On High Dimensional Ribbon Knots

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神戸大学博士論文

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ON HIGH DIMENSIONAL RIBBON KNOTS

(高次元リボン結び目について)

by

Tomoyuki Yasuda

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Preface

A locally flat n-sphere embedded in the Euclidean $(n+2)$-space is called an n-knot. If $n \geq 2$, it is called a high dimensional knot. There is the so-called "motion picture method" to describe an n-manifold in the $(n+2)$-space. In this dissertation, by this method we will study an important class of n-knots which are called ribbon n-knots $(n \geq 2)$, which has been studied by T. Yajima, T. Yanagawa and others since 1960's.

In Chapter 1, we will give a simple method to calculate Alexander polynomials of ribbon n-knots and an estimation of the ribbon genus for ribbon n-knots.

A ribbon n-knot K^n are constructed by attaching m bands to $m+1$ n-spheres in the Euclidean $(n+2)$-space. There are many ways of attaching them; as a result, K^n has many presentations which are called ribbon presentations. From this point of view, we will study ribbon n-knots in Chapters 2 and 3.

I would like to express my sincere appreciation to Professor Fujitsugu Hosokawa and Professor Takaaki Yanagawa, who have been encouraging and taking care of me.

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Contents

Chapter 0

INTRODUCTION

0.1. Contents of the Dissertation

In Chapter 1, we will consider presentations and the genus for ribbon n-knots. Concerning n-knots K^n in the oriented Euclidean $(n+2)$-space $\mathbf{R}^{n+2}$, there is a quention "what type of $(n+1)$-manifolds can be bounded by K^n?". T. Yanagawa gave a partial answer that any ribbon n-knot bounds an $(n+1)$-manifold W^{n+1} homeomorphic to an $(n+1)$-disk D^{n+1} or $\#_{i=1}^{r}(S^n \times S^1)_i - {}^{\circ}\Delta^{n+1}$ with a trivial system in [Y1] and [Y2]. Furthermore, in [Y4] he defined the ribbon genus, $g(K^n)$, of a ribbon n-knot K^n by the lower bound of such integers r.

Until now, we have not known a method to compute $g(K^n)(n \geq 2)$. In Chapter 1, we will give an estimation of the ribbon genus by the degree of the Alexander polynomial, $deg\Delta_{K^n}(t)$, and the *width index* $wid(K^n)$ of K^n, which will be introduced in Section 1.2. We will also show how to construct an $(n+1)$-manifold bounded by K^n, which is homeomorphic to $\#_{i=1}^{wid(K^n)}(S^n \times S^1)_i - {}^{\circ}\Delta^{n+1}$. The main theorem in Chapter 1 is the following.

Theorem A. *For a ribbon n-knot $K^n(n \geq 2)$, we have*
$$deg\Delta_{K^n}(t) \leq g(K^n) \leq wid(K^n).$$

Moreover, for the case that $g(K^n)$ is greater than 1, we will give examples of inequivalent ribbon n-knots with the same Alexander polynomial and the same ribbon genus.

To give the above Theorem A, we will introduce a new presentation of a ribbon n-knot, which is called the (± 1)-*distribution presentation.* It has the following utilities:

(1) We can restore K^n (see 1.2.5).

(2) We can easily calculate the Alexander polynomial of K^n (see 1.2.6).

(3) We can estimate the ribbon genus of K^n (see the above Theorem A).

In Chapter 2, we will consider ribbon types of ribbon knots. Ribbon types are kinds of geometrical equivalence classes for ribbon presentations of ribbon knots. It has been a fundamental problem to study whether a ribbon knot has a unique ribbon type or not. For the trivial n-knot, it is known that it has the unique ribbon type as a ribbon knot of 1-fusion because we can interpret Scharlemann's theorem in [Sc] and Marumoto's theorem in [M1] as solutions in the cases of $n = 1$ and $n \geq 2$, respectively. For the case of non-trivial ribbon n-knots, Y. Nakanishi and Y. Nakagawa [NN] constructed ribbon 1-knots with distinct ribbon types in the case of $n = 1$. But the technique in [NN] cannot be applied in the case of $n \geq 2$. For an arbitrary integer $n \geq 2$, the above problem has been remained open. In Chapter 2, we will solve this problem for an arbitrary integer $n \geq 2$.

Theorem B. *For an arbitrary integer $n \geq 2$, there are infinitely many ribbon n-knots of 1-fusion, each of which has two ribbon types.*

We will spin two arcs which are obtained by removing different segments of a 2-bridge knot to construct ribbon n-knots of 1-fusion, which have distinct ribbon type. We will show that these two ribbon n-knots have distinct ribbon types by pointing out that group presentations of their knot guoups are not Nielsen equivalent.

In Chapter 3, we will consider a classification for ribbon presentations of ribbon knots. In the first place, we will induce a notion to classify ribbon presentations for ribbon n-knots of 2-fusions, which is applicable in general. In the second place, we will show that such classes form a totally ordered set by natural inclusion relation as follows:

Theorem C. *Let* $\mathbf{K}_m^n$ *be the set of all ribbon presentations for ribbon n-knots of m-fusions. There are four classes in* $\mathbf{K}_2^n$*, denoted by* $\mathbf{K}_{2,k}^n$ $(n \geq 2;\ k = 1, 2, 3, 4)$*, such that*

(i) $\mathbf{K}_1^n = \mathbf{K}_{2,1}^n \subsetneq \mathbf{K}_{2,2}^n \subsetneq \mathbf{K}_{2,3}^n \subsetneq \mathbf{K}_{2,4}^n = \mathbf{K}_2^n$*, and*

(ii) for any $k = 2, 3, 4$*, there exists a ribbon presentation* $(K^n, \{b_i\}_{i=1}^2)$ *in* $\mathbf{K}_{2,k}^n$ *such that every ribbon presentation in* $\mathbf{K}_{2,k-1}^n$ *is of distinct ribbon type to the given one.*

In the same way, we can also give a partially order in general cases. (see the corollary (3.3.2)). Moreover, we can give a negative answer to the following question which is a higher dimensional version of Gordon's conjecture [Go] (see the corollary (3.3.3)): Can a ribbon concordance be decomposed into a finite sequence of ribbon concordances with only one suddle-point and one minimal-point?

0.2. Notations and Preliminaries

Definition 0.2.1. An *n-link with* μ *components* is the image of locally flat embedding $e : \bigcup_{j=1}^{\mu} S_j^n \to \mathbf{R}^{n+2}$ of a disjoint union of μ copies of an oriented n-sphere into the oriented Euclidean $(n+2)$-space. In particular, an n-link with one component is called an *n-knot*. An n-link, which is the image of $e : \bigcup_{j=1}^{\mu} S_j^n \to \mathbf{R}^{n+2}$ with μ components, is called *trivial* if and only if there exists a locally flat embedding $e^* : \bigcup_{j=1}^{\mu} D_j^{n+1} \to \mathbf{R}^{n+2}$ with $e^*(\partial D_j^{n+1}) = e(S_j^n)$ $(j = 1, 2, \ldots, \mu)$, where D_j^{n+1} is the $(n+1)$-disk and ∂D_j^{n+1} is the boundary of D_j^{n+1}. Two n-links $K_1^n \cup K_2^n \cup \cdots \cup K_\mu^n$ in $\mathbf{R}^{n+2}$

and $K_1^{*n} \cup K_2^{*n} \cup \cdots \cup K_\mu^{*n}$ in $\mathbf{R}^{n+2}$ are said to be *equivalent* (or of *the same type*) if and only if there is an orientation preserving homeomorphism f of $\mathbf{R}^{n+2}$ onto itself with $f(K_j^n) = K_j^{*n} (j = 1, 2, \cdots, \mu)$.

Definition 0.2.2. We say K^n a *ribbon n-knot of m-fusions* if and only if the following condition $C_{n,m}$ is satisfied:

$C_{n,m}$: (1) there exists a trivial link $S_0^n \cup S_1^n \cup \cdots \cup S_m^n$ in $\mathbf{R}^{n+2}$,
(2) there exists an embedding of the direct product of D^n and the unit interval $I = [0,1]$, $f_i : D^n \times I \to \mathbf{R}^{n+2}$, for each i $(i = 1, 2, \ldots, m)$, such that

$$\text{(a) } f_i(D^n \times I) \cap S_j^n = \begin{cases} f_i(D^n \times \{0\}) & \text{if } j = 0, \\ f_i(D^n \times \{1\}) & \text{if } j = i, \\ \phi & \text{otherwise,} \end{cases}$$

(b) $f_i(D^n \times I) \cap f_j(D^n \times I) = \phi$ for $i \neq j$, and
(3) $K^n = S_0^n \cup S_1^n \cup \cdots \cup S_m^n \cup T^* - {}^\circ T$,
where $T^* = \bigcup_{j=1}^m f_j(\partial D^n \times I)$ and
${}^\circ T$ is the interior of $T = \bigcup_{j=1}^m f_j(D^n \times \partial I)$.

We call $f_i(D^n \times I)$ $(= b_i)$ an *i*th *band* of K^n. A ribbon n-knot of m-fusions $(m \geq 1)$ is sometimes called a ribbon n-knot shortly. We say $(K^n, \{b_i\}_{i=1}^m)$ a *ribbon presentation with m-bands* of K^n. Two ribbon presentations with m-bands of K^n, $(K^n, \{b_i\}_{i=1}^m)$ and $(K^n, \{b_i^*\}_{i=1}^m)$ are said to be *equivalent* (or of the *same ribbon type*) if and only if there exists an orientation preserving homeomorphism of $\mathbf{R}^{n+2}$ onto itself which maps K^n onto K^n and $\{b_i\}_{i=1}^m$ onto $\{b_i^*\}_{i=1}^m$.

The fundamental group $\pi_1(\mathbf{R}^{n+2} - K^n)$ is called the *knot group* of K^n. For an arbitrary ribbon n-knot of m-fusions, K^n, we construct a group presentation of the knot group for a ribbon presentation with m-bands of K^n as follows (cf. [Ya]). Let K^n be a ribbon n-knot of m-fusions and $(K^n, \{f_i(D^n \times I)\}_{i=1}^m)$ a ribbon presentation with m-bands of K^n. We denote $l_i = f_i(\{0\} \times I)$ $(i = 1, 2, \cdots, m)$, where

$\{0\}$ is the central point of D^n. l_i has its initial point $f_i(\{0\} \times \{0\})$ in S_0^n, and its terminal point $f_i(\{0\} \times \{1\})$ in S_i^n. For S_j^n in the definition (0.2.2), there are disjoint $(n+1)$-balls D_j^{n+1} such that $\partial D_j^{n+1} = S_j^n$ $(j = 0, 1, 2, \cdots, m)$.

We can assume that l_i intersects $\bigcup_{j=0}^m D_j^{n+1}$ transversely at finite points and we denote these points by $a_{i1}, a_{i2}, \cdots, a_{is_i}$ according to the direction of l_i, we obtain the word w_i, consisting of s_i letters as follows. If l_i intersects D_j^{n+1} at a_{ik} from its positive side (or negative side), we denote the kth letter of w_i is x_j (or x_j^{-1}, respectively). Here, x_j is corresponding to the meridian generator of S_j^n, and $x_0 w_i x_i^{-1} w_i^{-1}$ ($i = 1, 2, \cdots, m$) are the defining relators of the knot group of K^n. Then we obtain a group presentation of the knot group with $(m+1)$-generators and m-relators as follows:

$$(*) \qquad [\ x_j; j = 0, 1, 2, \cdots, m \mid x_0 w_i x_i^{-1} w_i^{-1}; i = 1, 2, \cdots, m\ ].$$

Definition 0.2.3. We call the group presentation obtained from a ribbon presentation $(K^n,\ \{b_i\}_{i=1}^m)$, by the above construction, a *group presentations associated with* $(K^n,\ \{b_i\}_{i=1}^m)$, denoted by $G(K^n, \{b_i\}_{i=1}^m)$. On the other hand, we can construct a ribbon presentation $(K^n,\ \{b_i\}_{i=1}^m)$ in the inverse procedure, from the group presentation $G(K^n, \{b_i\}_{i=1}^m)$. We call this ribbon presentation $(K^n,$ $\{b_i\}_{i=1}^m)$, a *ribbon presentation associated with* $G(K^n, \{b_i\}_{i=1}^m)$.

For a ribbon presentation $(K^n,\ \{b_i\}_{i=1}^m)$, we cannot have the unique group presentation. But this fact does not affect the following argument.

Chapter 1

RIBBON GENUS

Section 1.1 gives a notion of the *ribbon genus.* Section 1.2 includes notions and results on the (± 1)-*distribution presentation* and the *width index*, and in addition utilities of (± 1)-distribution presentation. Section 1.3 is devoted to a proof of the following theorem.

Theorem A. *For a ribbon n-knot $K^n (n \geq 2)$, we have*

$$deg\Delta_{K^n}(t) \leq g(K^n) \leq wid(K^n),$$

where $deg\Delta_{K^n}(t)$ is the degree of the Alexander polynomial of K^n, $g(K^n)$ is the ribbon genus of K^n, and $wid(K^n)$ is the width idex of K^n.

Section 1.3 gives an application of the Theorem A.

1.1. Ribbon Genus

The following theorem is known. By "$A \cong B$", we denote that A is homeomorphic to B.

Theorem 1.1.1([Y1],[Y2]). *Let K^n be a ribbon n-knot in $\mathbf{R}^{n+2}$. Then, K^n bounds an $(n+1)$-manifold W^{n+1} for K^n such that W^{n+1} is homeomorphic to an $(n+1)$-disk D^{n+1} or a connected-sum of some copies of $S^n \times S^1$ with an open $(n+1)$-disk missing, $\#_{i=1}^r (S^n \times S^1)_i - {}^\circ\Delta^{n+1}$. Moreover, when $W^{n+1} \cong \#_{i=1}^r (S^n \times S^1)_i - {}^\circ\Delta^{n+1}$, W^{n+1} has a trivial system of n-spheres which is defined as below.*

A collection $S_1^n, S_2^n, \cdots, S_{2r-1}^n, S_{2r}^n$ of mutually disjoint n-spheres in an $(n+1)$-manifold $W^{n+1} \cong \#_{i=1}^r (S^n \times S^1)_i - {}^\circ\Delta^{n+1}$ is called *a trivial system of n-spheres in W^{n+1} if and only if it satisfies the following;*

(1) $S_1^n \cup S_2^n \cup \cdots \cup S_{2r-1}^n \cup S_{2r}^n$ is a trivial n-link in $\mathbf{R}^{n+2}$,

(2) $S_i^n \cup S_{i+r}^n$ bounds a spherical-shell $N_i \cong S^n \times W^{n+1}$ where $N_i \cap N_j = \emptyset$ for $1 \leq i \leq j \leq r$,

(3) the closure $Cl(W^{n+1} - N_1 - N_2 - \cdots - N_r)$ is homeomorphic to

an $(n+1)$-sphere S^{n+1} with $2r+1$ open $(n+1)$-disks missing.

(The converse of this theorem is valid, too. See, [Y1] and [Y2].)

Definition 1.1.2. For an $(n+1)$ -manifold W^{n+1} in Theorem (1.1.1), we define $g(W^{n+1})$ by $g(W^{n+1}) = 0$ if $W^{n+1} \cong D^{n+1}$ and $g(W^{n+1}) = r$ if $W^{n+1} \cong \#_{i=1}^r (S^n \times S^1)_i - {}^\circ\Delta^{n+1}$. For a ribbon n-knot K^n, we define the *ribbon genus* $g(K^n)$ of K^n by the lower bound of integers $g(W^{n+1})$ for W^{n+1} bounded by K^n in $\mathbf{R}^{n+2}$.

1.2. (± 1)-distribution presentation

In this section, we will introduce a new presentation for a ribbon n-knot K^n which will be called (± 1)-*distribution presentation.* By making use of it, we will introduce the *width index* and utilities of (± 1)-distribution presentation.

Definition 1.2.1. By making use of words in $(*)$ in Section 0.2, We construct a set of words $w = (w_1, w_2, \cdots, w_m)$. We call it a *word presentation* of a ribbon n-knot K^n, where w_i $(i = 1, 2, \cdots, m)$ is the word corresponding to the i-th band of K^n. For the i-th word w_i, if it includes $x_s x_s^{-1}$ or $x_s^{-1} x_s$ $(s = 0, 1, 2, \cdots, m)$, we can reduce it because it has no sense geometrically. And because of the same reason, if the first letter of the word is x_0 or x_0^{-1}, we can reduce it, and if the last letter of the word is x_i or x_i^{-1}, we can reduce it. If the above reductions make a word empty, we redefine this word an one-letter word x_i, for the sake of convinience. If a word w is equal to w^* by the above reductions, we say that $w = w^*$ *modulo reductions of words.*

We define a (± 1)-distribution presentation of a ribbon n-knot by using a word presentation of K^n as follows. Let K^n be any ribbon n-knot of m-fusions.

Definition 1.2.2. We make correspondences between x_0^{-1}, x_0, x_s^{-1}, and x_s $(s = 1, 2, \cdots, m)$ and four (2×2) -lattices with directions z_1 and z_2 as shown in Fig. 1, Fig. 2, Fig. 3, and Fig. 4, respectively.

A (± 1)-distribution presentation of the i-th component w_i for a word presentation of K^n is composed by putting these four sorts of (2×2)-lattices on the infinite lattice with directions z_1 and z_2 according to the given word of K^n under the following rule. Now, we may assume that the i-th word w_i has u letters.

Putting rule is the following;
(1) we choose a (2×2)-lattice on the infinite lattice,
(2) on it, we put the (2×2)-lattice corresponding to the first letter of w_i such as the directions of the z_1- and z_2-axises coinside with those of the infinite lattice,
(3) we put the (2×2)-lattice corresponding to the k-th letter of the given word w_i as the directions of z_1- and z_2-axises coincide with those of the infinite lattice, and the initial point of its direct graph coincides with the terminal point of the direct graph in the (2×2)-lattice corresponding to the $(k - 1)$-th letter of the given word $w_i (2 \leq k \leq u)$, and
(4) lastly, we write '-1' in the square with the terminal point of the direct graph corresponding to the u-th letter.

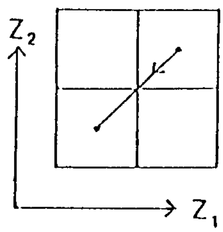


Fig. 1

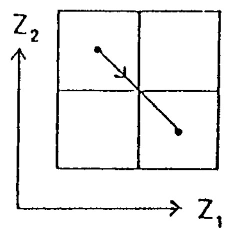


Fig. 2

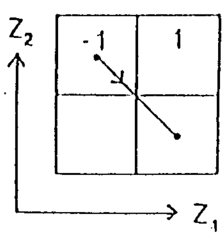


Fig. 3

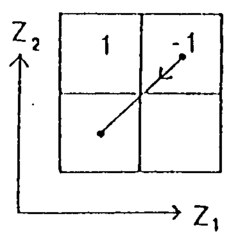


Fig. 4

In this way, we get a direct graph and a distribution of ± 1 on the infinite lattice, which is called the (± 1)-*distribution presentation* $D(w_i)$ *corresponding to* w_i $(i = 1{,}2, \cdots, m)$. And we call $(D(w_1), D(w_2), \cdots, D(w_m))$ a (± 1)-*distribution presentation* of K^n. We denote $D(K^n) = (D(w_1), D(w_2), \cdots, D(w_m))$. And the graph in $D(w_i)$ is denoted by $G(w_i)$.

Example 1.2.3. Let a set of words for K^n be

$$w = (w_1, w_2) = (x_1 x_2^{-1} x_0^{-1} x_2^3 x_1^{-1} x_0^{-1},\ x_2 x_0^{-1} x_1 x_0)$$

where K^n is a ribbon n-knot of 2-fusions. Fig. 5 shows sketch of K^n. And Fig. 6 shows the (± 1) -distribution presentation of K^n.

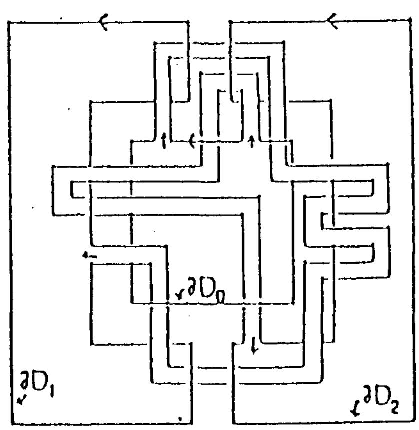


Fig. 5

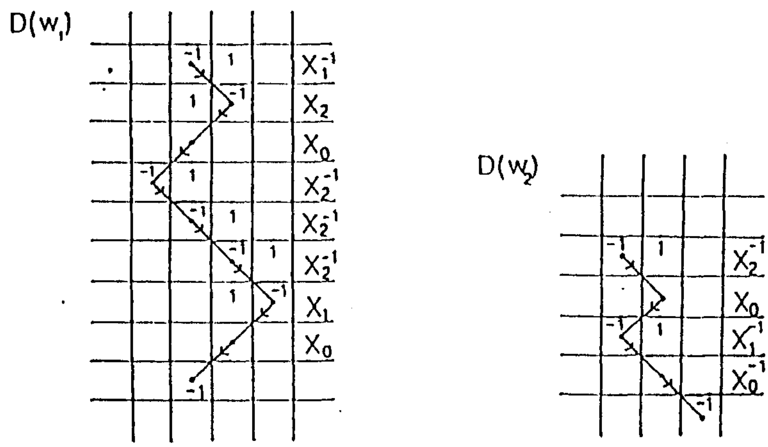


Fig. 6

Definition 1.2.4. The column in the infinite lattice is called the *the base column* of $D(w_i)$ if the initial point of $G(w_i)$ is in the column $(i = 1, 2, \cdots, m)$. If the terminal point of the graph $G(w_i)$ is in the cth column of the infinite lattice counting from the base column in the direction of z_1-axis, take the word $w_i^* = w_i x_i^{-c}$. Then, $G(w_i^*)$ has the terminal point in the base column and $w_i^* = w_i$ modulo reductions of words. For a fixed integer $j (j = 1, 2, \cdots, m)$, let $v_{ij1}, v_{ij2}, \cdots, v_{ijy_{ij}}, \cdots, v_{ijq_{ij}}$ be the vertices in (2×2)-lattices which are the initial and terminal points of edges corresponding to letters x_j and x_j^{-1} in $D(w_i)$. Now, for each $v_{ijy_{ij}}$, we define the integer $z(v_{ijy_{ij}})(y_{ij} = 1, 2, \cdots, q_{ij}; j = 1, 2, \cdots, m)$ as $(v_{ijy_{ij}})$ is in the $z(v_{ijy_{ij}})$th column of the infinite lattice counting from the base column in the direction of z_1-axis. For $D(K^n)$, we define:

$$
\begin{aligned}
&wid(x_j, x_j^{-1}) \\
&= max\{\ z_1(v_{ijy_{ij}}) \ \mid \ z_1(v_{ijy_{ij}}) \geq 0,\ 1 \leq y_{ij} \leq q_{ij},\ 1 \leq i \leq m\} \\
&+max\{\ |z_1(v_{ijy_{ij}})| \ \mid z_1(v_{ijy_{ij}}) \leq 0,\ 1 \leq y_{ij} \leq q_{ij},\ 1 \leq i \leq m\ \}
\end{aligned}
$$

and the *width number* $wid(D(K^n)) = \sum_{j=1}^{m} wid(x_j, x_j^{-1})$.

The minimum number of width numbers for all (± 1)- distribution presentations of K^n is called *width index* of K^n, and denoted by $wid(K^n)$.

For example, K^n in the example(1.2.3) has the width number 5 because
$wid(x_1, x_1^{-1})$=2+|0|=2 and $wid(x_2, x_2^{-1})$=2+| − 1|=3. In fact, this n-knot K^n has the width index 5, which will be shown by the corollary(1.3.4).

Fact 1.2.5 From a (± 1)-distribution presentation $D(K^n)$, we can restore a ribbon n-knot equivalent to K^n.

A method restoring K^n is clear according to the method of constructing $(*)$ in Section 0.2 and the inverse procedure.

Theorem 1.2.6. *Let K^n be a ribbon n-knot of m-fusions and $D(K^n)$ its (± 1)- distribution presentation where $D(K^n) =$ $(\ (D(w_1),\ D(w_2), \cdots, D(w_m)\ ))$. Get the sum of the numbers lining up in the columun where there are the initial and terminal points of edges corresponding to letters x_j and x_j^{-1} of $D(w_i)$ $(i, j = 1, 2, \cdots,$ $m)$. Let $n(v_{ijy_{ij}})$ $(y_{ij} = 1, 2, \cdots, q_{ij})$ be the sum of numbers corresponding to $z_1(v_{ijy_{ij}})$th column and $a_{ij} = \sum_{y_{ij}=1}^{q_{ij}} n(v_{ijy_{ij}}) t^{z_1(v_{ijy_{ij}})}$ where $v_{ij1}, v_{ij2}, \cdots, v_{ijy_{ij}}, \cdots, v_{ijq_{ij}}$ are the vertices which are the initial and terminal points of edges corresponding to letters x_j and x_j^{-1} in $D(w_i)$. Then, $A = (a_{ij})(i, j = 1, 2, \cdots, m)$ is an Alexander matrix and $|A|$ is the Alexander polynomial of K^n.*

Proof. Let K^n be a ribbon n-knot of m-fusions. Then, by $(*)$ in Section 0.2 the fundamental group $\pi_1(\mathbf{R}^{n+2} - K^n)$ has a group presentation

$$[\ x_j;\ j = 0, 1, \cdots, m \quad | \quad x_0 w_i x_i^{-1} w_i^{-1};\ i = 1, 2, \cdots, m\ ],$$

where x_j is the same generator as in $(*)$ of Section 0.2, $w_i = \prod_{k=1}^{u} x_{s(k)}^{\delta_{ik}} x_0^{\epsilon_{ik}-\delta_{ik}}$ $(1 \leq s(k) \leq m)$ and $(\epsilon_{ik}, \delta_{ik}) = (1,0)$, $(-1,0)$, $(1,1)$, or $(-1,-1)$ according as the kth letter in the word w_i of the ith band is x_0, x_0^{-1}, $x_{s(k)}$, or $x_{s(k)}^{-1}$, respectively $(i = 1, 2, \cdots, m)$ (cf.[Ki], [M1], [Y3]). Let r_i be $x_0(w_i)x_i^{-1}(w_i^{-1})$ $(i = 1, 2, \cdots, m)$. By making use of Fox's free differential calculus([F1], [F2], [Ki]), we have

$$\begin{aligned}
\Psi\Phi\left(\frac{\partial r_i}{\partial x_j}\right)_{1\leq i,j\leq m} &= \Psi\Phi\ \Bigg(\Big(x_0 \frac{\partial x_{s(1)}^{\delta_{i1}}}{\partial x_j} x_{s(1)}^{\delta_{i1}} \frac{\partial x_{s(1)}^{-\delta_{i1}}}{\partial x_j}\Big) \\
&+ \big(x_{s(1)}^{\delta_{i1}} x_0^{(\epsilon_{i1}-\delta_{i1})}\big)\Big(x_0 \frac{\partial x_{s(2)}^{\delta_{i2}}}{\partial x_j} + x_{s(2)}^{\delta_{i2}} \frac{\partial x_{s(2)}^{-\delta_{i2}}}{\partial x_j}\Big) \\
&+ \big(x_{s(1)}^{\delta_{i1}} x_0^{(\epsilon_{i1}-\delta_{i1})}\big)\big(x_{s(2)}^{\delta_{i2}} x_0^{(\epsilon_{i2}-\delta_{i2})}\big)\Big(x_0 \frac{\partial x_{s(3)}^{\delta_{i3}}}{\partial x_j} + x_{s(3)}^{\delta_{i3}} \frac{\partial x_{s(3)}^{-\delta_{i3}}}{\partial x_j}\Big) \\
&+ \cdots \\
&+ \big(x_{s(1)}^{\delta_{i1}} x_0^{(\epsilon_{i1}-\delta_{i1})}\big) \cdots \big(x_{s(u-1)}^{\delta_{iu-1}} x_0^{(\epsilon_{iu-1}-\delta_{iu-1})}\big)\Big(x_0 \frac{\partial x_{s(u)}^{\delta_{iu}}}{\partial x_j} + x_{s(u)}^{\delta_{iu}} \frac{\partial x_{s(u)}^{-\delta_{iu}}}{\partial x_j}\Big) \\
&+ \big(x_{s(1)}^{\delta_{i1}} x_0^{(\epsilon_{i1}-\delta_{i1})}\big) \cdots \big(x_{s(u)}^{\delta_{iu}} x_0^{(\epsilon_{iu}-\delta_{iu})}\big)\big(x_0 x_i^{-1}\big)(-1)\nu_{ij}\ \Bigg)_{1\leq i,j\leq m}
\end{aligned}$$

where $\nu_{ij} = \begin{cases} 0 & (i \neq j) \\ 1 & (i = j) \end{cases}$, and where Φ and Ψ are the following homomorphisms: Let F be the free group on generators $x_1, x_2, \cdots, x_m$ and φ the natural homomorphism of F onto $G = \pi_1(\mathbf{R}^{n+2} - K^n)$. Let H be the abelianization of G and ψ the natural homomorphism of G onto H. Let JF, JG, and JH be the group-rings of F, G, and H with integral coefficients, respectively. Then these homomorphisms φ and ψ can be linearly extended to homomorphisms of JF to JG and of JG to JH, respectively. We denote these extended homomorphisms by Φ and Ψ, respectively.

Therefore, we can obtain an Alexander matrix, $\left(f_{ij}(t)\right)_{1 \leq i,j \leq m}$ of K^n, where

$$
\begin{aligned}
(1.2.7) \qquad f_{ij}(t) = {} & t^{\delta_{i1}} - 1 \\
& + t^{\epsilon_{i1}}(\ t^{\delta_{i2}} - 1\ ) \\
& + t^{\epsilon_{i1}+\epsilon_{i2}}(\ t^{\delta_{i3}} - 1\ ) \\
& + \cdots \\
& + t^{\epsilon_{i1}+\epsilon_{i2}+\cdots+\epsilon_{iu-1}}(\ t^{\delta_{iu}} - 1\ ) \\
& + t^{\epsilon_{i1}+\epsilon_{i2}+\cdots+\epsilon_{iu-1}+\epsilon_{iu}}(-1)\nu_{ij}.
\end{aligned}
$$

Consequently, the coefficients of $f_{ij}(t)$ is calculated by entering 1 and/or -1 in the infinite lattice as in Fig. 7.

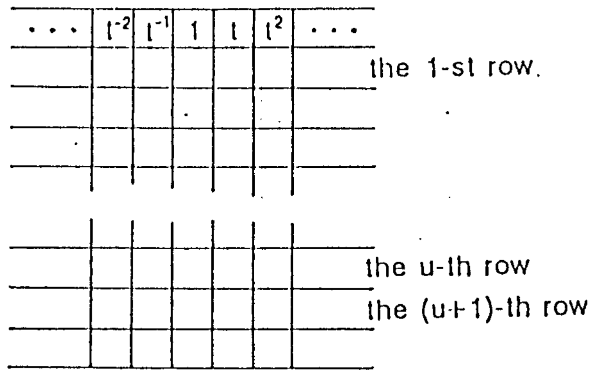


Fig. 7

We entry the coefficients -1 and/or 1 corresponding to the polynomial of the kth term in the formula(1.2.7) in the kth row ($k = 1, 2, \cdots, u, u+1$). When all the coefficients of a term in the formula(1.2.7) are 0, we do not entry any number. Hence, the coefficients of the term in the formula(1.2.7) corresponding to the kth letter of the word w_i is entried in the kth row ($k = 1, 2, \cdots, u$). Then, the distribution of ± 1 on the infinite lattice coincides with that of ± 1 corresponding to only x_j^{-1} and x_j in $D(w_i)$. The proof is complete.

Example 1.2.8. K^n in the example(1.2.3) has the Alexander polynomial

$$\Delta_{K^n}(t) = \begin{vmatrix} -2 + 2t - t^2 & -t^{-1} + 1 - t + t^2 \\ -1 + t & -1 + t - t^2 \end{vmatrix}$$
$$= -1 + 4t - 6t^2 + 7t^3 - 4t^4 + t^5 \quad (mod \pm t^r).$$

See Figs. 6 and 8.

$f_{11}(t)$	t^{-1}	1	t		
		-1	1		X_1^{-1}
					X_2
					X_0
					X_2^{-1}
					X_2^{-1}
					X_2^{-1}
			1	-1	X_1
					X_0
		-1			
		-2	2	-1	

$f_{12}(t)$	t^{-1}	1	t	t^2	
					X_1^{-1}
		1	-1		X_2
					X_0
	-1	1			X_2^{-1}
		-1	1		X_2^{-1}
			-1	1	X_2^{-1}
					X_1
					X_0
		-1			
	-1	0	-1	1	

$f_{21}(t)$	1	t	t^2	
	-1	1		X_2^{-1}
				X_0
				X_1^{-1}
				X_0^{-1}
			-1	
	-1	1	-1	

$f_{22}(t)$	1	t	t^2	
				X_2^{-1}
				X_0
	-1	1		X_1^{-1}
				X_0^{-1}
			-1	
	-1	1	-1	

Fig. 8

1.3. Proof of Theorem A

Definition 1.3.1. We define that x_i^h (or x_i^{-h}) is consisted of h "x_i" (x_i^{-1} resp.) ($h \geq 0; i = 1, 2, \cdots, m$). The following words are called the *basic words*;

$$
\begin{aligned}
w_{h_i}^{x_i, x_0^{-1}} &= x_i^{h_i - 1} x_i x_0^{-1} x_0^{-(h_i - 1)}, \\
w_{h_i}^{x_0, x_i^{-1}} &= x_0^{h_i - 1} x_0 x_i^{-1} x_i^{-(h_i - 1)}, \\
w_{h_i}^{x_i, x_i^{-1}} &= x_i^{h_i - 1} x_i x_0^{-1} x_i^{-(h_i - 1)}, \\
w_{h_i}^{x_0, x_0^{-1}} &= x_0^{h_i - 1} x_0 x_i^{-1} x_0^{-(h_i - 1)}, \\
w_{-h_i}^{x_i^{-1}, x_0} &= x_i^{-(h_i - 1)} x_i x_0^{-1} x_0^{-(h_i - 1)}, \\
w_{-h_i}^{x_0^{-1}, x_i} &= x_0^{-(h_i - 1)} x_0 x_i^{-1} x_i^{-(h_i - 1)}, \\
w_{-h_i}^{x_i^{-1}, x_i} &= x_i^{-(h_i - 1)} x_i x_0^{-1} x_i^{-(h_i - 1)}, \text{ and} \\
w_{-h_i}^{x_0^{-1}, x_0} &= x_0^{-(h_i - 1)} x_0 x_i^{-1} x_0^{-(h_i - 1)} \quad (h_i \geq 0, i = 1, 2, \cdots, m).
\end{aligned}
$$

The above h_i and $-h_i$ are called the *width* of each basic word and the graph corresponding to basic words is called *basic graphs.*

Lemma 1.3.2. *Let K^n be a ribbon n-knot of m-fusions and $D(K^n)$ $= (D(w_1), D(w_2), \cdots, D(w_m))$ a (± 1)-distribution presentation of K^n where each w_i is a word consisting of $2m + 2$ letters x_0, x_0^{-1}, x_1, x_1^{-1}, $\cdots$, x_m, x_m^{-1}. And let the graph in $D(K^n)$ be $G(K^n) =$ $(G(w_1), G(w_2), \cdots, G(w_m))$. Then, there exists a (± 1)-distribution presentation $D^*(K^n) = (D^*(w_1^*), D^*(w_2^*), \cdots, D^*(w_m^*))$ of K^n such that*

(i) the graph $G^*(w_i^*)$ in $D^*(w_i^*)$ is the union of basic graphs,
(ii) $w_i = w_i^*$ modulo reductions of words $(i = 1, 2, \cdots, m)$ and

(iii) $wid(D^*(K^n)) = wid(D(K^n))$.

Proof. We assume that the terminal point of $G(w_i)$ is the base column by replacing w_i by $w_i x_i^h$ for an appropriate integer h if it is necessary ($i = 1, 2, \cdots, m$). For $w = (w_1, w_2, \cdots, w_m)$, we can write

$w_i = x_0^{r_{10}} x_1^{r_{11}} \cdots x_m^{r_{1m}} x_0^{r_{20}} x_1^{r_{21}} \cdots x_m^{r_{2m}} \cdots x_0^{r_{s0}} x_1^{r_{s1}} \cdots x_m^{r_{sm}}$

$(r_{kj} = -1, 0, 1)$.

In w_i, we replace $x_j^{r_{kj}} x_{j+1}^{r_{kj+1}}$ by $x_j^{r_{kj}} x_0^{R_{kj}} x_0^{-R_{kj}} x_{j+1}^{r_{kj+1}}$ by turns from the first and second letters where $R_{kj} = \sum_{p=1}^{k-1}(\sum_{l=1}^{m}) r_{pl} + \sum_{l=1}^{j} r_{kl}$ ($i, j = 0, 1, \cdots, m$; $k = 1, 2, \cdots, s$). The resulting words satisfy (i), (ii), and (iii) ($i = 1, 2, \cdots, m$). The proof is complete.

Lemma 1.3.3. *Let K^n be a ribbon n-knot of m-fusions. If K^n has a (± 1)-distribution presentation $D(K^n)$ whose graphs are a union of basic graphs and its width number $wid(D(K^m))$ is r, then K^n bounds an $(n+1)$-manifold $W_{(r)}^{n+1} \simeq \#_{i=1}^{r}(S^n \times S^1)_i - {}^{\circ}\Delta^{n+1}$, where S^j is a j-sphere ($j = 1, n$), $\#$ means the connected sum, and ${}^{\circ}\Delta^{n+1}$ is the interior of an $(n+1)$-simplex Δ^{n+1}.*

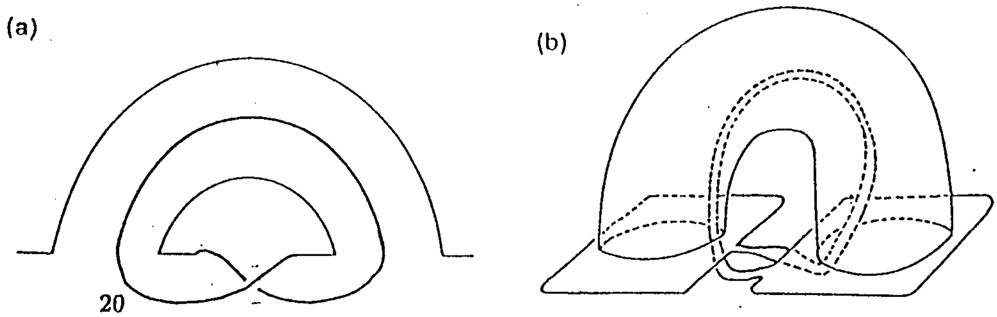
Proof. (Step 1) Let K^n be a ribbon n-knot of 1-fusion and w_i its word where $w_i = w_{11}\, w_{12} \cdots w_{1s_1}$, each w_{1j} is a basic word, and its width is h_{1j} ($j = 1, 2, \cdots, s_1$). We assume that

$p_1 = max\{\ h_{1j} \ \mid\ h_{1j} \geq 0\ (j = 1, 2, \cdots, s_1)\}$

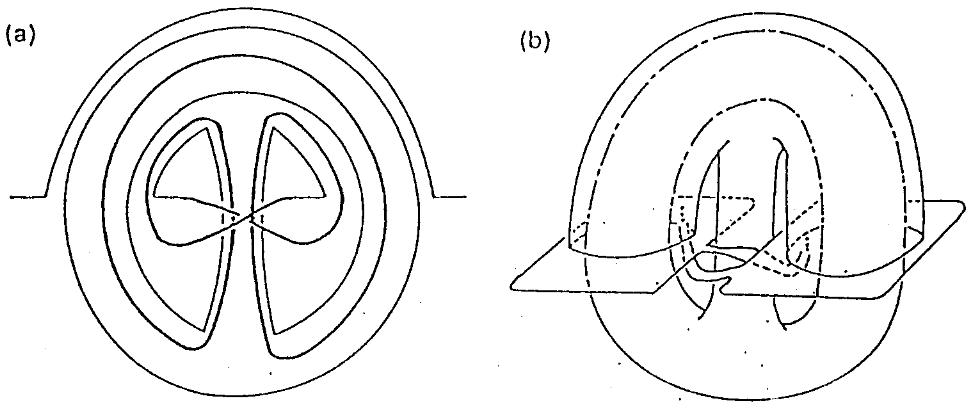
$q_1 = max\{\ |h_{1j}| \ \mid\ h_{1j} \leq 0\ (j = 1, 2, \cdots, s_1)\}$.

By the definition(1.3.1), $wid(K^n) \leq wid(x_1, x_1^{-1}) = p_1 + q_1$. If $w = w_1^{x_1, x_0^{-1}}$, $w_1^{x_0, x_1^{-1}}$, $w_1^{x_1, x_1^{-1}}$, and $w_1^{x_0, x_0^{-1}}$, the ribbon n-knot of 1-fusion with the word w bounds an $(n+1)$-manifold $W_1^{n+1} \simeq (S^n \times S^1) - {}^{\circ}\Delta^{n+1}$, where W_1^{n+1} is constructed by attaching a tube homeomorphic to $S^n \times I$ to n-disk as shown in Fig. 9(i). For the case $w = w_k^{x_1, x_0^{-1}}$, $w_k^{x_0, x_1^{-1}}$, $w_k^{x_1, x_1^{-1}}$, and $w_k^{x_0, x_0^{-1}}$ ($k \geq 2$), the ribbon n-knot of 1-fusion with the word w bounds an $(n+1)$-manifold

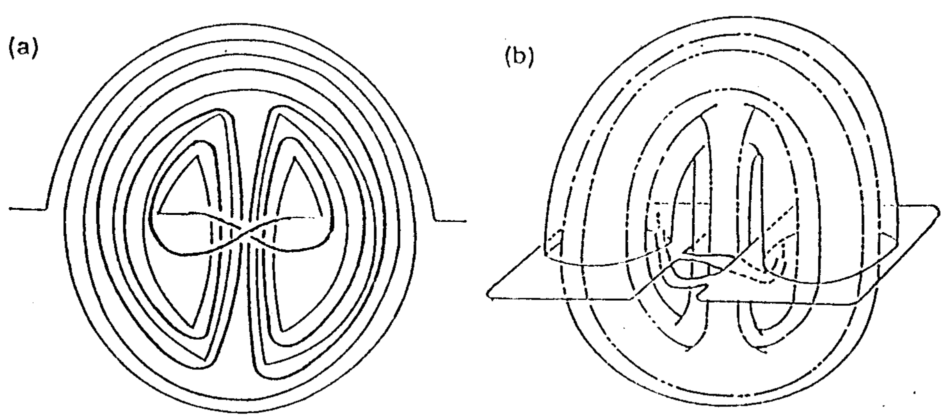
$W_k^{n+1} \simeq \#_{i=1}^k (S^n \times S^1)_i - {}^\circ\Delta^{n+1}$, where W_k^{n+1} is constructed by carrying out a connected-sum of $S^n \times S^1$ for W_{k-1}^{n+1} as shown in Fig. 9. Here, the heavy lines means the band for cases (i)-(a), (ii)-(a), and (iii)-(a), and we omit the bands corresponding to those heavy lines in (ii)-(b) and (iii)-(b). Moreover, (i)-(a), (ii)-(a), and (iii)-(a) are cross sections for (i)-(b), (ii)-(b), and (iii)-(b), respectively.



(i) The case that $w = w_{-1}^{x_1} \cdot x_0^{-1}$.



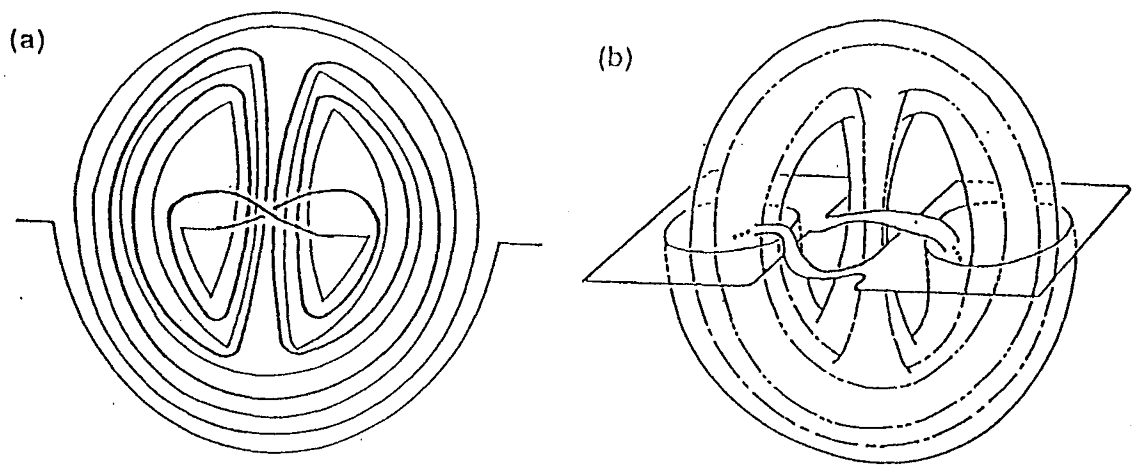
(ii) The case that $w = w_{-2}^{x_1} \cdot x_0^{-1}$



(iii) The case that $w = w_{-3}^{x_1} \cdot x_0^{-1}$

Fig. 9

On the other hand, in the similar way we can show that if $w = w_{-k}^{x_1^{-1},x_0}$, $w_{-k}^{x_0^{-1},x_1}$, $w_{-k}^{x_1^{-1},x_1}$, or $w_{-k}^{x_0^{-1},x_0}$, the ribbon n-knot of 1-fusion with the word w bounds an $(n+1)$-manifold $W_{-k}^{n+1} \simeq \#_{i=1}^{k}(S^n \times S^1)_i - {}^\circ\Delta^{n+1}$ as shown in Fig. 10. We show the band by heavy line in (a) and omit it in (b).

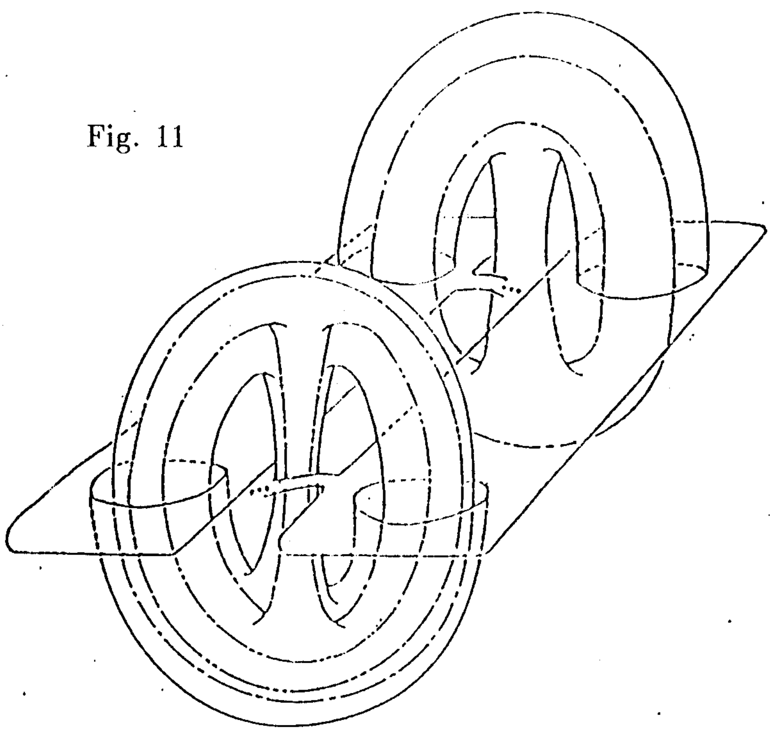


The case that $w = w_3^{x_1^{-1},x_0}$

Fig. 10

Seeing the above form of W_k^{n+1} $(k = 1, 2, \cdots)$, we find that K^n bounds $W_{p_1,q_1}^{n+1} \simeq \#_{i=1}^{p_1+q_1}(S^n \times S^1)_i - {}^\circ\Delta^{n+1}$ as shown in Fig. 11. Hence, if $D(K^n)$ has the width number $wid(D(K^n)) = p_1 + q_1 = r$, K^n bounds an $(n+1)$-manifold $W_{(r)}^{n+1} \simeq \#_{i=1}^{r}(S^n \times S^1)_i - {}^\circ\Delta^{n+1}$.

Fig. 11



The case that $p_1 = 3$ and $q_1 = 2$.

(Step 2) We assume that $D(K^n) = (D(w_1), D(w_2), \cdots, D(w_m))$, $w_i = w_{i1}\ w_{i2} \cdots w_{is_i}$ is a basic word consisting of $2m+2$ letters $x_0, x_0^{-1}, x_1, x_1^{-1}, \cdots, x_m, x_m^{-1}$, and its width number is h_{ij} ($j = 1, 2, \cdots, s_i$). In the similar way to (Step 1), if $wid(x_j, x_j^{-1}) = p_j + q_j$ where

$p_j = max\{\ h_{ij} \ \mid\ h_{ij} \geq 0;\ j = 1, 2, \cdots, s_i; i = 1, 2, \cdots, m\}$ and
$q_j = max\{\ |h_{ij}| \ \mid\ h_{ij} \leq 0;\ j = 1, 2, \cdots, s_i; i = 1, 2, \cdots, m\}$.

K^n bounds $W^{n+1} \simeq \#_{i=1}^{\sum_{i=1}^{m}(p_i+q_i)}(S^n \times S^1)_i - {}^\circ\Delta^{n+1}$. Therefore, if $D(K^n)$ has the width number $wid(D(K^n)) = \sum_{j=1}^{m} \{wid(x_j, x_j^{-1})\}$ $= r$, K^n bounds an $(n+1)$-manifold $W_{(r)}^{n+1} \simeq \#_{i=1}^{r}(S^n \times S^1)_i - {}^\circ\Delta^{n+1}$. The proof is complete.

Proof of Theorem A. Let D be a (± 1) -distribution presentation attaining the width index r of K^n. We may assume that its graph is an union of basic graphs by the lemma(1.3.2). And then, by the lemma(1.3.3), if $wid(D) = r$, K^n bounds $W_{(r)}^{n+1}$. Hence, $g(K^n) \leq wid(D) = wid(K^n)$.

On the other hand, we shall prove the other inequality in the Theorem 1. The following has been suggested by T. Yanagawa and Y. Nakanishi to the author.

Let $\tilde{X}$ be the infinite cyclic covering of $X = S^{n+2} - K^n$ and W_r^{n+1} an $(n+1)$-manifold homeomorphic to $\#_{i=1}^{r}(S^n \times S^1)_i - {}^\circ\Delta^{n+1}$ which is bounded by K^n. We have

$$H_1(S^{n+2} - W_r^{n+1};\ \mathbb{Z}) \simeq H^{(n+1)-1}(W_r^{n+1};\ \mathbb{Z}) \simeq \mathbb{Z}^r$$

by the Alexander duality and $H_1(W_r^{n+1};\ \mathbb{Z}) \simeq \mathbb{Z}^r$. Therefore, $H_1(\tilde{X})$ has r generators and r relations by the method of the construction for $\tilde{X}$, and then we have an $(r \times r)$-matrix as an Alexander matrix where every entry is a polynomial with the degree of at most one. Hence, for any ribbon n-knot K^n and the degree of $\Delta_{K^n}(t)$, $deg\Delta_{K^n}(t)$, we have $deg\Delta_{K^n}(t) \leq g(K^n)$. The proof of Theorem 1 is complete.

By a (± 1)-distribution presentation of K^n, we can easily obtain $deg\Delta_{K^n}(t)$ and $wid(K^n)$.

Corollary 1.3.4. *Let K^n be a ribbon n-knot of m-fusions.*
If $deg\Delta_{K^n}(t) = wid(K^n) = g$, then $g(K^n) = g$.

Example 1.3.5. The knot in the example(1.2.3) is a ribbon n-knot with the ribbon genus 5.

1.4. Application

The following theorem is known.

Theorem 1.4.1([Y4]). *Suppose that K_1^n and K_2^n are ribbon n-knots with $g(K_1^n) = g(K_2^n) = 1$. If their Alexander polynomials are identical with together, K_1^n belong to the same knot type as either K_2^n or $-K_2^n$, where $-K_2^n$ denotes an n-knot with the reversed orientation of K_2^n.*

We obtain the following theorem.

Theorem 1.4.2. *There exist ribbon n-knots K_1^n and K_2^n such that*
(i) $g(K_1^n) = g(K_2^n) = g$ $(g \geq 2)$,
(ii) $\Delta_{K_1^n} = \Delta_{K_2^n}$ and
(iii) K_1^n and K_2^n are inequivalent and is not the inverse of the other.

Proof. (Case 1) $g = 2k + 1 (k \geq 1)$.
Let $K_{2k+1,1}^n$ and $K_{2k+1,2}^n$ be ribbon n-knots of 1-fusion such that their words are

$$\begin{aligned} w_{2k+1,1} &= x_1 x_0^{-1} x_0^{-2k+1} x_1^{-1} x_0^{2k} x_1 x_0^{-1} \\ &= w_1^{x_1, x_0^{-1}} w_{-2k+1}^{x_0^{-1}, x_1} w_{-2k}^{x_1^{-1}, x_0} w_1^{x_1, x_0^{-1}}, \text{and} \\ w_{2k+2,2} &= x_1 x_0^{-1} x_1 x_0^{-1} x_0^{-2k+1} x_1^{-1} x_0^{2k} \\ &= w_1^{x_1, x_0^{-1}} w_1^{x_1, x_0^{-1}} w_{-2k+1}^{x_0^{-1}, x_1} w_{-2k}^{x_1^{-1}, x_0}, \end{aligned}$$

respectively. By their (± 1)-distribution presentations

$$\Delta_{K_{2k+1,i}^n}(t) = 2 - 3t^{-1} - t^{-2k} + t^{-2k-1} \quad \text{and}$$
$$wid(D(W_{2k+1,i})) = 2k + 1 \qquad (i = 1, 2).$$

Hence, $g(K_{2k+1,i}^n) = 2k + 1$ by the corollary(1.3.4) $(i = 1, 2)$.
The knot group of $K_{2k+1,i}^n$ has a group presentation:

$K^n_{2k+1,1}: [x_0, x_1 \mid x_0 x_1 x_0^{-2k} x_1^{-1} x_0^{2k} x_1 x_0^{-1} x_1^{-1} x_0 x_1^{-1} x_0^{-2k} x_1 x_0^{2k} x_1^{-1}]$,
$K^n_{2k+1,2}: [x_0, x_1 \mid x_0 x_1 x_0^{-1} x_1 x_0^{-2k} x_1^{-1} x_0^{2k} x_1^{-1} x_0^{-2k} x_1 x_0^{2k} x_1^{-1} x_0 x_1^{-1}]$.
Let $(M^{n+2}_{2k+1,i}, L^n_{2k+1,i})$ be the 3-fold irregular branched covering space of $(S^{n+2}, K^n_{2k+1,i})$ $(i = 1, 2)$. Then $L^n_{2k+1,i}$ in $M^{n+2}_{2k+1,i}$ has the distinct reduced Alexander polynomial $\tilde{\nabla}_{2k+1,i}(t)$ (which is called the $\tilde{\nabla}$-polynomial of the knot $K^n_{2k+1,i}$ by K. Murasugi([Mu]) such that

$$\begin{aligned} &\tilde{\nabla}_{2k+1,1}(-1) = 3k + 3 \quad \text{and} \\ &\tilde{\nabla}_{2k+1,2}(-1) = -6k - 1 \quad (i = 1, 2). \end{aligned}$$

This fact shows that $K^n_{2k+1,1}$ and $K^n_{2k+1,2}$ are inequivalent and one is not the inverse of the other.

(Case 2) $g = 2k$ $(k \geq 1)$.

Let $K^n_{2k,1}$ and $K^n_{2k,2}$ be ribbon n-knots of 1-fusion such that their words are

$$\begin{aligned} w_{2k,1} &= x_1 x_0^{-1} x_1 x_0^{-2k+1} x_1^{-1} x_0^{2k-1} x_1 x_0^{-1} \\ &= w_1^{x_1, x_0^{-1}} w_1^{x_1, x_0^{-1}} w_{-2k+2}^{x_0^{-1}, x_1} w_{-2k+1}^{x_1^{-1}, x_0} w_1^{x_1, x_0^{-1}}, \quad \textit{and} \\ w_{2k,2} &= x_1 x_0^{-1} x_1 x_0^{-1} x_1 x_0^{-2k+1} x_1^{-1} x_0^{2k-1} \\ &= w_1^{x_1, x_0^{-1}} w_1^{x_1, x_0^{-1}} w_{-2k+2}^{x_0^{-1}, x_1} w_{-2k+1}^{x_1^{-1}, x_0}, \end{aligned}$$

respectively. By their (± 1)-distribution presentations

$$\begin{aligned} &\Delta_{K^n_{2k,i}}(t) = 3 - 4t^{-1} - t^{-2k+1} + t^{-2k} \quad \text{and} \\ &wid(D(W_{2k,i})) = 2k \quad (i = 1, 2). \end{aligned}$$

Hence, $g(K^n_{2k,i}) = 2k$ by the corollary(1.3.4) $(i = 1, 2)$.

The knot group of $K^n_{2k,i}$ has a group presentation:

$K^n_{2k,1}: [x_0, x_1 \mid x_0 x_1 x_0^{-1} x_1 x_0^{-2k+1} x_1^{-1} x_0^{2k-1} x_1 x_0^{-1} x_1^{-1} x_0 x_1^{-1} x_0^{-2k+1}$

$x_1 x_0^{2k-1} x_1^{-1} x_0 x_1^{-1}]$,
$K^n_{2k,2} : [x_0, x_1 \mid x_0 x_1 x_0^{-1} x_1 x_0^{-1} x_1 x_0^{-2k+1} x_1^{-1} x_0^{2k-1} x_1^{-1} x_0^{-2k+1}$
$x_1 x_0^{2k-1} x_1^{-1} x_0 x_1^{-1} x_0 x_1^{-1}]$.

Let $(M^{n+2}_{2k,i}, L^n_{2k,i})$ be the 3-fold irregular branched covering space of $(S^{n+2}, K^n_{2k,i})$ $(i = 1,2)$. Then $L^n_{2k,i}$ in $M^{n+2}_{2k,i}$ has the distinct reduced Alexander polynomial $\tilde{\nabla}_{2k,i}(t)$ (which is called the $\tilde{\nabla}$-polynomial of the knot $K^n_{2k+1,i}$ by K. Murasugi([Mu]) such that

$$\begin{aligned} &\tilde{\nabla}_{2k,1}(-1) = -9k + 9 \quad \text{and} \\ &\tilde{\nabla}_{2k,2}(-1) = -7k + 7 \quad (i = 1,2). \end{aligned}$$

This fact shows that $K^n_{2k,1}$ and $K^n_{2k,2}$ are inequivalent and one is not the inverse of the other. The proof is complete.

Chapter 2

RIBBON KNOTS WITH TWO RIBBON TYPES

Section 2.1 gives a notion of the *Nielsen equivalence*, which plays an important part in discriminating whether two ribbon presentations are same ribbon type. Section 2.2 is devoted to a proof of the following theorem.

Theorem B. *For an arbitrary integer $n \geq 2$, there are infinitely many ribbon n-knots of 1-fusion, each of which has two ribbon types.*

2.1. Nielsen Equivalence

There is the following equivalence relation for group presentations with two generators and one defining relator ([LM],[MKS]).

Definition 2.1.1. Let $[\ x, y \mid \ R\ ]$ and $[\ x^*, y^* \mid \ R^*\ ]$ be group presentations of a given group with two generators and one relator. These two group presentations are said to be *Nielsen equivalent* if and only if there exists an isomorphism for free groups, $\varphi : [\ x, y \mid \quad] \to [\ x^*, y^* \mid \quad]$, such that $\varphi(R)$ is conjugate to $R^{*\pm 1}$.

We call the group presentation obtained from (K^n, b) by construction of the group presentation $(*)$ in Section 0.2, the *group presentation associated with* (K^n, b). By the definition(2.1.1) and the construction of $(*)$, we can obtain the following lemma.

Lemma 2.1.2. *Let (K^n, b) and (K^n, b^*) be ribbon presentations of a ribbon n-knot of 1-fusion, K^n. And let $G(K^n, b)$ and $G(K^n, b^*)$*

the group presentations associated with (K^n, b) *and* (K^n, b^*), *respectively. If* (K^n, b) *and* (K^n, b^*) *are of the same ribbon type, then* $G(K^n, b)$ *and* $G(K^n, b^*)$ *are Nielsen equivalent.*

2.2. Proof of Theorem B.

Concerning arbitrary two group presentations with two generators and one relator for the knot group of a 2-bridge knot, Funcke [Fu] gave a necessary and sufficient condition to determine whether these group presentations are Nielsen equivalent or not. After Schubert's notation ([S]), let $S(\alpha,\beta)$ be a 2-bridge knot, where α and β are coprime integers such that $\alpha > 0, -\alpha < \beta < \alpha$, and β is odd.

Lemma 2.2.1([S]). *$S(\alpha,\beta)$ and $S(\alpha^*,\beta^*)$ are of the same knot type, if and only if (i) $\alpha = \alpha^*$, and (ii) $\beta = \beta^*$ or $\beta\beta^* \equiv 1 \bmod 2\alpha$.*

Let $G(S(\alpha,\beta))$ and $G(S(\alpha,\beta^*))$ be the group presentations for the knot groups of $S(\alpha,\beta)$ and $S(\alpha,\beta^*)$, respectively, each of which is a group presentation with two generators and one relator, where the elements corresponding to two overbridges of the knot diagram.

Lemma 2.2.2([Fu]). *If $\beta\beta^* \equiv 1 \bmod 2\alpha$ and $\beta^* \neq \pm\beta$, then $G(S(\alpha,\beta))$ and $G(S(\alpha,\beta^*))$ are Nielsen equivalent.*

In the following, we construct an n-knot by spinning an arbitrary 1-knot (cf. [A], [C], [Su]). $S(\alpha,\beta)$ has two underbridges, γ_{u1} and γ_{u2}, and two overbridges, γ_{o1} and γ_{o2}. Let $\mathbf{R}^2_t$ be the plane defined by $\{\ (z_1,z_2,\ldots,z_{n+2}) \in \mathbf{R}^{n+2} \mid z_3 = t,\ z_4 = \cdots = z_{n+2} = 0\ \}$. We can put γ_{u1} and γ_{u2} on $\mathbf{R}^2_1$, and put γ_{o1} and γ_{o2} in $\mathbf{H}^3_1$ (see Fig. 1), where $\mathbf{H}^3_t$ is the half space difined by $\{\ (z_1,z_2,\ldots,z_{n+2}) \in \mathbf{R}^{n+2} \mid z_3 \geq t,\ z_4 = \cdots = z_{n+2} = 0\ \}$. We obtain the arc, denoted by γ, by removing $^\circ\gamma_{u2}$ from $S(\alpha,\beta)$. Pull the boundary of γ down $\mathbf{R}^2_0$ parallel to the z_3-axis in $\mathbf{H}^3_0$, then we have the proper arc in $\mathbf{H}^3_0$, denoted by $\gamma(S(\alpha,\beta))$, with its boundary in $\mathbf{R}^2_0$. Both γ_{o1} and γ_{o2} have one endpoint in $\mathbf{R}^2_0$ and the other endpoint in $\mathbf{R}^2_1$. We spin $\gamma(S(\alpha,\beta))$ about $\mathbf{R}^2_0$. Then, $\gamma(S(\alpha,\beta))$ sweep out an n-sphere. We call this n-sphere the spun n-knot of $S(\alpha,\beta)$, denoted by $\mathbf{K}_s^n(S(\alpha,\beta))$. It is a ribbon n-knot of 1-fusion because the loca of $\gamma_{o1}\cup\gamma_{o2}$ and γ_{u1} correspond to $S_0^n\cup S_1^n - {}^\circ T$ and T^* in the definition (0.2.2), respectively. Therefore, we can obtain the following lemma.

Lemma 2.2.3. *For an arbitrary 2-bridge knot, $S(\alpha,\beta)$, its spun n-knot, denoted by $K_s^n(S(\alpha,\beta))$, is a ribbon n-knot, and has a ribbon presentation with 1-band.*

For the spun n-knot of the lemma (2.2.3), we denote its ribbon presentation by $K_s^n(S(\alpha,\beta),b_s)$, and the group presentation associated with $G(K_s^n(\alpha,\beta),b_s))$. By the construction of $K_s^n(S(\alpha,\beta))$ we can obtain the following lemma.

Lemma 2.2.4. $G(S(\alpha,\beta)) = G(K_s^n(S(\alpha,\beta),b_s))$.

Two 2-bridge knots, denoted by $k^1 = S(\alpha,\beta)$ and $k^{*1} = S(\alpha,\beta^*)$, satisfying the condition in the lemma (2.2.2) are of the same knot type by the lemma (2.2.1). Therefore, their spun n-knots, $K_s^n(k^1)$ and $K_s^n(k^{*1})$ are also of the same knot type. By the lemma (2.2.3), they are ribbon n-knots of 1-fusion, and by the lemma (2.2.2), $G(K_s^n(k^1),b_s)$ and $G(K_s^n(k^{*1}),b_s^*)$ are Nielsen inequivalent. Then, by the lemma (2.1.2), these two ribbon presentations with 1-band are distinct ribbon types.

Therefore, we have infinitely ribbon n-knots with two ribbon types because there are infinitely many pairs of 2-bridge knots satisfying the condition in the lemma (2.2.2). The proof of Theorem is complete.

Example 2.5. The 5_2-knot ([AB], [R]) has two knot diagrams as shown in Figs. 12 and 14, the spun 2-knot of 5_2-knot has two ribbon types, which are $K_s^2(S(7,-3),b_s)$ and $K_s^2(S(7,-5),b_s^*)$. Their equatorial sections are indicated in Figs. 13 and 15, respectively, and their group presentations are the following:

$G(K_s^2(S(7,-3),b_s))$
$= [\ x_0, x_1 \mid x_0x_1^{-1}x_0^{-1}x_1x_0x_1^{-1}x_0^{-1}x_1^{-1}x_0x_1x_0^{-1}x_1^{-1}x_0x_1\ ],$
$G(K_s^2(S(7,-5),b_s^*))$
$= [\ x_0^*, x_1 \mid x_0^*x_1^{-1}x_0^*x_1^{-1}x_0^{*-1}x_1x_0^{*-1}x_1^{-1}x_0^*x_1^{-1}x_0^*x_1x_0^{*-1}x_1\ ].$

These two group presentations are not Nielsen equivalent by the lemma (2.2.2), but there exists an isomorphism $\varphi : G(K_s^2(S(7,-5),b_s^*)) \to G(K_s^2(S(7,-3),b_s))$ such that $\varphi(x_0^*) = x_0^{-1}x_1x_0$.

$\tilde{S}(7,-3)$

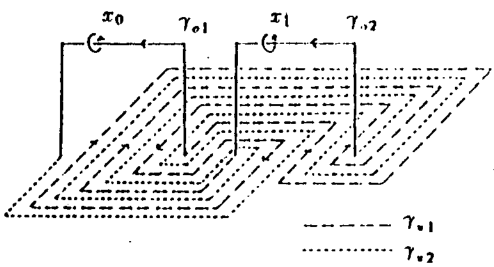


Fig. 12

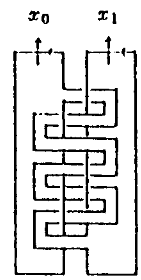


Fig. 13

$S(7,-5)$

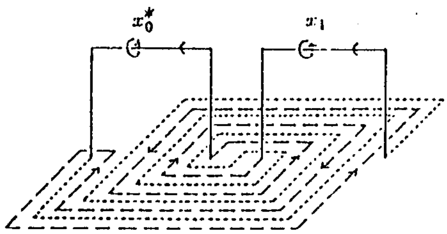


Fig. 14

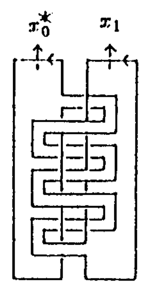


Fig. 15

Chapter 3

RIBBON PRESENTATIONS

Section 3.1 gives a notion on classification of ribbon presentations, and Section 3.2 is devoted to a proof of the following theorem.

Theorem C. *Let* $\mathbf{K}^n_m$ *be the set of all ribbon presentations for ribbon* n*-knots of* m*-fusions. There are four classes in* $\mathbf{K}^n_2$*, denoted by* $\mathbf{K}^n_{2,k}$ $(n \geq 2;\ k = 1, 2, 3, 4)$*, such that*

(i) $\mathbf{K}^n_1 = \mathbf{K}^n_{2,1} \subsetneq \mathbf{K}^n_{2,2} \subsetneq \mathbf{K}^n_{2,3} \subsetneq \mathbf{K}^n_{2,4} = \mathbf{K}^n_2$*, and*

(ii) for any $k = 2, 3, 4$*, there exists a ribbon presentation* $(K^n, \{b_i\}^2_{i=1})$ *in* $\mathbf{K}^n_{2,k}$ *such that every ribbon presentation in* $\mathbf{K}^n_{2,k-1}$ *is of distinct ribbon type to the given one.*

Section 3.3 includes an extension of the above Theorem C and answers to problems on the ribbon concordance.

3.1. Classes on Ribbon Presentations

A useful set of invariants of knot types is the chain of elementary ideals and the equivalence classes on Alexander matrices ([CF]), which also include informations of the ribbon types if we calculate them from a group presentation associated with a ribbon presentation. From such a viewpoint, we calculate the chain of elementary ideals and Alexander matrices, and classify ribbon presentations.

Definition 3.1.1. Let $(K^n, \{b_i\}^m_{i=1})$ be a ribbon presentation with m-bands of a ribbon n-knot K^n, G a group presentation associated with $(K^n, \{b_i\}^m_{i=1})$, and
r_i $(i = 1, 2, \cdots, m)$ defining relators of G. By Fox's free differential calculus ([F1],[F2],[Ki]), we have $\Psi\Phi\left(\frac{\partial r_i}{\partial x_j}\right)_{1 \leq i,j \leq m}$ where Ψ and Φ are the homomorphisms in 'Proof' of the theorem(1.2.6).

We define the *Alexander matrix associated with* G by

$$A(G) = (a_{ij}) = \Psi\Phi\left(\frac{\partial r_i}{\partial x_j}\right)_{1 \leq i,j \leq m}.$$

For two matrices with entries in JH, A and A^*, A is said to be *equivalent* to A^*, if there exists a finite sequence of matrices, $A = A_1, A_2, \cdots, A_n = A^*$, such that A_{i+1} is obtained from A_i, or vice-versa, by one of the following operations:

(i) Permuting rows or permuting columns.

(ii) Adjoining a row of zeros; $A \to \begin{pmatrix} A \\ 0 \end{pmatrix}$.

(iii) Adding to a row a linear combination of other rows.

(iv) Adding to a column a linear combination of other columns.

(v) Adjoining a new row and column such that the entry in the intersection of the new row and column is 1, and the remaining entries in the new row and column are all 0; $A \to \begin{pmatrix} A & 0 \\ 0 & 1 \end{pmatrix}$.

The following lemma is known.

Lemma 3.1.2 ([CF]). *Let K^n be an n-knot, and G and G^* group presentations of K^n. Then, $A(G)$ and $A(G^*)$ are equivalent.*

Definition 3.1.3. For an arbitrary non-negative integer d, we define the dth *elementary ideal* of $A(G)$, denoted by $E_d(A(G))$, by the ideal of JH generated by the $(m-d) \times (m-d)$ minors of $A(G)$, with the conventions: (i) $m - d \leq 0$ then $E_d(A(G)) = JH$. (ii) $m - d \geq m + 1$ then $E_d(A(G)) = 0$.

The following lemmas are known.

Lemma 3.1.4([CF]). *The elementary ideals of $A(G)$ form an ascending chain,*

$E_0(A(G)) \subset E_1(A(G)) \subset \cdots \subset E_m(A(G)) = \cdots = JH.$

Lemma 3.1.5([CF]). *Equivalent matrices define the same chain of elementary ideals.*

Definition 3.1.6. Let $(K^n, \{b_i\}_{i=1}^2)$ be an arbitrary ribbon presentation of a ribbon n-knot of 2-fusions K^n. We apply the construction of $(*)$ in Section 1 to $(K^n, \{b_i\}_{i=1}^2)$. Then $(K^n, \{b_i\}_{i=1}^2)$ is said to belong to $\begin{pmatrix} * & \\ & * \end{pmatrix}$ -*class*, $\begin{pmatrix} * & \\ * & * \end{pmatrix}$ -*class*, and $\begin{pmatrix} * & * \\ * & * \end{pmatrix}$ -*class*, if and only if $(K^n, \{b_i\}_{i=1}^2)$ is a ribbon presentation such that l_1 and l_2 satisfy the following conditions (i),(ii), and (iii), respectively;

(i) l_1 does not intersect D_2^{n+1}.
And l_2 does not intersect D_1^{n+1}.

(ii) l_1 does not intersect D_2^{n+1}.

(iii) No restriction.

By the definitions(3.1.3) and (3.1.6), we obtain the following propositions.

Proposition 3.1.7. *Let K^n be a ribbon knot of 1-fusion and (K^n, b) its ribbon presentation. Then, $E_1(A(G(K^n, b)))$ is a principal ideal and $E_2(A(G(K^n, b)))$ is JH.*

Proposition 3.1.8. *Let K^n be a ribbon n-knot of 2-fusions, $(K^n, \{b_i\}_{i=1}^2)$ its ribbon presentation, and G a group presentation associated with $(K^n, \{b_i\}_{i=1}^2)$.*
Then, $A(G)$ is a 2×2 matrix such that

(i) the $(1,2)$- and $(2,1)$-components are 0, if $(K^n, \{b_i\}_{i=1}^2)$ belongs to

$\begin{pmatrix} * & \\ & * \end{pmatrix}$ -*class,* *(ii) the* $(1,2)$*-component is* 0, *if* $(K^n, \{b_i\}_{i=1}^2)$ *belongs to* $\begin{pmatrix} * & \\ * & * \end{pmatrix}$ *-class.*

3.2. Proof of Theorem C

By making use of the chain of elementary ideals obtained from the group presentation of a ribbon n-knot, we can find a difference between ribbon n-knots of 1-fusion and ribbon n-knots of 2-fusions, and also among three classes on ribbon types in n-knots of 2-fusions. Let $\mathbf{K}_m^n$ be the set of all ribbon presentations for ribbon n-knots of m-fusions ($m = 1, 2$), and $\mathbf{K}_{2,2}^n, \mathbf{K}_{2,3}^n$, and $\mathbf{K}_{2,4}^n$ the set of all ribbon presentations belonging to $\begin{pmatrix} * & \\ & * \end{pmatrix}$ -class $\begin{pmatrix} * & \\ * & * \end{pmatrix}$ -class, and $\begin{pmatrix} * & * \\ * & * \end{pmatrix}$ -class, respectively. By using the conditions (i),(ii), and (iii) in the definition (3.1.6), we have concluded that $\mathbf{K}_{2,2}^n \subset \mathbf{K}_{2,3}^n \subset \mathbf{K}_{2,4}^n = \mathbf{K}_2^n$. On the other hand, for any element (K^n, b) belonging to $\mathbf{K}_1^n$, we can interpret (K^n, b) as an element in $\mathbf{K}_{2,2}^n$ by the following procedure: We can consider that (K^n, b) is constructed by using D_0^{n+1}, D_1^{n+1}, and l_1 in the construction of $(*)$ in Section 0.2. There is an n-disk D_2^{n+1} which does not intersect $D_0^{n+1} \cup D_1^{n+1} \cup l_1$. There is also a 1-disk l_2 which has its initial point p_0 in ∂D_0^{n+1} and its terminal point p_2 in ∂D_2^{n+1}, and intersects $D_0^{n+1} \cup D_1^{n+1} \cup l_1$ only at $p_0 \cup p_2$. By applying the construction of $(*)$ in Section 0.2 to the above $D_0^{n+1} \cup D_1^{n+1} \cup D_2^{n+1}$ and $l_1 \cup l_2$, we obtain a ribbon presentation of K^n as an element in $\mathbf{K}_{2,2}^n$. Therefore, we can identify $\mathbf{K}_1^n$ with $\mathbf{K}_{2,1}^n$, and obtain the following lemma. Moreover, we can obtain the following lemmas (3.2.2), (3.2.3), and (3.2.4).

Lemma 3.2.2. $\mathbf{K}^n_{2,1} \subsetneqq \mathbf{K}^n_{2,2}$.

Proof. Let $G_{2,2}$ be the following group presentation:
$[\ x_0, x_1, x_2 \mid x_0x_1^{-1}x_0x_1^{-1}x_0^{-1}x_1,\ \ x_0x_2^{-1}x_0x_2^{-1}x_0^{-1}x_2\ ]$.
Let $(K^n_{2,2}, \{b_i\}^2_{i=1})$ be a ribbon presentation associated with $G_{2,2}$. Then, $E_1(A(G_{2,2})) = (\ (1-2t)^2\ )$ and $E_2(A(G_{2,2})) = (\ 1-2t\ )$. Therefore, by the proposition (2,7), $(K^n, \{b_i\}^2_{i=1})$ does not belong to $\mathbf{K}^n_1 = \mathbf{K}^n_{2,1}$.

Lemma 3.2.3. $\mathbf{K}^n_{2,2} \subsetneqq \mathbf{K}^n_{2,3}$.

Proof. Let $G_{2,3}$ be the following group presentation:
$[\ x_0, x_1, x_2 \mid x_0x_1^{-1}x_0x_1^{-1}x_0^{-1}x_1,\ \ x_0wx_2^{-1}w^{-1}\ ]$,
where $w = x_1^{-1}x_0x_1^{-1}x_0x_1^{-1}x_0x_2^{-1}x_0$. Let $(K^n_{2,3}, \{b^*_i\}^2_{i=1})$ be a ribbon presentation associated with $G_{2,3}$.

Then $A(G_{2,3}) = \begin{pmatrix} 1-2t & 0 \\ 3(1-t) & 1-2t \end{pmatrix}$.

$E_2(A(G_{2,3})) = (\ 3\ ,\ 1+t\ )$ is not a principal ideal. On the other hand, if $(K^n_{2,3}, \{b^*_i\}^2_{i=1})$ belongs to $K^n_{2,2}$, the diagonal components are 1 and $(1-2t)^2$, or both of them are $1-2t$, because $E_1(A(G_{2,3})) = (\ (1-2t)^2\ )$. In both cases, $E(A(G_{2,3}))$ is a principal ideal. This is a contradiction. Therefore, $(K^n_{2,3}, \{b^*_i\}^2_{i=1})$ does not belong to $K^n_{2,2}$.

Lemma 3.2.4. $\mathbf{K}^n_{2,3} \subsetneqq \mathbf{K}^n_{2,4}$.

Proof. Let $G_{2,4}$ be the following group presentation:
$[\ x_0, x_1, x_2 \mid x_0w_1x_1^{-1}w_1^{-1},\ \ x_0w_2x_2^{-1}w_2^{-1}\ ]$,
where $w_1 = x_1^{-1}x_0x_2^{-1}x_0x_2^{-1}x_0x_2^{-1}x_0$ and
$w_2 = x_2^{-1}x_0x_1^{-1}x_0x_1^{-1}x_0x_1^{-1}x_0x_1^{-1}x_0x_1^{-1}x_0x_1^{-1}x_0$.

where $w_1 = x_1^{-1}x_0x_2^{-1}x_0x_2^{-1}x_0x_2^{-1}x_0$ and
$w_2 = x_2^{-1}x_0x_1^{-1}x_0x_1^{-1}x_0x_1^{-1}x_0x_1^{-1}x_0x_1^{-1}x_0x_1^{-1}x_0$.

Let $(K^n_{2,3},\ \{b_i^{**}\}_{i=1}^2)$ be a ribbon presentation associated with $G_{2,4}$. Then, $A(G_{2,4}) = \begin{pmatrix} 1-2t & 3(1-t) \\ 6(1-t) & 1-2t \end{pmatrix}$ and $E_1(A(G_{2,4})) =$ $(\ -14t^2 + 32t - 17\ )$. Here, the generator of $E_1(A(G_{2,4}))$ is an irreducible element in JH. On the other hand, by the condition (ii) in the proposition(3.1.8), an arbitrary ribbon presentation belonging to $\mathbf{K}^n_{2,3}$ has the first elementary ideal generated by a reducible element in JH, or there is 1 in the orthogonal components of the Alexander matrix. In the latter case, the second elemetary ideal is JH. Therefore, if $(K^n_{2,4}, \{b_i^{**}\}_{i=1}^2)$ does not belong to $\mathbf{K}^2_{2,3}$, then $E_2(A(G_{2,4}))$ must be JH. But by the substitution of $t = -1$, there exists an onto homomorphism
$h : E_2(A(G_{2,4})) = (\ 1-2t, 3(1-t)\ ) \to (\ 3\ )$.
Then $E_2(A(G_{2,4})) \neq JH$. This is a contradiction.

Therefore, $(K^n_{2,4},\ \{b_i^{**}\}_{i=1}^2)$ does not belong to $\mathbf{K}^n_{2,3}$.

3.3. Concluding Remarks

We can generalize the definition (3.1.6) as follows.

Definition 3.3.1. Let $(K^n,\ \{b_i\}_{i=1}^m)$ be a ribbon presentation of a ribbon n-knot of m-fusions K^n constructed by $\bigcup_{i=1}^m l_i$ and $\bigcup_{j=0}^m D_j^{n+1}$ as in the construction of $(*)$ in Section 1. We define the following $m \times m$ matrix R from $(K^n, \{b_i\}_{i=1}^m)$ by the rules as in the definition (3.1.6): For a pair of integers i and j such that l_i intersects D_j^{n+1}, the (i,j)-component of R is entried by " $*$ ". And the other components of R are blank. Then, $(K^n, \{b_i\}_{i=1}^m)$ is said to belong to *R-class*.

We can obtain a partially ordered relation in $\mathbf{K}_m^n$ in the same way as in Section 3.2. That is, for a pair of classes in $\mathbf{K}_m^n$, say R_1-class and R_2-class, R_2-class is said to include R_1-class if R_2 is entried by " $*$ " in the same components as R_1's. Furthermore, let R-class be an arbitrary class in $\mathbf{K}_m^n$, where R is an $m \times m$ matrix. R-class can be identified with some class in $\mathbf{K}_{m+1}^n$, R^*-class, where R^* is an $(m+1) \times (m+1)$ matrix entried by " $*$ " in the same component as R's ($m = 1, 2, \cdots$). Therefore, as a corollary of the theorem in §0, we can obtain the following.

Corollary 3.3.2. *A family of R-classes is a partially ordered set.*

C. McA. Gordon [Go] has given the following conjecture: For two knots K_1 and K_2 in the 3-sphere S^3, we write $K_2 \geq K_1$ if there exists a ribbon concordance from K_2 to K_1. Then, " $\geq$ " is a partial ordering on the set of knots in S^3. In relation to this conjecture, there is the following question:
($**$) For two knots K_1 and K_2 in S^3, we write $K_2 \underset{1}{\geq} K_1$ if there exists a ribbon concordance with only one saddle-point and one minimal-point from K_2 to K_1. Then, if $K_2 \geq K_1$, do there exist t knots $K_{11}, K_{12}, \cdots, K_{1t}$ in S^3 such that $K_2 \underset{1}{\geq} K_{1t} \underset{1}{\geq} \cdots \underset{1}{\geq} K_{12} \underset{1}{\geq} K_{11} \underset{1}{\geq} K_1$, for some positive integer t ?

As a parallel question, there is the following question:
($***$) For two n-knots K_1^n and K_2^n in S^{n+2}, we write $K_2^n \underset{1}{\geq} K_1^n$ if there exists a ribbon concordance with only one saddle-point and one minimal-point from K_2^n to K_1^n. Then, If $K_2^n \geq K_1^n$, do there exist t knots $K_{11}^n, K_{12}^n, \cdots, K_{1t}^n$ in S^{n+2} such that $K_2^n \underset{1}{\geq} K_{1t}^n \underset{1}{\geq} \cdots \underset{1}{\geq} K_{12}^n \underset{1}{\geq} K_{11}^n \underset{1}{\geq} K_1^n$, for some positive integer t ($n \geq 2$) ?

An answer to ($***$) is negative by the theorem in this paper (see Theorem C). For example, there exists a ribbon concordance from $K_{2,4}^n$ in the lemma(3.2.4) to the trivial n-knot O^n. But, by the lemma(3.2.4), there does not exist a ribbon n-knot of 1-fusion K^{n*}

$= O^n \#_b O^n$ such that $K_{2,4}^n \cong K^{n*} \#_b O^n$ where we mean a band connected sum by "$\#_b$". Therefore, as a corollary of Theorem C, we can obtain the following.

Corollary 3.3.3. *There exist two knots K_1^n and K_2^n in S^{n+2} such that (i)$K_2^n \underset{1}{\geq} K_1^n$, and (ii)there do not exist t knots $K_{11}^n, K_{12}^n, \cdots,$ K_{1t}^n in S^{n+2} such that $K_2^n \underset{1}{\geq} K_{1t}^n \underset{1}{\geq} \cdots \underset{1}{\geq} K_{12}^n \underset{1}{\geq} K_{11}^n \underset{1}{\geq} K_1^n$, for any positive integer t $(n \geq 2)$.*

Moreover, an answer to the question $(**)$ is also negative by the following argument. T. Kobayashi introduce the following argument of K. Miyazaki to the author. We mean a connected sum by "$\#$", trivial 1-knot by O, the mirror image of 1-knot L by rL, the Alexander plynomial of L by $\Delta_L(t)$, and the genus of L by $g(L)$.

Classical Knot Case 3.3.4. Let K be prime, fibered, not a 2-bridge knot, and have an irreducible Alexander polynomial. If $K_1^1 = K\#rK$ and $K_2^1 = O$, then K_1^1 and K_2^1 satisfy the conditions (i) and (ii) in the corollary(3.3.3) for $n = 1$.

A proof is the following. It is easily seen that $K\#rK \geq O$. Moreover, $K\#rK$ satisfies the condition(ii) in the corollary(3.3.3) for $n = 1$. That is, there is no ribbon concordance such that $K\#rK$ $\underset{1}{\geq}$ K^* for any 1-knot $K^* \not\cong K\#rK$. Because if there exists such an 1-knot K^*, we have a contradictionary result by the following argument.

Lemma A. $K\#rK \cong K^*\#_b O$.

Proof. We have the lemma(A) by the assumption that $K\#rK \underset{1}{\geq}$ K^*.

Lemma B. *K^* is fibered.*

Proof. By the assumption that K is fibered, $K\#rK$ is fibered. Therefore, K^* is fibered by the lemma(A) and the following theorem in [Ko]:

For two knots L_1 and L_2 in S^3, if $L_1 \#_b L_2$ is fibered, then L_1 and L_2 are fibered.

The following fact is well known (cf. [FM]):
For two knots L_1 and L_2 in S^3, if $L_2 \geq L_1$, then $\Delta_{L_2}(t) = \Delta_{L_1}(t) \cdot p(t) \cdot p(1/t)$ where $p(t)$ is a polynomial with integral coefficients.
Therefore, by the assumption that there exists an 1-knot K^* such that $K\#rK \underset{1}{\geq} K^*$, we have $\Delta_{K\#rK}(t) = \Delta_{K^*}(t) \cdot q(t) \cdot q(1/t)$ for some polynomial $q(t)$ with integral coefficients. Moreover, it is easily seen that $\Delta_{K\#rK}(t) = \Delta_K(t) \cdot \Delta_K(t)$. By the assumption that $\Delta_K(t)$ is an irreducible polynomial, we have the following two cases; (C1) $\Delta_{K^*}(t) = 1$ and $\Delta_K(t) = q(t)$, or (C2) $\Delta_{K^*}(t) = \Delta_K(t) \cdot \Delta_K(t)$ and $q(t) = 1$.

In the case of (C1), since $\Delta_{K^*}(t) = 1$ by the assumption and K^* is fibered by the lemma(B), K^* is a genus zero fibered knot, that is, the trivial 1-knot O. Therefore, $K\#rK \cong O\#_b O$ by the lemma(A). That is, $K\#rK$ is a ribbon number one knot. Moreover, there is following theorem in [BM]:
A composite ribbon number one knot has a 2-bridge summand.
Therefore, $K\#rK$ has a 2-bridge summand. This is in contradiction with the assumption that K is prime and not a 2-bridge knot.

In the case of (C2), since $\Delta_{K^*}(t) = \Delta_K(t) \cdot \Delta_K(t) = \Delta_{K\#rK}(t)$ by the assumption. Therefore, we can obtain $g(K^*) = g(K\#rK)$ because K^* and $K\#rK$ are fibered by the lemma(B) and its proof. Moreover, $g(K\#rK) = g(K^*\#_b O)$ by the lemma(A). Therefore, $g(K^*) = g(K^*\#_b O)$. The following theorem in [Ga] is known:
For two knots L_1 and L_2 in S^3, if $L = L_1\#_b L_2$, then $g(L) \geq g(L_1) + g(L_2)$. Equality holds if and only if there exists a Seifert surface for L which is a band connected sum (using the same band) of minimal Seifert surface for L_1 and L_2.
Therefore, $K^*\#_b O = K^*\#O \cong K^*$. That is, $K\#rK \cong K^*$ by the lemma(A). This is in contradiction with the assumption that $K\#rK \not\cong K^*$.

By the above argument, a ribbon concordance from $K\#rK$ to

O cannot be decomposed by " $\underset{1}{\geq}$ ". That is, an answer to the question(**) is negative. For example, we can take the 8_{16}-knot ([AB],[R]) as K. This is prime, not a 2-bridge knot and has an irreducible Alexander polynomial ([C],[R]), whose leading coefficient is ± 1. Therefore, the 8_{16}-knot is fibered ([Ka]).

We must take notice that the above two arguments for the two questions (**) and (* * *) are independent at the present time. Because ribbon presentations for a ribbon n-knot is not always unique by the theorems in [NN] and [Ys2] ($n \geq 1$). That is, even if K^2 is the ribbon 2-knot associated with the ribbon 1-knot $K\#rK$ in the argument of Miyazaki, then by making use of another ribbon presentation for K^2, we may construct a ribbon concordance from K^2 to the trivial 2-knot O^2 which can be decomposed by " $\underset{1}{\geq}$ ". Therefore, at the present time, we cannot extend the above argument for (**) to the argument for (* * *), and also in the inverse extension, we cannot do so by the same reason.

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