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Extensions of Semimodules I

By Michihiro TAKAHASHI

Given Λ-semimodules A and B over a fixed semiring Λ, we want to consider all possible Λ-semimodules E such that B is a regular Λ-subsemimodule of E and $E/B \simeq A$. We can define a set of equivalence classes of such Λ-semimodules, written $E_\Lambda(A,B)$.

Let $SMod_\Lambda$, Mod_Λ and Set be the category of Λ-semimodules, the category of Λ-modules and the category of sets, respectively.

In this paper, we prove that $E_\Lambda(-,-):SMod_\Lambda \times Mod_\Lambda \to Set$ is a bifunctor.

In a subsequent paper, we will introduce notions of Λ-projective and Λ-injective of semimodules in order to compute the set $E_\Lambda(A,B)$.

For the definitions and notions used here we refer to [5],[6] and [7].

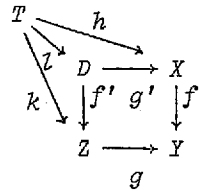
1. Pullback and pushout

(1.1) A diagram in a category C

$$(*):\ Z \xrightarrow{g} Y \xleftarrow{f} X$$

is called an *angle* in C.

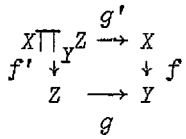
A triple (T,h,k) is called a *source* of the angle $(*)$ if $h:T \to X$ and $k:T \to Z$ are morphisms such that $f\circ h = g\circ k$. A source (D,g',f') is a *limit* of the angle $(*)$ if for any source (T,h,k) there exists a unique morphism $l:T \to D$ such that the diagram



is commutative.

Such a limit (D,g',f') is called a *pullback* of the angle (*) and D is denoted by $X\prod_Y Z$.

The diagram



is called a *pullback square*. The category C is said to *have pullbacks* if every angle in C has a pullback.

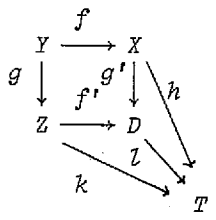
(1.2) A diagram in C

$$(*):\ Z \xleftarrow{g} Y \xrightarrow{f} X$$

is called a *coangle* in C.

A triple (k,h,T) is called a *sink* of the coangle (*) if $k:Z \to T$ and $h:X \to T$ are morphisms such that $h\circ f = k\circ g$.

A sink (f',g',D) is a *colimit* of the coangle (*) if for any sink (k,h,T) there exists a unique morphism $l:D \to T$ such that the diagram



is commutative.

Such a colimit (f',g',D) is called a *pushout* of the coangle (*), and D is

denoted by $X\coprod_Y Z$.

The diagram

$$\begin{array}{ccc} Y & \xrightarrow{f} & X \\ {\scriptstyle g}\downarrow & & \downarrow{\scriptstyle g'} \\ Z & \xrightarrow[f']{} & X\coprod_Y Z \end{array}$$

is called a *pushout square*.

The category C is said to *have pushouts* if every coangle in C has a pushout.

(1.3) The following Proposition is well known.

Proposition. *Let diagrams*

$$\begin{array}{ccccc} X & \xrightarrow{x} & Y & \xrightarrow{y} & Z \\ {\scriptstyle r}\downarrow & \text{I} & {\scriptstyle s}\downarrow & \text{II} & {\scriptstyle t}\downarrow \\ A & \xrightarrow[a]{} & B & \xrightarrow[b]{} & C \end{array} \quad , \quad \begin{array}{ccc} X & \xrightarrow{y\circ x} & Z \\ {\scriptstyle r}\downarrow & \text{III} & {\scriptstyle t}\downarrow \\ A & \xrightarrow[b\circ a]{} & C \end{array}$$

be commutative diagrams.

1) Assume II *is a pullback square. Then* III *is a pullback square if and only if* I *is a pullback square.*

2) Assume I *is a pushout square. Then* III *is a pushout square if and only if* II *is a pushout square.*

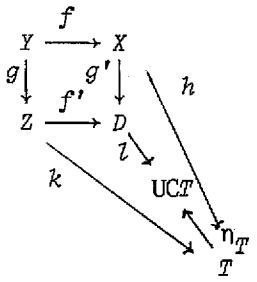
(1.4) Let A,C be categories with an adjunction $\langle C,U,\varphi\rangle: C \to A$. We denote the unit of the adjunction by η. Then it is well known that for each object $X\in|C|$ the morphism $\eta_X = \varphi_{X,CX}(1_{CX}): X \to UCX$ has the following property:

Proposition. *Let* $X\in|C|$, $A\in|A|$.

For every morphism $f: X \to UA$ *in* C , *there is a unique morphism* $\overline{f}: CX \to A$ *in* A *such that* $f = U(\overline{f})\circ\eta_X$.

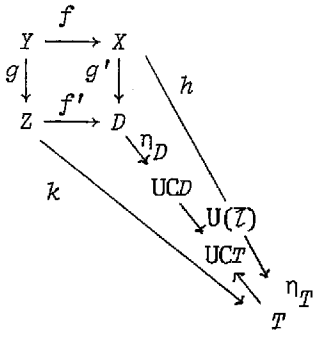
(1.5) A sink (f',g',D) of coangle $(*): Z \xleftarrow{g} Y \xrightarrow{f} X$ is a C-*colimit* if for any

sink (k,h,T) there exists a unique morphism $l:D \to UCT$ such that the diagram



is commutative (cf. [7]).

By Proposition (1.4), we can say that for every sink (k,h,T) there exists a unique morphism $\bar{l}:CD \to CT$ in A such that the diagram



is commutative.

Such a C-colimit (f',g',D) is called a C-*pushout* of the coangle (*).

Even if there exists a C-pushout, it is not unique in general.

The category C is said to *have* C-*pushouts* if every coangle in C has a C-pushout.

(1.6) Let Λ be a fixed semiring with 1.

We denote by $SMod_\Lambda$ the category whose objects are Λ-semimodules and whose morphisms are Λ-homomorphisms. By $CSMod_\Lambda$ we denote the subcategory of $SMod_\Lambda$ whose objects are cancellative Λ-semimodules.

Let $U: CSMod_\Lambda \to SMod_\Lambda$ be the inclusion functor, and let $c: SMod_\Lambda \to CSMod_\Lambda$ be the functor defined in (7.7) of [5]. Then Theorem (7.7) in [5] shows that c is a left adjoint functor of U. We denote by η the unit. It is easily checked that the morphism $\eta_X: X \to UcX$ coincides with the epimorphism $c: X \to c(X)$ defined in (7.2) of [5]. We denote $[x]$ the image of c of x in X.
By Theorem (3.2) and Theorem (3.4) in [7], the category $SMod_\Lambda$ is complete. Theorem (3.5) in [7] shows the subcategory $CSMod_\Lambda$ is complete and cocomplete.

Thus we have

Proposition.

The category $SMod_\Lambda$ has pullbacks and c-pushouts.

In particular, the subcategory $CSMod_\Lambda$ has pullbacks and pushouts.

(1.7) Given an angle $(*)$: $Z \xrightarrow{g} Y \xleftarrow{f} X$ in $SMod_\Lambda$, we will construct a pushout as follow.

Define a source (D,g',f') by

$$D = \{(x,y) \in X \times Z \mid f(x) = g(x)\} \subset X \times Z,$$

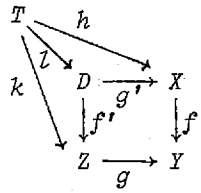
$g'(x,z) = x$ and $f'(x,z) = z$ for every (x,z) in D.

Then the square

$$\begin{array}{ccc} D & \xrightarrow{g'} & X \\ {\scriptstyle f'}\downarrow & & \downarrow{\scriptstyle f} \\ Z & \xrightarrow{g} & Y \end{array}$$

is commutative. By the definition, D is a Λ-subsemimodule of $X \times Z$, and g', f' are Λ-homomorphisms. Therefore (D,g',f') is a source of the angle $(*)$.
We will show the source (D,g',f') is a pullback of the angle $(*)$.
For any source (T,h,k) define $l: T \to D$ by $l(a) = (h(a),k(a))$ for all a in T.
Since $f \circ h(a) = g \circ k(a)$, $(h(a),k(a)) \in D$. This shows l is well defined.
It is easily checked that this map l is a unique Λ-homomorphism such that

the diagram



is commutative. Thus (D,g',f') is a required pullback of the given angle: $D = X\overset{\frown}{\prod}_Y Z$.

(1.8) Given a coangle $(*): Z \overset{g}{\leftarrow} Y \overset{f}{\rightarrow} X$ in $SMod_\Lambda$, we will construct a c-pushout as follow.

Define a subset $R \subset (X\times Z)\times(X\times Z)$ by

$$R = \{<(f(y),g(y')),(f(y'),g(y))> \mid y,y' \in Y\}.$$

Then R is a symmetric Λ-subsemimodule of $(X\times Z)\times(X\times Z)$. Let $\rho(R)$ be the congruence of $X\times Z$ associated with R.

Define a Λ-semimodule D by

$$D = X\times Z/\rho(R).$$

We denote by $\overline{(x,z)}$ the class of $(x,z)\in X\times Z$ in D. By the definition, $\overline{(x,z)} = \overline{(x',z')}$ if and only if $x+f(y) = x'+f(y')$ and $z+g(y') = z'+g(y)$ for some y,y' in Y.

We define $g': X \to D$ and $f': Z \to D$ by

$$g'(x) = \overline{(x,0)} \text{ and } f'(z) = \overline{(0,z)}, \text{ respectively.}$$

Then g',f' are Λ-homomorphisms.

Next we will show (f',g',D) is a sink of the given coangle $(*)$.

Since $g'\circ f(y) = \overline{(f(y),0)}$ and $f'\circ g(y) = \overline{(0,g(y))}$ for each y in Y, we must show $\overline{(f(y),0)} = \overline{(0,g(y))}$.

Since $f(y)+f(0) = 0+f(y)$ and $0+g(y) = g(y)+g(0)$ for $0,y$ in Y, we have $\overline{(f(y),0)} = \overline{(0,g(y))}$.

Finally we will show the sink (f',g',D) is a c-pushout.

For any sink (k,h,T) define $l:D \to UcT$ by

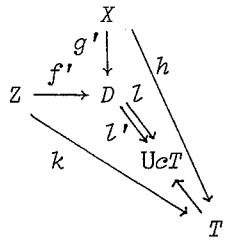
$$\begin{aligned} l(\overline{x,z}) &= \eta_T \circ h(x) + \eta_T \circ k(z) \\ &= [h(x)+k(z)]. \end{aligned}$$

We must show the map l is well defined.

Suppose $(\overline{x,z}) = (\overline{x',z'})$. Then $x+f(y) = x'+f(y')$ and $z+g(y') = z'+g(y)$ for some y,y' in Y. It follows that $h(x+f(y))+k(z+g(y')) = h(x'+f(y'))+k(z'+g(y))$, so that $h(x)+k(z)+h\circ f(y)+k\circ g(y') = h(x')+k(z')+h\circ f(y')+k\circ g(y)$. Where $h\circ f(y)+k\circ g(y') = h\circ f(y')+k\circ g(y)$ since $h\circ f(y) = k\circ g(y)$ and $h\circ f(y')= k\circ g(y')$. Thus we have $[h(x)+k(z)] = [h(x')+k(z')]$. This shows the map l is well defined. By the definition, l is a Λ-homomorphism.

It is left to show that such l is unique.

Suppose $l':D \to UcT$ is a Λ-homomorphism such that the diagram



is commutative.

For each $(\overline{x,z}) \in D$,

$$\begin{aligned} l'(\overline{x,z}) &= l'((\overline{x,0})+(\overline{0,z})) \\ &= l'(\overline{x,0})+l'(\overline{0,z}) \\ &= l'\circ g'(x)+l'\circ f'(z) \\ &= \eta_T \circ h(x)+\eta_T \circ k(z) \\ &= [h(x)+k(z)] \\ &= l(\overline{x,z}). \end{aligned}$$

Hence such l is unique.

The above consideration shows that the sink (f',g',D) is a required c-pushout of the given coangle $(*)$: $Z \overset{g}{\leftarrow} Y \overset{f}{\rightarrow} X$. We denote D by $\overline{X \amalg_Y Z}$.

(1.9) We notice that for cancellative Λ-semimodules X and Z, $X \overline{\prod}_Y Z$ and $\overline{X \amalg_Y Z}$ are also cancellative since $X \times Z$ is cancellative (cf.Proposition (7.1) in [5]). By the constructions in (1.7) and (1.8), we have

Proposition.

In the category $SMod_\Lambda$, for a given angle $Z \overset{g}{\rightarrow} Y \overset{f}{\leftarrow} X$ the triple $(X \overline{\prod}_Y Z, g', f')$ is a pullback and for a given coangle $Z \overset{g}{\leftarrow} Y \overset{f}{\rightarrow} X$ the triple $(f', g', \overline{X \amalg_Y Z})$ is a c-pushout.

In particular, in the category $CSMod_\Lambda$, $(X \overline{\prod}_Y Z, g', f')$ and $(f', g', \overline{X \amalg_Y Z})$ are a pullback and a pushout, respectively.

(1.10) In the category $SMod_\Lambda$, a c-pushout is not necessarily a pushout. But when Y is a Λ-module the c-pushout $(f', g', \overline{X \amalg_Y Z})$ is always a pushout. That is,

Proposition.

Let $Z \overset{g}{\leftarrow} Y \overset{f}{\rightarrow} X$ be a given coangle in $SMod_\Lambda$.

If Y is a Λ-module, then the c-pushout $(f', g', \overline{X \amalg_Y Z})$ is a pushout of the coangle.

Proof. Consider the diagram

$$\begin{array}{ccc} Y & \xrightarrow{f} & X \\ {\scriptstyle g}\downarrow & & \downarrow{\scriptstyle g'} \\ Z & \xrightarrow{f'} & D = \overline{X \amalg_Y Z}. \end{array}$$

Since Y is a Λ-module, $(\overline{x,z}) = (\overline{x',z'})$ if and only if $x = x' + f(y)$ and

$z = z'+g(-y)$ for some y in Y.

For any sink (k,h,T) define $l:D \to T$ by $l(\overline{x,z}) = h(x)+k(z)$.

We must show it is well defined.

Suppose $(\overline{x,z}) = (\overline{x',z'})$. Then $x = x'+f(y)$ and $z = z'+g(-y)$ for some y in Y.

$$\begin{aligned} h(x)+k(z) &= h(x')+h\circ f(y)+k(z')+k\circ g(-y) \\ &= h(x')+k(z')+h\circ f(y)+k\circ g(-y) \\ &= h(x')+k(z')+k\circ g(y)+k\circ g(-y) \\ &= h(x')+k(z')+k\circ g(y-y) \\ &= h(x')+k(z')+k\circ g(0) \\ &= h(x')+k(z'). \end{aligned}$$

This shows the map l is well defined. Since h and k are Λ-homomorphisms, the map l is also a Λ-homomorphism. The uniqueness of such l follows easily.

2. Extensions of semimodules

(2.1) Let Λ be a semiring with 1, and let A,B be two Λ-semimodules. We want to consider all possible Λ-semimodules E such that B is a regular Λ-subsemimodule of E and $E/B \simeq A$. We then have a regular exact sequence $0 \to B \rightarrowtail E \twoheadrightarrow A \to 0$.

A regular exact sequence

$$0 \to B \overset{\kappa}{\rightarrowtail} E \overset{\nu}{\twoheadrightarrow} A \to 0$$

is called an *extension* of A by B.

Then by Proposition (4.6) in [5] and the regularity of the exact sequence, we have followings:

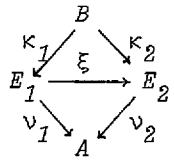
1) $\kappa_*:B \simeq \mathrm{Ker}(\nu)$,

2) $\nu_*:\mathrm{Coker}(\kappa) \simeq A$,

3) κ is injective and i-regular,and

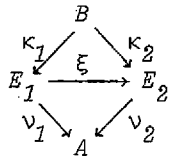
4) ν is surjective and k-regualr.

(2.2) We say that the extension $B \overset{\kappa_1}{\rightarrowtail} E_1 \overset{\nu_1}{\twoheadrightarrow} A$ is *equivalent* to the extension $B \overset{\kappa_2}{\rightarrowtail} E_2 \overset{\nu_2}{\twoheadrightarrow} A$ if there is a regular Λ-homomorphism $\xi: E_1 \to E_2$ such that the diagram



is commutative.

(2.3) Lemma. *Let* $B \overset{\kappa_1}{\rightarrowtail} E_1 \overset{\nu_1}{\twoheadrightarrow} A$, $B \overset{\kappa_2}{\rightarrowtail} E_2 \overset{\nu_2}{\twoheadrightarrow} A$ *be two extensions of A by B. If there is a* Λ*-homomorphism* $\xi: E_1 \to E_2$ *such that the diagram*



is commutative, then the sequence $0 \to E_1 \overset{\xi}{\to} E_2 \to 0$ *is exact.*

Proof. Consider the exact sequence

$$0 \to \mathrm{Ker}(\xi) \rightarrowtail E_1 \overset{\xi}{\longrightarrow} E_2 \twoheadrightarrow E_2/\xi \to 0 \ .$$

It is sufficent to show that $\mathrm{Ker}(\xi) = 0$ and $E_2/\xi = 0$.

(1°): Let x be any element of $\mathrm{Ker}(\xi)$. Then $\xi(x) = 0$.

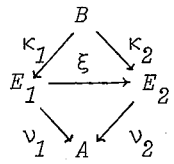
Since $\nu_1(x) = \nu_2 \circ \xi(x) = \nu_2(0) = 0$, $x \in \mathrm{Ker}(\nu_1) = \mathrm{Im}(\kappa_1) = \kappa_1(B)$.

It follows $x = \kappa_1(b)$ for some b in B. By the commutativity we have $\kappa_2(b) = \xi \circ \kappa_1(b) = \xi(x) = 0$. Since κ_2 is injective, $b = 0$. Hence $x = 0$. This shows $\mathrm{Ker}(\xi) = 0$.

(2°): Let y be any element of E_2. Then $\nu_2(y) \in A$.

Since ν_1 is surjective, $\nu_2(y) = \nu_1(x)$ for some x in E_1.
By the commutativity $\nu_1(x) = \nu_2(\xi(x))$. Thus we have $\nu_2(\xi(x)) = \nu_2(y)$.
Since ν_2 is k-regular, $\xi(x)+k_1 = y+k_2$ for some k_1, k_2 in $\mathrm{Ker}(\nu_2)$. By the regular exactness $k_1 = \kappa_2(\bar{k}_1)$ and $k_2 = \kappa_2(\bar{k}_2)$ for some $\bar{k}_1, \bar{k}_2$ in B.
By the commutativity $k_1 = \xi\circ\kappa_1(\bar{k}_1)$ and $k_2 = \xi\circ\kappa_1(\bar{k}_2)$.
Thus we have $\xi(x)+\xi(\kappa_1(\bar{k}_1)) = y+\xi(\kappa_1(\bar{k}_2))$, so that $\xi(x+\kappa_1(\bar{k}_1)) = y+\xi(\kappa_1(\bar{k}_2))$.
This shows $y\in\mathrm{Im}(\xi)$, so that $\mathrm{Im}(\xi) = E_2$. This means $E_2/\xi = 0$.

(2.4) Proposition. *Let* $B \overset{\kappa_1}{\rightarrowtail} E_1 \overset{\nu_1}{\twoheadrightarrow} A$, $B \overset{\kappa_2}{\rightarrowtail} E_2 \overset{\nu_2}{\twoheadrightarrow} A$ *be two extensions of* A *by* B. *If there is a regular* Λ*-homomorphism* $\xi: E_1 \to E_2$ *such that the diagram*



is commutative, then $\xi: E_1 \to E_2$ *is a* Λ*-isomorphism.*

Conversely if there is a Λ*-isomorphism* $\xi: E_1 \to E_2$ *which satisfys the above commutative diagram, then two extensions are equivalent.*

Proof. By Lemma (2.3) the sequence $0 \to E_1 \overset{\xi}{\to} E_2 \to 0$ is exact.
Since ξ is regular, ξ is a Λ-isomorphism.

Conversely if ξ is a Λ-isomorphism, then ξ is regular (cf. Proposition (4.4) in [5]). Hence two extensions are equivalent.

(2.5) Proposition. *The equvalence of extensions is an equivalence relation.*

Proof. This follows easily from Proposition (2.4).

(2.6) We denote by $\mathrm{E}_\Lambda(A,B)$ the set of equivalence classes of extensions of A by B.

(2.7) $E_\Lambda(A,B)$ contains at least one element : The Λ-semimodule $A\times B$, together with the maps q_B and p_A, yiels an extension $B \overset{q_B}{\rightarrowtail} A\times B \overset{p_A}{\twoheadrightarrow} A$.
We call any extension equivalent to the above extension a *split extension* of A by B.

(2.8) Let S be any semimodule.
If we denote by $G(S)$ the maximal subgroup of S, then $G(S)$ is a regular subsemimodule of S. Denote by $Q(S)$ the quotient semimodule $S/G(S)$. Then the sequence $0 \to G(S) \rightarrowtail S \twoheadrightarrow Q(S) \to 0$ is a regular exact sequence. Thus any semimodule S is an extension of the semimodule $Q(S)$ by the group $G(S)$.

3. An induced map $\alpha^*:E_\Lambda(A,B) \to E_\Lambda(A',B)$

(3.1) Let $\alpha:A' \to A$ be a Λ-homomorphism and let $B \overset{\kappa}{\rightarrowtail} E \overset{\nu}{\twoheadrightarrow} A$ be a representative of an element in $E_\Lambda(A,B)$.
We want to construct an extension

$$B \overset{\kappa'}{\rightarrowtail} E^\alpha \overset{\nu'}{\twoheadrightarrow} A'$$

such that the diagram

$$\begin{array}{ccc} B & = & B \\ {\scriptstyle\kappa'}\downarrow & & \downarrow{\scriptstyle\kappa} \\ E^\alpha & \xrightarrow{\xi} & E \\ {\scriptstyle\nu'}\downarrow & & \downarrow{\scriptstyle\nu} \\ A' & \xrightarrow{\alpha} & A \end{array}$$

is commutative.

Define (E^α,ξ,ν') as a pullback of the angle $A' \overset{\alpha}{\to} A \overset{\nu}{\leftarrow} E$. That is,

$$E^\alpha = \{(x,a)\in E\times A' \mid \nu(x) = \alpha(a)\} \subset E\times A',$$

$$\xi(x,a) = x \text{ and } \nu'(x,a) = a.$$

Define $\kappa':B \to E^\alpha$ by $\kappa'(b) = (\kappa(b),0)$.

Then E^{α} is a Λ-semimodule and ν', ξ are Λ-homomorphisms. Moreover the under square is commutative. Since $\nu(\kappa(b)) = 0 = \alpha(0)$, κ' is well defined and it is a Λ-homomorphism. Since $\xi \circ \kappa'(b) = \xi(\kappa(b), 0) = \kappa(b)$, the upper square is commutative.

Together with the following Proposition we obtain a reqired extension.

(3.2) Proposition.

The sequence $B \xrightarrow{\kappa'} E^{\alpha} \xrightarrow{\nu'} A'$ *is an extension of* A' *by* B.

Proof. (1°): We will show the sequence $0 \to B \xrightarrow{\kappa'} E^{\alpha} \xrightarrow{\nu'} A' \to 0$ is exact.

Since $\nu' \circ \kappa'(b) = \nu'(\kappa(b), 0) = 0$, we have $\mathrm{Im}(\kappa') \subset \mathrm{Ker}(\nu')$.

Let (x, a) be any element of $\mathrm{Ker}(\nu')$. Then $\nu'(x, a) = a = 0$.

Since $\nu(x) = \alpha(a) = \alpha(0) = 0$, $x \in \mathrm{Ker}(\nu) = \mathrm{Im}(\kappa) = \kappa(B)$. This means $x = \kappa(b)$ for some b in B. Therefore $(x, a) = (\kappa(b), 0) = \kappa'(b)$, so that $(x, a) \in \mathrm{Im}(\kappa')$.

This shows $\mathrm{Im}(\kappa') \supset \mathrm{Ker}(\nu')$. Thus we have proved that the sequence $B \xrightarrow{\kappa'} E^{\alpha} \xrightarrow{\nu'} A'$ is exact.

Now consider the exact sequence $\mathrm{Ker}(\kappa') \to B \xrightarrow{\kappa'} E^{\alpha}$.

Let b be any element of $\mathrm{Ker}(\kappa')$. Then $\kappa'(b) = (\kappa(b), 0) = (0, 0)$.

Hence $\kappa(b) = 0$. Since κ is injective, we have $b = 0$.

This shows the sequence $0 \to B \xrightarrow{\kappa'} E^{\alpha}$ is exact.

Let a be any element of A'. Then $\alpha(a) \in A$. Since ν is surjective, $\alpha(a) = \nu(x)$ for some x in E. This shows $(x, a) \in E^{\alpha}$ and $\nu'(x, a) = a$. Hence ν' is surjective.

Thus we have proved that the sequence

$$0 \to B \xrightarrow{\kappa'} E^{\alpha} \xrightarrow{\nu'} A' \to 0$$

is exact and ν' is surjective.

(2°): We will show κ' is regular.

Suppose $\kappa'(b) = \kappa'(b')$. Then $(\kappa(b),0) = (\kappa(b'),0)$. Therefore $\kappa(b) = \kappa(b')$. Since κ is injective, $b = b'$. This shows κ' is injective, so that κ' is k-regular.

Let (x,a) be any element of $\mathrm{Im}(\kappa')$. Then $(x,a)+\kappa'(b) = \kappa'(b')$ for some $b,b' \in B$. It follows $(x+\kappa(b),a) = (\kappa(b'),0)$. This shows $a = 0$ and $x+\kappa(b) = \kappa(b')$. Hence $x \in \mathrm{Im}(\kappa)$. Since κ is i-regular, $x = \kappa(\bar{b})$ for some $\bar{b}$ in B. Thus we have $(x,a) = (\kappa(\bar{b}),0) = \kappa'(\bar{b})$. This means $\mathrm{Im}(\kappa') = \kappa'(B)$. Hence κ' is i-regular. Thus we have proved κ' is a regular injection.

(3°): Finally we must prove that ν' is regular.

Since ν' is surjective, ν' is i-regular. So it is left to prove that ν' is k-regular.

Suppose $\nu'(x,a) = \nu'(x',a')$. Then $a = a'$. Since $(x,a),(x',a) \in E^{\alpha}$, $\nu(x) = \nu(x') = \alpha(a)$. By the regularity of ν, $x+k_1 = x'+k_2$ for some k_1,k_2 in $\mathrm{Ker}(\nu)$. It follows $(x,a)+(k_1,0) = (x+k_1,a) = (x'+k_2,a) = (x',a)+(k_2,0)$. Since $(k_1,0),(k_2,0) \in \mathrm{Ker}(\nu')$, ν' is k-regular. Thus we have proved that ν' is a regular surjection.

The assertions (1°),(2°) and (3°) show $B \overset{\kappa'}{\rightarrowtail} E^{\alpha} \overset{\nu'}{\twoheadrightarrow} A'$ is an extension of A' by B.

(3.3) For any Λ-homomorphism $\alpha: A' \to A$, we define the induced map

$$\alpha^* = \mathrm{E}_{\Lambda}(\alpha,B): \mathrm{E}_{\Lambda}(A,B) \to \mathrm{E}_{\Lambda}(A',B)$$

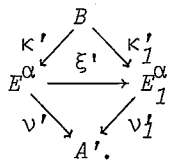
by assigning to the class of $B \rightarrowtail E \twoheadrightarrow A$ the class of $B \rightarrowtail E^{\alpha} \twoheadrightarrow A'$. Then we must prove that this definition is independent of the chosen representative $B \rightarrowtail E \twoheadrightarrow A$.

Let $B \overset{\kappa}{\rightarrowtail} E \overset{\nu}{\twoheadrightarrow} A$ be equivalent to $B \overset{\kappa_1}{\rightarrowtail} E_1 \overset{\nu_1}{\twoheadrightarrow} A$ with $\xi: E \to E_1$.

Then we define $\xi': E^{\alpha} \to E_1^{\alpha}$ by $\xi'(x,a) = (\xi(x),a)$.

We can easily check that ξ' is a Λ-homomorphism satisfying the commutative

diagram



It is left to prove that ξ' is a Λ-isomorphism.

(1°): We will show ξ' is injective.

Suppose $\xi'(x,a) = \xi'(x',a')$. Then $(\xi(x),a) = (\xi(x'),a')$.

Therefore $\xi(x) = \xi(x')$ and $a = a'$. Since ξ is injective, $x = x'$.

Thus $(x,a) = (x',a')$. This shows ξ' is injective.

(2°): We will show ξ' is surjective.

Let $(\bar{x},\bar{a})$ be any element of E_1^{α}, where $\bar{x}\in E_1, \bar{a}\in A'$. Since ξ is surjective, $\bar{x} = \xi(x)$ for some x in E. Hence $(\bar{x},\bar{a}) = (\xi(x),\bar{a}) = \xi'(x,\bar{a})$.

This shows ξ' is surjective.

The assertions (1°) and (2°) show ξ' is a Λ-isomorphism.

(3.4) Let *Set* be the category of sets.

We define $E_\Lambda(-,B):SMod_\Lambda \to Set$ as a map whose object function assigns to each object $A\in|SMod_\Lambda|$ the set $E_\Lambda(A,B)\in|Set|$ and whose morphism function assigns to each morphism $\alpha:A' \to A$ the map $\alpha^* = E_\Lambda(\alpha,B):E_\Lambda(A,B) \to E_\Lambda(A',B)$.

Theorem. *For each Λ-semimodule B,*

$$E_\Lambda(-,B):SMod_\Lambda \to Set$$

is a contravariant functor.

Proof. Let $\alpha = 1_A:A \to A$ be the identity of A.

Consider the diagram

$$\begin{array}{ccc} B & = & B \\ \downarrow{\scriptstyle \kappa'} & & \downarrow{\scriptstyle \kappa} \\ E^{\alpha} & \xrightarrow{\xi} & E \\ \downarrow{\scriptstyle \nu'} & & \downarrow{\scriptstyle \nu} \\ A & \xrightarrow[\alpha = 1_A]{} & A. \end{array}$$

Then ξ is a Λ-homomorphism.

We will show $E_{\Lambda}(1_A, B) = 1_{E_{\Lambda}(A,B)}$.

It is sufficient to prove that ξ is an isomorphism.

Suppose $\xi(x,a) = \xi(x',a')$. Then $x = x'$. It follows $a = \nu(x) = \nu(x') = a'$. Thus $(x,a) = (x',a')$. This shows ξ is injective.

Let x be any element of E. Put $a = \nu(x)$. Then $(x,a) \in E^{\alpha}$. Hence $\xi(x,a) = x$. This shows ξ is surjective, so that ξ is an isomorphism.

(2°): Let $\alpha' : A'' \to A'$, $\alpha : A' \to A$ be two Λ-homomorphisms.

Consider the diagram

$$\begin{array}{ccccc} (E^{\alpha})^{\alpha'} & \xrightarrow{\xi'} & E^{\alpha} & \xrightarrow{\xi} & E \\ \downarrow{\scriptstyle \nu''} & \text{I} & \downarrow{\scriptstyle \nu'} & \text{II} & \downarrow{\scriptstyle \nu} \\ A'' & \xrightarrow[\alpha']{} & A' & \xrightarrow[\alpha]{} & A, \end{array}$$

where I,II are pullback squares, respectively.

By Proposition (1.3), the diagram

$$\begin{array}{ccc} (E^{\alpha})^{\alpha'} & \xrightarrow{\xi \circ \xi'} & E \\ \nu''\downarrow & \text{III} & \downarrow\nu \\ A'' & \xrightarrow[\alpha\circ\alpha']{} & A \end{array}$$

is a pullback square.

On the other hand we consider a pullback square for the composition $\alpha\circ\alpha' : A'' \to A$

$$\begin{array}{ccc} E^{\alpha\circ\alpha'} & \xrightarrow{\bar{\xi}} & E \\ \bar{\nu}\downarrow & \alpha\circ\alpha' & \downarrow\nu \\ A'' & \longrightarrow & A. \end{array}$$

By the universal property of the pullback there exists a Λ-isomorphism $l:(E^{\alpha})^{\alpha'} \to E^{\alpha\circ\alpha'}$. Then we will prove that the diagram

$$\begin{array}{ccc} & B & \\ {\scriptstyle \kappa''}\swarrow & & \searrow{\scriptstyle \bar{\kappa}} \\ (E^{\alpha})^{\alpha'} & \xrightarrow[l]{} & E^{\alpha\circ\alpha'} \end{array}$$

is commutative.

From the construction in (1.7), l is given by

$$l((x,a'),a'') = (x,a''),$$

where $a' = \alpha'(a'')$.

For every b in B,

$$\begin{aligned} l\circ\kappa''(b) &= l(\kappa'(b),0) \\ &= l((\kappa(b),0),0) \\ &= (\kappa(b),0) \\ &= \bar{\kappa}(b). \end{aligned}$$

This shows the diagram is commutative.

Thus $B \overset{\kappa''}{\rightarrowtail} (E^{\alpha})^{\alpha'} \overset{\nu''}{\twoheadrightarrow} A''$ is equivalent to $B \overset{\bar{\kappa}}{\rightarrowtail} E^{\alpha\circ\alpha'} \overset{\bar{\nu}}{\twoheadrightarrow} A''$ with $l:(E^{\alpha})^{\alpha'} \to E^{\alpha\circ\alpha'}$. That is, we have proved

$$E_{\Lambda}(\alpha\circ\alpha',B) = E_{\Lambda}(\alpha',B)\circ E_{\Lambda}(\alpha,B).$$

By (1°) and (2°), $E_{\Lambda}(-,B)$ is a contravariant functor.

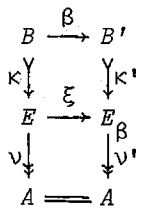
4. An induced map $\beta_*:E_{\Lambda}(A,B) \to E_{\Lambda}(A,B')$

(4.1) Let $\beta:B \to B'$ be a Λ-homomorphism and let $B \overset{\kappa}{\rightarrowtail} E \overset{\nu}{\twoheadrightarrow} A$ be a representative of an element in $E_{\Lambda}(A,B)$.

We want to construct an extension

$$B' \overset{\kappa'}{\rightarrowtail} E_{\beta} \overset{\nu'}{\twoheadrightarrow} A$$

such that the diagram



is commutative. But the fact is rather delicate than the construction in the previous section.

For the coangle $E \overset{\kappa}{\leftarrow} B \overset{\beta}{\rightarrow} B'$ we denote by E_β the Λ-semimodule $\overline{B' \amalg_B E}$ constructed in (1.8). That is, we define

$$E_\beta = B' \times E / \rho(R),$$

where R is the symmetric Λ-subsemimodule of $(B' \times E) \times (B' \times E)$ defined by

$$R = \{ <(\beta(b), \kappa(b')), (\beta(b'), \kappa(b))> \mid b, b' \in B \}$$

and $\rho(R)$ is the congruence associated with R.

And we define κ', ξ and ν' by

$$\kappa'(b) = \overline{(b, 0)},$$

$$\xi(x) = \overline{(0, x)} \text{ and}$$

$$\nu'(\overline{b, x}) = \nu(x), \text{ respectively.}$$

Then E_β is a Λ-semimodule and κ', ξ are Λ-homomorphisms.

(4.2) In $E_\beta = B' \times E / \rho(R)$, $\overline{(b, x)} = \overline{(b', x')}$ if and only if $b + \beta(\bar{b}) = b' + \beta(\bar{b}')$ and $x + \kappa(\bar{b}') = x' + \kappa(\bar{b})$ for some $\bar{b}, \bar{b}'$ in B.

(4.3) We must show $\nu' : E_\beta \to A$ is well defined.

Suppose $\overline{(b, x)} = \overline{(b', x')}$. Then $b + \beta(\bar{b}) = b' + \beta(\bar{b}')$ and $x + \kappa(\bar{b}') = x' + \kappa(\bar{b})$ for some $\bar{b}, \bar{b}'$ in B. It follows $\nu(x) + \nu\circ\kappa(\bar{b}') = \nu(x') + \nu\circ\kappa(\bar{b})$. Since $\nu\circ\kappa = 0$, we have $\nu(x) = \nu(x')$. Thus ν' is well defined. Plainly ν' is a Λ-homomorphism.

(4.4) **Proposition.**

For any Λ-homomorphism $\beta : B \to B'$, the diagram

$$\begin{array}{ccc} B & \xrightarrow{\beta} & B' \\ {\scriptstyle\kappa}\downarrow & \xi & \downarrow{\scriptstyle\kappa'} \\ E & \longrightarrow & E_\beta \\ {\scriptstyle\nu}\downarrow & & \downarrow{\scriptstyle\nu'} \\ A & = & A \end{array}$$

is commutative such that

1) κ' *is i-regular, and*

2) ν' *is a regular surjection.*

Proof. By the discussion in (1.8) the diagram

$$\begin{array}{ccc} B & \xrightarrow{\beta} & B' \\ {\scriptstyle\kappa}\downarrow & \xi & \downarrow{\scriptstyle\kappa'} \\ E & \longrightarrow & E_\beta \end{array}$$

is commutative.

Next the commutativity of the square

$$\begin{array}{ccc} E & \xrightarrow{\xi} & E_\beta \\ {\scriptstyle\nu}\downarrow & & \downarrow{\scriptstyle\nu'} \\ A & = & A \end{array}$$

follows from the definition of ν'.

1): We will show κ' is i-regular.

Let $(\overline{b,x}) \in \mathrm{Im}(\kappa')$. Then $(\overline{b,x}) + \kappa'(c) = \kappa'(c')$ for some c, c' in B'.

It follows $(\overline{b+c,x}) = (\overline{c',0})$. Therefore $b+c+\beta(\vec{d}) = c'+\beta(\vec{d}')$ and $x+\kappa(\vec{d}') = \kappa(\vec{d})$ for some $\vec{d}, \vec{d}'$ in B, so that $x \in \mathrm{Im}(\kappa)$. Since $\mathrm{Im}(\kappa) = \kappa(B)$, $x = \kappa(\overline{\overline{b}})$ for some $\overline{\overline{b}}$ in B.

Thus we have

$$\begin{aligned} (\overline{b,x}) &= (\overline{b,0}) + (\overline{0,x}) \\ &= \kappa'(b) + \overline{(0,\kappa(\overline{\overline{b}}))} \\ &= \kappa'(b) + \xi(\kappa(\overline{\overline{b}})) \\ &= \kappa'(b) + \kappa'(\beta(\overline{\overline{b}})) \end{aligned}$$

$$= \kappa'(b+\beta(\bar{\bar{b}})).$$

This shows κ' is i-regular.

2): First we show ν' is surjective.

Let a be any element of A. Since ν is surjective, $a = \nu(x)$ for some x in E. Then $\nu'(\overline{0,x}) = \nu(x) = a$. Thus ν' is surjective, and so ν' is i-regular.

Next we show ν' is k-regular.

Suppose $\nu'(\overline{b,x}) = \nu'(\overline{b',x'})$. Then $\nu(x) = \nu(x')$. Since ν is k-regular, $x+k = x'+k'$ for some k,k' in $\mathrm{Ker}(\nu)$.

Then we have

$$\begin{aligned}(\overline{b,x})+(\overline{b',0})+(\overline{0,k}) &= (\overline{b,0})+(\overline{0,x})+(\overline{b',0})+(\overline{0,k})\\ &= (\overline{b',0})+(\overline{0,x+k})+(\overline{b,0})\\ &= (\overline{b',0})+(\overline{0,x'+k'})+(\overline{b,0})\\ &= (\overline{b',0})+(\overline{0,x'})+(\overline{0,k'})+(\overline{b,0})\\ &= (\overline{b',x'})+(\overline{0,k'})+(\overline{b,0}).\end{aligned}$$

That is, $(\overline{b,x})+(\overline{b',0})+(\overline{0,k}) = (\overline{b',x'})+(\overline{0,k'})+(\overline{b,0})$, where $(\overline{b',0}),(\overline{b,0}) \in \mathrm{Ker}(\nu')$. Since $k,k' \in \mathrm{Ker}(\nu)$, we have $\nu'(\overline{0,k}) = \nu(k) = 0$ and $\nu'(\overline{0,k'}) = \nu(k') = 0$. Therefore $(\overline{0,k}),(\overline{0,k'}) \in \mathrm{Ker}(\nu')$. Thus we have $(\overline{b',0})+(\overline{0,k})$, $(\overline{0,k'})+(\overline{b,0}) \in \mathrm{Ker}(\nu')$. This means ν' is k-regular.

(4.5) Proposition. *The sequence* $B' \xrightarrow{\kappa'} E_\beta \xrightarrow{\nu'} A \to 0$ *is exact.*

Proof. Since ν' is surjective, it is left to prove that $B' \xrightarrow{\kappa'} E_\beta \xrightarrow{\nu'} A$ is exact. Since $\nu' \circ \kappa'(b) = \nu'(\overline{b,0}) = \nu(0) = 0$, $\mathrm{Im}(\kappa') \subset \mathrm{Ker}(\nu')$.

Now let $(\overline{b,x}) \in \mathrm{Ker}(\nu')$. Then $\nu'(\overline{b,x}) = \nu(x) = 0$. Thus $x \in \mathrm{Ker}(\nu) = \mathrm{Im}(\kappa) = \kappa(B)$, so that $x = \kappa(\bar{b})$ for some $\bar{b}$ in B.

Then we have

$$\begin{aligned}(\overline{b,x}) &= (\overline{b,0})+(\overline{0,x})\\ &= \kappa'(b)+\xi(x)\end{aligned}$$

$$= \kappa'(b)+\xi(\kappa(\bar{b}))$$

$$= \kappa'(b)+ \kappa'(\beta(\bar{b}))$$

$$= \kappa'(b+\beta(\bar{b})).$$

This means $(\overline{b,x})\in \mathrm{Im}(\kappa')$. Thus we have $\mathrm{Im}(\kappa')\supset \mathrm{Ker}(\nu')$, so that $\mathrm{Im}(\kappa') = \mathrm{Ker}(\nu')$.

(4.6) Proposition. *Let B' be a cancellative Λ-semimodule. Then for any Λ-homomorphism $\beta:B \to B'$ the sequence*

$$0 \to B' \xrightarrow{\kappa'} E_\beta \xrightarrow{\nu'} A \to 0$$

is an extension of A by B'.

Proof. Since by Proposition (4.4) ν' is a regular surjection and κ' is i-regular, it is left to prove that κ' is injective.

Suppose $\kappa'(b) = \kappa'(b')$. Then $(\overline{b,0}) = (\overline{b',0})$. This means $b+\beta(c) = b'+\beta(c')$ and $\kappa(c') = \kappa(c)$ for some c,c' in B. Since κ is injective, $c = c'$. Therefore $b+\beta(c) = b'+\beta(c)$. Since B' is cancellative, we have $b = b'$. This shows κ' is injective.

(4.7) Let B' be a cancellative Λ-semimodule. Then for any Λ-homomorphism $\beta:B \to B'$, we define the induced map

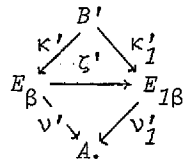
$$\beta_* = E_\Lambda(A,\beta):E_\Lambda(A,B) \to E_\Lambda(A,B')$$

by assigning to the class of $B \rightarrowtail E \twoheadrightarrow A$ the class of $B' \rightarrowtail E_\beta \twoheadrightarrow A$. Then we must prove that this definition is independent of the chosen representative $B \rightarrowtail E \twoheadrightarrow A$.

Let $B \xrightarrow{\kappa} E \xrightarrow{\nu} A$ be equivalent to $B \xrightarrow{\kappa_1} E_1 \xrightarrow{\nu_1} A$ with $\zeta:E \to E_1$. Then we define $\zeta':E_\beta \to E_{1\beta}$ by

$$\zeta'(\overline{b,x}) = (\overline{b,\zeta(x)}).$$

Plainly ζ' is a Λ-homomorphism satisfying the commutative diagram



We must prove ζ' is an isomorphism.

(1°): We will show ζ' is surjective.

Let $(\overline{b,x_1})\epsilon E_{1\beta}$. Then $b\epsilon B', x_1\epsilon E_1$. Since ζ is surjective, $x_1 = \zeta(x)$ for some x in E. Therefore $(\overline{b,x_1}) = (\overline{b,\zeta(x)}) = \zeta'(\overline{b,x})$. This shows ζ' is surjective.

(2°): We will show ζ' is injective.

Suppose $\zeta'(\overline{b,x}) = \zeta'(\overline{b',x'})$. Then $(\overline{b,\zeta(x)}) = \overline{(b',\zeta(x'))}$ in $E_{1\beta}$. This means $b+\beta(\bar{c}) = b'+\beta(\bar{c}')$ and $\zeta(x)+\kappa_1(\bar{c}') = \zeta(x')+\kappa_1(\bar{c})$ for some $\bar{c},\bar{c}'$ in B. Since $\kappa_1(\bar{c}'),\kappa_1(\bar{c})\epsilon E_1$ and ζ is a Λ-isomorphism, $\kappa_1(\bar{c}') = \zeta(\bar{y}')$ and $\kappa_1(\bar{c}) = \zeta(\bar{y})$ for some $\bar{y},\bar{y}'$ in E. Then,

$$\begin{aligned}\zeta(x+\bar{y}') &= \zeta(x)+\zeta(\bar{y}')\\ &= \zeta(x)+\kappa_1(\bar{c}')\\ &= \zeta(x')+\kappa_1(\bar{c})\\ &= \zeta(x')+\zeta(\bar{y})\\ &= \zeta(x'+\bar{y}).\end{aligned}$$

Since ζ is a Λ-isomorphism, $x+\bar{y}' = x'+\bar{y}$.

On the other hand by the commutativity $\zeta(\bar{y}') = \kappa_1(\bar{c}') = \zeta(\kappa(\bar{c}'))$ and $\zeta(\bar{y}) = \kappa_1(\bar{c}) = \zeta(\kappa(\bar{c}))$. Since ζ is a Λ-isomorphism, $\bar{y}' = \kappa(\bar{c}')$ and $\bar{y} = \kappa(\bar{c})$. Thus we have $b+\beta(\bar{c}) = b'+\beta(\bar{c}')$ and $x+\kappa(\bar{c}') = x'+\kappa(\bar{c})$. This means $(\overline{b,x}) = (\overline{b',x'})$. Hence ζ' is injective.

Thus we have proved β_* is well defined.

To summarize, we have proved :

Proposition. *Let B' be a cancellative Λ-semimodule.*

Then for any Λ*-homomorphism* $\beta:B \to B'$ *we can define the induced map*

$$\beta_* = E_\Lambda(A,\beta):E_\Lambda(A,B) \to E_\Lambda(A,B')$$

by assigning to the class of $B \rightarrowtail E \twoheadrightarrow A$ *the class of* $B' \rightarrowtail E_\beta \twoheadrightarrow A$.

5. A bifunctor $E_\Lambda(-,-):SMod_\Lambda \times Mod_\Lambda \to Set$

(5.1) We denote by Mod_Λ the category of Λ-modules.

Define $E_\Lambda(A,-):Mod_\Lambda \to Set$ as a map whose object function assigns to each object $B \in |Mod_\Lambda|$ the set $E_\Lambda(A,B) \in |Set|$ and whose morphism function assigns to each morphism $\beta:B \to B'$ the induced map $\beta_* = E_\Lambda(A,\beta):E_\Lambda(A,B) \to E_\Lambda(A,B')$.

Our first aim is to make $E_\Lambda(A,-)$ into a functor.

(5.2) Our extensions considered in this section are always extensions of a Λ-semimodule by a Λ-module.

We notice that every Λ-homomorphism $f:B \to E$ of a Λ-module B to a Λ-semimodule E is always regular.

(5.3) Let B,B' be Λ-modules and let $\beta:B \to B'$ be a Λ-homomorphism. Since B' is cancellative, by Proposition (4.7) we have the induced map

$$\beta_* = E_\Lambda(A,\beta):E_\Lambda(A,B) \to E_\Lambda(A,B').$$

Let $B \overset{\kappa}{\rightarrowtail} E \overset{\nu}{\twoheadrightarrow} A$ be a representative of an element of $E_\Lambda(A,B)$.

In the commutative diagram

$$\begin{array}{ccc} B & \xrightarrow{\beta} & B' \\ {\scriptstyle\kappa}\downarrow & & \downarrow{\scriptstyle\kappa'} \\ E & \xrightarrow{\xi} & E_\beta \\ {\scriptstyle\nu}\downarrow & & \downarrow{\scriptstyle\nu'} \\ A & = & A, \end{array}$$

$E_\beta = B' \times E/\rho(R)$, $\kappa'(b) = \overline{(b,0)}$, $\nu'\overline{(b,x)} = \nu(x)$ and $\xi(x) = \overline{(0,x)}$, where $R = \{<(\beta(\bar{b}),\kappa(\bar{b}')),(\beta(\bar{b}'),\kappa(\bar{b}))> \mid \bar{b},\bar{b}' \in B\}$.

Since B is a Λ-module, $(\overline{b,x}) = (\overline{b',x'})$ in E_β if and only if $b = b'+\beta(c)$ and $x = x'+\kappa(-c)$ for some c in B.

By Proposition (1.10), the Λ-semimodule E_β is determined uniquely as a pushout.

(5.4) Lemma. *Let* $\beta = 1_B:B \to B$ *be the identity. Then* $B \overset{\kappa}{\rightarrowtail} E \overset{\nu}{\twoheadrightarrow} A$ *is equivalent to* $B \overset{\kappa'}{\rightarrowtail} E_\beta \overset{\nu'}{\twoheadrightarrow} A$.

Proof.

Since B is a Λ-module, E_β is determined uniquely by Proposition (1.10) as a pushout satisfying the commutative diagram

$$\begin{array}{ccc} B & \xrightarrow{1_B} & B \\ {\scriptstyle\kappa}\downarrow & & \downarrow{\scriptstyle\kappa'} \\ E & \xrightarrow{\xi} & E_\beta \\ {\scriptstyle\nu}\downarrow & & \downarrow{\scriptstyle\nu'} \\ A & = & A. \end{array}$$

On the other hand, since the square

$$\begin{array}{ccc} B & \xrightarrow{1_B} & B \\ {\scriptstyle\kappa}\downarrow & & \downarrow{\scriptstyle\kappa} \\ E & \xrightarrow{1_E} & E \end{array}$$

means that the triple $(1_E,\kappa,E)$ is a sink of the coangle $E \overset{\kappa}{\leftarrow} B \overset{1_B}{\rightarrow} B$, there exists a unique morphism $l:E_\beta \to E$ such that the diagram

$$\begin{array}{ccc} B & \xrightarrow{1_B} & B \\ {\scriptstyle\kappa}\downarrow & & \downarrow{\scriptstyle\kappa'} \\ E & \xrightarrow{\xi} & E_\beta \\ & {\scriptstyle 1_E}\searrow & \swarrow{\scriptstyle\kappa} \\ & & E \end{array}$$

is commutative.

Thus $1_E = l\circ\xi$. This shows ξ is injective.

By the construction in (1.10), l is given by $l(\overline{b,x}) = \kappa(b)+x$.

For $(\overline{b,x}) \in E_\beta$, $\xi \circ l(\overline{b,x}) = \xi(\kappa(b)+1_E(x)) = (\overline{0,\kappa(b)+x})$.

Since $b = 0+1_B(b)$ and $x = \kappa(b)+x+\kappa(-b)$ for b in B, we have $(\overline{b,x}) = (\overline{0,\kappa(b)+x})$. This shows $\xi \circ l = 1_{E_\beta}$. Therefore ξ is surjective. Thus ξ is a required equivalence.

(5.5) Theorem.

For any Λ-semimodule A, $E_\Lambda(A,-):Mod_\Lambda \to Set$ is a covariant functor.

Proof. (1°): For the identity $\beta = 1_B:B \to B$, by Lemma (5.4) we have

$$E_\Lambda(A,1_B) = 1_{E_\Lambda(A,B)}.$$

(2°): For given two Λ-homomorphisms $\beta:B \to B'$ and $\beta':B' \to B''$, consider the commutative diagram

$$\begin{array}{ccccc} B & \xrightarrow{\beta} & B' & \xrightarrow{\beta'} & B'' \\ {\scriptstyle\kappa}\downarrow & \text{I} & \downarrow{\scriptstyle\kappa'} & \text{II} & \downarrow{\scriptstyle\kappa''} \\ E & \underset{\xi}{\longrightarrow} & E_\beta & \underset{\xi'}{\longrightarrow} & (E_\beta)_{\beta'} \end{array},$$

where I,II are pushout squares, respectively.

By Proposition (1.3), the diagram

$$\begin{array}{ccc} B & \xrightarrow{\beta'\circ\beta} & B'' \\ {\scriptstyle\kappa}\downarrow & \text{III} & \downarrow{\scriptstyle\kappa''} \\ E & \underset{\xi'\circ\xi}{\longrightarrow} & (E_\beta)_{\beta'} \end{array}$$

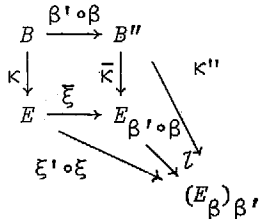
is a pushout square.

On the other hand, we consider a pushout square for the composition $\beta'\circ\beta:B \to B''$

$$\begin{array}{ccc} B & \xrightarrow{\beta'\circ\beta} & B'' \\ {\scriptstyle\kappa}\downarrow & & \downarrow{\scriptstyle\bar{\kappa}} \\ E & \underset{\bar{\xi}}{\longrightarrow} & E_{\beta'\circ\beta} \end{array}.$$

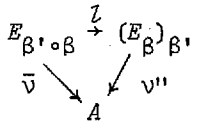
By the universal property of the pushout there exists a unique Λ-isomorphism

$l: E_{\beta' \circ \beta} \to (E_\beta)_{\beta'}$ such that the diagram



is commutative.

It is left to prove that the diagram



is commutative.

From the construction in (1.10), l is given by

$$l(\overline{b'',x}) = \kappa''(b'') + \xi' \circ \xi(x) \text{ for } (\overline{b'',x}) \text{ in } E_{\beta' \circ \beta} .$$

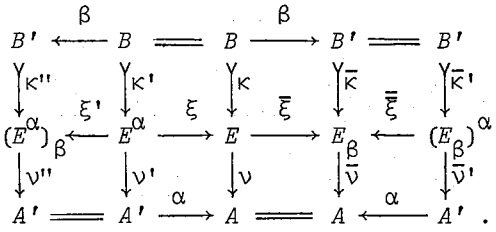
Since
$$\begin{aligned} \nu'' \circ l(\overline{b'',x}) &= \nu''(\kappa''(b'') + \xi' \circ \xi(x)) \\ &= \nu''((\xi' \circ \xi)(x)) \\ &= \nu''(\xi'(\overline{0,x})) \\ &= \nu''(0, \overline{(0,x)}) \\ &= \nu'(\overline{0,x}) \\ &= \nu(x) \\ &= \bar{\nu}(\overline{b'',x}), \end{aligned}$$

the diagram is commutative.

(5.6) Theorem.

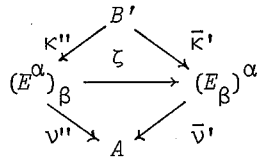
$E_\Lambda(-,-): SMod_\Lambda \times Mod_\Lambda \to Set$ is a bifunctor.

Proof. For given two Λ-homomorphisms $\beta: B \to B'$ and $\alpha: A' \to A$, consider the diagram



It remains to check that $\beta_*\alpha^* = \alpha^*\beta_* : E_\Lambda(A,B) \to E_\Lambda(A',B')$.

We have to show the existence of a Λ-isomorphism $\zeta : (E^\alpha)_\beta \to (E_\beta)^\alpha$ such that the diagram



is commutative.

Define $\zeta : (E^\alpha)_\beta \to (E_\beta)^\alpha$ by

$$\zeta(\overline{b',(x,a')}) = (\overline{(b',x)},a') \quad \text{for each } \overline{(b',(x,a'))} \in (E^\alpha)_\beta .$$

(1°): We first have to check that the definition is well defined.

Suppose $\overline{(b',(x,a'))} = \overline{(\hat{b}',(\hat{x},\hat{a}'))}$ in $(E^\alpha)_\beta$.

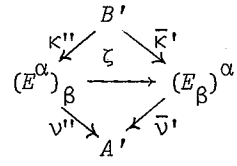
Then $b' = \hat{b}' + \beta(c)$ and $(x,a') = (\hat{x},\hat{a}') + \kappa'(-c)$ for some c in B.

It follows $(x,a') = (\hat{x},\hat{a}') + (\kappa(-c),0)$, that is, $x = \hat{x} + \kappa(-c)$ and $a' = \hat{a}'$.

Thus we have $b' = \hat{b}' + \beta(c)$ and $x = \hat{x} + \kappa(-c)$ for some c in B.

This shows $\overline{(b',x)} = \overline{(\hat{b}',\hat{x})}$ in E_β. Therefore $(\overline{(b',x)},a') = (\overline{(\hat{b}',\hat{x})},\hat{a}')$ in $(E_\beta)^\alpha$. Hence ζ is well defined.

(2°): We next check that the diagram



is commutative.

For any element b' of B',

$$\begin{aligned}\zeta\circ\kappa''(b') &= \zeta(\overline{b',(0,0)}) \\ &= (\overline{(b',0)0}) \\ &= (\bar{\kappa}(b'),0) \\ &= \bar{\kappa}'(b').\end{aligned}$$

For any element $\overline{(b',(x,a'))}$ of $(E^{\alpha})_{\beta}$,

$$\begin{aligned}\bar{\nu}'\circ\zeta(\overline{b',(x,a')}) &= \bar{\nu}'((\overline{b',x}),a') \\ &= a' \\ &= \nu'(x,a') \\ &= \nu''(\overline{b',(x,a')}).\end{aligned}$$

Thus the diagram is commutative.

(3°): The fact that ζ is a Λ-homomorphism is checked as follow.

$$\begin{aligned}&\zeta(\overline{(b',(x,a'))}+\overline{(\hat{b}',(\hat{x},\hat{a}'))} \\ &\quad= \zeta(\overline{b'+\hat{b}',(x+\hat{x},a'+\hat{a}')}) \\ &\quad= (\overline{(b'+\hat{b}',(x+\hat{x}))},a'+\hat{a}') \\ &\quad= (\overline{(b',x)}+\overline{(\hat{b}',\hat{x})},a'+\hat{a}') \\ &\quad= (\overline{(b',x)},a')+(\overline{(\hat{b}',\hat{x})},\hat{a}') \\ &\quad= \zeta(\overline{b',(x,a')})+\zeta(\overline{\hat{b}',(\hat{x},\hat{a}')}).\end{aligned}$$

$$\begin{aligned}&\zeta(\lambda\overline{(b',(x,a'))} \\ &\quad= \zeta(\overline{\lambda b',\lambda(x,a')}) \\ &\quad= \zeta(\overline{\lambda b',(\lambda x,\lambda a')}) \\ &\quad= (\overline{(\lambda b',\lambda x)},\lambda a') \\ &\quad= (\lambda\overline{(b',x)},\lambda a') \\ &\quad= \lambda((\overline{b',x)},a') \\ &\quad= \lambda\zeta(\overline{b',(x,a')}).\end{aligned}$$

Finally we must prove that ζ is an isomorphism.

(4°): We will show ζ is surjective.

Let $(\overline{(b',x)},a') \in (E_\beta)^\alpha$. Then $\nu(x) = \bar{\nu}\overline{(b',x)} = \alpha(a')$. Therefore $(x,a') \in E^\alpha$. Since $b' \in B'$, $\overline{(b',(x,a'))} \in (E^\alpha)_\beta$. This shows ζ is surjective.

(5°): We will show ζ is injective.

Suppose $\zeta\overline{(b',(x,a'))} = \zeta\overline{(\hat{b}',(\hat{x},\hat{a}'))}$. Then $(\overline{(b',x)},a) = (\overline{(\hat{b}',\hat{x})},\hat{a}')$. This means $\overline{(b',x)} = \overline{(\hat{b}',\hat{x})}$ and $a' = \hat{a}'$. It follows $b' = \hat{b}' + \beta(c)$ and $x = \hat{x} + \kappa(-c)$ for some c in B. Then $(x,a') = (\hat{x},\hat{a}') + (\kappa(-c),0) = (\hat{x},\hat{a}') + \kappa'(-c)$. Thus we have $b' = \hat{b}' + \beta(c)$ and $(x,a') = (\hat{x},\hat{a}') + \kappa'(-c)$ for some c in B. This means $\overline{(b',(x,a'))} = \overline{(\hat{b}',(\hat{x},\hat{a}'))}$. Hence ζ is injective.

This completes the proof.

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