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Lajos, Sándor

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Characterizations of Semigroups by Principal (m, n)-Ideals

By Sándor LAJOS

In this paper we follow the notation and terminology of the books [2] and [8]. In particular, for an element a of a semigroup S, the principal left, right, quasi-, bi-, and (m, n)-ideal generated by a will be denoted by $L(a)$, $R(a)$, $Q(a)$, $B(a)$, and $\langle a\rangle_{(m, n)}$, respectively. We recall that a subsemigroup A of a semigroup S is called an (m, n)-ideal of S if the following inclusion holds:

$$A^m S A^n \subseteq A, \tag{1}$$

where m, n are arbitrary nonnegative integers and A^0 is the empty symbol.

Theorem 1. *A semigroup S is regular if and only if the relation*

$$B(a) = B(a)L(a) \tag{2}$$

holds for every element a of S.

Proof. Let S be a regular semigroup and a an arbitrary element in S. Then we have $B(a) = aSa$ and $L(a) = Sa$. Therefore we obtain that $B(a) = aSa = (aSa)(Sa) = B(a)L(a)$ holds for every element a of S.

Conversely, let S be a semigroup with property (2) for every element a of S. Then, for every element a of S, we have

$$a \in B(a) = B(a)L(a) \subseteq aS^1a.$$

Hence it follows that S is a regular semigroup.

The following result is the left-right dual of Theorem 1.

Theorem 2. *A semigroup S is regular if and only if the condition*

$$B(a) = R(a)B(a) \tag{3}$$

holds for every element a of S.

Now we give characterizations for semigroups that are unions of groups.

Theorem 3. *A semigroup S is completely regular if and only if*

$$\langle a\rangle_{(2,\,2)} = \langle a\rangle_{(2,\,2)}\ \langle a\rangle_{(0,\,2)} \tag{4}$$

holds for every element a of S.

Proof. First, let S be a completely regular semigroup. Then, for any element a of S, we have $a^2Sa^2 = a^2(Sa^2)^2$, that is, (4) holds.

Conversely, if (4) holds for all elements a in S, then

$$a \in \langle a\rangle_{(2,\,2)} = \langle a\rangle_{(2,\,2)}\ \langle a\rangle_{(0,\,2)} =$$
$$(\{a, a^2, a^3\} \cup a^2S^1a^2)(\{a\} \cup S^1a^2) \subseteq \{a^2\} \cup aS^1a^2,$$

whence it follows that S is a completely regular semigroup.

More generally, the following criterion may be obtained.

Theorem 4. *Suppose that k, m, n, are positive integers such that $m > k$, $n \geq 2$. Then a semigroup S is completely regular if and only if the condition*

$$\langle a\rangle_{(m,\,k)} \cap \langle a\rangle_{(m,\,n)} = \langle a\rangle_{(m,\,k)}\ \langle a\rangle_{(m,\,n)} \tag{5}$$

holds for every element a of S.

For a related result, see [1].

Theorem 5. *A semigroup S is completely simple if and only if*

(6) $$B(a) = R(ab)L(a)$$

holds for every couple a, b *of elements of* S.

Proof. If S is a completely simple semigroup, then $a \in abSa$ for all $a, b \in S$ (cf. [8], p. 114). Hence it follows that

$$B(a) = aSa \subseteq abSa = R(ab)L(a) \subseteq B(a)$$

holds for every couple a, b in S. Therefore (6) holds true.

Conversely, (6) implies $a \in a^2S^1a$ for every element a of S, i.e. S is completely regular. On the other hand, (6) implies $a \in SbS$ for any $a, b \in S$. Hence $S = SbS$ for every element b of S. Therefore S is a completely simple semigroup, indeed.

Now we shall give a new characterization for left groups.

Theorem 6. *A semigroup* S *is a left group if and only if the condition*

(7) $$B(a) = B(ab)$$

holds for every couple a, b *of elements in* S.

Proof. Let S be a left group. Then it is completely regular and (left) simple. Hence, by our Theorem 5, it follows that (6) holds for any two elements a, b in S. Hence

$$B(a) \subseteq abS = R(ab) = B(ab) \subseteq B(a)$$

for every couple $a, b \in S$.

Conversely, if (7) holds for any two elements of a semigroup S, then $B(a) = B(a^2)$ for every element a of S, whence it follows that S is completely regular (cf. [3], Theorem 2). On the other hand, (7) implies $ab \in B(ab) = B(a) = aSa$ for any elements a, b in S. Hence $aS = aSa$ for every element a of S, i.e. every principal *bi*-ideal of S is a prin-

cipal right ideal of S.

Next we show that the set $B_p(S)$ of all principal bi-ideals of S is a left zero semigroup under set product. If $a, b \in S$, then

$$aS.bS \subseteq aS = abS \subseteq aS.bS,$$

whence $aS.bS = aS$ holds for any two elements of S. Thus the multiplicative semigroup of all principal bi-ideals of S is a left zero band. Let $e, f \in E(S)$. We have $e \in B(e) = B(e)B(f)$, whence it follows that $e = esf (s \in S)$. Hence $ef = esf = e$. Thus $E(S)$ is a left zero band and S is a left group (cf. [8], Theorem IV. 3. 9).

Corollary. *For a semigroup S the following conditions are equivalent:*

(A) S is a left group.

(B) The set $B_p(S)$ of all principal bi-ideals of S is a left zero band with respect to set product.

(C) $B(a) = B(ab)$ for every couple a, b in S.

(D) $Q(a) = Q(ab)$ for every couple a, b in S.

(E) $\langle a \rangle_{(m, n)} = \langle ab \rangle_{(m, n)}$ for all a, b in S $(m, n > 0)$.

(F) $B = BR$ for every bi-ideal B and every right ideal R of S.

(G) $B = BL$ for every bi-ideal B and every left ideal L of S.

(H) $B = BQ$ for every bi-ideal B and quasi-ideal Q of S.

(I) $Q = QL$ for every left ideal L and quasi-ideal Q of S.

(J) $Q = QB$ for every quasi-ideal Q and bi-ideal B of S.

(K) $Q_1 = Q_1Q_2$ for every couple Q_1, Q_2 of quasi-ideals of S.

(L) The set of all quasi-ideals of S is a left zero band with respect to set product.

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Institute for Mathematics and Computer Science
K. Marx University for Economics
H-1828 Budapest, Hungary

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