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On Some Common Fixed Point Theorems for Contractive Mappings

By Shouro KASAHARA*

Various common fixed point theorems for self-mappings f, g on a metric space (X, d) satisfying some contractive conditions are known. It was shown in a previous paper [5] that certain contractive conditions can be reduced to the pairwise contractive condition

$$(\dagger) \qquad \begin{cases} d(gf(x),\ f(x)) \le \alpha d(f(x),\ x) \\ d(fg(x),\ g(x)) \le \beta d(g(x),\ x) \end{cases}$$

and those mappings which satisfy pairwise contractive conditions are studied in [5]-[7]. However, in these earlier works, we mainly concerned with continuous mappings. The purpose of this paper is to offer, by relaxing slightly the pairwise contractive condition (†), a unified treatment of some common fixed point theorems for two self-mappings which satisfy certain contractive conditions but need not be continuous. The common fixed point theorems due to Fisher [1]-[3], Khan [9], Rhoades [11], and a fixed point theorem of Pal and Maiti [10] will be derived from a common fixed point theorem (see Theorem 1 below) in L-spaces. Moreover, the author's results obtained in [8] will be also derived from a similar theorem (Theorem 2 below).

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1. Basic theorems

We shall denote by ω the set of all nonnegative integers. Let X be a set. A family D of extended real valued functions on $X \times X$ is said to be *separating* if $d(x, y) = 0$ $(x, y \in X)$ for all $d \in D$ implies $x = y$.

Let f and g be self-mappings of an L-space $(X, \to)$, D a nonempty family of nonnegative extended real valued functions on $X \times X$, and $x \in X$. We say that $(X, \to)$ is *(x, D, f, g)-complete* if every sequence $\{x_n\}_{n \in \omega}$ in the set $O(x, gf) \cup f(O(x, gf))$ satisfying

$$\sum_{n=0}^{\infty} d(x_{n+1}, x_n) < \infty$$

for all $d \in D$, converges to at least one point of X, where $O(x, f)$ denotes the set $\{f^n(x) \mid n \in \omega\}$. Let d be a nonnegative extended real valued function on $X \times X$. By a *d-semifixer* on X, we mean an extended real valued function φ on X^4 satisfying the following conditions:

(1) If $\varphi(x, y, x, x) \geq 0$ then $d(x, y) = 0$.

(2) If $u_n \to u$, $v_n \to u$ in $(X, \to)$, $d(v_n, u_n) \to 0$, and

$$\varphi(x, y, u_n, v_n) \geq \lambda$$

for all $n \in \omega$, then $\varphi(x, y, u, u) \geq \lambda$.

A d-semifixer φ on X is called a *d-fixer* on X if $\varphi(x, x, x, y) \geq 0$ implies $d(x, y) = 0$.

We need the following

Lemma. *Let f and g be mappings of an* L*-space $(X, \to)$ into itself, and D a nonempty family of nonnegative extended real valued functions on $X \times X$. Suppose that there exists a point $x_0 \in X$ satisfying the following conditions:*

(∞) $d(f(x_0), x_0) < \infty$ *for all* $d \in D$.

(O) *There exist functions* α, β *of* D *into* $[0, 1)$ *such that, for all* $d \in D$ *and* $x \in O(x_0, gf)$,

$$d(gf(x), f(x)) \leq \alpha(d)d(f(x), x)$$

$$d(fgf(x), gf(x)) \leq \beta(d)d(gf(x), f(x)).$$

Then the sequence $\{x_n\}_{n\in\omega}$ *defined by*

$$(*) \qquad x_{2n} = (gf)^n(x_0) \quad \text{and} \quad x_{2n+1} = f(x_{2n}) \qquad \text{for all } n \in \omega,$$

satisfies the inequality

$$\sum_{n=0}^{\infty} d(x_{n+1}, x_n) < \infty$$

for all $d \in D$; *consequently if* $(X, \to)$ *is* (x_0, D, f, g)*-complete, then the sequence* $\{x_n\}_{n\in\omega}$ *converges to a point in* $\overline{O}(x_0, gf)$, *the closure of the set* $O(x_0, gf)$.

Proof. Let d be in D. Then for each positive integer n, we have

$$\begin{aligned} d(x_{2n+1}, x_{2n}) &= d(fgf(x_{2n-2}), gf(x_{2n-2})) \\ &\leq \beta(d)d(gf(x_{2n-2}), f(x_{2n-2})) \\ &\leq \alpha(d)\beta(d)d(f(x_{2n-2}), x_{2n-2}) \\ &= \alpha(d)\beta(d)d(x_{2n-1}, x_{2n-2}), \end{aligned}$$

and hence

$$d(x_{2n+1}, x_{\;n}) \leq (\alpha(d)\beta(d))^n d(x_1, x_0).$$

Therefore, for each $n \in \omega$,

$$\begin{aligned} d(x_{2n+2}, x_{2n+1}) &\leq d(gf(x_{2n}), f(x_{2n})) \\ &\leq \alpha(d)d(f(x_{2n}), x_{2n}) \\ &\leq \alpha(d)(\alpha(d)\beta(d))^n d(x_1, x_0). \end{aligned}$$

It follows that $\sum_{n=0}^{\infty} d(x_{n+1}, x_n) < \infty$.

Theorem 1. *Let* f *and* g *be mappings of an* L-*space* $(X, \to)$ *into itself, and* D *a nonempty separating family of nonnegative extended real* *valued functions on* $X \times X$. *Suppose that there exists a point* $x_0 \in X$ *satisfying the conditions* (∞), (O), *and*

(c) $(X, \rightarrow)$ *is* (x_0, D, f, g)*-complete.*

Then the following statements hold:

1° *If for each* $d \in D$, *there exists a d-fixer* φ_d *on* X *such that*

$$\varphi_d(x, f(x), f(y), gf(y)) \geq 0$$

for all $x \in \overline{O}(x_0, gf)$ *and* $y \in O(x_0, gf)$ *with*

$$d(x, f(x)) \neq 0 \quad \text{or} \quad d(f(y), gf(y)) \neq 0,$$

then the sequence $\{x_n\}_{n\in\omega}$ *defined by* (*) *converges to a common fixed point of* f *and* g.

2° *If for each* $d \in D$, *there exists a d-fixer* φ_d *on* X *such that*

$$\varphi_d(x, f(x), f(y), gf(y)) \geq 0$$

for all $x, y \in X$ *and that* $\varphi_d(x, x, y, y) \geq 0$ *implies* $d(x, y) = 0$, *then the sequence* $\{x_n\}_{n\in\omega}$ *defined by* (*) *converges to a unique common fixed point of* f *and* g *which is a unique fixed point of* f.

3° *If for each* $d \in D$ *there exists a d-fixer* φ_d *on* X *such that*

$$\varphi_d(x, f(x), y, g(y)) \geq 0$$

for all $x, y \in X$ *and that* $\varphi_d(x, x, y, y) \geq 0$ *implies* $d(x, y) = 0$, *then the sequence* $\{x_n\}_{n\in\omega}$ *defined by* (*) *converges to a unique fixed point of* f *which is also a unique fixed point of* g.

Proof. By Lemma, the sequence $\{x_n\}_{n\in\omega}$ converges to a point a in the set $\overline{O}(x_0, gf)$. In order to prove 1°, suppose $f(a) \neq a$. Then $d(a, f(a)) \neq 0$ for some $d \in D$. Hence

$$\varphi_d(a, f(a), x_{2n+1}, x_{2n+2}) = \varphi_d(a, f(a), f(x_{2n}), gf(x_{2n})) \geq 0$$

for all $n \in \omega$. Since $x_{2n+2} \rightarrow a$, $x_{2n+1} \rightarrow a$ and since $d(x_{2n+2}, x_{2n+1}) \rightarrow 0$ by Lemma, we have

$$\varphi_d(a,\ f(a),\ a,\ a) \geq 0$$

which yields $d(a, f(a)) = 0$, a contradiction. Therefore $f(a) = a$. Suppose $g(a) \neq a$. Then $d(f(a), gf(a)) = d(a, g(a)) \neq 0$ for some $d \in D$, and so we have

$$\varphi_d(a,\ a,\ a,\ g(a)) = \varphi_d(a,\ f(a),\ f(a),\ gf(a)) \geq 0,$$

which gives $d(a, g(a)) = 0$. This contradiction establishes $g(a) = a$.

From 1° it follows that, under the hypothesis of 2°, a is a common fixed point of f and g. Hence, to prove 2°, it suffices only to show that a is a unique fixed point of f. Let $b \in X$ be a fixed point of f. Then for all $d \in D$, we have

$$\varphi_d(b,\ b,\ a,\ a) = \varphi_d(b,\ f(b),\ f(a),\ gf(a)) \geq 0,$$

which implies $d(b, a) = 0$. Thus $a = b$.

By virtue of 2°, to establish 3°, we have only to prove that a is a unique fixed point of g. Let $b \in X$ be a fixed point of g. Then for all $d \in D$, we have

$$\varphi_d(a,\ a,\ b,\ b) = \varphi_d(a,\ f(a),\ b,\ g(b)) \geq 0.$$

Hence $d(a, b) = 0$ for all $d \in D$. Thus $a = b$. This completes the proof.

By a *pseudo-o-metric* on a set X, we mean a nonnegative extended real valued function d on $X \times X$ such that $d(x, x) = 0$ for all $x \in X$.

Theorem 2. *Let* f *and* g *be mappings of an* L-*space* $(X, \to)$ *into itself, and* D *a nonempty separating family of pseudo-o-metrics on* X *such that the function* $x \to d(a, x)$ *is continuous for each* $(d, a) \in D \times X$. *Suppose* (∞), (O), *and* (c) *hold for some* $x_0 \in X$ *and suppose for each* $d \in D$, *there exist a* $\rho_d \in (0, \infty]$ *and a d-fixer* φ_d *on* X *such that*

$$\varphi_d(x,\ f(x),\ y,\ g(y)) \geq 0$$

for all $x, y \in \overline{O}(x_0, gf)$ *with* $d(x, y) < \rho_d$ *and either* $d(x, f(x)) \neq 0$ *or* $d(y, g(y)) \neq 0$. *Then the sequence* $\{x_n\}_{n\in\omega}$ *defined by* (*) *converges to a common fixed point of* f *and* g.

Proof. By Lemma, $\{x_n\}_{n\in\omega}$ converges to a point $a \in \overline{O}(x_0, gf)$. Suppose $f(a) \neq a$. Then $d(a, f(a)) \neq 0$ for some $d \in D$. Since $x_{2n+1} \to a$ and the function $x \to d(a, x)$ is continuous, we have $d(a, x_{2n_i+1}) \to d(a, a) = 0$ for some subsequence $\{x_{2n_i+1}\}_{i\in\omega}$ of $\{x_{2n+1}\}_{n\in\omega}$. Hence we can choose a $k \in \omega$ such that $d(a, x_{2n_i+1}) < \rho_d$ for all $i \geq k$. Put $u_i = x_{2n_{k+i}+1}$ and $v_i = x_{2n_{k+i}+2}$ for all $i \in \omega$. Then $d(v_i, u_i) \to 0$ by Lemma, and

$$\mathcal{F}_d(a, f(a), u_i, v_i) = \mathcal{F}_d(a, f(a), u_i, g(u_i))$$

for all $i \in \omega$. So we have $\mathcal{F}_d(a, f(a), a, a) \geq 0$ which implies a contradiction $d(a, f(a)) = 0$. Therefore $f(a) = a$. Suppose $g(a) \neq a$. Then we have $d(a, g(a)) \neq 0$ for some $d \in D$. Since $d(a, a) = 0 < \rho_d$, we have

$$\mathcal{F}_d(a, a, a, g(a)) = \mathcal{F}_d(a, f(a), a, g(a)) \geq 0,$$

which gives a contradiction $d(a, g(a)) = 0$. Therefore $g(a) = a$.

Moreover we have the following result which is similar to Theorem 1.

Theorem 3. *Let* f *and* g *be mappings of an* L-*space* $(X, \to)$ *into itself, and* D *a nonempty separating family of pseudo-o-metrics on* X. *Suppose that there exists a point* $x_0 \in X$ *satisfying* (∞), (c), *and the following condition*:

($\overline{O}$) *There exist functions* α *and* β *of* D *into* $[0, 1)$ *such that, for all* $d \in D$ *and* $x \in \overline{O}(x_0, gf)$,

$$d(gf(x), f(x)) \leq \alpha(d)d(f(x), x)$$

$$d(fgf(x), gf(x)) \leq \beta(d)d(gf(x), f(x)).$$

Then the following statements hold:

1° *If for each* $d \in D$ *there exists a d-semifixer* φ_d *on* X *such that*

$$\varphi_d(x,\ f(x),\ f(y),\ gf(y)) \geq 0$$

for all $x \in \overline{O}(x_0, gf)$ *and* $y \in O(x_0, gf)$ *with* $d(x, f(x)) \neq 0$, *then the sequence* $\{x_n\}_{n\in\omega}$ *defined by* (*) *converges to a common fixed point of* f *and* g.

2° *If for each* $d \in D$ *there exists a d-semifixer* φ_d *on* X *such that*

$$\varphi_d(x,\ f(x),\ f(y),\ gf(y)) \geq 0$$

for all $x, y \in X$ *and that* $\varphi_d(x, x, y, y) \geq 0$ *implies* $d(x, y) = 0$, *then the sequence* $\{x_n\}_{n\in\omega}$ *defined by* (*) *converges to a unique common fixed point of* f *and* g *which is a unique fixed point of* f.

3° *If for each* $d \in D$ *there exists a d-semifixer* φ_d *on* X *such that*

$$\varphi_d(x,\ f(x),\ y,\ g(y)) \geq 0$$

for all $x, y \in X$ *and that* $\varphi_d(x, x, y, y) \geq 0$ *implies* $d(x, y) \neq 0$, *then the sequence* $\{x_n\}_{n\in\omega}$ *defined by* (*) *converges to a unique fixed point of* f *which is also a unique fixed point of* g.

Proof. By Lemma, the sequence $\{x_n\}_{n\in\omega}$ converges to a point a in the set $\overline{O}(x_0, gf)$. The same argument used in the proof of Theorem 1 establishes that a is a fixed point of f. Now for each $d \in D$, we have

$$d(g(a),\ a) = d(gf(a),\ f(a)) \leq \alpha(d)d(f(a),\ a) = 0.$$

Thus $g(a) = a$. This completes the proof of the statement 1°. The proofs of 2° and 3° are the same as those of 2° and 3° of Theorem 1.

The following result corresponds to Theorem 2. The proof is easy, and is therefore omitted.

Theorem 4. *Let* g *and* g *be mappings of an* L-*space* $(X, \to)$ *into itself, and* D *a nonempty separating family of pseudo-o-metrics on* X *such* *that the function* $x \to d(a, x)$ *is continuous for each* $(d, a) \in D \times X$. *Suppose* (∞), (c) *and* $(\overline{O})$ *hold for some* $x_0 \in X$, *and suppose for each* $d \in D$ *there exist a* $\rho_d \in (0, \infty]$ *and a d-semifixer* φ_d *on* X *such that*

$$\varphi_d(x, f(x), y, g(y)) \geq 0$$

for all $x, y \in O(x_0, gf)$ *with* $d(x, y) < \rho_d$ *and either* $d(x, f(x)) \neq 0$ *or* $d(y, g(y)) \neq 0$. *Then the sequence* $\{x_n\}_{n\in\omega}$ *defined by* (*) *converges to a common fixed point of* f *and* g.

2. Remarks on completeness

In [4], we note that if a premetric space (X, d) is complete, then the L-space $(X, \overset{d}{\to})$ is d-complete, where $x_n \overset{d}{\to} x$ means that $d(x, x_n) \to 0$ as $n \to \infty$. Although this gives the "only if" part of the following theorem, we shall state here a proof for the sake of completeness. Recall that an L-space $(X, \to)$ is *d-complete* if each sequence $\{x_n\}_{n\in\omega}$ in X with

(**) $$\sum_{n=0}^{\infty} d(x_{n+1}, x_n) < \infty$$

converges to at least one point of X.

Theorem 5. *A metric space* (X, d) *is complete if and only if the* L-*space* $(X, \overset{d}{\to})$ *is d-complete, where* $x_n \overset{d}{\to} x$ *means that* $d(x, x_n) \to 0$ *as* $n \to \infty$.

Proof. To prove the "only if" part, consider a sequence $\{x_n\}_{n\in\omega}$ in X satisfying (**). Let $\varepsilon > 0$. Since $\{\sum_{i=0}^{n} d(x_{i+1}, x_i)\}_{n\in\omega}$ is a Cauchy sequence of real numbers, there exists an integer $n_0 \geq 1$ such that

$$\sum_{i=0}^{m} d(x_{i+1}, x_i) - \sum_{i=0}^{n} d(x_{i+1}, x_i) < \varepsilon$$

whenever $m > n \geq n_0$. Hence if $m > n \geq n_0$ then

$$d(x_m, x_n) \leq \sum_{i=0}^{m-1} d(x_{i+1}, x_i) - \sum_{i=0}^{n-1} d(x_{i+1}, x_i) < \varepsilon.$$

Therefore $\{x_n\}_{n\in\omega}$ is a Cauchy sequence, and hence $x_n \overset{d}{\to} x$ for some $x \in X$.

Conversely let $\{x_n\}_{n\in\omega}$ be a Cauchy sequence in (X, d). Then for each $i \in \omega$, the set S_i of all integers $n > i$ such that $d(x_m, x_n) < 1/2^i$ for all $m \geq n$, is nonempty. Denote by $\varphi(i)$ the least element of S_i. Since φ is a mapping of ω into itself, the finite recursion theorem ensures existence of a sequence $\{n_i\}_{i\in\omega}$ in ω such that $n_0 = \varphi(0)$ and $n_{i+1} = \varphi(n_i)$ for all $i \in \omega$. Hence we have $d(x_{n_{i+1}}, x_{n_i}) < 1/2^{n_i} \leq 1/2^i$ for all $i \in \omega$. Therefore $\sum_{i=0}^{\infty} d(x_{n_{i+1}}, x_{n_i}) < 2$. Since the L-space $(X, \overset{d}{\to})$ is d-complete, this implies that $x_{n_i} \overset{d}{\to} x$ for some $x \in X$. But, since $\{x_{n_i}\}_{i\in\omega}$ is a subsequence of the Cauchy sequence $\{x_n\}_{n\in\omega}$, it follows that the sequence $\{x_n\}_{n\in\omega}$ converges to x. This completes the proof.

Let f and g be mappings of a metric space (X, d) into itself. Following Rhoades [12], we say that (X, d) is (g, f)-*orbitally complete* if for each $x \in X$, every Cauchy sequence in the set $O(x, gf) \cup f(O(x, gf))$ is convergent. Obviously (X, d) is (g, f)-orbitally complete if and only if $(X, \overset{d}{\to})$ is $(x, \{d\}, f, g)$-complete for each $x \in X$.

Let f and g be mappings of an L-space $(X, \to)$ into itself, D a nonempty family of extended real valued functions on $X \times X$, and $x \in X$. The following example shows that a (x, D, f, g)-complete L-space need not be D-complete:

Example. Let X denote the set of all rational numbers, and put

$$d(x, y) = |x - y|$$

for all $x, y \in X$. Consider the identity mapping f of X into itself and the mapping g defined by $g(x) = 2x$ for all $x \in X$. Then clearly $(X, \overset{d}{\to})$

is $(0, \{d\}, f, g)$-complete but not $\{d\}$-complete.

3. Consequences

First of all, we shall derive the following result which improves the main result of Fisher [1].

Corollary 1. *If* S *and* T *are mappings of a nonempty complete metric space* (X, d) *into itself satisfying the inequality*

$$d(Sx, TSy) \le \alpha\cdot\max\{d(x, Sy), d(x, Sx), d(Sy, TSy), \frac{1}{2}[d(x, TSy) + d(Sy, Sx)]\}$$

for all x, y *in* X*, where* $0 \le \alpha < 1$*, then* S *and* T *have a unique common fixed point.*

Proof. Consider the L-space $(X, \overset{d}{\to})$ which is d-complete by Theorem 5. Since $1/(2 - \alpha) < 1$, it is easy to see that

$$d(TSx, Sx) \le \alpha d(x, Sx)$$

and

$$d(STSx, TSx) \le \alpha d(TSx, Sx)$$

for all $x \in X$. For each $(x, y, u, v) \in X^4$, put

$$\varphi(x, y, u, v) = \alpha\cdot\max\{d(x, u), d(x, y), d(u, v), \frac{1}{2}[d(x, v) + d(y, u)]\} - d(y, v).$$

It is clear that φ is a d-fixer on X, and $\varphi(x, Sx, Sy, TSy) \ge 0$ for all $x, y \in X$. Moreover if $\varphi(x, x, y, y) \ge 0$, then since $\alpha d(x, y) - d(x, y) \ge 0$, we have $d(x, y) = 0$. Therefore, the statement 1° of Theorem 1 yields the desired conclusion.

The following common fixed point theorem due to Fisher [2] is also a consequence of Theorem 1.

Corollary 2. *If* S *and* T *are mappings of a nonempty complete metric space* (X, d) *into itself satisfying the inequality*

$$\{d(Sx, Ty)\}^2 \leq \alpha d(x, Sx)d(y, Ty) + \beta d(x, Ty)d(y, Sx)$$

for all $x, y \in X$, *where* $0 \leq \alpha < 1$ *and* $0 \leq \beta$, *then* S *and* T *have a common fixed point. If, in addition* $0 \leq \beta < 1$, *then each of* S *and* T *has a unique fixed point and these two points coincide.*

Proof. Consider, as before, the L-space $(X, \overset{d}{\to})$ which is d-complete by Theorem 5. It can be readily seen that

$$d(TSx, Sx) \leq \alpha d(Sx, x)$$

and

$$d(STx, Tx) \leq \alpha d(Tx, x)$$

for all $x \in X$. For each $(x, y, u, v) \in X^4$, put

$$\varphi(x, y, u, v) = \alpha d(x, y)d(u, v) + \beta d(x, v)d(y, u) - \{d(y, v)\}^2.$$

Obviously φ is a d-fixer on X and $\varphi(x, Sx, y, Ty) \geq 0$ for all $x, y \in X$. Moreover if $0 \leq \beta < 1$ then $\varphi(x, x, y, y) \geq 0$ yields

$$0 \leq (\beta - 1)\{d(x, y)\}^2 \leq 0,$$

proving $d(x, y) = 0$. Therefore, the desired conclusion follows from the statements 1° and 3° of Theorem 1.

Note that, in the proofs of the above corollaries, we may use Theorem 3 instead of Theorem 1.

The following corollary extends a result of Pal and Maiti [10] to two mappings:

Corollary 3. *Let* S *and* T *be mappings of a metric space* (X, d) *into itself, and let* $1 \leq \alpha < 2$, $\frac{1}{2} \leq \beta < \frac{2}{3}$, $1 \leq \gamma < \frac{3}{2}$, *and* $0 \leq \delta < 1$. *Suppose that* (X, d) *is* (T, S)*-orbitally complete and there exists an* $x_0 \in X$ *such* *that, for each* $(x, y) \in \overline{O}(x_0, TS) \times S(O(x_0, TS))$, *at least one of the following inequalities holds*:

(1) $d(Sx, x) + d(Ty, y) \le \alpha d(x, y)$.

(2) $d(Sx, x) + d(Ty, y) \le \beta[d(Ty, x) + d(Sx, y) + d(x, y)]$.

(3) $d(Sx, x) + d(Ty, y) + d(Sx, Ty) \le \gamma[d(Ty, x) + d(Sx, y)]$.

(4) $d(Sx, Ty) \le \delta \cdot \max\{d(x, y), d(Sx, x), d(Ty, y),$
$$\tfrac{1}{2}[d(Sx, y) + d(Ty, x)]\}.$$

Then the sequence $\{x_n\}_{n\in\omega}$ *defined by*

$$x_{2n} = (TS)^n(x_0) \quad \text{and} \quad x_{2n+1} = S(x_{2n})$$

for all $n \in \omega$, *converges to a common fixed point of* S *and* T.

Proof. Consider the L-space $(X, \overset{d}{\to})$ which is $(x_0, \{d\}, S, T)$-complete. Let x be in $O(x_0, TS)$. Then, at least one of (1)-(4) holds for (x, Sx). A straightforward computation shows that

$$d(TSx, Sx) \le \lambda d(Sx, x),$$

where

$$\lambda = \begin{cases} \alpha - 1, & \text{if (1) holds for } (x, Sx), \\ \dfrac{2\beta - 1}{1 - \beta}, & \text{if (2) holds for } (x, Sx), \\ \dfrac{\gamma - 1}{2 - \gamma}, & \text{if (3) holds for } (x, Sx), \\ \delta, & \text{if (4) holds for } (x, Sx). \end{cases}$$

Hence by putting $\mu = \max\{\alpha - 1, \frac{2\beta - 1}{1 - \beta}, \frac{\gamma - 1}{2 - \gamma}, \delta\}$, we obtain

$$d(TSx, Sx) \le \mu d(Sx, x)$$

for all $x \in O(x_0, TS)$. Clearly $0 \le \mu < 1$. Let again $x \in O(x_0, TS)$. Then a similar computation gives

$$d(STSx, TSx) \le \mu d(TSx, Sx).$$

For every $(x, y, u, v) \in X^4$, define $\varphi(x, y, u, v)$ to be the maximum of four real numbers

$$\alpha d(x, u) - d(x, y) - d(u, v),$$

$$\beta[d(x, v) + d(y, u) + d(x, u)] - d(x, y) - d(u, v),$$

$$\gamma[d(x, v) + d(y, u)] - d(x, y) - d(u, v) - d(y, v), \text{ and}$$

$$\delta\cdot\max\{d(x, u), d(x, y), d(u, v), \tfrac{1}{2}[d(y, u) + d(x, v)]\} - d(y, v).$$

Then $\varphi(x, y, x, x) \geq 0$ implies

$$0 \leq \max\{-d(x, y), (\beta - 1)d(x, y), (\gamma - 2)d(x, y), (\delta - 1)d(x, y)\} \leq 0,$$

and so $d(x, y) = 0$. Similarly if $\varphi(x, x, x, y) \geq 0$ then we have $d(x, y) = 0$. Thus φ is a d-fixer on X since it is continuous on the product space X^4. Obviously $\varphi(x, Sx, Sy, TSy) \geq 0$ for all $x \in \overline{O}(x_0, TS)$ and $y \in O(x_0, TS)$. Therefore from 1° of Theorem 1, we have the required conclusion.

Now, from this corollary, we can derive the following result of Rhoades [11]:

Corollary 4. *Let f and g be mappings of a nonempty complete metric space (X, d) into itself satisfying*

$$d(f(x), g(y)) \leq \alpha\cdot\max\{d(x, y), d(x, f(x)), d(y, g(y)), \tfrac{1}{2}[d(x, g(y)) + d(y, f(x))]\}$$

for all $x, y \in X$, where $0 \leq \alpha < 1$. Then each of f and g has a unique fixed point and these two points coincide. Moreover both the sequences $\{(fg)^n(x\ )\}_{n\in\omega}$ and $\{(gf)^n(x_0)\}_{n\in\omega}$ converges to the point for every x_0 in X.

Proof. Let $x_0 \in X$. Then, by Corollary 3, the sequence $\{(fg)^n(x_0)\}_{n\in\omega}$ converges to a common fixed point a of f and g. Hence interchanging f

and g, we see that the sequence $\{(gf)^n(x_0)\}_{n\in\omega}$ converges also to a common fixed point b of f and g. Now if $c \in X$ is a fixed point of f, then we have

$$d(c, a) = d(f(c), g(a)) \leq \alpha \cdot \max\{d(c, a), 0, 0, d(c, a)\},$$

which implies $c = a$. This shows that a is the unique fixed point of f. Therefore $a = b$. Moreover, since the circumstance is symmetric with respect to f and g, we see that the point a is the unique fixed point of g.

As an immediate consequence of this corollary we have the following result obtained by Khan in [9]:

Corollary 5. *Let* S *and* T *be mappings of a nonempty complete metric space* (X, d) *into itself satisfying*

$$d(STx, TSy) \leq \alpha \cdot \max\{d(x, y), d(x, STx), d(y, TSy), \frac{1}{2}[d(x, TSy) + d(y, STx)]\}$$

for all $x, y \in X$, *where* $0 \leq \alpha < 1$. *Then* S *and* T *have a unique common fixed point.*

Proof. By Corollary 4, each of ST and TS has a unique fixed point and these two points coincide. Denote the point by a. Then $ST(Sa) = S(TSa) = Sa$, and so $Sa = a$. Thus by symmetry, we have $Ta = a$. Since a common fixed point of S and T is a common fixed point of ST and TS, the common fixed point of S and T is unique.

Further, the following result of Fisher [3] can be derived from Theorem 1.

Corollary 6. *Let S and T be mappings of a nonempty complete metric space (X, d) into itself such that*

$$d(Sx, Ty) \le c\,\frac{\{d(x, Sx)\}^2 + \{d(y, Ty)\}^2}{d(x, Sx) + d(y, Ty)}$$

for all $x, y \in X$ with $d(x, Sx) + d(y, Ty) \neq 0$, where $0 \le c < 1$. Then S and T have a common fixed point z. If in addition $d(x, Sx) + d(y, Ty) = 0$ implies $d(Sx, Ty) = 0$, then z is the unique fixed point of S and T.

Proof. Consider the L-space $(X, \overset{d}{\to})$ which is d-complete by Theorem 5. Put

$$\alpha = \frac{-1 + \sqrt{1 + 4c(1 - c)}}{2(1 - c)}.$$

If $Tx = x$ and $Sx \neq x$, then we have

$$d(Sx, x) = d(Sx, Tx) \le cd(x, Sx) < d(x, Sx),$$

a contradiction. Hence if $Tx = x$, then $Sx = x$, and so

$$d(STx, Tx) \le \alpha d(x, Tx).$$

Suppose $Tx \neq x$. Then

$$d(STx, Tx) \le c\,\frac{\{d(Tx, STx)\}^2 + \{d(x, Tx)\}^2}{d(Tx, STx) + d(x, Tx)}.$$

Hence by letting $t = d(STx, Tx)/d(x, Tx)$, we have $(1 - c)t^2 + t - c \le 0$, which implies $t \le \alpha$, i.e., $d(STx, Tx) \le \alpha d(x, Tx)$. Thus we have

$$d(STx, Tx) \le \alpha d(Tx, x)$$

for all $x \in X$. Since S and T may be interchanged, we have

$$d(TSx, Sx) \le \alpha d(Sx, x)$$

for all $x \in X$. From $0 \le c < 1$, it is easy to see that $0 \le \alpha < 1$. Now for each $(x, y, u, v) \in X^4$, define

$$\varphi(x, y, u, v) = c\cdot\frac{\{d(x, y)\}^2 + \{d(u, v)\}^2}{d(x, y) + d(u, v)} - d(y, v)$$

if $d(x, y) + d(u, v) \neq 0$; and

$$\varphi(x, y, u, v) = -d(y, v),$$

otherwise. As can readily be seen, φ is a d-fixer on X, and

$$\varphi(x, Sx, y, Ty) \geq 0$$

for all $x, y \in X$ with $d(x, Sx) \neq 0$ of $d(y, Ty) \neq 0$. Therefore by 1° of Theorem 1, S and T have a common fixed point. Now suppose that $d(x, Sx) + d(y, Ty) = 0$ implies $d(Sx, Ty) = 0$. Then $\varphi(x, Sx, y, Ty) \geq 0$ for all $x, y \in X$. Moreover if $\varphi(x, x, y, y) \geq 0$ then $\varphi(x, x, y, y) = -d(x, y)$, and therefore $d(x, y) = 0$. Thus 3° of Theorem 1 yields the desired conclusion.

We shall now derive from Theorem 2 the results obtained in [8].

Corollary 7. *Let $(X, \rightarrow)$ be an* L-*space which is D-complete for a nonempty separating family D of separately continuous pseudo-symmetrics on X. Let f and g be mappings of X into itself satisfying the following conditions*:

(1) *$d(f(x), x) < \infty$ and $d(g(x), x) < \infty$ for all $d \in D$ and $x \in X$.*

(2) *There exists functions $\alpha: D \rightarrow [0, 1)$ and $\beta: D \rightarrow (0, \infty]$ such that $d(x, y) < \beta(d)$ implies*

$$d(f(x), g(y)) \leq \alpha(d) \max\{d(x, y), d(f(x), x), d(g(y), y)\}$$

for each $d \in D$.

If there exists an $x_0 \in X$ with $d(f(x_0), x_0) < \beta(d)$ for all $d \in D$, then f and g have a common fixed point, and the sequence $\{x_n\}_{n \in \omega}$ defined by

(*) $\qquad x_{2n} = (gf)^n(x_0)$ *and* $x_{2n+1} = f(x_{2n})$ *for all* $n \in \omega$,

converges to a common fixed point of f and g.

Proof. Since $d(f(x_0), x_0) < \beta(d)$ for all $d \in D$, the condition (∞) is satisfied. Let d be in D, and suppose $d(f(gf)^n(x_0), (gf)^n(x_0)) < \beta(d)$. Then since

$$d(f(gf)^n(x_0), (gf)^{n+1}(x_0)) \le \alpha(d)d(f(gf)^n(x_0), (gf)^n(x_0)) < \beta(d),$$

we have

$$d(f(gf)^{n+1}(x_0), (gf)^{n+1}(x_0)) \le \alpha(d)d(f(gf)^n(x_0), (gf)^{n+1}(x_0)) < \beta(d).$$

Thus $d(f(x), x) < \beta(d)$ for all $x \in O(x_0, gf)$. Consequently, for each x in $O(x_0, gf)$,

$$d(gf(x), f(x)) \le \alpha(d)d(f(x), x) < \beta(d),$$

and so

$$d(fgf(x), gf(x)) \le \alpha(d)d(gf(x), f(x)).$$

Therefore the condition (O) holds. For each $d \in D$, define

$$\varphi_d(x, y, u, v) = \alpha(d)\cdot\max\{d(x, u), d(x, y), d(u, v)\} - d(y, v)$$

for all $(x, y, u, v) \in X^4$. Assume $u_n \to u$, $v_n \to u$ in $(X, \to)$, $d(v_n, u_n) \to 0$ and $\varphi_d(x, y, u_n, v_n) \ge \lambda$ for all $n \in \omega$. Since $d(x, u_{n_i}) \to d(x, u)$ and $d(y, v_{n_i}) \to d(y, u)$ for some subsequence $\{n_i\}_{i\in\omega}$ of $\{n\}_{n\in\omega}$, it follows that $\varphi_d(x, y, u, u) \ge \lambda$. It is clear that $\varphi_d(x, y, x, x) \ge 0$ or $\varphi_d(x, x, x, y) \ge 0$ implies $d(x, y) = 0$. Hence φ_d is a d-fixer on X. Since $\varphi_d(x, f(x), y, g(y)) \ge 0$ whenever $d(x, y) < \beta(d)$, the conclusion follows from Theorem 2.

Note that Theorem 2 of [8] is not concerned with common fixed point. We shall prove here its revised version:

Corollary 8. *Let* $(X, \to)$ *be an* L-*space which is* D-*complete for a non-empty separating family* D *of separately continuous pseudo-symmetrics on* X. *Let* f *be a mapping of* X *into itself for which there exist functions*

$\alpha:D \to [0, 1)$ *and* $\beta:D \to (0, \infty]$ *such that* $d(x, y) < \beta(d)$ *implies*

$$d(f(x), f(y)) \le \alpha(d)d(x, y)$$

for each $d \in D$. *If there exists an* $x_0 \in X$ *with* $d(f(x_0), x_0) < \beta(d)$ *for all* $d \in D$ *then the sequence* $\{f^n(x_0)\}_{n\in\omega}$ *converges to a fixed point of* f.

Proof. The condition (∞) clearly holds. Let $d \in D$. If

$$d(f^{2n+1}(x_0), f^{2n}(x_0)) < \beta(d),$$

then

$$d(f^{2n+2}(x_0), f^{2n+1}(x_0)) \le \alpha(d)d(f^{2n+1}(x_0), f^{2n}(x_0)) < \beta(d),$$

and so

$$d(f^{2n+3}(x_0), f^{2n+2}(x_0)) \le \alpha(d)d(f^{2n+2}(x_0), f^{n+1}(x_0)) < \beta(d).$$

Hence $d(f(x), x) < \beta(d)$ for all $x \in O(x_0, f^2)$. Therefore, we have for each $x \in O(x_0, f^2)$,

$$d(f^2(x), f(x)) \le \alpha(d)d(f(x), x) < \beta(d),$$

and consequently

$$d(f^3(x), f^2(x)) \le \alpha(d)d(f^2(x), f(x)).$$

Thus the condition (O) is satisfied. For each $d \in D$, put

$$\mathcal{F}_d(x, y, u, v) = \alpha(d)d(x, u) - d(y, v)$$

for all $(x, y, u, v) \in X^4$. It is easily shown that $\mathcal{F}_d$ is a d-fixer on X. Further, if $d(x, y) < \beta(d)$ then $\mathcal{F}_d(x, f(x), y, g(y)) \ge 0$. Therefore by virtue of Theorem 2, we have the desired conclusion.

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