SOME ERGODIC THEOREMS AND ITS APPLICATIONS TO LIMIT THEOREMS FOR ADDITIVE FUNCTIONALS OF COMPLEX BROWNIAN MOTION

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Doctoral Dissertation

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(いくつかのエルゴード定理とその複素ブラウン運動の加法的汎関数の極限定理への応用)

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Chapter 1

Introduction

Let z_t, $z_0 = 0$, be a complex Brownian motion starting at the origin. Many works have been done on the limit theorems for additive functionals of z_t. The main purpose of this paper is to present some extentions of the old results and related theorems. Well-known classical results are due to G. Kallianpur and H. Robbins [5] for occupation times and to F. Spitzer [11] for winding number of z_t around a given non-zero point: Since z_t is neighborhood recurrent, for any open domain D in $\mathbf{C}$, the random occupation time of D by z_t before time t, $\int_0^t 1_D(z_s)\,ds$, has limit ∞ almost surely as $t \to \infty$. Then Kallianpur-Robbins showed that if $0 < \text{area}(D) < \infty$, then

$$(1.1) \qquad \frac{1}{\log t}\int_0^t 1_D(z_s)\,ds \quad \xrightarrow{d} \quad \frac{1}{2\pi}\text{area}(D)\,H \qquad (t \to \infty),$$

where H is a random variable with the standard exponential distribution and $\xrightarrow{d}$ indicates convergence in distribution. On the other hand, since z_t does not hit points, we can define a winding process $\Theta_a(t)$, the continuous total angle wound by z_t around the point $a \in \mathbf{C}\backslash\{0\}$ up to time t. More precisely, we define $X_a(t)$ and $\Theta_a(t)$ by

$$(1.2) \qquad \int_0^t \frac{dz_s}{z_s - a} = \log\left(\frac{z_t - a}{-a}\right) := X_a(t) + \sqrt{-1}\,\Theta_a(t).$$

Then Spitzer showed that

$$(1.3) \qquad \frac{2}{\log t}\,\Theta_a(t) \quad \xrightarrow{d} \quad C \qquad (t \to \infty),$$

where C is a random variable with the standard Cauchy distribution.

Y. Kasahara and S. Kotani [6] generalized these classical results from the standpoint of the convergence as *stochastic processes*: In the following, we denote the local time at 0 of one-dimensional Brownian motion $\xi(t)$ by $l(t, 0, \xi)$, the maximum process $\max_{0\le s\le t}\xi(s)$ of ξ by $\mu(t,\xi)$ and the Legesgue measure on $\mathbf{C}$ by $m(dz)$. Set

$$\tau(t) = \int_0^t \frac{ds}{|z_s - a|^2}.$$

Since $\tau(t) = \langle X_a \rangle_t = \langle \Theta_a \rangle_t$ (Generally, $\langle M \rangle_t$ is the usual quadratic variation process of a conformal (local) martingale $M(t)$) , it is clear from the Knight theorem that

$$(1.4) \qquad \frac{1}{\lambda} \int_0^{\tau^{-1}(\lambda^2 t)} \frac{dz_s}{z_s - a} \quad \xrightarrow{L} \quad \xi(t) + \sqrt{-1}\,\eta(t) \qquad (\lambda \to \infty),$$

where $\xi(t)$ and $\eta(t)$ are mutually independent one-dimensional Brownian motions and $\xrightarrow{L}$ indicates convergence in law on the continuous path-space. Kasahara-Kotani showed that if f is a bounded function from $\mathbf{C}$ to $\mathbf{C}$ satisfying $\int_{\mathbf{C}} |f(z)||z|^\varepsilon \, m(dz) < \infty$ for some $\varepsilon > 0$, then it hold that

$$(1.5) \qquad \frac{1}{\lambda} \int_0^{\tau^{-1}(\lambda^2 t)} f(z_s)\, ds \quad \xrightarrow{L} \quad 2l(t, 0, \xi)\, \overline{f} \qquad (\lambda \to \infty)$$

and

$$(1.6) \qquad \frac{1}{2\lambda} \log\left(\tau^{-1}(\lambda^2 t) + 1\right) \quad \xrightarrow{L} \quad \mu(t, \xi) \qquad (\lambda \to \infty)$$

jointly with (1.4), where $\overline{f} = (1/2\pi) \int_{\mathbf{C}} f(z)\, m(dz)$. Denoting the right continuous time inverse $\mu^{-1}(t, \xi)$ by $\sigma(t, \xi)$, We obtain from (1.5) and (1.6) that

$$(1.7) \qquad \frac{1}{\lambda} \int_0^{e^{2\lambda t} - 1} f(z_s)\, ds \quad \xrightarrow{f.d.} \quad 2l(\sigma(t, \xi), 0, \xi)\, \overline{f} \qquad (\lambda \to \infty)$$

and from (1.4) and (1.6) that

$$(1.8) \qquad \frac{1}{\lambda} \int_0^{e^{2\lambda t} - 1} \frac{dz_s}{z_s - a} \quad \xrightarrow{f.d.} \quad t + \sqrt{-1}\,\eta(\sigma(t, \xi)) \qquad (\lambda \to \infty),$$

where $\xrightarrow{f.d.}$ indicates convergence in the sense of finite dimensional distributions. Note that $\sigma(t, \xi) = \inf\{u; \xi(u) = t\}$. Now we can understand (1.1) and (1.3) as the section at $t = 1$ of (1.7) and (1.8), respectively. Indeed, $l(\sigma(t, \xi), 0, \xi)$ has the exponential distribution with mean t for fixed $t > 0$ and $\eta(\sigma(t, \xi))$ is a Cauchy process.

Kasahara-Kotani [6] discussed also the case that $\overline{f} = 0$. In this case, the study is intimately related by Itô's formula to the study of an additive functional of z_t given in the form

$$(1.9) \qquad \frac{1}{\sqrt{\lambda}} \int_0^{\tau^{-1}(\lambda^2 t)} F(z_s)\, dz_s$$

for some function F. P. Messulam and M. Yor [7] discussed (1.9) itself when F is bounded with compact support. The limit law of (1.9) as $\lambda \to \infty$ is $B(2l(t,0,\xi)\overline{F})$, where $B(t)$ is a complex Brownian motion independent of ξ.

On the other hand, J. Pitman and M. Yor [8],[9] gave another extention of Spitzer's law (1.3): Let $a_1, \cdots, a_n$ be given distinct points on $\mathbf{C}\setminus\{0\}$. We write the processes X_{a_i} and Θ_{a_i} given in (1.2) corresponding to a_i simply by X_i and Θ_i, respectively. Then Pitman-Yor showed that

$$(1.10) \qquad \left\{\frac{2}{\log t}\Theta_i(t),\ l(t,0,X_i)\right\}_{1\le i\le n} \xrightarrow{d} \{W_i + W_+,\ \Lambda\}_{1\le i\le n} \qquad (t\to\infty),$$

where for each i the triple $(W_i,\ W_+,\ \Lambda)$ is equivalent in law to

$$(1.11) \qquad \left(\int_0^{\sigma(1)} 1_{(\xi(s)<0)}\, d\eta(s),\ \int_0^{\sigma(1)} 1_{(\xi(s)>0)}\, d\eta(s),\ l(\sigma(1),0,\xi)\right)$$

and the $n+1$ random variables $W_1, \cdots, W_n,\ W_+$ are mutually conditionally independent given Λ. Here $\xi(t)$ and $\eta(t)$ are mutially independent one-dimensional Brownian motions and $\sigma(1) = \inf\{u; \xi(u) = 1\}$. The $W_1, \cdots, W_n$ are individual components for each point a_i, attributable to *small windings* about a_i, and the W_+ is a component in common to all points, attributable to *big windings* made when z_t is far from all points.

We can investigate their result (1.10)–(1.11) from the viewpoint of (1.4)–(1.6). The facts (1.4)–(1.6) are that

$$(1.12) \qquad \left\{\begin{array}{c} \dfrac{1}{\lambda}\displaystyle\int_0^{\tau_i^{-1}(\lambda^2 t)} \frac{dz_s}{z_s - a_i} \\ \dfrac{1}{\lambda}\displaystyle\int_0^{\tau_i^{-1}(\lambda^2 t)} f(z_s)\, ds \\ \dfrac{1}{2\lambda}\log\left(\tau_i^{-1}(\lambda^2 t) - 1\right) \end{array}\right\} \xrightarrow{L} \left\{\begin{array}{c} \xi_i(t) + \sqrt{-1}\,\eta_i(t) \\ 2l(t,0,\xi_i)\,\overline{f} \\ \mu(t,\xi_i) \end{array}\right\} \qquad (\lambda\to\infty)$$

for each i, where $\tau_i(t) = \int_0^t |z_s - a_i|^{-2}\, ds$ and (ξ_i, η_i) is a two-dimensional Brownian motion. If we consider these processes jointly for $i = 1, \cdots, n$, then the Brownian motions $\xi = (\xi_1, \cdots, \xi_n)$ and $\eta = (\eta_1, \cdots, \eta_n)$ have the following structure:

$$(1.13) \qquad \xi_i(t) = \alpha_i\left(\int_0^t 1_{(\xi_i(s)<0)}\, ds\right) + \alpha_+\left(\int_0^t 1_{(\xi_i(s)>0)}\, ds\right)$$

$$(1.14) \qquad \eta_i(t) = \beta_i\left(\int_0^t 1_{(\xi_i(s)<0)}\, ds\right) + \beta_+\left(\int_0^t 1_{(\xi_i(s)>0)}\, ds\right),$$

where $\alpha_1,\cdots,\alpha_n,\alpha_+,\beta_1,\cdots,\beta_n,\beta_+$ are $n+2$ mutually independent 1-dimensional Brownian motions. [1] Then as a corollary to (1.12), we have

$$\left\{\frac{1}{\lambda}\Theta_i(e^{2\lambda t}-1)\right\}_{1\le i\le n} \quad \xrightarrow{f.d.} \quad \left\{\beta_i\left(\int_0^{\sigma_i(t)} 1_{(\xi_i(s)<0)}\,ds\right)+\beta_+\left(\int_0^{\sigma_i(t)} 1_{(\xi_i(s)>0)}\,ds\right)\right\}_{1\le i\le n}$$

as $\lambda\to\infty$, where $\sigma_i(t)=\inf\{u;\xi_i(u)=t\}$. Thus we can understand Pitman-Yor's result (1.10)–(1.11) as the section at $t=1$ of the above convergence. These facts were pointed out and proved by S. Watanabe in an unpublished note [12].

Pitman-Yor [10] extended their results (1.10)–(1.11) to the case of

$$\left\{\frac{2}{\log t}\int_0^t f_i(e^{\sqrt{-1}\Theta_i(s)})\,d\Theta_i(s)\right\}_{1\le i\le n}, \tag{1.15}$$

where $f_1,\cdots,f_n$ are bounded Borel functions on $\mathbf{C}$. In this case, another Brownian motions (or Gaussian random measures) independent of (ξ,η) appear in the limit process. Can we reproduce and extend also this result in the context of Kasahara-Kotani-Watanabe?

In Chapter 2, we execute this program above: We extend (1.12) to additive functionals of z_t given in the form

$$\begin{aligned}\frac{1}{\lambda N_{ij}(\lambda)}\int_0^{\tau_i^{-1}(\lambda^2 t)}\frac{f_{ij}(z_s)}{z_s-a_i}\,dz_s \\ = \frac{1}{\lambda N_{ij}(\lambda)}\int_0^{\tau_i^{-1}(\lambda^2 t)} f_{ij}\left(a_i-a_ie^{X_i(s)+\sqrt{-1}\Theta_i(s)}\right)d(X_i(s)+\sqrt{-1}\,\Theta_i(s)),\end{aligned} \tag{1.16}$$

where $N_{ij}(\lambda)$ are some normalizing functions. We call the above additive functionals "winding-type". The case that $f_{ij}\equiv 1$ corresponds to windings, and the case that $f_{ij}(a_i-a_ie^{x+\sqrt{-1}\theta})$ depends only on θ corresponds to (1.15). We do not assume that each f_{ij} is bounded, but we treat, roughly speaking, the case that the asymptotic behaviour of $f_{ij}(a_i-a_ie^{\lambda x+\sqrt{-1}\theta})$ as $\lambda\to\infty$ is $|x|^{\rho_{ij}}c_{ij}(\theta)$ where $\rho_{ij}>-1/2$ and $c_{ij}\in L^2(0,2\pi)$.

In our proof of the limit theorem, an ergodic theorem such that

[1] The processes such as (1.13) exist uniquely in the sense of law. Indeed, since by Tanaka's formula we can express the processes in the right hand side of (1.13) as

$$\begin{cases}\alpha_i\left(\int_0^t 1_{(\xi_i(s)<0)}\,ds\right)=r_i\left(\int_0^t 1_{(\xi_i(s)<0)}\,ds\right)+l(t,0,\xi_i)\\ \alpha_+\left(\int_0^t 1_{(\xi_i(s)>0)}\,ds\right)=r_+\left(\int_0^t 1_{(\xi_i(s)>0)}\,ds\right)-l(t,0,\xi_i),\end{cases}$$

where $r_1,\cdots,r_n,r_+$ are mutually independent reflecting Brownian motions, using these Skorohod equations and the excursion theory, we can construct $(\xi_i)_{1\le i\le n}$ satisfying (1.13) by the Poisson point process p^+ of positive Brownian excursion corresponding to r_+ and the Poisson point processes p_i^-, $i=1,\cdots,n$, of negative Brownian excursions corresponding to r_i. We can also see from this construction that $l(\mu^{-1}(t,\xi_i),0,\xi_i)$, $i=1,\cdots,n$, are identical to $l(\sigma_+(t),0,r_+)$ where $\sigma_+(t)=\inf\{u;r_+(u)=t\}$. See remark 2.4.1 in the last part of Chapter 2, section 2.4.

$$E \sup_{0\le t\le T}\left|\int_0^t |X_i(s)|^{p_{ij}} c_{ij}\big(\Theta_i(\lambda s)\big)\,ds - \overline{c_{ij}}\int_0^t |X_i(s)|^{p_{ij}}\,ds\right| \longrightarrow 0 \qquad (\lambda\to\infty), \tag{1.17}$$

where $\overline{c_{ij}} = (1/2\pi)\int_0^{2\pi} c_{ij}(\theta)\,d\theta$, plays an essential role. We prove this ergodic theorem for a class of diffusion processes $(X(t),\Theta(t))$ on $\mathbf{R}^d \times M$ where M is a compact C^∞-Riemannian manifold.

In Chapter 3, we prove another ergodic theorem for $(X(t),\Theta(t))$ on $\mathbf{R}^d \times M$ such that

$$N(\lambda)\, E \sup_{0\le t\le T}\left|\int_0^t g\big(X(\lambda s),\Theta(\lambda s)\big)\,ds - \int_0^t \overline{g}\big(X(\lambda s)\big)ds\right| \longrightarrow 0 \qquad (\lambda\to\infty), \tag{1.18}$$

where $g:\mathbf{R}^d \times M \mapsto \mathbf{R}^1$ is a function satisfying some conditions for integrability, $\overline{g}(x) = (1/2\pi)\int_0^{2\pi} g(x,\theta)\,d\theta$ and $N(\lambda)$ is some normalizing function corresponding to g. Using this ergodic theorem, we extend naturally the class of functionals in the limit theorems for (1.5) and (1.9). More precisely, we prove the convergence (1.5) for $f \in L^1(\mathbf{C})\cap L^p(\mathbf{C})$ $(1 < p \le \infty)$ and the convergence of (1.9) for $F \in L^2(\mathbf{C}) \cap L^p(\mathbf{C})$ $(2 < p \le \infty)$.

In Chapter 4, we extend the ergodic theorem (1.17) to a similar form as (1.18). Moreover, we show that these ergodic theorems are also valid in the case that M is $\mathbf{R}^m$ which is endowed with the normal distribution $\nu(d\theta)$ instead of $d\theta$ and Θ_t is an Ornstein-Uhlenbeck process.

Chapter 2

On Limit Theorems Related to a Class of "Winding-Type" Additive Functionals of Complex Brownian Motion

2.1 Introduction

Let $z(t) = x(t) + \sqrt{-1}\,y(t)$, $z(0) = 0$, be a complex Brownian motion starting at the origin. Main purpose of this chapter is to reproduce and extend some results of Pitman-Yor by the method of Kasahara-Kotani: In particular we discuss the convergence as *stochastic processes* of time scaled additive functionals belonging to a little more general class.

First, we describe briefly the main idea of Kasahara-Kotani. In order to study the limit process as $\lambda \to \infty$ of additive functionals $A^\lambda(t)$, $\lambda > 0$, given in the form

$$A^\lambda(t) = \frac{1}{\lambda N(\lambda)} \int_0^{u(\lambda t)} f(z_s) dz_s$$

where $u(t) = e^{2t} - 1$ and $N(\lambda)$ is some normalizing function, we set

$$Z(t) = \log(z(t) + 1)$$

and introduce an increasing process

$$\tau^\lambda(t) = \frac{1}{\lambda} u^{-1}(\langle Z \rangle^{-1}(\lambda^2 t)).$$

(Generally, $\langle M \rangle (t)$ is the usual quadratic variation process of a conformal (local) martingale $M(t)$ and $g^{-1}(t)$ is the right continuous inverse function of a continuous increasing function $g(t)$.) Then, by the time substitution, we have

$$A^\lambda(\tau^\lambda(t)) = \frac{1}{N(\lambda)} \int_0^t f(e^{\lambda \widehat{Z}^\lambda(s)} - 1) e^{\lambda \widehat{Z}^\lambda(s)} d\widehat{Z}^\lambda(s)$$

where $\widehat{Z}^\lambda(t) = \frac{1}{\lambda} Z(\langle Z \rangle^{-1}(\lambda^2 t))$. Note that $\widehat{Z}^\lambda(t)$ is a complex Brownian motion for every $\lambda > 0$. The limit process of $A^\lambda(t)$ can be found if we can obtain the limit process as $\lambda \to \infty$ of the joint continuous processes $\{ A^\lambda(\tau^\lambda(t)),\ \widehat{Z}^\lambda(t),\ \tau^\lambda(t) \}$. The limit process of $\{ \widehat{Z}^\lambda(t),\ \tau^\lambda(t) \}$ is given by $\{ b(t),\ \mu(t) \}$ where $b(t)$ is a complex Brownian motion and $\mu(t) = \max_{0 \le s \le t} \mathcal{R}e[b(s)]$ (*cf.* Lemma 3.1 of [6]). The study of convergence for the above joint processes is therefore reduced to that for

$$\frac{1}{N(\lambda)} \int_0^t f(e^{\lambda b(s)} - 1) e^{\lambda b(s)} db(s)$$

as $\lambda \to \infty$. If we represent $b(t)$ as

$$b(t) = x(t) + \sqrt{-1} \int_0^t d\theta(s),$$

where $\theta(t)$ is a Brownian motion on the unit circle $\mathbf{T} = \mathbf{R}/2\pi\mathbf{Z} \simeq [0, 2\pi]$ so that $(x(t), \theta(t))$ is a Brownian motion on the Riemannian manifold $\mathbf{R} \times \mathbf{T}$, then, in this study, the ergodic property of $\theta(t)$ plays an important role; indeed, it is a homogenization problem for $(x(t), \theta(t))$.

We would apply this method of Kasahara-Kotani to some problems discussed by Pitman-Yor, namely to the study of joint limit distribution, as $\lambda \to \infty$, of the processes $(A_{ij}{}^\lambda)$ given by

$$\text{(2.1)} \qquad A_{ij}{}^\lambda(t) = \frac{1}{\lambda N_{ij}(\lambda)} \int_0^{u(\lambda t)} \frac{f_j(z_s)}{z_s - a_i} dz_s,$$

where $a_1, \cdots, a_n$ are distinct points on $\mathbf{C} \backslash \{0\}$ and f_j, $j = 1, \cdots, m$, are some Borel functions on $\mathbf{C}$. If $f_j \equiv 1$, then $\mathcal{I}m[A_{ij}{}^\lambda(t)]$ is a normalized algebraic total angle wound by $z(t)$ around a_i up to the time $e^{2\lambda t} - 1$. Writing

$$f_j(a_i - a_i e^{x + \sqrt{-1}\theta}) = g_{ij}(x, \theta), \qquad (x, \theta) \in \mathbf{R} \times \mathbf{T},$$

Pitman-Yor discussed the case when g_{ij} depend only on θ. Here we consider a more general case by introducing a notion of functions regularly varying at point a_i and also at the point at infinity. This class of functions was introduced by S. Watanabe in an unpublished note. In order to apply Kasahara-Kotani's method to this class of additive functionals, we need an ergodic theorem for Brownian motion $(x(t), \theta(t))$ on $\mathbf{R} \times \mathbf{T}$ which we establish in section 2.2 by using the method of eigenfunction expansions.

Finally, we summarize the contents of this chapter. In section 2.2, we consider a class of diffusion processes on $\mathbf{R}^d \times M$ where M is a compact Riemannian manifold and obtain an ergodic theorem for them. In section 2.3, we apply the result of section 2.2 to a homogenization problem for Brownian motion $(x(t), \theta(t))$ on $\mathbf{R} \times \mathbf{T}$ and thereby describe the limit process as $\lambda \to \infty$ of the joint processes

$$\left\{ \frac{1}{N_i(\lambda)} \int_0^t f_i(a - ae^{\lambda z(s)}) dz(s) \right\}_{1 \leq i \leq m},$$

where $a \in \mathbf{C} \backslash \{0\}$, $z(t) = x(t) + \sqrt{-1} \int_0^t d\theta(s)$ so that $z(t)$ is a complex Brownian motion, and f_i are taken from the class of regularly varying functions in the sense given by Definition 2.3.1. Here, the asymptotic Knight's theorem of Pitman-Yor [10] for a class of conformal martingales also plays an important role. In section 2.4, we obtain the joint limit theorem for additive functionals of the form (2.1) by applying the results in section 2.3.

2.2 An ergodic theorem for some class of diffusion processes on compact manifolds

Let M be an m-dimensional compact (connected) C^∞-Riemannian manifold without boundary and $(\Theta_t)_{t \geq 0}$ be a Brownian motion on M (see Ikeda and Watanabe [4], Chapter 5, section 4). The generator of (Θ_t) is $(1/2)\Delta_M$, where Δ_M is the Laplace-Beltrami operator for M. Since M is compact, Δ_M has pure point spectrum

$$(2.2) \qquad 0 = \lambda_0 > -\lambda_1 \geq -\lambda_2 \geq \cdots$$

and we denote the corresponding normalized eigenfunctions by $\{\varphi_n\}$. It is known that the transition density $q(t, \theta, \eta)$ of (Θ_t) has the following expansion:

$$(2.3) \qquad q(t, \theta, \eta) = \sum_{n=0}^{\infty} e^{-\lambda_n t} \varphi_n(\theta) \varphi_n(\eta),$$

which converges uniformly in (θ, η) for every $t > 0$ (see Chavel [1] p.140).

Let $(X_t)_{t \geq 0}$ be an $\mathbf{R}^d$-valued diffusion process determined by the stochastic differential equation

$$(2.4) \qquad dX_t = \sigma(X_t)\, dB_t + b(X_t)\, dt,$$

where $\sigma(x)$ and $b(x)$ are bounded and smooth, $\sigma(x)$ is uniformly non-degenerate and $(B_t)_{t\geq 0}$ is a $d-$dimensional Brownian motion.

We assume that X and Θ are independent and $X_0 = 0$ and $\Theta_0 = \theta_0$ ($\theta_0 \in M$) throughout this section.

Our main result in this section is as follows:

Theorem 2.2.1 *Let h be a Borel measurable function from $\mathbf{R}^d$ to $\mathbf{R}^1$ and f be a Borel measurable function from M to $\mathbf{R}^1$ satisfying the following conditions:*

(1) $|h(x)| \leq \text{const.}\, |x|^\alpha$ *for every* $x \in \mathbf{R}^d$ *for some* $\alpha > -\min(2, d)$,

(2) f *is in* $L^p(M) = L^p(M, d\theta)$ *for some* p *with* $p \geq 1$ *and* $p > m/(\alpha + 2)$, *where* $d\theta$ *is the volume element of* M ,

(3) f *is null charged i.e.*

$$\int_M f(\theta)\, d\theta = 0.$$

Then for every $T > 0$, *it holds that*

$$E_{(0,\theta_0)}\left[\sup_{0\leq t\leq T}\left|\int_0^t h(X_s) f(\Theta_{\lambda s})\, ds\right|\right] \longrightarrow 0$$

as $\lambda \to \infty$.

To prove our theorem, we prepare some estimates for $E_x|h(X_t)|$ and $E_\theta|f(\Theta_t)|$.

Lemma 2.2.1 *Suppose that* $h\colon \mathbf{R}^d \mapsto \mathbf{R}^1$ *satisfies* $|h(x)| \leq \text{const.}\, |x|^\alpha$ *for every* $x \in \mathbf{R}^d$, *where* $\alpha > -d$. *Then for every* $x \in \mathbf{R}^d$ *and* $t > 0$,

$$E_x|h(X_t)| \leq \text{const.}\, t^{\alpha/2} + \text{const.}\, |x|^\alpha\, 1_{(\alpha>0)}.$$

Proof. From the assumption of X_t, we have the following estimate for the transition density $p(t, x, y)$ of X_t :

$$(2.5) \qquad p(t,x,y) \leq \text{const.}\, t^{-d/2} \exp\left(-\frac{\text{const.}\, |x-y|^2}{2t}\right).$$

(See Friedman [3], p.141, Theorem 4.5.)

Then, from the assumption for $h(x)$,

$$(2.6) \qquad E_x|h(X_t)| \leq \text{const.}\, t^{-d/2} \int_{\mathbf{R}^d} \exp\left(-\frac{\text{const.}\, |x-y|^2}{2t}\right) |y|^\alpha\, dy$$

$$= \text{const.} \int_{\mathbf{R}^d} \exp\left(-\frac{\text{const.}\, |\xi|^2}{2}\right) |\sqrt{t}\,\xi + x|^\alpha\, d\xi.$$

If $\alpha > 0$, the right hand side (RHS) of (2.6) is bounded by

$$\begin{aligned}
&\text{const.} \int_{\mathbf{R}^d} \exp\left(-\frac{\text{const.}\,|\xi|^2}{2}\right) (\sqrt{t}\,|\xi|)^\alpha \, d\xi \\
&\qquad + \text{const.} \int_{\mathbf{R}^d} \exp\left(-\frac{\text{const.}\,|\xi|^2}{2}\right) |x|^\alpha \, d\xi \\
&= \text{const.}\, t^{\alpha/2} + \text{const.}\, |x|^\alpha.
\end{aligned}$$

If $-d < \alpha \leq 0$, the RHS of (2.6) is bounded by

$$\begin{aligned}
&\text{const.}\, t^{\alpha/2} \int_{\mathbf{R}^d} \exp\left(-\frac{\text{const.}\,|\xi|^2}{2}\right) |\xi + x/\sqrt{t}|^\alpha \, d\xi \\
&= \text{const.}\, t^{\alpha/2} \int_{|\xi + x/\sqrt{t}| < 1} \exp\left(-\frac{\text{const.}\,|\xi|^2}{2}\right) |\xi + x/\sqrt{t}|^\alpha \, d\xi \\
&\qquad + \text{const.}\, t^{\alpha/2} \int_{|\xi + x/\sqrt{t}| \geq 1} \exp\left(-\frac{\text{const.}\,|\xi|^2}{2}\right) |\xi + x/\sqrt{t}|^\alpha \, d\xi \\
&\leq \text{const.}\, t^{\alpha/2} \int_{|\xi + x/\sqrt{t}| < 1} |\xi + x/\sqrt{t}|^\alpha \, d\xi \\
&\qquad + \text{const.}\, t^{\alpha/2} \int_{|\xi + x/\sqrt{t}| \geq 1} \exp\left(-\frac{\text{const.}\,|\xi|^2}{2}\right) d\xi \\
&\leq \text{const.}\, t^{\alpha/2}.
\end{aligned}$$

Q.E.D.

Lemma 2.2.2 *Suppose $f \in L^p(M \mapsto \mathbf{R}^1, d\theta)$ with some $p \geq 1$. Then for every $\theta \in M$ and $t > 0$,*

$$E_\theta |f(\Theta_t)| \leq (\text{const.}\, t^{-m/2p} + \text{const.}) \|f\|_p.$$

Proof. For a moment, let $p > 1$. First note that

$$E_\theta |f(\Theta_t)| = \int_M q(t,\theta,\eta) |f(\eta)| \, d\eta \leq \left(\int_M q(t,\theta,\eta)^q \, d\eta\right)^{1/q} \|f\|_p,$$

where $q(t,\theta,\eta)$ is the transition density of Θ_t and $1/q + 1/p = 1$.
Since we have the uniform estimate

$$q(t,\theta,\eta) \leq \text{const.}\, t^{-m/2} \qquad (t \downarrow 0)$$

(see Chavel [1], $p.154 \sim 155$), setting $\delta > 0$ small enough,

$$\begin{aligned}\left(\int_M q(t,\theta,\eta)^q \, d\eta\right)^{1/q} 1_{(t<\delta)} &\leq \left(\text{const.} \int_M t^{-m(q-1)/2} q(t,\theta,\eta)\, d\eta\right)^{1/q} 1_{(t<\delta)} \\ &= \text{const.}\, t^{-m/2p}\, 1_{(t<\delta)}.\end{aligned}$$

On the other hand, from (2.3) we have

$$\begin{aligned} q(t,\theta,\eta)\, 1_{(t>\delta)} &\leq \left(\sum_{n=0}^{\infty} e^{-\lambda_n t}\varphi_n(\theta)^2\right)^{1/2} \left(\sum_{n=0}^{\infty} e^{-\lambda_n t}\varphi_n(\eta)^2\right)^{1/2} 1_{(t>\delta)} \\ &\leq \left(\sum_{n=0}^{\infty} e^{-\lambda_n \delta}\varphi_n(\theta)^2\right)^{1/2} \left(\sum_{n=0}^{\infty} e^{-\lambda_n \delta}\varphi_n(\eta)^2\right)^{1/2} \\ &= q(\delta,\theta,\theta)^{1/2} q(\delta,\eta,\eta)^{1/2} \\ &\leq \text{const.} \end{aligned}$$

Hence

$$\left(\int_M q(t,\theta,\eta)^q \, d\eta\right)^{1/q} 1_{(t>\delta)} \leq \text{const.}\, V(M)^{1/q},$$

where $V(M) = \int_M d\theta$.
Therefore,

$$\begin{aligned} E_\theta|f(\Theta_t)| &\leq (\text{const.}\, t^{-m/2p}\, 1_{(t<\delta)} + \text{const.}\, V(M)^{1/q})\|f\|_p \\ &\leq (\text{const.}\, t^{-m/2p} + \text{const.})\|f\|_p. \end{aligned}$$

In the case that $p = 1$, we can prove the lemma similarly by replacing $(\int_M q(t,\theta,\eta)^q \, d\eta)^{1/q}$ with $\sup_\eta q(t,\theta,\eta)$. Q.E.D.

Proof of Theorem 2.2.1

First we will prove the theorem in the case that $f = \varphi_n$ for some $n \geq 1$. From now on we write the expectation $E_{(0,\theta_0)}$ simply by E.

Set

$$u_\lambda(x,\theta) = \int_0^\infty E_{(x,\theta)}\Big(h(X_s)\varphi_n(\Theta_{\lambda s})\Big)\, ds$$

and

$$M_t^\lambda = u_\lambda(X_t,\Theta_{\lambda t}) + \int_0^t h(X_s)\varphi_n(\Theta_{\lambda s})\, ds.$$

In order to prove that

$$E \sup_{0 \le t \le T} \left| \int_0^t h(X_s)\varphi_n(\Theta_{\lambda s})\, ds \right| \longrightarrow 0 \qquad (\lambda \to \infty), \tag{2.7}$$

it is clearly sufficient to prove that

$$E_{0 \le t \le T} \sup |u_\lambda(X_t, \Theta_{\lambda t})| \longrightarrow 0 \qquad (\lambda \to \infty) \tag{2.8}$$

and

$$E \sup_{0 \le t \le T} \left| M_t^\lambda \right| \longrightarrow 0 \qquad (\lambda \to \infty). \tag{2.9}$$

The convergence (2.8) is proved as follows. By the orthonormality of $\{\varphi_k\}$ and (2.3), we see the following identity:

$$\begin{aligned} E_\theta(\varphi_n(\Theta_{\lambda s})) &= \int_M q(\lambda s, \theta, \eta)\varphi_n(\eta)\, d\eta \\ &= e^{-\lambda_n \lambda s}\varphi_n(\theta) \qquad \text{for every } \theta \in M\ . \end{aligned}$$

Clearly $\varphi_n(\theta)$ is bounded since φ_n is continuous and M is compact and hence we have the basic estimate

$$E_\theta(\varphi_n(\Theta_{\lambda s})) \le \text{const.}\, e^{-\lambda_n \lambda s}. \tag{2.10}$$

By Lemma 2.2.1 and (2.10), we obtain the following estimate for u_λ:

$$\begin{aligned} |u_\lambda(x,\theta)| &\le \int_0^\infty E_x|h(X_s)| \left| E_\theta\big(\varphi_n(\Theta_{\lambda s})\big) \right| ds \\ &\le \text{const.} \int_0^\infty s^{\alpha/2} e^{-\lambda_n \lambda s}\, ds + \text{const.}\, |x|^\alpha 1_{(\alpha>0)} \int_0^\infty e^{-\lambda_n \lambda s}\, ds \\ &= \text{const.}\, \lambda^{-\alpha/2-1} + \text{const.}\, |x|^\alpha\, 1_{(\alpha>0)}\, \lambda^{-1}. \end{aligned} \tag{2.11}$$

Hence

$$E \sup_{0 \le t \le T} |u_\lambda(X_t, \Theta_{\lambda t})| \le \text{const.}\, \lambda^{-\alpha/2-1} + \text{const.}\, \lambda^{-1}\, E \sup_{0 \le t \le T} |X_t|^\alpha\, 1_{(\alpha>0)}.$$

Here

$$E \sup_{0 \le t \le T} |X_t|^\alpha < +\infty \tag{2.12}$$

holds for $\alpha > 0$. Indeed, if $\alpha > 1$, we have by (2.4) and the martingale inequality that

$$\begin{aligned}
E \sup_{0\le t\le T} |X_t|^\alpha &= E \sup_{0\le t\le T} \left| \int_0^t \sigma(X_s)\, dB_s + \int_0^t b(X_s)\, ds \right|^\alpha \\
&\le \text{const.}\, E \sup_{0\le t\le T} \left| \int_0^t \sigma(X_s)\, dB_s \right|^\alpha + \text{const.} \\
&\le \text{const.}\, E \left| \int_0^T \sigma(X_s)\, dB_s \right|^\alpha + \text{const.} \\
&\le \text{const.}\, E|X_T|^\alpha + \text{const.},
\end{aligned}$$

and the finiteness of $E|X_T|^\alpha$ follows from (2.5). It is easy to see that (2.12) is also valid for $0 < \alpha \le 1$ since

$$E \sup_{0\le t\le T} |X_t|^\alpha \le \text{const.} \left(E \sup_{0\le t\le T} |X_t|^2 \right)^{\alpha/2}$$

by Hölder's inequality. Thus (2.8) is proved.

We now show (2.9). Fixing $\lambda > 0$ and setting $\mathcal{F}_t = \sigma\{(X_s, \Theta_{\lambda s}); s \le t\}$, we can prove that $M_t{}^\lambda$ becomes an $(\mathcal{F}_t)$-martingale by a repeated use of Fubini's theorem. (Note by Lemma 2.2.1 and (2.10) that

$$\begin{aligned}
&E \left[\int_0^\infty \left| E_{(X_t, \Theta_{\lambda t})} \big(h(X_u) \varphi_n(\Theta_{\lambda u}) \big) \right| du \right] \\
&= E \left[\int_0^\infty \left| E_{X_t} \big(h(X_u) \big) \right| \left| E_{\Theta_{\lambda t}} (\varphi_n(\Theta_{\lambda u})) \right| du \right] < +\infty \qquad .)
\end{aligned}$$

Then we have

$$\begin{aligned}
&E \sup_{0\le t\le T} |M_t{}^\lambda| \le (E \sup_{0\le t\le T} |M_t{}^\lambda|^2)^{1/2} \le \text{const.}\, (E|M_T{}^\lambda|^2)^{1/2} \\
&\le \left(\text{const.}\, E\left| u_\lambda(X_T, \Theta_{\lambda T}) \right|^2 + \text{const.}\, E \left| \int_0^T h(X_s) \varphi_n(\Theta_{\lambda s}) ds \right|^2 \right)^{1/2}
\end{aligned}$$

by the martingale inequality. Hence it is only necessary to show that

$$I_1 = E \left| u_\lambda(X_T, \Theta_{\lambda T}) \right|^2 \longrightarrow 0 \qquad \text{as } \lambda \to \infty$$

and

$$I_2 = E \left(\int_0^T h(X_s) \varphi_n(\Theta_{\lambda s})\, ds \right)^2 \longrightarrow 0 \qquad \text{as } \lambda \to \infty.$$

We can easily see that $I_1 \to 0$ as $\lambda \to \infty$. Indeed, (2.11) implies that

$$I_1 \le \text{const.}\, \lambda^{-\alpha-2} + \text{const.}\, \lambda^{-\alpha/2-2} E|X_T|^{\alpha}\, 1_{(\alpha>0)} + \text{const.}\, \lambda^{-2} E|X_T|^{2\alpha}\, 1_{(\alpha>0)}$$

and the finiteness of $E|X_T|^{\alpha}\, 1_{(\alpha>0)}$ and $E|X_T|^{2\alpha}\, 1_{(\alpha>0)}$ follows from (2.5).

Finally we shall prove that $I_2 \to 0$ as $\lambda \to \infty$. By Lemma 2.2.1, (2.10) and Fubini's theorem, we have

$$\begin{aligned} I_2 &= 2E\left[\int_0^T ds \int_0^s du\, h(X_s)h(X_u)\varphi_n(\Theta_{\lambda s})\varphi_n(\Theta_{\lambda u})\right] \\ &= 2\int_0^T ds \int_0^s du\, E\left[h(X_u)E_{X_u}\left(h(X_{s-u})\right)\right] E\left[\varphi_n(\Theta_{\lambda u})E_{\Theta_{\lambda u}}\left(\varphi_n(\Theta_{\lambda(s-u)})\right)\right]. \end{aligned}$$

Since Lemma 2.2.1 implies that

$$\begin{aligned} &\left|E\left[h(X_u)E_{X_u}\left(h(X_{s-u})\right)\right]\right| \le E\left[|h(X_u)|\, E_{X_u}\left(|h(X_{s-u})|\right)\right] \\ &\le \text{const.}\,(s-u)^{\alpha/2}u^{\alpha/2} + u^{\alpha}\, 1_{(\alpha>0)} \end{aligned}$$

and (2.10) implies that

$$\left|E\left[\varphi_n(\Theta_{\lambda u})E_{\Theta_{\lambda u}}\left(\varphi_n(\Theta_{\lambda(s-u)})\right)\right]\right| \le \text{const.}\, e^{-\lambda_n\lambda(s-u)},$$

$$\begin{aligned} I_2 &\le \text{const.} \int_0^T ds \int_0^s du\, e^{-\lambda_n\lambda(s-u)}(s-u)^{\alpha/2}u^{\alpha/2} \\ &\qquad + \text{const.} \int_0^T ds \int_0^s du\, e^{-\lambda_n\lambda(s-u)}u^{\alpha}\, 1_{(\alpha>0)} \\ &\le \text{const.}\, \lambda^{-\alpha/2-1} + \text{const.}\, \lambda^{-1}\, 1_{(\alpha>0)} \\ &\longrightarrow 0 \qquad (\lambda \to \infty). \end{aligned}$$

Thus the proof of (2.7) is complete.

Next we will show Theorem 2.2.1 for general f satisfying the conditions (2) and (3). Let $\mathcal{L}$ be the set of all linear combinations of finite number of $\varphi_1, \varphi_2, \cdots$. We know by (2.7) that Theorem 2.2.1 holds for $f \in \mathcal{L}$. Furthermore, by Lemma 2.2.1 and Lemma 2.2.2 we have that

$$\begin{aligned} &E \sup_{0\le t\le T}\left|\int_0^t h(X_s)f(\Theta_{\lambda s})\, ds\right| \le \int_0^T \left(E|h(X_s)|\right)\left(E|f(\Theta_{\lambda s})|\right) ds \\ &\le \left(\text{const.} \int_0^T s^{\alpha/2}(\lambda s)^{-m/2p}\, ds + \text{const.} \int_0^T s^{\alpha/2}\, ds\right) \|f\|_p \\ &\le (\text{const.}\, \lambda^{-m/2p} + \text{const.})\|f\|_p. \end{aligned}$$

Therefore,

$$E \sup_{0\le t\le T} \left| \int_0^t h(X_s) f(\Theta_{\lambda s})\, ds \right| \le \Big(o(1) + \text{const.} \Big) \|f\|_p \qquad (\lambda \to \infty).$$

To complete the proof we have only to note the following facts: Since M is compact, any continuous function f on M satisfying the null charged condition (3) is uniformly approximated by functions of $\mathcal{L}$ (*cf.* Chavel [1], p.139-140), and continuous functions are dense in $L^p(M)$. Q.E.D.

2.3 Some limit theorem for additive functionals of a Brownian motion on the cylinder

In this section, we will prove some limit theorem (Theorem 2.3.1) for additive functionals of a Brownian motion on the cylinder $\mathbf{R} \times \mathbf{T}$, $\mathbf{T} = \mathbf{R}/2\pi\mathbf{Z} \simeq [0, 2\pi]$, as an application of Theorem 2.2.1 in the previous section.

First of all we prepare some notations for conformal martingales. Let $z(t) = x(t) + \sqrt{-1}\, y(t)$ be a conformal martingale *i.e.* $\langle x\rangle(t) = \langle y\rangle(t)$ and $\langle x, y\rangle(t) = 0$. We denote these common processes $\langle x\rangle(t)$ and $\langle y\rangle(t)$ by $\langle z\rangle(t)$. Throughout this paper we always denote by $\langle z\rangle^{-1}(t)$ the process obtained by the right continuous inverse function of $t \mapsto \langle z\rangle(t)$. If $\langle z\rangle(t) \to \infty$ $(t \to \infty)$ a.s., then the time changed process $z(\langle z\rangle^{-1}(t))$ becomes a complex Brownian motion by the Knight theorem. We always denote this Brownian motion by $\widehat{z}(t)$.

If $z_1(t) = x_1(t) + \sqrt{-1}\, y_1(t)$ and $z_2(t) = x_2(t) + \sqrt{-1}\, y_2(t)$ are conformal martingales, then we denote by $\langle z_1, z_2\rangle(t)$ the matrix of quadratic variation processes

$$\begin{pmatrix} \langle x_1, x_2\rangle(t) & \langle x_1, y_2\rangle(t) \\ \langle y_1, x_2\rangle(t) & \langle y_1, y_2\rangle(t) \end{pmatrix}.$$

Note that $\langle z, z\rangle(t) = \langle z\rangle(t) \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ and

$$\begin{aligned} &\left\langle \int_0^{\cdot} \Phi_1(s)\, dz_s, \int_0^{\cdot} \Phi_2(s)\, dz_s \right\rangle(t) \\ &= \int_0^t \mathcal{R}e(\Phi_1\Phi_2{}^*)(s)\, d\langle z\rangle_s \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \int_0^t \mathcal{I}m(\Phi_1\Phi_2{}^*)(s)\, d\langle z\rangle_s \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}. \end{aligned}$$

(Here Φ^* represents the complex conjugate of Φ .)

Let $(S, \mathcal{B}(S), \mu)$ be a measure space and set $\mathcal{F} = \{\, A \in \mathcal{B}(S);\ \mu(A) < +\infty \,\}$. A family of random variables $M = \{M(A); A \in \mathcal{F}\}$ is called a (real) Gaussian random measure on

S with mean 0 and variance measure μ if and only if M is a Gaussian system such that $E[M(A)] = 0$ and $E[M(A)M(B)] = \mu(A \cap B)$ hold for any $A, B \in \mathcal{F}$. Furthermore, a complex Gaussian random measure M on S with mean 0 and variance measure μ is by definition a family of complex random variables $M(A)$ which can be expressed in the form $M(A) = M_1(A) + \sqrt{-1}\, M_2(A)$ where M_1 and M_2 are mutually independent Gaussian random measures with mean 0 and the same variance measure μ.

Throughout this section, we always denote $L^2(\mathbf{T} \mapsto \mathbf{C}, d\theta/2\pi)$ by $L^2(0, 2\pi)$. Let us introduce a definition of regularly varying functions of a complex variable:

Definition 2.3.1. A function $f(z)$ defined on $0 < |z - a| < R$ is called *regularly varying at $a(\neq 0)$ with order $\rho(> -1/2)$* if there exist some slowly varying (at ∞) function $L(\lambda)$, $c(\theta) \in L^2(0, 2\pi)$ and $r > (\log |a/R|) \vee 0$, which have the following two properties:

1°) There exist some constants $\varepsilon \geq 0$, $K > 0$, and $\lambda_0 > 0$ such that $\varepsilon < \rho + 1/2$ and

$$\int_0^{2\pi} d\theta \int_{-\infty}^{-r/\lambda} \left|(\lambda^\rho L(\lambda))^{-1} f(a - ae^{\lambda x + \sqrt{-1}\theta})\right|^2 e^{-x^2/s}\, dx$$
$$\leq\ K \cdot \left(s^{\rho - \varepsilon + 1/2}\, 1_{(0<s<1)} + s^{\rho + \varepsilon + 1/2}\, 1_{(s \geq 1)}\right)$$

for all $\lambda \geq \lambda_0$ and $s > 0$.

2°) For any $s > 0$,

$$\int_0^{2\pi} d\theta \int_{-\infty}^{-r/\lambda} \left|(\lambda^\rho L(\lambda))^{-1} f(a - ae^{\lambda x + \sqrt{-1}\theta}) - c(\theta)(-x)^\rho\right|^2 e^{-x^2/s}\, dx$$
$$\longrightarrow 0 \qquad \text{as } \lambda \to \infty.$$

For $a = 0$, we substitute the condition $r > (\log |a/R|) \vee 0$ with the condition $r > (-\log R) \vee 0$ and a $-ae^{\lambda x + \sqrt{-1}\theta}$ with $e^{\lambda x + \sqrt{-1}\theta}$ in the above definition.

We call $N(\lambda) = \lambda^\rho L(\lambda)$ and $c(\theta)$ *the regular normalizing function of f at a* and *the asymptotic angular component of f at a*, respectively.

Furthermore, we call a function $f(z)$ defined on $|z| > R$ *regularly varying at ∞ with order ρ* if $\tilde{f}(z) = f(1/z)$ is regularly varying at 0 with order ρ. The regular normalizing function of f at ∞ and the asymptotic angular component of f at ∞ are those of $\tilde{f}$ at 0, respectively.

Remark 2.3.1. The class of functions regularly varying both at a and at ∞ defined above contains the original class of functions regularly varying at a defined by Watanabe([12]).

Example 1. For any given domain $D \subset \mathbf{C}$ such that D or D^c is bounded, the function $f(z) = 1_D(z)$ is regularly varying at a with order 0 for any $a \in \mathbf{C} \cup \{\infty\} \setminus \partial D$. The regular normalizing function of f at a is 1 and the asymptotic angular component of f at a is 1 if $a \in D$ and 0 if $a \notin D$. (Here we consider that $\infty \in D$ when D^c is bounded.)

Example 2. Let $g(\theta) \in L^2(0, 2\pi)$ and let $h(x)$ be an ordinary regularly varying function at ∞ with exponent $\rho(< \infty)$ such that

$$\left|\frac{h(\lambda x)}{h(\lambda)}\right| \leq K \cdot \left(x^{\rho-\varepsilon} 1_{(|x|<1)} + x^{\rho+\varepsilon} 1_{(|x|\geq 1)}\right)$$

for all λ, where $K > 0$ and $\varepsilon \geq 0$ are some constants satisfying $\varepsilon < \rho + 1/2$. Then

$$f(z) = g\left(\arg \frac{z-a}{-a}\right) \cdot h\left(-\log \left|\frac{z-a}{-a}\right|\right)$$

is regularly varying at a with order ρ. The regular normalizing function of f at a is $h(\lambda)$ and the asymptotic angular component of f at a is $g(\theta)$.

When $f(z)$ is regularly varying at ∞, the asymptotic behavoir of $f(a - ae^{\lambda x + \sqrt{-1}\theta})\, 1_{(x>0)}$ as $\lambda \to \infty$ for every $a \neq 0$ can be described using that of $f(e^{\lambda x + \sqrt{-1}\theta})\, 1_{(x>0)}$:

Proposition 2.3.1 *Suppose that a function $f(z)$ defined on $|z| > R$ be regularly varying at ∞ with order ρ. Then for any $a \in \mathbf{C} \setminus \{0\}$, there exists $r' > \log(1 + R/|a|)$ such that the following two properties hold:*

1°) *There exist some constants $\varepsilon \geq 0$, $K > 0$ and $\lambda_0 > 0$ such that $\varepsilon < \rho + 1/2$ and*

$$\begin{aligned} (2.13) \qquad I_1 &:= \int_0^{2\pi} d\theta \int_{r'/\lambda}^{+\infty} \left|\frac{1}{N(\lambda)} f(a - ae^{\lambda x + \sqrt{-1}\theta})\right|^2 e^{-x^2/s} dx \\ &\leq K \cdot \left(s^{\rho-\varepsilon+1/2}\, 1_{(0<s<1)} + s^{\rho+\varepsilon+1/2}\, 1_{(s\geq 1)}\right) \end{aligned}$$

for all $\lambda \geq \lambda_0$ and $s > 0$.

2°) *For any $s > 0$,*

$$\begin{aligned} (2.14) \qquad I_2 &:= \int_0^{2\pi} d\theta \int_{r'/\lambda}^{+\infty} \left|\frac{1}{N(\lambda)} f(a - ae^{\lambda x + \sqrt{-1}\theta}) - c(-\theta - \arg(-a)) \cdot x^\rho\right|^2 e^{-x^2/s}\, dx \\ &\longrightarrow 0 \qquad \text{as } \lambda \to \infty, \end{aligned}$$

where $N(\lambda)$ and $c(\theta)$ are the regular normalizing function of f at ∞ and the asymptotic angular component of f at ∞, respectively.

Proof. By the assumptions, there exists some $r > (\log R) \vee 0$ which satisfies the following two properties:

1°) There exist some constants $\varepsilon \geq 0$, $K > 0$ and $\lambda_0 > 0$ such that $\varepsilon < \rho + 1/2$ and

$$\int_0^{2\pi} d\theta \int_{r/\lambda}^{+\infty} \left| \frac{1}{N(\lambda)} f(e^{\lambda x + \sqrt{-1}\theta}) \right|^2 e^{-x^2/s}\, dx \tag{2.15}$$
$$\leq K \cdot \left(s^{\rho-\varepsilon+1/2}\, 1_{(0<s<1)} + s^{\rho+\varepsilon+1/2}\, 1_{(s\geq 1)} \right)$$

for all $\lambda \geq \lambda_0$ and $s > 0$.

2°) For any $s > 0$,

$$\int_0^{2\pi} d\theta \int_{r/\lambda}^{+\infty} \left| \frac{1}{N(\lambda)} f(e^{\lambda x + \sqrt{-1}\theta}) - c(-\theta) \cdot x^\rho \right|^2 e^{-x^2/s}\, dx \tag{2.16}$$
$$\longrightarrow 0 \qquad \text{as } \lambda \to \infty.$$

Now denoting $\max(r,\ \log(|a|^2 + 2|a|),\ \log(1 + 1/|a|))$ by r again, we see that (2.15) and (2.16) clearly hold for this new r. Therefore we may assume that $r \geq \log(|a|^2 + 2|a|)$ and $r \geq \log(1 + 1/|a|)$. Set $r' = \log(1 + e^r/|a|)$. We have that $r' > \log(1 + R/|a|)$ since $r > \log R$.

In order to change the variables of the integrals I_1 and I_2 above, we set

$$a - ae^{\lambda x' + \sqrt{-1}\theta'} = e^{\lambda x + \sqrt{-1}\theta} = z.$$

Then

$$x' = x - \frac{1}{\lambda} \log |aJ^\lambda|,$$

$$\theta' = \theta - \arg(-aJ^\lambda)$$

and

$$dx' \wedge d\theta' = (-2\sqrt{-1}\lambda |z - a|^2)^{-1} \cdot dz \wedge d\bar{z} = |J^\lambda|^2 dx \wedge d\theta,$$

where

$$J^\lambda(x, \theta) = \frac{z}{z - a} = (1 - ae^{-\lambda x - \sqrt{-1}\theta})^{-1}.$$

Hence

$$I_1 = \int_0^{2\pi} d\theta \int_{\mathbf{R}} 1_{(\lambda x - \log|aJ^\lambda| > r')} \cdot \left| N(\lambda)^{-1} f(e^{\lambda x + \sqrt{-1}\theta}) \right|^2$$
$$\times \exp\left(-(x - \frac{1}{\lambda}\log|aJ^\lambda|)^2/s \right) |J^\lambda|^2 \, dx.$$

Noting that $|J^\lambda| \geq (1+|a|e^{-\lambda x})^{-1}$ and $r' = \log(1+e^r/|a|)$, we see that if $\lambda x - \log|aJ^\lambda| > r'$, then $\lambda x > r$. So we have

$$I_1 \leq \int_0^{2\pi} d\theta \int_{r/\lambda}^{+\infty} \left| \frac{1}{N(\lambda)} f(e^{\lambda x + \sqrt{-1}\theta}) \right|^2 \exp\left(-(x - \frac{1}{\lambda}\log|aJ^\lambda|)^2/s \right) |J^\lambda|^2 dx.$$

Moreover by the inequality $r \geq \log(|a|^2 + 2|a|)$ it holds that

$$|J^\lambda| \leq (1 - |a|e^{-\lambda x})^{-1} < (1 - |a|e^{-r})^{-1} < |a|^{-1}e^{r/2}.$$

This implies that

$$\text{(2.17)} \qquad \frac{1}{\lambda}\log|aJ^\lambda| < \frac{x}{2}$$

for $x > r/\lambda$. Therefore,

$$I_1 \leq |a|^{-2}e^r \int_0^{2\pi} d\theta \int_{r/\lambda}^{+\infty} \left| \frac{1}{N(\lambda)} f(e^{\lambda x + \sqrt{-1}\theta}) \right|^2 e^{-x^2/4s} \, dx$$

which proves (2.13) together with (2.15).

Similarly,

$$\text{(2.18)} \qquad I_2 \leq |a|^{-2}e^r \int_0^{2\pi} d\theta \int_{r/\lambda}^{+\infty} \left| \frac{1}{N(\lambda)} f(e^{\lambda x + \sqrt{-1}\theta}) \right.$$
$$\left. - c(-\theta + \arg J^\lambda) \cdot (x - \frac{1}{\lambda}\log|aJ^\lambda|)^\rho \right|^2 e^{-x^2/4s} \, dx.$$

On the other hand, by $r \geq \log(1 + 1/|a|)$ it holds that

$$|J^\lambda| \geq (1 + |a|e^{-\lambda x})^{-1} > (1 + |a|e^{-r})^{-1} > |a|^{-1}e^{-r}.$$

This implies that

$$\frac{1}{\lambda}\log|aJ^\lambda| > -x$$

for $x > r/\lambda$. Noting this and (2.17) we have the estimate

$$\left| x^\rho - (x - \frac{1}{\lambda}\log|aJ^\lambda|)^\rho \right| 1_{(x > r/\lambda)} \leq \text{const.}\, x^\rho \, 1_{(x>0)}$$

for any $\rho > -1/2$. Hence we can easily prove that

$$(2.19)\qquad \int_0^{2\pi} d\theta \int_{r/\lambda}^{+\infty} \left| c(-\theta)x^\rho - c(-\theta)(x - \frac{1}{\lambda}\log|aJ^\lambda|)^\rho \right|^2 e^{-x^2/4s}dx$$

$$\longrightarrow 0 \qquad \text{as } \lambda \to \infty$$

by Lebesgue's convergence theorem and the fact that

$$J^\lambda(x,\theta) \longrightarrow 1 \qquad \text{as } \lambda \to \infty$$

uniformly in θ for any $x > 0$.

Since $\arg J^\lambda \longrightarrow 0$ as $\lambda \to \infty$ uniformly in θ for any $x > 0$, we can also prove that

$$(2.20)\qquad \int_0^{2\pi} d\theta \int_{r/\lambda}^{+\infty} (x - \frac{1}{\lambda}\log|aJ^\lambda|)^\rho |c(-\theta) - c(-\theta + \arg J^\lambda)|^2 e^{-x^2/4s}$$

$$\longrightarrow 0 \qquad \text{as } \lambda \to \infty$$

by Lebesgue's convergence theorem and the fact that

$$\int_0^{2\pi} |c(-\theta) - c(-\theta + \arg J^\lambda)|^2 d\theta \longrightarrow 0 \qquad \text{as } \lambda \to \infty$$

for fixed $x > 0$.

Combining (2.18), (2.19), (2.20) and (2.16), we obtain (2.14). Q.E.D.

Let (x_t, θ_t) be a Brownian motion on the cylinder $\mathbf{R} \times \mathbf{T}$ matisfying $x_0 = 0$ and $\theta_0 = 0$ a.s. Clearly

$$z_t = x_t + \sqrt{-1}\int_0^t d\theta_s$$

becomes a complex Brownian motion. Our main theorem in this section is as follows:

Theorem 2.3.1 (1) *Suppose that the functions $f_1, \cdots, f_m$ defined on $0 < |z - a| < R$ are regularly varying at a with order $\rho_1, \cdots, \rho_m$, respectively. Denote the regular normalizing function of f_i at a and the asymptotic angular component of f_i at a by $N_i(\lambda)$ and $c_i(\theta)$, respectively for $i = 1, \cdots, m$. Then there exists some $r > (\log|a/R|) \vee 0$ and we have*

$$\Big\{ N_i(\lambda)^{-1} \int_0^\cdot f_i(a - ae^{\lambda z_s})\, 1_{(\lambda x_s < -r)}\, dz_s,$$

$$N_i(\lambda)^{-2} \int_0^\cdot |f_i(a - ae^{\lambda z_s})|^2\, 1_{(\lambda x_s < -r)}\, ds \Big\}_{1 \le i \le m}$$

$$\longrightarrow \Big\{ \overline{c_i} \int_0^\cdot (-x_s)^{\rho_i}\, 1_{(x_s < 0)}\, dz_s + \int_0^\cdot \int_0^{2\pi} (c_i(\theta) - \overline{c_i})(-x_s)^{\rho_i} M(1_{(x_s<0)}ds, d\theta),$$

$$\overline{|c_i|^2} \int_0^\cdot (-x_s)^{2\rho_i}\, 1_{(x_s<0)}\, ds \Big\}_{1 \le j \le m}$$

as $\lambda \longrightarrow \infty$ *in law, where* $\overline{c} = (1/2\pi)\int_0^{2\pi} c(\theta)d\theta$ *and* M *is a complex Gaussian random measure on* $[0,\infty) \times [0, 2\pi]$ *with mean 0 and variance measure* $dt \cdot (d\theta/2\pi)$ *which is independent of* $z(t)$.

rm (2) Suppose that the functions $f_1, \cdots, f_m$ *defined on* $|z| > R$ *are regularly varying at* ∞ *with order* $\rho_1, \cdots, \rho_m$*, respectively. Denote the regular normalizing function of* f_i *at* ∞ *and the asymptotic angular component of* f_i *at* ∞ *by* $N_i(\lambda)$ *and* $c_i(\theta)$*, respectively for* $i = 1, \cdots, m$ *. Then, for every* $a \in \mathbf{C} \setminus \{0\}$*, there exists some* $r > (\log(1 + R/|a|)) \vee 0$ *and we have*

$$\Big\{ \; N_i(\lambda)^{-1} \int_0^{\cdot} f_i(a - ae^{\lambda z_s})\, 1_{(\lambda x_s > r)}\, dz_s,$$

$$N_i(\lambda)^{-2} \int_0^{\cdot} |f_i(a - ae^{\lambda z_s})|^2\, 1_{(\lambda x_s > r)}\, ds \; \Big\}_{1 \le i \le m}$$

$$\longrightarrow \Big\{ \; \overline{c_i} \int_0^{\cdot} (x_s)^{\rho_i}\, 1_{(x_s > 0)}\, dz_s + \int_0^{\cdot}\int_0^{2\pi} (c_i(\theta) - \overline{c_i})(x_s)^{\rho_i} M(1_{(x_s>0)}ds, d\theta),$$

$$\overline{|c_i|^2} \int_0^{\cdot} (x_s)^{2\rho_i}\, 1_{(x_s > 0)}\, ds \; \Big\}_{1 \le j \le m}$$

as $\lambda \longrightarrow \infty$ *in law, where* $\overline{c} = (1/2\pi)\int_0^{2\pi} c(\theta)d\theta$ *and* M *is a complex Gaussian random measure on* $[0,\infty) \times [0, 2\pi]$ *with mean 0 and variance measure* $dt \cdot (d\theta/2\pi)$ *which is independent of* $z(t)$.

Proof. We will prove (1) only, because by Proposition 2.3.1, the proof of (2) proceeds similarly. (Note that

$$\Big\{ \int_0^{\cdot}\int_0^{2\pi} (c_i(-\theta - \arg(-a)) - \overline{c_i})(x_s)^{\rho_i}\, M(1_{(x_s>0)}ds, d\theta) \Big\}_{1 \le i \le m}$$

is equivalent in law to

$$\Big\{ \int_0^{\cdot}\int_0^{2\pi} (c_i(\theta) - \overline{c_i})(x_s)^{\rho_i}\, M(1_{(x_s>0)}ds, d\theta) \Big\}_{1 \le i \le m} . \;\;)$$

Let $\{e_0 \equiv 1, e_1, \cdots, e_p\}$ be some orthonormal system in $L^2(0, 2\pi)$ such that

$$c_i(\theta) = \sum_{k=0}^{p} \alpha_i^{(k)} e_k(\theta), \qquad \alpha_i^{(k)} \in \mathbf{C} \qquad (k = 0, \cdots, p)$$

for $i = 1, \cdots, m$. Define

$$V_k{}^{\lambda}(t) = \int_0^t e_k(\lambda\theta_s)\, 1_{(\lambda x_s < -r)}\, dz_s \qquad (k = 0, \cdots, p) \tag{2.21}$$

for some $r > (\log|a/R|) \vee 0$. Then it holds that

$$(2.22) \quad I_\lambda = E \sup_{0\le t\le T} \Big| N_i(\lambda)^{-1} \int_0^t f_i(a - ae^{\lambda z_s})\, 1_{(\lambda x_s < -r)}\, dz_s - \sum_{k=0}^{p} \alpha_i^{(k)} \int_0^t (-x_s)^{\rho_i}\, dV_k{}^\lambda(s) \Big|^2 \longrightarrow 0 \qquad \text{as } \lambda \to \infty.$$

The proof of (2.22) is as follows. Let $q(t,\theta,\eta)$ be the transition density of $\theta(t)$. Then

$$\begin{aligned}
I_\lambda &= E \sup_{0\le t\le T} \left| \int_0^t \left(N_i(\lambda)^{-1} f_i(a - ae^{\lambda z_s}) - c_i(\lambda\theta_s)(-x_s)^{\rho_i} \right) 1_{(\lambda x_s < -r)}\, dz_s \right|^2 \\
&\le \text{const.}\, E \int_0^T \left| N_i(\lambda)^{-1} f_i(a - ae^{\lambda z_s}) - c_i(\lambda\theta_s)(-x_s)^{\rho_i} \right|^2 1_{(\lambda x_s < -r)}\, ds \\
&= \text{const.}\, E \int_0^T \left| N_i(\lambda)^{-1} f_i(a - ae^{\lambda x_s + \sqrt{-1}\theta(\lambda^2 s)}) - c_i(\theta(\lambda^2 s))(-x_s)^{\rho_i} \right|^2 1_{(\lambda x_s < -r)}\, ds \\
&= \text{const.} \int_0^T ds \int_0^{2\pi} d\theta \quad q(\lambda^2 s, 0, \theta) \\
&\qquad \times \int_{-r/\lambda}^{\infty} \left| N_i(\lambda)^{-1} f_i(a - ae^{\lambda x + \sqrt{-1}\theta}) - c_i(\theta)(-x)^{\rho_i} \right|^2 \frac{1}{\sqrt{2\pi s}} e^{-x^2/2s}\, dx.
\end{aligned}$$

Hence noting the inequality

$$q(s,\theta,\eta) \le \text{const.}\, s^{-1/2} + \text{const.}$$

which we have seen in the proof of Lemma 2.2.2, we have

$$\begin{aligned}
I_\lambda &\le \text{const.} \int_0^T ds (\text{const.}\, \lambda^{-1} s^{-1} + \text{const.}\, s^{-1/2}) \\
&\qquad \times \int_0^{2\pi} d\theta \int_{-r/\lambda}^{\infty} \left| N_i(\lambda)^{-1} f_i(a - ae^{\lambda x + \sqrt{-1}\theta}) - c_i(\theta)(-x)^{\rho_i} \right|^2 e^{-x^2/2s}\, dx.
\end{aligned}$$

This last expression clearly tends to 0 as $\lambda \to \infty$ for some $r > (\log|a/R|) \vee 0$ by the definition of regularly varying functions at a and Lebesgue's convergence theorem.

Similarly we have

$$(2.23) \quad J_\lambda = E \sup_{0\le t\le T} \left| N_i(\lambda)^{-2} \int_0^t |f_i(a - ae^{\lambda z_s})|^2 1_{(\lambda x_s < -r)}\, ds - \overline{|c_i|^2} \int_0^t (-x_s)^{2\rho_i}\, d\left\langle V_0{}^\lambda \right\rangle_s \right| \longrightarrow 0 \qquad \text{as } \lambda \to \infty.$$

Actually,

$$J_\lambda = E \sup_{0\le t\le T} \left| \int_0^t \left(N_i(\lambda)^{-2} |f_i(a - ae^{\lambda z_s})|^2 - \overline{|c_i|^2}(-x_s)^{2\rho_i} \right) 1_{(\lambda x_s < -r)}\, ds \right|$$

$$\begin{aligned}
&\le E\int_0^T \left|N_i(\lambda)^{-2}|f_i(a-ae^{\lambda z_s})|^2 - |c_i(\lambda\theta_s)|^2(-x_s)^{2\rho_i}\right| 1_{(\lambda x_s<-r)}\,ds \\
&\quad + E\sup_{0\le t\le T}\left|\int_0^t \left(|c_i(\lambda\theta_s)|^2 - \overline{|c_i|^2}\right)(-x_s)^{2\rho_i} 1_{(\lambda x_s<-r)}\,ds\right| \\
&:= J_\lambda{}^{(1)} + J_\lambda{}^{(2)}.
\end{aligned}$$

By Theorem 2.2.1, we have that $J_\lambda{}^{(2)} \to 0$ as $\lambda \to \infty$. As for $J_\lambda{}^{(1)}$,

$$\begin{aligned}
J_\lambda{}^{(1)} &\le \left(E\int_0^T \left|N_i(\lambda)^{-1}|f_i(a-ae^{\lambda z_s})| + |c_i(\lambda\theta_s)|(-x_s)^{\rho_i}\right|^2 1_{(\lambda x_s<-r)}\,ds\right)^{1/2} \\
&\quad \times \left(E\int_0^T \left|N_i(\lambda)^{-1}|f_i(a-ae^{\lambda z_s})| - |c_i(\lambda\theta_s)|(-x_s)^{\rho_i}\right|^2 1_{(\lambda x_s<-r)}\,ds\right)^{1/2}
\end{aligned}$$

by Schwartz' inequality. The first expectation in the last form is bounded by a constant by the definition of the regularly varying functions. The second expectation in the last form is bounded by the expectation

$$E\int_0^T \left|N_i(\lambda)^{-1} f_i(a-ae^{\lambda z_s}) - c_i(\lambda\theta_s)(-x_s)^{\rho_i}\right|^2 1_{(\lambda x_s<-r)}\,ds$$

which tends to 0 as $\lambda \to \infty$ as we have seen above in the proof of (2.22).

Therefore if we can prove that the joint processes

$$\Big\{ \int_0^\cdot (-x_s)^{\rho_i}\,dV_0{}^\lambda(s),\ \int_0^\cdot (-x_s)^{\rho_i}\,dV_k{}^\lambda(s),$$
$$\int_0^\cdot (-x_s)^{2\rho_i}\,d\left\langle V_0{}^\lambda\right\rangle_s,\ \int_0^\cdot (-x_s)^{2\rho_i}\,d\left\langle V_k{}^\lambda\right\rangle_s \Big\}_{1\le i\le m}^{1\le k\le p}$$

converge to

$$\Big\{ \int_0^\cdot (-x_s)^{\rho_i} 1_{(x_s<0)}\,dz_s,\ \int_0^\cdot\int_0^{2\pi} e_k(\theta)(-x_s)^{\rho_i} M(1_{(x_s<0)}ds, d\theta),$$
$$\int_0^\cdot (-x_s)^{2\rho_i} 1_{(x_s<0)}\,ds,\ \int_0^\cdot (-x_s)^{2\rho_i} 1_{(x_s<0)}\,ds \Big\}_{1\le i\le m}^{1\le k\le p}$$

as $\lambda \to \infty$ in law, then we can finish the proof of our theorem. This follows at once from Lemma 2.3.3 and Lemma 2.3.4 below. Q.E.D.

Before stating these lemmas, we introduce the following two general lemmas which have been obtained in Watanabe [12].

Lemma-W 1 *Let M_λ be a continuous conformal martingale for any λ $(1 \le \lambda \le \infty)$ satisfying the following properties:*

$$(2.24) \qquad E(\langle M_\lambda\rangle(t))^2 \le K_1(t) \qquad \textit{for any } t>0 \textit{ and } 1\le\lambda<\infty,$$

$$(2.25) \qquad E\left(\int_0^t |\Phi_\lambda(s)|^2 d\langle M_\lambda\rangle(s)\right)^2 \le K_2(t) \qquad \textit{for any } t>0 \textit{ and } 1\le\lambda<\infty,$$

$$(2.26) \qquad \int_0^t |\Phi_\lambda(s)|^2 d\langle M_\lambda\rangle(s) \longrightarrow \infty \qquad \textit{as } t\to\infty \textit{ a.s. for any } \lambda \quad (1\le\lambda\le\infty),$$

where $K_1(t)$ *and* $K_2(t)$ *are some positive functions independent of* λ, *and* $\Phi_\lambda(t)$ $(1\le\lambda\le\infty)$ *are some* $({F_t}^{M_\lambda})$*-predictable real or complex valued processes.*

If

$$\left\{ M_\lambda,\ \langle M_\lambda\rangle,\ \left\langle \int_0^\cdot \Phi_\lambda(s)\,dM_\lambda(s), M_\lambda\right\rangle,\ \int_0^\cdot |\Phi_\lambda(s)|^2\,d\langle M_\lambda\rangle(s) \right\}$$

$$\longrightarrow \left\{ M_\infty,\ \langle M_\infty\rangle,\ \left\langle \int_0^\cdot \Phi_\infty(s)\,dM_\infty(s), M_\infty\right\rangle,\ \int_0^\cdot |\Phi_\infty(s)|^2\,d\langle M_\infty\rangle(s) \right\}$$

as $\lambda\longrightarrow\infty$ *in law on* $C([0,\infty)\mapsto \mathbf{C}\times\mathbf{R}\times\mathbf{R}^4\times\mathbf{R})$, *then*

$$\left\{ M_\lambda,\ \int_0^\cdot \Phi_\lambda(s)\,dM_\lambda(s),\ \int_0^\cdot |\Phi_\lambda(s)|^2\,d\langle M_\lambda\rangle(s) \right\}$$

$$\longrightarrow \left\{ M_\infty,\ \int_0^\cdot \Phi_\infty(s)\,dM_\infty(s),\ \int_0^\cdot |\Phi_\infty(s)|^2\,d\langle M_\infty\rangle(s) \right\}$$

as $\lambda\longrightarrow\infty$ *in law on* $C([0,\infty)\mapsto \mathbf{C}^2\times\mathbf{R})$.

Proof. We will prove the lemma assuming that M_λ and Φ_λ are real valued, because the proof of the general case follows at once from this case. Set

$$N_\lambda(t) = \int_0^t \Phi_\lambda(s)dM_\lambda(s) \qquad (1\le\lambda\le\infty).$$

By the condition (2.26) and the Knight theorem, we see that $\widehat{N_\lambda}$ ($1\le\lambda\le\infty$) becomes a Brownian motion. Thus the laws induced by $N_\lambda = \widehat{N_\lambda}(\langle N_\lambda\rangle)$ form a tight family, which implies that the family of laws induced by

$$\{\ M_\lambda,\ N_\lambda,\ \langle M_\lambda\rangle,\ \langle M_\lambda, N_\lambda\rangle,\ \langle N_\lambda\rangle\ \}$$

is tight. Hence we may choose one of the limit points of the above family which we may assume to be the law of

$$\{\ M_\infty,\ X,\ \langle M_\infty\rangle,\ \langle M_\infty, N_\infty\rangle,\ \langle N_\infty\rangle\ \},$$

where

$$N_\infty = \int_0^\cdot \Phi_\infty(s)\,dM_\infty(s)$$

and X is some continuous process. Then we can conclude that $X = N_\infty$ as follows. We see from the condition (2.24) that both $\{M_\lambda^2(t)\}_{\lambda\geq 1}$ and $\{\langle M_\lambda\rangle(t)\}_{\lambda\geq 1}$ are uniformly integrable for any $t > 0$. Similarly we see from the condition (2.25) that both $\{N_\lambda^2(t)\}_{\lambda\geq 1}$ and $\{\langle N_\lambda\rangle(t)\}_{\lambda\geq 1}$ are uniformly integrable for any $t > 0$. Therefore $\{M_\lambda(t)N_\lambda(t)\}_{\lambda\geq 1}$ and $\{\langle M_\lambda, N_\lambda\rangle(t)\}_{\lambda\geq 1}$ are also uniformly integrable for any $t > 0$. Consequently, we see that M_∞ and X are $(F^{M_\infty,X})$-martingales and that

$$\langle X\rangle = \langle N_\infty\rangle = \int_0^\cdot |\Phi_\infty(s)|^2\, d\langle M_\infty\rangle(s),$$

$$\langle X, M_\infty\rangle = \langle N_\infty, M_\infty\rangle = \int_0^\cdot \Phi_\infty(s)\, d\langle M_\infty\rangle(s)$$

from the Skorohod theorem realizing a sequence of random variables converging in law by an almost sure convergent sequence. From these we have

$$\begin{aligned}
\langle X - N_\infty\rangle &= \langle X\rangle + \langle N_\infty\rangle - 2\langle X, N_\infty\rangle \\
&= 2\int_0^\cdot |\Phi_\infty(s)|^2\, d\langle M_\infty\rangle(s) - 2\int_0^\cdot \Phi_\infty(s)\, d\langle X, M_\infty\rangle(s) \\
&= 2\int_0^\cdot |\Phi_\infty(s)|^2\, d\langle M_\infty\rangle(s) - 2\int_0^\cdot |\Phi_\infty(s)|^2\, d\langle M_\infty\rangle(s) \\
&= 0 \qquad \text{a.s.,}
\end{aligned}$$

which implies that $X = N_\infty$ a.s. Q.E.D.

Lemma-W 2 *Let M_λ be a continuous conformal martingale such that* $\lim_{t\uparrow\infty}\langle M_\lambda\rangle(t) = \infty$ *a.s. for every λ $(1 \leq \lambda \leq +\infty)$.*

If

$$\{\, M_\lambda,\ \langle M_\lambda\rangle \,\} \longrightarrow \{\, M_\infty,\ \langle M_\infty\rangle \,\} \qquad \text{as } \lambda \to \infty$$

in law on $C([0,\infty) \mapsto \mathbf{C}\times\mathbf{R})$, then

$$\{\, M_\lambda,\ \langle M_\lambda\rangle,\ \widehat{M_\lambda} \,\} \longrightarrow \{\, M_\infty,\ \langle M_\infty\rangle,\ \widehat{M_\infty} \,\} \qquad \text{as } \lambda \to \infty$$

in law on $C([0,\infty) \mapsto \mathbf{C}\times\mathbf{R}\times\mathbf{C})$.

Proof. Let $X(t)$ be a process such that

$$\{\, M_\lambda(t),\ \langle M_\lambda\rangle(t),\ \widehat{M_\lambda}(t) \,\} \longrightarrow \{\, M_\infty(t),\ \langle M_\infty\rangle(t),\ X(t) \,\}$$

as $\lambda \longrightarrow \infty$ in law and realize this sequence by an almost sure convergent sequence. Since $\widehat{M_\lambda}(\langle M_\lambda \rangle (t)) = M_\lambda(t)$, we have that $X(\langle M_\infty \rangle (t)) = M_\infty(t)$. Hence $X(t) = M_\infty(\langle M_\infty \rangle^{-1} (t)) = \widehat{M_\infty}(t)$. Q.E.D.

Now we state our lemmas which are essential in our proof.

Lemma 2.3.1 *If $c \in L^1(0, 2\pi)$ and $\rho > -1$, then*

$$\begin{aligned} I_\lambda &= E \sup_{0 \le t \le T} \left| \int_0^t c(\lambda\theta_s)(-x_s)^\rho 1_{(\lambda x_s < -r)} \, ds - \bar{c} \int_0^t (-x_s)^\rho 1_{(x_s < 0)} \, ds \right| \\ &\longrightarrow 0 \qquad \text{as } \lambda \to \infty \end{aligned}$$

for any $r \ge 0$.

Proof.

$$\begin{aligned} I_\lambda &\le \text{const.} \, E \int_0^T \left| c(\lambda\theta_s)(-x_s)^\rho \left(1_{(\lambda x_s < -r)} - 1_{(x_s < 0)}\right) \right| ds \\ &\qquad + E \sup_{0 \le t \le T} \left| \int_0^t (c(\lambda\theta_s) - \bar{c}(-x_s)^\rho 1_{(x_s < 0)} ds \right| \\ &:= I_\lambda{}^{(1)} + I_\lambda{}^{(2)}, \qquad \text{say.} \end{aligned}$$

By Lemma 2.2.1 and Lemma 2.2.2 we have

$$\begin{aligned} &E \left| c(\lambda\theta_s)(-x_s)^\rho \left(1_{(\lambda x_s < -r)} - 1_{(x_s < 0)}\right) \right| \le E \left| 2c(\lambda\theta_s)(-x_s)^\rho \right| \\ &\le 2E|x_s|^\rho E|c(\lambda\theta_s)| = 2E|x_s|^\rho E|c(\theta(\lambda^2 s))| \\ &\le \text{const.} \, \|c\|_1 s^{\rho/2}(\lambda^{-1}s^{-1/2} + \text{const.}) < +\infty. \end{aligned}$$

Then we can see easily that

$$E \left| c(\lambda\theta_s)(-x_s)^\rho \left(1_{(\lambda x_s < -r)} - 1_{(x_s < 0)}\right) \right| \longrightarrow 0 \qquad \text{as } \lambda \to \infty$$

for any $s > 0$ by Lebesgue's convergence theorem. Since

$$\int_0^T s^{\rho/2}(\lambda^{-1}s^{-1/2} + \text{const.}) \, ds < +\infty,$$

we have that $I_\lambda{}^{(1)} \to 0$ as $\lambda \to \infty$ using Lebesgue's convergence theorem again.

On the other hand, it follows from Theorem 2.2.1 that $I_\lambda{}^{(2)} \to 0$ as $\lambda \to \infty$ since $\lambda\theta(t)$ has the same law as $\theta(\lambda^2 t)$. Q.E.D.

Lemma 2.3.2 *If $c(\not\equiv 0) \in L^1(0, 2\pi)$ and $\rho > -1$, then for any λ $(1 \le \lambda < \infty)$ and any $r \ge 0$,*

$$\int_0^t |c(\lambda\theta_s)|(-x_s)^\rho \, 1_{(\lambda x_s < -r)} \, ds \longrightarrow \infty \qquad a.s. \qquad as\ t \to \infty.$$

Proof. Fix $K > 0$, $t > 0$, and $1 \le \lambda < \infty$. Then for any $\alpha > 0$ we have

$$\begin{aligned} & P\left[\int_0^{\alpha^2 t} |c(\lambda\theta_s)|(-x_s)^\rho \, 1_{(\lambda x_s < -r)} \, ds > K\right] \\ = \;& P\left[\int_0^t |c(\lambda\theta(\alpha^2 s))|(-x(\alpha^2 s))^\rho \, 1_{(\lambda x(\alpha^2 s) < -r)} \, ds > K/\alpha^2\right] \\ = \;& P\left[\int_0^t |c(\lambda\alpha\theta_s)|(-x_s)^\rho \, 1_{(\lambda\alpha x_s < -r)} \, ds > K/\alpha^{2+\rho}\right] \end{aligned}$$

This, together with Lemma 2.3.1, gives an inequality

$$\begin{aligned} & \liminf_{\alpha\to\infty} P\left[\int_0^{\alpha^2 t} |c(\lambda\theta_s)|(-x_s)^\rho \, 1_{(\lambda x_s < -r)} \, ds > K\right] \\ \ge \;& P\left[\overline{|c|} \int_0^t (-x_s)^\rho \, 1_{(x_s < 0)} \, ds > \varepsilon\right] \end{aligned}$$

for any $\varepsilon > 0$. The last expression obviously converges to 1 as $\varepsilon \to 0$ because $x_0 = 0$. Therefore, noting that the process involved is increasing in t, we obtain the lemma. Q.E.D.

Lemma 2.3.3 *Let $V_k{}^\lambda(t)$ $(k = 0, \cdots, p)$ be as (2.21). Then*

$$\begin{aligned} & \{\, V_0{}^\lambda,\ V_k{}^\lambda,\ \langle V_0{}^\lambda \rangle,\ \langle V_k{}^\lambda \rangle \,\}_{1\le k\le p} \\ & \longrightarrow \Big\{ \int_0^\cdot 1_{(x_s<0)} \, dz_s,\ \int_0^\cdot \int_0^{2\pi} e_k(\theta) \, M(1_{(x_s<0)} ds, d\theta), \\ & \qquad \int_0^\cdot 1_{(x_s<0)} \, ds,\ \int_0^\cdot 1_{(x_s<0)} \, ds \Big\}_{1\le k\le p} \end{aligned}$$

as $\lambda \to \infty$ in law.

Proof. First note by Lemma 2.3.1 that

$$\begin{aligned} \langle V_k{}^\lambda, V_l{}^\lambda \rangle (t) \;=\; & \int_0^t \mathcal{R}e(e_k e_l{}^*)(\lambda\theta_s) \, 1_{(\lambda x_s < -r)} \, ds \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \\ & + \int_0^t \mathcal{I}m(e_k e_l{}^*)(\lambda\theta_s) \, 1_{(\lambda x_s < -r)} \, ds \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \\ & \longrightarrow \delta_{kl} \int_0^t 1_{(x_s<0)} \, ds \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \qquad \text{for } k, l = 0, \cdots, p \end{aligned}$$

as $\lambda \to \infty$ on $C([0, \infty) \mapsto \mathbf{R}^4)$ in probability for any $t > 0$ and, also by Lemma 2.3.2 that

$$\left\langle V_k{}^\lambda \right\rangle (t) \longrightarrow \infty \qquad \text{a.s.} \qquad \text{as } t \to \infty \qquad \text{for } k = 0, \cdots, p.$$

Fix $t > 0$ and $\varepsilon > 0$. Since the facts stated above imply that

$$P\left[\left\langle V_k{}^\lambda \right\rangle (n) < t\right] \longrightarrow 0 \qquad \text{as } n \to \infty,$$

$$P\left[\int_0^n 1_{(x_s<0)}\, ds < t\right] \longrightarrow 0 \qquad \text{as } n \to \infty$$

and

$$P\left[\left\langle V_k{}^\lambda \right\rangle (n) < t\right] \longrightarrow P\left[\int_0^n 1_{(x_s<0)}\, ds < t\right] \qquad \text{as } \lambda \to \infty$$

for any $n > 0$, there exist $\lambda_0 > 0$ and $n_0 > 0$ such that

$$P\left[\left\langle V_k{}^\lambda \right\rangle^{-1} (t) > n_0\right] = P\left[\left\langle V_k{}^\lambda \right\rangle (n_0) < t\right] < \varepsilon$$

for all $\lambda \geq \lambda_0$. Therefore there exists $\lambda_1 > 0$ such that

$$\begin{aligned} & P\left[\left|\left\langle V_k{}^\lambda, V_l{}^\lambda \right\rangle (\left\langle V_k{}^\lambda \right\rangle^{-1} (t)) > \varepsilon\right] \\ \leq \ & P\left[\left\langle V_k{}^\lambda \right\rangle^{-1} (t) > n_0\right] + P\left[\sup_{t \leq n_0} \left|\left\langle V_k{}^\lambda, V_l{}^\lambda \right\rangle\right|(t) > \varepsilon\right] \\ < \ & 2\varepsilon \end{aligned}$$

for all $\lambda \geq \lambda_1$. Consequently we have

$$\text{(2.27)} \qquad \left\langle V_k{}^\lambda, V_l{}^\lambda \right\rangle (\left\langle V_k{}^\lambda \right\rangle^{-1} (t)) \longrightarrow \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \qquad \text{if } k \neq l$$

as $\lambda \to \infty$ in probability for any $t > 0$, from which we obtain that $\{ \widehat{V_0{}^\lambda}, \widehat{V_1{}^\lambda}, \cdots, \widehat{V_p{}^\lambda} \}$ converges in law to a $(p+1)$-dimensional complex Brownian motion as $\lambda \to \infty$ by the '!symptotic Knight's theorem" in Pitman and Yor [10] (p.1008).

On the other hand, we easily see by Lemma-W1 and Lemma-W2 that the limit law of $\{ V_0{}^\lambda, \left\langle V_0{}^\lambda \right\rangle, \widehat{V_0{}^\lambda} \}$ is that of

$$\left\{ \int_0^\cdot 1_{(x_s<0)}\, dz_s,\ \int_0^\cdot 1_{(x_s<0)}\, ds,\ \int_0^\cdot \widehat{1_{(x_s<0)}\, dz_s} \right\}.$$

Hence we can conclude that the limit law of $\widehat{V_k{}^\lambda}(t)$ ($k = 1, \cdots, p$) can be represented by the law of

$$\int_0^t \int_0^{2\pi} e_k(\theta)\, M(ds, d\theta) \qquad (k = 1, \cdots, p).$$

Thus we have

$$\{ \widehat{V_0{}^\lambda},\ \widehat{V_k{}^\lambda},\ \left\langle V_j{}^\lambda \right\rangle \}_{0\le j\le p}^{1\le k\le p}$$

$$\longrightarrow \left\{ \widehat{\int_0^\cdot 1_{(x_s<0)}\, dz_s},\ \int_0^\cdot \int_0^{2\pi} e_k(\theta)\, M(ds, d\theta),\ \int_0^t 1_{(x_s<0)}\, ds \right\}_{0\le j\le p}^{1\le k\le p}$$

as $\lambda \to \infty$ in law. This implies the assertion of the lemma. Q.E.D.

Lemma 2.3.4 *Let $V_k{}^\lambda(t)$ $(k = 0, \cdots, p)$ be as (2.21). If $\rho > -1/2$, then*

$$\left\{ V_0{}^\lambda,\ \int_0^\cdot (-x_s)^\rho\, dV_0{}^\lambda(s),\ \int_0^\cdot (-x_s)^{2\rho}\, d\left\langle V_0{}^\lambda \right\rangle_s \right\} \tag{2.28}$$

$$\longrightarrow \left\{ \int_0^\cdot 1_{(x_s<0)}\, dz_s,\ \int_0^\cdot (-x_s)^\rho\, 1_{(x_s<0)}\, dz_s,\ \int_0^\cdot (-x_s)^{2\rho}\, 1_{(x_s<0)}\, ds \right\}$$

as $\lambda \to \infty$ in law and

$$\left\{ V_k{}^\lambda,\ \int_0^\cdot (-x_s)^\rho\, dV_k{}^\lambda(s),\ \int_0^\cdot (-x_s)^{2\rho}\, d\left\langle V_k{}^\lambda \right\rangle_s \right\} \tag{2.29}$$

$$\longrightarrow \left\{ \int_0^\cdot \int_0^{2\pi} e_k(\theta)\, M(1_{(x_s<0)}ds, d\theta), \right.$$

$$\int_0^\cdot \int_0^{2\pi} e_k(\theta)(-x_s)^\rho\, M(1_{(x_s<0)}ds, d\theta),$$

$$\left. \int_0^\cdot (-x_s)^{2\rho}\, 1_{(x_s<0)}\, ds \right\}$$

for $k = 1, \cdots, p$ as $\lambda \to \infty$ in law.

Proof. Set

$$V_0{}^\infty(t) = \int_0^t 1_{(x_s<0)}\, dz_s$$

and

$$V_k{}^\infty(t) = \int_0^t \int_0^{2\pi} e_k(\theta)\, M(1_{(x_s<0)}ds, d\theta).$$

By Lemma 2.3.1, we have

$$E \sup_{0\le t\le T} \left| \int_0^t (-x_s)^{2\rho}\, d\left\langle V_k{}^\lambda \right\rangle_s - \int_0^t (-x_s)^{2\rho}\, d\langle V_k{}^\infty \rangle_s \right|$$

$$\longrightarrow 0 \qquad \text{as } \lambda \to \infty$$

and

$$E \sup_{0\le t\le T}\left|\int_0^t (-x_s)^\rho \, d\left\langle V_k{}^\lambda\right\rangle_s - \int_0^t (-x_s)^\rho \, d\langle V_k{}^\infty\rangle_s\right|$$

$$\longrightarrow 0 \qquad \text{as } \lambda \to \infty$$

for $k = 0, 1, \cdots, p$. On the other hand, Lemma 2.3.3 implies that

$$\{\, V_k{}^\lambda,\ \left\langle V_k{}^\lambda\right\rangle \,\} - \to \{\, V_k{}^\infty,\ \langle V_k{}^\infty\rangle \,\} \qquad \text{as } \lambda \to \infty$$

in law for each k. Therefore,

$$\left\{\, V_k{}^\lambda,\ \left\langle V_k{}^\lambda\right\rangle,\ \left\langle \int_0^\cdot (-x_s)^\rho \, dV_k{}^\lambda(s), V_k{}^\lambda(s)\right\rangle,\ \left\langle \int_0^\cdot (-x_s)^\rho \, dV_k{}^\lambda(s)\right\rangle \,\right\}$$

$$= \left\{\, V_k{}^\lambda,\ \left\langle V_k{}^\lambda\right\rangle,\ \int_0^\cdot (-x_s)^\rho \, d\left\langle V_k{}^\lambda\right\rangle_s \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix},\ \int_0^\cdot (-x_s)^{2\rho} \, d\left\langle V_k{}^\lambda\right\rangle_s \,\right\}$$

$$\longrightarrow \left\{\, V_k{}^\infty,\ \langle V_k{}^\infty\rangle,\ \int_0^\cdot (-x_s)^\rho \, d\langle V_k{}^\infty\rangle_s \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix},\ \int_0^\cdot (-x_s)^{2\rho} \, d\langle V_k{}^\infty\rangle_s \,\right\}$$

as $\lambda \to \infty$ in law for each k.

Thus if we can prove that the above processes satisfy the conditions (2.24)–(2.26) in Lemma-W1 , then (2.28) and (2.29) follow from Lemma-W1. It is easy to show that

$$\text{(2.30)} \qquad E\left\langle V_k{}^\lambda\right\rangle(t)^2 \le \text{const.}\,(t^{1/2} + \text{const.}\,t)^2, \qquad 1 \le \lambda < \infty.$$

for each k. Indeed,

$$E\left\langle V_k{}^\lambda\right\rangle(t)^2 = 2E\int_0^t ds \int_s^t |e_k(\lambda\theta_s)|^2 |e_k(\lambda\theta_u)|^2 \, 1_{(x_s<0)} \, 1_{(x_u<0)} \, du$$

$$\le \ 2E\int_0^t ds \int_s^t |e_k(\lambda\theta_s)|^2 |e_k(\lambda\theta_u)|^2 \, du$$

$$= \ 2E\int_0^t ds \int_s^t |e_k(\theta(\lambda^2 s))|^2 |e_k(\theta(\lambda^2 u))|^2 \, du$$

$$\le \ \text{const.} \int_0^t ds \int_s^t (\lambda^{-1}s^{-1/2} + \text{const.}\,)\{\lambda^{-1}(u-s)^{-1/2} + \text{const.}\,\}\, du.$$

Here the last inequality follows from Lemma 2.2.2. Then we have (2.30).

We can also prove that

$$E\left|\int_0^t (-x_s)^{2\rho} \, d\left\langle V_k{}^\lambda\right\rangle_s\right|^2 \le \text{const.}\, t^{2\rho}(t^{1/2} + \text{const.}\,t)^2, \qquad 1 \le \lambda < \infty$$

for each k by a similar argument as above using Lemma 2.2.1 and Lemma 2.2.2.

Further it has already been shown in Lemma 2.3.2 that

$$\int_0^t (-x_s)^{2\rho}\, d\left\langle V_k{}^\lambda \right\rangle_s \longrightarrow \infty \qquad (t \to \infty) \quad \text{a.s.}, \qquad 1 \le \lambda < \infty$$

and

$$\int_0^t (-x_s)^{2\rho}\, d\left\langle V_k{}^\infty \right\rangle_s = \int_0^t (-x_s)^{2\rho}\, 1_{(x_s<0)}\, ds \longrightarrow \infty \qquad (t \to \infty) \qquad \text{a.s.}$$

for each k. Consequently we have completed the proof of the lemma. Q.E.D.

2.4 Application to a limit theorem for "winding-type" additive functionals

Throughout this section let $z(t) = x(t) + \sqrt{-1}\, y(t)$, $z(0) = 0$, be a complex Brownian motion starting at the origin. Let $a_1, a_2, \cdots, a_n$ be given distinct points on $\mathbf{C} \setminus \{0\}$ and $a_\infty = \infty$. For $i = 1, \cdots, n, \infty$, let $f_{i1}, f_{i2}, \cdots, f_{im}$ be some regularly varying functions at a_i with order $\rho_{i1}, \rho_{i2}, \cdots, \rho_{im}$, respectively. (See Definition 3.1.) We denote the regular normalizing function of f_{ij} at a_i by $N_{ij}(\lambda)$ and the asymptotic angular component of f_{ij} at a_i by $c_{ij}(\theta)$ for $i = 1, \cdots, n, \infty$ and $j = 1, \cdots, m$.

The main purpose of this section is to give the joint limit processes, as $\lambda \to \infty$, of the processes $\{ A_{ij-}{}^\lambda,\ A_{ij+}{}^\lambda \}$ defined by

$$(2.31) \qquad \begin{cases} A_{ij-}{}^\lambda(t) &= \dfrac{1}{\lambda N_{ij}(\lambda)} \displaystyle\int_0^{u(\lambda t)} \frac{f_{ij}(z_s)}{z_s - a_i}\, 1_{D(i-)}(z_s)\, dz_s \\ A_{ij+}{}^\lambda(t) &= \dfrac{1}{\lambda N_{\infty j}(\lambda)} \displaystyle\int_0^{u(\lambda t)} \frac{f_{\infty j}(z_s)}{z_s - a_i}\, 1_{D(i+)}(z_s)\, dz_s, \end{cases}$$

where $u(t) = e^{2t} - 1$, $D(i-)$ is some bounded domain containing a_i and $D(i+)$ is some domain such that $D(i+)^c$ is bounded and $a_i \notin \overline{D(i+)}$. As we shall see, a particular choice of $D(i-)$ and $D(i+)$ is immaterial in the limit theorem.

First, we introduce the notion of $K-$*convergence* for stochastic processes:

Definition 2.4.1. Let $D_1 = D_1([0,\infty) \mapsto \mathbf{R}^d)$ be the space of all $\mathbf{R}^d$-valued right continuous functions with left limits. A sequence of D_1-valued stochastic processes $\{X_n(t)\}$ is said to be K-convergent to $X_\infty(t)$ if there exist a sequence of $\mathbf{R}^d \times \mathbf{R}$-valued stochastic processes $\{(Y_n(t), \varphi_n(t))\}$ and $(Y_\infty(t), \varphi_\infty(t))$ such that

1°) $Y_n(t)$ $(1 \le n \le \infty)$ and $\varphi_n(t)$ $(1 \le n \le \infty)$ are all continuous stochastic processes,

2°) $\varphi_n(t)$ is non-decreasing a.s., $\varphi_n(0) = 0$ and $\varphi_n(t) \to \infty$ as $t \to \infty$ a.s. for all $1 \leq n \leq \infty$,

3°) $X_n(t) = Y_n(\varphi_n^{-1}(t)) \qquad (1 \leq n \leq \infty)$,

4°) $\{(Y_n, \varphi_n)\} \longrightarrow (Y_\infty, \varphi_\infty)$ as $n \to \infty$ in law on $C([0,\infty) \mapsto \mathbf{R}^d \times \mathbf{R})$.

We remark that the main limit theorems by Kasahara and Kotani [6] are in the sense of K-convergence. If $\{X_n(t)\}$ is K-convergent to $X_\infty(t)$ as $n \to \infty$ and $X_\infty(t)$ is non-decreasing *w.p.*1, then $\{X_n(t)\}$ is weakly M_1-convergent to $X_\infty(t)$. Generally, M_1-convergence does not follow from K-convergence but, if $\{X_n(t)\}$ is K-convergent to $X_\infty(t)$ as $n \to \infty$ and φ_∞^{-1} has no fixed discontinuous point, then $\{X_n(t)\}$ converges to $X_\infty(t)$ as $n \to \infty$ *in the sense of finite dimensional distributions.* This fact is obviously derived from the following real variable proposition:

Proposition 2.4.1 *Let $\{y_n(t)\}$ and $\{\varphi_n(t)\}$ be sequences of continuous functions on $[0,\infty)$ such that $\varphi_n(t)$ is non-decreasing and $\varphi_n(t) \to \infty \quad (t \to \infty) \quad (n = 1, 2, \cdots)$. Suppose $y_n(t) \to y(t)$ and $\varphi_n(t) \to \varphi(t)$ uniformly in t on each compact sets as $n \to \infty$ and $\varphi(t) \to \infty \quad (t \to \infty)$.*

If $y(t)$ is constant on $(\varphi^{-1}(t_0-), \varphi^{-1}(t_0))$ for some $t_0 \in [0,\infty)$, then we have

$$(2.32) \qquad y_n(\varphi_n^{-1}(t_0)) \longrightarrow y(\varphi^{-1}(t_0)) \qquad (n \to \infty).$$

Particularly, if $\varphi^{-1}(t_0-) = \varphi^{-1}(t_0)$ then we have (2.32) *also.*

We omit the proof.

Next, in order to describe the joint limit processes, we introduce a particular system of n complex Brownian motions and $n+1$ complex Gaussian random measures. As in the preceding section, we always denote by $\widehat{M}(t)$ the time-changed process $M(\langle M\rangle^{-1}(t))$ for a conformal martingale $M(t)$.

Let $\zeta = (\zeta_1, \cdots, \zeta_n)$ be a $\mathbf{C}^n$-valued continuous process which has the following properties:

(1) Each $\zeta_i = \xi_i + \sqrt{-1}\,\eta_i$ is a complex Brownian motion starting at the origin for $i = 1, \cdots, n$.

(2) Setting

$$\begin{cases} \zeta_{i-}(t) &= \int_0^t 1_{(\xi_i(s)<0)}\, d\zeta_i(s) \\ \zeta_{i+}(t) &= \int_0^t 1_{(\xi_i(s)>0)} d\zeta_i(s), \end{cases}$$

the family $\{\widehat{\zeta_{1-}}, \cdots, \widehat{\zeta_{n-}}, \widehat{\zeta_{1+}}\}$ is mutually independent and $\widehat{\zeta_{1+}} = \widehat{\zeta_{2+}} = \cdots = \widehat{\zeta_{n+}}$.

An important fact is that a $\mathbf{C}^n$-valued process with these properties exists uniquely in the sense of law. We will explain the structure of ζ in Remark 3.1 in the last part of this section.

Furthermore we take $n+1$ complex Gaussian random measures $M_1, \cdots, M_n, M_+$ with the following properties:

(3) Each M_i is a complex Gaussian random measure on $[0, \infty) \times [0, 2\pi]$ with mean 0 and variance measure $dt \cdot d\theta/2\pi$ for $i = 1, \cdots, n, +$.

(4) The family $\{\zeta, M_1, \cdots, M_n, M_+\}$ is mutually independent.

Now define, for $i = 1, 2, \cdots, n$,

$$Z_i(t) = X_i(t) + \sqrt{-1}\, Y_i(t) = \int_0^t \frac{dz_s}{z_s - a_i},$$

$$\widehat{Z_i}^\lambda(t) = \widehat{X_i}^\lambda(t) + \sqrt{-1}\,\widehat{Y_i}^\lambda(t) = \frac{1}{\lambda} Z_i(\langle Z_i\rangle^{-1}(\lambda^2 t))$$

and

$$\begin{aligned} \tau_i{}^\lambda(t) &= \frac{1}{\lambda} u^{-1}(\langle Z_i\rangle^{-1}(\lambda^2 t)) \\ &= \frac{1}{2\lambda} \log\left[\lambda^2 |a_i|^2 \int_0^t e^{2\lambda \widehat{X_i}^\lambda(s)}\, ds + 1\right]. \end{aligned}$$

Then our theorem can be stated as follows:

Theorem 2.4.1

$$\{\, \widehat{Z_i}^\lambda,\ \tau_i{}^\lambda,\ A_{ij-}{}^\lambda(\tau_i{}^\lambda),\ A_{ij+}{}^\lambda(\tau_i{}^\lambda) \,\}_{1\le i\le n}^{1\le j\le m} \longrightarrow \{\, \zeta_i,\ \mu_i,\ \mathcal{L}_{ij-},\ \mathcal{L}_{ij+} \,\}_{1\le i\le n}^{1\le j\le m}$$

as $\lambda \to \infty$ *in law on* $C([0,\infty) \to \mathbf{C}^n \times \mathbf{R}^n \times \mathbf{C}^{mn} \times \mathbf{C}^{mn})$, *where*

$$\mu_i(t) = \max_{0\le s\le t} \xi_i(s),$$

$$\mathcal{L}_{ij-}(t) = \overline{c_{ij}} \int_0^t (-\xi_i(s))^{\rho_{ij}}\, d\zeta_{i-}(s) + \int_0^t \int_0^{2\pi} (c_{ij}(\theta) - \overline{c_{ij}})(-\xi_i(s))^{\rho_{ij}} M_i(d\langle \zeta_{i-}\rangle(s), d\theta), \tag{2.33}$$

$$\mathcal{L}_{ij+}(t) = \overline{c_{\infty j}} \int_0^t \xi_i(s)^{\rho_{\infty j}}\, d\zeta_{i+}(s) + \int_0^t \int_0^{2\pi} (c_{\infty j}(\theta) - \overline{c_{\infty j}})\xi_i(s)^{\rho_{\infty j}} M_+(d\langle \zeta_{i+}\rangle(s), d\theta) \tag{2.34}$$

and $\bar{c} = (1/2\pi)\int_0^{2\pi} c(\theta)\, d\theta$ *, in general.*

As a corollary to Theorem 2.4.1, we can conclude the following:

Theorem 2.4.2

$$\{\, \widehat{Z}_i^{\lambda},\ A_{ij-}{}^{\lambda},\ A_{ij+}{}^{\lambda}\,\}_{1\le i\le n}^{1\le j\le m} \longrightarrow \{\, \zeta_i,\ \mathcal{L}_{ij-}(\mu_i^{-1}),\ \mathcal{L}_{ij+}(\mu_i^{-1})\,\}_{1\le i\le n}^{1\le j\le m}$$

as $\lambda \to \infty$ *in the sense of* K*-convergence.*

Proof of theorem 2.4.1

The fact that

$$\{\, \widehat{Z}_i^{\lambda},\ \tau_i^{\lambda}\,\} \longrightarrow \{\, \zeta_i,\ \max_{0\le s\le \cdot} \xi_i(s)\,\} \qquad \text{as } \lambda \to \infty$$

in law on $C([0,\infty) \mapsto \mathbf{C}\times\mathbf{R})$ for each i was obtained by Kasahara and Kotani ([6], Lemma 3.1).

The first important step in our proof is the following transformation:

$$\begin{aligned} &\frac{1}{\lambda}\int_0^{(Z_i)^{-1}(\lambda^2 t)} \frac{f(z_s)}{z_s - a_i}\, dz_s \\ &= \frac{1}{\lambda}\int_0^{(Z_i)^{-1}(\lambda^2 t)} f(a_i - a_i e^{Z_i(s)})\, dZ_i(s) \\ &= \int_0^t f(a_i - a_i e^{\lambda \widehat{Z}_i^{\lambda}(s)})\, d\widehat{Z}_i^{\lambda}(s). \end{aligned} \tag{2.35}$$

By this transformation, we have

$$\begin{cases} A_{ij-}{}^{\lambda}(\tau_i^{\lambda}(t)) = \dfrac{1}{N_{ij}(\lambda)}\displaystyle\int_0^t (f_{ij}\cdot 1_{D(i-)})(a_i - a_i e^{\lambda \widehat{Z}_i^{\lambda}(s)})\, d\widehat{Z}_i^{\lambda}(s) \\ A_{ij+}{}^{\lambda}(\tau_i^{\lambda}(t)) = \dfrac{1}{N_{\infty j}(\lambda)}\displaystyle\int_0^t (f_{\infty j}\cdot 1_{D(i+)})(a_i - a_i e^{\lambda \widehat{Z}_i^{\lambda}(s)})\, d\widehat{Z}_i^{\lambda}(s). \end{cases}$$

Fix sufficiently large $r > 0$ and set

$$\begin{cases} F_{ij-}{}^{\lambda}(t) &= \dfrac{1}{N_{ij}(\lambda)} \displaystyle\int_0^t f_{ij}(a_i - a_i e^{\lambda \widehat{Z}_i{}^{\lambda}(s)}) 1_{(\lambda \widehat{X}_i{}^{\lambda}(s) < -r)} d\widehat{Z}_i{}^{\lambda}(s) \\ F_{ij+}{}^{\lambda}(t) &= \dfrac{1}{N_{\infty j}(\lambda)} \displaystyle\int_0^t f_{\infty j}(a_i - a_i e^{\lambda \widehat{Z}_i{}^{\lambda}(s)}) 1_{(\lambda \widehat{X}_i{}^{\lambda}(s) > r)} d\widehat{Z}_i{}^{\lambda}(s). \end{cases}$$

Since

$$\sup_{0 \le \theta \le 2\pi} |1_{D(i-)}(a_i - a_i e^{\lambda x + \sqrt{-1}\theta}) - 1_{(\lambda x < -r)}| \longrightarrow 0$$

and

$$\sup_{0 \le \theta \le 2\pi} |1_{D(i+)}(a_i - a_i e^{\lambda x + \sqrt{-1}\theta}) - 1_{(\lambda x > r)}| \longrightarrow 0$$

as $\lambda \to \infty$, we can easily deduce that

$$(2.36) \qquad E \sup_{0 \le t \le T} |A_{ij\pm}{}^{\lambda}(\tau_i{}^{\lambda}(t)) - F_{ij\pm}{}^{\lambda}(t)|^2 \longrightarrow 0 \qquad \text{as } \lambda \to \infty$$

and

$$(2.37) \qquad E \sup_{0 \le t \le T} | \left\langle A_{ij\pm}{}^{\lambda}(\tau_i{}^{\lambda}) \right\rangle_t - \left\langle F_{ij\pm}{}^{\lambda} \right\rangle_t | \longrightarrow 0 \qquad \text{as } \lambda \to \infty$$

by a similar argument as in the proof of Theorem 2.3.1.

Therefore the joint processes

$$\left\{ \widehat{Z}_i{}^{\lambda},\ A_{ij-}{}^{\lambda}(\tau_i{}^{\lambda}),\ \left\langle A_{ij-}{}^{\lambda}(\tau_i{}^{\lambda}) \right\rangle,\ A_{ij+}{}^{\lambda}(\tau_i{}^{\lambda}),\ \left\langle A_{ij+}{}^{\lambda}(\tau_i{}^{\lambda}) \right\rangle \right\}_{1 \le i \le n}^{1 \le j \le m}$$

have the same limit law as the joint processes

$$\left\{ \widehat{Z}_i{}^{\lambda},\ F_{ij-}{}^{\lambda},\ \left\langle F_{ij-}{}^{\lambda} \right\rangle,\ F_{ij+}{}^{\lambda},\ \left\langle F_{ij+}{}^{\lambda} \right\rangle \right\}_{1 \le i \le n}^{1 \le j \le m}.$$

We know by Theorem 2.3.1 that the joint limit processes as $\lambda \to \infty$ of $\left\{ \widehat{Z}_i{}^{\lambda},\ F_{ij-}{}^{\lambda},\ \left\langle F_{ij-}{}^{\lambda} \right\rangle \right\}_{1 \le j \le m}$ and $\left\{ \widehat{Z}_i{}^{\lambda},\ F_{ij+}{}^{\lambda},\ \left\langle F_{ij+}{}^{\lambda} \right\rangle \right\}_{1 \le j \le m}$ are $\{ \zeta_i,\ \mathcal{L}_{ij-},\ \langle \mathcal{L}_{ij-} \rangle \}_{1 \le j \le m}$ and $\{ \zeta_i,\ \mathcal{L}_{ij+},\ \langle \mathcal{L}_{ij+} \rangle \}_{1 \le j \le m}$ respectively for each i, where $\mathcal{L}_{ij-}$ and $\mathcal{L}_{ij+}$ are defined by (2.33) and (2.34). Then the laws of

$$\left\{ \widehat{Z}_i{}^{\lambda},\ A_{ij-}{}^{\lambda}(\tau_i{}^{\lambda}),\ \left\langle A_{ij-}{}^{\lambda}(\tau_i{}^{\lambda}) \right\rangle,\ A_{ij+}{}^{\lambda}(\tau_i{}^{\lambda}),\ \left\langle A_{ij+}{}^{\lambda}(\tau_i{}^{\lambda}) \right\rangle \right\}_{1 \le j \le m}^{1 \le i \le n},$$

$\lambda > 0$, form a tight family because each component converges in law. Further it is clear from the above argument that we may assume for any limit point of this family that it is the law of

$$\{\ \zeta_i,\ \mathcal{A}_{ij-},\ \langle\mathcal{A}_{ij-}\rangle\,,\ \mathcal{A}_{ij+},\ \langle\mathcal{A}_{ij+}\rangle\ \}_{1\le j\le m}^{1\le i\le n},$$

where $\zeta_1, \zeta_2, \cdots, \zeta_n$ are *some* complex Brownian motions,

$$\begin{aligned}\mathcal{A}_{ij-}(t) &= \overline{c_{ij}}\int_0^t (-\xi_i(s))^{\rho_{ij}}\, d\zeta_{i-}(s) \\ &\quad + \int_0^t\int_0^{2\pi} (c_{ij}(\theta) - \overline{c_{ij}})(-\xi_i(s))^{\rho_{ij}}\, M_i(d\langle\zeta_{i-}\rangle(s), d\theta),\end{aligned}$$

$$\begin{aligned}\mathcal{A}_{ij+}(t) &= \overline{c_{\infty j}}\int_0^t \xi_i(s)^{\rho_{\infty j}}\, d\zeta_{i+}(s) \\ &\quad + \int_0^t\int_0^{2\pi} (c_{\infty j}(\theta) - \overline{c_{\infty j}})\xi_i(s)^{\rho_{\infty j}}\, \tilde{M}_i(d\langle\zeta_{i+}\rangle(s), d\theta)\end{aligned}$$

and $M_1, M_2, \cdots, M_n, \tilde{M}_1, \tilde{M}_2, \cdots, \tilde{M}_n$ are *some* complex Gaussian random measures on $[0,\infty)\times[0,2\pi]$ with mean 0 and variance measure $dt\cdot d\theta/2\pi$. We fix these $\zeta_1, \cdots, \zeta_n, M_1, \cdots, M_n, \tilde{M}_1, \cdots, \tilde{M}_n$ below. It remains to prove the identity

$$\text{(2.38)}\qquad \widehat{\zeta_{1+}} = \widehat{\zeta_{2+}} = \cdots = \widehat{\zeta_{n+}},$$

the identity

$$\text{(2.39)}\qquad \tilde{M}_1 = \tilde{M}_2 = \cdots = \tilde{M}_n := M_+$$

and the mutual independence of

$$\text{(2.40)}\qquad \widehat{\zeta_{1-}},\ \widehat{\zeta_{2-}},\ \cdots,\ \widehat{\zeta_{n-}},\ \widehat{\zeta_{1+}},\ M_1,\ M_2,\ \cdots,\ M_n,\ M_+.$$

Firstly we prove the identity (2.38). As a consequence of (2.36) and (2.37), we may replace $D(i+)$ by $D(1+)\cap D(2+)\cap\cdots\cap D(n+)$. Therefore we may assume that

$$D(1+) = D(2+) = \cdots = D(n+) := D(\infty).$$

Set

$$W_{i+}{}^{\lambda}(t) = \frac{1}{\lambda}\int_0^{u(\lambda t)} \frac{1}{z_s - a_i}\, 1_{D(\infty)}(z_s)\, dz_s.$$

This is the particular case of $A_{ij+}{}^{\lambda}(t)$. We remark that

$$\text{(2.41)}\qquad E\sup_{0\le t\le T}\left|W_{i+}{}^{\lambda}(\tau_1{}^{\lambda}(t)) - W_{1+}{}^{\lambda}(\tau_1{}^{\lambda}(t))\right|^2 \longrightarrow 0 \qquad \text{as } \lambda\to\infty$$

and

$$\text{(2.42)}\qquad E\sup_{0\le t\le T}\left|\left\langle W_{i+}{}^{\lambda}(\tau_1{}^{\lambda})\right\rangle_t - \left\langle W_{1+}{}^{\lambda}(\tau_1{}^{\lambda})\right\rangle_t\right| \longrightarrow 0 \qquad \text{as } \lambda\to\infty$$

for any i. To prove (2.41), note that

$$\begin{aligned} W_{i+}{}^{\lambda}(\tau_1{}^{\lambda}(t)) &= \frac{1}{\lambda}\int_0^{(Z_1)^{-1}(\lambda^2 t)} -\frac{1}{z_s - a_i}\, 1_{D(\infty)}(z_s)\, dz_s \\ &= \frac{1}{\lambda}\int_0^{(Z_1)^{-1}(\lambda^2 t)} -\frac{1}{z_s - a_1}\, 1_{D(\infty)}(z_s)\cdot\frac{z_s - a_1}{z_s - a_i}\, dz_s \\ &= \int_0^t 1_{D(\infty)}(a_1 - a_1 e^{\lambda \widehat{Z_1}^{\lambda}(s)}) R_i(\lambda \widehat{X_1}^{\lambda}, \lambda \widehat{Y_1}^{\lambda})\, d\widehat{Z_1}^{\lambda}(s), \end{aligned}$$

where

$$R_i(x,\theta) = -a_1 e^{x+\sqrt{-1}\theta} \big/ (a_1 - a_i - a_1 e^{x+\sqrt{-1}\theta}).$$

Hence

$$\begin{aligned} &E \sup_{0\le t\le T} \left| W_{i+}{}^{\lambda}(\tau_1{}^{\lambda}(t)) - W_{1+}{}^{\lambda}(\tau_1{}^{\lambda}(t)) \right|^2 \\ &\le \text{ const. } E\int_0^T 1_{D(\infty)}(a_1 - a_1 e^{\lambda \widehat{Z_1}^{\lambda}(s)}) |R_i(\lambda \widehat{X_1}^{\lambda}, \lambda \widehat{Y_1}^{\lambda}) - 1|^2\, ds. \end{aligned}$$

Since $1_{D(\infty)}(a_1 - a_1 e^{x+\sqrt{-1}\theta})|R_i(x,\theta)|$ is bounded in $(x,\theta) \in \mathbf{R}\times\mathbf{T}$ and $\sup_{0\le\theta\le 2\pi} 1_{D(\infty)}(a_1 - a_1 e^{\lambda x+\sqrt{-1}\theta})|R_i(\lambda x,\theta) - 1| \longrightarrow 0$ as $\lambda \to \infty$ for any $x \ne 0$, we can deduce the convergence (2.41). The proof of (2.42) can be given similarly.

Then the laws of P_λ, $\lambda > 0$, of

$$\left\{ \widehat{Z_i}^{\lambda},\ W_{i+}{}^{\lambda}(\tau_i{}^{\lambda}),\ \left\langle W_{i+}{}^{\lambda}(\tau_i{}^{\lambda})\right\rangle,\ W_{i+}{}^{\lambda}(\tau_1{}^{\lambda}),\ \left\langle W_{i+}{}^{\lambda}(\tau_1{}^{\lambda})\right\rangle \right\}_{1\le i\le n}$$

form a tight family and we may assume one limit point P_∞ of $\{P_\lambda\}$ to be the law of

$$\{ \zeta_i,\ \zeta_{i+},\ \langle\zeta_{i+}\rangle,\ \zeta_{1+},\ \langle\zeta_{1+}\rangle \}_{1\le i\le n}.$$

Let $P_{\lambda_\nu} \to P_\infty$ for some subsequence and write λ_ν as λ for the notational simplicity. We can prove that $\left\langle W_{i\pm}{}^{\lambda}(\tau_i{}^{\lambda})\right\rangle_t \to \infty$ and $\left\langle W_{i+}{}^{\lambda}(\tau_1{}^{\lambda})\right\rangle_t \to \infty$ as $t \to \infty$ by a similar argument as in the proof of Lemma 2.3.2, and hence we have, by Lemma-W2, that

$$\begin{aligned} &\{ \widehat{Z_i}^{\lambda},\ W_{i+}{}^{\lambda}(\tau_i{}^{\lambda}),\ \widehat{W_{i+}{}^{\lambda}(\tau_i{}^{\lambda})},\ W_{i+}{}^{\lambda}(\tau_1{}^{\lambda}),\ \widehat{W_{i+}{}^{\lambda}(\tau_1{}^{\lambda})} \}_{1\le i\le n} \\ &\qquad \longrightarrow \{ \zeta_i,\ \zeta_{i+},\ \widehat{\zeta_{i+}},\ \zeta_{1+},\ \widehat{\zeta_{1+}} \}_{1\le i\le n} \end{aligned}$$

as $\lambda \to \infty$ in law. We may assume by the Skorohod theorem that this convergence is uniform on each compact interval a.s. Then we see that $\widehat{\zeta_{i+}}$ is identical to $\widehat{\zeta_{1+}}$ for $i = 1, \cdots, n$ because $\widehat{W_{i+}{}^{\lambda}(\tau_i{}^{\lambda})} = \widehat{W_{i+}{}^{\lambda}(\tau_1{}^{\lambda})}$. Thus the identity (2.38) is now proved.

Secondly, we prove the identity (2.39). We can prove similarly to (2.41) and (2.42) that

$$E \sup_{0\le t\le T} \left|A_{ij+}{}^{\lambda}(\tau_1{}^{\lambda}(t)) - A_{1j+}{}^{\lambda}(\tau_1{}^{\lambda}(t))\right|^2 \longrightarrow 0 \qquad \text{as } \lambda \to \infty$$

and

$$E \sup_{0\le t\le T} \left|\left\langle A_{ij+}{}^{\lambda}(\tau_1{}^{\lambda})\right\rangle_t - \left\langle A_{1j+}{}^{\lambda}(\tau_1{}^{\lambda})\right\rangle_t\right| \longrightarrow 0 \qquad \text{as } \lambda \to \infty$$

for any i and j. Let P_∞ be one limit point of the tight family of the laws P_λ, $\lambda > 0$, of

$$\left\{ \widehat{Z_i}^{\lambda},\ A_{ij+}{}^{\lambda}(\tau_i{}^{\lambda}),\ \left\langle A_{ij+}{}^{\lambda}(\tau_i{}^{\lambda})\right\rangle,\ A_{ij+}{}^{\lambda}(\tau_1{}^{\lambda}),\ \left\langle A_{ij+}{}^{\lambda}(\tau_1{}^{\lambda})\right\rangle \right\}_{1\le i\le n}^{1\le j\le m}.$$

We may assume the law of P_∞ to be the law of

$$\{ \zeta_i,\ \mathcal{A}_{ij+},\ \langle \mathcal{A}_{ij+}\rangle,\ \mathcal{A}_{1j+},\ \langle \mathcal{A}_{1j+}\rangle \}_{1\le i\le n}^{1\le j\le m}.$$

Let $P_{\lambda_\nu} \to P_\infty$ for some subsequence and write λ_ν simply as λ. Since we can prove that $\left\langle A_{ij+}(\tau_i{}^{\lambda})\right\rangle_t \to \infty$ and $\left\langle A_{ij+}(\tau_1{}^{\lambda})\right\rangle_t \to \infty$ as $t \to \infty$ by a similar argument as in the proof of Lemma 2.3.2, and hence by Lemma-W2 we have that

$$\{ \widehat{Z_i}^{\lambda},\ A_{ij+}{}^{\lambda}(\tau_i{}^{\lambda}),\ \widehat{A_{ij+}{}^{\lambda}}(\tau_i{}^{\lambda}),\ A_{ij+}{}^{\lambda}(\tau_1{}^{\lambda}),\ \widehat{A_{ij+}{}^{\lambda}}(\tau_1{}^{\lambda}) \}_{1\le i\le n}^{1\le j\le m}$$

$$\longrightarrow \{ \zeta_i,\ \mathcal{A}_{ij+},\ \widehat{\mathcal{A}_{ij+}},\ \mathcal{A}_{1j+},\ \widehat{\mathcal{A}_{1j+}} \}_{1\le i\le n}^{1\le j\le m}$$

as $\lambda \to \infty$ in law. We may assume by the Skorohod theorem that this convergence is uniform on each compact interval a.s. Then we have that

$$\widehat{\mathcal{A}_{1j+}}(t) = \widehat{\mathcal{A}_{2j+}}(t) = \cdots = \widehat{\mathcal{A}_{nj+}}(t) \qquad (j = 1, \cdots, m) \tag{2.43}$$

because $\widehat{A_{ij+}{}^{\lambda}}(\tau_i{}^{\lambda}) = \widehat{A_{ij+}{}^{\lambda}}(\tau_1{}^{\lambda})$.

Set

$$\mathcal{N}_{ij+}(t) = \mathcal{A}_{ij+}(\langle \zeta_{i+}\rangle^{-1}(t)).$$

The identity (2.43) implies that

$$\widehat{\mathcal{N}_{1j+}}(t) = \widehat{\mathcal{N}_{2j+}}(t) = \cdots = \widehat{\mathcal{N}_{nj+}}(t) \qquad (j = 1, \cdots, m). \tag{2.44}$$

On the other hand, note that

$$\begin{aligned} \mathcal{N}_{ij+}(t) &= \overline{c_{\infty j}} \int_0^t \left(\xi_i(\langle \zeta_{i+}\rangle^{-1}(s)) \vee 0\right)^{\rho_{\infty j}} d\widehat{\zeta_{i+}}(s) \\ &\quad + \int_0^t \int_0^{2\pi} (c_{\infty j}(\theta) - \overline{c_{\infty j}}) \left(\xi_i(\langle \zeta_{i+}\rangle^{-1}(s)) \vee 0\right)^{\rho_{\infty j}} \tilde{M}_i(ds, d\theta) \end{aligned}$$

and

$$\langle \mathcal{N}_{ij+}\rangle(t) = \overline{|c_{\infty j}|^2} \int_0^t \left(\xi_i(\langle\zeta_{i+}\rangle^{-1}(s)) \vee 0\right)^{2\rho_{\infty j}} ds.$$

Since $\xi_i(\langle\zeta_{i+}\rangle^{-1}(t)) \vee 0$, $i = 1, \cdots, n$, are the same reflecting Brownian motion by the identity (2.38) (See remark 2.4.1 below), we have that

$$(2.45) \qquad \langle \mathcal{N}_{1j+}\rangle(t) = \langle \mathcal{N}_{2j+}\rangle(t) = \cdots = \langle \mathcal{N}_{nj+}\rangle(t) \qquad (j = 1, \cdots, m).$$

Combining (2.44) and (2.45), we obtain the identity

$$\mathcal{N}_{1j+}(t) = \mathcal{N}_{2j+}(t) = \cdots = \mathcal{N}_{nj+}(t) \qquad (j = 1, \cdots, m).$$

This clearly shows the identity (2.39).

Finally, we prove the mutual independence of (2.40). Let $\{e_0 \equiv 1, e_1, \cdots, e_p\}$ be some orthonormal system in $L^2(0, 2\pi)$ such that

$$c_{ij}(\theta) = \sum_{k=0}^{p} \alpha_{ij}{}^{(k)} e_k(\theta), \qquad \alpha_{ij}{}^{(k)} \in \mathbf{C} \qquad (k = 0, \cdots, p)$$

for $i = 1, \cdots, n, \infty$ and $j = 1, \cdots, m$. Set

$$\begin{cases} V_{ik-}{}^{\lambda}(t) &= \displaystyle\int_0^t e_k(\lambda \widehat{Y}_i{}^{\lambda}(s))\, 1_{(\lambda \widehat{X}_i{}^{\lambda}(s) < -r)}\, d\widehat{Z}_i{}^{\lambda}(s) \\ V_{ik+}{}^{\lambda}(t) &= \displaystyle\int_0^t e_k(\lambda \widehat{Y}_i{}^{\lambda}(s))\, 1_{(\lambda \widehat{X}_i{}^{\lambda}(s) > r)}\, d\widehat{Z}_i{}^{\lambda}(s). \end{cases}$$

By (2.22) and (2.36), we have

$$E \sup_{0\le t\le T} \left| A_{ij-}{}^{\lambda}(\tau_i{}^{\lambda}(t)) - \sum_{k=0}^{p} \alpha_{ij}{}^{(k)} \int_0^t (-\widehat{X}_i{}^{\lambda}(s))^{\rho_{ij}}\, dV_{ik-}{}^{\lambda}(s) \right|^2 \longrightarrow 0 \qquad \text{as } \lambda \to \infty$$

and

$$E \sup_{0\le t\le T} \left| A_{ij+}{}^{\lambda}(\tau_i{}^{\lambda}(t)) - \sum_{k=0}^{p} \alpha_{\infty j}{}^{(k)} \int_0^t \widehat{X}_i{}^{\lambda}(s)^{\rho_{\infty j}}\, dV_{ik+}{}^{\lambda}(s) \right|^2 \longrightarrow 0 \qquad \text{as } \lambda \to \infty.$$

Hence by Lemma 2.3.3, Lemma 2.3.4 and Lemma-W2 we may assume the law of one limit point of the tight family of the laws of

$$\Big\{ \widehat{Z}_i{}^{\lambda},\ A_{ij-}{}^{\lambda}(\tau_i{}^{\lambda}),\ A_{ij+}{}^{\lambda}(\tau_i{}^{\lambda}),\ \int_0^{\cdot} (-\widehat{X}_i{}^{\lambda}(s))^{\rho_{ij}}\, dV_{ik-}{}^{\lambda}(s),\ V_{ik-}{}^{\lambda},\ \widehat{V_{ik-}}{}^{\lambda},$$

$$\int_0^{\cdot} (\widehat{X}_i{}^{\lambda}(s))^{\rho_{\infty j}}\, dV_{ik+}{}^{\lambda}(s),\ V_{ik+}{}^{\lambda},\ \widehat{V_{ik+}}{}^{\lambda} \Big\}_{1\le i\le n}^{1\le j\le m;\ 0\le k\le p},$$

$\lambda > 0$, to be the law of

$$\Big\{ \zeta_i,\ \mathcal{A}_{ij-},\ \mathcal{A}_{ij+},\ \int_0^{\cdot} (-\xi_i(s))^{\rho_{ij}}\, d\mathcal{V}_{ik-}(s),\ \mathcal{V}_{ik-},\ \widehat{\mathcal{V}_{ik-}},$$

$$\int_0^{\cdot} \xi_i(s)^{\rho_{\infty j}}\, d\mathcal{V}_{ik+}(s),\ \mathcal{V}_{ik+},\ \widehat{\mathcal{V}_{ik+}} \Big\}_{1\le i\le n}^{1\le j\le m;\ 0\le k\le p},$$

where

$$\begin{cases} \mathcal{V}_{i0-}(t) &= \zeta_{i-}(t) \\ \mathcal{V}_{ik-}(t) &= \int_0^t \int_0^{2\pi} e_k(\theta)\, M_i(d\langle \zeta_{i-}\rangle(s), d\theta) \qquad (k = 1, \cdots, p) \\ \mathcal{V}_{i0+}(t) &= \zeta_{i+}(t) \\ \mathcal{V}_{ik+}(t) &= \int_0^t \int_0^{2\pi} e_k(\theta)\, \tilde{M}_i(d\langle \zeta_{i+}\rangle(s), d\theta) \qquad (k = 1, \cdots, p). \end{cases}$$

Therefore if we can prove that

$$\{\, \widehat{\mathcal{V}_{1k-}},\ \widehat{\mathcal{V}_{2k-}}, \cdots,\ \widehat{\mathcal{V}_{nk-}},\ \widehat{\mathcal{V}_{1k+}}\ \}_{0\le k\le p}$$

is an $(n+1)\cdot(p+1)$-dimensional Brownian motion, then the mutual independence of (2.40) follows at once.

To prove this, set

$$\begin{cases} G_{ik-}{}^{\lambda}(t) &= \dfrac{1}{\lambda}\displaystyle\int_0^{\lambda^2 t} e_k\left(\arg\frac{z_s - a_i}{-a_i}\right) 1_{(\log|z_s - a_i/-a_i| < -r)} \frac{dz_s}{z_s - a_i} \\ G_{ik+}{}^{\lambda}(t) &= \dfrac{1}{\lambda}\displaystyle\int_0^{\lambda^2 t} e_k\left(\arg\frac{z_s - a_i}{-a_i}\right) 1_{(\log|z_s - a_i/-a_i| > r)} \frac{dz_s}{z_s - a_i}. \end{cases}$$

By the transformation (2.35), we have

$$G_{ik\pm}{}^{\lambda}\left(\lambda^{-2}\langle Z_i\rangle^{-1}(\lambda^2 t)\right) = V_{ik\pm}{}^{\lambda}(t).$$

This implies that

$$\text{(2.46)} \qquad \left\langle G_{ik-}{}^{\lambda}, G_{il-}{}^{\lambda}\right\rangle\left(\left\langle G_{ik-}{}^{\lambda}\right\rangle^{-1}(t)\right) = \left\langle V_{ik-}{}^{\lambda}, V_{il-}{}^{\lambda}\right\rangle\left(\left\langle V_{ik-}{}^{\lambda}\right\rangle^{-1}(t)\right).$$

By (2.27), the right hand side of (2.46) converges to $\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$ in probability as $\lambda \to \infty$ for any $t > 0$ if $k \neq l$. On the other hand, since

$$1_{(\log|z_s - a_i/-a_i| < -r)} \cdot 1_{(\log|z_s - a_j/-a_j| < -r)} \equiv 0 \qquad \text{if } i \neq j$$

for sufficiently large r, we have that

$$\text{(2.47)} \qquad \left\langle G_{ik-}{}^{\lambda}, G_{jl-}{}^{\lambda}\right\rangle(t) \equiv \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \qquad \text{if } i \neq j$$

for any k, l. Combining (2.46), (2.47) and the obvious relation

$$\left\langle G_{1k+}{}^{\lambda}, G_{il-}{}^{\lambda}\right\rangle(t) \equiv \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

for any i, k, l, we can conclude by the asymptotic Knight's theorem in Pitman-Yor [10] that $\{ \widehat{G_{1k-}}^{\lambda}, \cdots, \widehat{G_{nk-}}^{\lambda}, \widehat{G_{1k+}}^{\lambda} \}_{0\le k\le p}$ converges in law to an $(n+1)\cdot(p+1)$-dimensional Brownian motion. Then noting that $\widehat{G_{ik\pm}}^{\lambda}(t) = \widehat{V_{ik\pm}}^{\lambda}(t)$, we arrive at the needed conclusion.

Now the proof is complete. Q.E.D.

Remark 2.4.1. (due to S.Watanabe)

The $\mathbf{C}^n$-valued process $\zeta = (\zeta_1, \cdots, \zeta_n)$ can be constructed as follows: We follow the notions and notations concerning Brownian excursions to [4], Chapter 3, section 4.3. Take n Poisson point processes of Brownian negative excursions $p_1{}^-, p_2{}^-, \cdots, p_n{}^-$ (*i.e.* stationary Poisson point processes on $\mathcal{W}^-$ with the characteristic measure $\mathbf{n}^-$), a Poisson point process of Brownian positive excursion p^+ (*i.e.* a stationary Poisson point process on $\mathcal{W}^+$ with the characteristic measure $\mathbf{n}^+$) and $n+1$ one-dimensional Brownian motions $\beta_1, \beta_2, \cdots, \beta_n, \beta_+$ such that the family $(p_1{}^-, \cdots, p_n{}^-, p^+, \beta_1, \cdots, \beta_n, \beta_+)$ is mutually independent. The sum p_i of $p_i{}^-$ and p^+ is a Poisson point process of Brownian excursions (*i.e.* a stationary Poisson point process on $\mathcal{W} = \mathcal{W}^- \cup \mathcal{W}^+$ with the characteristic measure $\mathbf{n} = \mathbf{n}^- + \mathbf{n}^+$) and we can construct a Brownian motion ξ_i from p_i as in Chapter 3, section 4.3 of [4], $i = 1, \cdots, n$. Set

$$\eta_i(t) = \beta_i\left(\int_0^t 1_{(\xi_i(s)<0)}\, ds\right) + \beta_+\left(\int_0^t 1_{(\xi_i(s)>0)}\, ds\right)$$

and define finally

$$\zeta_i(t) = \xi_i(t) + \sqrt{-1}\,\eta_i(t), \qquad i = 1, \cdots, n.$$

Then it is easy to see that $\{\zeta_1, \cdots, \zeta_n\}$ satisfies the conditions (1) and (2) above.

Conversely, suppose we are given a family $\{\zeta_1, \cdots, \zeta_n\}$ possessing the properties (1) and (2). Set

$$\begin{cases} \zeta_{i-}(t) &=& \displaystyle\int_0^t 1_{(\xi_i(s)<0)}\, d\zeta_i(s) \qquad (i = 1, \cdots, n) \\ \zeta_{i+}(t) &=& \displaystyle\int_0^t 1_{(\xi_i(s)>0)}\, d\zeta_i(s) \qquad (i = 1, \cdots, n) \end{cases}$$

and write

$$\text{(2.48)} \qquad \begin{cases} \zeta_{i-}(t) &:=& \xi_{i-}(t) + \sqrt{-1}\,\eta_{i-}(t) \qquad (i = 1, \cdots, n) \\ \zeta_{i+}(t) &:=& \xi_{i+}(t) + \sqrt{-1}\,\eta_{i+}(t) \qquad (i = 1, \cdots, n) \end{cases}$$

and

$$(2.49)\quad \begin{cases} \widehat{\zeta_{i-}}(t) := \alpha_i(t) + \sqrt{-1}\,\beta_i(t) \qquad (i = 1, \cdots, n) \\ \widehat{\zeta_{1+}}(t) = \widehat{\zeta_{2+}}(t) = \cdots = \widehat{\zeta_{n+}}(t) := \alpha_+(t) + \sqrt{-1}\,\beta_+(t). \end{cases}$$

By the assumptions, $\alpha_1, \cdots, \alpha_n, \alpha_+, \beta_1, \cdots, \beta_n, \beta_+$ are mutually independent one-dimensional Brownian motions. By Tanaka's formula, we have

$$(2.50)\quad \xi_i(t) \wedge 0 = \xi_{i-}(t) - l_i(t) \qquad (i = 1, \cdots, n)$$

and

$$(2.51)\quad \xi_i(t) \vee 0 = \xi_{i+}(t) + l_i(t) \qquad (i = 1, \cdots, n),$$

where $l_i(t)$ is the local time at 0 of one-dimensional Brownian motion $\xi_i(t)$. If we make a time change $t \mapsto \langle \xi_{i-} \rangle^{-1}(t)$ for (2.50) and $t \mapsto \langle \xi_{i+} \rangle^{-1}(t)$ for (2.51), then $\xi_i(\langle \xi_{i-} \rangle^{-1}(t)) \wedge 0$, $i = 1, \cdots, n$, are mutually independent reflecting Brownian motions on $(-\infty, 0]$ and $\xi_i(\langle \xi_{i+} \rangle^{-1}(t)) \vee 0$, $i = 1, \cdots, n$, are the same reflecting Brownian motion on $[0, \infty)$. That is, from (2.50) and (2.51) we have $n+1$ equations

$$(2.52)\quad \begin{cases} r_i(t) = \alpha_i(t) - \phi_i(t) \qquad (i = 1, \cdots, n) \\ r_+(t) = \alpha_+(t) + \phi_+(t), \end{cases}$$

where $r_i(t) = \xi_i(\langle \xi_{i-} \rangle^{-1}(t)) \wedge 0$, $i = 1, \cdots, n$, $r_+(t) = \xi_i(\langle \xi_{i+} \rangle^{-1}(t)) \vee 0$, $\phi_i(t) = l_i(\langle \xi_{i-} \rangle^{-1}(t))$, $i = 1, \cdots, n$ and $\phi_+(t) = l_i(\langle \xi_{i+} \rangle^{-1}(t))$. These equations give the Skorohod decompositions of $r_i(t)$, $i = 1, \cdots, n, +$; in particular,

$$\phi_i(t) = \lim_{\varepsilon \downarrow 0} \frac{1}{2\varepsilon} \int_0^t 1_{(0 < r_i(s) < \varepsilon)}\, ds \qquad (i = 1, \cdots, n, +).$$

If p^+ is the Poisson point process of positive Brownian excursion corresponding to r_+ and $p_i{}^-$, $i = 1, \cdots, n$, are the Poisson point processes of negative Brownian excursions corresponding to r_i, then $p_1{}^-, \cdots, p_n{}^-, p^+, \beta_1, \cdots, \beta_n, \beta_+$ are mutually independent. Thus we have recovered this independent family from $\{\zeta_i\}_{1 \le i \le n}$ and hence, the uniqueness in law of $\{\zeta_i\}_{1 \le i \le n}$ is now obvious.

Set

$$\mu_i(t) = \max_{0 \le s \le t} \xi_i(s) \qquad (i = 1, \cdots, n)$$

and

$$\sigma_+(t) = \left(\max_{0\le s\le \cdot} r_+(s)\right)^{-1}(t) = \inf\{u;\ r_+(u) = t\}.$$

Then we have the following:

$$l_i(\mu_i^{-1}(t)) = \phi_+(\sigma_+(t)) := e(t), \tag{2.53}$$

$$\begin{cases} \langle \xi_{i-}\rangle (l_i^{-1}(t)) &= \phi_i^{-1}(t) \\ \langle \xi_{i+}\rangle (l_i^{-1}(t)) &= \phi_+^{-1}(t) \end{cases} \tag{2.54}$$

and

$$\begin{cases} \langle \xi_{i-}\rangle (\mu_i^{-1}(t)) &= \phi_i^{-1}(\phi_+(\sigma_+(t))) \qquad (\ = \phi_i^{-1}(e(t))\ ) \\ \langle \xi_{i+}\rangle (\mu_i^{-1}(t)) &= \sigma_+(t) \qquad (\ \neq \phi_+^{-1}(e(t))\ ). \end{cases} \tag{2.55}$$

These properties are easily deduced by our way of construction of $\{\zeta_i(t)\}_{1\le i\le n}$, *cf.*[4]. The structure of the process $t \mapsto e(t)$ is well known: It is the inverse of the Dwass's extremal process (*cf.*[2]), in particular, for fixed $t > 0$, $e(t)$ has the exponential distribution with mean t.

Putting together (2.48), (2.49), (2.52), (2.54) and (2.55), and noting that $r_i(\phi_i^{-1}(t)) \equiv 0$ $(i = 1, \cdots, n, +)$ and $r_+(\sigma_+(t)) \equiv t$, we can express $\zeta_{i\pm}(l_i^{-1}(t))$ and $\zeta_{i\pm}(\mu_i^{-1}(t))$ as follows:

$$\begin{cases} \zeta_{i-}(l_i^{-1}(t)) &= t + \sqrt{-1}\, C_i(t) \\ \zeta_{i+}(l_i^{-1}(t)) &= -t + \sqrt{-1}\, C_+(t) \end{cases} \tag{2.56}$$

and

$$\begin{cases} \zeta_{i-}(\mu_i^{-1}(t)) &= e(t) + \sqrt{-1}\, C_i(e(t)) \\ \zeta_{i+}(\mu_i^{-1}(t)) &= t - e(t) + \sqrt{-1}\, \beta_+(\sigma_+(t)), \end{cases} \tag{2.57}$$

where

$$C_i(t) = \beta_i(\phi_i^{-1}(t)) \qquad (i = 1, \cdots, n, +).$$

Note that $C_1, \cdots, C_n, C_+$ are mutually independent Cauchy processes in (2.56). Note also that $C_1, \cdots, C_n, r_+, \beta_+$ are mutually independent in (2.57).

These processes appear as components of limit process of windings of $z(t)$: Theorem 2.4.2 implies that

$$\{ W_{i-}{}^{\lambda},\ W_{i+}{}^{\lambda} \}_{1\leq i\leq n} \longrightarrow \{ \zeta_{i-}(\mu_i^{-1}),\ \zeta_{i+}(\mu_i^{-1}) \}_{1\leq i\leq n}$$

as $\lambda \to \infty$ in the sense of K-convergence, where

$$W_{i\pm}{}^{\lambda}(t) = \frac{1}{\lambda}\int_0^{u(\lambda t)} \frac{1}{z_s - a_i} 1_{D(i\pm)}(z_s)\, dz_s \qquad (i = 1, \cdots, n).$$

Taking $D(i+) = D(i-)^c$, the process $\mathcal{I}m[W_{i-}{}^{\lambda}(t) + W_{i+}{}^{\lambda}(t)]$ is a normalized algebraic total angle wound by $z(t)$ around a_i up to the time $u(\lambda t) = e^{2\lambda t} - 1$. Then the imaginary parts of (2.57) clearly show that the primary description by Pitman and Yor([8]) of the asymptotic joint distribution of windings of z_t.

In addition, using above analysis, we give an another description of the joint limit process of windings of z_t below. Let $g(z)$ be a bounded function such that

$$\int_{\mathbf{C}} |g(z)||z|^{\varepsilon}\, m(dz) < \infty$$

for some $\varepsilon > 0$, where $m(dz)$ denotes the Lebesgue integral. Set

$$\bar{g} = \frac{1}{2\pi}\int_{\mathbf{C}} |g(z)|\, m(dz)$$

and

$$T^{\lambda}(t) = \frac{1}{\lambda}\int_0^{u(\lambda t)} g(z_s)\, ds.$$

Then, by Kasahara-Kotani's result (see [6]), we have

$$\{ \widehat{Z}_i^{\lambda},\ T^{\lambda}(\tau_i^{\lambda}) \}_{1\leq i\leq n} \longrightarrow \{ \zeta_i,\ 2\bar{g} l_i(\mu_i^{-1}) \}_{1\leq i\leq n} = \{ \zeta_i,\ 2\bar{g}\, e \}_{1\leq i\leq n} \tag{2.58}$$

as $\lambda \to \infty$ in the sense of K-convergence. Combining (2.58) and Theorem 2.4.1, we have

$$\{ W_{i-}{}^{\lambda}((T^{\lambda})^{-1}),\ W_{i+}{}^{\lambda}((T^{\lambda})^{-1}) \}_{1\leq i\leq n} \longrightarrow \{ \zeta_{i-}(l_i^{-1}(\cdot/(2\bar{g}))),\ \zeta_{i+}(l_i^{-1}(\cdot/(2\bar{g}))) \}_{1\leq i\leq n}$$

as $\lambda \to \infty$ in the sense of K-convergence if $g(z) > 0$. By (2.56), we can express this last limit process as

$$\begin{cases} \zeta_{i-}(l_i^{-1}(t/(2\bar{g}))) &= \quad t/(2\bar{g}) + \sqrt{-1}\, C_i(t/(2\bar{g})) \\ \\ \zeta_{i+}(l_i^{-1}(t/(2\bar{g}))) &= -t/(2\bar{g}) + \sqrt{-1}\, C_+(t/(2\bar{g})). \end{cases}$$

This is one of natural (symmetric) descriptions for the joint limit process of windings of $z(t)$ in the compact Riemannian surface $\mathbf{C} \cup \{\infty\}$.

Chapter 3

An Ergodic Theorem Related to Some Limit Theorems for Additive Functionals of Complex Brownian Motion

3.1 Introduction

Let $z(t) = x(t) + \sqrt{-1}\, y(t)$, $z(0) = 0$, be a complex Brownian motion starting at the origin. In the previous chapter, we proved an ergodic theorem for diffusion processes (X_t, Θ_t) on $\mathbf{R}^d \times M$ where M is a compact Riemannian manifold, and then using it and Kasahara-Kotani's method, obtained scaling limit processes for some class of "winding-type" additive functionals of $z(t)$, which is an extention of the results of F. Spitzer [11], J. Pitman and M. Yor [9], [10].

The aim of this chapter is to prove another ergodic theorem for $(X(t), \Theta(t))$ using a similar method in Chapter 1, and as its applications, to extend naturally the class of functionals for which the result of Kasahara-Kotani and that of Messulam-Yor hold.

For example, it is shown in [6] that if $f: \mathbf{C} \mapsto \mathbf{R}^1$ is a bounded Borel function such that

$$\int_{\mathbf{C}} |f(z)||z|^{\varepsilon}\, m(dz) < \infty \qquad \text{for some } \varepsilon > 0,$$

then

$$\lambda^{-1/2} \int_0^{u(\sqrt{\lambda}t)} f(z(s))\, ds$$

converges as $\lambda \to \infty$ in the sense of finite dimensional distributions, where $m(dz)$ denotes the Lebesgue measure and $u(t) = e^{2t} - 1$. In this chapter we will show the same convergence for $f \in L^1(\mathbf{C}) \cap L^p(\mathbf{C}) \quad (1 < p \leq \infty)$.

For another example, it is shown in [7] and [10] that if $h_1, \cdots, h_n, k_1, \cdots, k_n$ are bounded Borel functions from $\mathbf{C}$ to $\mathbf{R}^1$ such that $h_i, k_i \in L^2(\mathbf{C}) \quad (i = 1, \cdots, n)$, then

$$(\log t)^{-1/2} \int_0^t \{h_i(z(s))\, dx(s) + k_i(z(s))\, dy(s)\} \qquad (i = 1, \cdots, n)$$

converge jointly in distribution as $t \to \infty$. In this chapter we will show that if $f_1, \cdots, f_n$ are Borel functions from $\mathbf{C}$ to $\mathbf{C}$ such that $f_1, \cdots, f_n \in L^2(\mathbf{C}) \cap L^p(\mathbf{C}) \quad (2 < p \leq \infty)$, then

$$\lambda^{-1/4} \int_0^{u(\sqrt{\lambda} t)} f_i(z(s))\, dz(s) \qquad (i = 1, \cdots, n)$$

converge jointly as $\lambda \to \infty$ in the sense of finite dimensional distributions, where $u(t) = e^{2t} - 1$.

Before closing this section, we explain the contents of this chapter. In section 3.2 we consider a class of diffusion processes on $\mathbf{R}^d \times M$ where M is a compact Riemannian manifold and state our ergodic theorem for them. As corollaries to this theorem, we obtain some ergodic theorems for Brownian motion on $\mathbf{R}^1 \times [0, 2\pi]$. Then we give, as applications of these corollaries, some limit theorems for additive functionals of $z(t)$, *e.g.*, occupation times, square integrable martingale additive functionals and occupation times in the null charged case. In section 3.3 we prove the main theorem stated in section 3.2 by using the method of eigenfunction expansions.

3.2 An ergodic theorem for some class of diffusion processes on compact manifolds and its applications

Let M be an m-dimensional compact (connected) C^∞-Riemannian manifold without boundary and $(\Theta_t)_{t \geq 0}$ be the Brownian motion on M (see Ikeda and Watanabe [4], Chapter 5, section 4).

Let $(X_t)_{t \geq 0}$ be an $\mathbf{R}^d$-valued diffusion process determined by the stochastic differential equation

$$dX_t = \sigma(X_t)\, dB_t + b(X_t)\, dt,$$

where $\sigma(x)$ and $b(x)$ are bounded and smooth, $\sigma(x)$ is uniformly non-degenerate and $(B_t)_{t\geq 0}$ is a d-dimensional Brownian motion. We assume that X_t and Θ_t are independent, $X_0 = 0$ and $\Theta_0 = \theta_0$ ($\theta_0 \in M$). In the following, we consider L_p-spaces $L^p(\mathbf{R}^d)$, $L^p(M)$, $L^p(G)$ where $\mathbf{R}^d$ and M are endowed with Lebesgue measure dx and the Riemannian volume $d\theta$, respectively and $G := \mathbf{R}^d \times M$ is endowed with the product measure. The norms are denoted by $\| \quad \|_{p(R^d)}$, $\| \quad \|_{p(M)}$, $\| \quad \|_{p(G)}$, respectively, to distinguish the spaces. Then our main theorem is as follows:

Theorem 3.2.1 *Let $g(x,\theta)$ be a Borel measurable function from G to $\mathbf{R}^1$ satisfying the following conditions:*

(1) $g(x,\theta) \in L^r(G)$ *for some r with $1 \leq r < \infty$ and $r > \max\{d/2, m/2\}$,*

(2) $\|g(x,\cdot)\|_{r(M)} e^{-\beta|x|} \in L^p(\mathbf{R}^d)$ *for some p and $\beta \geq 0$ with $r \leq p \leq \infty$ and $p > d/(2-m/r)$,*

(3) *For almost all $x \in \mathbf{R}^d$, $g(x,\theta)$ is null charged on M i.e. $\int_M g(x,\theta)d\theta = 0$.*

Then for every $T > 0$ and any $N > 0$, it holds that

$$\sup_{0\leq\mu\leq N\lambda} \mu^{d/2r} E_{(0,\theta_0)}\left[\sup_{0\leq t\leq T}\left|\int_0^t g(X_{\mu s}, \Theta_{\lambda s})\, ds\right|\right] \longrightarrow 0$$

as $\lambda \longrightarrow \infty$.

Proof will be given in section 2.

Remark 3.2.1. The choice of β and p in Theorem 3.2.1 is not essential; The assumption (2) is a sufficient condition for the existence of the expectation $E_{(0,\theta_0)}\left[\int_0^1 g(X_{\mu s}, \Theta_{\lambda s})\, ds\right]$. In the case that $r \leq (d+m)/2$ the assumption (1) does not guarantee the existence of this expectation, while in the case that $r > (d+m)/2$ the condition (2) follows from (1) with $p = r$.

Corollary 1 *In the particular case of $d = 1$ and $M = \mathbf{R}^1/2\pi\mathbf{Z}$ being the 1-dimensional circle with radius 1, if $g(x,\theta) \in L^1(G)$, $g(x,\theta)e^{-\beta|x|} \in L^p(G)$ for some p and $\beta \geq 0$ with $1 < p \leq \infty$ and $g(x,\theta)$ is null charged on M in the sense that $\int_M g(x,\theta)\, d\theta = 0$ for a.a.$x \in \mathbf{R}^1$, then for every $T > 0$ and any $N > 0$, it holds that*

$$\sup_{0\leq\mu\leq N\lambda} \sqrt{\mu} E_{(0,\theta_0)}\left[\sup_{0\leq t\leq T}\left|\int_0^t g(X_{\mu s}, \Theta_{\lambda s})\, ds\right|\right] \longrightarrow 0$$

as $\lambda \longrightarrow \infty$.

Proof. Apply Theorem 3.2.1 with $r = d = m = 1$. (Note that $\|g(x,\cdot)\|_{1(M)}e^{-\beta|x|} \leq$ const. $\|g(x,\cdot)\|_{p(M)}e^{-\beta|x|} \in L^p(R^1)$.) Q.E.D.

Corollary 2 *Let $g(x,\theta)$ be a Borel measurable function from $G = \mathbf{R}^d \times M$ to $\mathbf{R}^1$ such that $g \in L^r(G)$ for some r with $(d+m)/2 < r < \infty$. If $g(x,\theta)$ is null charged on M for a.a. $x \in \mathbf{R}^d$, then for every $T > 0$ and $N > 0$, it holds that*

$$\sup_{0\leq\mu\leq N\lambda} \mu^{d/2r} E_{(0,\theta_0)}\left[\sup_{0\leq t\leq T}\left|\int_0^t g(X_{\mu s}, \Theta_{\lambda s})\, ds\right|\right] \longrightarrow 0$$

as $\lambda \to \infty$.

Proof. Apply Theorem 3.2.1 with $r = p$. Q.E.D.

Let $z(t) = x(t) + \sqrt{-1}\, y(t)$, $z(0) = 0$, be a complex Brownian motion starting at the origin. As applications of these ergodic theorems above, we give some limit theorems for additive functionals of $z(t)$ which have been discussed by many authors.

Application 1. (*Occupation time*). Let $f\colon \mathbf{C} \mapsto \mathbf{C}$ be a function such that

$$(3.1) \qquad f \in L^1(\mathbf{C}) \cap L^p(\mathbf{C}) \qquad \text{for some} \quad 1 < p \leq \infty.$$

Consider the following additive functional of $z(t)$:

$$A^\lambda(t) = \lambda^{-1}\int_0^{u(\sqrt{\lambda}t)} f(z(s))\, ds,$$

where $u(t) = e^{2t} - 1$. The study of the limit process of $A^\lambda(t)$ as $\lambda \to \infty$ can be reduced to that of a homogenization problem for a Brownian motion on the cylinder $G = \mathbf{R}^1 \times \mathbf{T}$, $\mathbf{T} = \mathbf{R}^1/2\pi\mathbf{Z} \simeq [0, 2\pi]$ as follows: This method is due to Kasahara and Kotani [6]. Fix $a \in \mathbf{C}\setminus\{0\}$ and set

$$Z(t) = X(t) + \sqrt{-1}\, Y(t) = \int_0^t \frac{1}{z(s) - a}\, dz(s),$$

$$\widehat{Z}(t) = \widehat{X}(t) + \sqrt{-1}\, \widehat{Y}(t) = Z(\langle Z\rangle^{-1}(t)),$$

$$\widehat{Z}^\lambda(t) = \widehat{X}^\lambda(t) + \sqrt{-1}\, \widehat{Y}^\lambda(t) = \lambda^{-1/2}\widehat{Z}(\lambda t)$$

and

$$\tau^\lambda(t) = \lambda^{-1/2} u^{-1}(\langle Z\rangle^{-1}(\lambda t)).$$

Note that $\widehat{Z}^\lambda(t)$ is a complex Brownian motion for every $\lambda > 0$ by the Knight theorem. (Generally, $\langle M\rangle(t)$ is the usual quadratic variation process of a conformal (local) martingale $M(t)$ and $h^{-1}(t)$ is the right continuous inverse function of a continuous increasing function $h(t)$.) Then, by the time substitution, we have that

$$\begin{aligned} A^\lambda(\tau^\lambda(t)) &= \lambda^{-1}\int_0^{\langle Z\rangle^{-1}(\lambda t)} f(a - ae^{Z(s)})\,ds \\ &= \lambda^{-1}\int_0^{\lambda t} f(a - ae^{\widehat{Z}(s)})|a|^2 e^{2\widehat{X}(s)}\,ds \\ &= \int_0^t g(\widehat{X}(\lambda s), \widehat{Y}(\lambda s))\,ds, \end{aligned}$$

where $g(x,\theta) = |a|^2 f(a - ae^{x+\sqrt{-1}\theta})e^{2x}$. Set

$$\bar{f} = \frac{1}{2\pi}\int_{\mathbf{C}} f(z)\,m(dz) = \frac{1}{2\pi}\int_G g(x,\theta)\,dx d\theta.$$

Here $m(dz)$ denotes the Lebesgue measure. We shall now prove that

$$\{\ \widehat{Z}^\lambda,\ \sqrt{\lambda}A^\lambda(\tau^\lambda)\ \} \longrightarrow \{\ \zeta,\ 2l(\cdot,\xi)\bar{f}\ \} \tag{3.2}$$

as $\lambda \to \infty$ in law on $C([0,\infty) \mapsto \mathbf{C}^2)$, where $\zeta(t) = \xi(t) + \sqrt{-1}\,\eta(t)$ is a complex Brownian motion and $l(t,\xi)$ is the local time at 0 of ξ. Indeed, since $g(x,\theta) \in L^1(G)$ and $g(x,\theta)e^{-2(1-1/p)|x|} \in L^p(G)$ by the assumption (3.1), setting $\bar{g}(x) = (1/2\pi)\int_0^{2\pi} g(x,\theta)\,d\theta$ ($\in L^1(\mathbf{R}^1)$), we have, using Corollary 1, that

$$\begin{aligned} &E\sup_{0\le t\le T}\left|\sqrt{\lambda}A^\lambda(\tau^\lambda(t)) - \sqrt{\lambda}\int_0^t \bar{g}(\sqrt{\lambda}\widehat{X}^\lambda(s))\,ds\right| \\ &= E\sup_{0\le t\le T}\left|\sqrt{\lambda}\int_0^t (g-\bar{g})(\widehat{X}(\lambda s),\widehat{Y}(\lambda s))\,ds\right| \longrightarrow 0 \end{aligned} \tag{3.3}$$

as $\lambda \to \infty$. Moreover it is easy to see that if $\bar{g}(x) \in L^1(\mathbf{R}^1)$ and $\xi(t)$ is a 1-dimensional Brownian motion, then

$$\sup_{0\le t\le T}\left|\sqrt{\lambda}\int_0^t \bar{g}(\sqrt{\lambda}\xi(s))\,ds - 2l(t,\xi)\int_{-\infty}^\infty \bar{g}(x)\,dx\right| \longrightarrow 0 \qquad \text{a.s.}$$

as $\lambda \to \infty$ (see [6]). Thus noting that $\bar{f} = \int_{-\infty}^\infty \bar{g}(x)\,dx$, we obtain (3.2).

On the other hand, the limit process of $\{\ \widehat{Z}^\lambda,\ \tau^\lambda\ \}$ as $\lambda \to \infty$ in law on $C([0,\infty) \mapsto \mathbf{C}\times\mathbf{R}^1)$ is given by $\{\ \zeta,\ \mu\ \}$ where $\mu(t) = \max_{0\le s\le t}\xi(s)$ ([6], Lemma 3.1). Therefore we can conclude that

$$(3.4)\qquad \sqrt{\lambda}A^{\lambda}(t) = \lambda^{-1/2}\int_0^{u(\sqrt{\lambda}t)} f(z(s))\,ds \longrightarrow 2l(\mu^{-1}(t),\xi)\bar{f}$$

as $\lambda \to \infty$ *in the sense of finite dimensional distributions.* (This follows from the fact that $\mu^{-1}(t)$ has no fixed discontinuous point. For example, *cf.* Chapter 2, Proposition 2.4.1.

Remark 3.2.2. In [6], Kasahara-Kotani showed (3.3) by proving that if $g \in L^1(G)\cap L^p(G)$ for some $1 < p \leq \infty$, then

$$\frac{1}{t}E\left|\int_0^t g(\widehat{X}(s),\widehat{Y}(s))\,ds\right|^2 \longrightarrow \left|\frac{1}{2\pi}\int_G g(x,\theta)\,dx d\theta\right|^2$$

as $t \to \infty$. Using this, they obtained the limit theorem (3.2) for bounded functions f satisfying $\int_{\mathbf{C}} |f(z)||z|^{\varepsilon}\, m(dz) < \infty$ for some $\varepsilon > 0$. Thus the result of Application 1 is an extension of their result to unbounded functions.

Application 2. (*Square integrable martingale additive functionals*). Let $f_i\colon \mathbf{C} \mapsto \mathbf{C}$, $i = 1,\cdots,n$, be functions such that

$$(3.5)\qquad f_i(z) \in L^2(\mathbf{C})\cap L^p(\mathbf{C}) \qquad \text{for some } 2 < p \leq \infty$$

for $i = 1,\cdots,n$. Consider the following additive functionals of $z(t)$:

$$A_i^{\lambda}(t) = \lambda^{-1/2}\int_0^{u(\sqrt{\lambda}t)} f_i(z(s))\,dz(s) \qquad (i = 1,\cdots,n),$$

where $u(t) = e^{2t}-1$. Here, it should be noted that these stochastic integrals can be defined because $f_i \in L^p(\mathbf{C})\quad (p > 2)$. Define $Z(t)$, $\widehat{Z}(t)$, $\widehat{Z}^{\lambda}(t)$ and $\tau^{\lambda}(t)$ as in Application 1. Then by the same transformation, we have that

$$A_i^{\lambda}(\tau^{\lambda}(t)) = \int_0^t g_i(\widehat{X}(\lambda s),\widehat{Y}(\lambda s))d\widehat{Z}^{\lambda}(s),$$

where $g_i(x,\theta) = -af_i(a - ae^{x+\sqrt{-1}\theta})e^{x+\sqrt{-1}\theta}$, $i = 1,\cdots,n$.

Let $\zeta(t) = \xi(t)+\sqrt{-1}\,\eta(t)$ be a complex Brownian motion, $l(t,\xi)$ be the local time at 0 of ξ and N be a complex Gaussian random measure on $[0,\infty)\times\mathbf{C}$ with mean 0 and variance measure $dt\cdot m(dz)/2\pi$ which is independent of ζ. (Generally, for measurable space $(S,\mathcal{B}(S),\mu)$ and $\mathcal{F} = \{A \in \mathcal{B}(S);\ \mu(A) < +\infty\}$, a family of random variables $M = \{M(A);\ A \in \mathcal{F}\}$ is called a Gaussian random measure on S with mean 0 and variance measure μ if M is a Gaussian system such that $E[M(A)] = 0$ and $E[M(A)M(B)] = \mu(A\cap B)$ for any $A,B \in \mathcal{F}$. Furthermore, a complex Gaussian random measure M on S with mean 0 and variance measure μ is by definition a family of complex random variables $M(A)$ which can be expressed in the form $M(A) = M_1(A) + \sqrt{-1}\,M_2(A)$ where M_1 and M_2 are mutually independent Gaussian random measures with mean 0 and variance measure μ.) Now we shall prove that

$$(3.6) \qquad \{ \widehat{Z}^{\lambda},\ \lambda^{1/4} A_i^{\lambda}(\tau^{\lambda}) \}_{1\le i\le n} \longrightarrow \left\{ \zeta,\ \int_0^{2l(\cdot,\xi)} \int_{\mathbf{C}} f_i(z)\, N(ds,dz) \right\}_{1\le i\le n}$$

as $\lambda \to \infty$ in law on $C([0,\infty) \mapsto \mathbf{C}^2)$. By the same arguments as in Application 1, we have from (3.6) that

$$(3.7) \qquad \{\lambda^{-1/4} \int_0^{u(\sqrt{\lambda} t)} f_i(z(s))\, dz(s)\}_{1\le i\le n} \longrightarrow \left\{ \int_0^{2l(\mu^{-1}(\cdot),\xi)} \int_{\mathbf{C}} f_i(z) N(ds,dz) \right\}_{1\le i\le n}$$

as $\lambda \to \infty$ in the sense of finite dimensional distributions, where $\mu(t) = \max_{0\le s\le t} \xi(s)$.

For the proof of (3.6), let $\{e_1,\cdots,e_r\}$ be an orthonormal system in $L^2(G)$ $(G = \mathbf{R}^1 \times [0,2\pi])$ obtained by Schmidt's method from $\{g_1,\cdots,g_n\}$. Since $g_i(x,\theta) \in L^2(G)$ and $g_i(x,\theta)e^{-(1-2/p)|x|} \in L^p(G)$ by the assumption (3.5), each e_k is so. Set

$$V_k^{\lambda}(t) = \lambda^{1/4} \int_0^t e_k(\widehat{X}(\lambda s), \widehat{Y}(\lambda s)) d\widehat{Z}^{\lambda}(s) \qquad (k = 1,\cdots,r).$$

Then

$$\begin{aligned} \left\langle V_j^{\lambda}, V_k^{\lambda} \right\rangle_t &= \lambda^{1/2} \int_0^t \mathcal{R}e(e_j e_k^*)(\widehat{X}(\lambda s), \widehat{Y}(\lambda s))\, ds \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \\ &\quad + \lambda^{1/2} \int_0^t \mathcal{I}m(e_j e_k^*)(\widehat{X}(\lambda s), \widehat{Y}(\lambda s))\, ds \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}. \end{aligned}$$

Here e_k^* represents the complex conjugate of e_k .
(Generally, if $z_1(t) = x_1(t) + \sqrt{-1}\, y_1(t)$ and $z_2(t) = x_2(t) + \sqrt{-1}\, y_2(t)$ are conformal martingales with the same filtration, then we denote by $\langle z_1, z_2 \rangle_t$ the matrix of quadratic variation processes $\begin{pmatrix} \langle x_1, x_2\rangle(t) & \langle x_1, y_2\rangle(t) \\ \langle y_1, x_2\rangle(t) & \langle y_1, y_2\rangle(t) \end{pmatrix}$.)
Hence just as in Application 1, we have by Corollary 1 that

$$(3.8) \qquad \sup_{0\le t\le T} \left| \left\langle V_j^{\lambda}, V_k^{\lambda} \right\rangle_t - 2\delta_{jk}\, l(t, \widehat{X}^{\lambda}) \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \right| \longrightarrow \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

in probability as $\lambda \to \infty$ for $j,k = 1,\cdots,r$.

Moreover, it holds that

$$(3.9) \qquad E \sup_{0\le t\le T} \left| \left\langle V_k^{\lambda}, \widehat{Z}^{\lambda} \right\rangle_t \right| = E \sup_{0\le t\le T} \left| \lambda^{1/4} \int_0^t e_k(\widehat{X}(\lambda s), \widehat{Y}(\lambda s))\, ds \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \right| \longrightarrow \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

as $\lambda \to \infty$ for $k = 1,\cdots,r$. Indeed, thanks to Corollary 2 (note that e_k is *not* in $L^1(G)$ but in $L^2(G)$), we have that

$$E \sup_{0\le t\le T} \left| \lambda^{1/4} \int_0^t e_k(\widehat{X}(\lambda s), \widehat{Y}(\lambda s))\, ds - \lambda^{1/4} \int_0^t \overline{e_k}(\widehat{X}(\lambda s))\, ds \right| \longrightarrow 0$$

as $\lambda \to \infty$ for $k = 1, \cdots, n$, where $\overline{e_k}(x) = (1/2\pi) \int_0^{2\pi} e_k(x, \theta)\, d\theta$. On the other hand, for any continuous function φ on $\mathbf{R}^1$ with compact support, we know that

$$E \sup_{0 \le t \le T} \left| \lambda^{1/4} \int_0^t \varphi(\widehat{X}(\lambda s))\, ds \right| \longrightarrow 0 \qquad \text{as } \lambda \to \infty.$$

Then noting that $\overline{e_k}(x) \in L^2(\mathbf{R}^1)$, by the inequality

$$\begin{aligned} E \sup_{0 \le t \le T} \left| \lambda^{1/4} \int_0^t \overline{e_k}(\widehat{X}(\lambda s))\, ds \right| &\le \lambda^{1/4} \int_0^T E |\overline{e_k}(\widehat{X}(\lambda s))|\, ds \\ \le \ \text{const.}\, \lambda^{1/4} \int_0^T (\lambda s)^{-1/4}\, ds\, \|\overline{e_k}\|_2 &= \text{const.}\, \|\overline{e_k}\|_2 \end{aligned}$$

which is obtained from Lemma 3.3.1 in section 2 below, we can conclude that

$$E \sup_{0 \le t \le T} \left| \lambda^{1/4} \int_0^t \overline{e_k}(\widehat{X}(\lambda s))\, ds \right| \longrightarrow 0 \qquad \text{as } \lambda \to \infty$$

for $k = 1, \cdots, n$ because continuous functions with compact support are dense in $L^2(\mathbf{R}^1)$. Thus (3.9) is proved.

From (3.8), (3.9) and the "asymptotic Knight's theorem" in Pitman-Yor [10], we obtain that

$$\{\, \widehat{Z}^\lambda,\ V_k^\lambda \,\}_{1 \le k \le r} \longrightarrow \left\{ \zeta,\ \int_0^{2l(\cdot,\xi)} \int_G \frac{e_k(x, \theta)}{-ae^{x+\sqrt{-1}\theta}}\, N(ds, dx d\theta) \right\}_{1 \le k \le r}$$

as $\lambda \to \infty$ in law. (Note that $|-ae^{x+\sqrt{-1}\theta}|^2\, dx d\theta = m(dz)$.)
This implies that

$$\{\, \widehat{Z}^\lambda,\ \lambda^{1/4} A_i^\lambda(\tau^\lambda) \,\}_{1 \le i \le n} \longrightarrow \left\{ \zeta,\ \int_0^{2l(\cdot,\xi)} \int_G \frac{g_i(x, \theta)}{-ae^{x+\sqrt{-1}\theta}}\, N(ds, dx d\theta) \right\}_{1 \le i \le n}.$$

as $\lambda \to \infty$ in law. Thus we arrive at the assertion (3.6).

Remark 3.2.3. Messulam and Yor [7] proved that if h and k are in $L^2(\mathbf{C} \mapsto \mathbf{R}^1)$ and bounded, then

$$M(t; h, k) = \left(\frac{2}{\log t} \right)^{1/2} \int_0^t \{ h(z(s))\, dx(s) + k(z(s))\, dy(s) \}$$

converges in distribution as $t \to \infty$. (See also [10], section 6.) This follows immediately from the result of Application 2. Indeed, the result (3.7) implies that if h and k are in $L^2(\mathbf{C} \mapsto \mathbf{R}^1)$ and in $L^p(\mathbf{C} \mapsto \mathbf{R}^1)$ for some $2 < p \le \infty$, then

$$\begin{aligned} &\lambda^{-1/4} \int_0^{e^{2\sqrt{\lambda}t}-1} \{ h(z(s))\, dx(s) + k(z(s))\, dy(s) \} \\ &\qquad \longrightarrow \int_0^{2l(\mu^{-1}(t),\xi)} \int_{\mathbf{C}} \{ h(z) N_1(ds, dz) + k(z) N_2(ds, dz) \} \end{aligned}$$

as $\lambda \to \infty$ in the sense of finite dimensional distributions, where $N_1 = \mathcal{R}e(N)$ and $N_2 = \mathcal{I}m(N)$. Taking $t = 1$ and $\lambda = ((1/2)\log T)^{1/2}$, we see that

$$M(T; h, k) \longrightarrow \int_0^{2l(\sigma,\xi)} \int_{\mathbf{C}} \{h(z)N_1(ds, dz) + k(z)N_2(ds, dz)\}$$

as $T \to \infty$ in distribution, where $\sigma = \mu^{-1}(1) = \inf\{u; \xi(u) = 1\}$.

Application 3. (*Occupation time; null charged case*). In [6], Kasahara-Kotani showed that if $f: \mathbf{C} \mapsto \mathbf{R}^1$ is a bounded function in $L^1(\mathbf{C})$ such that

$$\int_{\mathbf{C}} |f(z)||z|^{\varepsilon}\, m(dz) < \infty$$

for some $\varepsilon > 2$ and $\int_{\mathbf{C}} f(z)\, m(dz) = 0$, then

$$\lambda^{-1/4} \int_0^{u(\sqrt{\lambda} t)} f(z(s))\, ds \longrightarrow B(\langle f \rangle\, l(\mu^{-1}(t), \xi)))$$

as $\lambda \to \infty$ in the sense of finite dimensional distributions, where

$$u(t) = e^{2t} - 1,$$

$$\langle f \rangle = -(2/\pi^2) \int_{\mathbf{C}} \int_{\mathbf{C}} \log|z - z'|\, f(z) f(z')\, m(dz) m(dz'),$$

$B(t)$ is a 1-dimensional Brownian motion, $l(t, \xi)$ is the local time at 0 of a Brownian motion $\xi(t)$ which is independent of B and $\mu(t) = \max_{0 \le s \le t} \xi(s)$. This result is closely connected to that of Application 2. In fact, in a similar way as Application 2, we can extend this result to a function $f: \mathbf{C} \mapsto \mathbf{R}^1$ such that

$$f \in L^1(\mathbf{C}) \cap L^2(\mathbf{C}),$$

$$\int_{|z| \ge 1} |f(z)|\, \log|z|\, m(dz) < \infty,$$

$$\int_{\mathbf{C}} |f(z)|^2 |z|^2\, m(dz) < \infty$$

and

$$\int_{\mathbf{C}} f(z)\, m(dz) = 0.$$

We explain this briefly below.

By the same argument as in Application 1, it is sufficient to show that

$$\text{(3.10)} \qquad \left\{ \widehat{X}^{\lambda},\ \lambda^{3/4} \int_0^{\cdot} g(\widehat{X}(\lambda s), \widehat{Y}(\lambda s))\, ds \right\} \longrightarrow \{\, \xi,\ B(\langle f \rangle\, l(\cdot, \xi))\, \}$$

as $\lambda \to \infty$ in law on $C([0,\infty) \to \mathbf{R}^2)$, where $(\widehat{X}(t), \widehat{Y}(t))$ is a Brownian motion on $G = \mathbf{R}^1 \times [0, 2\pi]$, $\widehat{X}^\lambda(t) = \lambda^{-1/2}\widehat{X}(\lambda t)$ and

$$g(x,\theta) = |a|^2 f(a - ae^{x+\sqrt{-1}\theta})\, e^{2x} \qquad (a \in \mathbf{C} \setminus \{0\}).$$

Set

$$\|g\|_K = \int_{-\infty}^{\infty} |x| \left| \int_0^{2\pi} g(x,\theta)\, d\theta \right| dx,$$

$$\bar{\bar{g}} = \frac{1}{2\pi} \int_G g(x,\theta)\, dx d\theta,$$

$$\varphi_g(x) = \int_{-\infty}^{x} \int_0^{2\pi} g(u,\theta)\, du d\theta$$

and

$$\Phi_g(x) = \int_{-\infty}^{x} \varphi_g(y)\, dy.$$

From the assumptions for $f(z)$, we have that $g \in L^1(G) \cap L^2(G)$, $\|g\|_K < \infty$ and $\bar{\bar{g}} = 0$. Also it is easy to see that

$$|\varphi_g(x)| \le \|g\|_1$$

and

$$|x \cdot \varphi_g(x)| \le \|g\|_K.$$

Indeed, $\varphi_g(x) = -\int_x^\infty \int_0^{2\pi} g(u,\theta)\, du d\theta$ since $\bar{\bar{g}} = 0$, and hence

$$|x \cdot \varphi_g(x)| \le |x| \int_x^\infty \left| \int_0^{2\pi} g(u,\theta)\, d\theta \right| du \le \int_x^\infty |u| \left| \int_0^{2\pi} g(u,\theta)\, d\theta \right| du \le \|g\|_K.$$

Moreover, using these estimates, we have that

$$\begin{aligned} (3.11) \qquad \|\varphi_g\|_p &= \left(\int_{-1}^{1} |\varphi_g(x)|^p\, dx + \int_{|x|>1} |\varphi_g(x)|^p\, dx \right)^{1/p} \\ &\le \left(\|g\|_1{}^p \int_{-1}^{1} dx + \|g\|_K{}^p \int_{|x|>1} |x|^{-p}\, dx \right)^{1/p} \\ &\le \text{const.}\, \|g\|_1 + \text{const.}\, \|g\|_K \qquad (1 < p < \infty) \end{aligned}$$

and

$$\begin{aligned} (3.12) \qquad |\Phi_g(x)| &\le \left| [u\varphi_g(u)]_{-\infty}^{x} - \int_{-\infty}^{x} u \int_0^{2\pi} g(u,\theta)\, d\theta du \right| \\ &\le 2\|g\|_K + \int_{-\infty}^{x} |u| \left| \int_0^{2\pi} g(u,\theta)\, d\theta \right| du \\ &\le 3\|g\|_K. \end{aligned}$$

Define

$$\Gamma_0(x,\theta) = -\frac{1}{2\pi}\log|e^{\sqrt{-1}\theta} - e^{-|x|}|^2$$

and

$$\Gamma g(x,\theta) = -\frac{1}{\pi}\Phi_g(x) + (\Gamma_0 * g)(x,\theta).$$

Then it holds that

$$(3.13) \qquad \Gamma_0 \in L^p(G) \qquad (1 \le p < \infty),$$

$$(3.14) \qquad \Gamma_1 = \partial\Gamma_0/\partial x \in L^p(G), \qquad \Gamma_2 = \partial\Gamma_0/\partial\theta \in L^p(G) \qquad (0 < p < 2)$$

and $\Delta\Gamma g = -2g$ for any suffisiently smooth function g. (See [6].) We deduce from (3.12) and (3.13) that

$$(3.15) \qquad |\Gamma g| \le \text{const.}\, \|g\|_2 + \text{const.}\, \|g\|_K$$

and from (3.11) and (3.14) that

$$(3.16) \qquad \frac{\partial\Gamma g}{\partial x} \in L^p(G), \qquad \frac{\partial\Gamma g}{\partial\theta} \in L^p(G) \qquad (1 < p < \infty).$$

Then we can put

$$M(t) = \int_0^t \frac{\partial\Gamma g}{\partial x}(\widehat{X}(s),\hat{Y}(s))d\widehat{X}(s) + \int_0^t \frac{\partial\Gamma g}{\partial\theta}(\widehat{X}(s),\hat{Y}(s))d\hat{Y}(s).$$

(Generally, if $F \in L^p(G)$ for some p with $1 < p \le \infty$, then $E_0 \int_0^t F(\widehat{X}(s),\hat{Y}(s))ds \le$ const. $\|F\|_p < \infty$, *cf.* Lemma 3.3.1 and Lemma 3.3.2 in section 2 below.)

If g is a sufficiently smooth function, Itô's formula gives an identity

$$(3.17) \qquad \int_0^t g(\widehat{X}(s),\hat{Y}(s))\,ds = \Gamma g(0,0) - \Gamma g(\widehat{X}(t),\hat{Y}(t)) + M(t) \qquad \text{a.s.}$$

For a non-smooth g, by the method of approximation we see that (3.17) is still valid. Since Γg is bounded by (3.15), we can consider $M^\lambda(t) = \lambda^{-1/4}M(\lambda t)$ in place of

$$\lambda^{3/4}\int_0^t g(\widehat{X}(\lambda s),\hat{Y}(\lambda s))\,ds.$$

The fact (3.16) allows us to apply Corollary 1 for $\left\langle M^\lambda\right\rangle_t = \lambda^{1/2}\int_0^t h(\widehat{X}(\lambda s),\hat{Y}(\lambda s))\,ds$ where $h(x,\theta) = (\partial\Gamma g/\partial x)^2 + (\partial\Gamma g/\partial\theta)^2$, and Corollary 2 for

$$\left\langle M^\lambda, \widehat{X}^\lambda\right\rangle_t = \lambda^{1/4}\int_0^t \frac{\partial\Gamma g}{\partial x}(\widehat{X}(\lambda s),\hat{Y}(\lambda s))\,ds$$

and

$$\left\langle M^\lambda, \hat{Y}^\lambda \right\rangle_t = \lambda^{1/4} \int_0^t \frac{\partial \Gamma g}{\partial \theta}(\widehat{X}(\lambda s), \hat{Y}(\lambda s))\, ds.$$

Then, as in Application 2, we arrive at the conclusion that

$$\{\, \widehat{X}^\lambda,\ \hat{Y}^\lambda,\ M^\lambda \,\} \longrightarrow \{\, \xi,\ \eta,\ B(2l(\cdot, \xi)\bar{\bar{h}}) \,\}$$

as $\lambda \to \infty$ in law, where $(\xi(t), \eta(t), B(t))$ is a 3-dimensional Brownian motion, $l(t, \xi)$ is the local time at 0 of ξ and $\bar{\bar{h}} = \frac{1}{2\pi} \int_G h(x, \theta)\, dx d\theta$. The assertion (1.10) now follows if we notice that $\langle f \rangle = 2\bar{\bar{h}}$.

The equality $\langle f \rangle = 2\bar{\bar{h}}$ can be seen as follows. If g is a sufficiently smooth function, noting that $\Delta \Gamma g = -2g$, we have

$$(3.18) \qquad 2\bar{\bar{h}} = \frac{1}{\pi} \int_G \left[\left(\frac{\partial \Gamma g}{\partial x} \right)^2 + \left(\frac{\partial \Gamma g}{\partial \theta} \right)^2 \right] dx d\theta = \frac{2}{\pi} \int_G g(x, \theta) \Gamma g(x, \theta)\, dx d\theta$$

by the integration by parts. For non-smooth g, by the method of approximation we see that (3.18) is still valid. On the other hand, using the fact that $\bar{\bar{g}} = 0$ and $\|g\|_K < \infty$, we have

$$\Phi_g(x) = [y \varphi_g(y)]_{-\infty}^x - 2\pi \int_{-\infty}^x y \bar{g}(y)\, dy = \text{const.} + 2\pi \int_{-\infty}^x (x - y) \bar{g}(y)\, dy,$$

where $\bar{g}(y) = \frac{1}{2\pi} \int_0^{2\pi} g(y, \theta)\, d\theta$. Also we have

$$\begin{aligned}
\Gamma_0 * g(x, \theta) &= \int_G \Gamma_0(x - y, \theta - \psi) g(y, \psi)\, dy d\psi \\
&= -\frac{1}{\pi} \int_x^\infty \int_0^{2\pi} \log |e^{\sqrt{-1}(\theta - \psi)} - e^{x-y}| \cdot g(y, \psi)\, dy d\psi \\
&\quad -\frac{1}{\pi} \int_{-\infty}^x \int_0^{2\pi} \log |e^{\sqrt{-1}(\theta - \psi)} - e^{y-x}| \cdot g(y, \psi)\, dy d\psi \\
&= -\frac{1}{\pi} \int_x^\infty \int_0^{2\pi} \left(\log |e^{y - \sqrt{-1}\psi} - e^{x - \sqrt{-1}\theta}| - y \right) g(y, \psi)\, dy d\psi \\
&\quad -\frac{1}{\pi} \int_{-\infty}^x \int_0^{2\pi} \left(\log |e^{x + \sqrt{-1}\theta} - e^{y + \sqrt{-1}\psi}| - x \right) g(y, \psi)\, dy d\psi \\
&= -\frac{1}{\pi} \int_x^\infty \int_0^{2\pi} \log |e^{x + \sqrt{-1}\theta} - e^{y + \sqrt{-1}\psi}| \cdot g(y, \psi)\, dy d\psi + 2 \int_x^\infty y \bar{g}(y)\, dy \\
&\quad -\frac{1}{\pi} \int_{-\infty}^x \int_0^{2\pi} \log |e^{x + \sqrt{-1}\theta} - e^{y + \sqrt{-1}\psi}| \cdot g(y, \psi)\, dy d\psi + 2x \int_{-\infty}^x \bar{g}(y)\, dy \\
&= -\frac{1}{\pi} \int_G \log |e^{x + \sqrt{-1}\theta} - e^{y + \sqrt{-1}\psi}| \cdot g(y, \psi)\, dy d\psi + 2 \int_{-\infty}^x (x - y) \bar{g}(y) dy + \text{const.}.
\end{aligned}$$

Then recalling the definition of $\Gamma g(x, \theta)$ we have

$$(3.19)\qquad \Gamma g(x,\theta) = -\frac{1}{\pi}\int_G \log|e^{x+\sqrt{-1}\theta} - e^{y+\sqrt{-1}\psi}|\cdot g(y,\psi)\,dyd\psi + \text{const.}$$

Combining (3.18) and (3.19) and using $\bar{\bar{g}} = 0$ again, we obtain

$$2\bar{\bar{h}} = -\frac{2}{\pi^2}\int_G\int_G \log|e^{x+\sqrt{-1}\theta} - e^{y+\sqrt{-1}\psi}|\cdot g(x,\theta)g(y,\psi)\,dxdyd\theta d\psi = \langle f\rangle .$$

Remark 3.2.4. In Theorem 2.1 of [6] , it is proved that if $g \in L^1(G)\cap L^p(G)$ for some p with $2 < p \leq \infty$, $\|g\|_K < +\infty$ and $\bar{\bar{g}} = 0$, then (3.10) holds. The above proof shows that we can refine Theorem 2.1 of [6] by weakening the condition $g \in L^1(G)\cap L^p(G)\quad (2 < p \leq \infty)$ to the condition $g \in L^1(G)\cap L^p(G)\quad (2 \leq p \leq \infty)$. (It should be noted that if $p > 2$, then $(\partial\Gamma g/\partial x)$ and $(\partial\Gamma g/\partial\theta)$ are bounded, while in case that $p = 2$, we have only (3.16).)

3.3 The proof of Theorem 3.2.1

For the proof of Theorem 3.2.1, first note that the generator of Θ_t is $(1/2)\Delta_M$, where Δ_M is the Laplace-Beltrami operator for M. Since M is compact, Δ_M has pure point spectrum

$$0 = \lambda_0 > -\lambda_1 \geq -\lambda_2 \geq \cdots$$

and we denote the corresponding normalized eigenfunctions by $\{\varphi_n\}$. It is known that the transition density $q(t,\theta,\eta)$ of Θ_t has the following expansion:

$$(3.20)\qquad q(t,\theta,\eta) = \sum_{n=0}^{\infty} e^{-\lambda_n t}\varphi_n(\theta)\varphi_n(\eta),$$

which converges uniformly in (θ,η) for every $t > 0$ (see Chavel [1], p.140).

Before proving our theorem, we prepare some estimates for expectations of the functionals of X_t and Θ_t.

Lemma 3.3.1 *Suppose that $h\colon \mathbf{R}^d \mapsto \mathbf{R}^1$ satisfies the condition:*

$$h(x)e^{-\beta|x|} \in L^p(\mathbf{R}^d) \qquad \textit{for some } 1 \leq p \leq \infty \textit{ and } \beta \geq 0\ .$$

Then for every $x \in \mathbf{R}^d$ and $t > 0$,

$$E_x|h(X_t)| \leq \text{const.}\, t^{-d/2p}\, e^{\text{const.}\,\beta^2 t + \beta|x|}\,\|H\|_p,$$

where $H(x) = h(x)e^{-\beta|x|}$.

Proof. From the assumption for X_t, we have the following estimate for the transition density $p(t,x,y)$ of X_t :

$$(3.21) \qquad p(t,x,y) \le \text{const.}\, t^{-d/2} \exp\left(-\frac{\text{const.}\,|x-y|^2}{2t}\right).$$

(See Friedman [3], p.141, Theorem 4.5.)

Then, by the assumption for $h(x)$ and Hölder's inequality,

$$\begin{aligned}
E_x|h(X_t)| &\le \text{const.}\, t^{-d/2} \int_{\mathbf{R}^d} \exp\left(-\frac{\text{const.}\,|x-y|^2}{2t}\right) |H(y)|e^{\beta|y|}\, dy \\
&\le \text{const.} \int_{\mathbf{R}^d} \exp\left(-\frac{\text{const.}\,|\xi|^2}{2} + \beta|\sqrt{t}\xi| + \beta|x|\right) |H(\sqrt{t}\xi + x)|\, d\xi \\
&\le \text{const.}\, e^{\beta|x|} \left(\int_{\mathbf{R}^d} \exp\left(-\frac{\text{const.}\,|\xi|^2 - 2\beta\sqrt{t}|\xi|}{2} q\right) d\xi\right)^{1/q} \\
&\qquad \times \left(\int_{\mathbf{R}^d} |H(\sqrt{t}\xi + x)|^p\, d\xi\right)^{1/p},
\end{aligned}$$

where $1/p + 1/q = 1$.

The integral in the second factor of the last expression is bounded by

$$\begin{aligned}
&\text{const.} \int_0^\infty |\xi|^{d-1} \exp\left(-\frac{\text{const.}\,|\xi|^2 - 2\beta\sqrt{t}|\xi|}{2} q\right) d|\xi| \\
&\le \text{const.}\, e^{\text{const.}\,\beta^2 t} \int_0^\infty |\xi|^{d-1} \exp\left(-\frac{\text{const.}\,(|\xi| - \text{const.}\,\beta\sqrt{t})^2}{4} q\right) d|\xi| \\
&\le \text{const.}\, e^{\text{const.}\,\beta^2 t},
\end{aligned}$$

and the third factor is equal to $t^{-d/2p}\|H\|_p$. Q.E.D.

Lemma 3.3.2 *Suppose that $f \in L^r(M, d\theta)$ with some $1 \le r \le \infty$. Then for every $\delta > 0$ there exist positive constants C and C_δ such that*

$$E_\theta|f(\Theta_t)| \le Ct^{-m/2r}\, 1_{(t\le\delta)}\|f\|_r + C_\delta\, 1_{(t>\delta)}\|f\|_1$$

for every $\theta \in M$ and $t > 0$.

Proof. If $\delta > 0$, we have from (3.20) that

$$\begin{aligned} q(t,\theta,\eta)\,1_{(t>\delta)} &\le \left(\sum_{n=0}^{\infty} e^{-\lambda_n t}\varphi_n(\theta)^2\right)^{1/2}\left(\sum_{n=0}^{\infty} e^{-\lambda_n t}\varphi_n(\eta)^2\right)^{1/2} 1_{(t>\delta)} \\ &\le \left(\sum_{n=0}^{\infty} e^{-\lambda_n \delta}\varphi_n(\theta)^2\right)^{1/2}\left(\sum_{n=0}^{\infty} e^{-\lambda_n \delta}\varphi_n(\eta)^2\right)^{1/2} \\ &= q(\delta,\theta,\theta)^{1/2}q(\delta,\eta,\eta)^{1/2} \le \sup_{\theta} q(\delta,\theta,\theta) \equiv C_\delta. \end{aligned}$$

On the other hand, the uniform estimate

$$q(t,\theta,\eta) \le \text{const.}\, t^{-m/2} \qquad (t \downarrow 0)$$

holds. (see Chavel [1], $p.154 \sim 155$). Then noting that $\int_M q(t,\theta,\eta)\,d\eta = 1$ and $f \in L^r(M) \subset L^1(M)$, we have that

$$\begin{aligned} E_\theta|f(\Theta_t)| &= \int_M q(t,\theta,\eta)|f(\eta)|\,d\eta \\ &= \int_M q(t,\theta,\eta)|f(\eta)|\,d\eta\, 1_{(t\le\delta)} + \int_M q(t,\theta,\eta)|f(\eta)|\,d\eta\, 1_{(t>\delta)} \\ &\le \left(\int_M q(t,\theta,\eta)^q\,d\eta\right)^{1/q} \|f\|_r\, 1_{(t\le\delta)} + C_\delta\|f\|_1\, 1_{(t>\delta)} \\ &\le \left(\int_M (\text{const.}\, t^{-m(q-1)/2} q(t,\theta,\eta)\,d\eta\right)^{1/q} \|f\|_r\, 1_{(t\le\delta)} + C_\delta\|f\|_1\, 1_{(t>\delta)} \\ &= \text{const.}\, t^{-m/2r}\|f\|_r\, 1_{(t\le\delta)} + C_\delta\|f\|_1\, 1_{(t>\delta)} \end{aligned}$$

by Hölder's inequality, where $1/r + 1/q = 1$. (In the case that $r = 1$, the above estimate is still valid by replacing $\left(\int_M q(t,\theta,\eta)^q\,d\eta\right)^{1/q}$ with $\sup_\eta q(t,\theta,\eta)$.) Q.E.D.

Now we are ready to prove Theorem 3.2.1.

Proof of Theorem 3.2.1. We shall first prove the special case that $g(x,\theta)$ is of the form $h(x)\varphi_n(\theta)$ and reduce the general case to this special case by approximations.

1°) *The case that* $g(x,\theta) = h(x)\varphi_n(\theta)$ *for some* $n \ge 1$ *where* $h(x) \in L^r(\mathbf{R}^d)$ *for some* r *with* $1 \le r \le \infty$ *and* $r > d/2$.

In this case, as seen by the proof below, the conditions $r < \infty$, $r > m/2$ and (2) are not necessary. Moreover, the convergence is uniform in μ.

From now on we write the expectation $E_{(0,\theta_0)}$ simply by E. Set

$$u_{\mu\lambda}(x,\theta) = \mu^{d/2r} \int_0^\infty E_{(x,\theta)} \left[h(X_{\mu s})\varphi_n(\Theta_{\lambda s})\right] ds$$

and

$$M_t^{\mu\lambda} = u_{\mu\lambda}(X_{\mu t}, \Theta_{\lambda t}) + \mu^{d/2r} \int_0^t h(X_{\mu s})\varphi_n(\Theta_{\lambda s})\, ds.$$

In order to show that

$$\mu^{d/2r} E \sup_{0\le t\le T} \left| \int_0^t h(X_{\mu s})\varphi_n(\Theta_{\lambda s})\, ds \right| \longrightarrow 0 \tag{3.22}$$

as $\lambda \to \infty$ uniformly in μ, it is clearly sufficient to prove that

$$E \sup_{0\le t\le T} |u_{\mu\lambda}(X_{\mu t}, \Theta_{\lambda t})| \longrightarrow 0 \tag{3.23}$$

and

$$E \sup_{0\le t\le T} |M_t^{\mu\lambda}| \longrightarrow 0 \tag{3.24}$$

as $\lambda \to \infty$ uniformly in μ.

The convergence (3.23) is proved as follows. By the orthonormality of $\{\varphi_k\}$ and (3.20), it holds that

$$E_\theta\left[\varphi_n(\Theta_{\lambda s})\right] = \int_M q(\lambda s, \theta, \eta)\varphi_n(\eta)\, d\eta = e^{-\lambda_n \lambda s}\varphi_n(\theta) \qquad \text{for every } \theta \in M. \tag{3.25}$$

By Lemma 3.3.1 and (3.25), we obtain the estimate for $u_{\mu\lambda}$:

$$\begin{aligned} |u_{\mu\lambda}(x,\theta)| &\le \mu^{d/2r} \int_0^\infty E_x |h(X_{\mu s})|\, |E_\theta[\varphi_n(\Theta_{\lambda s})]|\, ds \\ &\le \text{const.}\, \mu^{d/2r} \int_0^\infty (\mu s)^{-d/2r} e^{-\lambda_n \lambda s}\, ds\, |\varphi_n(\theta)| \\ &= \text{const.}\, \lambda^{d/2r-1} |\varphi_n(\theta)|. \end{aligned} \tag{3.26}$$

Hence

$$E \sup_{0\le t\le T} |u_{\mu\lambda}(X_{\mu t}, \Theta_{\lambda t})| \le \text{const.}\, \lambda^{d/2r-1} E \sup_{0\le t\le T} |\varphi_n(\Theta_{\lambda t})|.$$

Because of the boundedness of φ_n (note that φ_n is continuous and M is compact), it holds that $E \sup_{0\le t\le T} |\varphi_n(\Theta_{\lambda t})| \le \text{const.}$.

Therefore,

$$E \sup_{0\le t\le T} |u_{\mu\lambda}(X_{\mu t}, \Theta_{\lambda t})| \le \text{const.}\, \lambda^{d/2r-1} \longrightarrow 0 \tag{3.27}$$

as $\lambda \longrightarrow \infty$ uniformly in μ.

We now prove (3.24). Fixing λ, μ and setting $\mathcal{F}_t = \sigma\{(X_{\mu s}, \Theta_{\lambda s});\ s \leq t\}$, we can see that $M_t^{\mu\lambda}$ is an (F_t)-martingale by a repeated use of Fubini's theorem. (Note by Lemma 3.3.1 and (3.25) that

$$
\begin{aligned}
& E\left[\int_0^\infty \left|E_{(X_{\mu t},\Theta_{\lambda t})}(h(X_{\mu u})\varphi_n(\Theta_{\lambda u}))\right| du\right] \\
&= E\left[\int_0^\infty \left|E_{X_{\mu t}}(h(X_{\mu u}))\right| \left|E_{\Theta_{\lambda t}}(\varphi_n(\Theta_{\lambda u}))\right| du\right] < +\infty. \quad)
\end{aligned}
$$

Then we have that

$$
\begin{aligned}
& E \sup_{0\leq t\leq T} |M_t^{\mu\lambda}| \leq (E \sup_{0\leq t\leq T} |M_t^{\mu\lambda}|^2)^{1/2} \leq \text{const.}\,(E|M_T^{\mu\lambda}|^2)^{1/2} \\
&\leq \left(\text{const.}\, E|u_{\mu\lambda}(X_{\mu T}, \Theta_{\lambda T})|^2 + \text{const.}\, E\left|\mu^{d/2r}\int_0^T h(X_{\mu s})\varphi_n(\Theta_{\lambda s})\,ds\right|^2\right)^{1/2}
\end{aligned}
$$

by the martingale inequality. Hence it is sufficient to show that

$$
(3.28) \qquad I_1 = E|u_{\mu\lambda}(X_{\mu T}, \Theta_{\lambda T})|^2 \longrightarrow 0
$$

and

$$
(3.29) \qquad I_2 = \mu^{d/r} E\left(\int_0^T h(X_{\mu s})\varphi_n(\Theta_{\lambda s})\,ds\right)^2 \longrightarrow 0
$$

as $\lambda \longrightarrow \infty$ uniformly in μ. We can easily deduce (3.28) from (3.26) in a similar way as the proof of (3.23). (3.29) can be proved as follows. By Lemma 3.3.1, (3.25) and Fubini's theorem, we have that

$$
\begin{aligned}
I_2 &= 2\mu^{d/r} E\left[\int_0^T ds \int_0^s du\, h(X_{\mu s})h(X_{\mu u})\varphi_n(\Theta_{\lambda s})\varphi_n(\Theta_{\lambda u})\right] \\
&= 2\mu^{d/r}\int_0^T ds \int_0^s du\, E\left[h(X_{\mu u})E_{X_{\mu u}}[h(X_{\mu(s-u)})]\right] \\
&\qquad \times E\left[\varphi_n(\Theta_{\lambda u})E_{\Theta_{\lambda u}}[\varphi_n(\Theta_{\lambda(s-u)})]\right].
\end{aligned}
$$

Lemma 3.3.1 implies that

$$
\begin{aligned}
& \left|E\left[h(X_{\mu u})E_{X_{\mu u}}[h(X_{\mu(s-u)})]\right]\right| \leq E\left[|h(X_{\mu u})| \cdot E_{X_{\mu u}}\left|h(X_{\mu(s-u)})\right|\right] \\
&\leq \text{const.}\left(\mu(s-u)\right)^{-d/2r} \cdot (\mu u)^{-d/2r}
\end{aligned}
$$

and (3.25) implies that

$$(3.30) \quad \left|E\left[\varphi_n(\Theta_{\lambda u})E_{\Theta_{\lambda u}}\left(\varphi_n(\Theta_{\lambda(s-u)})\right)\right]\right| \le e^{-\lambda_n\lambda(s-u)}E|\varphi_n(\Theta_{\lambda u})|^2$$
$$\le \text{const.}\, e^{-\lambda_n\lambda(s-u)}.$$

Hence

$$I_2 \le \text{const.} \int_0^T ds \int_0^s du\, e^{-\lambda_n\lambda(s-u)}(s-u)^{-d/2r}u^{-d/2r}$$
$$\le \text{const.}\, \lambda^{d/2r-1} + \text{const.}\, \lambda^{d/2r-1} \longrightarrow 0$$

as $\lambda \to \infty$ uniformly in μ. This completes the proof of (3.22).

2°) *General case.*

Since X_t and Θ_t are mutually independent, by Lemma 3.3.1 and Lemma 3.3.2, we have for fixed $\delta > 0$, that

$$E|g(X_{\mu t},\Theta_t)|\, 1_{(t>\delta)} \le \text{const.}\, E\|g(X_{\mu t},\cdot)\|_{1(M)}\, 1_{(t>\delta)} \le \text{const.}\, E\|g(X_{\mu t},\cdot)\|_{r(M)}$$
$$\le \text{const.}\, (\mu t)^{-d/2r}\|g\|_{r(G)}.$$

Similarly, using the assumption (2), we have by Lemma 3.3.1 and Lemma 3.3.2 that

$$E|g(X_{\mu t},\Theta_t)|\, 1_{(t\le\delta)} \le \text{const.}\, t^{-m/2r}E\|g(X_{\mu t},\cdot)\|_{r(M)}\, 1_{(t\le\delta)}$$
$$\le \text{const.}\, t^{-m/2r}(\mu t)^{-d/2p}e^{\text{const.}\,\mu t}\left\|\|g(x,\cdot)\|_{r(M)}e^{-\beta|x|}\right\|_{p(\mathbf{R}^d)}$$
$$= \text{const.}\, \mu^{-d/2p}t^{-m/2r-d/2p}e^{\text{const.}\,\mu t}.$$

Putting these estimates together, we obtain that

$$E\sup_{0\le t\le T}\left|\int_0^t g(X_{\mu s},\Theta_{\lambda s})\,ds\right| \le \int_0^T E|g(X_{\mu s},\Theta_{\lambda s})|\,ds = \frac{1}{\lambda}\int_0^{\lambda T} E|g(X_{\mu s/\lambda},\Theta_s)|\,ds$$
$$= \frac{1}{\lambda}\int_0^{\delta} E|g(X_{\mu s/\lambda},\Theta_s)|\,ds + \frac{1}{\lambda}\int_\delta^{\lambda T} E|g(X_{\mu s/\lambda},\Theta_s)|\,ds$$
$$\le \text{const.}\, \frac{1}{\lambda}\int_0^\delta \left(\frac{\mu}{\lambda}\right)^{-d/2p} s^{-m/2r-d/2p}e^{\text{const.}\,\mu s/\lambda}\,ds$$
$$+ \text{const.}\, \frac{1}{\lambda}\int_\delta^{\lambda T}\left(\frac{\mu}{\lambda}s\right)^{-d/2r}ds\,\|g\|_{r(G)}$$
$$\le \text{const.}\, \mu^{-d/2p}\lambda^{d/2p-1}e^{\text{const.}\,\mu/\lambda} + \mu^{-d/2r}\left(\text{const.} + \text{const.}\, \lambda^{d/2r-1}\right)\|g\|_{r(G)}.$$

Here for the existence of the above integrals by s, the conditions $r > d/2$, $r > m/2$ and $p > d/(2 - m/r)$ are needed.

Therefore, using the condition that $p \geq r$, we have

$$(3.31) \qquad \sup_{0 \leq \mu \leq N\lambda} \mu^{d/2r} E \sup_{0 \leq t \leq T} \left| \int_0^t g(X_{\mu s}, \Theta_{\lambda s})\, ds \right|$$

$$\leq \text{const.}\, N^{d/2r - d/2p} e^{\text{const.}\, N} \lambda^{d/2r-1} + \left(\text{const.} + \text{const.}\, \lambda^{d/2r-1}\right) \|g\|_{r(G)}$$

$$= o(1) + (\text{const.} + o(1)) \|g\|_{r(G)} \qquad (\lambda \to \infty).$$

Now define

$$\mathcal{L} := \{f;\ f \text{ is a finite linear combination of } \varphi_1, \varphi_2, \cdots\}$$

$$A := \{hf;\ h \in L^r(\mathbf{R}^d),\ h(x)e^{-\beta|x|} \in L^p(\mathbf{R}^d) \text{ and } f \in \mathcal{L}\}$$

$$A' := \{g;\ g \text{ is a finite linear combination of functions of } A\}$$

$$B := \{hf;\ h \in L^r(\mathbf{R}^d),\ h(x)e^{-\beta|x|} \in L^p(\mathbf{R}^d),\ f \in L^r(M) \ s.t.\ \textstyle\int_M f(\theta)d\theta = 0\}$$

$$B' := \{g;\ g \text{ is a finite linear combination of functions of } B\}$$

$$S := \{g;\ g \text{ is a Borel measurable function from } G \text{ to } \mathbf{R}^1 \ s.t.\ g \in L^r(G),$$
$$\|g(x,\cdot)\|_{r(M)} e^{-\beta|x|} \in L^p(\mathbf{R}^d) \text{ and } \textstyle\int_M g(x,\theta)d\theta = 0 \text{ for a.a. } x \in \mathbf{R}^d\}\ .$$

Every $f \in L^r(M)$ satisfying $\int_M f(\theta)d\theta = 0$ can be $\| \quad \|_{r(M)}$-approximated by $f' \in \mathcal{L}$, because any continuous function ψ on M satisfying $\int_M \psi(\theta)d\theta = 0$ is uniformly approximated by functions of $\mathcal{L}$ since M is compact (*cf.* Chavel [1], p.139-140), and continuous functions are dense in $L^r(M)$ $(1 \leq r < \infty)$. Hence every $hf \in B$ can be
$\| \quad \|_{r(G)}$-approximated by $hf' \in A$. On the other hand, every $g \in S$ can be $\| \quad \|_{r(G)}$-approximated by $g' \in B'$. Therefore, every $g \in S$ can be $\| \quad \|_{r(G)}$-approximated by $g' \in A'$.

Consequently, noting that by (3.22) Theorem 3.2.1 holds for every $hf \in A$ (and so for every $g' \in A'$), we complete the proof by an approximation argument using (3.31). Q.E.D.

Chapter 4

On Some Ergodic Theorems for a Brownian Motion on a Compact Manifold and an Ornstein-Uhlenbeck Process

4.1 Introduction

Let M be an m-dimensional compact (connected) C^∞-Riemannian manifold without boundary. We consider a diffusion process (X_t, Θ_t) on $G = \mathbf{R}^d \times M$, where $(\Theta_t)_{t\geq 0}$ is the Brownian motion on M (*i.e.*, the generator of Θ_t is $\frac{1}{2}\Delta_M$, where Δ_M is the Laplace-Beltrami operator for M, see Ikeda and Watanabe [4], chapter 5, section 4) and $(X_t)_{t\geq 0}$ is an $\mathbf{R}^d$-valued diffusion process with the generator L;

$$Lf(x) = \frac{1}{2}\sum_{i,j=1}^{d} a^{ij}(x)\frac{\partial^2 f}{\partial x^i \partial x^j}(x) + \sum_{i=1}^{d} b^i(x)\frac{\partial f}{\partial x^i}(x),$$

where $a^{ij}(x) = \sum_{k=1}^{d} \sigma_k^i(x)\sigma_k^j(x)$ for $i, j = 1, \cdots, d$. We assume that $\sigma(x) = (\sigma_k^i(x)) \in \mathbf{R}^d \times \mathbf{R}^d$ and $b(x) = (b^i(x)) \in \mathbf{R}^d$ are bounded and smooth and $\sigma(x)$ is uniformly non-degenerate. Moreover we assume that X_t and Θ_t are independent and $X_0 = 0$ and $\Theta_0 = \theta_0$ ($\theta_0 \in M$). In the following, we consider L_p-spaces $L^p(\mathbf{R}^d)$, $L^p(M)$, $L^p(G)$ where $\mathbf{R}^d$ and M are endowed with Lebesgue measure dx and the Riemannian volume $d\theta$, respectively and G is endowed with the product measure $dxd\theta$. The norms are denoted by $\| \ \|_{p(\mathbf{R}^d)}$, $\| \ \|_{p(M)}$, $\| \ \|_{p(G)}$, respectively, to distinguish the spaces.

In Chapter 2 we proved, using the method of eigenfunction expantions, the following ergodic theorem to obtain some limit theorems for additive functionals of 2-dimensional Brownian motion.

Theorem 4.1.1 *Let $F(x,\theta)$ be a Borel measurable function from G to $\mathbf{R}^1$ satisfying the following conditions:*

(A1) *For almost all $x \in \mathbf{R}^d$, $F(x,\theta)$ is null charged on M; i.e., $\int_M F(x,\theta)d\theta = 0$,*

(A2) *$F(x,\theta) \in L^r(G)$ for some r with $1 \leq r < \infty$ and $r > \max\{d/2, m/2\}$,*

(A3) *$\|F(x,\cdot)\|_{r(M)} e^{-\beta|x|} \in L^p(\mathbf{R}^d)$ for some p and $\beta \geq 0$ with $r \leq p \leq \infty$ and $p > d/(2-m/r)$.*

Then for every $T > 0$, it holds that

$$\sup_{0\leq\mu\leq\lambda} \mu^{d/2r} E_{(0,\theta_0)} \left[\sup_{0\leq t\leq T} \left| \int_0^t F(X_{\mu s}, \Theta_{\lambda s}) ds \right| \right] \longrightarrow 0$$

as $\lambda \longrightarrow \infty$.

Here $E_{(0,\theta_0)}$ denotes the expectation specifying the starting point of (X_t, Θ_t).

On the other hand, in Chapter 1 we also proved the following ergodic theorem to obtain some limit theorems for "winding-type" additive functionals of a 2-dimensional Brownian motion.

Theorem 4.1.2 *Let h be a Borel measurable function from $\mathbf{R}^d$ to $\mathbf{R}^1$ and f be a Borel measurable function from M to $\mathbf{R}^1$ satisfying the following conditions:*

(B1) *f is null charged on M; i.e., $\int_M f(\theta)d\theta = 0$,*

(B2) *f is in $L^r(M)$ for some r with $1 \leq r < \infty$,*

(B3) *For every $x \in \mathbf{R}^d$, $|h(x)| \leq \text{const.}\,|x|^\alpha$ for some α with $\alpha > -d$ and $\alpha > m/r - 2$.*

Then for every $T > 0$, it holds that

$$E_{(0,\theta_0)} \left[\sup_{0\leq t\leq T} \left| \int_0^t h(X_s) f(\Theta_{\lambda s}) ds \right| \right] \longrightarrow 0$$

as $\lambda \longrightarrow \infty$.

Our first purpose of this paper is to extend Theorem 4.1.2 as Theorem 4.1.1. That is, we will prove the following Theorem 4.1.3 in section 4.2.

Theorem 4.1.3 *Let* $F(x,\theta)$ *be a Borel measurable function from* G *to* $\mathbf{R}^1$ *satisfying the following conditions:*

($C1$) *For almost all* $x \in \mathbf{R}^d$, $F(x,\theta)$ *is null charged on* M,

($C2$) *For almost all* $x \in \mathbf{R}^d$, $F(x,\theta) \in L^r(M)$ *for some* r *with* $1 \leq r \leq \infty$ *and* $r > d/2$,

($C3$) *For every* $x \in \mathbf{R}^d$, $\|F(x,\cdot)\|_{r(M)} \leq \text{const.}\,|x|^\alpha$ *for some* α *with* $\alpha > -d$ *and* $\alpha > m/r - 2$.

Then for every $T > 0$, *it holds that*

$$E_{(0,\theta_0)}\left[\sup_{0 \leq t \leq T}\left|\int_0^t F(X_s, \Theta_{\lambda s})ds\right|\right] \longrightarrow 0$$

as $\lambda \longrightarrow \infty$.

Our second purpose of this paper is to show that these theorems are also valid in the case that M is $\mathbf{R}^m$ which is endowed with the normal distribution $\nu(d\theta) = (2\pi)^{-m/2}\exp(-|\theta|^2/2)d\theta$ instead of $d\theta$ and $(\Theta_t)_{t\geq 0}$ is an Ornstein-Uhlenbeck process; *i.e.*, $(\Theta_t)_{t \geq 0}$ is a $\mathbf{R}^m$-valued diffusion process with generator A;

$$Af(x) = \sum_{i=1}^d \frac{\partial^2 f}{(\partial x^i)^2}(x) - \sum_{i=1}^d x^i \frac{\partial f}{\partial x^i}(x).$$

We will prove this in section 4.3. In this case, the argument becomes subtler than in the case of Brownian motions on compact manifolds, because $E_0 \sup_{0 \leq t \leq T} |\varphi_n(\Theta_t)| \longrightarrow \infty$ as $T \longrightarrow \infty$ where φ_n, $n = 1, 2, \cdots$, are the Hermite polynomials, which are the eigenfunctions of A.

4.2 The proof of Theorem 4.1.3

First of all, we will recall that the generator $\frac{1}{2}\Delta_M$ of Θ_t has pure point spectrum

(4.1) $\quad 0 = \lambda_0 > -\lambda_1 \geq -\lambda_2 \geq \cdots,$

since M is compact. We denote the corresponding normalized eigenfunctions by $\{\varphi_n\}$. It is known that the transition density $q(t,\theta,\eta)$ of Θ_t has the following expansion:

$$(4.2) \qquad q(t,\theta,\eta) = \sum_{n=0}^{\infty} e^{-\lambda_n t}\varphi_n(\theta)\varphi_n(\eta),$$

which converges uniformly in (θ,η) for every $t > 0$ by Mercer's theorem (see Chavel [1] p.140).

We will also recall that X_t is determined by the stochastic differential equation:

$$(4.3) \qquad dX_t = \sigma(X_t)\,dB_t + b(X_t)\,dt,$$

where $(B_t)_{t\geq 0}$ is a d-dimensional Brownian motion (see Ikeda and Watanabe [4], Chapter 4, section 6).

Before proving our theorem, we prepare some estimates for expectations of the functionals of X_t and Θ_t.

Lemma 4.2.1 *Suppose that $h : \mathbf{R}^d \to \mathbf{R}^1$ satisfies the condition:*

$$h(x)e^{-\beta|x|} \in L^p(\mathbf{R}^d) \qquad \text{for some } 1 \leq p \leq \infty \text{ and } \beta \geq 0.$$

Then for every $x \in \mathbf{R}^d$ and $t > 0$,

$$E_x|h(X_t)| \leq \text{const.}\, t^{-d/2p}\, e^{\text{const.}\,\beta^2 t + \beta|x|}\, \|g\|_p,$$

where $g(x) = h(x)e^{-\beta|x|}$.

Proof. From the assumption for X_t, we have the following estimate for the transition density $p(t,x,y)$ of X_t :

$$(4.4) \qquad p(t,x,y) \leq \text{const.}\, t^{-d/2} \exp\left(-\frac{\text{const.}\,|x-y|^2}{2t}\right).$$

(See Friedman [3], p.141, Theorem 4.5.)

Then, by the assumption for $h(x)$ and Hölder's inequality,

$$\begin{aligned}
E_x|h(X_t)| &\leq \text{const.}\, t^{-d/2} \int_{\mathbf{R}^d} \exp\left(-\frac{\text{const.}\,|x-y|^2}{2t}\right) |g(y)|e^{\beta|y|}dy \\
&\leq \text{const.} \int_{\mathbf{R}^d} \exp\left(-\frac{\text{const.}\,|\xi|^2}{2} + \beta|\sqrt{t}\xi| + \beta|x|\right) |g(\sqrt{t}\xi + x)|d\xi \\
&= \text{const.}\, e^{\beta|x|} \left(\int_{\mathbf{R}^d} \exp\left(-\frac{\text{const.}\,|\xi|^2 - 2\beta\sqrt{t}|\xi|}{2}q\right) d\xi\right)^{1/q} \\
&\quad \times \left(\int_{\mathbf{R}^d} |g(\sqrt{t}\xi + x)|^p d\xi\right)^{1/p},
\end{aligned}$$

where $1/p + 1/q = 1$.

The integral in the first factor of the last expression is bounded by

$$\begin{aligned}
&\text{const.} \int_0^\infty |\xi|^{d-1} \exp\left(-\frac{\text{const.}\, |\xi|^2 - 2\beta\sqrt{t}|\xi|}{2} q\right) d|\xi| \\
\leq\ & \text{const.}\, e^{\text{const.}\, \beta^2 t} \int_0^\infty |\xi|^{d-1} \exp\left(-\frac{\text{const.}\, (|\xi| - \text{const.}\, \beta\sqrt{t})^2}{2} q\right) d|\xi| \\
\leq\ & \text{const.}\, e^{\text{const.}\, \beta^2 t},
\end{aligned}$$

and the second factor is equal to $t^{-d/2p}\|g\|_p$. Q.E.D.

Lemma 4.2.2 *Suppose that $f \in L^r(M, d\theta)$ with some $1 \leq r \leq \infty$. Then, for every $\delta > 0$ there exist positive constants C and C_δ such that*

$$E_\theta|f(\Theta_t)| \leq Ct^{-m/2r}\, 1_{(t\leq\delta)}\, \|f\|_r + C_\delta\, 1_{(t>\delta)}\, \|f\|_1$$

for every $\theta \in M$ and $t > 0$.

Proof. From (4.2) we have that

$$\begin{aligned}
q(t, \theta, \eta)\, 1_{(t>\delta)} &\leq \left(\sum_{n=0}^\infty e^{-\lambda_n t} \varphi_n(\theta)^2\right)^{1/2} \left(\sum_{n=0}^\infty e^{-\lambda_n t} \varphi_n(\eta)^2\right)^{1/2} 1_{(t>\delta)} \\
&\leq \left(\sum_{n=0}^\infty e^{-\lambda_n \delta} \varphi_n(\theta)^2\right)^{1/2} \left(\sum_{n=0}^\infty e^{-\lambda_n \delta} \varphi_n(\eta)^2\right)^{1/2} \\
&= q(\delta, \theta, \theta)^{1/2} q(\delta, \eta, \eta)^{1/2} \\
&\leq \sup_\theta q(\delta, \theta, \theta) \equiv C_\delta.
\end{aligned}$$

On the other hand, the uniform estimate

$$q(t, \theta, \eta) \leq \text{const.}\, t^{-m/2} \quad (t \downarrow 0) \tag{4.5}$$

holds. (see Chavel [1], p.154 ~ 155). Then noting that $\int_M q(t, \theta, \eta) d\eta = 1$ and $f \in L^r(M) \subset L^1(M)$, we have that

$$\begin{aligned}
E_\theta|f(\Theta_t)| &= \int_M q(t, \theta, \eta)|f(\eta)| d\eta \\
&= \int_M q(t, \theta, \eta)|f(\eta)| d\eta\, 1_{(t\leq\delta)} + \int_M q(t, \theta, \eta)|f(\eta)| d\eta\, 1_{(t>\delta)}
\end{aligned}$$

$$\leq \left(\int_M q(t,\theta,\eta)^q d\eta\right)^{1/q} \|f\|_r 1_{(t\leq\delta)} + C_\delta \|f\|_1 1_{(t>\delta)}$$

$$\leq \left(\int_M (\text{const.}\, t^{-m(q-1)/2} q(t,\theta,\eta) d\eta\right)^{1/q} \|f\|_r 1_{(t\leq\delta)} + C_\delta \|f\|_1 1_{(t>\delta)}$$

$$= \text{const.}\, t^{-m/2r} \|f\|_r 1_{(t\leq\delta)} + C_\delta \|f\|_1 1_{(t>\delta)}$$

by Hölder's inequality, where $1/r + 1/q = 1$. (In the case that $r = 1$, the above estimate is still valid by replacing $(\int_M q(t,\theta,\eta)^q d\eta)^{1/q}$ with $\sup_\eta q(t,\theta,\eta)$.) Q.E.D.

Proof of Theorem 4.1.3. The proof of this theorem is very similar to the proof of Theorem 4.1.1 (*cf.* Chapter 2). We shall first prove the special case that $F(x,\theta)$ is of the form $h(x)\varphi_n(\theta)$ and reduce the general case to this special case by approximations.

1°) *The case that $F(x,\theta) = h(x)\varphi_n(\theta)$ for some $n \geq 1$, where $h(x)e^{-|x|} \in L^p(\mathbf{R}^d)$ for some p with $1 \leq p \leq \infty$ and $p > d/2$.*

From now on we write the expectation $E_{(0,\theta_0)}$ simply by E. Set

$$u_\lambda(x,\theta) = \int_0^\infty E_{(x,\theta)}[h(X_s)\varphi_n(\Theta_{\lambda s})]ds$$

and

$$M_t^\lambda = u_\lambda(X_t, \Theta_{\lambda t}) + \int_0^t h(X_s)\varphi_n(\Theta_{\lambda s})ds.$$

In order to show that

$$(4.6) \qquad E \sup_{0\leq t\leq T} \left|\int_0^t h(X_s)\varphi_n(\Theta_{\lambda s})ds\right| \longrightarrow 0$$

as $\lambda \to \infty$, it is clearly sufficient to prove that

$$(4.7) \qquad E \sup_{0\leq t\leq T} |u_\lambda(X_t, \Theta_{\lambda t})| \longrightarrow 0$$

and

$$(4.8) \qquad E \sup_{0\leq t\leq T} |M_t^\lambda| \longrightarrow 0$$

as $\lambda \to \infty$.

The convergence (4.7) is proved as follows. By the orthonormality of $\{\varphi_k\}$ and (4.2), it holds that

$$(4.9) \qquad E_\theta[\varphi_n(\Theta_{\lambda s})] = \int_M q(\lambda s, \theta, \eta)\varphi_n(\eta)d\eta = e^{-\lambda_n \lambda s}\varphi_n(\theta)$$

for every $\theta \in M$. By Lemma 4.2.1 and (4.9), we obtain the estimate for u_λ:

$$
\begin{aligned}
(4.10) \qquad |u_\lambda(x,\theta)| &\leq \int_0^\infty E_x|h(X_s)||E_\theta[\varphi_n(\Theta_{\lambda s})]|ds \\
&\leq \text{const.}\, e^{|x|}\int_0^\infty s^{-d/2p}e^{\text{const.}\, s}e^{-\lambda_n\lambda s}ds|\varphi_n(\theta)| \\
&\leq \text{const.}\, e^{|x|}(\lambda_n\lambda-\text{const.})^{d/2p-1}|\varphi_n(\theta)|,
\end{aligned}
$$

for sufficiently large λ. Hence

$$
\begin{aligned}
(4.11) \qquad & E\sup_{0\leq t\leq T}|u_\lambda(X_t,\Theta_{\lambda t})| \\
&\leq \text{const.}\,(\lambda_n\lambda-\text{const.})^{d/2p-1}E\sup_{0\leq t\leq T}(|\varphi_n(\Theta_{\lambda t})|)\,E\sup_{0\leq t\leq T}\exp(|X_t|).
\end{aligned}
$$

Here

$$
(4.12) \qquad E\sup_{0\leq t\leq T}\exp(|X_t|)<\infty
$$

holds. To prove this, set

$$
M_i(t,a,a') = \exp\left(a\sum_{j=1}^d\int_0^t\sigma_j^i(X_s)dB_s^j - a'\sum_{j=1}^d\int_0^t\sigma_j^i(X_s)^2ds\right).
$$

Since the quadratic process

$$
\langle a\sum_{j=1}^d\int_0^\cdot\sigma_j^i(X_s)dB_s^j\rangle_t = a^2\sum_{j=1}^d\int_0^t\sigma_j^i(X_s)^2ds\leq \text{const.}\, t,
$$

$M(t,a,a^2/2)$ is a continuous exponential martingale for any $a\in\mathbf{R}^1$ (see Ikeda and Watanabe [3], p.154). Hence

$$
\begin{aligned}
E\sup_{0\leq t\leq T}M_i(t,a,a^2/4) &= E\sup_{0\leq t\leq T}M_i(t,a/2,a^2/8)^2 \\
&\leq 4EM_i(T,a/2,a^2/8)^2 \\
&= 4EM_i(T,a,a^2/4)
\end{aligned}
$$

by the martingale inequality. Therefore noting that the coefficients $\sigma(x)$ and $b(x)$ of the stochastic differential equation (4.3) are bounded, for any $a\in\mathbf{R}^1$ and each i we have

$$
\begin{aligned}
E\sup_{0\leq t\leq T}\exp(aX_t^i) &\leq \text{const.}\, E\sup_{0\leq t\leq T}M_i(t,a,a^2/4) \\
&\leq \text{const.}\, EM_i(T,a,a^2/4) \\
&\leq \text{const.}\, E\exp(aX_T^i) \\
&\leq \text{const.}\, E\exp(|aX_T|).
\end{aligned}
$$

From this we have

$$\begin{aligned} E \sup_{0\le t\le T} \exp(a|X_t^i|) &\le E \sup_{0\le t\le T} \exp(aX_t^i) + E \sup_{0\le t\le T} \exp(-aX_t^i) \\ &\le \text{const.}\, E\exp(|aX_T|) \end{aligned}$$

and hence

$$\begin{aligned} E \sup_{0\le t\le T} \exp(|X_t|) &\le E \sup_{0\le t\le T} \exp(\sum_{i=1}^{d} |X_t^i|) \le \sum_{i=1}^{d} E \sup_{0\le t\le T} \exp(d\cdot |X_t^i|) \\ &\le \text{const.}\, E\exp(d\cdot |X_T|). \end{aligned}$$

Since Lemma 4.2.1 implies that the last expectation is bounded, we have (4.12).

On the other hand, because of the boundedness of φ_n (note that φ_n is continuous and M is compact), it holds that

$$(4.13) \qquad E \sup_{0\le t\le T} |\varphi_n(\Theta_{\lambda t})| \le \text{const.}$$

Combining (4.11), (4.12) and (4.13), we have

$$\begin{aligned} (4.14) \qquad & E \sup_{0\le t\le T} |u_\lambda(X_t,\Theta_{\lambda t})| \le \text{const.}\,(\lambda_n\lambda - \text{const.}\,)^{d/2p-1} E \sup_{0\le t\le T} |\varphi_n(\Theta_{\lambda t})| \\ & \quad \longrightarrow 0 \quad (\lambda\to\infty). \end{aligned}$$

That is, (4.7) holds.

We now prove (4.8). Fixing λ and setting $\mathcal{F}_t = \sigma\{(X_s,\Theta_{\lambda s}); s\le t\}$, we can see that M_t^λ is an $(\mathcal{F}_t)$-martingale by a repeated use of Fubini's theorem and the Markov property of (X_t,Θ_t). (Note by Lemma 4.2.1 and (4.9) that

$$\begin{aligned} & E\left[\int_0^\infty \left|E_{(X_t,\Theta_{\lambda t})}(h(X_u)\varphi_n(\Theta_{\lambda u}))\right| du\right] \\ &= E\left[\int_0^\infty |E_{X_t}(h(X_u))|\,|E_{\Theta_{\lambda t}}(\varphi_n(\Theta_{\lambda u}))|\, du\right] < +\infty.) \end{aligned}$$

Then we have

$$\begin{aligned} & E \sup_{0\le t\le T} |M_t^\lambda| \le (E \sup_{0\le t\le T} |M_t^\lambda|^2)^{1/2} \le \text{const.}\,(E|M_T^\lambda|^2)^{1/2} \\ &\le \left(\text{const.}\, E|u_\lambda(X_T,\Theta_{\lambda T})|^2 + \text{const.}\, E\left|\int_0^T h(X_s)\varphi_n(\Theta_{\lambda s})ds\right|^2\right)^{1/2} \end{aligned}$$

by the martingale inequality. Hence it is sufficient to show that

$$(4.15)\qquad I_1 = E|u_\lambda(X_T, \Theta_{\lambda T})|^2 \longrightarrow 0$$

and

$$(4.16)\qquad I_2 = E\left(\int_0^T h(X_s)\varphi_n(\Theta_{\lambda s})\,ds\right)^2 \longrightarrow 0$$

as $\lambda \to \infty$. We can easily deduce (4.15) from (4.10) in a similar way as the proof of (4.7). (4.16) can be proved as follows. By Lemma 4.2.1, (4.9), Markov property and Fubini's theorem, we have that

$$\begin{aligned} I_2 &= 2E\left[\int_0^T ds \int_0^s du\, h(X_s)h(X_u)\varphi_n(\Theta_{\lambda s})\varphi_n(\Theta_{\lambda u})\right] \\ &= 2\int_0^T ds \int_0^s du\, E\Big[h(X_u)E_{X_u}[h(X_{s-u})]\Big] \\ &\qquad \times E\left[\varphi_n(\Theta_{\lambda u})E_{\Theta_{\lambda u}}[\varphi_n(\Theta_{\lambda(s-u)})]\right]. \end{aligned}$$

Lemma 4.2.1 implies that

$$\begin{aligned} \Big|E\Big[h(X_u)E_{X_u}[h(X_{s-u})]\Big]\Big| &\leq E\Big[|h(X_u)|\cdot E_{X_u}|h(X_{s-u})|\Big] \\ &\leq \text{const.}\,(s-u)^{-d/2p}e^{\text{const.}\,(s-u)}u^{-d/2p}e^{\text{const.}\,u} \end{aligned}$$

and (4.9) implies that

$$(4.17)\qquad \begin{aligned} \left|E\left[\varphi_n(\Theta_{\lambda u})E_{\Theta_{\lambda u}}\left[\varphi_n(\Theta_{\lambda(s-u)})\right]\right]\right| &\leq e^{-\lambda_n\lambda(s-u)}E|\varphi_n(\Theta_{\lambda u})|^2 \\ &\leq \text{const.}\,e^{-\lambda_n\lambda(s-u)}. \end{aligned}$$

Hence

$$\begin{aligned} I_2 &\leq \text{const.}\int_0^T ds\int_0^s du\, e^{-\lambda_n\lambda(s-u)}(s-u)^{-d/2p}e^{\text{const.}\,(s-u)}u^{-d/2p}e^{\text{const.}\,u} \\ &\leq \text{const.}\,(\lambda_n\lambda - \text{const.})^{d/2p-1} \longrightarrow 0 \quad (\lambda \to \infty). \end{aligned}$$

This completes the proof of (4.6).

2°) *General case.*

For fixed r and α in the conditions (*C2*) and (*C3*), there exists some $p \in [1, \infty)$ such that $d/2 < p \leq r$ and $\alpha p > -d$. We fix one of such p. Then it holds that $F(x,\theta)e^{-|x|} \in L^p(G)$.

Indeed,

$$\begin{aligned}\left\|F(x,\theta)e^{-|x|}\right\|_{p(G)} &= \left\|\|F(x,\cdot)\|_{p(M)}\, e^{-|x|}\right\|_{p(\mathbf{R}^d)} \\ &\le \left\|\text{const.}\, \|F(x,\cdot)\|_{r(M)}\, e^{-|x|}\right\|_{p(\mathbf{R}^d)} \\ &\le \text{const.}\, \left\||x|^\alpha e^{-|x|}\right\|_{p(\mathbf{R}^d)} \quad < \infty.\end{aligned}$$

In another words, $F(x,\theta) \in L^p(G, e^{-p|x|}dxd\theta)$. We denote the norm of this space by $\| \quad \|_P$.

Since X_t and Θ_t are mutually independent, by Lemma 4.2.1 and Lemma 4.2.2, we have, for fixed $\delta > 0$, that

$$\begin{aligned}E|F(X_t,\Theta_{\lambda t})|\, 1_{(\lambda t>\delta)} &\le \text{const.}\, E\|F(X_t,\cdot)\|_{1(M)} \le \text{const.}\, E\|F(X_t,\cdot)\|_{p(M)} \\ &\le \text{const.}\, t^{-d/2p} e^{\text{const.}\, t}\, \|F\|_P.\end{aligned}$$

Similarly we have by Lemma 2.1 and Lemma 2.2 that

$$\begin{aligned}E|F(X_t,\Theta_{\lambda t})|\, 1_{(\lambda t\le\delta)} &\le \text{const.}\, (\lambda t)^{-m/2r} E\|F(X_t,\cdot)\|_{r(M)} \\ &\le \text{const.}\, (\lambda t)^{-m/2r} E|X_t|^\alpha \\ &\le \text{const.}\, (\lambda t)^{-m/2r} t^{-d/2} \int_{\mathbf{R}^d} e^{-|x|^2/2t}|x|^\alpha dx \\ &= \text{const.}\, (\lambda t)^{-m/2r} \int_{\mathbf{R}^d} e^{-|\xi|^2/2}|\sqrt{t}\xi|^\alpha d\xi \\ &= \text{const.}\, (\lambda t)^{-m/2r} t^{\alpha/2} \int_{\mathbf{R}^d} e^{-|\xi|^2/2}|\xi|^\alpha d\xi \\ &= \text{const.}\, \lambda^{-m/2r} t^{-m/2r+\alpha/2}.\end{aligned}$$

In deriving the third inequality in the above we used (4.4).

Putting these estimates together, we obtain

$$\begin{aligned}&E \sup_{0\le t\le T}\left|\int_0^t F(X_s,\Theta_{\lambda s})ds\right| \le \int_0^T E|F(X_s,\Theta_{\lambda s})|ds \\ &= \int_0^{\delta/\lambda} E|F(X_s,\Theta_{\lambda s})|ds + \int_{\delta/\lambda}^T E|F(X_s,\Theta_{\lambda s})|ds \\ &\le \text{const.}\, \lambda^{-m/2r} \int_0^{\delta/\lambda} s^{-m/2r+\alpha/2} ds \\ &\qquad + \text{const.} \int_{\delta/\lambda}^T s^{-d/2p} e^{\text{const.}\, s} ds\, \|F\|_P \\ &\le \text{const.}\, \lambda^{-\alpha/2-1} + (\text{const.} + \text{const.}\, \lambda^{d/2p-1})\|F\|_P.\end{aligned}$$

Hence

$$(4.18) \qquad E \sup_{0\le t\le T} \left| \int_0^t F(X_s, \Theta_{\lambda s})\, ds \right| \le o(1) + (\text{const.} + o(1))\|F\|_P$$

as $\lambda \to \infty$.

Now define

$$\mathcal{L} := \{f;\ f \text{ is a finite linear combination of } \varphi_1, \varphi_2, \cdots\}$$

$$A := \{hf;\ h(x)e^{-|x|} \in L^p(\mathbf{R}^d) \text{ and } f \in \mathcal{L}\}$$

$$A' := \{F;\ F \text{ is a finite linear combination of functions of } A\}$$

$$B := \{hf;\ h(x)e^{-|x|} \in L^p(\mathbf{R}^d),\ f \in L^p(M) \text{ and } \textstyle\int_M f(\theta)d\theta = 0\}$$

$$B' := \{F;\ F \text{ is a finite linear combination of functions of } B\}$$

$$S := \{F;\ F(x,\theta)e^{-|x|} \in L^p(G) \text{ and } \textstyle\int_M F(x,\theta)d\theta = 0 \text{ for a.a. } x \in \mathbf{R}^d\}.$$

Every $f \in L^p(M)$ satisfying $\int_M f(\theta)d\theta = 0$ can be $\|\ \|_{p(M)}$-approximated by $f' \in \mathcal{L}$, because any continuous function ψ on M satisfying $\int_M \psi(\theta)d\theta = 0$ is uniformly approximated by functions of $\mathcal{L}$ since M is compact (*cf.* Chavel [1], p.139-140), and continuous functions are dense in $L^p(M)$. Hence every $h\cdot f \in B$ can be $\|\ \|_P$-approximated by $h\cdot f' \in A$. On the other hand, every $F \in S$ can be $\|\ \|_P$-approximated by $F' \in B'$; Note that $S \subset L^p(G, e^{-p|x|}dxd\theta)$ and $B' \subset L^p(G, e^{-p|x|}dxd\theta)$. Therefore, every $F \in S$ can be $\|\ \|_P$-approximated by $F' \in A'$.

Consequently, noting that by (4.6) Theorem 4.1.3 holds for every $hf \in A$ (and so for every $F' \in A'$), we complete the proof by an approximation argument using (4.18). Q.E.D.

4.3 Extention to the Ornstein-Uhlenbeck process

In this section, we shall show that Theorem 4.1.1 and Theorem 4.1.3 (and so Theorem 4.1.2) can be proved in the case that $M = \mathbf{R}^m$ and (Θ_t) is an Ornstein-Uhlenbeck process:

$$\Theta_t = (\Theta_t^1, \Theta_t^2, \cdots, \Theta_t^m),$$

$$d\Theta_t = -\Theta_t dt + \sqrt{2}\, dB_t,$$

where B_t is an m-dimensional Brownian motion. Here, of course, in the statement of the theorems, we take the normal distribution $N(d\theta) = (2\pi)^{-m/2}\exp(-|\theta|^2/2)d\theta$ on $\mathbf{R}^m$ instead of $d\theta$.

For the proof, note first that the generator of Θ_t has also pure point spectrum (4.1). The corresponding eigenfunctions $\{\varphi_n\}$ are of the form;

$$\varphi_n(\theta) = H_{k_1}(\theta^1)H_{k_2}(\theta^2)\cdots H_{k_m}(\theta^m)$$

for some non-negative integers $k_1, k_2, \cdots, k_m$ where $H_{k_i}(\theta^i)$ is the normalized Hermite polynomial of degree k_i and $\theta = (\theta^1, \theta^2, \cdots, \theta^m)$.
The transition density $q(t, \theta, \eta)$ of Θ_t is given by

$$(4.19) \quad \prod_{i=1}^{m}(1-e^{-2t})^{-1/2}\exp\left(-\frac{e^{-2t}(\theta^i)^2 - 2e^{-t}\theta^i\eta^i + e^{-2t}(\eta^i)^2}{2(1-e^{-2t})}\right)$$
$$= \prod_{i=1}^{m}(1-e^{-2t})^{-1/2}\exp\left(\frac{(\theta^i)^2}{2} - \frac{e^{-2t}(e^t\theta^i - \eta^i)^2}{2(1-e^{-2t})}\right).$$

Since $\mathbf{R}^m$ is not compact, the necessary changes for this case in the course of the proof appear where we have used the compactness of M. What we have to check are the following points in the proof of Theorem 3 (since the proof of Theorem 1, Theorem 2 and Theorem 3 are very similar, these points are the same as in the case of Theorem 1 and Theorem 2):

(1) The assertion of Lemma 4.2.2.

(2) $\lambda^{-\varepsilon} E \sup_{0\le t\le T} |\varphi_n(\Theta_{\lambda t})| \to 0$ as $\lambda \to \infty$ for any $\varepsilon > 0$ in (4.14).

(3) $E|\varphi_n(\Theta_{\lambda u})|^2 \le$ const. in (4.15) and (4.17).

(4) For $1 \le r < \infty$, $\mathcal{L}$ is dense in $L^r(\mathbf{R}^m, N(d\theta))$ where $\mathcal{L}$ is the set of all linear combinations of finite number of $\varphi_1, \varphi_2, \cdots$.

Firstly we consider (1). We see from the expression (4.19) that $q(t, \theta, \eta)\, 1_{(t>\delta)}$ is bounded and that $q(t, \theta, \eta)$ satisfies the uniform estimate

$$q(t, \theta, \eta) \le \text{const.}\, t^{-m/2}\, e^{|\theta|^2/2} \qquad (t \downarrow 0)$$

instead of (4.5), and hence Lemma 4.2.2 is also valid for the Ornstein-Uhlenbeck process Θ_t if we substitute $t^{-m/2r}e^{|\theta|^2/2r}$ for $t^{-m/2r}$ and $N(d\theta)$ for $d\theta$, respectively. Since we used Lemma 4.2.2 only in the case that $\theta = 0$ to prove (4.18) in the proof of Theorem 3, (4.18) is also valid for the Ornstein-Uhlenbeck process Θ_t.

Secondly we consider (3). The proof of (3) is easy. Indeed, since

$$E|\varphi_n(\Theta_{\lambda t})|^2 = \prod_{i=1}^{m} E|H_{k_i}(\Theta^i_{\lambda t})|^2$$

and

(4.20) $\qquad H_{k_i}(\theta^i) \leq \text{const.}\{1 + (\theta^i)^{k_i}\} \qquad (i = 1, \cdots, m),$

it is sufficient to show that

(4.21) $\qquad E|\Theta^i_{\lambda t}|^k \leq \text{const.} \qquad \text{for any } k \in \mathbf{N}$

But this is obvious because

$$E|\Theta^i_{\lambda t}|^k = (1 - e^{-2\lambda t})^{-1/2} \int_{-\infty}^{\infty} |\theta|^k e^{-e^{-2\lambda t}\theta^2/2(1-e^{-2\lambda t})} N(d\theta) \leq \text{const.}$$

Thirdly we consider (2). It should be noted that (4.13) does not hold for the Ornstein-Uhlenbeck process Θ_t. Indeed, it is known that $E\sup_{0\leq t\leq T}|\Theta_{\lambda t}| \to \infty$ as $\lambda \to \infty$ (see *e.g.* Friedman [2], chapter 8, section 1). But the assertion of (2) holds, which implies (4.14). The proof of (2) is as follows. By the inequality

$$E \sup_{0\leq t\leq T} |\varphi_n(\Theta_{\lambda t})| \leq \prod_{i=1}^{m} E \sup_{0\leq t\leq T} |H_{k_i}(\Theta^i_{\lambda t})|$$

and (4.20), it is sufficient to show that

$$\lambda^{-\varepsilon} E \sup_{0\leq t\leq T} |\Theta^i_{\lambda t}|^k \longrightarrow 0$$

as $\lambda \to \infty$ for any $\varepsilon > 0$ and every $k \in \mathbf{N}$. This follows from Lemma 4.3.1 below.

Lemma 4.3.1 *Suppose $(\Theta_t)_{t\geq 0}$ is a 1-dimensional Ornstein-Uhlenbeck process, i.e., an $\mathbf{R}^1$-valued process satisfying the stochastic differential equation*

$$d\Theta_t = \sqrt{2}\, dB_t - \Theta_t dt, \qquad \Theta_0 = 0,$$

where B_t is a 1-dimensional Brownian motion. Then for any $k \in \mathbf{N}$ and any $\varepsilon > 0$, it holds that

$$E_0 \sup_{0\leq t\leq T} |\Theta_t|^k = o(T^\varepsilon) \qquad (T \to \infty).$$

Proof. Fix $\varepsilon > 0$ and let $n \in \mathbf{N}$ be an even number such that $n > 1/\varepsilon$. Since $|x|^n \in C^2(\mathbf{R}^1)$, by Itô's formula we have that

$$\begin{aligned} |\Theta_t|^{nk} &= \sqrt{2}\, nk \int_0^t |\Theta_s|^{nk-1} sgn(\Theta_s)\, dB_s \\ &\quad + \int_0^t \left(nk(nk-1)|\Theta_s|^{nk-2} - nk|\Theta_s|^{nk}\right) ds \\ &:= I_1 + I_2. \end{aligned}$$

By Jensen's inequality and the moment inequality, we have

$$\begin{aligned}\left(E \sup_{0\le t\le T} |I_1|\right)^2 &\le E\left(\sup_{0\le t\le T}|I_1|\right)^2 \le 8E\left|n^2k^2\int_0^T |\Theta_s|^{2nk-2}ds\right| \\ &\le 8n^2k^2\int_0^T E|\Theta_s|^{2nk-2}ds.\end{aligned}$$

As for I_2, we have

$$E\sup_{0\le t\le T}|I_2| \le nk(nk-1)\int_0^T E|\Theta_s|^{nk-2}ds + nk\int_0^T E|\Theta_s|^{nk}ds.$$

Combining these estimates and (4.21), we obtain that

$$E\sup_{0\le t\le T}|\Theta_t|^{nk} \le (\text{const.}\,T)^{1/2} + \text{const.}\,T.$$

On the other hand, by Jensen's inequality, we conclude that

$$\left(E\sup_{0\le t\le T}|\Theta_t|^k\right)^n \le E\left(\sup_{0\le t\le T}|\Theta_t|^k\right)^n = E\sup_{0\le t\le T}|\Theta_t|^{nk}.$$

Hence we have that

$$\begin{aligned}T^{-\varepsilon}E\sup_{0\le t\le T}|\Theta_t|^k &\le T^{-\varepsilon}(\text{const.}\,T^{1/2}+\text{const.}\,T)^{1/n} \\ &= (\text{const.}\,T^{1/2-n\varepsilon}+\text{const.}\,T^{1-n\varepsilon})^{1/n} \\ &\longrightarrow 0 \qquad (T\to\infty).\end{aligned}$$

Q.E.D.

Remark 4.3.1. By a similar argument as in Lemma 4.3.1, we can see that

$$E\sup_{0\le t\le T}|\Theta_t| \le \log(1+\text{const.}\,T^{1/2}+\text{const.}\,T). \tag{4.22}$$

Indeed, since $E[\exp(\Theta_t)] \le \text{const.}$ and $E[|\Theta_t|\exp(\Theta_t)] \le \text{const.}$, applying Itô's formula to $\exp(x)$, we have that

$$\exp\left(E\sup_{0\le t\le T}|\Theta_t|\right) \le E\left(\sup_{0\le t\le T}\exp(\Theta_t)\right) \le 1+\text{const.}\,T^{1/2}+\text{const.}\,T,$$

which implies (4.22).

Finally we consider (4). This fact is proved as the following lemma due to I.Siegal (I learned this from professor K.Itô).

Lemma 4.3.2 *Let $\mathcal{P}$ be the set of all polynomials of $\mathbf{R}^m$. If a probability measure μ on $\mathbf{R}^m$ satisfies that $\int_{\mathbf{R}^m} \exp(\xi|\theta^i|)\mu(d\theta) < \infty$, $i = 1, \cdots, m$, for any $\xi > 0$, then $\mathcal{P}$ is dense in $L^p_\mu = L^p(\mathbf{R}^m, \mu(d\theta))$ for every $1 \leq p < \infty$.*

Proof. Firstly we consider the case that $1 < p < \infty$. Since $\mathcal{P}$ is a vector subspace of L^p_μ, its $\| \quad \|_p$-closure in L^p_μ , written $\mathcal{C}$, is a closed linear subspace of L^p_μ . It suffices to prove that $\mathcal{C} = L^p_\mu$. Suppose that this were not true. Since $1 < p < \infty$, we can find a non-zero element $y(\theta) \in L^q_\mu$ $(1/p + 1/q = 1)$ satisfying

$$(4.23) \qquad \int_{\mathbf{R}^m} y(\theta) f(\theta) \mu(d\theta) = 0 \qquad \text{for any } f \in \mathcal{C}$$

by using the Hahn-Banach theorem and noting that L^q_μ is dual to L^p_μ . Since $\exp(|\xi\theta^i|) \in L^p_\mu$, $i = 1, \cdots, m$, for any $\xi \in \mathbf{R}^1$ by the assumption, we obtain that $\int_{\mathbf{R}^m} |y(\theta)| \exp(|\xi\theta^i|)\mu(d\theta) < \infty$, *i.e.*,

$$(4.24) \qquad \sum_{k=0}^{\infty} \int_{\mathbf{R}^m} |y(\theta)| \frac{1}{k!} |\xi\theta^i|^k \mu(d\theta) < \infty \qquad (i = 1, \cdots, m).$$

Observe that

$$(4.25) \qquad \begin{aligned} \int_{\mathbf{R}^m} y(\theta) e^{\sqrt{-1}\xi\theta} \mu(d\theta) &= \sum_{k=1}^{\infty} \frac{(\sqrt{-1})^k}{k!} \int_{\mathbf{R}^m} y(\theta)(\xi_1\theta^1 + \cdots + \xi_m\theta^m)^k \mu(d\theta) \\ &= 0. \end{aligned}$$

Here the first equality in the above was derived from (4.24); the last equality was derived from (4.23) and the fact that $(\xi_1\theta^1 + \cdots + \xi_m\theta^m)^k \in \mathcal{P} \subset \mathcal{C}$. Since $y \in L^q_\mu \subset L^1_\mu$, $y(\theta)\mu(d\theta)$ is a signed measure of bounded variation on $\mathbf{R}^m$ whose Fourier transform vanishes identically by (4.25). Hence $y(\theta)\mu(d\theta) = 0$ a.e.(μ). This implies $y = 0$ a.e. in contradictio with our assumption $y \neq 0$.

Secondly we prove that $\mathcal{P}$ is dense in L^1_μ . Every $f \in L^1_\mu$ can be $\| \quad \|_1$-approximated by $f' \in L^1_\mu \cap L^2_\mu$ ($\subset L^2_\mu$) and f' can be $\| \quad \|_2$-approximated (and so $\| \quad \|_1$-approximated) by $f'' \in \mathcal{P}$ as seen above. Q.E.D.

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