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On Compactly Packed Lattices and Autometrized Algebras

By K.L.N. SWAMY and V. SWAMINATHAN

C. Reis and T. Viswanathan [6] introduced the notion of a C.P.ring. A commutative ring with unity is said to be compactly packed by primes (briefly a C.P.ring) if whenever an ideal I of R is contained in the union of a family of prime ideals of R, then I is actually contained in one of the primes of the family. Theorem 1 of [3] gives an ideal theoretic characterization of C.P.rings. In this paper, we introduce the notion of C.P.lattices and C.P.autometrized algebras. We first observe that lattices and autometrized algebras share the following properties which are true in commutative rings with unity namely (i) If I is an ideal such that $I \subset \bigcup_{i=1}^{n} P_i$ where P_i's are prime ideals then $I \subset P_i$ for some i and (ii) If $I_1, I_2, \ldots, I_n$ are ideals and if a prime ideal $P \supset \bigcap_{j=1}^{n} I_j$ then $P \supset I_j$ for some j. If $P = \bigcap_{j=1}^{n} I_j$ then $P = I_j$ for some j. We show in this paper that a distributive lattice is compactly packed by primes if and only if it is Noetherian, a distributive join-semilattice is compactly packed by primes if and only if every ideal is principal and a semi-regular semi-normal autometrized algebra satisfying decomposition property is compactly packed by primes if and only if every ideal of the algebra is principal. Thus, we get a characterization for C.P.1-groups, C.P.D.R.1.semi-groups, C.P.semi-Brouwerian algebras and C.P.Boolean rings. In particular, we obtain that in a distributive lattice with 1, if every

prime ideal is principal then every ideal is principal, which is similar to the property proved for commutative rings with unity (see [1]).

We start with the following.

Definition 1. Let L be a lattice. L is said to be *compactly packed by primes* if whenever I is an ideal of L such that $I \subset \bigcup_{\alpha\in\Delta} P_\alpha$ where P_α are prime ideals of L, then $I \subset P_\alpha$ for some $\alpha \in \Delta$.

Theorem 2. A distributive lattice with 1 is compactly packed if and only if every prime ideal of the lattice is principal.

Proof. Let L be a C.P. distributive lattice with 1. Let I be an ideal of L. Let $D = \{P \mid P \text{ is a prime ideal of } L \text{ and } I \not\subset P\}$. If I contains only one element then obviously I is principal. If I contains more than one element, then D is non-empty (by corollary 17, P. 64 of [2]). As L is C.P., we get that $I \not\subset \bigcup_{P\in D} P$. Therefore, there exists an $x \in I$ such that $x \notin \bigcup_{P\in D} P$. Let $< x >_L$ denote the ideal generated by x in L and let $V(I) = \{P \mid P \text{ a prime ideal of } L \text{ and } I \subset P\}$. We claim that $V(I) = V(< x >_L)$. If $P \in V(I)$ then $P \supset I \supset < x >_L$ and hence $P \in V(< x >_L)$. If $P \in V(< x >_L)$ then $P \not\supset I$ implies $P \in D$ and hence $x \in \bigcup_{P\in D} P$ which is a contradiction. Thus, $P \supset I$. Therefore $V(I) = V(< x >_L)$. As every ideal of a distributive lattice is the intersection of all prime ideals containing it, we get that $I = < x >_L$. Thus, if L is C.P.distributive lattice then every ideal of L is principal and so in particular, every prime ideal of L is principal. Conversely, let every prime ideal of L be principal. Let I be an ideal of L such that $I \subset \bigcup_{\alpha\in\Delta} P_\alpha$ where P_α's are prime ideals of L. Let

$D = L - \bigcup_{\alpha\in\Delta} P_\alpha$. D is nonempty since $1 \in L$. We can easily check that D is a dual ideal of L. As $I \subset \bigcup_{\alpha\in\Delta} P_\alpha$, we get that $I \cap D = \phi$. Therefore, there exists a prime ideal P in L such that $P \supset I$ and $P \cap D = \phi$. Hence, $I \subset P \subset \bigcup_{\alpha\in\Delta} P_\alpha$. As every prime ideal is principal, $P = \langle x \rangle_L$ for some $x \in L$. Hence $x \in P_\alpha$ for some $\alpha \in \Delta$ and so $I \subset P \subset P_\alpha$. This shows that L is compactly packed.

Remark 2.1. If every prime ideal of a distributive lattice with 1 is principal, then L is compactly packed and hence every ideal of L is principal.

Remark 2.2. If every prime ideal of a distributive lattice L is principal, then proper ideal of L is principal.

Remark 2.3. A distributive lattice L is compactly packed if and only if every ideal of L is principal, i.e. if and only if L is Noetherian.

Remark 2.4. A C.P.distributive lattice is complete.

We recall the following definitions and results from [2].

Definition 3. A join-semilattice L is called *distributive* if and only if $a \leq b_0 \vee b_1$ $(a,b_0,b_1 \in L)$ implies the existence of $a_0,a_1 \in L$, $a_i \leq b_i$ $i = 0, 1$ with $a = a_0 \vee a_1$ (Note that a_0 and a_1 need not be unique).

Definition 4. A subset D of a join-semilattice L is called a *dual ideal* if and only if $a \in D$ and $x \geq a$ imply that $x \in D$ and $a,b \in D$ implies that there exists a lower bound d of $\{a, b\}$ such that $d \in D$.

Definition 5. An ideal I of L is *prime* if and only if $I \neq L$ and $L - I$ is a dual ideal of L.

Theorem 6. If I is an ideal and D a non-void dual ideal of a distributive join-semilattice and if $I \cap D = \phi$, then there exists a prime ideal P of the lattice with $P \supset I$ and $P \cap D = \phi$.

Theorem 7. Every ideal I of a distributive join-semilattice L is the intersection of all prime ideals containing it.

Using the above results and proceeding as in theorem 1.2, we obtain the following.

Theorem 8. A complete distributive join-semilattice L is compactly packed if and only if every prime ideal of L is principal.

Corollary 8.1. In a complete distributive join-semilattice, if every prime ideal is principal, then every ideal is principal.

Corollary 8.2. A distributive join semilattice is compactly packed if and only if every ideal is principal in which case L can be made into a Noetherian distributive lattice.

We recall the following definitions and results from [5] and [7].

Definition 9. A system $A = (A, +, \leq, *)$ is called an *autometrized algebra* if and only if (A_1) $(A, +, \leq)$ is a commutative partially ordered semigroup with additive identity 0 and (A_2) $*$ is a metric operation on A, i.e., $*$ is a mapping from $A \times A$ to A such that (M_1) $a * b \geq 0$ for all a, b in A and $a * b = 0$ if and only if $a = b$

(M_2) $a * b = b * a$ for all $a, b \in A$ (M_3) $a * c \leq (a * b) + (b * c)$ for all a, b, c in A.

Definition 10. An autometrized algebra $A = (A, +, \leq, *)$ is called a *seminormal autometrized algebra* if and only if (i) $(a + c) * (b + d) \leq (a * b) + (c * d)$ for any $a, b, c, d \in A$ (ii) $(a * c) * (b * d) \leq 2[(a * b) + (c * d)]$ for any $a, b, c, d \in A$ (iii) $a * b = (a * b) * 0$ for any $a, b \in A$ and (iv) $(a * 0) + (b * 0) = [(a * 0) + (b * 0)] * 0$ for any a, b in A.

Definition 11. An autometrized algebra $A = (A, +, \leq, *)$ is called a *semiregular autometrized algebra* if and only if $a \geq 0$ implies $a * 0 = a$.

Definition 12. A non-empty subset I of an autometrized algebra $A = (A, +, \leq, *)$ is called an *ideal* of A if and only if (i) $a, b \in I$ implies $a + b \in I$ and (ii) $a \in I$, $b \in A$, and $b * 0 \leq a * 0$ imply $b \in I$.

Definition 13. By a *dual ideal* D of an autometrized algebra $A = (A, +, \leq, *)$ we mean a non-empty subset D of A satisfying (i) $x \in D$, $x * 0 \leq m(y * 0)$ for some positive integer m imply that $y \in D$ and (ii) $x \in D$, $y \in D$ imply that there exists $z \in D$ such that $(z) \subset (x) \wedge (y)$.

Definition 14. A proper ideal of an autometrized algebra A is called a *prime ideal* if $A - P$ is a dual ideal.

Definition 15. Let $A = (A, +, \leq, *)$ be an autometrized algebra. A is said to have *decomposition property* if $a, b, c \leq A$, $a \leq b + c$,

$a, b, c \geq 0$ implies that there exists b_1, c_1 in A such that $a = b_1 + c_1$ and $0 \leq b_1 \leq b$, $0 \leq c_1 \leq c$.

Theorem 16. If I is an ideal and D is a dual ideal of a semiregular seminormal autometrized algebra A with decomposition property, and if $I \cap D = \phi$, then there exists a prime ideal P of A such that $I \subset P$, $P \cap D = \phi$.

Theorem 17. Let A be a semiregular seminormal autometrized algebra with decomposition property. Each ideal I of A is the intersection of all prime ideals of A containing I.

Using the above definitions and results, we have the following.

Theorem 18. A semiregular seminormal autometrized algebra satisfying decomposition property is C.P. if and only if every ideal of the algebra is principal.

Proof. Proceeding as in Theorem 1.2 we get that if A is C.P. then every ideal of the algebra is principal. Conversely, let every ideal of A be principal. Let I be an ideal of A and let $I \subset \bigcup_{\alpha \in \Delta} P_\alpha$ where P_α's are prime ideals of A. As I is principal, $I = (x) \subset \bigcup_{\alpha \in \Delta} P_\alpha$. Hence $x \in P_\alpha$ for some α and therefore $I \subset P_\alpha$. Thus A is a C.P. algebra.

Corollary 18.1. As l-groups, D.R.l.semigroups (see [8], [9] and [10]), semi-Brouwerian algebras [4] and Boolean algebras are all semiregular seminormal autometrized algebras with decomposition property, we get that they are compactly packed if and only if every ideal of the algebra is principal.

In particular an l-group G is compactly packed if and only if every l-ideal of G is principal and a Boolean algebra is compactly packed by primes if and only if it is finite.

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