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Application of a Common Fixed Point Theorem*

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Motivated by a paper of K. Iséki [1] we shall prove the following

Theorem 1. *Let* (X, d) *be a nonempty complete metric space, and* S, T *be continuous mappings of the product space* $X \times X$ *into* X *such that*

$$S(X \times X) \subset T(X \times X) \quad \text{and} \quad S(T(x, y), y) = T(S(x, y), y)$$

for all x, y *in* X. *If there exists an* α *such that* $0 \leq \alpha < 1$ *and*

$$(*) \quad d(S(x, y), S(x', y')) \leq \alpha \cdot \max \{ d(T(x, y), T(x', y')), d(T(x, y), S(x, y)), d(T(x', y'), S(x', y')), \frac{1}{2}[d(T(x, y), S(x', y')) + d(T(x', y'), S(x, y))] \}$$

for all $(x, y), (x', y')$ *in* $X \times X$, *then there exists exactly one point* a *in* X *such that* $S(a, y) = a = T(a, y)$ *for all* y *in* X.

The proof requires the following common fixed point theorem:

Theorem 2. *Let* S *and* T *be continuous commuting mappings of a nonempty complete metric space* (X, d) *into itself such that* $S(X) \subset T(X)$. *If there exists an* α *such that* $0 \leq \alpha < 1$ *and*

$$(**) \quad d(Sx, Sy) \leq \alpha \cdot \max \{ d(Tx, Ty), d(Sx, Tx), d(Sy, Ty), \frac{1}{2}[d(Sx, Ty) + d(Sy, Tx)] \}$$

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for all x, y *in* X, *then* S *and* T *have a unique common fixed point.*

Proof. Let φ be a choice function for the family $\{T^{-1}y : y \in S(X)\}$, and $x_0 \in X$. Then there is a sequence $s = \{x_n\}_{n\in\omega}$ in X such that $x_{n+1} = \varphi(T^{-1}Sx_n)$ for all n in ω, the set of nonnegative integers. Hence by (**),

$$\begin{aligned} d(Tx_{n+1}, Tx_n) &= d(Sx_n, Sx_{n-1}) \\ &\le \alpha d(Tx_n, Tx_{n-1}) \\ &\le \alpha^{n-1} d(Tx_2, Tx_1). \end{aligned}$$

Therefore Ts is a Cauchy sequence. Thus Ts converges to some $z \in X$. Since Ss is a subsequence of Ts, the sequence Ss also converges to z. Then by the continuity of S and T, we have $TSs \to Tz$ and $STs \to Sz$, which imply $Tz = Sz$ since S and T are commuting. Hence using (**) it is easy to see that

$$d(Sz, SSz) \le \alpha d(Sz, SSz),$$

proving $SSz = Sz$. Thus we have $Sz = SSz = STz = TSz$. Therefore Sz is a common fixed point of S and T. To show that S and T can have only one common fixed point, let $u = Su = Tu$ and $v = Sv = Tv$. Then by (**),

$$d(u, v) = d(Su, Sv) \le \alpha d(u, v),$$

implying $u = v$.

We remark that in case $Tx = x$ for every x in X, the continuity condition on S in Theorem 2 can be waived. We shall apply Theorem 2 to prove Theorem 1. Hence if $T(x, y) = x$ for every (x, y) in $X \times X$, the continuity condition on S in the first theorem can also be waived, and consequently, in such a situation, the result of the first theorem pre-

sents a stronger version of the result of K. Iséki [1].

Proof of Theorem 1. Let $y = y'$ and $x \neq x'$. Then by (*),

$$d(S(x, y), S(x', y)) \leq \alpha \max \{d(T(x, y), T(x', y)),$$
$$d(T(x, y), S(x, y)), d(T(x', y), S(x', y)),$$
$$\frac{1}{2}[d(T(x, y), S(x', y)) + d(T(x', y), S(x, y))]\}.$$

This holds for every y in X. Therefore by Theorem 2, for each y in X, there exists only one $x(y)$ in X such that

$$S(x(y), y) = x(y) = T(x(y), y).$$

We claim $x(y) = x(y')$ for $y \neq y'$. For if not, then by (*),

$$d(x(y), x(y')) = d(S(x(y), y), S(x(y'), y'))$$
$$\leq \alpha d(x(y), x(y')).$$

Hence $x(\cdot)$ is some constant $a \in X$, and so $S(a, y) = a = T(a, y)$ for all y in X. To show the uniqueness of a, let there be another such b. Then by (*),

$$d(a, b) = d(S(a, y), S(b, y)) \leq \alpha d(a, b),$$

proving $a = b$.

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Reference

[1] K. Iséki: Application of Zamfirescu's fixed point theorem, these NOTES, 4 (1976), 215-216.

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