1)n\times(N-1)n}$, $e_{2-N}(t) \triangleq [e_2^T(t), \dots, e_N^T(t)]$.

Let $P \in \mathbb{C}^{(N-1)n\times(N-1)n}$ and $P^{-1} \in \mathbb{C}^{(N-1)n\times(N-1)n}$ be the matrices such that $P^{-1}\,\Pi P = J(\Pi)$, where $J(\Pi)$ is the Jordan canonical form of the matrix Π. From the definition of Π, it is obvious that if all matrices $A + c\lambda_i BK$, where $\lambda_i \neq 0$, are Hurwitz, the matrix Π is surely Hurwitz and all $\mathrm{Re}\big(\lambda_i(\Pi)\big) < 0$. Since the triggering function $f_i\big(t, e_i(t)\big)$ for agent i is reset to zero when an event is triggered. Before the next event is triggered, $f_i\big(t, e_i(t)\big)$ will not cross zero, that is, $\|e_i(t)\| \leq \mathrm{c}_1 e^{-\alpha t}$ is satisfied until the next event is triggered. Hence $\|e_{2-N}(t)\| \leq \sqrt{N-1}\mathrm{c}_1 e^{-\alpha t}$ and $\|e_{2-N}(t)\| \to 0$, as $t \to \infty$. It follows from (7.6) and the input-to-state stability argument that $\varepsilon_{2-N}(t)$ approaches zero. Then, it follows that the disagreement vector $\delta(t)$ of the closed-loop system (7.4) asymptotically converges to zero for all initial conditions, that is, the controller (3.6) solves the event-triggered consensus problem.

Next, we will show that under the controller (3.6), the closed-loop system (7.4) does not exhibit the Zeno behavior. The solution of $\varepsilon_{2-N}(t)$ can be obtained as

$$\varepsilon_{2-N}(t) = e^{\Pi t}\varepsilon_{2-N}(0) + \int_0^t e^{\Pi(t-s)}(c\Delta W \otimes BK)\, e_{2-N}(s)ds. \quad (7.7)$$

Then, for $0 \leq s \leq t$,

$$\left\|e^{\Pi(t-s)}(c\Delta W \otimes BK)e_{2-N}(s)\right\| \leq \mathrm{c}_\Pi c_1\sqrt{N-1}\big((N-1)n-1\big)\|P\|\|P^{-1}\|$$

$$* \|c\Delta W \otimes BK\| e^{a_\Pi(t-s)}e^{-\alpha s}, \quad (7.8)$$

where c_Π is a positive constant with respect to Π and $\max_i Re\big(\lambda_i(\Pi)\big) < a_\Pi < 0$.

For the sake of simplicity, let $a_1 = c_\Pi\big((N-1)n-1\big)\|P\|\|P^{-1}\|\|\varepsilon_{2-N}(0)\|$ and $a_2 = \mathrm{c}_\Pi c_1\sqrt{N-1}\big((N-1)n-1\big)\|P\|\|P^{-1}\|\|c\Delta W \otimes BK\|$. It follows from (7.7) and (7.8) that $\|\varepsilon(t)\| = \|\varepsilon_{2-N}(t)\| \leq \left(a_1 + \frac{a_2}{|a_\Pi+\alpha|}\right)e^{a_\Pi t} + \frac{a_2}{|a_\Pi+\alpha|}e^{-\alpha t}$. Then it follows from (7.8) that $\|\delta(t)\|$ satisfies $\|\delta(t)\| \leq \|T \otimes I_n\|\|\varepsilon(t)\| \leq k_1 e^{a_\Pi t} + k_2 e^{-\alpha t}$, $k_1 = \|T\|\left(a_1 + \frac{a_2}{|a_\Pi+\alpha|}\right)$ and $k_2 = \frac{\|T\|a_2}{|a_\Pi+\alpha|}$.

Let $u(t)$ be the column stack vector of $u_i(t)$. Using the property of the

Laplacian matrix: $L1_N \equiv 0_N$, we conclude that $(cL \otimes BK)\big(\delta(t) + e(t)\big) = (I_N \otimes B)u(t)$. Similarly, $\|(I_N \otimes B)u(t)\|$ is upper bounded by

$$\|(I_N \otimes B)u(t)\| \leq \|cL \otimes BK\|(\|\delta(t)\| + \|e(t)\|) = b_1 e^{a_\Pi t} + b_2 e^{-\alpha t}, \quad (7.9)$$

where $b_1 = \|cL \otimes BK\| k_1$ and $b_2 = \|cL \otimes BK\|\big(k_2 + \sqrt{N} c_1\big)$.

Recall that we assume that agent i is triggered at the latest event instant t^*. Hence the value $x_i\big(t_{k_i}^i\big)$ remains constant from $t_{k_i}^i$ to t^*. It follows from (3.7) that $\dot{e}_i(t) = Ae^{A(t-t^*)}x_i(t^*) - Ax_i(t) - Bu_i(t) = Ae_i(t) - Bu_i(t)$. Moreover, with the fact that $\|e_i(t)\| \leq c_1 e^{-\alpha t}$ before the next event is triggered, $\|u_i(t)\| \leq \|u(t)\|$, and (7.9), we can get the upper bound of $\|\dot{e}_i(t)\|$ between the two triggered events for agent i as $\|\dot{e}_i(t)\| \leq \|A\|\|e_i(t)\| + \|Bu_i(t)\| \leq b_1 e^{a_\Pi t} + d_2 e^{-\alpha t} \triangleq g(t)$, where $d_2 = \|A\| c_1 + b_2$. Note that b_1 and d_2 are both positive constants here.

Since the latest event instant, it follows that $\|e_i(t)\| = \left\|\int_{t^*}^{t} \dot{e}_i(s)ds\right\| \leq \int_{t^*}^{t} g(s)ds$. From the definition of the triggering function (3.7), we know that the next event of agent i will not be triggered before $f_i\big(t, e_i(t)\big) = 0$ or equivalently $\|e_i(t)\| = c_1 e^{-\alpha t}$. Hence the next event will not be triggered before $\int_{t^*}^{t} g(s)ds = c_1 e^{-\alpha t}$. Since $t \geq t^*$ and both a_Π and $-\alpha$ are negative, we have $e^{a_\Pi t} \leq e^{a_\Pi t^*}$ and $e^{-\alpha t} \leq e^{-\alpha t^*}$. Let $\tau = t - t^*$ be the time-interval between the two triggered events. So τ is greater than or equal to the solution of the implicit equation $\big(b_1 e^{a_\Pi t^*} + d_2 e^{-\alpha t^*}\big)\tilde{\tau} = \sqrt{N} c_1 e^{-\alpha(t^* + \tilde{\tau})}$, which is equivalent to $\big(b_1 e^{(\alpha + a_\Pi)t^*} + d_2\big)\tilde{\tau} = \sqrt{N} c_1 e^{-\alpha\tilde{\tau}}$. Because $\alpha < -\max_i \mathrm{Re}\big(\lambda_i(\Pi)\big)$, there must exist a negative constant a_Π such that $\max_i \mathrm{Re}\big(\lambda_i(\Pi)\big) < a_\Pi < -\alpha < 0$. As $\alpha < -a_\Pi$, we know the term $b_1 e^{(\alpha + a_\Pi)t^*} + d_2$ is upper bounded by $b_1 + d_2$. So the solution of the implicit equation is greater than or equal to the solution of $(b_1 + d_2)\bar{\tau} = \sqrt{N} c_1 e^{-\alpha\bar{\tau}}$, which is strictly positive. It means that if the coefficients c_1 and α in (3.7) satisfy $c_1 > 0$ and $0 < \alpha < -\mathrm{Re}\big(\lambda_1(\Pi)\big)$, there is a positive lower bound $\bar{\tau}$ on the

inter-event times for agent i. So, the event-triggered consensus problem of the general linear MAS is solved with no Zeno behavior exhibited.
(Necessity) The necessity is obvious. Note that the initial state-based measurement error might not be zero. If at least one matrix $A + c\lambda_i BK$ is not Hurwitz, where $\lambda_i \neq 0$, $\varepsilon_{2-N}(t)$ will go to infinity as $t \to \infty$ and so will $\delta(t)$. Then, the states of the N agents will not reach consensus for all initial conditions. □
Theorem 4.1[27]. Suppose the topology is undirected and connected. Then, with the controller (4.2) and the threshold of triggering function (4.3) satisfies

$$0 < c_{\max} = \max_i(c_i) < \sqrt{\frac{\lambda_2}{\lambda_N^3 N^{\frac{1-\mu}{1+\mu}}}}, \tag{7.10}$$

(i) The MAS (4.1) achieves consensus within the estimated time

$$T_s = \frac{1}{1-\mu} \frac{V(0)^{\frac{1-\mu}{2}}}{\hat{\alpha} 2^{\frac{\mu-1}{2}} \lambda_2^{\frac{\mu+1}{2}}}, \tag{7.11}$$

(ii) The MAS does not exhibit Zeno behavior on time interval $[0, T_s)$,

where $\hat{\alpha} = \alpha \frac{\mu}{1+\mu}\left[1 - N^{\frac{1-\mu}{2}} \left(\frac{\lambda_N^3 c_{\max}^2}{\lambda_2}\right)^{\frac{\mu+1}{2}}\right] > 0,\ \alpha > 0,\ 0 < \mu < 1.$

Proof. (i) The first part verifies the finite-time convergence for consensus. $V(t)$ along the state trajectories of MAS (4.1) satisfies

$$\begin{aligned}\dot{V}(t) &= -\alpha \sum_{i=1}^N \left[\sum_{j=1}^N a_{ij}\left(x_i(t) - x_j(t)\right)\right] \\ &\quad * \left[\sum_{j=1}^N a_{ij}\left(x_i(t) - x_j(t) + e_i(t) - e_j(t)\right)\right]^{[\mu]} \\ &= -\alpha \sum_{i=1}^N \{X_i [X_i - E_i]^{[\mu]}\},\end{aligned}$$

where $X_i = L_i x(t) = \sum_{j=1}^N a_{ij}\left(x_i(t) - x_j(t)\right)$, $E_i = L_i e(t) = \sum_{j=1}^N a_{ij}\left(e_i(t) - e_j(t)\right)$, $x(t) = [x_1(t), \dots, x_N(t)]^T$, $e(t) = [e_1(t), \dots, e_N(t)]^T$ and L_i is the ith row of matrix L.

It is now to show that $|E_i| \leq |X_i|$, $i = 1,2,\dots,N$, under the event condition (4.3) with the threshold satisfying (7.10). Since $\sum_{i=1}^N |E_i|^2 = \sum_{i=1}^N e^T L_i^T L_i e =$

$e^T L^T Le$, it follows that

$$\sum_{i=1}^{N}|E_i|^2 \leq \lambda_N^2 \mathrm{c}_{\max}^2 \sum_{i=1}^{N}|X_i|^2 < \frac{\lambda_2}{\lambda_N N^{\frac{1-\mu}{1+\mu}}} \sum_{i=1}^{N}|X_i|^2 < \sum_{i=1}^{N}|X_i|^2. \quad (7.12)$$

By the definitions: $e_i(t) = x_i(t_{k_i}^i) - x_i(t)$, $E_i = L_i e(t)$ and $X_i = L_i x(t)$, without loss of generality, let $E_i = q_i X_i$. The inequality (7.12) is then equivalent to $\sum_{i=1}^{N} q_i^2 |X_i|^2 < \sum_{i=1}^{N}|X_i|^2$. The matrix form is $x^T L^T Q^2 Lx < x^T L^T Lx$, where $Q = \mathrm{diag}(q_1, \dots, q_N)$ is the diagonal matrix. Note that the initial state of MAS (4.1) is arbitrary. That is, the inequality $x^T L^T Q^2 Lx < x^T L^T Lx$ holds for all $x \in \mathbb{R}^N$. Thus, one has $0 < |q_i| < 1$, implying $|E_i| \leq |X_i|$.

Using **Lemma 2.12**, one has

$$\begin{aligned}
\dot{V}(t) &\leq -\alpha \sum_{i=1}^{N} X_i [X_i]^{[\mu]} + \alpha \sum_{i=1}^{N}|X_i||E_i|^{\mu} \\
&\leq -\alpha \sum_{i=1}^{N} \left|\sum_{j=1}^{N} a_{ij}\left(x_i(t) - x_j(t)\right)\right|^{\mu+1} \\
&\quad +\alpha \sum_{i=1}^{N} \frac{1}{1+\mu} \left|\sum_{j=1}^{N} a_{ij}\left(x_i(t) - x_j(t)\right)\right|^{\mu+1} \\
&\quad +\alpha \sum_{i=1}^{N} \frac{\mu}{1+\mu} \left|\sum_{j=1}^{N} a_{ij}\left(e_i(t) - e_j(t)\right)\right|^{\mu+1}, \quad (7.13)
\end{aligned}$$

where the last inequality is based on the Young's inequality.

Recalling the event condition (4.3), one obtains

$$\begin{aligned}
\sum_{i=1}^{N}\left[\left|\sum_{j=1}^{N} a_{ij}\left(e_i(t) - e_j(t)\right)\right|^2\right]^{\frac{\mu+1}{2}} &\leq N^{\frac{1-\mu}{2}} \left\{\sum_{i=1}^{N}\left[\sum_{j=1}^{N} a_{ij}\left(e_i(t) - e_j(t)\right)\right]^2\right\}^{\frac{\mu+1}{2}} \\
&\leq N^{\frac{1-\mu}{2}} (2\lambda_N^3 c_{\max}^2)^{\frac{\mu+1}{2}} \left(V(x)\right)^{\frac{\mu+1}{2}}.
\end{aligned}$$

Substituting the above inequality into (7.13) gives

$$\begin{aligned}
\dot{V}(t) &\leq -\alpha\left(1 - \frac{1}{1+\mu}\right) \sum_{i=1}^{N} \left|\sum_{j=1}^{N} a_{ij}\left(x_i(t) - x_j(t)\right)\right|^{\mu+1} \\
&\quad +\alpha \frac{\mu}{1+\mu} \left\{\sum_{i=1}^{N}\left[\sum_{j=1}^{N} a_{ij}\left(e_i(t) - e_j(t)\right)\right]^2\right\}^{\frac{\mu+1}{2}} \\
&\leq -\alpha \frac{\mu}{1+\mu} (2\lambda_2)^{\frac{\mu+1}{2}} \left(V(x)\right)^{\frac{\mu+1}{2}} + \alpha \frac{\mu}{1+\mu} N^{\frac{1-\mu}{2}} (2c_{\max})^{\frac{\mu+1}{2}} \left(V(x)\right)^{\frac{\mu+1}{2}}
\end{aligned}$$

$$= -\frac{\alpha\mu}{1+\mu}(2\lambda_2)^{\frac{\mu+1}{2}}\left[1 - N^{\frac{1-\mu}{2}}\left(\frac{c_{\max}^2\lambda_N^3}{\lambda_2}\right)^{\frac{\mu+1}{2}}\right](V(x))^{\frac{\mu+1}{2}}$$

$$= -\hat{\alpha}(2\lambda_2)^{\frac{\mu+1}{2}}(V(x))^{\frac{\mu+1}{2}}. \tag{7.14}$$

Since $0 < \frac{\mu+1}{2} < 1$ and $M = \{x: V(x) = 0\} = \{x: Lx = 0\}$, it follows from **Theorem 2.8** that the set M is finite-time attractive for MAS (4.1) under the event condition (4.3). Therefore, MAS reaches the finite-time consensus within the estimated settling time (7.11).

(ii) This part shows the exclusion of Zeno behavior. It will be verified that each inter-event time $t_{k_i+1}^i - t_{k_i}^i$ that implicitly defined by (4.3) is positively lower bounded.

Consider that an event of agent i occurs at time $t_{k_i}^i$. According to the event-driven scheme, one has $\left|e_i\left(t_{k_i}^i\right)\right| = 0$, and only when the error $|e_i(t)|$ is about to exceed the twisted threshold $c_i|X_i|$, for $X_i \neq 0$, agent i will be reactivated. Thus, before the next event time, one has $\frac{|e_i(t)|}{|X_i|} \leq c_i$.

Similar to [16], the comparison principle of differential equations is used to obtain a positive lower bound for $t_{k_i+1}^i - t_{k_i}^i$. Clearly, $|e_i(t)| \leq \|e(t)\|$, $i = 1,2,\dots,N$. For $X_i \neq 0$ and $|X_i| < \infty$, there exists a finite constant $\vartheta_i \geq 1$ such that $\vartheta_i|X_i| \geq \frac{\|X\|}{\sqrt{N}}$, where $X = [X_1,\dots,X_N]^T$. One obtains $\frac{|e_i(t)|}{|X_i|} \leq \vartheta_i\sqrt{N}\frac{\|e(t)\|}{\|X\|}$. Then, the time interval for which $\frac{|e_i(t)|}{|X_i|}$ ranges from 0 to c_i is greater than that $\vartheta_i\sqrt{N}\frac{\|e(t)\|}{\|X\|}$ needs.

The time derivative of $\frac{\|e\|}{\|X\|}$ satisfies

$$\frac{d}{dt}\frac{\|e\|}{\|X\|} \leq \frac{\|\dot{x}\|\|X\| + \|e\|\|\dot{X}\|}{\|X\|^2} \leq \frac{\|\dot{x}\|\|X\| + \|L\|\|e\|\|\dot{x}\|}{\|X\|^2}. \tag{7.15}$$

Based on **Lemma 2.9** and **Lemma 2.11**, one gets

$$\|\dot{x}\| \leq \sum_{i=1}^N|\dot{x}_i| = \alpha\sum_{i=1}^N\left|X_i + \sum_{j=1}^N a_{ij}\left(e_i(t) - e_j(t)\right)\right|^\mu$$

$$\leq \alpha N^{1-\frac{\mu}{2}}(\sum_{i=1}^{N} X_i^2)^{\frac{\mu}{2}} + \alpha N^{1-\frac{\mu}{2}}\left[\sum_{i=1}^{N}\left(\sum_{j=1}^{N} a_{ij}\left(e_i(t) - e_j(t)\right)\right)^2\right]^{\frac{\mu}{2}}$$

$$= \alpha N^{1-\frac{\mu}{2}}(\|X\|^{\mu} + \|Le\|^{\mu}). \tag{7.16}$$

According to the event-driven scheme, one has $\|e\| \leq c_{\max}\|X\|$ and e_i can be detected at any time t. Let $\mathcal{E}_{ik} = \max\{|e_i(t): t > t_{k_i}^i, e_i \neq 0|\}$. Substituting (7.16) into (7.15) then gives

$$\frac{d}{dt}\frac{\|e\|}{\|X\|} \leq \alpha N^{1-\frac{\mu}{2}}\left(1 + \|L\|\frac{\|e\|}{\|X\|}\right)\left[1 + \|L\|^{\mu}\left(\frac{\|e\|}{\|X\|}\right)^{\mu}\right]\frac{1}{\|X\|^{1-\mu}}$$

$$\leq \alpha N^{1-\frac{\mu}{2}}\frac{c_{\max}^{1-\mu}}{\mathcal{E}_{ik}^{1-\mu}}\left(1 + \|L\|\frac{\|e\|}{\|X\|}\right)\left[1 + \|L\|^{\mu}\left(\frac{\|e\|}{\|X\|}\right)^{\mu}\right]. \tag{7.17}$$

By the comparison principle, the inequality (7.17) yields $\frac{\|e\|}{\|X\|} \leq \psi(t)$, where $\psi(t)$ is the solution of differential equation

$$\dot{\psi}(t) = b_{ik}\left(1 + \|L\|\psi(t)\right),\ \psi\left(t_{k_i}^i\right) = 0,\ t \in \left[t_{k_i}^i, t_{k_i+1}^i\right),$$

with $b_{ik} = \alpha N^{1-\frac{\mu}{2}}\frac{c_{\max}^{1-\mu} + \|L\|^{\mu}c_{\max}}{\mathcal{E}_{ik}^{1-\mu}} > 0$.

Solving the above differential equation gives $\psi(t) = \frac{\exp\left(\|L\|b_{ik}\left(t - t_{k_i}^i\right)\right) - 1}{\|L\|}$. Denote $y_i(t) = \frac{|e_i(t)|}{|X_i|}$. Before the next event instant, one has $y_i(t) \leq \vartheta_i\sqrt{N}\int_{t_{k_i}^i}^{t}\dot{\psi}(s)ds$. Under the event condition (4.3), the next event instant is no less than $t_{k_i}^i + \tau_{ik}$, where τ_{ik} satisfies $\vartheta_i\sqrt{N}\int_{t_{k_i}^i}^{t}\dot{\psi}(s)ds = c_i$. Thus

$$t_{k_i+1}^i - t_{k_i}^i \geq \tau_{ik} = \frac{1}{\lambda_N b_{ik}}\ln\left(1 + \frac{\lambda_N c_i}{\vartheta_i\sqrt{N}}\right) > 0. \ \square$$

Acknowledgments

I would like to express my gratitude to Prof. Luo to give me the opportunity for this research. His enthusiastic guidance and rigorous academic attitude guide me to complete each task in a down-to-earth and unswerving manner. I would also like to thank associate Prof. Quan, assistant Prof. Mori and assistant Prof. Cao, for their guidance and help throughout the process of my research. I would like to acknowledge all the members of the CS11 System Control Research Laboratory, both past and present, for encouragement and helpful discussions.

I would like to express my deep gratitude to Prof. Ohta and Prof. Masubuchi, for their efforts to the peer review of this paper.

Finally, my heartfelt thanks to my family for providing the caring and supports. I would not concentrate on my studies in these years without you.

Research Achievements

A：Journal

1. Ziyang Cheng, Changqin Quan, Kohei Mori, Sheng Cao, Zhiwei Luo, “Research on Consensus State of Multi-agent System with Event-triggered Controller,” システム制御情報学会論文誌, Vol. 35, No.4, 2022.
2. Ziyang Cheng, Changqin Quan, Kohei Mori, Sheng Cao, Zhiwei Luo, “On Distributed Event-triggered Controller with Aperiodic Event Detection for Finite-time Consensus of Multi-agent Systems,” システム制御情報学会論文誌 (accept).
3. Ziyang Cheng, Sheng Cao, Changqin Quan, Kohei Mori, Zhiwei Luo, “Finite-time Consensus of Linear Multi-agent Systems via Time-based Event-triggered Control,” システム制御情報学会論文誌 (accept).

B：Conference

1. 程子洋, 曹晟, 羅志偉, 森耕平, 全昌勤, “Adaptive Control Design of Multi-robot System with Connection Uncertainty,” 第 63 回システム制御情報学会研究発表講演会論文集, 2019
2. 程子洋, 全昌勤, 森耕平, 曹晟, 羅志偉, “Research on Consensus State of Multi-agent System with Event-triggered Controller,” 第 65 回システム制御情報学会研究発表講演会論文集, 2021
3. Ziyang Cheng, Changqin Quan, Kohei Mori, Sheng Cao, Zhiwei Luo, “Research on Self-triggered Controller for Finite-time Consensus of Multi-agent System,” International Conference on Electrical, Computer, Communications and Mechatronics Engineering (IEEE ICECCME), 2021.

Bibliography

[1] M. P. Singh and A. K. Chopra, “The internet of things and multiagent systems: Decentralized intelligence in distributed computing,” 2017 IEEE 37th International Conference on Distributed Computing Systems (ICDCS). IEEE, pp. 1738–1747, 2017.

[2] J. Zhou, G. Gu, and X. Chen, “Distributed Kalman filtering over wireless sensor networks in the presence of data packet drops,” IEEE Transactions on Automatic Control, Vol. 64, No. 4, pp. 1603–1610, 2018.

[3] E. Montijano, E. Cristofalo, D. Zhou, M. Schwager, and C. Saguees, “Vision-based distributed formation control without an external positioning system,” IEEE Transactions on Robotics, Vol. 32, No. 2, pp. 339–351, 2016.

[4] A. Bidram, F. L. Lewis, and A. Davoudi, “Distributed control systems for small-scale power networks: Using multiagent cooperative control theory,” IEEE Control systems magazine, Vol. 34, No. 6, pp. 56–77, 2014.

[5] K. Hajar, A. Hably, S. Bacha, A. Elrafhi, and Z. Obeid, “Optimal centralized control application on microgrids,” 2016 3rd International Conference on Renewable Energies for Developing Countries (REDEC). IEEE, pp. 1–6, 2016.

[6] Y. Guo, D. J. Hill, and Y. Wang, “Nonlinear decentralized control of large-scale power systems,” Automatica, Vol. 36, No. 9, pp. 1275–1289, 2000.

[7] D. Trentesaux, “Distributed control of production systems,” Engineering Applications of Artificial Intelligence, Vol. 22, No. 7, pp. 971–978 , 2009.

[8] K. J. Åström and B. Wittenmark, “Computer-controlled systems: theory and design,” Courier Corporation, 2013.

[9] G. Franklin, J. Powel, and A. Emami-Naeini, “Feedback control of dynamical systems,” Prentice Hall, 2010.

[10] L. Hetel, C. Fiter, H. Omran, A. Seuret, E. Fridman, J. P. Richard, and S. I. Niculescu, “Recent developments on the stability of systems with aperiodic sampling: An overview,” Automatica, Vol. 76, pp. 309–335, 2017.

[11] H. G. Beyer and B. Sendhoff, “Robust optimization–a comprehensive survey,” Computer methods in applied mechanics and engineering, Vol. 196, No. 33–34, pp.

3190-3218, 2007.

[12] P. Kolios, C. Panayiotou, G. Ellinas, and M. Polycarpou, “Data-driven event triggering for IoT applications,” IEEE Internet of Things Journal, Vol. 3, No. 6, pp. 1146–1158, 2016.

[13] R. Olfati-Saber, J. A. Fax, and R. M. Murray, “Consensus and cooperation in networked multi-agent systems,” Proceedings of the IEEE, Vol. 95, No.1, pp. 215–233, 2007.

[14] K. E. Åarzén, “A simple event-based PID controller,” IFAC Proceedings Volumes, Vol. 32, No.2, pp. 8687–8692, 1999.

[15] W. P. Heemels, K. H. Johansson, and P. Tabuada, “An introduction to event-triggered and self-triggered control,” In 2012 IEEE 51st IEEE conference on decision and control (cdc). IEEE, pp. 3270–3285, 2012.

[16] D. V. Dimarogonas, E. Frazzoli, and K. H. Johansson, “Distributed event-triggered control for multi-agent systems,” IEEE Transactions on Automatic Control, Vol. 57, No. 5, pp. 1291–1297, 2011.

[17] D. P. Borgers and W. P. M. H. Heemels, “Event-separation properties of event-triggered control systems,” IEEE Transactions on Automatic Control, Vol. 59, No. 10, pp. 2644–2656, 2014.

[18] C. Nowzari, E. Garcia, and J. Cortés, “Event-triggered communication and control of networked systems for multi-agent consensus,” Automatica, Vol. 105, pp.1–27, 2019.

[19] G. S. Seyboth, D. V. Dimarogonas, and K. H. Johansson, “Event-based broadcasting for multi-agent average consensus,” Automatica, Vol. 49, No.1, pp. 245–252, 2013.

[20] X. Yi, K. Liu, D. V. Dimarogonas, and K. H. Johansson, “Distributed dynamic event-triggered control for multi-agent systems,” 2017 IEEE 56th annual conference on decision and control (CDC). IEEE, pp. 6683–6698, 2017.

[21] R. Olfati-Saber, “Flocking for multi-agent dynamic systems: Algorithms and theory,” IEEE Transactions on automatic control, Vol. 51, No. 3, pp.401–420, 2006.

[22] Y. Kim and M. Mesbahi, “On maximizing the second smallest eigenvalue of a state-dependent graph Laplacian,” In Proceedings of the 2005, American Control Conference, pp. 99–103, 2005.

[23] V. T. Haimo, “Finite time controllers,” SIAM Journal on Control and Optimization, Vol. 24, No. 4, pp.760–770, 1986.

[24] S. P. Bhat and D. S. Bernstein, “Finite-time stability of continuous autonomous systems,” SIAM Journal on Control and Optimization, Vol. 38, No. 3, pp. 751–766, 2000.

[25] F. Xiao, L. Wang, J. Chen, and Y. Gao, “Finite-time formation control for multi-agent systems,” Automatica, Vol. 45, No. 11, pp. 2605–2611, 2009.

[26] H. Zhang, D. Yue, X. Yin, S. Hu, and C. X. Dou, “Finite-time distributed event-triggered consensus control for multi-agent systems,” Information Sciences, Vol. 339, pp. 132–142, 2016.

[27] B. Hu, Z. H. Guan, and M. Fu, “Distributed event-driven control for finite-time consensus,” Automatica, Vol. 103, pp. 88–95, 2019.

[28] J. Liu, Y. Yu, H. He, and C. Sun, “Team-triggered practical fixed-time consensus of double-integrator agents with uncertain disturbance,” IEEE Transactions on Cybernetics, Vol. 51, No.6, pp. 3263–3272, 2020.

[29] G. S. Seyboth, D. V. Dimarogonas, and K. H. Johansson, “Control of multi-agent systems via event-based communication,” IFAC Proceedings Volumes, Vol. 44, No. 1, pp. 10086–10091, 2011.

[30] P. Dorato, “An overview of finite-time stability,” Current Trends in Nonlinear Systems and Control, pp. 185–194, 2006.

[31] J. L. W. V. Jensen, “Sur les fonctions convexes et les inégalités entre les valeurs moyennes,” Acta mathematica, Vol. 30, No. 1, pp. 175–193, 1906.

[32] W. Hu, L. Liu, and G. Feng, “Consensus of linear multi-agent systems by distributed event-triggered strategy,” IEEE transactions on cybernetics, Vol. 46, No. 1, pp. 148–157, 2015.

[33] H. Yan, Y. Shen, H. Zhang, and H. Shi, “Decentralized event-triggered consensus control for second-order multi-agent systems,” Neurocomputing, Vol. 133, pp. 18–24, 2014.

[34] J. Dai and G. Guo, “Event-triggered leader-following consensus for multi-agent systems with semi-Markov switching topologies,” Information Sciences, Vol. 459, pp. 290–301, 2018.

[35] D. Yang, W. Ren, X. Liu, and W. Chen, “Decentralized event-triggered consensus for linear multi-agent systems under general directed graphs,”

Automatica, Vol. 69, pp. 242–249, 2016.

[36] Olfati-Saber and R. M. Murray, “Consensus problems in networks of agents with switching topology and time-delays,” IEEE Transactions on automatic control, Vol. 49, No. 9, pp. 1520–1533, 2004.

[37] Z. Zhang, F. Hao, L. Zhang, and L. Wang, “Consensus of linear multi-agent systems via event-triggered control,” International Journal of Control, Vol. 87, No. 6, pp. 1243–1251, 2014.

[38] Q. Lu, Q. L. Han, B. Zhang, D. Liu, and S. Liu, “Cooperative control of mobile sensor networks for environmental monitoring: An event-triggered finite-time control scheme,” IEEE transactions on cybernetics, Vol. 47, No. 12, pp. 4134–4147, 2016.

[39] Z. Cao, C. Li, X. Wang, and T. Huang, “Finite-time consensus of linear multi-agent system via distributed event-triggered strategy,” Journal of the Franklin Institute, Vol. 355, No. 3, pp. 1338–1350, 2018.

[40] Z. Zuo, “Nonsingular fixed-time consensus tracking for second-order multi-agent networks,” Automatica, Vol. 54, pp. 305–309, 2015.

[41] G. Ran, J. Liu, C. Li, L. Chen, and D. Li, “Event-based finite-time consensus control of second-order delayed multi-agent systems,” IEEE Transactions on Circuits and Systems II: Express Briefs, Vol. 68, No. 1, pp. 276–280, 2020.

[42] J. Wang, Y. Xu, and D. Yang, “Time-varying formation for high-order multi-agent systems with external disturbances by event-triggered integral sliding mode control,” Applied Mathematics and Computation, Vol. 359, pp. 333–343, 2019.

[43] Z. G. Wu, Y. Xu, R. Lu, Y. Wu, and T. Huang, “Event-triggered control for consensus of multiagent systems with fixed/switching topologies,” IEEE Transactions on Systems, Man, and Cybernetics: Systems, Vol. 48, No. 10, pp. 1736–1746, 2017.

[44] D. Xie, S. Xu, Y. Chu, and Y. Zou, “Event-triggered average consensus for multi-agent systems with nonlinear dynamics and switching topology,” Journal of the Franklin Institute, Vol. 352, No. 3, pp. 1080–1098, 2015.

[45] T. H. Cheng, Z. Kan, J. R. Klotz, J. M. Shea, and W. E. Dixon, “Event-triggered control of multiagent systems for fixed and time-varying network topologies,” IEEE Transactions on Automatic Control, Vol. 62, No. 10, pp. 5365–5371, 2017.

[46] X. Ge, Q. L. Han, L. Ding, Y. L. Wang, and X. M. Zhang, "Dynamic event-triggered distributed coordination control and its applications: A survey of trends and techniques," IEEE Transactions on Systems, Man, and Cybernetics: Systems, Vol. 50, No. 9, pp. 3112–3125, 2020.

[47] A. Girard, "Dynamic triggering mechanisms for event-triggered control," IEEE Transactions on Automatic Control, Vol. 60, No. 7, pp. 1992–1997, 2014.

[48] J. Berneburg and C. Nowzari, "Distributed dynamic event-triggered coordination with a designable minimum inter-event time," In 2019 American Control Conference (ACC), pp. 1424–1429, 2019.

[49] A. Anta and P. Tabuada, "To sample or not to sample: Self-triggered control for nonlinear systems," IEEE Transactions on automatic control, Vol. 55, No. 9, pp. 2030–2042, 2010.

[50] E. Kharisov, X. Wang, and N. Hovakimyan, "Distributed event-triggered consensus algorithm for uncertain multi-agent systems," In AIAA guidance, navigation, and control conference, pp. 8325, 2010.

[51] C. De Persis and P. Frasca, "Robust self-triggered coordination with ternary controllers," IEEE Transactions on Automatic Control, Vol. 58, No.12, pp. 3024–3038, 2013.

[52] G. Duan, F. Xiao, and L. Wang, "Asynchronous periodic edge-event triggered control for double-integrator networks with communication time delays," IEEE transactions on cybernetics, Vol. 48, No. 2, pp.675–688, 2017.

[53] J. Hoelscher, "Diffused art and diffracted objecthood: Painting in the distributed field," CAA: College Art Association, 2014.

[54] N. A. Lynch, "Distributed algorithms," Elsevier, 1996.

Doctor Thesis, Kobe University
“Analysis of Consensus State and Design of Triggering Functions for Event-triggered Control of Multi-agent Systems”, 87 pages
Submitted on January, 20, 2022
The date of publication is printed in cover of repository version published in Kobe University Repository Kernel.