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Common fixed points of commuting mappings in 2-metric spaces

By S. L. SINGH and B. RAM

The concept of 2-metric spaces has been investigated by S.Gähler in a series of papers [1] ∿ [4] . An extensive bibliography regarding related works by himself and others appears in [31]. For literature in field of 2-metric spaces which is not contained in [31], we refer to [5], [7] ∿ [23] and [25] ∿ [33].

A 2-metric space is a space X with a real valued function d defined on $X \times X \times X$ satisfying the following conditions (i),(ii),(iii) and (iv);

(i) for two distinct points a,b there is a point c such that $d(a, b, c) \neq 0$,

(ii) $d(x, y, z) = 0$ if at least two of three points are equal,

(iii) $d(x, y, z) = d(x, z, y) = d(y, z, x)$ *(symmetry about three variable)*,

(iv) $d(x, y, z) \leq d(x, y, u) + d(x, u, z) + d(u, y, z)$.

A sequence $\{x_n\}$ in X is called *Cauchy* if $\lim d(x_m, x_n, a) = 0$ for all $a \in X$. The sequence $\{x_n\}$ is convergent to $x \in X$ and x is the limit of this sequence if $\lim d(x_n, x, a) = 0$ for each $a \in X$. A 2-metric space in which every Cauchy sequence is convergent is called *complete*. A mapping $T:X \to X$ is *sequentially continuous* if for every sequence $\{y_n\}$ such that $y_n \to y$, $Ty_n \to Ty$.

The 2-metric d is a continuous function of each of its three arguments, Gähler [1, Th.4] has shown that if d is continuous in two arguments simultaneously, then it is continuous in all three arguments. So, if a 2-metric is continuous in any two arguments then we shall call it *continuous*. Recently common fixed point theorems have been established for a pair and triplet of self-mappings on 2-metric spaces satisfying contraction type conditions (e.g. see [8], [9], [25], [27], [28], [30], [31]).

The purpose of this paper is to prove common fixed point theorems for commuting mappings on a 2-metric space. As a consequence, a multitude of fixed point theorems may be obtained as corollaries to our results. The contraction type conditions used in this paper are motivated by a functional condition [6, p.123] for four self-mappings of an L-space.

The following lemma in [24] will be used frequently.

Lemma (Lemma 1. in [24]). *Let* $\{y_n\}$ *be a sequence in a complete 2-metric space* X. *If there exists* $h \in (0, 1)$ *such that*

(*) $d(y_n, y_{n+1}, a) \leq hd(y_{n-1}, y_n, a)$

for all n *and for all* $a \in X$, *then* $\{y_n\}$ *converges to a point in* X.

A simple consequence of this lemma is that any sequence $\{y_n\}$ in a 2-metric space satisfying (*) is Cauchy.

Theorem 1. *Let* (X, d) *be a complete 2-metric space,* d *continuous and* P, Q, S *and* T *mappings from* X *to itself such that*

(1. 1) $PT = TP$, $PS = SP$, $QT = TQ$, $QS = SQ$, $TS = ST$, *and*

(1. 2) $PT(X) \cup QS(X) \subseteq ST(X)$.

If S *and* T *are sequentially continuous and there exists an* h *such that* $0 < h < 1$ *and*

(1. 3) $d(Px, Qy, a) \leq h \times max\{ d(Sx, Ty, a),\ d(Px, Sx, a),$

$d(Qy, Ty, a),\ \frac{1}{2}\ [d(Px, Ty, a) + d(Qy, Sx, a)]\}$

for all x, y, a *in* X, *then four mappings* P, Q, S *and* T *have a unique common fixed point.*

Proof. Pick $x_0 \in X$. Construct a sequence $\{x_n\}$ as follows ;

$PTx_{2n} = STx_{2n+1}$, $QSx_{2n+1} = STx_{2n+2}$, $n = 0, 1, 2, \ldots$.

We can do this since $PT(X)$ and $QS(X)$ are contained in $ST(X)$. By *(1. 3)*, we have that

$$d(STx_{2n+1}, STx_{2n+2}, a)$$

$$= d(PTx_{2n}, QSx_{2n+1}, a)$$

$$\leq h \times max\{ d(STx_{2n}, STx_{2n+1}, a),\ \tfrac{1}{2}\, d(STx_{2n+2}, STx_{2n}, a)\}$$

and $d(STx_{2n+1}, STx_{2n+2}, STx_{2n}) = 0$. So one of the following relations must hold.

$$d(STx_{2n+1}, STx_{2n+2}, a) \leq hd(STx_{2n}, STx_{2n+1}, a) \text{ , and}$$

$$d(STx_{2n+1}, STx_{2n+2}, a) \leq \tfrac{1}{2}\, hd(STx_{2n+2}, STx_{2n}, a)$$

$$\leq \tfrac{1}{2}\, h[d(STx_{2n+2}, STx_{2n+1}, a)$$

$$+ d(STx_{2n+1}, STx_{2n}, a) + d(STx_{2n+2}, STx_{2n}, STx_{2n+1})],$$

that is $d(STx_{2n+1}, STx_{2n+2}, a) \leq \frac{h}{2-h}\, d(STx_{2n+1}, STx_{2n}, a)$.

Consequently $d(STx_{2n+1}, STx_{2n+2}, a) \leq hd(STx_{2n}, STx_{2n+1}, a)$ since $h > h/(2 - h)$. Similarly $d(STx_{2n+2}, STx_{2n+3}, a) \leq hd(STx_{2n+1}, STx_{2n+2}, a)$. Hence $d(STx_{n+1}, STx_{n+2}, a) \leq hd(ST_n, STx_{n+1}, a)$ for all a in X. By the above lemma, $\{STx_n\}$ converges to some point z in X . $\{PTx_{2n}\}$ *and*

$\{QSx_{2n+1}\}$ being the subsequences of $\{STx_n\}$ also converge to the same point z. By the sequential continuity of S and T, $PSTx_{2n} = SPTx_{2n} \to Sz$, $QSTx_{2n+1} = TQSx_{2n+1} \to Tz$, $SSTx_{2n} \to Sz$ and $TSTx_{2n+1} \to Tz$. Now taking $x = STx_{2n}$ and $y = STx_{2n+1}$ in *(1. 3)*, making $n \to \infty$ and noting that d is continuous we get $d(Sz, Tz, a) \leq hd(Sz, Tz, a)$ for all a in X. So $Sz = Tz$. Again taking $x = z$ and $y = STx_{2n+1}$ in *(1. 3)*, making $n \to \infty$ and remembering that d is continuous and moreover $Sz = Tz$, we obtain $d(Pz, Tz, a) \leq hd(Pz, Tz, a)$ for all a in X. So $Pz = Tz$. Similarly $Qz = Sz$. Now taking $x = Tx_{2n}$ and $y = z$ in *(1. 3)*, making $n \to \infty$ and using $Qz = Tz$, we obtain $d(z, Qz, a) \leq hd(z, Qz, a)$ for all a in X. So $z = Qz = Tz = Sz = Pz$. The uniqueness of z as a common fixed point of P, Q, S and T follows easily.

Remark 1. *If $P = Q$ and $S = T$ then condition (1. 1) says that P and T are commuting, and (1. 2) should be replaced by $P(X) \subseteq T(X)$. Moreover, in such a situation, the continuity of d will not be required.*

In view of this remark, Theorem 1 of [31] is obtained as a particular case of Theorem 1. As a variant of the above theorem we have the following result.

Theorem 2. *Let (X, d) be a complete 2-metric space, d continuous and P, Q, S and T mappings from X to itself such that (1. 1) holds. Suppose that there exist positive integers p, q, s, t and a positive number $h < 1$ such that*

(2. 1) $PT(X) \cup QS(X) \subseteq S^s T^t(X)$ *and*

(2. 2) $d(Ux, Vy, a) \leq h \cdot max\{d(Mx, Ny, a), d(Ux, Mx, a), d(Vy, Ny, a), \frac{1}{2}[d(Ux, Ny, a) + d(Vy, Mx, a)]\}$

for all x, y, a *in* X, *where* $U = P^p$, $V = Q^q$, $M = S^s$ *and* $N = T^t$. *If* M *and* N *are sequentially continuous then* P, Q, S *and* T *have a unique common fixed point.*

Proof. Since P commutes with each of S and T, $P^p = U$ also commutes with each of $S^s = M$ and $T^t = N$. Similarly, M commutes with N, and V commutes with each of M and N. Also $UM(X) \cup VM(X) \subseteq MN(X)$. Thus Theorem 1 pertains to U, V, M and N. So there exists a unique u in X such that $u = Uu = Vu = Mu = Nu$. From this, $Su = USu = VSu = MSu = NSu$. Therefore Su and similarly Tu are common fixed points of U, V, M and N. Also $Pu = UPu = MPu = NPu$. So Pu is a common fixed point of U, M and N. Similarly Qu is a common fixed point of V, M and N. Putting $x = Pu$ and $y = Qu$ in *(2. 2)*, we obtain $d(Pu, Qu, a) \leq hd(Pu, Qu, a)$ for all a in X. So $Pu = Qu$, Su and Tu are common fixed points of U, V, M and N. The uniqueness of u implies that $u = Pu = Qu = Su = Tu$.

Remark 2. *In case* $S = T$ *and* $s = t$ *in* Theorem 1 and 2, *it can be shown that the continuity condition on* d *will not be needed. In fact, in such a situation fixed point theorems are presented in* [25].

Now we present an extension of the above theorems to non-complete 2-metric space.

Theorem 3. *Let* (X, d) *be a 2-metric space,* d *continuous and* P, Q, S *and* T *mappings from* X *to itself satisfying (1. 1), (2. 2) and*

the following

(3. 1) there exists a sequnece $\{x_n\}$ *in* X *such that* $UNx_{2n} = MMx_{2n+1}$, $VMx_{2n+1} = MNx_{2n+2}$, $n = 0, 1, 2, \ldots$,

(3. 2) the sequence $\{MNx_n\}$ *has a subsequence converging to a point* $z \in X$,

(3. 3) M *and* N *are sequentially continuous at* z .

Then z *is the unique common fixed point of* P, Q, S *and* T .

Proof. Proofs of Theorem 1 and 2 work.

It is interesting to note that if U and V are also continuous at z then the continuity condition on d in Theorem 3 and hence in Theorem 1 and 2 can be avoided. Now we do this.

Theorem 4. *Let* P, Q, S *and* T *be mappings from a 2-metric space* X *to itself satisfying (1. 1), (2. 2), (3. 1) and (3. 2). If* U, V, M *and* N *are continuous at* z *then* Uz $(= Vz = Mz = Nz)$ *is the unique common fixed point of* P, Q, S *and* T.

Proof. In view of the proof of Theorem 2, we need prove that Uz is the unique common fixed point of U, V, M and N. Proceeding as in the proof of Theorem 1 we obtain that $\{MNx_n\}$ is a Cauchy sequence. By *(3. 2)* $MNx_n \to z$. So the subsequences $\{MNx_{2n}\}$ and $\{UNx_{2n}\}$ converge to the same point z . The sequential continuity of U and M at z implies that $UMNx_{2n} \to Uz$ and $MUNx_{2n} \to Mz$. Since U and M are commuting we obtain $Uz = Mz$. Similarly $Vz = Nz$. From *(2. 2)*, $d(Mz, Nz, a) = d(Uz, Vz, a) \leq hd(Mz, Nz, a)$ for all a in X , proving $Mz = Nz$. So

(4. 1) $UUz = UMz = MUz = MMz = MNz = NMz = NNz = NVz = VNz = VVz = VMz$.

So from *(2. 2)*, $d(Mz, MMz, a) = d(Uz, VMz, a) \leq hd(Mz, MMz, a)$ for all a in X, yielding $Mz = MMz$. In view of *(4. 1)*, Mz is a common fixed point of U, V, M and N. Uniqueness of the common fixed point follows easily.

In the above results S and T are commuting while $PQ \neq QP$ but if we take P and Q commuting then the commutativity of S and T can be avoided. In what follows we do not reqire $ST = TS$.

Theorem 5. *Let P, Q, S and T be mappings from a 2-metric space X to itself satisfying the condition (2. 2) and the following*

(5. 1) $PS = SP$, $PT = TP$, $QS = SQ$, $QT = TQ$ *and* $PQ = QP$,

(5. 2) there exists a sequence $\{x_n\}$ *in* X *such that* $MVx_{2n+1} = UVx_{2n}$, $NUx_{2n+2} = UVx_{2n+1}$, $n = 0, 1, 2, \ldots$,

(5. 3) the sequence $\{UVx_n\}$ *has a subsequence converging to a point* z *in* X,

(5. 4) U, V, M *and* N *are sequentially continuous at* z.

Then Uz $(= Vz = Mz = Nz)$ *is the unique common fixed point of* P, Q, S *and* T.

Proof. Note that $UM = MU$, $UN = NU$, $VM = MV$, $VN = NV$ and $UV = VU$. If the mappings U, V, M and N have a unique common fixed point then proceeding as in Theorem 2 it can be seen that the same point is also the unique common fixed point of P, Q, S and T. Moreover, if U, V, M and N have a common fixed point then in view of the condition *(2. 2)* it will be unique. Thus we need show that Uz is a common fixed point of U, V, M and N. By *(2. 2)*, $d(UVx_{2n}, UVx_{2n+1}, a) = d(UVx_{2n+1}, VUx_{2n}, a)$

$\leq h \times \max\{d(UVx_{2n}, UV_{2n-1}, a) , \frac{1}{2} d(UVx_{2n+1}, UVx_{2n-1}, a)\}$ and

$d(UVx_{2n}, UVx_{2n+1}, UVx_{2n-1}) = 0$. So as in Theorem 1,

$$d(UVx_{2n}, UVx_{2n+1}, a) \leq \max\{h, \frac{h}{2-h}\} d(UVx_{2n-1}, UVx_{2n}, a) .$$

Similarly by *(2. 2)*,

$$d(UVx_{2n+1}, UVx_{2n+2}, a) = d(UVx_{2n+1}, VUx_{2n+2}, a)$$
$$\leq h \times \max\{d(UVx_{2n}, UVx_{2n+1}, a) ,$$
$$\frac{1}{2} d(UVx_{2n+2}, UVx_{2n}, a)\}$$

which gives

$$d(UVx_{2n+1}, UVx_{2n+2}, a) \leq \max\{h, \frac{h}{2-h}\} d(UVx_{2n}, UVx_{2n+1}, a) .$$

Consequently $d(UVx_{n+1}, UVx_{n+2}, a) \leq h \cdot d(UVx_n, UVx_{n+1}, a)$ for all a in X, and the sequence $\{UVx_n\}$ is Cauchy. So, in view of *(5. 3)*, $UVx_n \to z$, $MVx_{2n+1} \to z$ and $UVx_{2n+1} \to z$. By *(5. 4)*, $UMVx_{2n+1} \to Uz$ and $MUVx_{2n+1} \to Mz$. Since the mappings U and M are commuting, we obtain $Uz = Mz$. Similarly $Vz = Nz$. Using *(2. 2)*, it is easy to see that $d(Uz, Vz, a) \leq h \cdot d(Uz, Vz, a)$ for all a in X , yielding $Uz = Vz$. Therefore

(5. 5) $UUz = UMz = MUz = MMz = MVz = VMz = VVz = VNz = NVz = NNz$.

From *(2. 2)*, $d(Uz, UUz, a) = d(Uz, VVz, a) \leq hd(Uz, UUz, a)$ for all a in X , proving $Uz = UUz$. So in view of *(5. 5)*, $Uz = Vz = Mz = Nz$ is the common fixed point of U, V, M and N .

If d is continuous then the continuity condition on the mappings U and V in Theorem 5 can be avoided.

Theorem 6. *Let (X, d) be a 2-metric space, d continuous and P, Q, S and T mappings from X to itself satisfying the conditions (2. 2) (5. 1) ∿ (5. 3) and the following*

(6. 1) M *and* N *are sequentially continuous at* z .

Then z *is the unique common fixed point of* P, Q, S *and* T .

Proof. It suffices to show that U, V, M and N have a common fixed point. Proceeding as the proof of Theorem 5 we have $UVx_n \to z$, $UVx_{2n+1} \to z$ and $NUx_{2n+2} \to z$. By *(6. 1)*, $UMVx_{2n+1} = MUVx_{2n+1} \to Mz$, $MMVx_{2n+1} \to Mz$, $VNUx_{2n+2} = NUVx_{2n+2} \to Nz$ and $NNUx_{2n\ 2} \to Nz$. By *(2. 2)*,

$$d(UMVx_{2n+1}, VNUx_{2n+2}, a) \le h \cdot max\Big[d(MMVx_{2n+1}, NNUx_{2n+2}, a), d(UMVx_{2n+1}, MMVx_{2n+1}, a), d(VNUx_{2n+2}, NNUx_{2n+2}, a), \frac{1}{2}\{d(UMVx_{2n+1}, NNVx_{2n+2}, a) + d(VNUx_{2n+2}, MMVx_{2n+1}, a)\}\Big] .$$

Since d is continuous, making $n \to \infty$ we obtain that $d(Mz, Nz, a) \le hd(Mz, Nz, a)$ for all a in X , yielding $Mz = Nz$. Similarly, in view of *(2. 2)*, it follows from

$$d(UMVx_{2n+1}, Vz, a) \le h \times max\{d(MMVx_{2n+1}, Nz, a) , d(UMVx_{2n+1}, MMVx_{2n+1}, a), d(Vz, Nz, a) \frac{1}{2}[d(UMVx_{2n+1}, Nz, a) + d(Vz, MMVx_{2n+1}, a)]\}$$

that $d(Mz, Vz, a) \le hd(Mz, Vz, a)$ for all a in X . So $Nz = Uz$. Similar;y $Nz = Uz$. Thus $Uz = Vz = Mz = Nz$. Now taking $x = Vx_{2n+1}$ and $y = z$ in *(2. 2)* and making $n \to \infty$ we get $d(z, Vz, a) \le hd(z, Vz, a)$ for all a in X , proving $z = Vz$. So z is a common fixed point of U, V, M and N , and hence the proof is completed.

Remark 3. *Condition (1. 3) and (2. 2) contains a multitude of*

contraction type conditions. In fact, choosing P, Q, S and T suitably, several general type of contraction conditions may be obtained from (1, 3) *and* (2, 2). *For example, if we let* $P = Q$ *in* (1, 3) *with* (5, 1), *then we obtain a definition for three mappings introduced in* [27], *and if we set* $S = T$ *and* $s = t$ *in* (2, 2) *with* (1, 1) *then we obtain a condition used in* Theorem 2 *of* [25].

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