Viscosity solutions of nonlinear second order elliptic PDEs with constraints

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Viscosity solutions of nonlinear second order elliptic PDEs with constraints

（制約条件をもつ非線形２階楕円型偏微分方程式の粘性解）

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Notations and Remarks

In the following we list the notations which are often used throughout this thesis. Let $N \in \mathbb{N}$.

$\mathbb{R}^N = N$-dimensional Euclidean space,

$\langle \cdot, \cdot \rangle =$ the Euclidean inner product in $\mathbb{R}^N$,

$B(x,r) =$ the open ball of radius r centered at x,

$K(r,s,n) = \bigcup_{0<t\leqq r} \overline{B(tn,st)}$ for $r,\ s>0$ and $n \in \mathbb{R}^N$ with $|n|=1$.

As to matrices, we define

$\mathbb{S}^N =$ the set of all $N \times N$ real symmetric matrices,

$\|A\| =$ the norm of $A \in \mathbb{S}^N$ as a self adjoint operator,

$I =$ the identity matrix,

${}^tA =$ the transposed matrix of A.

Let $\mathcal{O} \subset \mathbb{R}^N$. When its boundary $\partial\mathcal{O}$ is smooth, we denote $\nu(x)$ by

$$\nu(x) = (\nu_1(x), \cdots, \nu_N(x)) = \text{the outward unit normal to } \mathcal{O} \text{ at } x \in \partial\mathcal{O}.$$

For any function $u : \mathcal{O} \to \mathbb{R}$, we define

$$u^*(x) = \lim_{r\to 0} \sup\{u(y) \mid y \in B(x,r) \cap \mathcal{O}\},$$
$$u_*(x) = \lim_{r\to 0} \inf\{u(y) \mid y \in B(x,r) \cap \mathcal{O}\}.$$

We call u^* (resp., u_*) the upper semicontimuous (u.s.c.) (resp., lower semicontinuous (l.s.c.)) envelope of u. It is easily seen that $u_* \leq u \leq u^*$ on $\mathcal{O}$ and that u^* is u.s.c. in $\mathcal{O}$ and u_* is l.s.c. in $\mathcal{O}$. We observe that u is u.s.c. (resp., l.s.c.) at $x \in \mathcal{O}$ if and only if $u(x) = u^*(x)$ (resp., $= u_*(x)$).

We denote by $J_{\mathcal{O}}^{2,+}u(x)$, $J_{\mathcal{O}}^{2,-}u(x)$ the super 2-jet of u and the sub 2-jet of u in $\mathcal{O}$, respectively:

$$J_{\mathcal{O}}^{2,+}u(x) = \left\{ (p,X) \in \mathbb{R}^N \times \mathbb{S}^N \;\middle|\; u(x+h) \leq u(x) + \langle p, h\rangle \right.$$
$$\left. + \frac{1}{2}\langle Xh, h\rangle + o(|h|^2) \quad \text{as } x+h \in \mathcal{O} \text{ and } h \to 0 \right\},$$
$$J_{\mathcal{O}}^{2,-}u(x) = \left\{ (p,X) \in \mathbb{R}^N \times \mathbb{S}^N \;\middle|\; u(x+h) \geq u(x) + \langle p, h\rangle \right.$$
$$\left. + \frac{1}{2}\langle Xh, h\rangle + o(|h|^2) \quad \text{as } x+h \in \mathcal{O} \text{ and } h \to 0 \right\}.$$

Furthermore $\bar{J}_{\mathcal{O}}^{2,+}u(x)$, $\bar{J}_{\mathcal{O}}^{2,-}u(x)$ are the graph closure of $J_{\mathcal{O}}^{2,+}u(x)$, $J_{\mathcal{O}}^{2,-}u(x)$, respectiverly:

$$\bar{J}_{\mathcal{O}}^{2,+}u(x) = \{(p,X) \in \mathbb{R}^N \times \mathbb{S}^N \mid \text{there exist } \{x_n\}_{n\in\mathbb{N}} \subset \mathcal{O}$$
$$\text{and } (p_n, X_n) \in J^{2,+}u(x_n) \text{ such that}$$
$$(x_n, u(x_n), p_n, X_n) \to (x, u(x), p, X) \text{ as } n \to +\infty\},$$
$$\bar{J}_{\mathcal{O}}^{2,-}u(x) = \{(p,X) \in \mathbb{R}^N \times \mathbb{S}^N \mid \text{there exist } \{x_n\}_{n\in\mathbb{N}} \subset \mathcal{O}$$
$$\text{and } (p_n, X_n) \in J^{2,-}u(x_n) \text{ such that}$$
$$(x_n, u(x_n), p_n, X_n) \to (x, u(x), p, X) \text{ as } n \to +\infty\}.$$

We note that if $\varphi \in C^2(\mathcal{O})$ and $u - \varphi$ attains a local maximum (resp., local minimum) at $x_0 \in \mathcal{O}$, then $(D\varphi(x_0), D^2\varphi(x_0)) \in J_{\mathcal{O}}^{2,+}u(x_0)$ (resp., $(D\varphi(x_0), D^2\varphi(x_0)) \in J_{\mathcal{O}}^{2,-}u(x_0)$). Conversely, it is easily verified that, for any $x_0 \in \mathcal{O}$, if $(p,X) \in J_{\mathcal{O}}^{2,+}u(x_0)$ (resp., $(p,X) \in J_{\mathcal{O}}^{2,-}u(x_0)$), then there exists a function $\varphi \in C^2(\mathcal{O})$ such that $u - \varphi$ attains a local maximum (resp., local minimum) at x_0 and $(D\varphi(x_0), D^2\varphi(x_0)) = (p, X)$.

Let $\mathcal{O} \subset \mathbb{R}^N$. We define the sets of functions as follows.

$$USC(\mathcal{O}) = \{u : \mathcal{O} \to \mathbb{R} \cup \{\pm\infty\} : \text{u.s.c.}\},$$

$$LSC(\mathcal{O}) = \{u : \mathcal{O} \to \mathbb{R} \cup \{\pm\infty\} : \text{l.s.c.}\},$$

$C(\mathcal{O}) = \{u : \mathcal{O} \to \mathbb{R} : \text{continuous}\}$ with the norm

$$\|u\|_{C(\mathcal{O})} = \sup_{x \in \mathcal{O}} |u(x)|,$$

$L^\infty(\mathcal{O}) = \{u : \mathcal{O} \to \mathbb{R} : \text{bounded and measurable}\}$ with the norm

$$\|u\|_{L^\infty(\mathcal{O})} = \text{ess.sup}_{x \in \mathcal{O}} |u(x)|.$$

We introduce the notion of degenerate ellipticity, which plays an important role to assure that the classical solutions of elliptic PDEs are viscosity solutions.

Definition. *Let $\mathcal{O} \subset \mathbb{R}^N$ and let $F \in C(\mathcal{O} \times \mathbb{R} \times \mathbb{R}^N \times \mathbb{S}^N)$. Then we say the F is degenerate elliptic provided*

$$F(x, r, p, X + Y) \leqq F(x, r, p, X)$$

for all $x \in \mathcal{O}$, $r \in \mathbb{R}$, $p \in \mathbb{R}^N$, X, $Y \in \mathbb{S}^N$ and $Y \geqq O$.

Finally throughout this thesis we use the usual summation convention on repeated indices. We surpress the term "viscosity" whenever we do not give rise to confusions since we are mainly concerned with viscosity sub-, super- and solutions.

Introduction

In this thesis we consider the Dirichlet problems for the nonlinear second order degenerate elliptic partial differential equations (PDEs) with constraints. Mainly we are concerned with the following problems:

$$(1)\qquad \begin{cases} \max\{F(x,u,Du,D^2u), u - Mu\} = 0 & \text{in} \quad \Omega, \\ \max\{u - g, u - Mu\} = 0 & \text{on} \quad \partial\Omega, \end{cases}$$

$$(2)\qquad \begin{cases} \min\{\max\{F(x,u,Du,D^2u), u - Mu\}, u - Nu\} = 0 & \text{in} \quad \Omega, \\ \min\{\max\{u - g, u - Mu\}, u - Nu\} = 0 & \text{on} \quad \partial\Omega. \end{cases}$$

Here $\Omega \subset \mathbb{R}^N$ is a bounded domain, F is a nonlinear degenerate elliptic operator, Du, D^2u are, respectively, the gradient and the Hessian matrix of the function u and M, N denote the nonlocal operators defined below. These equations arise in the Dynamic Programming approach for the optimal control problems and differential games for diffusion processes governed by stochastic differential equations. See W. H. Fleming - R. W. Rishel [12], A. Benssousan - J. Lions [4], [5], N. V. Krylov [30], B. Perthame [40] and G. Barles [1] etc. for the backgrounds. However we consider these problems from the analytical viewpoints.

We easily observe by simple examples that, in general, the above problems do not have classical solutions even if the F is uniformly elliptic and the coefficients of F are smooth. Thus we must consider the weak solutions for them. However we cannot use the weak solutions in the sense of Schwarz' distributions because we cannot integrate by parts the expressions which we have by multiplying the equations above by test functions. Hence, in this thesis, we adapt the notion of viscosity solutions as the weak solutions.

In 1983 the notion of viscosity solutions was introduced by M. G. Crandall - P. L. Lions [9] as the weak solutions of Hamilton-Jacobi equations. We also refer M. G. Crandall - L. C. Evans - P. L. Lions [6]. Here we briefly explain the derivation of viscosity solutions. We consider the following Hamilton-Jacobi equation:

$$(3)\qquad H(x,u,Du) = 0 \qquad \text{in} \quad \Omega.$$

To show the existence of solutions of (3), we approximate it by semilinear elliptic PDEs as follows:

$$(3)_\varepsilon \qquad -\varepsilon\Delta u_\varepsilon + H(x, u_\varepsilon, Du_\varepsilon) = 0 \quad \text{in} \quad \Omega \qquad (\varepsilon > 0).$$

Let u_ε be a classical solution of $(3)_\varepsilon$. Assume the sequence $\{u_\varepsilon\}_{\varepsilon>0}$ converges to some function u uniformly on $\overline{\Omega}$ as $\varepsilon \to 0$. Then in what sense does the u satisfy (3) ? By this motivation the notion of viscosity solutions was derived. In the same year P. L. Lions [35] extended this notion to nonlinear second order elliptic PDEs. Afterwords by the results in R. Jensen [25], H. Ishii [16], H. Ishii - P. L. Lions [19] we can interpret the notion of viscosity solutions as the weak solutions in the sense of pointwise derived from Taylor expansion and the maximum principle in calculus and general theories on viscosity solutions have been developped. See M. G. Crandall - H. Ishii - P. L. Lions [8] for the survay.

In their results, they assumed the strict monotonicity with respect to the 0-th order terms. But we cannot apply them to the problems (1) and (2) directly because the equations in (1) and (2) have the monotonicity with respect to the 0-th order terms, not the strict monotonicity by the nonlocal terms Mu and Nu. Furthermore, in these problems the implicit bounbary conditions are imposed, which are natural from the viewpoint of the impulse control problems. We study (1) and (2) with paying the attentions to these points.

This thesis consists of three chapters. In Chapter I we consider the problem (1). A. Benssousan and J. L. Lions treated this problem from the viewpoint of quasi-variational inequality for the first time when F is linear. (See [5].) Since then, it was studied as the usual Dirichlet problem under some compatibility conditions by which we can get $g \leqq Mu$ on $\partial\Omega$. We refer to [5] and B. Perthame [38] etc. In B. Perthame [39] we obtained the existence and uniqueness of viscosity solutions of (1). In this chapter we have the comparison principle and existence of viscosity solutions of (1) under the weaker assumptions than those in [39].

The main strategies are similar to [8]. However, we cannot apply them to the problem (1) directly because of the nonlocal term Mu. In proving the comparison, we regard the term $u - Mu$ as a function and deal with it. Then the equation in (1) has only the monotonicity with respect to the 0-th order term. But, by using the concavity of the operator M and the convex structure of the equation we can prove the comparison principle. In the proof of the existence of solutions it is very difficult to construct a subsolution and a supersolution of (1) satisfying the boundary condition because of the term Mu. To overcome this difficulty we construct a solution of the equation by Perron's method and show that it satisfies the boundary condition in (2) by the barrier argument and the comparison principle.

Chapter II is devoted the problem (2). This can be regarded an extension of the problem (1). Once the existence of this problem was shown by using L^2 theory under some compatibility conditions by which we can get $Nu \leqq g \leqq Mu$ on $\partial\Omega$. However, the uniqueness has not been proved. See [5] for the detail. Although it seems that we can prove the uniqueness of solutions in a class of smooth functions (e.g., $W^{2,p}(\Omega), p > n$), the regularity of solutions have not been obtained since the operators does not neccesarily preserve the regularity of functions. Hence, by using the notion of viscosity solutions we can obtain the comparison principle and existence of solutions of (2).

The strategies of their proofs are similar to those in Chapter I. But we need to remark that the operators M, N have different properties from each other and the equation does not have the convex structure. Thus we use the definitions of M, N and the idea seen in H. Ishii - S. Koike [18] to obtain the comparison principle. As to the existence of solutions of (2) we construct one by Perron's method and show that it satisfies the boundary condition by the barrier argument and the comparison principle. Especially, we must discuss the latter fact carefully because the boundary condition is more complicated than that in the problem (1).

Finally, in Chapter III we return the problem (1) and consider the case F is degenerate on the boundary $\partial\Omega$. In Chapter I, since the principal parts of the equations are not degenerate on the boundary, we can observe that a unique solution satisfies the boundary conditions in the classical sense. However, in the case F is degenerate on the boundary this fails. Hence we interpret the boundary conditions in the viscosity sense, which is weaker than that in the classical sense. This was introduced by H. Ishii [15] and is derived naturally from the Dynamic Programming principle in the optimal control theory. Concerning the Bellman equations without constraints, see [15] and M. A. Katsoulakis [26], [27]. In this chapter we obtain the comparison principle and existence of solutions of (1) satisfying the boundary conditions in the viscosity sense.

In proving the comparison principle we cannot help assuming the nontangential semicontinuity for a subsolution u and a supersolution v of (1) because we interpret the boundary condition in the viscosity sense and do not know whether $u \leqq v$ on $\partial\Omega$. As to the existence, if we get the nontangential semicontinuity of solutions, we have the continuity of solutions of (1) by the comparison principle. However, it is difficult to analyze the solutions of (1) directly because of the nonlocality of the operator M. Hence we apply the iterative approximation scheme by B. Hanouzet - J. L. Joly [14] to have the existence of solutions of (1) in $C(\overline{\Omega})$. Of course, the solution constructed by this method is a unique solution. We see the uniqueness and existence of approximate solutions by [15], [26] and [27]. Moreover, we can get the representation formula of the solution of (1). By the similar arguments we can prove the comparison principle and existence of solutions of the boundary value problem of oblique type containing the nonlocal term Mu.

To conclude, we have made clear the uniqueness and existence of viscosity solutions of the Dirichlet problems for nonlinear second order degenerate elliptic PDEs with constraints such as (1) and (2).

Chapter I

Viscosity solutions of nonlinear second order elliptic PDEs associated with impulse control problems

§1. Introduction

In this chapter we study the nonlinear second order elliptic PDEs with implicit obstacles.

Let $\Omega \subset \mathbb{R}^N$ be a bounded domain with smooth boundary $\partial\Omega$. For any $u : \overline{\Omega} \to \mathbb{R}$, we define the nonlocal operator M as the following:

$$Mu(x) = \inf_{\substack{\xi \geqq 0 \\ x+\xi \in \overline{\Omega}}} \{k(\xi) + u(x+\xi)\},$$

where $k(\xi)$ is a nonnegative and continuous function on $(\mathbb{R}^+)^N$ and $\xi \geqq 0$ means $\xi \in (\mathbb{R}^+)^N$.

We consider the following nonlinear elliptic PDE with the implicit boundary condition:

$$\max\{Lu - f, u - Mu\} = 0 \quad \text{in} \quad \Omega, \tag{1.1}$$

$$\max\{u - g, u - Mu\} = 0 \quad \text{on} \quad \partial\Omega, \tag{1.2}$$

where L is a linear second order elliptic operator of the form:

$$Lu = -a_{ij}u_{x_i x_j} + b_i u_{x_i} + cu.$$

The problem (1.1)-(1.2) is associated with optimal impulse control problems, whose state is governed by stochastic differential equations with impulsive jumps and whose value function has $k(\xi)$ as an impulsive cost. (For the details, see A. Bensoussan - J. L. Lions [5].)

In general the equation (1.1) with $u = g$ on $\partial\Omega$ has no solution because we don't know *a priori* whether $g \leq Mu$ on $\partial\Omega$ or not. So we put the implicit obstacle in (1.2). (cf. B. Perthame [39] etc.)

From the view point of the impulse control, [5] treated one for the nondegenerate diffusions and J. L. Menaldi [36] did the degenerate case. They characterized the value function of impulse control problems as the maximum solution of the corresponding quasi-variational inequality (QVI) in some Sobolev spaces. Using the notion of viscosity solutions by M. G. Crandall - P. L. Lions [9], G. Barles [1] showed that the value function for deterministic impulse control problems is a unique viscosity solution of the corresponding first order Hamilton-Jacobi QVI in $\mathbb{R}^N$.

By an analytical treatment, B. Perthame [38] proved the existence and uniqueness of solutions in the class $W^{2,\infty}_{loc}(\Omega) \cap C(\overline{\Omega})$ under the assumption that (1.1) has a subsolution $\underline{u}$ satisfying $\underline{u} \leq g \leq M\underline{u}$ on $\partial\Omega$. Moreover, B. Perthame [39] remarked that a unique maximal subsolution of (1.1) with the usual Dirichlet condition is a unique viscosity solution of the problem (1.1)-(1.2). G. Barles [2] extended his results in [1] to the general Hamiltonian case and obtained the existence and uniqueness of viscosity solutions of Hamilton-Jacobi QVI. J. Yong [56] treated a system of Hamilton-Jacobi QVI associated with switching and impulsive control problems in $\mathbb{R}^N$.

Our main purpose here is to obtain the comparison principle and existence of viscosity solutions of the problem (1.1)-(1.2) under the assumptions weaker than [39]. Although (1.2) is not the usual Dirichlet condition, we see in Sections 3 and 4 that the problem (1.1)-(1.2) can be treated similarly to that.

Our plan of this chapter is the following. In Section 2 we state our assumptions and recall the definition of viscosity solutions to general PDEs and the properties of the operator M. In order to take the boundary condition (1.2) into account, we introduce the notion of *strong* viscosity solutions (see M. G. Crandall - H. Ishii - P. L. Lions [8; Section 7]). Section 3 is devoted to the proof of the comparison principle of viscosity solutions. Our argument is based upon H. Ishii - P. L. Lions [19]. In Section 4 we construct a strong viscosity solution of the problem (1.1)-(1.2) by Perron's method. Because of the strongness, we can show the existence of viscosity solutions satisfying (1.2) for each point on $\partial\Omega$ without using the iterative process by B. Hanouzet - J. L. Joly [14] and [41].

Finally we refer to H. Ishii - S. Koike [18] and S. M. Lenhart - N. Yamada [31], [32] for some problems and results related to ours.

§2. Preliminaries

In this section we shall state our assumptioms and shall recall the definition of viscosity solutions of nonlinear elliptic PDEs and the properties of the operator M. We make the following assumptions.

(A.1) $\Omega \subset \mathbb{R}^N$ is a bounded domain with smooth boundary $\partial\Omega$.

(A.2) There exists a mapping $P : \overline{\Omega} \times (\mathbb{R}^+)^N \to (\mathbb{R}^+)^N$ satisfying

$$\begin{aligned} &x + P(x,\xi) \in \overline{\Omega} && \text{for any } x \in \overline{\Omega},\ \xi \in (\mathbb{R}^+)^N, \\ &P(x,\xi) = \xi && \text{if } x + \xi \in \overline{\Omega}, \\ &P(\cdot,\xi) \in C(\overline{\Omega}) && \text{for each } \xi \geqq 0. \end{aligned}$$

(A.3) For the matrix $(a_{ij}(x))$, there exists a nonnegative matrix $(\sigma_{ij}(x))$ such that

$$(a_{ij}) = {}^t(\sigma_{ij})(\sigma_{ij}) \text{ with } \sigma_{ij} \in W^{1,\infty}(\overline{\Omega}) \quad (i,j = 1,\cdots,N).$$

(A.4) $b_i \in W^{1,\infty}(\overline{\Omega})$ $(i = 1, \cdots, N)$.

(A.5) $c \in C(\overline{\Omega})$, $c \geqq c_0$ on $\overline{\Omega}$ for some $c_0 > 0$.

(A.6) $f \in C(\overline{\Omega})$.

(A.7) $k \in C((\mathbb{R}^+)^N)$, $k(\xi) \geqq k_0$ on $(\mathbb{R}^+)^N$ for some $k_0 > 0$.

(A.8) $g \in C(\overline{\Omega})$.

We denote by ω_c and ω_f the modulus of continuity of c and f, respectively.

Remark 2.1. The assumption (A.2) does not hold if we only suppose the smoothness of $\partial\Omega$. When Ω is convex and regular, we can take $P(x,\xi)$ as the projection of ξ on $(\mathbb{R}^+)^N \cap (\overline{\Omega} - \{x\})$. (See A. Bensoussan - J. L. Lions [5; Chapter 4, Remark 1.7] and J. L. Menaldi [36].)

Now we give the definition of solutions of the nonlinear degenerate elliptic PDEs with the implicit boundary condition:

$$\text{(2.1)} \qquad \begin{cases} \max\{F(x,u,Du,D^2u), u - Mu\} = 0 & \text{in} \quad \Omega, \\ \max\{u - g, u - Mu\} = 0 & \text{on} \quad \partial\Omega, \end{cases}$$

where $F \in C(\Omega \times \mathbb{R} \times \mathbb{R}^N \times \mathbb{S}^N)$ is a degenerate elliptic operator.

Definition 2.2. *Let $u : \overline{\Omega} \to \mathbb{R}$.*

(1) *We say u is a subsolution of* (2.1) *provided $u^* < +\infty$ on $\overline{\Omega}$ and for any $\varphi \in C^2(\overline{\Omega})$, if $u^* - \varphi$ attains a local maximum at $x \in \overline{\Omega}$, then*

$$\max\{F(x,u^*(x),D\varphi(x),D^2\varphi(x)), u^*(x) - Mu^*(x)\} \leqq 0.$$

(2) *We say u is a supersolution of* (2.1) *provided $u_* > -\infty$ on $\overline{\Omega}$ and for any $\varphi \in C^2(\overline{\Omega})$, if $u_* - \varphi$ attains a local minimum at $x \in \overline{\Omega}$, then*

$$\max\{F(x,u_*(x),D\varphi(x),D^2\varphi(x)), u_*(x) - Mu_*(x)\} \geqq 0.$$

(3) *We say u is a solution of (2.1) provided u is both a sub- and a supersolution of (2.1).*

Next we state the equivalent propositions of Definition 2.2. We refer the reader to M. G. Crandall - H. Ishii - P. L. Lions [8; Section 7] for general elliptic PDEs.

Proposition 2.3. *Let $u : \overline{\Omega} \to \mathbb{R}$.*

(1) *u is a subsolution of (2.1) if and only if $u^* < +\infty$ on $\overline{\Omega}$ and for all $x \in \overline{\Omega}$ and $(p, X) \in J_{\Omega}^{2,+}u^*(x)$, u^* satisfies*

$$\max\{F(x, u^*(x), p, X), u^*(x) - Mu^*(x)\} \leqq 0.$$

(2) *u is a supersolution of (2.1) if and only if $u_* > -\infty$ on $\overline{\Omega}$ and for all $x \in \overline{\Omega}$ and $(p, X) \in J_{\Omega}^{2,-}u_*(x)$, u_* satisfies*

$$\max\{F(x, u_*(x), p, X), u_*(x) - Mu_*(x)\} \geqq 0.$$

Proposition 2.4. *Assume $M : USC(\overline{\Omega}) \to USC(\overline{\Omega})$ and $M : LSC(\overline{\Omega}) \to LSC(\overline{\Omega})$. Let $u : \overline{\Omega} \to \mathbb{R}$.*

(1) *u is a subsolution of (2.1) if and only if $u^* < +\infty$ on $\overline{\Omega}$ and for all $x \in \overline{\Omega}$ and $(p, X) \in \bar{J}_{\Omega}^{2,+}u^*(x)$, u^* satisfies*

$$\max\{F(x, u^*(x), p, X), u^*(x) - Mu^*(x)\} \leqq 0.$$

(2) *u is a supersolution of (1.1) if and only if $u_* > -\infty$ on $\overline{\Omega}$ and for all $x \in \overline{\Omega}$ and $(p, X) \in \bar{J}_{\Omega}^{2,-}u_*(x)$, u_* satisfies*

$$\max\{F(x, u_*(x), p, X), u_*(x) - Mu_*(x)\} \geqq 0.$$

Since the proofs of the above propositions are similar to those in [8], we leave them to the reader.

The boundary condition (1.2) differs from the usual Dirichlet condition. So we introduce the notion of *strong* viscosity solutions. (cf. M. G. Crandall - H. Ishii - P. L. Lions [8; Section 7].)

Definition 2.5. *Let* $u : \overline{\Omega} \to \mathbb{R}$.

(1) *u is a strong subsolution of* (2.1) *if u is a subsolution of* (2.1) *and u satisfies*

$$\max\{u^*(x) - g(x), u^*(x) - Mu^*(x)\} \leqq 0 \qquad \text{for all } x \in \partial\Omega.$$

(2) *u is a strong supersolution of* (2.1) *if u is a supersolution of* (2.1) *and u satisfies*

$$\max\{u_*(x) - g(x), u_*(x) - Mu_*(x)\} \geqq 0 \qquad \text{for all } x \in \partial\Omega.$$

(3) *u is a strong solution of* (2.1) *if u is a strong subsolution and a strong supersolution of* (2.1).

Remark 2.6. It is easily seen that if the strong solution u of (2.1) is continuous on $\overline{\Omega}$, then u satisfies the boundary condition for all $x \in \partial\Omega$.

We recall the properties of the operator M.

Proposition 2.7. *Assume* (A.1), (A.3) *and* (A.8) *hold. Let* u, $v : \overline{\Omega} \to \mathbb{R}$. *Then we have the following properties.*

(1) *If* $u \leqq v$ *on* $\overline{\Omega}$, *then* $Mu \leqq Mv$ *on* $\overline{\Omega}$.
(2) $M(tu + (1-t)v) \geqq tMu + (1-t)Mv$ *for all* $t \in [0, 1]$.
(3) $M(u + c) = Mu + c$ *for all* $c \in \mathbb{R}$.
(4) *If* $u \in LSC(\overline{\Omega})$, *then* $Mu \in LSC(\overline{\Omega})$.
(5) *If* $u \in USC(\overline{\Omega})$, *then* $Mu \in USC(\overline{\Omega})$.
(6) $\|Mu - Mv\|_{C(\overline{\Omega})} \leqq \|u - v\|_{C(\overline{\Omega})}$ *for all* u, $v \in C(\overline{\Omega})$.

Proof. We only show (4) and (5) because it is obvious by the definition of M that (1)–(3) and (6) hold.

(4) We take $\{x_n\}_{n \in \mathbb{N}} \subset \overline{\Omega}$, $x \in \overline{\Omega}$ such that $x_n \to x$ $(n \to +\infty)$. The condition $u \in LSC(\overline{\Omega})$ implies that for each x_n there exists a $\xi_n \geqq 0$ such that

$$x_n + \xi_n \in \overline{\Omega}, \quad Mu(x_n) = k(\xi_n) + u(x_n + \xi_n).$$

Since $\{\xi_n\}_{n\in\mathbb{N}}$ is bounded, by taking a subsequence, if neccesary, we may consider that $\lim_{n\to+\infty}\xi_n = \xi \geqq 0$ such that $x+\xi \in \overline{\Omega}$. Hence we have

$$\begin{aligned}\liminf_{n\to+\infty} Mu(x_n) &\geqq \lim_{n\to+\infty} k(\xi_n) + \liminf_{n\to+\infty} u(x_n+\xi_n)\\ &\geqq k(\xi)+u(x+\xi)\\ &\geqq Mu(x),\end{aligned}$$

that is, $Mu \in LSC(\overline{\Omega})$.

(5) We take $\{x_n\}_{n\in\mathbb{N}} \subset \overline{\Omega}$ and $x\in\overline{\Omega}$ as in the proof of (4) and fix $\xi \geqq 0$ such that $x+\xi\in\overline{\Omega}$. Then we have by (A.3),

$$Mu(x_n) \leqq k(P(x_n,\xi)) + u(x_n + P(x_n,\xi)).$$

Thus we get

$$\begin{aligned}\limsup_{n\to+\infty} Mu(x_n) &\leqq \lim_{n\to+\infty} k(P(x_n,\xi)) + \limsup_{n\to+\infty} u(x_n+P(x_n,\xi))\\ &\leqq k(\xi)+u(x+\xi).\end{aligned}$$

Taking the infimum with respect to $\xi \geqq 0$ satisfying $x+\xi\in\overline{\Omega}$, $Mu\in USC(\overline{\Omega})$ is proved. ∎

§3. Comparison principle of solutions

In this section we shall prove the comparison principle of strong solutions of the problem (1.1)-(1.2).

Theorem 3.1. *Assume* (A.1)-(A.8). *Let* u, v *be a strong subsolution, a strong supersolution, respectively, of* (1.1)-(1.2). *Then* $u^* \leqq v_*$ *on* $\overline{\Omega}$.

In proving Theorem 3.1, we use some perturbation of strong subsolution to deal with the term $u - Mu$. (cf. H. Ishii - P. L. Lions [19; Section V.1].) Moreover the existence of certain *derivatives* plays an important role. So we prepare the following lemmas.

Lemma 3.2. *Let $u \in USC(\overline{\Omega})$ be a strong subsolution of* (1.1)-(1.2) *and let* $C = \max\{(\|f\|_{C(\overline{\Omega})}/c_0, \|g\|_{C(\overline{\Omega})}\} + 1$. *Then for each $m \in \mathbb{N}$, $u_m = (1 - 1/m)u - C/m$ is a strong subsolution of*

$$(3.1)_m \qquad \begin{cases} \max\{Lu_m - f, u_m - Mu_m\} + \dfrac{\gamma}{m} = 0 & \text{in} \quad \Omega, \\ \max\{u_m - g, u_m - Mu_m\} + \dfrac{\gamma}{m} = 0 & \text{on} \quad \partial\Omega, \end{cases}$$

where $\gamma = \min\{1, k_0\}$.

Proof. First, we note that $w \equiv -C$ on $\overline{\Omega}$ satisfies

$$\max\{Lw - f, w - Mw\} \leqq \max\{-c_0 C - f, -k\} \leqq -\gamma \qquad \text{in} \quad \Omega,$$
$$\max\{w - g, w - Mw\} \leqq \max\{-C - g, -k\} \leqq -\gamma \qquad \text{on} \quad \partial\Omega.$$

For any fixed $\varphi \in C^2(\Omega)$, we suppose $u_m - \varphi$ attains a local maximum at $x_0 \in \Omega$. Then we have

$$\begin{aligned} u_m(x_0) - \varphi(x_0) &= \left(1 - \frac{1}{m}\right) u(x_0) - \frac{C}{m} - \varphi(x_0) \\ &= \left(1 - \frac{1}{m}\right) \left\{u(x_0) - \left(\frac{m}{m-1}\varphi(x_0) + \frac{C}{m-1}\right)\right\} \end{aligned}$$

and see that $u - ((m/m-1)\varphi + C/(m-1))$ attains a local maximum at $x_0 \in \Omega$. Hence using the fact that u is a subsolution of (1.1), we observe

$$(3.2) \qquad \max\left\{-\frac{m}{m-1}a_{ij}(x_0)\varphi_{x_i x_j}(x_0) + \frac{m}{m-1}b_i(x_0)\varphi_{x_i}(x_0) \right.$$
$$\left. + c(x_0)u(x_0) - f(x_0), u(x_0) - Mu(x_0)\right\} \leqq 0$$

By (3.2) we have

$$(3.3) \qquad -\frac{m}{m-1}a_{ij}(x_0)\varphi_{x_i x_j}(x_0) + \frac{m}{m-1}b_i(x_0)\varphi_{x_i}(x_0) + c(x_0)u(x_0) - f(x_0) \leqq 0,$$

$$(3.4) \qquad u(x_0) - Mu(x_0) \leqq 0.$$

Thus multiplying (3.3) by $1 - 1/m$ and subtracting C/m from (3.3), we obtain

$$-a_{ij}(x_0)\varphi_{x_i x_j}(x_0) + b_i(x_0)\varphi_{x_i}(x_0) + c(x_0)u_m(x_0) - f(x_0) \leqq -\frac{1}{m},$$

and by a similar calculation and using (A.7) and Proposition 2.7 (2),(3), we get

$$\begin{aligned} u_m(x_0) - Mu_m(x_0) &= \left(1 - \frac{1}{m}\right)u(x_0) - \frac{C}{m} - M\left\{\left(1 - \frac{1}{m}\right)u(x_0) - \frac{C}{m}\right\} \\ &\leqq \left(1 - \frac{1}{m}\right)u(x_0) - M\left\{\left(1 - \frac{1}{m}\right)u(x_0) - \frac{1}{m}\cdot 0\right\} \\ &\leqq \left(1 - \frac{1}{m}\right)u(x_0) - \left(1 - \frac{1}{m}\right)Mu(x_0) - \frac{k_0}{m} \\ &\leqq -\frac{k_0}{m}. \end{aligned}$$

Thus we obtain

$$\begin{aligned} \max\{&-a_{ij}(x_0)\varphi_{x_i x_j}(x_0) + b_i(x_0)\varphi_{x_i}(x_0) + c(x_0)u_m(x_0) - f(x_0), \\ &u_m(x_0) - Mu_m(x_0)\} \leqq -\frac{1}{m}\min\{c_0 C + f(x_0), k_0\} \leqq -\frac{\gamma}{m}. \end{aligned}$$

It is easily verified that

$$\max\{u_m - g, u_m - Mu_m\} \leqq -\frac{1}{m}\min\{C + g, k_0\} \leqq -\frac{\gamma}{m} \qquad \text{on} \quad \partial\Omega.$$

Hence the proof is complete. ∎

Lemma 3.3. *Let $u \in USC(\overline{\Omega})$ and $v \in LSC(\overline{\Omega})$ and let $(\bar{x}, \bar{y}) \in \overline{\Omega \times \Omega}$ be a local maximum point of the function $u(x) - v(y) - \alpha|x-y|^2/2$ $(\alpha > 0)$. Then there exist $X, Y \in \mathbb{S}^N$ such that*

$$-3\alpha\begin{pmatrix} I & O \\ O & I \end{pmatrix} \leqq \begin{pmatrix} X & O \\ O & -Y \end{pmatrix} \leqq 3\alpha\begin{pmatrix} I & -I \\ -I & I \end{pmatrix}$$

and

$$(\alpha(\bar{x} - \bar{y}), X) \in \bar{J}^{2,+}u(\bar{x}), (\alpha(\bar{x} - \bar{y}), Y) \in \bar{J}^{2,-}v(\bar{y}).$$

This is proved in M. G. Crandall - H. Ishii [7; Example 1], so we omit the proof.

Proof of Theorem 3.1. We may assume that $u \in USC(\overline{\Omega})$ and $v \in LSC(\overline{\Omega})$, because, if otherwise, we replace u, v with u^*, v_*, respectively.

Let C be the same constant as in Lemma 3.2. For each $m \in \mathbb{N}$, the function $u_m = (1-1/m)u - C/m$ is a strong subsolution of $(3.1)_m$. To prove the comparison principle, it is sufficient to show $\max_{\overline{\Omega}}(u_m - v) \leqq 0$ for all $m \geqq 1$ because we obtain the desired result by letting $m \to +\infty$. To the contrary, we suppose $\max_{\overline{\Omega}}(u_{m_0} - v) = \theta > 0$ for some $m_0 \geqq 1$ and get a contradiction. Then there exists a point $z \in \overline{\Omega}$ such that $\theta = u_{m_0}(z) - v(z)$.

Case 1. $z \in \partial\Omega$.

In this case we have

$$\max\{u_{m_0}(z) - g(z), u_{m_0}(z) - Mu_{m_0}(z)\} \leqq -\frac{\gamma}{m_0},$$
$$\max\{v(z) - g(z), v(z) - Mv(z)\} \geqq 0.$$

When $v(z) - g(z) \geqq 0$, we obtain a contradiction easily. In the case $v(z) - Mv(z) \geqq 0$ we can find $\xi_z \geqq 0$ such that

$$z + \xi_z \in \overline{\Omega} \quad \text{and} \quad Mv(z) = k(\xi_z) + v(z + \xi_z)$$

Hence we get

$$\begin{aligned} \theta &\leqq k(\xi_z) + u_{m_0}(z + \xi_z) - k(\xi_z) - v(z + \xi_z) - \frac{\gamma}{m_0} \\ &\leqq \theta - \frac{\gamma}{m_0}, \end{aligned}$$

which is a contradiction.

Case 2. $z \in \Omega$.

We note that the function $u_{m_0}(x) - |x-z|^4 - v(x)$ takes the maximum θ and z is a unique maximum point of this function. For each $\alpha > 0$ we define the function Φ on $\overline{\Omega \times \Omega}$ by

$$\Phi(x,y) = u_{m_0}(x) - |x-z|^4 - v(y) - \frac{\alpha}{2}|x-y|^2,$$

and let $(\bar{x}, \bar{y}) \in \overline{\Omega \times \Omega}$ be a maximum point of Φ. We observe that the inequality $\Phi(z,z) \leqq \Phi(\bar{x}, \bar{y})$ implies that

$$(3.5)\quad \theta = u_{m_0}(z) - v(z) \leqq u_{m_0}(z) - v(z) + \frac{\alpha}{2}|\bar{x}-\bar{y}|^2 \leqq u_{m_0}(\bar{x}) - |\bar{x}-z|^4 - v(\bar{y}).$$

Since the functions u_{m_0} and $-v$ are bounded above, we have, from (3.5), $|\bar{x} - \bar{y}| \to 0$ as $\alpha \to +\infty$. By the compactness of $\overline{\Omega}$ we see that $\bar{x}_n,\ \bar{y}_n \to \bar{z}$ for a suitable sequence $\{\alpha_n\}_{n \in \mathbb{N}}$ tending to $+\infty$ and some $\bar{z} \in \overline{\Omega}$. Using (3.5) and semicontinuity of u_{m_0} and v, we get $\theta \leqq u_{m_0}(\bar{z}) - |\bar{z}-z|^4 - v(\bar{z})$. Hence we have $\bar{z} = z$ and $\bar{x},\ \bar{y} \to z$ $(\alpha \to +\infty)$ because z is a unique maximum point of the function $u_{m_0}(x) - |x-z|^4 - v(x)$. Moreover we observe

$$(3.6)\qquad \begin{aligned} u_{m_0}(z) - v(z) &\leqq \liminf_{\alpha \to +\infty}(u_{m_0}(\bar{x}) - v(\bar{y})) \\ &\leqq \limsup_{\alpha \to +\infty}(u_{m_0}(\bar{x}) - v(\bar{y})) \\ &\leqq \limsup_{\alpha \to +\infty} u_{m_0}(\bar{x}) - \liminf_{\alpha \to +\infty} v(\bar{y}) \\ &\leqq u_{m_0}(z) - v(z). \end{aligned}$$

Thus we have

$$\lim_{\alpha \to +\infty}(u_{m_0}(\bar{x}) - v(\bar{y})) = u_{m_0}(z) - v(z).$$

By this equality and the semicontinuity of u_{m_0} and v we obtain

$$\begin{aligned} u_{m_0}(z) &\geqq \limsup_{\alpha \to +\infty} u_{m_0}(\bar{x}) \\ &\geqq \liminf_{\alpha \to +\infty} u_{m_0}(\bar{x}) \\ &= \liminf_{\alpha \to +\infty}(u_{m_0}(\bar{x}) - v(\bar{y}) + v(\bar{y})) \\ &\geqq \liminf_{\alpha \to +\infty}(u_{m_0}(\bar{x}) - v(\bar{y})) + \liminf_{\alpha \to +\infty} v(\bar{y}) \\ &\geqq u_{m_0}(z) - v(z) + v(z) = u_{m_0}(z). \end{aligned}$$

Therefore we conclude

$$\text{(3.7)} \qquad \lim_{\alpha\to+\infty} u_{m_0}(\bar{x}) = u_{m_0}(z) \qquad \text{and} \qquad \lim_{\alpha\to+\infty} v(\bar{y}) = v(z).$$

It is easily seen by (3.5) and (3.7) that

$$\alpha|\bar{x}-\bar{y}|^2 \to 0 \quad (\alpha \to +\infty).$$

As $\bar{x}$, $\bar{y} \to z \in \Omega$ as $\alpha \to +\infty$, we have $\bar{x}$, $\bar{y} \in \Omega$ for large $\alpha > 0$. Then by Lemma 3.3 there exist $X, Y \in \mathbb{S}^N$ satisfying

$$\text{(3.8)} \qquad (\alpha(\bar{x}-\bar{y}), X) \in \bar{J}^{2,+}(u_{m_0}(\bar{x}) - |\bar{x}-z|^4),$$

$$\text{(3.9)} \qquad (\alpha(\bar{x}-\bar{y}), Y) \in \bar{J}^{2,-}v(\bar{y}),$$

and

$$\text{(3.10)} \qquad -3\alpha\begin{pmatrix} I & O \\ O & I \end{pmatrix} \leqq \begin{pmatrix} X & O \\ O & -Y \end{pmatrix} \leqq 3\alpha\begin{pmatrix} I & -I \\ -I & I \end{pmatrix}.$$

Furthermore, (3.8) implies

$$\left(\alpha(\bar{x}-\bar{y}) + 4|\bar{x}-z|^2(\bar{x}-z), X+Z\right) \in \bar{J}^{2,+}(u_{m_0}(\bar{x})),$$

where $Z = 4|\bar{x}-z|^2 I + 8(\bar{x}-z)\otimes(\bar{x}-z)$ and $Z \to O$ as $\alpha \to +\infty$. Hence using the facts that u_{m_0} is a strong subsolution of $(3.1)_{m_0}$ and that v is a strong supersolution of (1.1)-(1.2), we have the following inequalities:

$$\text{(3.11)} \qquad \max\{-tr\{{}^t\sigma(\bar{x})\sigma(\bar{x})(X+Z)\}$$
$$+b_i(\bar{x})\left\{\alpha(\bar{x}_i-\bar{y}_i) + 4|\bar{x}-z|^2(\bar{x}_i-z_i)\right\}$$
$$+c(\bar{x})u_{m_0}(\bar{x}) - f(\bar{x}), u_{m_0}(\bar{x}) - Mu_{m_0}(\bar{x})\} \leqq -\frac{\gamma}{m_0},$$

$$\text{(3.12)} \qquad \max\{-tr\{{}^t\sigma(\bar{y})\sigma(\bar{y})Y\} + \alpha b_i(\bar{y})(\bar{x}_i-\bar{y}_i)$$
$$+ c(\bar{y})v(\bar{y}) - f(\bar{y}), v(\bar{y}) - Mv(\bar{y})\} \geqq 0.$$

We divide our consideration into two cases.

(I) The case $v(\bar{y}) - Mv(\bar{y}) \geqq 0$ in (3.12).

(3.11) implies $u_{m_0}(\bar{x}) - Mu_{m_0}(\bar{x}) \leqq -\gamma/m_0$. Thus we get

$$\theta \leqq u_{m_0}(\bar{x}) - v(\bar{y}) - |\bar{x} - z|^4 \leqq Mu_{m_0}(\bar{x}) - Mv(\bar{y}) - \frac{\gamma}{m_0} - |\bar{x} - z|^4.$$

From Proposition 2.7 (4), (5), sending $\alpha \to +\infty$, we have

$$\begin{aligned}\theta &\leqq \limsup_{\alpha \to +\infty} Mu_{m_0}(\bar{x}) - \liminf_{\alpha \to +\infty} Mv(\bar{y}) - \frac{\gamma}{m_0} \\ &\leqq Mu_{m_0}(z) - Mv(z) - \frac{\gamma}{m_0}.\end{aligned}$$

As in *Case 1*, we get a contradiction.

(II) $-tr\{\,{}^t\sigma(\bar{y})\sigma(\bar{y})Y\} + \alpha b_i(\bar{y})(\bar{x}_i - \bar{y}_i) + c(\bar{y})v(\bar{y}) - f(\bar{y}) \geqq 0$ in (3.12).

By (3.11) we have

$$\begin{aligned}-tr\{\,{}^t\sigma(\bar{x})\sigma(\bar{x})(X + Z)\} + b_i(\bar{x})\left\{\alpha(\bar{x}_i - \bar{y}_i) + 4|\bar{x} - z|^2(\bar{x}_i - z_i)\right\} \\ + c(\bar{x})u_{m_0}(\bar{x}) - f(\bar{x}) \leqq -\frac{\gamma}{m_0}.\end{aligned}$$

Therefore noting that from (3.10)

$$\begin{aligned}&tr\{\,{}^t\sigma(\bar{x})\sigma(\bar{x})X\} - tr\{\,{}^t\sigma(\bar{y})\sigma(\bar{y})Y\} \\ &\qquad \leqq 3\alpha tr\{\,{}^t(\sigma(\bar{x}) - \sigma(\bar{y}))(\sigma(\bar{x}) - \sigma(\bar{y}))\},\end{aligned}$$

and using (A.3), (A.4) and (A.5) we obtain

$$\begin{aligned}c(\bar{x})u_{m_0}(\bar{x}) - c(\bar{y})v(\bar{y}) &\leqq 3\alpha tr\{\,{}^t(\sigma(\bar{x}) - \sigma(\bar{y}))(\sigma(\bar{x}) - \sigma(\bar{y}))\} \\ &\quad + tr\{\,{}^t\sigma(\bar{x})\sigma(\bar{x})Z\} \\ &\quad + \alpha(b_i(\bar{x}) - b_i(\bar{y}))(\bar{x} - \bar{y}) \\ &\quad - 4|\bar{x} - z|^2 b_i(\bar{x})(\bar{x}_i - z_i) \\ &\quad + f(\bar{x}) - f(\bar{y}) - \frac{\gamma}{m_0} \\ &\leqq K\alpha|\bar{x} - \bar{y}|^2 + K|\bar{x} - z|^2 + K|\bar{x} - z|^3 \\ &\quad + \omega_f(|\bar{x} - \bar{y}|) - \frac{\gamma}{m_0}.\end{aligned}$$

Here and hereafter K denotes a positive constant depending only on known constants. Moreover we have

$$\begin{aligned}
c_0\theta &\leqq c_0(u_{m_0}(\bar{x}) - v(\bar{y}) - |\bar{x}-z|^4) \\
&\leqq c(\bar{x})(u_{m_0}(\bar{x}) - v(\bar{y})) \\
&= c(\bar{x})u_{m_0}(\bar{x}) - c(\bar{y})v(\bar{y}) - v(\bar{y})(c(\bar{x}) - c(\bar{y})) \\
&\leqq K\alpha|\bar{x}-\bar{y}|^2 + K|\bar{x}-z|^2 + K|\bar{x}-z|^3 + \omega_f(|\bar{x}-\bar{y}|) \\
&\qquad - \frac{\gamma}{m_0} - v(\bar{y})(c(\bar{y}) - c(\bar{x})) \\
&\leqq K\alpha|\bar{x}-\bar{y}|^2 + K|\bar{x}-z|^2 + K|\bar{x}-z|^3 + \omega_f(|\bar{x}-\bar{y}|) \\
&\qquad + K\omega_c(|\bar{x}-\bar{y}|) - \frac{\gamma}{m_0},
\end{aligned}$$

because $-v$ is bounded above. Thus letting $\alpha \to +\infty$, we obtain a contradiction.

Thus we conclude that $\max_{\overline{\Omega}}(u_m - v) \leqq 0$ for all $m \geqq 1$. Letting $m \to +\infty$, we complete the proof. ∎

§4. Existence of solutions

In this section we shall show the existence of a strong solution of the problem (1.1)-(1.2).

Theorem 4.1. *Assume* (A.1)-(A.8). *Furthermore, assume* (A.9) *or* (A.10) *holds;*

(A.9) $(a_{ij}(x)) \geqq \mu I$ *on* $\overline{\Omega}$ *for some* $\mu > 0$,

(A.10) $(a_{ij}(x)) \geqq 0$ *on* $\overline{\Omega}$ *and*

$$b_i(x)\nu_i(x) < 0 \quad \text{on } \Gamma = \{x \in \partial\Omega \mid a_{ij}(x)\nu_i(x)\nu_j(x) = 0\}.$$

Then there exists a unique strong solution $u \in C(\overline{\Omega})$ *of the problem* (1.1)-(1.2), *which is a unique solution of* (1.1) *satisfying* (1.2).

Before proving Theorem 4.1, we show the existence of a strong supersolution of (1.1)-(1.2).

Lemma 4.2. *Assume* (A.1), (A.3), (A.4), (A.5), (A.6) *and* (A.8). *If* (A.9) *or* (A.10) *holds, there exists a solution* $\overline{u} \in C(\overline{\Omega})$ *of*

$$\text{(4.1)} \qquad \begin{cases} Lu - f = 0 & \text{in} \quad \Omega, \\ u = g & \text{on} \quad \partial\Omega. \end{cases}$$

Proof. First of all, we remark that the comparison principle of viscosity solutions of (4.1) holds. (See H. Ishii - P. L. Lions [19; Theorem II.2].) In the case where (A.9) holds, there exists a solution $\overline{u} \in W^{2,p}_{loc}(\Omega) \cap C(\overline{\Omega})$ $(n < p < +\infty)$ of (4.1) satisfying $\overline{u} = g$ on $\partial\Omega$ by D. Gilbarg - N. S. Trudinger [13; Corollary 9.18]. Hence it is also a unique solution of (4.1) satisfying $\overline{u} = g$ on $\partial\Omega$. (See P. L. Lions [34; Theorem I.2].)

In the case where (A.10) holds, we apply the barrier construction argument in A. O. Oleinik - E. V. Radkevic [37; Theorem 1.5.2]. (cf. H. Ishii - S. Koike [18; Proposition 4.3] and S. M. Lenhart - N. Yamada [31; Theorem 2.2].) Let $\psi \in C^2(\Omega) \cap C(\overline{\Omega})$ be a function such that $\psi = g$ on $\partial\Omega$. We consider the following degenerate linear elliptic PDE:

$$\text{(4.2)} \qquad \begin{cases} Lw - \hat{f} = 0 & \text{in} \quad \Omega, \\ w = 0 & \text{on} \quad \partial\Omega, \end{cases}$$

where $\hat{f} = a_{ij}\psi_{x_i x_j} - b_i\psi_{x_i} - c\psi + f$. Then for each $z \in \partial\Omega$, there exist a neighborhood V_z of z and a local barrier $\zeta_z \in C^2(\Omega \cap V_z) \cap C(\overline{\Omega \cap V_z})$ satisfying

$$\begin{aligned} &\zeta_z(z) = 0, \\ &\zeta_z \geqq 0 && \text{on} \quad \overline{\Omega \cap V_z}, \\ &\zeta_z \geqq \frac{\hat{C}}{c_0} && \text{on} \quad \Omega \cap \partial V_z, \\ &L\zeta_z - \hat{f} \geqq 0 && \text{in} \quad \Omega \cap V_z, \end{aligned}$$

where $\hat{C}$ is a constant depending on $\|a_{ij}\|_{C(\overline{\Omega})}$, $\|b_i\|_{C(\overline{\Omega})}$, $\|c\|_{C(\overline{\Omega})}$, $\|f\|_{C(\overline{\Omega})}$, $\|\psi\|_{C(\overline{\Omega})}$, $\|D\psi\|_{L^\infty(\Omega)}$, $\|D^2\psi\|_{L^\infty(\Omega)}$. Hence setting

$$\hat{\zeta}_z(x) = \begin{cases} \min\left\{\zeta_z(x), \dfrac{\hat{C}}{c_0}\right\} & \text{for } x \in \overline{V}_z, \\ \dfrac{\hat{C}}{c_0} & \text{otherwise,} \end{cases}$$
$$\zeta(x) = \inf\{\hat{\zeta}_z(x) | z \in \partial\Omega\},$$

we observe that $\hat{\zeta}_z \in C(\overline{\Omega})$ is a supersolution of (4.2) and thus ζ is a u.s.c supersolution of (4.2) such that $\zeta = 0$ on $\partial\Omega$. In the same way we can construct a l.s.c. subsolution ζ' of (4.2) satisfying $\zeta' = 0$ on $\partial\Omega$. Hence by Perron's method there exists a solution $w \in C(\overline{\Omega})$ of (4.2) satisfying $w = 0$ on $\partial\Omega$. Therefore $\overline{u} = w + \psi$ is a solution of (4.1) satisfying $\overline{u} = g$ on $\partial\Omega$. Indeed, for any $\varphi \in C^2(\Omega)$, suppose $\overline{u} - \varphi$ attains a local maximum at $x_0 \in \Omega$. Since $w - (\varphi - \psi)$ attains a local maximum at $x_0 \in \Omega$, we get

$$-a_{ij}(x_0)\{\varphi_{x_i x_j}(x_0) - \psi_{x_i x_j}(x_0)\} + b_i(x_0)\{\varphi_{x_i}(x_0) - \psi_{x_i}(x_0)\}$$
$$+ c(x_0)w(x_0) - \hat{f}(x_0) \leqq 0.$$

The definition of $\hat{f}$ implies that

$$-a_{ij}(x_0)\varphi_{x_i x_j}(x_0) + b_i(x_0)\varphi_{x_i}(x_0) + c(x_0)\{w(x_0) + \psi(x_0)\} - f(x_0) \leqq 0.$$

Thus $\overline{u}$ is a subsolution of (4.1). We can show similarly that $\overline{u}$ is a supersolution of (4.1). Hence we have obtained the result. ∎

By Lemma 4.2, it is easily seen that $\overline{u}$ is a strong supersolution of the problem (1.1)-(1.2). Next we show that Perron's method can be used for (1.1)-(1.2).

Proposition 4.3. *We define the set $\mathcal{S}$ and the function u as follows:*

$$\mathcal{S} = \left\{ v : \overline{\Omega} \to \mathbb{R} \mid v \text{ is a strong subsolution of } (1.1)\text{-}(1.2) \right\},$$
$$u(x) = \sup\{v(x) \mid v \in \mathcal{S}\} \qquad (x \in \overline{\Omega}).$$

Then the following properties hold.

(P.1) $u \in \mathcal{S}$.

(P.2) *If $v \in \mathcal{S}$ is not a strong supersolution of* (1.1), *then there exists $w \in \mathcal{S}$ such that $v(y) < w(y)$ for some $y \in \overline{\Omega}$.*

Proof. We note that $\mathcal{S} \neq \emptyset$ because, for the same constant C as in Lemma 3.2, $\underline{u} \equiv -C \in \mathcal{S}$. Moreover the function $\overline{u}$ in Lemma 4.2 is a strong supersolution of (1.1)-(1.2). Thus by the definition of u and Theorem 3.1 we observe that $\underline{u} \leqq u \leqq \overline{u}$ on $\overline{\Omega}$.

Now, we shall prove (P.1) holds. Suppose that for $\varphi \in C^2(\Omega)$, $u^* - \varphi$ attains a local maximum at $x_0 \in \Omega$. Without loss of generality, we may assume

$$u^*(x_0) - \varphi(x_0) = 0, \qquad u^*(x) - \varphi(x) \leqq 0 \qquad \text{in} \quad \Omega,$$

and

$$u^*(x) - \varphi(x) \leqq -|x - x_0|^4 \qquad \text{in} \quad B(x_0, r) \qquad \text{for some } r > 0.$$

Then there exists $\{x_n\}_{n \in \mathbb{N}} \subset B(x_0, r)$ such that

$$x_n \to x_0 \quad \text{and} \quad u^*(x_n) - \varphi(x_n) \to 0 \quad (n \to +\infty).$$

Because of the definition of u, for each $n \in \mathbb{N}$, there exists a sequence $\{u_n\}_{n \in \mathbb{N}} \subset \mathcal{S}$ satisfying

$$u_n^*(x_n) - \varphi(x_n) > u^*(x_n) - \varphi(x_n) - \frac{1}{n},$$
$$u_n^*(x) - \varphi(x) \leqq -|x - x_0|^4 \qquad\qquad \text{on} \quad \overline{B(x_0, r)}.$$

Let $y_n \in \overline{B(x_0,r)}$ be a maximum point of $u^* - \varphi$ on $\overline{B(x_0,r)}$. Then we see

$$u^*(x_n) - \varphi(x_n) - \frac{1}{n} < u_n^*(x_n) - \varphi(x_n) \leqq u_n^*(y_n) - \varphi(y_n)$$
$$\leqq u^*(y_n) - \varphi(y_n) \leqq -|y_n - x_0|^4.$$

Therefore we have $y_n \to x_0$ as $n \to +\infty$ and

$$\lim_{n\to+\infty} u^*(y_n) = \lim_{n\to+\infty} \varphi(y_n) = \varphi(x_0) = u^*(x_0).$$

Since u_n is a strong viscosity subsolution of (1.1)-(1.2), we obtain

$$\max\{-a_{ij}(y_n)\varphi_{x_ix_j}(y_n) + b_i(y_n)\varphi_{x_i}(y_n) + c(y_n)u_n^*(y_n)$$
$$- f(y_n), u_n^*(y_n) - Mu_n^*(y_n)\} \leqq 0$$

Remarking that $u_n \leqq u$ on $\overline{\Omega}$ and Proposition 2.7 (5), we have

$$\limsup_{n\to+\infty} Mu_n^*(y_n) \leqq \limsup_{n\to+\infty} Mu^*(y_n) \leqq Mu^*(x_0).$$

Sending $n \to +\infty$, we get

$$\max\{-a_{ij}(x_0)\varphi_{x_ix_j}(x_0) + b_i(x_0)\varphi_{x_i}(x_0)$$
$$+ c(x_0)u^*(x_0) - f(x_0), u^*(x_0) - Mu^*(x_0)\} \leqq 0.$$

As to the boundary condition (1.2), $u \leqq \overline{u}$ on $\overline{\Omega}$ and $\overline{u} \in C(\overline{\Omega})$ imply $u^* - g \leqq 0$ on $\partial\Omega$. Moreover, by the definition of u and Proposition 2.7 (1) we obtain

$$v^* - Mu^* \leqq v^* - Mv^* \leqq 0 \qquad \text{on} \quad \partial\Omega \quad \text{for } v \in \mathcal{S},$$

since $v^* - Mv^* \leqq 0$ on $\partial\Omega$. Thus we have $\max\{u^* - g, u^* - Mu^*\} \leqq 0$ on $\partial\Omega$. Hence $u \in \mathcal{S}$.

Next, we suppose $v \in \mathcal{S}$ is not a strong supersolution of (1.1)-(1.2). Then there exist $x_0 \in \overline{\Omega}$ and $\beta > 0$ such that

$$\begin{aligned}(4.3) \qquad & \max\{-a_{ij}(x_0)X_{ij} + b_i(x_0)p_i + c(x_0)v_*(x_0) - f(x_0), \\ & \qquad v_*(x_0) - Mv_*(x_0)\} \leqq -\beta \\ & \qquad \text{for some } (p, X)\bar{J}^{2,-}v_*(x_0) \quad \text{if } x_0 \in \Omega,\end{aligned}$$

$$(4.4) \qquad \max\{v_*(x_0) - g(x_0), v_*(x_0) - Mv_*(x_0)\} \leqq -\beta \quad \text{if } x_0 \in \partial\Omega.$$

We consider the following two cases.

(I) *The case* (4.3). We can find $\varphi \in C^2(\Omega)$ satisfying $D\varphi(x_0) = p$ and $D^2\varphi(x_0) = X$ and fix it. Furthermore, we may assume that

$$v_*(x_0) = \varphi(x_0), \quad v_*(x) \geqq \varphi(x) + |x - x_0|^4 \quad \text{for } x \in B(x_0, r)$$

for some $r > 0$.

We claim $v_*(x_0) < \overline{u}(x_0)$. If otherwise, $v_*(x_0) = \overline{u}(x_0)$ and $\overline{u}(x) - \varphi(x)$ attains its minimum at x_0. Therefore we get

$$\begin{aligned}\max\{-a_{ij}(x_0)\varphi_{x_i x_j}(x_0) + b_i(x_0)\varphi_{x_i}(x_0) + c(x_0)\overline{u}(x_0) \\ - f(x_0), \overline{u}(x_0) - M\overline{u}(x_0)\} \geqq 0,\end{aligned}$$

because $\overline{u} \in C(\overline{\Omega})$ is a strong supersolution of (1.1)-(1.2). This is a contradiction.

Thus there exists a $\delta_1 > 0$ such that

$$v_*(x_0) + \delta_1 < \overline{u}(x) \qquad \text{for } x \in B(x_0, \delta_1).$$

Using (A.3), (A.4), (A.5), (A.6), continuity of φ and lower semi-continuity of Mv_*, we have, for $0 < \exists\delta < \min\{r, \delta_1, 1\}/2$,

$$\begin{aligned}\max\left\{-a_{ij}(x)\varphi_{x_i x_j}(x) + b_i(x)\varphi_{x_i}(x) + c(x)\left\{\varphi(x) + \delta^4\right\}\right. \\ \left. -f(x), \varphi(x) + \delta^4 - Mv_*(x)\right\} \leqq 0, \quad \text{for } x \in B(x_0, 2\delta).\end{aligned}$$

Hence $\varphi(x) + \delta^4$ is a subsolution of

$$\max\{Lu - f, u - Mv_*\} = 0 \qquad \text{in} \quad B(x_0, 2\delta).$$

We define

$$w(x) = \begin{cases} \max\left\{\varphi(x) + \delta^4, v(x)\right\} & \text{for } x \in B\left(x_0, \delta\right), \\ v(x) & \text{otherwise.} \end{cases}$$

We note that if $\delta \leqq |x - x_0| \leqq 2\delta$, then $v_*(x) \geqq \varphi(x) + \delta^4$ and by $w^* = v^*$ on $\partial\Omega$ and $Mv^*(x) \leqq Mw^*(x)$ on $\overline{\Omega}$, we get

$$\max\{w^* - g, w^* - Mw^*\} \leqq 0 \qquad \text{on} \quad \partial\Omega.$$

Therefore by the similar argument to the proof of (P.1) we can observe that $w \in \mathcal{S}$. Since $0 = v_*(x_0) - \varphi(x_0) = \lim_{s\to 0} \inf\{v(x) - \varphi(x)|\ |x - x_0| < s,\ x \in \overline{\Omega}\}$, there exists an $\eta \in B(x_0, \delta)$ such that $\varphi(\eta) + \delta^4 > v(\eta)$.

(II) *The case* (4.4). We may assume that $\inf_{\overline{\Omega}} v > -\infty$. Because, if otherwise, there exists a point $y \in \overline{\Omega}$ such that $v(y) < \underline{u}$. Thus $\underline{u}$ is a desired function.

(4.4) implies

$$v_*(x_0) \leqq \min\{g(x_0), Mu_*(x_0)\} - \beta.$$

So we can find $\delta_0 > 0$ such that

$$v_*(x_0) + \frac{1}{2}\beta < \min\{g(x), Mv_*(x)\} \qquad \text{in} \quad B(x_0, \delta_0) \cap \overline{\Omega}.$$

Hence by the barrier argument there exist $0 < \varepsilon_0 < \delta_0$ and $\zeta \in C^2(B(x_0, \varepsilon_0) \cap \Omega) \cap C(B(x_0, \varepsilon_0) \cap \overline{\Omega})$ satisfying

$$\begin{aligned} &\zeta(x_0) = v_*(x_0) + \frac{1}{2}\beta, \\ &\zeta \leqq g && \text{in} \quad B(x_0, \varepsilon_0) \cap \partial\Omega, \\ &\zeta \leqq Mv_* && \text{in} \quad B(x_0, \varepsilon_0) \cap \overline{\Omega}, \\ &L\zeta - f \leqq 0 && \text{in} \quad B(x_0, \varepsilon_0) \cap \Omega, \\ &\zeta < \inf_{\overline{\Omega}} v && \text{on} \quad \partial B(x_0, \varepsilon_0) \cap \overline{\Omega}. \end{aligned}$$

We define

$$w(x) = \begin{cases} \max\{\zeta(x), v(x)\} & \text{for } x \in B(x_0, \varepsilon_0) \cap \overline{\Omega}, \\ v(x) & \text{otherwise.} \end{cases}$$

Noting $v_*(x) \geqq \zeta(x)$ if $|x - x_0| = \varepsilon_1$ for $0 < \exists\varepsilon_1 < \varepsilon_0$, we observe $w \in \mathcal{S}$ as in (I).

By the definition of ζ, we obtain

$$v_*(x_0) + \frac{1}{2}\beta = \zeta(x_0) = w_*(x_0) > v_*(x_0).$$

Thus we can find $y \in \overline{\Omega}$ such that $v(y) < w(y)$. ∎

Now, we can prove Theorem 4.1.

Proof of Theorem 4.1. Let u be as in Lemma 4.2. Then the assertions (P.1) and (P.2) in Proposition 4.3 imply that u is a sub and supersolution in the strong sense. Therefore by Theorem 3.1 we have $u^* \leqq u_*$ on $\overline{\Omega}$. Then combining this inequality with $u_* \leqq u \leqq u^*$, we obtain $u_* = u = u^*$ on $\overline{\Omega}$ and $u \in C(\overline{\Omega})$. Moreover u satisfies the boundary condition (1.2) for each $x \in \partial\Omega$. Using Theorem 3.1 again, we have the uniqueness of strong solutions. Thus the proof is complete. ∎

Remark 4.4. Of course we can extend Theorems 3.1 and 4.1 to Hamilton-Jacobi-Bellman equation with impulse control:

$$\begin{cases} \max\left\{ \sup_{v \in V}\{L^v u - f^v\}, u - Mu \right\} = 0 & \text{in} \quad \Omega, \\ \max\{u - g, u - Mu\} = 0 & \text{on} \quad \partial\Omega. \end{cases}$$

Chapter II

Viscosity solutions of nonlinear elliptic PDEs with nonlocal terms

1. Introduction

In this chapter we are concerned with nonlinear elliptic PDEs with nonlocal terms.

Let $\Omega \subset \mathbb{R}^N$ be a bounded domain. For any function $u : \overline{\Omega} \to \mathbb{R}$, we define the nonlocal operators M and N by

$$Mu(x) = \inf\{k_1(\xi) + u(x+\xi) \mid \xi \geqq 0,\ x+\xi \in \overline{\Omega}\},$$
$$Nu(x) = \sup\{-k_2(\xi) + u(x+\xi) \mid \xi \geqq 0,\ x+\xi \in \overline{\Omega}\},$$

where $k_1(\xi)$ and $k_2(\xi)$ are nonnegative and continuous functions on $(\mathbb{R}^+)^N$ and $\xi \geqq 0$ means $\xi \in (\mathbb{R}^+)^N$.

We consider the following nonlinear PDE:

$$\min\{\max\{Lu - f, u - Mu\}, u - Nu\} = 0 \quad \text{in} \quad \Omega, \tag{1.1}$$

under the implicit boundary condition:

$$\min\{\max\{u - g, u - Mu\}, u - Nu\} = 0 \quad \text{on} \quad \partial\Omega. \tag{1.2}$$

Here the functions f and g are given and the L is a linear second order elliptic operator of the form:

$$Lu = -a_{ij}(x)u_{x_i x_j} + b_i(x)u_{x_i} + c(x)u.$$

Formally the problem (1.1)-(1.2) is derived from impulsive games whose states are goverened by stochastic differential equations with impulsive jumps $\{\xi \geqq 0\}$

and whose value function has impulsive costs k_1 and k_2. If $k_2 \equiv +\infty$, then the equation (1.1) is equivalent to the following PDE of impulse controls for diffusion processes governed by stochastic differential equations:

$$(1.3) \qquad \max\{Lu - f, u - Mu\} = 0 \quad \text{in} \quad \Omega,$$

with the boundary condition:

$$(1.4) \qquad \max\{u - g, u - Mu\} = 0 \quad \text{on} \quad \partial\Omega.$$

For the derivations of the problems (1.1)-(1.2) and (1.3)-(1.4) and the related results, see A. Bensoussan - J. L. Lions [5], K. Ishii - N. Yamada [23], [24], S. M. Lenhart - N. Yamada [33] and Chapter I etc.

As to the problem (1.1)-(1.2), the existence of solutions in $H_0^1(\Omega) \cap L^\infty(\Omega)$ (the case $g \equiv 0$) was proved in [5; Chapter 3, Theorem 8.9] from the viewpoint of quasi-variational inequality. But the uniqueness of solutions was not obtained. In [23], [24] and [33], we have obtained the uniqueness and existence of viscosity solutions for general elliptic PDEs with nonlocal terms. In them, to prove the uniqueness we need some complicated conditions for equations and the one that all nonlocal terms are concave. (See, e.g., (F.4) and (M.3) in [23].) However, since the operator N is convex and the equation (1.1) has nonconvex structure, we cannot apply the perturbation techniques used in them. Our main purpose here is to get the uniqueness and existence of viscosity solutions of (1.1) satisfying (1.2) by modifying the arguments in [23], [24] and [33].

The plan is organized as follows. In Section 2 we state our assumptions and recall the notion of solutions of nonlinear PDEs whose principal part is a general elliptic operator. Then we give the properties of the operators M and N. Section 3 is devoted to the proof of the comparison principle of solutions of (1.1)-(1.2). We use the idea in H. Ishii - S. Koike [18; Section 3] to prove it. In Section 4 we show the existence of solutions of (1.1) satisfying (1.2) by Perron's method. Since we cannot construct viscosity sub- and supersolution of (1.1) satisfying (1.2) directly, we apply the argument in [23; Lemma 5.1] and the comparison principle in Section 3 to verify that the solution u satisfies (1.2).

2. Preliminaries

In this section we shall state our assumptions and shall recall the notion of solutions and the propreties of the operators M and N.

We make the following assumptions.

(A.1) $\Omega \subset \mathbb{R}^N$ is a bounded domain with smooth boundary $\partial\Omega$.

(A.2) There exists a mapping $P : \overline{\Omega} \times (\mathbb{R}^+)^N \to (\mathbb{R}^+)^N$ satisfying

$$
\begin{aligned}
&x + P(x,\xi) \in \overline{\Omega} && \text{for all } (x,\xi) \in \overline{\Omega} \times (\mathbb{R}^+)^N,\\
&P(x,\xi) = \xi && \text{if } x + \xi \in \overline{\Omega},\\
&P(\cdot,\xi) \in C(\overline{\Omega}) && \text{for each } \xi \geqq 0.
\end{aligned}
$$

(A.3) There exist $\sigma_{ij} \in W^{1,\infty}(\overline{\Omega})$ $(i,j = 1,\cdots N)$ such that

$$(a_{ij}(x)) = {}^t(\sigma_{ij}(x))(\sigma_{ij}(x)).$$

(A.4) $b_i(x) \in W^{1,\infty}(\overline{\Omega})$ for $i = 1,\cdots,N$.

(A.5) $c \in C(\overline{\Omega})$ and $c \geqq c_0$ on $\overline{\Omega}$ for some $c_0 > 0$.

(A.6) $f, g \in C(\overline{\Omega})$.

(A.7) $k_i \in C((\mathbb{R}^+)^N)$ and there exists a constant $k_0 > 0$ such that $k_i(\xi) \geqq k_0$ for all $\xi \in (\mathbb{R}^+)^N$. $(i = 1, 2.)$

Remark 2.1. As to the assumption (A.2), see Remark 2.1 in Chapter I.

Next we give the definition of solutions of the following nonlinear PDE:

$$\min\{\max\{F(x,u,Du,D^2u), u - Mu\}, u - Nu\} = 0 \quad \text{in} \quad \Omega, \tag{2.1}$$

where $F \in C(\overline{\Omega} \times \mathbb{R} \times \mathbb{R}^N \times \mathbb{S}^N)$ is a degenerate elliptic operator.

Definition 2.2. *Let $u : \overline{\Omega} \to \mathbb{R}$.*

(1) *We say u is a subsolution of (2.1) provided $u^* < \infty$ on $\overline{\Omega}$ and for any $\varphi \in C^2(\Omega)$, if $u^* - \varphi$ attains a local maximum at $x_0 \in \Omega$, then*

$$\min\{\max\{F(x_0, u^*(x_0), D\varphi(x_0), D^2\varphi(x_0)), u^*(x_0) - Mu^*(x_0)\}, \\ u^*(x_0) - Nu^*(x_0)\} \leqq 0.$$

(2) *We say u is a supersolution of (2.1) provided $u_* > -\infty$ on $\overline{\Omega}$ and for any $\varphi \in C^2(\Omega)$, if $u_* - \varphi$ attains a local minimum at $x_0 \in \Omega$, then*

$$\min\{\max\{F(x_0, u_*(x_0), D\varphi(x_0), D^2\varphi(x_0)), u_*(x_0) - Mu_*(x_0)\}, \\ u_*(x_0) - Nu_*(x_0)\} \geqq 0.$$

(3) *We say u is a solution of (2.1) if u is a subsolution and a supersolution of (2.1).*

We mention the propositions equivalent to Definition 2.2. We do not prove them here because the proofs are similar to those in M. G. Crandall - H. Ishii - P. L. Lions [8].

Proposition 2.3. *Let $u : \overline{\Omega} \to \mathbb{R}$.*

(1) *u is a subsolution of (2.1) if and only if $u^* < \infty$ on $\overline{\Omega}$ and*

$$\min\{\max\{F(x, u^*(x), p, X), u^*(x) - Mu^*(x)\}, u^*(x) - Nu^*(x)\} \leqq 0$$

for all $x \in \Omega$, $(p, X) \in J_{\Omega}^{2,+}u^*(x)$.

(2) *u is a supersolution of (2.1) if and only if $u_* > -\infty$ on $\overline{\Omega}$ and*

$$\min\{\max\{F(x, u_*(x), p, X), u_*(x) - Mu_*(x)\}, u_*(x) - Nu_*(x)\} \geqq 0$$

for all $x \in \Omega$, $(p, X) \in J_{\Omega}^{2,-} u_*(x)$.

Proposition 2.4. *Assume* M, N : $USC(\overline{\Omega})$ *(resp.,* $LSC(\overline{\Omega})$*)* $\to USC(\overline{\Omega})$ *(resp.,* $LSC(\overline{\Omega})$*). Let* $u : \overline{\Omega} \to \mathbb{R}$.

(1) u *is a subsolution of* (2.1) *if and only if* $u^* < \infty$ *on* $\overline{\Omega}$ *and*

$$\min\{\max\{F(x, u^*(x), p, X), u^*(x) - Mu^*(x)\}, u^*(x) - Nu^*(x)\} \leqq 0$$

for all $x \in \Omega$, $(p, X) \in \bar{J}_{\Omega}^{2,+} u^*(x)$.

(2) u *is a supersolution of* (2.1) *if and only if* $u_* > -\infty$ *on* $\overline{\Omega}$ *and*

$$\min\{\max\{F(x, u_*(x), p, X), u_*(x) - Mu_*(x)\}, u_*(x) - Nu_*(x)\} \geqq 0$$

for all $x \in \Omega$, $(p, X) \in \bar{J}_{\Omega}^{2,-} u_*(x)$.

We conclude this section by recalling the properties of M and N.

Proposition 2.5. *Suppose* (A.1), (A.2) *and* (A.7) *hold. Let* $T = M, N$ *and let* u, $v : \overline{\Omega} \to \mathbb{R}$. *Then the following properties hold.*

(1) *If* $u \leqq v$ *on* $\overline{\Omega}$, *then* $Tu \leqq Tv$ *on* $\overline{\Omega}$.

(2) $M(tu + (1-t)v) \geqq tMu + (1-t)Mv$ *and* $N(tu + (1-t)v) \leqq tNu + (1-t)Nv$ *for all* $t \in [0, 1]$.

(3) $T(u + \lambda) = Tu + \lambda$ *for all* $\lambda \in \mathbb{R}$.

(4) $T : USC(\overline{\Omega}) \to USC(\overline{\Omega})$ *and* $T : LSC(\overline{\Omega}) \to LSC(\overline{\Omega})$.

(5) $\|Tu - Tv\|_{C(\overline{\Omega})} \leqq \|u - v\|_{C(\overline{\Omega})}$ *for all* u, $v \in C(\overline{\Omega})$.

We omit the proof of this proposition. See the proof of Proposition 2.7 in Chapter I.

3. Comparison principle of solutions

In this section we establish the comparison principle of solutions of the problem (1.1)-(1.2). For general elliptic PDEs, see M. G. Crandall - H. ishii - P. L. Lions [8] and references therein.

Theorem 3.1. *Assume* (A.1)-(A.7). *Let* u, v *be, respectively, a subsolution and a supersolution of* (1.1). *If* u *and* v *satisfy*

$$\min\{\max\{u^* - g, u^* - Mu^*\}, u^* - Nu^*\} \leqq 0 \quad \text{on} \quad \partial\Omega, \tag{3.1}$$

$$\min\{\max\{v_* - g, v_* - Mv_*\}, v_* - Nv_*\} \geqq 0 \quad \text{on} \quad \partial\Omega, \tag{3.2}$$

then $u^* \leqq v_*$ *on* $\overline{\Omega}$.

To deal with the nonlocal terms $u - Mu$ and $u - Nu$, we need the following lemma.

Lemma 3.2. *Let* $\mathcal{O} \subset \mathbb{R}^N$ *be compact and* $u \in USC(\mathcal{O})$. *Then, for* a.a. $q \in \mathbb{R}^N$, *the function* $u(x) + \langle q, x\rangle$ *takes its strict maximum on* $\mathcal{O}$.

For the proof, see H. Ishii - S. Koike [18; Lemma 3.3].

Proof of Theorem 3.1. We may assume $u \in USC(\overline{\Omega})$ and $v \in LSC(\overline{\Omega})$. We suppose $\sup_{\overline{\Omega}}(u - v) = 3\theta > 0$ and get a contradiction.

It is easily seen from Definition 2.2 and (3.2) that $v \geqq Nv$ on $\overline{\Omega}$. By Lemma 3.2, for a.a.$q \in \mathbb{R}^N$, the function $u(x) - v(x) + \langle q, x\rangle$ on $\overline{\Omega}$ attains its strict maximum at $z (= z_q) \in \overline{\Omega}$. Thus we can take $q \geqq 0$ such that $0 < |q| < \min\{\theta/\gamma, c_0\theta/\|b\|_{C(\overline{\Omega})}\}$ and $\langle q, e_i\rangle \neq 0$ for $1 \leqq i \leqq N$, where $\gamma = \sup_{\overline{\Omega}}|x|$ and $\{e_i\}_{1 \leqq i \leqq N}$ is the standard basis for $\mathbb{R}^N$. We claim

$$u(z) > Nu(z) \qquad \text{and} \qquad v(z) < Mv(z). \tag{3.3}$$

First, suppose $u(z) \leqq Nu(z)$. Then we can find $\xi_z \geqq 0$ such that $\xi_z \neq 0$, $z + \xi_z \in \overline{\Omega}$ and $Nu(z) = -k_2(\xi_z) + u(z + \xi_z)$ by $u \in USC(\overline{\Omega})$ and (A.7). Therefore we have

$$\begin{aligned} u(z) - v(z) + \langle q, z\rangle &\leqq Nu(z) - Nv(z) + \langle q, x\rangle \\ &\leqq u(z + \xi_z) - v(z + \xi_z) + \langle q, z + \xi_z\rangle - \langle q, \xi_z\rangle. \\ &\leqq u(z) - v(z) + \langle q, z\rangle - \langle q, \xi_z\rangle, \end{aligned}$$

which is a contradiction because $\langle q, \xi_z \rangle > 0$. Thus we obtain $u(z) > Nu(z)$. Since u is a subsolution of (1.1) and satisfies (3.1), we get $u(z) \leqq Mu(z)$. Using this fact, we have $v(z) < Mv(z)$ in the similar way to the above argument. Hence we obtain the claim (3.3).

If $z \in \partial\Omega$, then it follows from (3.1), (3.2) and (3.3) that $u(z) \leqq g(z) \leqq v(z)$. Therefore we observe

$$2\theta \leqq u(z) - v(z) + \langle q, z \rangle \leqq |q||z| \leqq \theta,$$

which is a contradiction. Thus we may consider $z \in \Omega$.

For $\alpha > 1$ we define the function $\Phi(x, y)$ on $\overline{\Omega} \times \overline{\Omega}$ by

$$\Phi(x, y) = u(x) - v(y) + \langle q, y \rangle - \frac{\alpha}{2}|x - y|^2.$$

Let $(\bar{x}, \bar{y}) \in \overline{\Omega} \times \overline{\Omega}$ be a maximum point of Φ. By the similar calculation in [18; Section 3], we see the behaviors of $\bar{x}$, $\bar{y}$, $u(\bar{x})$ and $v(\bar{y})$ as $\alpha \to \infty$:

$$\bar{x},\ \bar{y} \to z,\ u(\bar{x}) \to u(z),\ v(\bar{y}) \to v(z),\ \alpha|\bar{x} - \bar{y}|^2 \to 0. \tag{3.4}$$

Then we apply the maximum principle for semicontinuous functions (cf. [8; Theorem 3.2]) to obtain X, $Y \in \mathbb{S}^N$ such that

$$(\alpha(\bar{x} - \bar{y}), X) \in \bar{J}^{2,+}u(\bar{x}),\ (\alpha(\bar{x} - \bar{y}) + q, Y) \in \bar{J}^{2,-}v(\bar{y}), \tag{3.5}$$

and

$$-3\alpha \begin{pmatrix} I & O \\ O & I \end{pmatrix} \leqq \begin{pmatrix} X & O \\ O & -Y \end{pmatrix} \leqq 3\alpha \begin{pmatrix} I & -I \\ -I & I \end{pmatrix}. \tag{3.6}$$

Remarking (3.4), we may assume $\bar{x}$, $\bar{y} \in \Omega$ for large $\alpha > 1$. Using the fact that u, v are, respectively, a subsolution and a supersolution of (1.1), we get the following inequalities:

$$\min\{\max\{-a_{ij}(\bar{x})X_{ij} + b_i(\bar{x})(\alpha(\bar{x}_i - \bar{y}_i)) + c(\bar{x})u(\bar{x}) - f(\bar{x}), u(\bar{x}) - Mu(\bar{x})\}, u(\bar{x}) - Nu(\bar{x})\} \leqq 0, \tag{3.7}$$

$$\min\{\max\{-a_{ij}(\bar{y})Y_{ij} + b_i(\bar{y})(\alpha(\bar{x}_i - \bar{y}_i) + q_i) + c(\bar{y})v(\bar{y}) - f(\bar{y}), v(\bar{y}) - Mv(\bar{y})\}, v(\bar{y}) - Nv(\bar{y})\} \geqq 0. \tag{3.8}$$

It is observed from (3.3), (3.4) and Proposition 2.5 (4) that

$$\liminf_{\alpha\to+\infty}(u(\bar{x}) - Nu(\bar{x})) \geqq u(z) - Nu(z) > 0,$$

$$\limsup_{\alpha\to+\infty}(v(\bar{y}) - Mv(\bar{y})) \leqq v(z) - Mv(z) < 0.$$

Hence there exists $\alpha_0 > 1$ such that

$$u(\bar{x}) - Nu(\bar{x}) > 0 \qquad \text{and} \qquad v(\bar{y}) - Mv(\bar{y}) < 0,$$

for all $\alpha > \alpha_0$. By (3.7), (3.8) and these inequalities, we conclude

$$-a_{ij}(\bar{x})X_{ij} + b_i(\bar{x})(\alpha(\bar{x}_i - \bar{y}_i)) + c(\bar{x})u(\bar{x}) - f(\bar{x}) \leqq 0, \tag{3.9}$$

$$-a_{ij}(\bar{y})Y_{ij} + b_i(\bar{y})(\alpha(\bar{x}_i - \bar{y}_i) + q_i) + c(\bar{y})v(\bar{y}) - f(\bar{y}) \geqq 0. \tag{3.10}$$

Subtracting (3.9) from (3.10) and using (3.6), (A.3)-(A.6), we obtain

$$\begin{aligned}
2c_0\theta &\leqq c(\bar{x})(u(\bar{x}) - u(\bar{y}) + \langle q, \bar{y}\rangle) \\
&\leqq c(\bar{x})u(\bar{x}) - c(\bar{y})u(\bar{y}) - (c(\bar{x}) - c(\bar{y}))u(\bar{y}) + c(\bar{x})\langle q, \bar{y}\rangle \\
&\leqq a_{ij}(\bar{x})X_{ij} - a_{ij}(\bar{y})Y_{ij} - (b_i(\bar{x}) - b_i(\bar{y}))(\alpha(\bar{x}_i - \bar{y}_i)) + b_i(\bar{y})q_i \\
&\qquad - (c(\bar{x}) - c(\bar{y}))u(\bar{y}) + c(\bar{x})\langle q, \bar{y}\rangle + f(\bar{x}) - f(\bar{y}) \\
&\leqq C\alpha|\bar{x} - \bar{y}|^2 + C\omega(|\bar{x} - \bar{y}|) + c_0\theta,
\end{aligned}$$

where ω is a modulus of continuity for c and f and C denotes the various constants depending only on known ones. Letting $\alpha \to +\infty$ and then $|q| \to 0$, we get a contradiction. Thus we have completed the proof. ▮

4. Existence of solutions

In this section we show the existence of solutions of (1.1) satisfying (1.2) by Perron's method. In addition to (A.1)-(A.7), we assume

(A.8) $b_i(x)\nu_i(x) < 0$ on $\{x \in \partial\Omega \mid a_{ij}(x)\nu_i(x)\nu_j(x) = 0\}$.

Then, as seen in Theorem 4.1 in Chapter I, there exists a unique solution $\underline{u} \in C(\overline{\Omega})$ of

$$\begin{cases} \max\{Lu - f, u - Mu\} = 0 & \text{in } \Omega, \\ \max\{u - g, u - Mu\} = 0 & \text{on } \partial\Omega, \end{cases}$$

which is a subsolution of (1.1) satisfying $\min\{\max\{\underline{u} - g, \underline{u} - M\underline{u}\}, \underline{u} - N\underline{u}\} \leqq 0$ on $\partial\Omega$. Furthermore, there exists a unique solution $\overline{u} \in C(\overline{\Omega})$ of

$$\begin{cases} \min\{Lu - f, u - Nu\} = 0 & \text{in } \Omega, \\ \min\{u - g, u - Nu\} = 0 & \text{on } \partial\Omega, \end{cases}$$

which is a supersolution of (1.1) satisfying $\min\{\max\{\overline{u} - g, \overline{u} - M\overline{u}\}, \overline{u} - N\overline{u}\} \geqq 0$ on $\partial\Omega$. (Note the existence of $\overline{u}$ can be proved simlarly to the proofs of Lemmas 4.2-4.4 below.) By using these functions we obtain the following theorem.

Theorem 4.1. *Assume* (A.1)-(A.8). *Then there exists a unique solution u of* (1.1) *satisfying*

$$\begin{cases} \min\{\max\{u^* - g, u^* - Mu^*\}, u^* - Nu^*\} \leqq 0, \\ \min\{\max\{u_* - g, u_* - Mu_*\}, u_* - Nu_*\} \geqq 0, \end{cases} \tag{4.1}$$

on $\partial\Omega$. Moreover the solution u is continuous on $\overline{\Omega}$ and satisfies the boundary condition (1.2).

We put

$$\mathcal{S} = \{v \ : \ \text{subsolution of (1.1)} \mid$$
$$\min\{\max\{v^* - g, v^* - Mv^*\}, v^* - Nv^*\} \leqq 0 \text{ on } \partial\Omega\},$$
$$u(x) = \sup\{v(x) \mid v \in \mathcal{S}\} \quad (x \in \overline{\Omega}).$$

We note $\mathcal{S} \neq \emptyset$ and $\underline{u} \leqq u \leqq \overline{u}$ on $\overline{\Omega}$ since $\underline{u} \in \mathcal{S}$ and Theorem 3.1 holds. Perron's method can be divided into the following lemmas. In what follows we always assume (A.1)-(A.8).

Lemma 4.2. $u \in \mathcal{S}$.

Lemma 4.3. *If $v \in \mathcal{S}$ is not a supersolution of* (1.1), *then there exist $w \in \mathcal{S}$ and $y \in \Omega$ such that $v(y) < w(y)$.*

By Lemma 4.2 and 4.3 we can easily see that the above function u is a solution of (1.1) satisfying $\min\{\max\{u^* - g, Mu^*\}, u^* - Nu^*\} \leqq 0$ on $\partial\Omega$. We need the next lemma to show that u is a unique solution and satisfies (1.2)

Lemma 4.4. $\min\{\max\{u_* - g, u_* - Mu_*\}, u_* - Nu_*\} \geqq 0$ *on* $\partial\Omega$.

First we admit Lemma 4.2-4.4 hold and show Theorem 4.1. After this, we prove them.

Proof of Theorem 4.1. Let u be the function defined above. From Lemma 4.2-4.4 it follows that u is a solution of (1.1) satisfying (4.1). Let v be any solution of (1.1) satisfying (4.1) with v in place of u. It follows from Theorem 3.1 that $u^* \leqq v_* \leqq v \leqq v^* \leqq u_* \leqq u \leqq u^*$ on $\overline{\Omega}$. Hence $u \equiv v \in C(\overline{\Omega})$. Combining this with (4.1), we see that u satisfies (1.2). The proof is complete. ∎

Proof of Lemma 4.2. Fix $x \in \Omega$ and $(p, X) \in J^{2,+}_{\Omega} u(x)$. From the definition of u^* there exists a sequence $\{x_n\}_{n \in \mathbf{N}} \subset \Omega$ such that

$$x_n \to x \quad \text{and} \quad u(x_n) \to u^*(x) \quad (n \to +\infty). \tag{4.2}$$

Moreover it is observed that, for each $n \in \mathbb{N}$, we can find $u_n \in \mathcal{S}$ such that

$$u(x_n) - \frac{1}{n} < u_n(x_n). \tag{4.3}$$

By (4.2) and (4.3) we get

$$u_n^*(x_n) \to u^*(x) \qquad (n \to +\infty). \tag{4.4}$$

From (4.2), (4.4) and M. G. Crandall - H. Ishii - P. L. Lions [8; Proposition 4.3] there exist $\{\hat{x}_n\}_{n\in\mathbb{N}} \subset \Omega$ and $(p_n, X_n) \in J_\Omega^{2,+}u_n^*(\hat{x}_n)$ satisfying

$$(\hat{x}_n, u_n^*(\hat{x}_n), p_n, X_n) \to (x, u^*(x), p, X) \quad (n \to +\infty). \tag{4.5}$$

Since $u_n \in \mathcal{S}$, we have

$$\min\{\max\{-a_{ij}(\hat{x}_n)X_{n,ij} + b_i(\hat{x}_n)p_{n,i} + c(\hat{x}_n)u_n^*(\hat{x}_n) - f(\hat{x}_n),$$
$$u_n^*(\hat{x}_n) - Mu_n^*(\hat{x}_n)\}, u_n^*(\hat{x}_n) - Nu_n^*(\hat{x}_n)\} \leqq 0.$$

In the case $u_n^*(\hat{x}_n) - Nu_n^*(\hat{x}_n) \leqq 0$, it follows from (4.5) and Proposition 2.5 (1), (4) that

$$u^*(x) - Nu^*(x) \leqq 0.$$

In the another case we observe

$$\max\{-a_{ij}(x)X_{ij} + b_i(x)p_i + c(x)u^*(x) - f(x), u^*(x) - Mu^*(x)\} \leqq 0$$

by (A.3)-(A.6) and Propsotion 2.5 (1), (4). Hence u is a subsolution of (1.1).

Next we prove $\min\{\max\{u^* - g, u^* - Mu^*\}, u^* - Nu^*\} \leqq 0$ on $\partial\Omega$. In order to do so, suppose to the contrary, i.e., $\exists x_0 \in \partial\Omega$ such that

$$\min\{\max\{u^*(x_0) - g(x_0), u^*(x_0) - Mu^*(x_0)\}, u^*(x_0) - Nu^*(x_0)\} = 2\beta > 0. \tag{4.6}$$

Then we can take the sequences $\{x_n\}_{n\in\mathbb{N}} \subset \overline{\Omega}$ and $\{u_n\}_{n\in\mathbb{N}} \subset \mathcal{S}$ satisfying (4.2) with x_0 in place of x. Suppose there exists a sequence $\{n_k\} \subset \mathbb{N}$ such that $n_k \to +\infty$ as $k \to +\infty$ and $x_{n_k} \in \partial\Omega$. Since $u_{n_k} \in \mathcal{S}$, we have

$$\min\{\max\{u^*_{n_k}(x_{n_k})-g(x_{n_k}), u^*_{n_k}(x_{n_k})-Mu^*_{n_k}(x_{n_k})\}, u^*_{n_k}(x_{n_k})-Nu^*_{n_k}(x_{n_k})\} \leqq 0.$$

Letting $k \to +\infty$, we obtain a contradiction to (4.6). Therefore we may consider $x_n \in \Omega$ for all $n \geqq 1$. By (4.2), (4.6) and Proposition 2.5 (4), we have

$$u^*_n(x_n) - Nu^*_n(x_n) > 0 \quad \text{for all } n \gg 1.$$

Using $u_n \in \mathcal{S}$, we get

$$\min\{\max\{-a_{ij}(x_n)X_{ij} + b_i(x_n)p_i + c(x_n)u^*_n(x_n) - f(x_n),$$
$$u^*_n(x_n) - Mu^*_n(x_n)\}, u^*_n(x_n) - Nu^*_n(x_n)\} \leqq 0$$

for all $(p, X) \in J^{2,+}_\Omega u^*_n(x_n)$. By this inequality we have $u^*_n(x_n) - Mu^*(x_n) \leqq 0$. Thus we obtain $u^*(x_0) - Mu^*(x_0) \leqq 0$. Consequently it is observed that

$$\min\{u^*(x_0) - g(x_0), u^*(x_0) - Nu^*(x_0)\} = 2\beta.$$

Then there exists a $\delta > 0$ such that

$$u^*(x_0) - \beta \geqq g(x) \qquad \text{for } x \in \overline{B(x_0,\delta)} \cap \partial\Omega,$$
$$u^*(x_0) - \beta \geqq Nu^*(x) \qquad \text{for } x \in \overline{B(x_0,\delta)} \cap \overline{\Omega}.$$

Furthermore, by using $u_n \in \mathcal{S}$, we observe that u_n is a subsolution of

$$\min\{Lu - f, u - (u^*(x_0) - \beta)\} = 0 \qquad \text{in} \quad B(x_0,\delta) \cap \Omega, \tag{4.7}$$

and satisfies

$$u^* \leqq u^*(x_0) - \beta \qquad \text{on} \quad \overline{B(x_0,\delta)} \cap \partial\Omega,$$
$$u^*_n \leqq \overline{u} \qquad \text{on} \quad \overline{\Omega}.$$

On the other hand, applying the barrier construction argument in A. O. Oleinik - E. V. Radkevic [37], there exist $\varepsilon_0 \in (0,\delta)$ and $\zeta \in C^2(B(x_0,\varepsilon_0) \cap \Omega) \cap C(\overline{B(x_0,\varepsilon_0)} \cap \overline{\Omega})$ satisfying

$$
\begin{aligned}
&\zeta(x_0) = u^*(x_0) - \beta, && \\
&\zeta \geqq u^*(x_0) - \beta && \text{in} \quad \overline{B(x_0,\varepsilon_0)} \cap \overline{\Omega}, \\
&L\zeta - f \geqq 0 && \text{in} \quad B(x_0,\varepsilon_0) \cap \Omega, \\
&\zeta > \sup_{\overline{\Omega}} \overline{u} && \text{on} \quad \partial B(x_0,\varepsilon_0) \cap \overline{\Omega}.
\end{aligned}
$$

We note ζ is a supersolution of (4.7) in $B(x_0,\varepsilon_0) \cap \Omega$. Thus it follows from the standard comparison argument that $u_n^* \leqq \zeta$ on $\overline{B(x_0,\varepsilon_0)} \cap \overline{\Omega}$ for all $n \geqq 1$. Since $x_n \in B(x_0,\varepsilon_0) \cap \Omega$ for large $n \in \mathbb{N}$ by (4.2), we obtain $u_n^*(x_n) \leqq \zeta(x_n)$. Sending $n \to +\infty$, we get

$$u^*(x_0) \leqq \zeta(x_0) = u^*(x_0) - \beta,$$

which is a contradiction. Hence we conclude $\min\{\max\{u^* - g, u^* - Mu^*\}, u^* - Nu^*\} \leqq 0$ on $\partial\Omega$ and $u \in \mathcal{S}$. ∎

Proof of Lemma 4.3. We suppose that $v \in \mathcal{S}$ is not a supersolution of (1.1). Then there exist $x_0 \in \Omega$ and $(p,X) \in J_{\Omega}^{2,-} v_*(x_0)$ such that

$$
\begin{aligned}
\min\{\max\{&-a_{ij}(x_0)X_{ij} + b_i(x_0)p_i + c(x_0)v_*(x_0) \\
&-f(x_0), v_*(x_0) - Mv_*(x_0)\}, v_*(x_0) - Nv_*(x_0)\} = -\beta < 0.
\end{aligned}
\tag{4.8}
$$

We claim $v_*(x_0) < \overline{u}(x_0)$. If not, $v_*(x_0) = \overline{u}(x_0)$ and $(p,X) \in J_{\Omega}^{2,-}\overline{u}(x_0)$. Since $\overline{u}$ is a supersolution of (1.1), we have

$$
\begin{aligned}
\min\{\max\{&-a_{ij}(x_0)X_{ij} + b_i(x_0)p_i + c(x_0)\overline{u}(x_0) \\
&-f(x_0), \overline{u}(x_0) - M\overline{u}(x_0)\}, \overline{u}(x_0) - N\overline{u}(x_0)\} \geqq 0.
\end{aligned}
\tag{4.9}
$$

Using $v_*(x_0) = \overline{u}(x_0)$, we obtain that (4.9) contradicts to (4.8). Hence we get the claim.

We take $\varphi \in C^2(\Omega)$ satisfying

$$\varphi(x_0) = v_*(x_0),\ D\varphi(x_0) = p,\ D^2\varphi(x_0) = X,$$
$$v_*(x) \geqq \varphi(x) + |x - x_0|^4 \quad \text{in} \quad B(x_0, r)(\subset\subset \Omega) \quad \text{for some } r > 0.$$

Noting (4.8) and Proposition 2.5 (4), there exists a $\delta > 0$ such that $\delta < \min(r, \beta)/2$ and

$$\min\{\max\{L\tilde{\varphi} - f, \tilde{\varphi} - Mv_*\}, \tilde{\varphi} - Nv_*\} \leqq 0 \quad \text{in} \quad B(x_0, 2\delta),$$

where $\tilde{\varphi}(x) = \varphi(x) + \delta^4$. Therefore $\tilde{\varphi}$ is a subsolution of

$$\min\{\max\{Lu - f, u - Mv^*\}, u - Nv^*\} \leqq 0 \quad \text{in} \quad B(x_0, 2\delta).$$

We define the function w by

$$w(x) = \begin{cases} \max\{\tilde{\varphi}(x), v(x)\} & \text{for } x \in B(x_0, \delta), \\ v(x) & \text{otherwise.} \end{cases}$$

We notice that $v_*(x) \geqq \tilde{\varphi}(x)$ if $|x - x_0| \geqq \delta$, which implies $w = \max\{\tilde{\varphi}, v\}$ in $B(x_0, 2\delta)$. Moreover we get $Mv^* \leqq Mw^*$ and $Nv^* \leqq Nw^*$ on $\overline{\Omega}$ because of Proposition 2.5 (1). Using the fact $w^* = v^*$ on $\partial\Omega$, we have $\min\{\max\{w^* - g, w^* - Mw^*\}, w^* - Nw^*\} \leqq 0$ on $\partial\Omega$. Thus by means of the similar argument to the proof of Lemma 4.2, we can show $w \in \mathcal{S}$. Since $w_*(x_0) = \tilde{\varphi}(x_0)$ by the definition of w, we can find $y \in \Omega$ such that $v(y) < \tilde{\varphi}(y) = w(y)$. We have completed the proof. ∎

Proof of Lemma 4.4. We prove the assertion by the similar argument to the proof of Lemma 4.2. We suppose that

$$\min\{\max\{u_*(x_0) - g(x_0), u_*(x_0) - Mu_*(x_0)\}, u_*(x_0) - Nu_*(x_0)\} = -2\beta < 0$$

for some $x_0 \in \partial\Omega$.

First we consider the case $\max\{u_*(x_0) - g(x_0), u_*(x_0) - Mu_*(x_0)\} = -2\beta$. By the barrier construction argument there exist $\varepsilon_0 > 0$ and $\zeta \in C^2(B(x_0, \varepsilon_0) \cap \Omega) \cap C(\overline{B(x_0, \varepsilon_0) \cap \Omega})$ satisfying

$$
\begin{aligned}
&\zeta(x_0) = u^*(x_0) + \beta, && \\
&\zeta \leqq g && \text{in } \ B(x_0, \varepsilon_0) \cap \partial\Omega, \\
&\zeta \leqq Mu_* && \text{in } \ B(x_0, \varepsilon_0) \cap \overline{\Omega}, \\
&L\zeta - f \leqq 0 && \text{in } \ B(x_0, \varepsilon_0) \cap \Omega, \\
&\zeta < \inf_{\overline{\Omega}} u && \text{on } \ \partial B(x_0, \varepsilon_0) \cap \overline{\Omega}.
\end{aligned}
$$

(See the proof of Proposition 4.3 in Chapter I.)

We define

$$
w(x) = \begin{cases} \max\{\zeta(x), u(x)\} & \text{for } x \in B(x_0, \varepsilon_0) \cap \overline{\Omega}, \\ u(x) & \text{otherwise.} \end{cases}
$$

Noting $\zeta(x) < u_*(x)$ if $|x - x_0| \geqq r$ for some $r \in (0, \varepsilon_0)$, we can show $w \in S$ by the same argument as in the proof of Lemma 4.2. Since $w_*(x_0) = u_*(x_0) + \beta$, we can find $y \in B(x_0, \varepsilon_0) \cap \overline{\Omega}$ such that $u(y) < w(y)$. This is a contradiction.

In the case $u_*(x_0) - Nu_*(x_0) = -2\beta$, we easily have a contradiction. Hence we obtain the result. ∎

Chapter III

Viscosity solutions of nonlinear second order elliptic PDEs associated with impulse control problems II

§1. Introduction

This chapter is concerned with the uniqueness and existence of viscosity solutions of nonlinear second order elliptic PDEs with implicit obstacles.

Let $\Omega \subset \mathbb{R}^N$ be a bounded domain. For any function $u : \overline{\Omega} \to \mathbb{R}$, we define the operator M as the following:

$$Mu(x) = \inf_{\substack{\xi \geqq 0 \\ x+\xi \in \overline{\Omega}}} \{k(\xi) + u(x+\xi)\},$$

where $k(\xi)$ is a nonnegative and continuous function on $(\mathbb{R}^+)^N$ and $\xi \geqq 0$ means $\xi \in (\mathbb{R}^+)^N$. We consider the nonlinear elliptic PDEs of the form:

$$(1.1) \qquad \begin{cases} \max\{F(x, u, Du, D^2u), u - Mu\} = 0 & \text{in } \Omega, \\ \max\{u - g, u - Mu\} = 0 & \text{on } \partial\Omega. \end{cases}$$

Here the g is a given function and the F is a second order degenerate elliptic operator. The problem (1.1) is associated with the impulse control problems for certain diffusion processes. For the formal derivation of (1.1) and some results on the impulse control problems, see A. Bensoussan - J. L. Lions [5], J. L. Menaldi [36], B. Perthame [40] and G. Barles [1] etc.

In the case where F is nondegenerate, we can interpret the boundary condition in (1.1) in the "classical" sense. When F is linear and $g \equiv 0$ on $\partial\Omega$, the existence and uniqueness of solutions of (1.1) in $H_0^1(\Omega) \cap C(\overline{\Omega})$ is discussed from the viewpoint of quasi-variational inequality in [5]. B. Perthame [38] obtained the existence and uniqueness of solutions of (1.1) in $W^{2,\infty}_{loc}(\Omega) \cap C(\overline{\Omega})$ under some compatibility conditions on g and Mu. After introducing the notion of viscosity solutions, B. Perthame [39] and Chapter I showed the uniqueness and existence of solutions of (1.1).

However, in the case F is degenerate (especially on $\partial\Omega$), we cannot interpret the boundary condition in the classical sense. H. Ishii [15] pointed out that in the degenerate case we should interpret the boundary condition in the "viscosity" sense and proved the comparison principle and existence of solutions of first order Hamilton-Jacobi equations by analytical methods. (Also see M. G. Crandall - H. Ishii - P. L. Lions [8] and references therein.) In order to get the comparison principle he assumed the continuity of sub- and supersolutions near $\partial\Omega$. Recently M. A. Katsoulakis [26] and [27] have obtained the comparison principle of solutions of nonlinear second order degenerate elliptic PDEs. To show the comparison principle he has assumed the nontangential semicontinuity of sub- and supersolutions, which is a weaker assumption than that in [15]. Moreover in [26] and [27] he has established the existence of such solutions by probabilistic arguments. As to the systems of elliptic PDEs, see S. Koike [29] and M. A. Katsoulakis - S. Koike [28].

Our main purpose here is to get the comparison principle and existence of solutions of the problem (1.1). Since we deal with the case where F is degenerate on $\partial\Omega$, we consider the boundary condition in the viscosity sense.

This chapter is organized in the following way. In Section 2 we give the definition of solutions of (1.1) and the equivalent propositions. In Section 3 we prove the comparison principle of solutions of (1.1). We remark that its proof is improved as compared with that of Theorem 3.1 in Chapter I. Sections 4 and 5 provide the existence of continuous solutions of (1.1). Since it is difficult to discuss it for general elliptic operators, we consider only the case F is the Hamilton-Jacobi-Bellman operator in these sections. In Section 4 we apply the iterative approximation scheme by B. Hanouzet - J. L. Joly [14] to obtain the existence result, assuming the existence of continuous solutions of the usual obstacle problems. In Section 5 we show it by using the results in [27]. In Section 6 we prove that the unique solution of (1.1) obtained in Section 4 can be represented as the optimal cost function associated with the impulse control problem. In Section 7 we treat the boundary value problem of oblique type involving the operator M. For the related problems, see P. L. Lions - B. Perthame [35], P. Dupuis - H. Ishii [10], [11] and H. Ishii [17].

§2. Definitions of solutions

In this section we shall give the definitions of solutions of (1.1) and the equivalent propositions. We set

$$G^*(x,r,p,X,m) = \begin{cases} \max\{F(x,r,p,X), r-m\} & (x \in \Omega), \\ \max\{\max\{F(x,r,p,X), r-m\}, \\ \qquad \max\{r-g(x), r-m\}\} & (x \in \partial\Omega), \end{cases}$$

$$G_*(x,r,p,X,m) = \begin{cases} \max\{F(x,r,p,X), r-m\} & (x \in \Omega), \\ \min\{\max\{F(x,r,p,X), r-m\}, \\ \qquad \max\{r-g(x), r-m\}\} & (x \in \partial\Omega), \end{cases}$$

where $F \in C(\overline{\Omega} \times \mathbb{R} \times \mathbb{R}^N \times \mathbb{S}^N)$ is a degenerate elliptic operator.

Definition 2.1. *Let* $u : \overline{\Omega} \to \mathbb{R}$.

(1) *We say u is a subsolution of* (1.1) *provided* $u^* < +\infty$ *on* $\overline{\Omega}$ *and for any* $\varphi \in C^2(\overline{\Omega})$, *if* $u^* - \varphi$ *attains a local maximum at* $x \in \overline{\Omega}$, *then*

$$G_*(x, u^*(x), D\varphi(x), D^2\varphi(x), Mu^*(x)) \leqq 0.$$

(2) *We say u is a supersolution of* (1.1) *provided* $u_* > -\infty$ *on* $\overline{\Omega}$ *and for any* $\varphi \in C^2(\overline{\Omega})$, *if* $u_* - \varphi$ *attains a local minimum at* $x \in \overline{\Omega}$, *then*

$$G^*(x, u_*(x), D\varphi(x), D^2\varphi(x), Mu_*(x)) \geqq 0.$$

(3) *We say* u *is a solution of* (1.1) *provided* u *is both a sub- and a supersolution of* (1.1).

Next we state the equivalent propositions of Definition 2.1. We refer the reader to M. G. Crandall - H. Ishii - P. L. Lions [8; Section 7] for general elliptic PDEs.

Proposition 2.2. *Let* $u : \overline{\Omega} \to \mathbb{R}$.

(1) u *is a subsolution of* (1.1) *if and only if* $u^* < +\infty$ *on* $\overline{\Omega}$ *and for all* $x \in \overline{\Omega}$ *and* $(p, X) \in J^{2,+}_{\overline{\Omega}} u^*(x)$, u^* *satisfies*

$$G_*(x, u^*(x), p, X, Mu^*(x)) \leqq 0.$$

(2) u *is a supersolution of* (1.1) *if and only if* $u_* > -\infty$ *on* $\overline{\Omega}$ *and for all* $x \in \overline{\Omega}$ *and* $(p, X) \in J^{2,-}_{\overline{\Omega}} u_*(x)$, u_* *satisfies*

$$G^*(x, u_*(x), p, X, Mu_*(x)) \geqq 0.$$

We note that, when $F \in C(\overline{\Omega} \times \mathbb{R} \times \mathbb{R}^N \times \mathbb{S}^N)$ and $g \in C(\overline{\Omega})$, G^* (resp., G_*) is the u.s.c. (resp., l.s.c.) envelope of the function G:

$$G(x, r, p, X, m) = \begin{cases} \max\{F(x, r, p, X), r - m\} & (x \in \Omega), \\ \max\{r - g(x), r - m\} & (x \in \partial\Omega). \end{cases}$$

Proposition 2.3. *Assume* $M : USC(\overline{\Omega}) \to USC(\overline{\Omega})$ *and* $M : LSC(\overline{\Omega}) \to LSC(\overline{\Omega})$. *Let* $u : \overline{\Omega} \to \mathbb{R}$.

(1) u *is a subsolution of* (1.1) *if and only if* $u^* < +\infty$ *on* $\overline{\Omega}$ *and for all* $x \in \overline{\Omega}$ *and* $(p, X) \in \overline{J}^{2,+}_{\overline{\Omega}} u^*(x)$, u^* *satisfies*

$$G_*(x, u^*(x), p, X, Mu^*(x)) \leqq 0.$$

(2) u *is a supersolution of* (1.1) *if and only if* $u_* > -\infty$ *on* $\overline{\Omega}$ *and for all* $x \in \overline{\Omega}$ *and* $(p, X) \in \overline{J}^{2,-}_{\overline{\Omega}} u_*(x)$, u_* *satisfies*

$$G^*(x, u_*(x), p, X, Mu_*(x)) \geqq 0.$$

Since the proofs of the above propositions are similar to those in [8], we leave them to the reader.

Finally, we state the definition and the equivalent propositions for the usual obstacle problems we treat in Sections 4 and 5.

$$(2.1) \qquad \begin{cases} \max\{F(x,u,Du,D^2u), u-\psi\} = 0 & \text{in } \Omega, \\ \max\{u-g, u-\psi\} = 0 & \text{on } \partial\Omega. \end{cases}$$

Definition 2.4. *Let $u : \overline{\Omega} \to \mathbb{R}$.*

(1) *We say u is a subsolution of (2.1) provided $u^* < +\infty$ on $\overline{\Omega}$ and for any $\varphi \in C^2(\overline{\Omega})$, if $u^* - \varphi$ attains a local maximum at $x \in \overline{\Omega}$, then*

$$G_*(x, u^*(x), D\varphi(x), D^2\varphi(x), \psi^*(x)) \leqq 0.$$

(2) *We say u is a supersolution of (2.1) provided $u_* > -\infty$ on $\overline{\Omega}$ and for any $\varphi \in C^2(\overline{\Omega})$, if $u_* - \varphi$ attains a local minimum at $x \in \overline{\Omega}$, then*

$$G^*(x, u_*(x), D\varphi(x), D^2\varphi(x), \psi_*(x)) \geqq 0.$$

(3) *We say u is a solution of (2.1) provided u is both a sub- and a supersolution of (2.1).*

Proposition 2.5. *Let $u : \overline{\Omega} \to \mathbb{R}$.*

(1) *u is a subsolution of (2.1) if and only if $u^* < +\infty$ on $\overline{\Omega}$ and for all $x \in \overline{\Omega}$ and $(p, X) \in J^{2,+}_{\overline{\Omega}} u^*(x)$, u^* satisfies*

$$G_*(x, u^*(x), p, X, \psi^*(x)) \leqq 0.$$

(2) *u is a supersolution of (2.1) if and only if $u_* > -\infty$ on $\overline{\Omega}$ and for all $x \in \overline{\Omega}$ and $(p, X) \in J^{2,-}_{\overline{\Omega}} u_*(x)$, u_* satisfies*

$$G^*(x, u_*(x), p, X, \psi_*(x)) \geqq 0.$$

Proposition 2.6. *Assume $\psi \in C(\overline{\Omega})$. Let $u : \overline{\Omega} \to \mathbb{R}$.*

(1) *u is a subsolution of (2.1) if and only if $u^* < +\infty$ on $\overline{\Omega}$ and for all $x \in \overline{\Omega}$ and $(p, X) \in \bar{J}^{2,+}_{\overline{\Omega}} u^*(x)$, u^* satisfies*

$$G_*(x, u^*(x), p, X, \psi(x)) \leqq 0.$$

(2) *u is a supersolution of (2.1) if and only if $u_* > -\infty$ on $\overline{\Omega}$ and for all $x \in \overline{\Omega}$ and $(p, X) \in \bar{J}^{2,-}_{\overline{\Omega}} u_*(x)$, u_* satisfies*

$$G^*(x, u_*(x), p, X, \psi(x)) \geqq 0.$$

We omit the proofs of the above propositions. See [8; Section 7].

§3. Comparison principle of solutions

In this section we shall prove the comparison principle of solutions of the problem (1.1). To do so, we use the similar techniques to those in H. M. Soner [41], H. Ishii [15] and M. A. Katsoulakis - S. Koike [28].

We make the following assumptions.

(A.1) $\Omega \subset \mathbb{R}^N$ is a bounded domain.

(A.2) There exist constants r, s, $t > 0$ and a mapping $n \in C(\overline{\Omega} : \mathbb{R}^N)$ with $|n| = 1$ on $\partial\Omega$ such that

$$z + K(r, s, n(z)) \subset \Omega \quad \text{for all } z \in \partial\Omega,$$
$$y + K\left(r, t, \frac{x}{|x|}\right) \subset \Omega \quad \text{for all } x \in K(r, s, n(z)),\ y \in B(z, r) \cap \overline{\Omega}.$$

(A.3) There exists a mapping $P : \overline{\Omega} \times (\mathbb{R}^+)^N \to (\mathbb{R}^+)^N$ satisfying

$$x + P(x, \xi) \in \overline{\Omega} \qquad \text{for all } (x, \xi) \in \overline{\Omega} \times (\mathbb{R}^+)^N,$$
$$P(x, \xi) = \xi \qquad \text{if } x + \xi \in \overline{\Omega},$$
$$P(\cdot, \xi) \in C(\overline{\Omega}) \qquad \text{for each } \xi \geqq 0.$$

(A.4) $F \in C(\overline{\Omega} \times \mathbb{R} \times \mathbb{R}^N \times \mathbb{S}^N)$.

(A.5) There exists a function $\omega_1 \in C(\mathbb{R}^+)$ such that $\omega_1(0) = 0$ for which

$$F(y,r,p,Y) - F(x,r,p,X) \leqq \omega_1(\alpha|x-y|^2 + |x-y|(|p|+1))$$
$$\text{if } -3\alpha \begin{pmatrix} I & O \\ O & I \end{pmatrix} \leqq \begin{pmatrix} X & O \\ O & -Y \end{pmatrix} \leqq 3\alpha \begin{pmatrix} I & -I \\ -I & I \end{pmatrix}$$

for all $x, y \in \overline{\Omega}$, $p \in \mathbb{R}^N$, $\alpha > 1$ and X, $Y \in \mathbb{S}^N$.

(A.6) There exists a function $\omega_2 \in C(\mathbb{R}^+)$ such that $\omega_2(0) = 0$ for which

$$|F(x,r,p,X) - F(x,r,q,X)| \leqq \omega_2(|p-q|)$$

for all $x \in \overline{\Omega}$, $r \in \mathbb{R}$, $p, q \in \mathbb{R}^N$ and $X \in \mathbb{S}^N$.

(A.7) There exists a constant $\lambda > 0$ such that

$$F(x,r,p,X) - F(x,s,p,X) \leqq \lambda(r-s) \quad \text{if } r \leqq s$$

for all $x \in \overline{\Omega}$, $r, s \in \mathbb{R}$, $p \in \mathbb{R}^N$, $X \in \mathbb{S}^N$.

(A.8) $k \in C((\mathbb{R}^+)^N)$ and there exists a constant $k_0 > 0$ such that $k(\xi) \geqq k_0$ for all $\xi \in (\mathbb{R}^+)^N$.

(A.9) $g \in C(\overline{\Omega})$.

Remark 3.1. (1) When $\partial\Omega$ is of class C^2, we take $r = s = t > 0$ sufficiently small and $n \in C(\overline{\Omega} : \mathbb{R}^N)$ such that $n(x)$ is the inner normal to Ω at $x \in \partial\Omega$. Then it is easily verified that (A.2) is satisfied.

(2) As to the assumption (A.3), see Remark 2.1 in Chapter I.

(3) If (A.6) holds, then the operator F is degenerate elliptic. (cf. [8; Remark 3.4].)

(4) A typical example of F satisfying (A.4)-(A.7) is the Hamilton-Jacobi-Bellman operator treated in Sections 4 and 5.

We notice that if (A.1), (A.3) and (A.8) hold, then Proposition 2.7 in Chapter I holds. The comparison principle of solutions of (1.1) is stated as follows.

Theorem 3.2. *Assume* (A.1)-(A.9) *hold. Let u and v, respectively, be a subsolution and a supersolution of* (1.1). *For each $z \in \partial\Omega$, let $K_z = z + K(r,s,n(z))$. If any one of the followings holds, then $u^* \leqq v_*$ on $\overline{\Omega}$.*

(1) $\limsup_{K_z \ni x \to z} u^*(x) = u^*(z)$ *and* $\liminf_{K_z \ni x \to z} v_*(x) = v_*(z)$ *for each* $z \in \partial\Omega$.

(2) $\limsup_{K_z \ni x \to z} u^*(x) = u^*(z)$ *and* $u^*(z) \leqq g(z)$ *for each* $z \in \partial\Omega$.

(3) $\liminf_{K_z \ni x \to z} v_*(x) = v_*(z)$ *and* $v_*(z) \geqq g(z)$ *for each* $z \in \partial\Omega$.

Remark 3.3. We call the properties in Theorem 3.3 (1) nontangential upper- and lower semicomtinuity, respectively. See M. A. Katoulakis [26], [27] and [28].

We need the following lemma to deal with the term $u - Mu$.

Lemma 3.4. *Let $\mathcal{O} \subset \mathbb{R}^N$ be compact and $u \in USC(\mathcal{O})$. Then, for a.a. $q \in \mathbb{R}^N$, the function $u(x) + \langle q, x\rangle$ takes its strict maximum on $\mathcal{O}$.*

For the proof, see H. Ishii - S. Koike [18; Lemma 3.3].

Proof of Theorem 3.2. We may assume $u \in USC(\overline{\Omega})$ and $v \in LSC(\overline{\Omega})$. We easily observe that $u \leqq Mu$ on $\overline{\Omega}$. First let the condition (1) hold.

We suppose $\sup_{x\in\overline{\Omega}}(u - v) = 5\theta > 0$ and shall get a contradiction. Let $L = \sup_{x\in\overline{\Omega}} |x|$ and let $\{e_i\}_{1\leqq i\leqq N}$ be the standard basis for $\mathbb{R}^N$. We take $q \geqq 0$ such that

$$0 < |q| \leqq \theta/L, \quad 0 \leqq \omega_2(|q|) \leqq \lambda\theta, \tag{3.1}$$

$$\langle q, e_i\rangle > 0 \quad \text{for each } i = 1, \cdots, N \tag{3.2}$$

and fix it. Then by Lemma 3.4 the function $u(x) - v(x) + \langle q, x\rangle$ attains its strict maximum at $z(= z_q) \in \overline{\Omega}$. We easily see

$$u(z) - v(z) + \langle q, z\rangle \geqq 4\theta, \quad u(z) > v(z). \tag{3.3}$$

We claim

$$v(z) < Mv(z). \tag{3.4}$$

To prove this, suppose $v(z) \geqq Mv(z)$. Since $v \in LSC(\overline{\Omega})$, using the definition of M and (A.8), we can find $\xi_z \geqq 0$ satisfying $\xi_z \neq 0$, $z + \xi_z \in \overline{\Omega}$ and $Mv(z) = k(\xi_z) + v(z + \xi_z)$. Thus $u(z) \leqq Mu(z)$ and $v(z) \geqq Mv(z)$ imply

$$u(z) - v(z) + \langle q, z\rangle \leqq u(z+\xi_z) - v(z+\xi_z) + \langle q, z+\xi_z\rangle - \langle q, \xi_z\rangle.$$

Then we obtain a contradiction because $\langle q, \xi_z\rangle > 0$ by (3.2). Therefore we get the claim (3.4).

We divide our consideration into three cases.

Case 1. $z \in \partial\Omega$ and $v(z) < g(z)$.

Let $\{z_n\}_{n\in\mathbb{N}} \subset K_z$ be a sequence such that

$$z_n \to z, \quad u^*(z_n) \to u^*(z) \qquad (n \to +\infty).$$

We define the function $\Phi(x, y)$ on $\overline{\Omega} \times \overline{\Omega}$ by

$$\Phi(x, y) = u(x) - v(y) + \langle q, y\rangle - \frac{\alpha_n}{2}|x - y - z_n + z|^2,$$

where $\alpha_n = s_0^2/|z_n - z|^2$ and $s_0 > 0$ satisfies $\omega_1(s_0^2) < \lambda\theta$.

Let $(x_n, y_n) \in \overline{\Omega} \times \overline{\Omega}$ be a maximum point of Φ. Since $\Phi(z_n, z) \leqq \Phi(x_n, y_n)$, we get

$$\begin{aligned} &u(z_n) - v(z) + \langle q, z\rangle \\ &\leqq u(z_n) - v(z) + \langle q, z\rangle + \frac{\alpha_n}{2}|x_n - y_n - z_n + z|^2 \\ &\leqq u(x_n) - v(y_n) + \langle q, y_n\rangle. \end{aligned} \tag{3.5}$$

The function $u(x) - v(y)$ is bounded above on $\overline{\Omega} \times \overline{\Omega}$ because u, $-v \in USC(\overline{\Omega})$ and $\overline{\Omega} \times \overline{\Omega}$ is compact in $\mathbb{R}^{2N}$. Hence (3.5) implies $|x_n - y_n - z_n + z| \to 0$ as

$n \to +\infty$. Moreover we easily observe $|x_n - y_n| \to 0$ as $n \to +\infty$. Then there exist a sequence $\{n_k\} \subset \mathbb{N}$ and a point $\bar{z} \in \overline{\Omega}$ such that x_{n_k}, $y_{n_k} \to \bar{z}$ as $k \to +\infty$. It follows from this, (3.5) and the semicontinuity of u and v that

$$u(z) - v(z) + \langle q, z \rangle \leqq u(\bar{z}) - v(\bar{z}) + \langle q, \bar{z} \rangle.$$

Since z is a unique maximum point of the function $u(x) - v(x) + \langle q, x \rangle$ on $\overline{\Omega}$, it follows from this inequality that $\bar{z} = z$ and

$$x_n,\ y_n \to z \qquad (n \to +\infty). \tag{3.6}$$

Thus, by (3.5) we get

$$\lim_{n \to \infty} (u(x_n) - v(y_n)) = u(z) - v(z).$$

Using (3.5), this equality and the semicontinuity of u and v, we have

$$u(x_n) \to u(z),\ v(y_n) \to v(z),\ \alpha_n |x_n - y_n - z_n + z|^2 \to 0 \ \ (n \to +\infty). \tag{3.7}$$

We may consider $x_n \in \Omega$ for sufficiently large $n \in \mathbb{N}$ because (3.7) implies $|x_n - y_n - z_n + z| < t|z_n - z|$ for large $n \in \mathbb{N}$, where t is the constant in (A.2). Furthermore, it is observed by the definition of α_n and (3.7) that

$$\sqrt{\alpha_n}|x_n - y_n| \to s_0 \qquad (n \to +\infty). \tag{3.8}$$

We can apply the maximum principle for semicontinuous functions to obtain $X, Y \in \mathbb{S}^N$ satisfying

$$(p_n, X) \in \bar{J}^{2,+}u(x_n),$$
$$(p_n + q, Y) \in \bar{J}^{2,+}v(y_n),$$

and

$$-3\alpha_n \begin{pmatrix} I & O \\ O & I \end{pmatrix} \leqq \begin{pmatrix} X & O \\ O & -Y \end{pmatrix} \leqq 3\alpha_n \begin{pmatrix} I & -I \\ -I & I \end{pmatrix},$$

where $p_n = \alpha_n(x_n - y_n - z_n + z)$. Using the fact that u and v are respectively, a subsolution and a supersolution of (1.1), we obtain the following inequalities:

$$G_*(x_n, u(x_n), p_n, X, Mu(x_n)) \leqq 0, \tag{3.9}$$

$$G^*(y_n, v(y_n), p_n + q, Y, Mv(y_n)) \geqq 0. \tag{3.10}$$

We note $v(y_n) < g(y_n)$ for large $n \in \mathbb{N}$ by (A.9), (3.6) and $v(z) < g(z)$. Moreover, since $Mv \in LSC(\overline{\Omega})$ by Proposition 2.7 (3) in Chapter I, using (3.4) we get

$$\limsup_{n\to+\infty}(v(y_n) - Mv(y_n)) \leqq v(z) - Mv(z) < 0$$

and conclude that $v(y_n) - Mv(y_n) < 0$ for sufficiently large $n \in \mathbb{N}$. Therefore, by (3.10) we obtain

$$F(y_n, v(y_n), p_n + q, Y) \geqq 0. \tag{3.11}$$

From (3.9) and $x_n \in \Omega$ for large $n \in \mathbb{N}$, we have

$$F(x_n, u(x_n), p_n, X) \leqq 0. \tag{3.12}$$

Subtracting (3.12) from (3.11) and using (A.5), (A.6), (A.7) and (3.3), we obtain

$$\begin{aligned} 4\lambda\theta &\leqq \lambda(u(x_n) - v(y_n) + \langle q, y_n\rangle) \\ &\leqq F(y_n, u(x_n), p_n + q, Y) - F(x_n, u(x_n), p_n, X) + \lambda\langle q, y_n\rangle \\ &\leqq \omega_1(\alpha_n|x_n - y_n|^2 + |x_n - y_n|(|p_n| + 1)) + \omega_2(|q|) + \lambda L|q|. \end{aligned}$$

Recalling (3.1), (3.7) and (3.8) and letting $n \to +\infty$, we get

$$4\lambda\theta \leqq \omega_1(s_0^2) + \omega_2(|q|) + \lambda\theta \leqq 3\lambda\theta,$$

which is a contradiction.

Case 2. $z \in \partial\Omega$, $u(z) > g(z)$.

As in *Case 1*, we define the function Φ by

$$\Phi(x,y) = u(x) - v(y) + \langle q, x\rangle - \frac{\alpha_n}{2}|x - y + z_n - z|^2 \qquad \text{on} \quad \overline{\Omega}\times\overline{\Omega}.$$

We can prove the remainder similarly to the above.

Case 3. $z \in \Omega$.

For $\alpha > 0$, we consider the function

$$\Phi(x,y) = u(x) - v(y) + \langle q, x\rangle - \frac{\alpha}{2}|x - y|^2 \qquad \text{on} \quad \overline{\Omega}\times\overline{\Omega}.$$

In this case the proof is standard. See [8; Section 3].

When the condition (2) (resp., (3)) holds, it is sufficiently to consider only *Case 2, 3* (resp., *Case 1, 3*) in the above proof. Thus we obtain the result. ∎

Remark 3.5. As compared with the proof of Theorem 3.1 in Chapter I, the above one is improved on the point that we do not need the uniform continuity in the variable $X \in \mathbb{S}^N$ and the convexity in $(r, p, X) \in \mathbb{R}\times\mathbb{R}^N\times\mathbb{S}^N$.

We conclude this section by stating the comparison principle of solutions of the usual obstacle problem (2.1). We omit the proof because it is similar to that of Theorem 3.2.

Theorem 3.6. *Assume* (A.1), (A.2), (A.4)-(A.7), (A.9) *and* $\psi \in C(\overline{\Omega})$. *Let* u, v *be, respectively, a subsolution and a supersolution of* (2.1). *For each* $z \in \partial\Omega$, *let* $K_z = z + K(r, s, n(z))$. *If any one of the followings holds, then* $u^* \leqq v_*$ *on* $\overline{\Omega}$.

(1) $\limsup_{K_z \ni x \to z} u^*(x) = u^*(z)$ *and* $\liminf_{K_z \ni x\to z} v_*(x) = v_*(z)$ *for each* $z \in \partial\Omega$,

(2) $\limsup_{K_z \ni x \to z} u^*(x) = u^*(z)$ *and* $u^*(z) \leqq g(z)$ *for each* $z \in \partial\Omega$,

(3) $\liminf_{K_z \ni x\to z} v_*(x) = v_*(z)$ *and* $v_*(z) \geqq g(z)$ *for each* $z \in \partial\Omega$,

Remark 3.7. Of course, in Theorems 3.2 and 3.6, if u, $v \in C(\overline{\Omega})$, then $u \leqq v$ on $\overline{\Omega}$.

§4. Existence of continuous solutions

In this and the next section we shall establish the existence of continuous solutions of (1.1). As mentioned in Section 1, it is difficult to show it for the general elliptic operator case. Hence in these sections we treat the case F is the Hamilton-Jacobi-Bellman operator:

$$F(x,r,p,X) = \sup_{\alpha\in\Lambda}\{ - tr({}^t\sigma(x,\alpha)\sigma(x,\alpha)X) + \langle b(x,\alpha),p\rangle + c(x,\alpha)r - f(x,\alpha)\},$$

where Λ is a compact metric space and trA and ${}^tA$ denote, respectively, the trace and the transposed matrix of A. In this and the next sections we assume $\partial\Omega$ is of class C^2. Then we note (A.2) is satisfied. Let $\rho(x) = dist\,(x,\Omega^c)$. We make the assumptions of the coefficients of F as follows.

(C.1) $\sup_{\alpha\in\Lambda}\left\{\|\sigma(\cdot,\alpha)\|_{W^{1,\infty}(\overline{\Omega})}, \|b(\cdot,\alpha)\|_{W^{1,\infty}(\overline{\Omega})}, \|c(\cdot,\alpha)\|_{C(\overline{\Omega})}, \|f(\cdot,\alpha)\|_{C(\overline{\Omega})}\right\}$
$< +\infty$.

(C.2) $\inf\{c(x,\alpha)\,|\,x\in\overline{\Omega},\ \alpha\in\Lambda\} \geqq c_0$ for some $c_0 > 0$.

(C.3) There exists a function $\alpha \in W^{1,\infty}(\overline{\Omega})$ satisfying

(i) $tr({}^t\sigma(x,\alpha(x))\sigma(x,\alpha(x))D^2\rho(x)) - \langle b(x,\alpha(x)), D\rho(x)\rangle \geqq \eta$ for some $\eta > 0$,

(ii) $\langle {}^t\sigma(x,\alpha(x))\sigma(x,\alpha(x))D\rho(x), D\rho(x)\rangle = 0$,

(iii) There are unit vectors $\{\hat{e}_l\}_{1\leqq l\leqq N-1} \subset \mathbb{R}^N$ by which the tangent plane at z is spanned such that

$$\langle {}^t\sigma(z,\alpha(x))\sigma(z,\alpha(x))\hat{e}_l, \hat{e}_l\rangle = 0$$

except at most two vectors $\{\hat{e}_{l_1}, \hat{e}_{l_2}\}$,

for all $x \in \partial\Omega$.

(C.4) There exist a constant $\eta > 0$ and a function $\beta \in W^{1,\infty}(\overline{\Omega})$ satisfying either

(i) $tr({}^t\sigma(x,\beta(x))\sigma(x,\beta(x))D^2\rho(x)) - \langle b(x,\beta(x)), D\rho(x)\rangle \leqq -\eta$

or

(ii) $\langle {}^t\sigma(x,\beta(x))\sigma(x,\beta(x))D\rho(x), D\rho(x)\rangle \geqq \eta$

for all $x \in \partial\Omega$.

Remark 4.1. (1) As to the assumption (C.3), see M. A. Katsoulakis [26] and [27].

(2) We consider the following operator:

$$F(x,r,p,X) = \max\{-trX + r - f^1(x), \langle b(x), p\rangle + r - f^2(x)\}.$$

Here $b \in W^{1,\infty}(\overline{\Omega})$, $b = -\nu$ on $\partial\Omega$ and f^1, $f^2 \in C(\overline{\Omega})$. Then the above F satisfies the assumptions (C.1)-(C.4).

(3) In the case $\sigma(x,\alpha) \equiv O$ for all $x \in \overline{\Omega}$, $\alpha \in \Lambda$, the existence of solutions was proved by H. Ishii [15; Section 4].

(4) In the case only (C.4) (i) or (ii) holds for all $z \in \partial\Omega$ and $\alpha \in \Lambda$, we have already proved the existence of solutions of (1.1) by Perron's method. See Section 4 in Chapter I.

Under the assumptions (A.1), (A.3), (A.8), (A.9), (C.1) and (C.2), Theorems 3.2 and 3.6 hold. We get the following theorem.

Theorem 4.2. *Assume* (A.1), (A.3), (A.8), (A.9) *and* (C.1)-(C.4). *Then there exists a unique solution* $u \in C(\overline{\Omega})$ *of the problem* (1.1).

In order to show this theorem, the following proposition plays an important role.

Proposition 4.3. *Assume* (A.1), (A.9) *and* (C.1)-(C.4). *Then, for each* $\psi \in C(\overline{\Omega})$, *there exists a unique solution* $u_\psi \in C(\overline{\Omega})$ *of the problem* (2.1).

Here we admit Proposition 4.3 is true and prove Theorem 4.2. We give the proof of Proposition 4.3 in the next section.

Proof of Theorem 4.2. We adopt the iterative approximation scheme introduced in B. Honouzet - J. L. Joly [14].

Let $C_1 = \max\{\sup_{\alpha\in\Lambda}\{\|f(\cdot,\alpha)\|_{C(\overline{\Omega})}\}, \|g\|_{C(\overline{\Omega})}\}$. By replacing $f(\cdot,\alpha)$, g with $f(\cdot,\alpha)+C_1$, $g+C_1$, respectively, we may assume $f(\cdot,\alpha) \geqq 0$ ($\alpha \in \Lambda$), $g \geqq 0$ on $\overline{\Omega}$. Using the results in [27], there exists a unique solution $u_0 \in C(\overline{\Omega})$ of

$$(4.2) \qquad \begin{cases} F(x,u,Du,D^2u) = 0 & \text{in} \quad \Omega, \\ u - g = 0 & \text{on} \quad \partial\Omega. \end{cases}$$

Since $Mu_0 \in C(\overline{\Omega})$ by Proposition 2.7 (4), (5) in Chapter I, there exists a unique solution $u_1 \in C(\overline{\Omega})$ of

$$(4.3)_1 \qquad \begin{cases} \max\{F(x,u,Du,D^2u), u - Mu_0\} = 0 & \text{in} \quad \Omega, \\ \max\{u-g, u-Mu_0\} = 0 & \text{on} \quad \partial\Omega. \end{cases}$$

by Proposition 4.3. For $n = 2,3,\cdots$, we denote by $u_n \in C(\overline{\Omega})$ a unique solution of

$$(4.3)_n \qquad \begin{cases} \max\{F(x,u,Du,D^2u), u - Mu_{n-1}\} = 0 & \text{in} \quad \Omega, \\ \max\{u-g, u-Mu_{n-1}\} = 0 & \text{on} \quad \partial\Omega. \end{cases}$$

(It is follows from Proposition 2.7 (4), (5) in Chapter I that $Mu_{n-1} \in C(\overline{\Omega})$.) Since u_1 is a subsolution of (4.2), we obtain $u_1 \leqq u_0$ on $\overline{\Omega}$ by Theorem 3.6. It is easily seen that $\underline{u} \equiv 0$ on $\overline{\Omega}$ is a subsolution of $(4.3)_1$. Thus Theorem 3.6 implies $u_1 \geqq 0$ on $\overline{\Omega}$. Since $0 \leqq Mu_1 \leqq Mu_0$ on $\overline{\Omega}$ by $0 \leqq u_1 \leqq u_0$ on $\overline{\Omega}$ and Proposition 2.7 (2) in Chapter I, u_2 is a subsolution of $(4.3)_1$. Then we get $u_2 \leqq u_1$ because of Theorem 3.6. In the similar way to the above, we have $u_2 \geqq 0$ on $\overline{\Omega}$. Continuing these processes, we conclude

$$(4.4) \qquad 0 \leqq \cdots \leqq u_n \leqq \cdots \leqq u_2 \leqq u_1 \leqq u_0 \qquad \text{on} \quad \overline{\Omega}.$$

Next we show an upper estimate. We take $\mu \in (0,1)$ such that $\mu\|u_0\|_{C(\overline{\Omega})} \leqq k_0$. For each $n \in \mathbb{N}$, there exist $\theta_n \in (0,1]$ such that

$$(4.5) \qquad u_n - u_{n+1} \leqq \theta_n u_n \qquad \text{on} \quad \overline{\Omega}.$$

It is observed by (A.8) and Proposition 2.7 (2) in Chapter I that

$$(4.6)\qquad \begin{aligned}(1-\theta_n)Mu_n+\theta_n k_0 &\leqq (1-\theta_n)Mu_n+\theta_n M\,0\\ &\leqq M(1-\theta_n)u_n \leqq Mu_{n+1} \qquad \text{on} \quad \overline{\Omega}.\end{aligned}$$

We define ψ, w and v_0 as follows:

$$\psi=(1-\theta_n)Mu_n+\theta_n k_0(\in C(\overline{\Omega})),$$

$w\in C(\overline{\Omega})$: a unique solution of

$$(4.7)\qquad \begin{cases}\max\{F(x,u,Du,D^2u),u-\psi\}=0 & \text{in} \quad \Omega,\\ \max\{u-g,u-\psi\}=0 & \text{on} \quad \partial\Omega,\end{cases}$$

$v_0\in C(\overline{\Omega})$: a unique solution of

$$(4.8)\qquad \begin{cases}\max\{F(x,u,Du,D^2u),u-k_0\}=0 & \text{in} \quad \Omega,\\ \max\{u-g,u-k_0\}=0 & \text{on} \quad \partial\Omega.\end{cases}$$

Noting $\psi\leqq Mu_{n+1}$ on $\overline{\Omega}$, we see that w is a subsolution of $(4.3)_{n+2}$. Hence we get $w\leqq u_{n+2}$ on $\overline{\Omega}$ by Theorem 3.6. It is observed by $f(\cdot,\alpha)\geqq 0$, $g\geqq 0$ on $\overline{\Omega}$ and (4.8) that $\theta_n v_0$ is a solution of

$$(4.9)\qquad \begin{cases}\max\{F_{\theta_n}(x,u,Du,D^2u),u-\theta_n k_0\}=0 & \text{in} \quad \Omega,\\ \max\{u-\theta_n g,u-\theta_n k_0\}=0 & \text{on} \quad \partial\Omega,\end{cases}$$

where

$$\begin{aligned}F_\theta(x,r,p,X)=\sup_{\alpha\in\mathcal{A}}\{&-tr({}^t\sigma(x,\alpha)\sigma(x,\alpha)X)+\langle b(x,\alpha),p\rangle\\ &+c(x,\alpha)-\theta f(x,\alpha)\}.\end{aligned}$$

It follows from $f(\cdot,\alpha)\geqq 0$, $g\geqq 0$, $\psi\geqq\theta_n k_0$ on $\overline{\Omega}$ and (4.7) that $\theta_n w$ is a supersolution of (4.9). Thus, using Theorem 3.6, we have $\theta_n v_0\leqq\theta_n w$ on $\overline{\Omega}$. Moreover, we easily see that $(1-\theta_n)u_{n+1}$ and $(1-\theta_n)w$ are, respectively, a subsolution and a solution of

$$\begin{cases}\max\{F_{1-\theta_n}(x,u,Du,D^2u),u-(1-\theta_n)\psi\}=0 & \text{in} \quad \Omega,\\ \max\{u-(1-\theta_n)g,u-(1-\theta_n)\psi\}=0 & \text{on} \quad \partial\Omega.\end{cases}$$

Therefore we obtain $(1-\theta_n)u_{n+1} \leqq (1-\theta_n)w$ on $\overline{\Omega}$ by Theorem 3.6. Consequently, we get

$$(4.10) \qquad (1-\theta_n)u_{n+1} + \theta_n v_0 \leqq u_{n+2} \qquad \text{on} \quad \overline{\Omega}.$$

From $\mu\|u_0\|_{C(\overline{\Omega})} \leqq k_0$ and $f(\cdot,\alpha) \geqq 0$, $g \geqq 0$ on $\overline{\Omega}$, we observe that μu_{n+1} is a subsolution of (4.8). Thus, by Theorem 3.6 we have $\mu u_{n+1} \leqq v_0$ on $\overline{\Omega}$. Hence (4.10) implies

$$(4.11) \qquad u_{n+1} - u_{n+2} \leqq \theta_n(1-\mu)u_{n+1} \qquad \text{on} \quad \overline{\Omega}.$$

By the way, since $u_1 - u_2 \leqq u_1$ on $\overline{\Omega}$, we obtain $u_2 - u_3 \leqq (1-\mu)u_2$ on $\overline{\Omega}$. Therefore we can take $\theta_2 = 1-\mu$ in (4.5) when $n = 2$. Then it is observed by (4.11) $u_3 - u_4 \leqq (1-\mu)^2 u_3$ on $\overline{\Omega}$. Therefore, using the above argument inductively, we conclude

$$(4.12) \qquad u_{n+1} - u_{n+2} \leqq (1-\mu)^n u_{n+1} \leqq (1-\mu)^n \|u_0\|_{C(\overline{\Omega})} \qquad \text{on} \quad \overline{\Omega},$$

which is our desired estimate.

Combining (4.4) with (4.12), we can find a function $u \in C(\overline{\Omega})$ such that $\|u_n - u\|_{C(\overline{\Omega})} \to 0$ as $n \to +\infty$. By Propositon 2.7 (6) in Chapter I and the stability of solutions (cf. P. L. Lions [34; Proposition I.3]), we conclude that u is a solution of (1.1). The uniqueness follows from Theorem 3.2. ∎

§5. Proof of Proposition 4.3

In this section we shall show Proposition 4.3. We always assume the assumptions in Proposition 4.3. We prepare some notations.

W_t = standard N − dimensional Brownian motion.

$\mathcal{A} = \{\alpha_t : [0, +\infty) \to \Lambda : \text{progressively measurable}\}$.

$\mathcal{B} = \{\theta : \text{stopping time}\}$.

X_t : solution of

$$\begin{cases} dX_t = -b(X_t, \alpha_t)dt + \sqrt{2}\sigma(X_t, \alpha_t)dW_t, \ t > 0, \\ X_0 = x \in \overline{\Omega}. \end{cases}$$

$\tau = \inf\{t \geqq 0 \,|\, X_t \notin \overline{\Omega}\}$.

1_A = characteristic function for A.

Let $\tilde{g} = \min\{g, \psi\}$ on $\partial\Omega$. We consider the penalized problem for (4.1).

$$(5.1) \qquad \begin{cases} F(x, u_n, Du_n, D^2u_n) + n(u_n - \psi)^+ = 0 & \text{in } \Omega, \\ u_n = \tilde{g} & \text{on } \partial\Omega, \end{cases}$$

where $n \in \mathbb{N}$ and $r^+ = \max\{r, 0\}$.

Noting $r^+ = \sup\{\gamma r \mid 0 \leqq \gamma \leqq 1\}$, it is easily seen that (5.1) is equivalent to the following PDE:

$$(5.2) \quad \begin{cases} \sup_{\substack{\alpha\in\Lambda \\ \gamma\in[0,1]}} \{-tr({}^t\sigma(x,\alpha)\sigma(x,\alpha)Du_n) + \langle b(x,\alpha), Du_n\rangle \\ \qquad\qquad + (c(x,\alpha) + n\gamma)\, u_n - f(x,\alpha) - n\gamma\psi\} = 0 & \text{in } \Omega, \\ u_n = \tilde{g} & \text{on } \partial\Omega. \end{cases}$$

Then applying the results in M. A. Katsoulakis [27], for each $n \in \mathbb{N}$, there exists a unique solution $u_n \in C(\overline{\Omega})$ of (5.2).

Next we consider the following problem:

$$(5.3) \qquad \begin{cases} F(x, v_n, Dv_n, D^2v_n) + n(u_n - \psi)^+ = 0 & \text{in } \Omega, \\ v_n = \tilde{g} & \text{on } \partial\Omega, \end{cases}$$

where u_n is the function obtained above. Using the results in [27] again, for each $n \in \mathbb{N}$, there exists a unique solution $v_n \in C(\overline{\Omega})$ of (5.3) and it is characterized as follows:

$$v_n(x) = \inf_{\alpha \in \mathcal{A}} E_x \left\{ \int_0^\tau (f(X_t, \alpha_t) - n(u_n(X_t) - \psi(X_t))^+) \cdot \exp\left(- \int_0^t c(X_s, \alpha_s) ds \right) dt + \tilde{g}(X_\tau) \exp\left(- \int_0^\tau c(X_s, \alpha_s) ds \right) \right\}.$$

Since (5.1) and (5.2) are equivalent to each other and the uniqueness of solutions of (5.1) holds in the class $C(\overline{\Omega})$, we get

$$u_n(x) = \inf_{\alpha \in \mathcal{A}} E_x \left\{ \int_0^\tau (f(X_t, \alpha_t) - n(u_n(X_t) - \psi(X_t))^+) \cdot \exp\left(- \int_0^t c(X_s, \alpha_s) ds \right) dt + \tilde{g}(X_\tau) \exp\left(- \int_0^\tau c(X_s, \alpha_s) ds \right) \right\}. \tag{5.4}$$

Using (C.4) and the barrier argument, we have

$$u_n \leqq \tilde{g} \quad \text{on} \quad \partial\Omega \quad \text{for all } n \in \mathbb{N}. \tag{5.5}$$

Since the operator nr^+ is monotone with respect to $n \in \mathbb{N}$ and $u_n \geqq -C$ for large $C > 0$, we obtain

$$-C \leqq \cdots \leqq u_n \leqq \cdots \leqq u_2 \leqq u_1 \quad \text{on} \quad \overline{\Omega} \tag{5.6}$$

by the comparison principle of solutions of (5.1). (cf. M. G. Crandall - H. Ishii - P. L. Lions [8; Theorem 7.9].) Hence we can define the function u by

$$u(x) = \lim_{n \to +\infty} u_n(x) \left(= \limsup_{\substack{n \to +\infty \\ y \to x}} u_n(y) \right). \tag{5.7}$$

Then we get the following lemma.

Lemma 5.1. *The above function u is a u.s.c. subsolution of* (2.1).

Proof. Since the sequence $\{u_n\}_{n\in\mathbb{N}}$ is decreasing by (5.6), we easily observe $u \in USC(\overline{\Omega})$. Using (5.5) and letting $n \to +\infty$, we have $u \leqq \tilde{g}$ on $\partial\Omega$.

For any $\varphi \in C^2(\overline{\Omega})$, we assume that $u - \varphi$ attains a local maximum at $x_0 \in \overline{\Omega}$. We may consider $x_0 \in \Omega$ and that x_0 is a strict local maximum point of $u - \varphi$. Then there exists a $\delta > 0$ such that

$$u(x_0) - \varphi(x_0) > u(x) - \varphi(x) \qquad \text{for all } x \in \overline{B(x_0, \delta)}(\subset \Omega),\ x \neq x_0. \tag{5.8}$$

Let x_n be a maximum point of $u_n - \varphi$ on $\overline{B(x_0, \delta)}$. Then there exists a subsequence $\{x_{n_k}\}_{k\in\mathbb{N}} \subset \{x_n\}_{n\in\mathbb{N}}$ such that

$$x_{n_k} \to \bar{x} \in \overline{B(x_0, \delta)},\quad u_{n_k}(x_{n_k}) \to \beta \in \mathbb{R}\quad (k \to +\infty).$$

Since

$$u_{n_k}(x) - \varphi(x) \leqq u_{n_k}(x_{n_k}) - \varphi(x_{n_k}) \qquad \text{for all } x \in B(x_0, \delta),$$

we get

$$\begin{aligned} u(x_0) - \varphi(x_0) &\leqq \limsup_{\substack{k\to+\infty \\ x\to x_0}} (u_{n_k}(x) - \varphi(x)) \\ &\leqq \limsup_{k\to+\infty} (u_{n_k}(x_{n_k}) - \varphi(x_{n_k})) \\ &= \beta - \varphi(\bar{x}) \\ &\leqq \limsup_{\substack{k\to+\infty \\ x\to \bar{x}}} (u_{n_k}(x) - \varphi(x)) \\ &= u(\bar{x}) - \varphi(\bar{x}). \end{aligned}$$

Therefore using (5.8) and the above inequality, we obtain

$$x_n \to x_0,\quad u_n(x_n) \to u(x_0) \qquad (n \to +\infty). \tag{5.9}$$

(cf. G. Barles - B. Perthame [3; Lemma A.3].) Since u_n is a subsolution of (5.1), we get

$$F(x_n, u_n(x_n), D\varphi(x_n), D^2\varphi(x_n)) + n(u_n(x_n) - \psi(x_n))^+ \leqq 0. \tag{5.10}$$

It follows from (C.1) and (5.9) that there exists a constant $C > 0$ such that

$$n(u_n(x_n) - \psi(x_n))^+ \leqq C \qquad \text{for all } n \in \mathbb{N}.$$

Thus passing to the limit as $n \to +\infty$, we have

$$u(x_0) - \psi(x_0) \leqq 0.$$

Moreover, (5.10) implies $F(x_n, u_n(x_n), D\varphi(x_n), D^2\varphi(x_n)) \leqq 0$. Sending $n \to +\infty$, we obtain

$$F(x_0, u(x_0), D\varphi(x_0), D^2\varphi(x_0)) \leqq 0.$$

Therefore we have completed the proof. ∎

Remark 5.2. We notice that we cannot apply the results for the limit operations in [8; Section 5] to (2.1) and (5.1) directly since the term $n(r - \psi(x))^+$ does not converge to 0 locally uniformly on $\overline{\Omega} \times \mathbb{R}$ as $n \to +\infty$.

We return to the formula (5.4). According to N. V. Krylov [30; p.37], we get the following lemma.

Lemma 5.3. *The formula* (5.4) *can be rewritten as follows.*

(5.11)

$$\begin{aligned} u_n(x) = \inf_{\substack{\alpha \in \mathcal{A} \\ \theta \in \mathcal{B}}} E_x \Bigg\{ & \int_0^{\tau \wedge \theta} f(X_t, \alpha_t) \exp\left(-\int_0^t c(X_s, \alpha_s) ds\right) dt \\ & + 1_{\theta < \tau} \psi_n(X_\theta) \exp\left(-\int_0^\theta c(X_s, \alpha_s) ds\right) \\ & + 1_{\theta \geqq \tau} \tilde{g}(X_\tau) \exp\left(-\int_0^\tau c(X_s, \alpha_s) ds\right) \Bigg\}, \end{aligned}$$

where $a \wedge b = \min(a, b)$ *and* $\psi_n = \psi + (u_n - \psi)^+$.

Proof. We remark that the function u_n satisfies the dynamic programming principle:

$$u_n(x) = \inf_{\alpha \in \mathcal{A}} E_x \left\{ \int_0^{\tau \wedge \theta} (f(X_t, \alpha_t) - n(u_n(X_t) - \psi(X_t))^+) \cdot \exp\left(- \int_0^t c(X_s, \alpha_s) ds \right) dt + 1_{\theta < \tau} u_n(X_\theta) \exp\left(- \int_0^\theta c(X_s, \alpha_s) ds \right) + 1_{\theta \geqq \tau} \tilde{g}(X_\tau) \exp\left(- \int_0^\tau c(X_s, \alpha_s) ds \right) \right\}$$

for any $\theta \in \mathcal{B}$. Since $u_n \leqq \psi_n$ on $\overline{\Omega}$, we get

(5.12)

$$u_n(x) \leqq \inf_{\substack{\alpha \in \mathcal{A} \\ \theta \in \mathcal{B}}} E_x \left\{ \int_0^{\tau \wedge \theta} f(X_t, \alpha_t) \exp\left(- \int_0^t c(X_s, \alpha_s) ds \right) dt + 1_{\theta < \tau} \psi_n(X_\theta) \exp\left(- \int_0^\theta c(X_s, \alpha_s) ds \right) + 1_{\theta \geqq \tau} \tilde{g}(X_\tau) \exp\left(- \int_0^\tau c(X_s, \alpha_s) ds \right) \right\}.$$

Let $\theta^n = \theta^{n,\alpha} = \inf\{t \geqq 0 \,|\, u_n(X_t) \geqq \psi(X_t)\}$. Then using $u_n(X_t) < \psi(X_t)$ $(0 \leqq t < \theta^n)$ and $u_n(X_{\theta^n}) = \psi(X_{\theta^n})$ with probability 1, we have

$$u_n(x) = \inf_{\alpha \in \mathcal{A}} E_x \left\{ \int_0^{\tau \wedge \theta^n} f(X_t, \alpha_t) \exp\left(- \int_0^t c(X_s, \alpha_s) ds \right) dt + 1_{\theta^n < \tau} \psi_n(X_\theta^n) \exp\left(- \int_0^\theta c(X_s, \alpha_s) ds \right) + 1_{\theta^n \geqq \tau} \tilde{g}(X_\tau) \exp\left(- \int_0^\tau c(X_s, \alpha_s) ds \right) \right\}.$$

Combining this with (5.12), we obtain the result. ∎

Using this lemma, we get

Lemma 5.4. *$u_n \rightrightarrows u$ on $\overline{\Omega}$ as $n \to +\infty$ and the function u is represented as*

$$u(x) = \inf_{\substack{\alpha\in\mathcal{A}\\ \theta\in\mathcal{B}}} E_x \left\{ \int_0^{\tau\wedge\theta} f(X_t,\alpha_t)\exp\left(-\int_0^t c(X_s,\alpha_s)ds\right)dt \right.$$
$$+1_{\theta<\tau}\psi(X_\theta)\exp\left(-\int_0^\theta c(X_s,\alpha_s)ds\right)$$
$$\left. +1_{\theta\geqq\tau}\tilde{g}(X_\tau)\exp\left(-\int_0^\tau c(X_s,\alpha_s)ds\right)\right\}.$$

Proof. It is easily seen by Lemma 5.1 that $u \leqq \psi$ on $\overline{\Omega}$. We observe that $\psi_n \in C(\overline{\Omega})$ for all $n \in \mathbb{N}$ and

$$\psi_n(x) \searrow \psi(x) \quad (n \to +\infty) \quad \text{for each } x \in \overline{\Omega}$$

by (5.6) and (5.7). Hence, using Dini's Theorem, we get

$$\psi_n \rightrightarrows \psi \qquad \text{on} \quad \overline{\Omega} \qquad (n \to +\infty).$$

Letting $n \to +\infty$, we conclude that

$$\text{RHS of (5.11)} \rightrightarrows \inf_{\substack{\alpha\in\mathcal{A}\\ \theta\in\mathcal{B}}} E_x \left\{ \int_0^{\tau\wedge\theta} f(X_t,\alpha_t)\exp\left(-\int_0^t c(X_s,\alpha_s)ds\right)dt \right.$$
$$+1_{\theta<\tau}\psi(X_\theta)\exp\left(-\int_0^\theta c(X_s,\alpha_s)ds\right)$$
$$\left. +1_{\theta\geqq\tau}\tilde{g}(X_\tau)\exp\left(-\int_0^\tau c(X_s,\alpha_s)ds\right)\right\}.$$

On the other hand, we have already obtained $u_n(x) \to u(x)$ as $n \to +\infty$ for each $x \in \overline{\Omega}$ by (5.7). Thus we have the result. ∎

We are now in a position to prove Proposition 4.3.

Proof of Proposition 4.3. We have only to show that u is a supersolution of (2.1).

For any $\varphi \in C^2(\overline{\Omega})$, we assume $u - \varphi$ takes a strict local minimum at $x_0 \in \overline{\Omega}$. We consider the case $x_0 \in \partial\Omega$. Then we may assume $u(x_0) < \tilde{g}(x_0)$, because, if otherwise, we get $u(x_0) = g(x_0)$ or $u(x_0) = \psi(x_0)$ and have nothing to prove. Since $u \in C(\overline{\Omega})$ by Lemma 5.4, there exists a $\delta > 0$ satisfying

$$u(x) < g(x) \qquad x \in \overline{B(x_0,\delta)} \cap \partial\Omega,$$
$$u(x) < \psi(x) \qquad x \in \overline{B(x_0,\delta)} \cap \overline{\Omega}.$$

Moreover, Lemma 5.3 implies there exists an $n_0 \in \mathbb{N}$ satisfying, for all $n > n_0$,

$$u_n(x) < g(x) \qquad x \in \overline{B(x_0,\delta)} \cap \partial\Omega, \tag{5.13}$$
$$u_n(x) < \psi(x) \qquad x \in \overline{B(x_0,\delta)} \cap \overline{\Omega}. \tag{5.14}$$

Let $x_n \in \overline{B(x_0,\delta)} \cap \overline{\Omega}$ be a minimum point of $u_n - \varphi$ on $\overline{B(x_0,\delta)} \cap \overline{\Omega}$. By the same argument as in the proof of Lemma 5.1, we have

$$x_n \to x_0, \quad u_n(x_n) \to u(x_0) \qquad (n \to +\infty).$$

Therefore, using (5.13), (5.14) and the fact that u_n is a supersolution of (5.1), we obtain

$$F(x_n, u_n(x_n), D\varphi(x_n), D^2\varphi(x_n)) \geq 0.$$

Sending $n \to +\infty$, we get

$$F(x_0, u(x_0), D\varphi(x_0), D^2\varphi(x_0)) \geq 0.$$

Thus the proof is completed. ∎

§6. Stochastic representation of solutions

In this section we shall prove that the unique solution of (1.1) is represented as the optimal cost function for the impulse control problem.

We call a collection (α, θ, ξ) an impulse control if

$\alpha \in \mathcal{A}$,

$\theta = \{\theta_i\}_{i=1}^{+\infty} \subset \mathcal{B}$ satisfies

$$\theta_1 < \theta_2 < \cdots < \theta_n < \cdots \to +\infty \quad (n \to +\infty),$$

$\xi = \{\xi_i\}_{i=1}^{+\infty}$: a sequence of random variables taking values on $(\mathbb{R}^+)^N$,

adapted with respect to $\{\theta_i\}_{i=1}^{+\infty}$.

The $\mathcal{C}$ denotes the set of all impulse controls.

We define the sequence of diffusions $\{X_t^n\}_{n=1}^{+\infty}$ with jumps by the Ito equation:

$$\begin{cases} dX_t^0 = -b(X_t^0, \alpha_t)dt + \sqrt{2}\sigma(X_t^0, \alpha_t)dW_t, & t \geqq 0, \\ X_0^0 = x \in \overline{\Omega}, \end{cases}$$

$$\begin{cases} dX_t^n = -b(X_t^n, \alpha_t)dt + \sqrt{2}\sigma(X_t^n, \alpha_t)dW_t, & t > \theta_n, \\ X_0^n = X_t^{n-1} + 1_{t=\theta_n}\xi_n & t \leqq \theta_n. \end{cases}$$

We set

$$X_t = \lim_{n \to +\infty} X_t^n, \qquad t \geqq 0.$$

Then the process X_t, which is right continuous and has left limits, satisfies the following stochastic differential equation:

$$\begin{cases} dX_t = -b(X_t, \alpha_t)dt + \sqrt{2}\sigma(X_t, \alpha_t)dW_t + \sum_{i=1}^{+\infty} \xi_i \delta(t - \theta_i)dt, & t \geqq 0, \\ X_0 = x, \end{cases}$$

where $\delta(t)$ is the Dirac measure. We put

$$\tau = \inf\{t \geqq 0 \mid X_t \notin \overline{\Omega}\}.$$

We call a collection $(\alpha, \theta, \xi) \in \mathcal{C}$ an admissible impulse control if it satisfies

$$X_\tau \in \overline{\Omega} \qquad \text{a.s. on} \quad \{\tau < +\infty\},$$

that is, no jump of the process X_t is outside of $\overline{\Omega}$ before τ. We denote by $\mathcal{C}_0$ the set of all admissible impulse controls.

Now, we can define the cost function for this system:

$$
\begin{aligned}
K &= (\alpha, \theta, \xi), \\
J(x, K) &= E_x \left\{ \int_0^\tau f(X_t, \alpha_t) \exp\left(- \int_0^t c(X_s, \alpha_s) ds \right) dt \right. \\
&\quad + \sum_{i=1}^{+\infty} 1_{\theta_i < +\infty} k(\xi_i) \exp\left(- \int_0^{\theta_i} c(X_s, \alpha_s) ds \right) \\
&\quad \left. + g(X_\tau) \exp\left(- \int_0^\tau c(X_s, \alpha_s) ds \right) \right\}
\end{aligned}
$$

and the optimal cost function:

$$
w(x) = \inf_{K \in \mathcal{C}_0} J(x, K).
$$

Then we have the following theorem.

Theorem 6.1. *Assume* (A.1), (A.3), (A.8), (A.9), (C.1)-(C.4) *and* $\partial\Omega$ *is of class* C^2. *Let* u *be a unique solution of* (1.1). *Then* $u = w$ *on* $\overline{\Omega}$.

We state some properties of the sequence $\{u_n\}_{n \in \mathbb{N}}$ of solutions of (5.1).

Lemma 6.2. *For each* $n \in \mathbb{N}$, *we have*

(6.1)

$$
\begin{aligned}
u_n(x) = \inf_{K \in C^n} E_x &\left\{ \int_0^\tau f(X_t, \alpha_t) \exp\left(- \int_0^t c(X_s, \alpha_s) ds \right) dt \right. \\
&\quad + \sum_{i=1}^{n} 1_{\theta_i < +\infty} k(\xi_i) \exp\left(- \int_0^{\theta_i} c(X_s, \alpha_s) ds \right) dt \\
&\quad \left. + g(X_\tau) \exp\left(- \int_0^\tau c(X_s, \alpha_s) ds \right) dt \right\},
\end{aligned}
$$

where $C^n = \{(\alpha, \{\theta_i\}_{i=1}^{+\infty}, \{\xi\}_{i=1}^{+\infty}) \in \mathcal{C}_0 \mid \theta_i = +\infty \text{ for } i \geqq n+1\}$.

Proof. By Lemma 5.4 the function u_n can be represented as follows:

$$
\begin{aligned}
u_n(x) = \inf_{\substack{\alpha \in \mathcal{A} \\ \theta \in \mathcal{B}}} E_x &\left\{ \int_0^{\tau \wedge \theta} f(X_t, \alpha_t) \exp\left(- \int_0^t c(X_s, \alpha_s) ds \right) dt \right. \\
&\quad + 1_{\theta < \tau} M u_{n-1}(X_\theta) \exp\left(- \int_0^\theta c(X_s, \alpha_s) ds \right) \\
&\quad \left. + 1_{\theta \geqq \tau} \tilde{g}(X_\tau) \exp\left(- \int_0^\tau c(X_s, \alpha_s) ds \right) \right\}.
\end{aligned}
$$

We prove the assertion by induction. Let w_n = RHS of (6.1).

For $n = 1$, it is trivial. We assume $u_n = w_n$ on $\overline{\Omega}$ for $n \geqq 1$ and show $u_{n+1} = w_{n+1}$ on $\overline{\Omega}$.

Fix $x \in \overline{\Omega}$ and $K \in \mathcal{C}^{n+1}$. We may consider $\theta_1 < \tau$, because, if otherwise, we have the result. It is clear that

$$\begin{aligned} w_{n+1}(x) \leqq & E_x \left\{ \int_0^\tau f(X_t, \alpha_t) \exp\left(- \int_0^t c(X_s, \alpha_s) ds \right) dt \right. \\ & + \sum_{i=1}^{n+1} 1_{\theta_i < +\infty} k(\xi_i) \exp\left(- \int_0^{\theta_i} c(X_s, \alpha_s) ds \right) dt \\ & \left. + g(X_\tau) \exp\left(- \int_0^\tau c(X_s, \alpha_s) ds \right) dt \right\}. \end{aligned}$$

We observe

$$\begin{aligned} w_{n+1}(x) \leqq & E_x \left[\int_0^{\theta_1} f(X_t, \alpha_t) \exp\left(- \int_0^t c(X_s, \alpha_s) ds \right) dt \right. \\ & + E_{X_{\theta_1 - 0} + \xi_1} \left\{ \int_0^{\tau - \theta_1} f(X_{t+\theta_1}, \alpha_{t+\theta_1}) \right. \\ & \cdot \exp\left(- \int_0^t c(X_{s+\theta_1}, \alpha_{s+\theta_1}) ds \right) dt \\ & + \sum_{i=2}^{n+1} 1_{\theta_i < +\infty} k(\xi_i) \exp\left(- \int_0^{\theta_i - \theta_1} c(X_{s+\theta_1}, \alpha_{s+\theta_1}) ds \right) dt \\ & \left. + g(X_\tau) \exp\left(- \int_0^{\tau - \theta_1} c(X_{s+\theta_1}, \alpha_{s+\theta_1}) ds \right) dt + k(\xi_1) \right\} \\ & \left. \cdot \exp\left(- \int_0^{\theta_1} c(X_s, \alpha_s) ds \right) dt \right]. \end{aligned}$$

Since $(\alpha_{t+\theta_1}, \{\theta_i - \theta_1\}_{i=2}^{n+1}, \{\xi_i\}_{i=2}^{n+1}) \in \mathcal{C}^n$, we take the infimum with respect to admissible controls in $\mathcal{C}^n$ to obtain

$$\begin{aligned} w_{n+1}(x) \leqq & E_x \left\{ \int_0^{\theta_1} f(X_t, \alpha_t) \exp\left(- \int_0^t c(X_s, \alpha_s) ds \right) dt \right. \\ & \left. + (u_n(X_{\theta_1 - 0} + \xi_1) + k(\xi_1)) \exp\left(- \int_0^{\theta_1} c(X_s, \alpha_s) ds \right) dt \right\}. \end{aligned}$$

Moreover, taking the infimum with respect to $\xi_1 \geqq 0$ satisfying $X_{\theta_1 - 0} + \xi \in \overline{\Omega}$, we have

$$w_{n+1}(x) \leqq E_x \left\{ \int_0^{\theta_1} f(X_t, \alpha_t) \exp\left(- \int_0^t c(X_s, \alpha_s) ds \right) dt \right.$$
$$\left. + M u_n(X_{\theta_1 - 0}) \exp\left(- \int_0^{\theta_1} c(X_s, \alpha_s) ds \right) dt \right\}.$$

Hence by taking the infimum with respect to $(\alpha, \theta_1) \in \mathcal{A} \times \mathcal{B}$ we get $w_{n+1}(x) \leqq u_{n+1}(x)$.

Next we prove the opposite inequality. For each $\varepsilon > 0$, there exists an impulse control $K = (\alpha, \theta, \xi) \in \mathcal{C}^{n+1}$ such that

$$w(x) + \varepsilon \geqq J(x, K).$$

We calculate

$$\begin{aligned} w_{n+1}(x) + \varepsilon \geqq E_x &\left\{ \int_0^{\theta_1} f(X_t, \alpha_t) \exp\left(- \int_0^t c(X_s, \alpha_s) ds \right) dt \right. \\ &\left. + (u_n(X_{\theta_1 - 0} + \xi_1) + k(\xi_1)) \exp\left(- \int_0^{\theta_1} c(X_s, \alpha_s) ds \right) dt \right\}. \\ \geqq E_x &\left\{ \int_0^{\theta_1} f(X_t, \alpha_t) \exp\left(- \int_0^t c(X_s, \alpha_s) ds \right) dt \right. \\ &\left. + M u_n(X_{\theta_1 - 0}) \exp\left(- \int_0^{\theta_1} c(X_s, \alpha_s) ds \right) dt \right\} \\ \geqq u_{n+1}(x). & \end{aligned}$$

Letting $\varepsilon \to 0$, we have $w_{n+1}(x) \geqq u_{n+1}(x)$. Thus we have completed the proof. ∎

Remark 6.3. We can show that the function w satisfies

$$\begin{aligned} w(x) = \inf_{\substack{\alpha \in \mathcal{A} \\ \theta \in \mathcal{B}}} E_x &\left\{ \int_0^{\tau} f(X_t, \alpha_t) \exp\left(- \int_0^t c(X_s, \alpha_s) ds \right) dt \right. \\ &+ 1_{\theta < \tau} M w(X_\theta) \exp\left(- \int_0^{\theta} c(X_s, \alpha_s) ds \right) dt \\ &\left. + g(X_\tau) \exp\left(- \int_0^{\tau} c(X_s, \alpha_s) ds \right) dt \right\} \end{aligned}$$

by the similar way. See G. Barles [1; Theorem 2.1].

Lemma 6.4. *We have*

$$\|w_n - w\|_{L^\infty(\Omega)} \leqq \frac{C}{k_0 n} \qquad \text{for some } C > 0. \tag{6.2}$$

Proof. First we remark that $w \leqq \cdots \leqq w_n \leqq \cdots \leqq w_1 \leqq u_0$ on $\overline{\Omega}$. Let $K = (\alpha, \theta, \xi) \in \mathcal{C}_0$. We set

$$\theta_i^n = \begin{cases} \theta_i & \text{if } i \leqq n, \\ +\infty & \text{if } i \geqq n+1, \end{cases}$$
$$\theta^n = \{\theta_i^n\}_{i=1}^{+\infty},$$
$$K^n = (\alpha, \theta^n, \xi) \in \mathcal{C}^n.$$

Let X_t^n be the process associated with K^n and $\tau^n = \inf\{t \geqq 0 \,|\, X_t^n \notin \overline{\Omega}\}$. Then we note that if $\tau < \theta_{n+1}$ or $\theta_{n+1} = +\infty$, then $\tau^n = \tau$. Hence we get

$$\begin{aligned} J(x,K) - J(x,K^n) \geqq E_x\Bigg[&\Bigg\{\int_{\theta_n}^{\tau} f(X_t,\alpha_t)\exp\left(-\int_0^t c(X_s,\alpha_s)ds\right)dt \\ &+g(X_\tau)\exp\left(-\int_0^\tau c(X_s,\alpha_s)ds\right)dt \\ &-\int_{\theta_n}^{\tau^n} f(X_t^n,\alpha_t)\exp\left(-\int_0^t c(X_s^n,\alpha_s)ds\right)dt \\ &+g(X_{\tau^n}^n)\exp\left(-\int_0^{\tau^n} c(X_s^n,\alpha_s)ds\right)dt\Bigg\} \\ &\cdot 1_{\theta_{n+1}<+\infty} 1_{\theta_{n+1}\leqq\tau}\Bigg] \\ \geqq E_x\Bigg[&1_{\theta_n<+\infty} u(X_{\theta_n})\exp\left(-\int_0^{\theta_n} c(X_s,\alpha_s)ds\right)dt \\ &-1_{\theta_n<+\infty}\frac{1}{c_0}(\|f(\cdot,\alpha)\|_{C(\overline{\Omega})} + \|g\|_{C(\overline{\Omega})}) \\ &\cdot\exp\left(-\int_0^{\theta_n} c(X_s^n,\alpha_s)ds\right)\Bigg] \\ \geqq -CE_x&\left\{\exp\left(-\int_0^{\theta_n} c(X_s^n,\alpha_s)ds\right)1_{\theta_n<+\infty}\right\}, \end{aligned}$$

where $C = \|u_0\|_{C(\overline{\Omega})} + (\sup_{\alpha\in\Lambda} \|f(\cdot,\alpha)\|_{C(\overline{\Omega})} + \|g\|_{C(\overline{\Omega})})/c_0$.

On the other hand, we may consider $J(x,K) \leqq \|u_0\|_{C(\overline{\Omega})}$ for all $K \in \mathcal{C}_0$. Therefore we have

$$E_x\left\{\sum_{i=1}^{+\infty} 1_{\theta_i<+\infty} k(\xi_i) \exp\left(-\int_0^{\theta_i} c(X_s,\alpha_s)ds\right)\right\} \leqq C.$$

Thus it is observed by this inequality and (A.8) that

$$k_0 n E_x\left\{1_{\theta_n<+\infty} \exp\left(-\int_0^{\theta_n} c(X_s,\alpha_s)ds\right)\right\} \leqq C.$$

Hence we obtain

$$\begin{aligned} J(x,K) &\geqq J(x,K^n) - \frac{C}{k_0 n} \\ &\geqq w_n - \frac{C}{k_0 n}. \end{aligned}$$

Taking the infimum with respect to $K \in \mathcal{C}_0$, we get

$$w(x) \geqq w_n(x) - \frac{C}{k_0 n} \qquad \text{for all } x \in \overline{\Omega}.$$

Thus we obtain (6.2). ▮

Proof of Theorem 6.1. Lemmas 6.2 and 6.4 imply $u_n \rightrightarrows w$ on $\overline{\Omega}$ as $n \to +\infty$. Hence it is clear that $u = w$ on $\overline{\Omega}$. ▮

§7. Boundary value problem of oblique type

In this section we shall treat the boundary value problem of oblique type:

$$(7.1) \qquad \begin{cases} \max\{F(x,u,Du,D^2u), u - Mu\} = 0 & \text{in } \Omega, \\ \max\left\{\dfrac{\partial u}{\partial\gamma}, u - Mu\right\} = 0 & \text{on } \partial\Omega. \end{cases}$$

Here $\partial\Omega$ is smooth and γ is a vector field on $\mathbb{R}^N$ "oblique" to $\partial\Omega$. The problem (7.1) is derived from the impulse control problem for the diffusion processes reflecting at the boundary $\partial\Omega$. See P. L. Lions - B. Perthame [35] for the related problems. P. Dupuis - H. Ishii [10], [11] and H. Ishii [17] has obtained the uniqueness and existence of solutions of some oblique derivative problems. But they do not contain the problem (7.1).

Here we prove the comparison principle and existence of solutions of (7.1) by the similar arguments to those in [10], [11] and [17]. Instead of (A.6) we assume the uniform continuity of the F with respect to the variable $(p, X) \in \mathbb{R}^N \times \mathbb{S}^N$.

(A.6)′ There exists a function $\omega_3 \in C(\mathbb{R}^+)$ such that $\omega_3(0) = 0$ for which

$$|F(x,r,p,X) - F(x,r,q,Y)| \leqq \omega_3(|p-q| + \|X - Y\|)$$

for all $x \in \overline{\Omega}$, $r \in \mathbb{R}$, p, $q \in \mathbb{R}^N$, X, $Y \in \mathbb{S}^N$.

Besides, we put the following assupmtion.

(A.10) $\gamma \in C^2(\partial\Omega)$ and there exists a constant $\eta > 0$ such that $\langle \nu(x), \gamma(x) \rangle \geqq \eta$ for all $x \in \partial\Omega$.

In order to give the definition of solutions of (7.1), we set

$$H^*(x,r,p,X,m) = \begin{cases} \max\{F(x,r,p,X), r-m\} & (x \in \Omega), \\ \max\{\max\{F(x,r,p,X), r-m\}, \\ \qquad \max\{\langle p, \gamma(x)\rangle, r-m\}\} & (x \in \partial\Omega), \end{cases}$$

$$H_*(x,r,p,X,m) = \begin{cases} \max\{F(x,r,p,X), r-m\} & (x \in \Omega), \\ \min\{\max\{F(x,r,p,X), r-m\}, \\ \qquad \max\{\langle p, \gamma(x)\rangle, r-m\}\} & (x \in \partial\Omega), \end{cases}$$

where F is the same function as in Section 2.

Definition 7.1. *Let* $u : \overline{\Omega} \to \mathbb{R}$.

(1) *We say* u *is a subsolution of* (7.1) *provided* $u^* < +\infty$ *on* $\overline{\Omega}$ *and for any* $\varphi \in C^2(\overline{\Omega})$, *if* $u^* - \varphi$ *attains a local maximum at* $x_0 \in \overline{\Omega}$, *then*

$$H_*(x_0, u^*(x_0), D\varphi(x_0), D^2\varphi(x_0), Mu^*(x_0)) \leqq 0.$$

(2) *We say* u *is a supersolution of* (7.1) *provided* $u_* > -\infty$ *on* $\overline{\Omega}$ *and for any* $\varphi \in C^2(\overline{\Omega})$, *if* $u_* - \varphi$ *attains a local minimum at* $x_0 \in \overline{\Omega}$, *then*

$$H^*(x_0, u_*(x_0), D\varphi(x_0), D^2\varphi(x_0), Mu_*(x_0)) \geqq 0.$$

(3) *We say u is a solution of (7.1) provided u is both a sub- and a supersolution of (7.1).*

We mention the equivalent propositions of Definition 7.1 wihtout their proofs.

Proposition 7.2. *Let $u : \overline{\Omega} \to \mathbb{R}$.*

(1) *u is a subsolution of (7.1) if and only if $u^* < +\infty$ on $\overline{\Omega}$ and for all $x \in \overline{\Omega}$ and $(p, X) \in J^{2,+}_{\overline{\Omega}} u^*(x)$, u^* satisfies*

$$H_*(x, u^*(x), p, X, Mu^*(x)) \leqq 0.$$

(2) *u is a supersolution of (7.1) if and only if $u_* > -\infty$ on $\overline{\Omega}$ and for all $x \in \overline{\Omega}$ and $(p, X) \in J^{2,-}_{\overline{\Omega}} u_*(x)$, u_* satisfies*

$$H^*(x, u_*(x), p, X, Mu_*(x)) \geqq 0.$$

Proposition 7.3. *Assume $M : USC(\overline{\Omega}) \to USC(\overline{\Omega})$ and $M : LSC(\overline{\Omega}) \to LSC(\overline{\Omega})$. Let $u : \overline{\Omega} \to \mathbb{R}$.*

(1) *u is a subsolution of (7.1) if and only if $u^* < +\infty$ on $\overline{\Omega}$ and for all $x \in \overline{\Omega}$ and $(p, X) \in \bar{J}^{2,+}_{\overline{\Omega}} u^*(x)$, u^* satisfies*

$$H_*(x, u^*(x), p, X, Mu^*(x)) \leqq 0.$$

(2) *u is a supersolution of (7.1) if and only if $u_* > -\infty$ on $\overline{\Omega}$ and for all $x \in \overline{\Omega}$ and $(p, X) \in \bar{J}^{2,-}_{\overline{\Omega}} u_*(x)$, u_* satisfies*

$$H^*(x, u_*(x), p, X, Mu_*(x)) \geqq 0.$$

Now, we state our main results in this section.

Theorem 7.4. *Assume* (A.1), (A.3)-(A.5), (A.6)$'$, (A.7), (A.8), (A.10) *and* $\partial\Omega$ *is smooth. Let* u, v *be, respectively, a subsolution and a supersolution of* (7.1). *Then* $u^* \leqq v_*$ *on* $\overline{\Omega}$.

Theorem 7.5. *Under the same assumptions as in Theorem 7.4, there exists a unique solution* u *of* (7.1). *Moreover* $u \in C(\overline{\Omega})$.

We need some lemmas to prove these theorems.

Lemma 7.6. *Assume* (A.10). *Let* $z \in \partial\Omega$. *Then there exist constants* $\delta > 0$, $C_0 > 0$ *and* $\{w_\alpha\}_{\alpha>0}$: $C^{1,1}$*-functions on* $\overline{B(z,\delta)} \times \overline{B(z,\delta)}$ *satisfying the following properties:*

$$
\begin{aligned}
&w_\alpha(x,x) \leqq \frac{1}{\alpha} && \text{on} \quad \overline{B(z,\delta)},\\
&w_\alpha(x,y) \geqq \frac{\alpha}{8}|x-y|^2 && \text{on} \quad \overline{B(z,\delta)} \times \overline{B(z,\delta)},\\
&\langle D_x w_\alpha(x,y), \gamma(x)\rangle \geqq -C_0\delta_\alpha && \text{if } x \in \partial\Omega \text{ and } y \in \overline{B(z,\delta)},\\
&\langle -D_y w_\alpha(x,y), \gamma(y)\rangle \leqq C_0\delta_\alpha && \text{if } y \in \partial\Omega \text{ and } x \in \overline{B(z,\delta)},\\
&|D_y w_\alpha(x,y)| \leqq C_0\,(\alpha|x-y|+1)\,, &&\\
&|D_x w_\alpha(x,y) + D_y w_\alpha(x,y)| \leqq C_0\delta_\alpha, &&\\
&\left(Dw_\alpha(x,y), \alpha C_0 \begin{pmatrix} I & -I \\ -I & I \end{pmatrix} + C_0\delta_\alpha \begin{pmatrix} I & O \\ O & I \end{pmatrix}\right) \in \bar{J}^{2,+} w_\alpha(x,y) &&\\
& && \text{for} \quad \alpha > 0,\ x, y \in \overline{B(z,\delta)},
\end{aligned}
$$

where $\delta_\alpha = (\alpha|x-y|^2 + 1/\alpha)$.

The above lemma is proved in [17; Section 4]. Hence we omit the proof.

Lemma 7.7. *Assume* (A.1), (A.3) *and* (A.8). *Let* $u \in USC(\overline{\Omega})$ *and* $v \in LSC(\overline{\Omega})$. *If* $u \leqq Mu$ *on* $\overline{\Omega}$, *then there exists a maximum point* $z \in \overline{\Omega}$ *of the function* $u - v$ *on* $\overline{\Omega}$ *such that* $v(z) - Mv(z) < 0$.

This lemma is mentioned in [35; Section 5] without its proof. For the sake of completeness we give the proof.

Proof. Let $z_0 \in \overline{\Omega}$ be any maximum point of the function $u - v$ on $\overline{\Omega}$. If the assertion in this lemma holds at z_0, we have nothing to prove. We suppose $v(z_0) - Mv(z_0) \geqq 0$. Then there exists a $\xi_0 \geqq 0$ such that $\xi_0 \neq 0$, $z_0 + \xi_0 \in \overline{\Omega}$ and $Mv(z_0) = k(\xi_0) + v(z_0 + \xi_0)$ by (A.8). Since $u \leqq Mu$ on $\overline{\Omega}$, we obtain

$$u(z_0) - v(z_0) \leqq u(z_0 + \xi_0) - v(z_0 + \xi_0). \tag{7.2}$$

Hence $z_0 + \xi_0$ is a maximum point of $u - v$. If the assertion holds at $z_0 + \xi_0$, the proof is completed. Here we suppose that the above process can be repeated indefinitely, that is, there exists a sequence $\{z_n\}_{n \in \mathbb{N}}$ of maximum points of $u - v$ on $\overline{\Omega}$ such that

$$z_1 = z_0 + \xi_0, \quad z_n = z_{n-1} + \xi_{n-1}, \quad v(z_n) \geqq Mv(z_n) \qquad (n \in \mathbb{N}),$$

where $\xi_n \geqq 0$ satisfies

$$\begin{aligned} &\xi_n \neq 0, \quad z_n + \xi_n \in \overline{\Omega}, \\ &Mv(z_n) = k(\xi_n) + v(z_n + \xi_n). \end{aligned}$$

Then we obtain

$$z_n \to \bar{z} \qquad (n \to +\infty) \quad \text{for some } \bar{z} \in \overline{\Omega},$$

because $\overline{\Omega}$ is compact and $z_0 \leqq z_1 \leqq \cdots \leqq z_n \leqq \cdots$ by the definition of $\{z_n\}_{n \in \mathbb{N}}$. ($z_n \geqq z_{n-1}$ means $z_n - z_{n-1} \in (\mathbb{R}^+)^N$.) Since the inequality (7.2) holds at $z_n + \xi_n$ in place of $z_0 + \xi_0$ and $u - v \in USC(\overline{\Omega})$, we have

$$u(z_0) - v(z_0) = \lim_{n \to +\infty} (u(z_n) - v(z_n)) = u(\bar{z}) - v(\bar{z}).$$

Thus it follows from the above equality and semicontinuity of u and v that $v(z_n) \to v(\bar{z})$ as $n \to +\infty$. Using the definition of $\{z_n\}_{n \in \mathbb{N}}$ and (A.8), we conclude that

$$\begin{aligned} v(\bar{z}) &= \lim_{n \to +\infty} v(z_n) \geqq \lim_{n \to +\infty} Mv(z_n) \\ &= \lim_{n \to +\infty} (k(\xi_n) + v(z_n + \xi_n)) \\ &\geqq k_0 + v(\bar{z}), \end{aligned}$$

which contradicts the fact $k_0 > 0$. Therefore we can find an $n_0 \in \mathbb{N}$ such that $v(z_{n_0}) - Mv(z_{n_0}) < 0$. ∎

Now, we can prove Theorem 7.4.

Proof of Theorem 7.4. We may assume $u \in USC(\overline{\Omega})$ and $v \in LSC(\overline{\Omega})$. We suppose $\sup_{\overline{\Omega}}(u - v) = \theta > 0$ and get a contradiction. Since u is a subsolution of (7.1), we get $u \leqq Mu$ on $\overline{\Omega}$. It is seen from Lemma 7.7 that there exists a maximum point $z \in \overline{\Omega}$ of $u - v$ satisfying $v(z) - Mv(z) < 0$. We divide our consideration into two cases.

Case 1. $z \in \partial\Omega$.

For simplicity we consider $|\gamma(x)| = 1$. Let $\varphi \in C^2(\overline{\Omega})$ be a function such that

$$\varphi = 0 \quad \text{on } \partial\Omega, \ \varphi > 0 \quad \text{in } \Omega, \text{ and } \langle D\varphi, \gamma\rangle \geqq \eta_0 \quad \text{on } \partial\Omega.$$

for some $\eta_0 > 0$. (cf. M. G. Crandall - H. Ishii - P. L. Lions [8; Section 7].) For each $\beta > 0$, the function $u(x) - v(x) - \beta(|x - z|^2 + 2\varphi(x))$ attains a strict maximum on $\overline{\Omega}$ at z. Thus we may restrict this function on $\overline{B(z,\delta)} \cap \overline{\Omega} (= W)$. For any $\alpha > 0$, we define the function $\Phi(x, y)$ on $W \times W$ by

$$\Phi(x, y) = u(x) - v(y) - w_\alpha(x, y) - \beta(|x - z|^2 + \varphi(x) + \varphi(y)),$$

where w_α is the function in Lemma 7.6. Let $(\bar{x}, \bar{y}) \in W \times W$ be a maximum point of Φ. By $\Phi(z, z) \leqq \Phi(\bar{x}, \bar{y})$ and (7.4) we get

$$\theta - \frac{1}{\alpha} \leqq u(\bar{x}) - v(\bar{y}) - \frac{\alpha}{8}|x - y|^2.$$

Thus we have

$$|\bar{x} - \bar{y}| \to 0 \qquad (\alpha \to +\infty).$$

As in the proof of Theorem 3.2, we obtain the behaviors of $\bar{x}$, $\bar{y}$, $u(\bar{x})$, $v(\bar{y})$ as $\alpha \to +\infty$:

(7.4) $$\bar{x}, \bar{y} \to z, \ u(\bar{x}) \to u(z), \ v(\bar{y}) \to v(z), \ \alpha|\bar{x} - \bar{y}|^2 \to 0.$$

Moreover, $\Phi(x,y) \leqq \Phi(\bar{x},\bar{y})$ on W implies, as $(x,y) \to (\bar{x},\bar{y})$,

$$\begin{aligned}
u(x) - u(y) \leqq\ & u(\bar{x}) - v(\bar{y}) - \beta|\bar{x} - z|^2 + \beta|x - z|^2 \\
& - \beta(\varphi(\bar{x}) - \varphi(x)) - \beta(\varphi(\bar{y}) - \varphi(y)) - w_\alpha(\bar{x},\bar{y}) - w_\alpha(x,y) \\
\leqq\ & u(\bar{x}) - v(\bar{y}) + 2\beta\langle \bar{x} - z, x - \bar{x}\rangle + \beta\langle x - \bar{x}, x - \bar{x}\rangle \\
& + \beta\{\langle D\varphi(\bar{x}), x - \bar{x}\rangle + \frac{1}{2}\langle D^2\varphi(\bar{x})(x - \bar{x}), x - \bar{x}\rangle\} \\
& + \beta\{\langle D\varphi(\bar{y}), y - \bar{y}\rangle + \frac{1}{2}\langle D^2\varphi(\bar{y})(y - \bar{y}), y - \bar{y}\rangle\} \\
& + \langle D_x w_\alpha(\bar{x},\bar{y}), x - \bar{x}\rangle + \langle D_y w_\alpha(\bar{x},\bar{y}), y - \bar{y}\rangle \\
& + \frac{1}{2}\alpha C_0|(x - \bar{x}) - (y - \bar{y})|^2 \\
& + \delta_\alpha(|x - \bar{x}|^2 + |y - \bar{y}|^2) \\
& + o(|x - \bar{x}|^2 + |y - \bar{y}|^2).
\end{aligned}$$

Thus we conclude

$$\begin{aligned}
\Bigg(\begin{pmatrix} 2\beta(\bar{x} - z) + D_x w_\alpha(\bar{x},\bar{y}) + \beta D\varphi(\bar{x}) \\ D_y w_\alpha(\bar{x},\bar{y}) + \beta D\varphi(\bar{y}) \end{pmatrix}, \alpha C_0 \begin{pmatrix} I & -I \\ -I & I \end{pmatrix} \\
+\delta_\alpha \begin{pmatrix} I & O \\ O & I \end{pmatrix} + \beta \begin{pmatrix} 2I + D^2\varphi(\bar{x}) & O \\ O & D^2\varphi(\bar{y}) \end{pmatrix} \Bigg) \\
\in J^{2,+}(u(\bar{x}) - v(\bar{y})).
\end{aligned}$$

Therefore by the maximum principle, there exist $X, Y \in \mathbb{S}^N$ such that

$$\begin{aligned}
& (2\beta(\bar{x} - z) + D_x w_\alpha(\bar{x},\bar{y}) + \beta D\varphi(\bar{x}), X) \in \bar{J}^{2,+}u(\bar{x}), \\
& (-D_y w_\alpha(\bar{x},\bar{y}) - \beta D\varphi(\bar{y}), Y) \in \bar{J}^{2,-}v(\bar{y}), \\
& - 3\alpha C_0 \begin{pmatrix} I & O \\ O & I \end{pmatrix} \leqq \begin{pmatrix} X - \delta_\alpha I & O \\ O & -Y - \delta_\alpha I \end{pmatrix} \\
& \qquad \leqq 3\alpha C_0 \begin{pmatrix} I & -I \\ -I & I \end{pmatrix} + \beta \begin{pmatrix} 2I + D^2\varphi(\bar{x}) & O \\ O & D^2\varphi(\bar{y}) \end{pmatrix}.
\end{aligned}$$

In the case $\bar{x} \in \partial\Omega$, we have

$$\begin{aligned}
\text{(7.5)} \qquad & \langle 2\beta(\bar{x} - z) + D_x w_\alpha(\bar{x},\bar{y}) + \beta D\varphi(\bar{x}), \gamma(\bar{x})\rangle \\
& \geqq 2\beta\langle \bar{x} - z, \gamma(\bar{x})\rangle - \delta_\alpha + \beta\eta_0 > 0
\end{aligned}$$

for sufficiently large $\alpha > 0$. Similarly, in the case $\bar{y} \in \partial\Omega$, we get

$$(7.6) \qquad \langle -D_y w_\alpha(\bar{x},\bar{y}) - \beta D\varphi(\bar{y}), \gamma(\bar{y})\rangle \leqq \delta_\alpha - \beta\eta_0 < 0$$

for sufficiently large $\alpha > 0$. Moreover, since it is easily observed from (7.4), $Mv \in LSC(\overline{\Omega})$ and Lemma 7.7 that

$$(7.7) \qquad \limsup_{\alpha\to+\infty}(v(\bar{y}) - Mv(\bar{y})) \leqq v(z) - Mv(z) < 0,$$

we obtain $v(\bar{y}) - Mv(\bar{y}) < 0$ for large $\alpha > 0$. Hence using (7.5), (7.6), (7.7) and the fact that u and v are, respectively, a subsolution and a supersolution of (7.1), we obtain the following inequalities:

$$F(\bar{x}, u(\bar{x}), 2\beta(\bar{x}-z) + D_x w_\alpha(\bar{x},\bar{y}) + \beta D\varphi(\bar{x}), X) \leqq 0,$$
$$F(\bar{y}, v(\bar{y}), -D_y w_\alpha(\bar{x},\bar{y}) - \beta D\varphi(\bar{y}), Y) \geqq 0.$$

By (A.5), (A.6)$'$, (A.7), (7.2) and Lemma 7.6 we observe

$$\begin{aligned}
\lambda\theta &\leqq \lambda(u(\bar{x}) - v(\bar{y})) \\
&\leqq F(\bar{y}, u(\bar{x}), -D_y w_\varepsilon(\bar{x},\bar{y}) - \beta D\varphi(\bar{y}), Y) \\
&\quad - F(\bar{x}, u(\bar{x}), 2\beta(\bar{x}-z) + D_x w_\varepsilon(\bar{x},\bar{y}) + \beta D\varphi(\bar{x}), X) \\
&\leqq F\left(\bar{y}, u(\bar{x}), -D_y w_\alpha(\bar{x},\bar{y}), Y + \delta_\alpha I + \beta D^2\varphi(\bar{y})\right) \\
&\quad - F\left(\bar{x}, u(\bar{x}), -D_y w_\alpha(\bar{x},\bar{y}), X - \delta_\alpha I - \beta(2I + D^2\varphi(\bar{x}))\right) \\
&\quad + \omega_3\left(\delta_\alpha + \beta(\|D\varphi\| + \|D^2\varphi\|)\right) \\
&\quad + \omega_3\left(2\delta_\alpha + \beta(2 + 2|\bar{x}-z| + \|D\varphi\| + \|D^2\varphi\|)\right) \\
&\leqq \omega_1\left(C_0|\bar{x}-\bar{y}|(\alpha|\bar{x}-\bar{y}| + 1) + \alpha C_0|\bar{x}-\bar{y}|^2\right) \\
&\quad + 2\omega_3\left(2\delta_\alpha + \beta(2 + 2|\bar{x}-z| + \|D\varphi\| + \|D^2\varphi\|)\right).
\end{aligned}$$

Letting $\alpha \to +\infty$ and then $\beta \to 0$, we obtain a contradiction.

Case 2. $z \in \Omega$.

We define the function $\Phi(x, y)$ on $\overline{\Omega} \times \overline{\Omega}$ by

$$\Phi(x, y) = u(x) - v(y) - \frac{\alpha}{2}|x - y|^2 - |x - z|^4.$$

By the same calculation as in *Case 1* with this function we also get a contradiction.

Thus we have completed the proof. ∎

Proof of Theorem 7.5. Let $C = \sup_{\overline{\Omega}} |F(x, 0, 0, O)|$. Then it is easily verified that $\underline{u}(x) \equiv -C$ and $\overline{u}(x) \equiv C$ are, respectively, a subsolution and a supersolution of (7.1). Thus by Perron's method and Theorem 7.4 we can show the existence of a unique solution u of (7.1) and $u \in C(\overline{\Omega})$. ∎

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