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Seminormal Autometrized Algebras*

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In [3] the present author and Swamy obtained an ideal theory for normal autometrized algebras. The class of normal autometrized algebras is fairly large, in fact, it contains the classes of Boolean algebras, Brouwerian algebras, semi-Brouwerian algebras, commutative lattice ordered groups, Boolean ℓ -algebras and commutative dually residuated lattice ordered semigroups as subclasses. But it does not contain the more recently studied classes of autometrized algebras - Brouwerian semigroups ([5]) and representable Griss algebras ([6]). In this paper we introduce the notion of a seminormal autometrized algebra which contains all the above mentioned autometrized algebras including Brouwerian semigroups and representable Griss algebras as special cases and we extend all the results of [3] to seminormal autometrized algebras.

1. For the sake of convenience, we recall the definition of an autometrized algebra ([3]).

Definition 1. A system $A = (A, +, \leq, *)$ is called an *autometrized al-*

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gebra if and only if

(1.1) $(A, +, \leq)$ is a commutative partially ordered semigroup with additive identity '0', i.e., $(A, +)$ is a commutative semigroup with additive identity 0 and $(A, \leq)$ is a partially ordered set such that $a \leq b$ implies $a + x \leq b + x$ for any x in A .

(1.2) $*$ is a metric operation on A , i.e., $*$ is a mapping from $A \times A$ to A such that

(1) $a * b \geq 0$ for all a, b in A and $a * b = 0$ if and only if $a = b$,

(2) $a * b = b * a$ for all a, b in A ,

(3) $a * c \leq (a * b) + (b * c)$ for all a, b, c in A .

Now we introduce the following:

Definition 2. An autometrized algebra $A = (A, +, \leq, *)$ is called a *seminormal* autometrized algebra if and only if

(1.3) $(a + c) * (b + d) \leq (a * b) + (c * d)$ for any a, b, c, d in A ,

(1.4) $(a * c) * (b * d) \leq 2[(a * b) + (c * d)]$ for any a, b, c, d in A ,

(1.5) $a * b = (a * b) * 0$ for any a, b in A ,

(1.6) $(a * 0) + (b * 0) = [(a * 0) + (b * 0)] * 0$ for any a, b in A .

The following example shows that every normal autometrized algebra is a seminormal autometrized algebra. Consequently Boolean algebras, Brouwerian algebras, semi-Brouwerian algebras autometrized with the symmetric difference and commutative ℓ -groups autometrized with $d(a, b) = |a - b|$ and commutative D. R. ℓ . semigroups autometrized with $d(a, b) = (a - b) \cup (b - a)$ are all seminormal autometrized algebras.

Example 1. Every normal autometrized algebra $A = (A, +, \leq, *)$ is a seminormal autometrized algebra. By definition ([3]), a normal autometrized algebra $A = (A, +, \leq, *)$ is an autometrized algebra satisfying

- (1) $a \leq a * 0$ for every a in A ,
- (2) $(a + c) * (b + d) \leq (a * b) + (c * d)$ for any a, b, c, d in A ,
- (3) $(a * c) * (b * d) \leq (a * b) + (c * d)$ for any a, b, c, d in A ,
- (4) $a \leq b$ implies there exists an element $x \geq 0$ such that $a + x = b$.

To show $(A, +, \leq, *)$ is a seminormal autometrized algebra it is enough to verify (1.5) and (1.6). Since $a * b \leq (a * b) * 0 = (a * b) * (a * a) \leq (a * a) + (b * a) = 0 + (b * a) = b * a = a * b$ for any a, b in A , we have $a * b = (a * b) * 0$ for every a, b in A . Hence (1.5) hold in A . Since $(a * 0) + (b * 0) \leq [(a * 0) + (b * 0)] * 0 = [(a * 0) + (b * 0)] * (0 + 0) \leq [(a * 0) * 0] + [(b * 0) * 0] = (a * 0) + (b * 0)$ (since 1.5 holds in A), we have $(a * 0) + (b * 0) = [(a * 0) + (b * 0)] * 0$ for any a, b in A . Hence (1.6) also holds in A .

The following example shows that the class of seminormal autometrized algebras is strictly wider than the class of normal autometrized algebras.

Example 2. Let A be the set consisting of elements $0, a, b, c, 1$.

Let the partial ordering

on A be such that $0 <$

$a < 1, 0 < b < 1, 0 < c$

< 1 . Let $+$ and $*$ be

defined by the tables at

right.

It is routine to ver-

$+$	0	a	b	c	1
0	0	a	b	c	1
a	a	a	1	1	1
b	b	1	b	1	1
c	c	1	1	c	1
1	1	1	1	1	1

$*$	0	a	b	c	1
0	0	b	b	b	b
a	b	0	b	b	b
b	b	b	0	b	b
c	b	b	b	0	b
1	b	b	b	b	0

ify that $A = (A, +, \leq, *)$ is a seminormal autometrized algebra. Since $a * 0 = b \neq a$, $A = (A, +, \leq, *)$ is not a normal autometrized algebra.

The following examples (Example 3 and Example 4) show that Brouwerian semigroups and representable Griss algebras are also examples of seminormal autometrized algebra.

Example 3. Every Brouwerian semigroup $A = (A, +, \leq, -)$ is a seminormal autometrized algebra with distance $a * b = (a - b) \cup 0 + (b - a) \cup 0$. By definition ([5]), a Brouwerian semigroup $A = (A, +, \leq, -)$ is a system where (1) $(A, +, \leq)$ is a commutative partially ordered semigroup with additive identity 0; (2) To each a, b in A there exists a least x in A such that $x + b \geq a$ (we denote this x by $a - b$); (3) For every a in A supremum of a and 0 (denoted by $a \cup 0$) exists; and (4) $a - a \geq 0$ for all a in A . By Theorem 5 of [5], $(A, +, \leq, *)$ is an autometrized algebra with distance $a * b = (a - b) \cup 0 + (b - a) \cup 0$. Since $x \geq 0$ implies $x * 0 = x$ for any x in A , (1.5) and (1.6) are obviously satisfied in A . Since $(a + c) * (b + d) = [(a + c) - (b + d)] \cup 0 + [(b + c) - (a + c)] \cup 0 \leq [(a - b) + (c - d)] \cup 0 + [(b - a) + (d - c)] \cup 0 \leq (a - b) \cup 0 + (c - d) \cup 0 + (b - a) \cup 0 + (d - c) \cup 0 = (a * b) + (c * d)$ for any a, b, c, d in A , (1.3) holds in A . Since

$$\begin{aligned}
 (a * c) * (b * d) &= (a * c - b * d) \cup 0 + (b * d - a * c) \cup 0 \\
 &= [\{(a - c) \cup 0 + (c - a) \cup 0\} - \{(b - d) \cup 0 + (d - b) \cup 0\}] \cup 0 \\
 &\quad + [\{(b - d) \cup 0 + (d - b) \cup 0\} - \{(a - c) \cup 0 + (c - a) \cup 0\}] \cup 0 \\
 &\leq [\{(a - c) \cup 0 - (b - d) \cup 0\} + \{(c - a) \cup 0 - (d - b) \cup 0\}] \cup 0 \\
 &\quad + [\{(b - d) \cup 0 - (a - c) \cup 0\} + \{(d - b) \cup 0 - (c - a) \cup 0\}] \cup 0 \\
 &\leq [\{(a - b) \cup 0 + (d - c) \cup 0\} + \{(c - d) \cup 0 + (b - a) \cup 0\}] \cup 0
 \end{aligned}$$

$$\begin{aligned}
& + [\{(b - a) \cup 0 + (c - d) \cup 0\} + \{(d - c) \cup 0 + (a - b) \cup 0\}] \cup 0 \\
& \leq (a - b) \cup 0 + (d - c) \cup 0 + (c - d) \cup 0 + (b - a) \cup 0 \\
& \quad + (b - a) \cup 0 + (c - d) \cup 0 + (d - c) \cup 0 + (a - b) \cup 0 \\
& = 2(a * b) + 2(c * d) \\
& = 2\{(a * b) + (c * d)\}
\end{aligned}$$

for any a, b, c, d in A , (1.4) also holds in A . Hence $(A, +, \leq, *)$ is a seminormal autometrized algebra.

Example 4. Every representable Griss algebra $G = (G, \cup, -, 0)$ is a seminormal autometrized algebra with distance $a * b = (a - b) \cup (b - a)$. By definition ([6]), a representable Griss algebra $G = (G, \cup, -, 0)$ is a Griss algebra satisfying (1) $a * b = (a * b) * 0$ for any a, b in G ; and (2) $(a - b) - (c - d) \leq (a - c) \cup (d - b)$ for any a, b, c, d in G . By Theorem 1 of [4], $(G, \cup, \leq, *)$ is an autometrized algebra with distance $a * b = (a - b) \cup (b - a)$. To show G is a seminormal autometrized algebra we have to verify (1.3), (1.4) and (1.6). Since $(a \cup b) * (c \cup d) \leq (a * c) \cup (b * d)$ for any a, b, c, d in a Griss algebra (see [4]), (1.3) is satisfied in G . Since $[(a * 0) \cup (b * 0)] * 0 = [(a - 0) \cup (b - 0)] * 0 = [(a \cup b) - 0] * 0 = [(a \cup b) * 0] * 0 = (a \cup b) * 0 = (a \cup b) - 0 = (a - 0) \cup (b - 0) = (a * 0) \cup (b * 0)$ for any a, b in G , the condition (1.6) of seminormality holds in G . Now

$$\begin{aligned}
(a * c) * (b * d) &= [(a * c) - (b * d)] \cup [(b * d) - (a * c)] \\
&= [\{(a - c) \cup (c - a)\} - \{(b - d) \cup (d - b)\}] \cup \\
&\quad [\{(b - d) \cup (d - b)\} - \{(a - c) \cup (c - a)\}] \\
&= [(a - c) - \{(b - d) \cup (d - b)\}] \cup \\
&\quad \cup [(c - d) - \{(b - d) \cup (d - b)\}] \cup [(b - d) - \{(a - c) \cup (c - a)\}] \\
&\quad \cup [(d - b) - \{(a - c) \cup (c - a)\}]
\end{aligned}$$

$$\begin{aligned}
&\leq [(a - c) - (b - d)] \cup [(c - a) - (d - b)] \\
&\quad \cup [(b - d) - (a - c)] \cup [(d - b) - (c - a)] \\
&\leq (a - b) \cup (d - c) \cup (c - d) \cup (b - a) \cup (b - a) \\
&\quad \cup (c - d) \cup (d - c) \cup (a - b) \\
&= (a - b) \cup (b - a) \cup (c - d) \cup (d - c) \\
&= (a * b) \cup (c * d) \\
&= 2[(a * b) \cup (c * d)]
\end{aligned}$$

for any a, b, c, d in G , and hence (1.4) holds in G .

2. Throughout this section $A = (A, +, \leq, *)$ stands for a seminormal autometrized algebra. We recall the following definition of an ideal in an autometrized algebra.

Definition 3. A nonempty subset I of an autometrized algebra $A = (A, +, \leq, *)$ is called an *ideal* if and only if

$$(3.1) \quad a \in I \text{ and } b \in I \text{ imply } a + b \in I,$$

$$(3.2) \quad a \in I, b \in A, \text{ and } b * 0 \leq a * 0 \text{ imply } b \in I.$$

Now we have

Theorem 1. *The set of all ideals of a seminormal autometrized algebra A is a complete algebraic lattice ordered by set inclusion.*

We need the following lemmas.

Lemma 1. *The intersection of any nonempty collection of ideals of A is an ideal of A .*

Proof. Obvious.

Lemma 2. For any a, b, c, d in A ,

$$(a * b) + (c * d) = [(a * b) + (c * d)] * 0.$$

Proof. Let a, b, c, d be in A . Then

$$\begin{aligned} [(a * b) + (c * d)] * 0 &= [(a * b) + (c * d)] * (0 + 0) \\ &\leq [(a * b) * 0] + [(c * d) * 0] \quad (\text{by 1.3}) \\ &= (a * b) + (c * d) \quad (\text{by 1.5}) \\ &= [(a * b) * 0] + [(c * d) * 0] \\ &= \{[(a * b) * 0] + [(c * d) * 0]\} * 0 \quad (\text{by 1.6}) \\ &= [(a * b) + (c * d)] * 0. \end{aligned}$$

Hence $[(a * b) + (c * d)] * 0 = (a * b) + (c * d)$.

Lemma 3. $(a_1 * 0) + \dots + (a_n * 0) = [(a_1 * 0) + \dots + (a_n * 0)] * 0$
for any $a_1, \dots, a_n$ in A .

Proof. Since $a_1 * 0 = (a_1 * 0) * 0$ (by 1.5), the equality is true for $n = 1$. Assume that the equality is true for a positive integer m .

Then

$$\begin{aligned} (a_1 * 0) + \dots + (a_{m+1} * 0) &= (a_1 * 0) + \dots + (a_m * 0) + (a_{m+1} * 0) \\ &= [\{(a_1 * 0) + \dots + (a_m * 0)\} * 0] + (a_{m+1} * 0) \\ &= [\{(a_1 * 0) + \dots + (a_m * 0)\} * 0 + (a_{m+1} * 0)] \\ &\quad * 0 \quad (\text{by 1.6}) \\ &= [\{(a_1 * 0) + \dots + (a_m * 0)\} + (a_{m+1} * 0)] * 0 \\ &= [(a_1 * 0) + \dots + (a_m * 0) + (a_{m+1} * 0)] * 0. \end{aligned}$$

This completes the proof.

Lemma 4. $n[(a * b) + (c * d)] = [n(a * b) + n(c * d)] * 0$ for any positive integer n .

Proof. Since $(a * b) + (c * d) = [(a * b) + (c * d)] * 0$ (by Lemma 2), the equality is true for $n = 1$. Assume that the equality is true for a positive integer m . Then

$$\begin{aligned}
 (m+1)[(a * b) + (c * d)] &= m[(a * b) + (c * d)] + [(a * b) + (c * d)] \\
 &= [\{m(a * b) + m(c * d)\} * 0] \\
 &\quad + \{[(a * b) + (c * d)] * 0\} \quad (\text{by our assumption and Lemma 2}) \\
 &= [\{m(a * b) + m(c * d)\} * 0 + \{(a * b) + (c * d)\} * 0] * 0 \quad (\text{by 1.6}) \\
 &= [\{m(a * b) + m(c * d)\} + \{(a * b) + (c * d)\}] * 0 \\
 &\quad (\text{by our assumption and Lemma 2}) \\
 &= [m(a * b) + m(c * d) + (a * b) + (c * d)] * 0 \\
 &= [(m+1)(a * b) + (m+1)(c * d)] * 0.
 \end{aligned}$$

This completes the proof.

Lemma 5. *For any nonempty subset S of A , the set $\{x \in A \mid x * 0 \leq (a_1 * 0) + \dots + (a_n * 0) \text{ for some } a_1, \dots, a_n \text{ in } S\}$ is the smallest ideal containing S .*

Proof. Let $G = \{x \in A \mid x * 0 \leq (a_1 * 0) + \dots + (a_n * 0) \text{ for some } a_1, \dots, a_n \text{ in } S\}$. Clearly $0 \in G$. Let $x \in G$ and $y \in G$. Then $x * 0 \leq (a_1 * 0) + \dots + (a_n * 0)$ for some $a_1, \dots, a_n$ in S and $y * 0 \leq (b_1 * 0) + \dots + (b_m * 0)$ for some $b_1, \dots, b_m$ in S . Since $(x+y) * 0 \leq (x * 0) + (y * 0)$ (by 1.3) $\leq (a_1 * 0) + \dots + (a_n * 0) + (b_1 * 0) + \dots + (b_m * 0)$ for some $a_1, \dots, a_n, b_1, \dots, b_m$ in S , it follows that $x + y \in G$. If $x \in G$ and $y * 0 \leq x * 0$ then clearly $y \in G$. Hence G is an ideal. It is clear that $S \subset G$. Now let I be an ideal containing S . Let $x \in G$. Then $x * 0 \leq (a_1 * 0) + \dots + (a_n * 0)$ for some $a_1, \dots, a_n$ in S . Since $S \subset I$, $a_1, \dots, a_n$ belong to I and hence $a_1 * 0, \dots,$

$a_n * 0$ belong to I (by 3.2). Hence $(a_1 * 0) + \dots + (a_n * 0) \in I$. Now, since $x * 0 \leq (a_1 * 0) + \dots + (a_n * 0) = [(a_1 * 0) + \dots + (a_n * 0)] * 0$ (by Lemma 3) it follows that $x \in I$ (by 3.2). So $G \subset I$. Hence G is the smallest ideal containing S .

Remark. From the above lemma, it is clear that the principal ideal generated by a , which is by definition the smallest ideal (a) containing a is $\{x \in A \mid x * 0 \leq m(a * 0) \text{ for some positive integer } m\}$. Also, from Lemmas 1 and 5 it is clear that the join of any collection A is the ideal generated by $\bigcup_{I \in A} I$. In particular we have the following:

Corollary 1. If I and J are ideals of A , $\{a \in A \mid a * 0 \leq (x * 0) + (y * 0) \text{ for some } x \in I \text{ and } y \in J\}$ is the join of I and J .

Lemma 6. The join of any two principal ideals in A is a principal ideal.

Proof. Let (a) and (b) be principal ideals of A . We show that $(a) \vee (b) = (a * 0 + b * 0)$. Let $x \in (a)$. Then $x * 0 \leq m(a * 0)$ for some positive integer m . Now $x * 0 \leq m(a * 0) + m(b * 0) = m[(a * 0) + (b * 0)] = m\{(a * 0) + (b * 0)\} * 0$ (by 1.6) and hence $x \in (a * 0 + b * 0)$. Hence $(a) \subset (a * 0 + b * 0)$. Similarly we obtain that $(b) \subset (a * 0 + b * 0)$. Now let J be any ideal containing (a) and (b) . Let $x \in (a * 0 + b * 0)$. Then $x * 0 \leq m\{(a * 0 + b * 0) * 0\}$ for some positive integer m . Hence

$$\begin{aligned} x * 0 &\leq m(a * 0 + b * 0) && \text{(by 1.6)} \\ &= [m(a * 0) + m(b * 0)] * 0 && \text{(by Lemma 4).} \end{aligned}$$

Now $a \in (a) \subset J$ and $b \in (b) \subset J$ show that $a \in J$ and $b \in J$. Hence $a * 0$ and $b * 0$ belong to J so that $t = m(a * 0) + m(b * 0) \in J$. Now $x * 0$

$\leq t * 0$ and $t \in J$. Hence it follows that $x \in J$. So $(a * 0 + b * 0) \in J$. Therefore $(a) \vee (b) = (a * 0 + b * 0)$. Thus $(a) \vee (b)$ is also a principal ideal.

Lemma 7. $(a_1) \vee \cdots \vee (a_n) = (a_1 * 0 + \cdots + a_n * 0)$ for any $a_1, \cdots, a_n$ in A .

Proof. Since x belongs to an ideal I if and only if $x * 0$ is in I , $(a_1) = (a_1 * 0)$ and hence the equality holds for $n = 1$. Assume that the equality holds for any positive integer m . Then

$$\begin{aligned} (a_1) \vee \cdots \vee (a_{m+1}) &= (a_1 * 0 + \cdots + a_m * 0) \vee (a_{m+1} * 0) \\ &= ((a_1 * 0 + \cdots + a_m * 0) * 0 + a_{m+1} * 0) \\ &= (a_1 * 0 + \cdots + a_m * 0 + a_{m+1} * 0) \end{aligned}$$

by Lemma 6 and Lemma 3.

Lemma 8. In A , an ideal I is compact if and only if I is a principal ideal.

Proof. Let (a) be the principal ideal generated by a . Suppose $(a) \subset \bigvee_{\alpha \in \Delta} I_\alpha$ where $\{I_\alpha \mid \alpha \in \Delta\}$ is a family of ideals of A . Then $a \in (a) \subset \bigvee_{\alpha \in \Delta} I_\alpha$ implies that $a * 0 \leq (x_1 * 0) + \cdots + (x_n * 0)$ where $x_1 \in I_{\alpha_1}, \cdots, x_n \in I_{\alpha_n}$ for some $\alpha_1, \cdots, \alpha_n$. Hence $a \in I_{\alpha_1} \vee \cdots \vee I_{\alpha_n}$ so that $(a) \subset I_{\alpha_1} \vee \cdots \vee I_{\alpha_n}$. Therefore (a) is a compact ideal. Now let I be any compact ideal. Since $I \subset \bigvee_{a \in I} (a)$, it follows that $I \subset (a_1) \vee \cdots \vee (a_n)$ for some $a_1, \cdots, a_n$ in I . Hence $I = (a_1) \vee \cdots \vee (a_n)$. From Lemma 7, it follows that I is a principal ideal. This completes the proof.

From Lemmas 1, 5 and 8, Theorem 1 is clear.

Corollary 2. *The set of all ideal of a normal autometrized algebra is a complete algebraic lattice. Consequently the ideal lattices of commutative ℓ -groups, the ideal lattices of Boolean algebras, the ideal lattices of Brouwerian algebras, the ideal lattices of semi-Brouwerian algebras and the ideal lattices of commutative D. R. ℓ . semigroups are all complete algebraic lattices*

Corollary 3. *The set of all ideals of a Brouwerian semigroup is a complete algebraic lattice.*

We have introduced the notion of a congruence relation in any autometrized algebra ([3]). Accordingly, we have the following

Definition 4. An equivalence relation θ on an autometrized algebra $S = (S, +, \leq, *)$ is called a *congruence relation* if and only if

$$(4.1) \quad a \equiv b (\theta), c \equiv d (\theta) \text{ imply that } a + c \equiv b + d (\theta) \text{ for any } a, b, c, d \text{ in } S,$$

$$(4.2) \quad a \equiv b (\theta), c \equiv d (\theta) \text{ imply that } a * c \equiv b * d (\theta) \text{ for any } a, b, c, d \text{ in } S,$$

$$(4.3) \quad a \equiv b (\theta) \text{ and } x * y \leq a * b \text{ imply that } x \equiv y (\theta) \text{ for any } a, b, x, y \text{ in } S.$$

We have the following

Theorem 2. *The set of all congruence relations of an autometrized algebra is a complete lattice.*

Proof. Let $C(S)$ denote the set of all congruence relations on an autometrized algebra $S = (S, +, \leq, *)$. Then $(C(S), \leq)$ is a poset with $\leq$

as set inclusion. Also, since the intersection of any family of congruence relations of S is a congruence relation of S , we see that $(\mathcal{C}(S), \leq)$ is a complete lattice by a well known elementary theorem (see Theorem 3, p. 112 of [1]).

Now, we have the following

Theorem 3. *The lattice of ideals of a seminormal autometrized algebra A is isomorphic with the lattice of congruence relations of A .*

Proof. Let S be an ideal of A . Define $a \equiv b (\theta_S)$ if and only if $a * b \in S$. Since $0 \in S$, $a \equiv a (\theta_S)$. Let $a \equiv b (\theta_S)$ and $b \equiv c (\theta_S)$. Then $a * b \in S$ and $b * c \in S$. Now

$$\begin{aligned} (a * c) * 0 &= a * c && \text{(by 1.5)} \\ &\leq (a * b) + (b * c) \\ &= [(a * b) + (b * c)] * 0 && \text{(by Lemma 2).} \end{aligned}$$

Since $(a * b) + (b * c) \in S$ and S is an ideal, this shows that $a * c$ is in S . Hence $a \equiv c (\theta_S)$. Thus θ_S is an equivalence relation. Let $a \equiv b (\theta_S)$ and $c \equiv d (\theta_S)$. Then $a * b \in S$ and $c * d \in S$. Now

$$\begin{aligned} [(a + c) * (b + d)] * 0 &= (a + c) * (b + d) \\ &\leq (a * b) + (c * d) && \text{(by 1.3)} \\ &= [(a * b) + (c * d)] * 0 && \text{(by Lemma 2).} \end{aligned}$$

Since $(a * b) + (c * d) \in S$ and S is an ideal, this implies that $(a + c) * (b + d) \in S$. Hence $a + c \equiv b + d (\theta_S)$. Moreover

$$\begin{aligned} [(a * c) * (b * d)] * 0 &= (a * c) * (b * d) \\ &\leq 2[(a * b) + (c * d)] && \text{(by 1.4)} \\ &= [2(a * b) + 2(c * d)] * 0 && \text{(by Lemma 4).} \end{aligned}$$

Since $a * b \in S$, $c * d \in S$ and S is an ideal, this shows that $(a * c) *$

$(b * d) \in S$. Hence $a * c \equiv b * d \pmod{\theta_S}$. Now let $a \equiv b \pmod{\theta_S}$ and $x * y \leq a * b$. Then $(x * y) * 0 = x * y \leq a * b = (a * b) * 0$ and $a * b \in S$. Since S is an ideal it follows that $x * y \in S$. Hence $x \equiv y \pmod{\theta_S}$. Thus θ_S is a congruence relation.

Now let f be the mapping from $I(A)$, the lattice of ideals of A into $C(A)$, the lattice of congruence relations on A defined by $(S)f = \theta_S$ for any $S \in I(A)$. To show f is onto, let θ be a congruence relation of A . Define $S_\theta = \{x \mid x \equiv 0 \pmod{\theta}\}$. Since $0 \in S_\theta$, S_θ is nonempty. Let $a \in S_\theta$ and $b \in S_\theta$. Then $a \equiv 0 \pmod{\theta}$ and $b \equiv 0 \pmod{\theta}$. Since θ is a congruence relation, it follows that $a + b \equiv 0 \pmod{\theta}$. Hence $a + b \in S_\theta$. Let $a \in S_\theta$ and $b * 0 \leq a * 0$. $a \in S_\theta$ implies that $a \equiv 0 \pmod{\theta}$. Since θ is a congruence relation it follows that $b \equiv 0 \pmod{\theta}$ (by 4.3). Hence $b \in S_\theta$. Hence S_θ is an ideal. Also $a \equiv b \pmod{\theta}$ if and only if $a * b \equiv 0$ which is equivalent to $a * b \in S_\theta$. Since $a * b \in S_\theta$ is equivalent to $a \equiv b \pmod{\theta_{(S_\theta)}}$, it follows that $(S_\theta)f = \theta_{(S_\theta)} = \theta$. Hence f is onto. To show that f is an isomorphism between $I(A)$ and $C(A)$, it is sufficient to show that $S \subset K$ if and only if $\theta_S \subset \theta_K$ for any ideals S and K of A . Let $S \subset K$. If $a \equiv b \pmod{\theta_S}$ then $a * b \in S \subset K$ so that $a * b \in K$. Hence $a \equiv b \pmod{\theta_K}$. Hence $\theta_S \subset \theta_K$ if $S \subset K$. Now let $\theta_S \subset \theta_K$. If $x \in S$ then $x * 0 \in S$ so that $x \equiv 0 \pmod{\theta_S}$. Hence $x \equiv 0 \pmod{\theta_K}$ which implies $x * 0 \in K$. Hence $x \in K$. Thus $S \subset K$ if $\theta_S \subset \theta_K$. Hence the lattice of ideals of A is isomorphic to the lattice of congruence relations of A .

Corollary 4. *In each of normal autometrized algebras (consequently of Boolean algebras, Brouwerian algebras, semi-Brouwerian algebras, commutative ℓ -groups and commutative D. R. ℓ semigroups) ideals correspond one to one to its congruence relations.*

Corollary 5. *The ideal lattice of a Brouwerian semigroup is isomorphic to the lattice of its congruence relations.*

Corollary 6. *The ideal lattice of a representable Griss algebra is isomorphic to the lattice of its congruence relations.*

If θ is any equivalence relation on a set S , as usual S/θ denotes the quotient set (i.e., the set of all θ -equivalence classes) and for any x in S , $\bar{x}$ denotes the equivalence class containing x .

Theorem 4. *Let θ be a congruence relation on a seminormal autometrized algebra A in which*

(1.7) *$a \leq b$ implies there exists an element $x \geq 0$ such that $a + x = b$ holds. For any $\bar{a}, \bar{b}$ in A/θ define $\bar{a} + \bar{b} = \overline{a + b}$; $\bar{a} * \bar{b} = \overline{a * b}$; $\bar{a} \leq \bar{b}$ if there is an $x \geq 0$ such that $a + x \equiv b (\theta)$. Then $(A/\theta, +, \leq, *)$ is a seminormal autometrized algebra in which (1.7) holds if and only if θ has the following property: For any a, b and $z_1 \geq 0, z_2 \geq 0$; $a + z_1 \equiv b (\theta)$ and $b + z_2 \equiv a (\theta)$ imply $a \equiv b (\theta)$.*

Proof. Suppose θ has the property: For any a, b and $z_1 \geq 0, z_2 \geq 0$, $a + z_1 \equiv b (\theta)$ and $b + z_2 \equiv a (\theta)$ imply that $a \equiv b (\theta)$. Now we show $(A/\theta, +, \leq, *)$ is a seminormal autometrized algebra in which (1.7) holds. To show that the operations $+$, $*$ are well defined, let $\bar{a} = \bar{c}$ and $\bar{b} = \bar{d}$. Then $a \equiv c (\theta)$ and $b \equiv d (\theta)$. Since θ has the substitution property for $+$ and $*$, it follows that $a + b \equiv c + d (\theta)$ and $a * b \equiv c * d (\theta)$. Hence $\overline{a + b} = \overline{c + d}$ and $\overline{a * b} = \overline{c * d}$. Therefore $+$ and $*$ are well defined. To show $\leq$ is well defined, let $\bar{a} \leq \bar{b}$ and $\bar{a} = \bar{c}, \bar{b} = \bar{d}$. Then there exists an element $x \geq 0$ such that $a + x \equiv b (\theta)$

and $a \equiv c \pmod{\theta}$, $b \equiv d \pmod{\theta}$. Now $c + x \equiv a + x \equiv b \equiv d \pmod{\theta}$. Hence $\bar{c} \leq \bar{d}$. Thus $\leq$ is well defined. Since $\bar{a} + \bar{b} = \overline{a + b} = \overline{b + a} = \bar{b} + \bar{a}$, $\bar{a} + \bar{0} = \overline{a + 0} = \bar{a}$ and $\bar{a} + (\bar{b} + \bar{c}) = \overline{a + (b + c)} = \overline{(a + b) + c} = (\bar{a} + \bar{b}) + \bar{c}$ for any $\bar{a}, \bar{b}, \bar{c}$ in A/θ , it follows that $(A/\theta, +)$ is a commutative semigroup with additive identity $\bar{0}$. Since $a + 0 = a \equiv a \pmod{\theta}$, we have $\bar{a} \leq \bar{a}$ for any $\bar{a}$ in A/θ . Let $\bar{a} \leq \bar{b}$ and $\bar{b} \leq \bar{a}$. Then there exist $z_1 \geq 0$ and $z_2 \geq 0$ such that $a + z_1 \equiv b \pmod{\theta}$ and $b + z_2 \equiv a \pmod{\theta}$. Hence $a \equiv b \pmod{\theta}$ by our assumption. Hence $\bar{a} = \bar{b}$. Let $\bar{a} \leq \bar{b}$ and $\bar{b} \leq \bar{c}$. Then there exist $z_1 \geq 0$ and $z_2 \geq 0$ such that $a + z_1 \equiv b \pmod{\theta}$ and $b + z_2 \equiv c \pmod{\theta}$. Now $c \equiv b + z_2 \equiv a + z_1 + z_2 \pmod{\theta}$. Since $z_1 + z_2 \geq 0$ it follows that $\bar{a} \leq \bar{c}$. Hence $\leq$ is a partial ordering on A/θ . Let $\bar{x} \leq \bar{y}$. Then there exists a $z \geq 0$ such that $x + z \equiv y \pmod{\theta}$. Hence for any a in A , $x + a + z \equiv y + a \pmod{\theta}$ which shows that $\bar{x} + \bar{a} = \overline{x + a} \leq \overline{y + a} = \bar{y} + \bar{a}$. Hence $(A/\theta, \leq)$ is a partially ordered set such that the semigroup translations preserve the ordering. Note that $x \leq y$ implies $\bar{x} \leq \bar{y}$. In fact, if $x \leq y$ then there exists a $z \geq 0$ such that $x + z = y$ (by 1.7) which implies that $x + z \equiv y \pmod{\theta}$, and so $\bar{x} \leq \bar{y}$. Since $0 \leq a * b$, we have $\bar{0} \leq \overline{a * b} = \bar{a} * \bar{b}$ for any $\bar{a}, \bar{b}$ in A/θ . Clearly $\bar{a} * \bar{a} = \overline{a * a} = \bar{0}$. Let $\bar{a} * \bar{b} = \bar{0}$. Then $\overline{a * b} = \bar{0}$ so that $a * b \equiv 0 \pmod{\theta}$ which implies that $a \equiv b \pmod{\theta}$ (by 1.5 and 4.3). Hence $\bar{a} = \bar{b}$. Also, $\bar{a} * \bar{b} = \overline{a * b} = \overline{b * a} = \bar{b} * \bar{a}$. Since $a * c \leq a * b + b * c$, it follows that $\bar{a} * \bar{c} = \overline{a * c} \leq \overline{a * b + b * c} = \overline{a * b} + \overline{b * c} = \bar{a} * \bar{b} + \bar{b} * \bar{c}$. Hence $*$ satisfies the formal properties of a metric. Since $(a + c) * (b + d) \leq a * b + c * d$, it follows that $(\bar{a} + \bar{c}) * (\bar{b} + \bar{d}) = \overline{(a + c) * (b + d)} \leq \overline{a * b + c * d} = (\bar{a} * \bar{b}) + (\bar{c} * \bar{d})$ for any $\bar{a}, \bar{b}, \bar{c}, \bar{d}$ in A/θ . Hence (1.3) holds in A/θ . Since $(a * c) * (b * d) \leq 2[(a * b)$

+ $(c * d)$], it follows that $(\bar{a} * \bar{c}) * (\bar{b} * \bar{d}) = \overline{(a * c) * (b * d)} \leq \overline{2[(a * b) + (c * d)]} = 2[(\bar{a} * \bar{b}) + (\bar{c} * \bar{d})]$ for any $\bar{a}, \bar{b}, \bar{c}, \bar{d}$ in A/θ . Hence (1.4) holds in A/θ . Since $\bar{a} * \bar{b} = \overline{a * b} = \overline{(a * b) * 0} = (\bar{a} * \bar{b}) * \bar{0}$ for any $\bar{a}, \bar{b}$ in A/θ , (1.5) also holds in A/θ . On similar lines it can be shown (1.6) holds in A/θ . Hence $(A/\theta, +, \leq, *)$ is a seminormal autometrized algebra. Now let $\bar{a} \leq \bar{b}$. Then there exists an $x \geq 0$ such that $a + x \equiv b \pmod{\theta}$. Hence $\bar{x} \geq \bar{0}$ and $\bar{a} + \bar{x} = \overline{a + x} = \bar{b}$. Therefore (1.7) also holds in A/θ . Thus $(A/\theta, +, \leq, *)$ is a seminormal autometrized algebra in which (1.7) holds. Let $(A/\theta, +, \leq, *)$ be a seminormal autometrized algebra in which (1.7) holds. Suppose for any a, b and $z_1 \geq 0, z_2 \geq 0$; $a + z_1 \equiv b \pmod{\theta}$ and $b + z_2 \equiv a \pmod{\theta}$. Then $\bar{a} \leq \bar{b}$ and $\bar{b} \leq \bar{a}$. Since $\leq$ is a partial ordering on A/θ it follows that $\bar{a} = \bar{b}$. Hence $a \equiv b \pmod{\theta}$. This completes the proof.

Now we introduce the following:

Definition 5. Let $S = (S, +, \leq, *)$ and $T = (T, +, \leq, *)$ be two autometrized algebras. We say that a mapping $f: S \rightarrow T$ is a *homomorphism* from S to T if and only if

$$(5.1) \quad (a + b)f = af + bf \text{ for any } a, b \text{ in } S,$$

$$(5.2) \quad (a * b)f = af * bf \text{ for any } a, b \text{ in } S,$$

$$(5.3) \quad a \leq b \text{ implies } af \leq bf \text{ for any } a, b \text{ in } S.$$

A homomorphism f from an autometrized algebra S onto T is called an *isomorphism* if $f^{-1}: T \rightarrow S$ is also a homomorphism.

For the sake of convenience we call those ideals which correspond to the congruence relations for which quotient algebras can be constructed as normal ideals. Accordingly, we have the following:

Definition 6. An ideal I of a seminormal autometrized algebra A in which (1.7) holds is called a *normal ideal* if it has the property that $(a + z_1) * b \in I$ and $(b + z_2) * a \in I$ for any a, b in A and $z_1 \geq 0$, $z_2 \geq 0$ imply $a * b \in I$.

We immediately have the following:

Theorem 5. In a seminormal autometrized algebra A in which (1.7) holds, normal ideals are just the kernels of epimorphisms.

Proof. Let f be an epimorphism from a seminormal autometrized algebra A in which (1.7) holds to a seminormal autometrized algebra B in which (1.7) holds. By definition $\ker f = \{x \in A \mid xf = \bar{0}\}$ where $\bar{0}$ is the zero element of B . Note that $0f = (0 * 0)f = (0f) * (0f) = \bar{0}$. Let $x \in \ker f$ and $y \in \ker f$. Then $xf = \bar{0}$ and $yf = \bar{0}$. Hence $(x + y)f = xf + yf = \bar{0} + \bar{0} = \bar{0}$. Hence $x + y \in \ker f$. Let $x \in \ker f$ and $y * 0 \leq x * 0$. Then $xf = \bar{0}$ and $(y * 0)f \leq (x * 0)f$. Now $\bar{0} \leq (yf) * \bar{0} = (yf) * (0f) = (y * 0)f \leq (x * 0)f = (xf) * (0f) = (xf) * \bar{0} = \bar{0} * \bar{0} = \bar{0}$. Hence $(yf) * \bar{0} = \bar{0}$ so that $yf = \bar{0}$. Therefore $\ker f$ is an ideal of A . To show that $\ker f$ is a normal ideal of A , let $(a + z_1) * b \in \ker f$ and $(b + z_2) * a \in \ker f$ and $z_1 \geq 0$, $z_2 \geq 0$. Then $[(a + z_1) * b]f = \bar{0}$ and $[(b + z_2) * a]f = \bar{0}$. Then $(a + z_1)f * (bf) = \bar{0}$ and $(b + z_2)f * (af) = \bar{0}$ which implies $(a + z_1)f = bf$ and $(b + z_2)f = af$. Since $0 \leq z_1$, we have $0f \leq z_1f$ and hence $af = af + 0f \leq af + z_1f = (a + z_1)f = bf$. Similarly, since $0 \leq z_2$ and $(b + z_2)f = af$, we have $bf \leq af$. Hence $af = bf$. Now we have $(a * b)f = (af) * (bf) = \bar{0}$ which implies $a * b \in \ker f$. So $\ker f$ is a normal ideal of A .

Let I be a normal ideal of A . Let θ be the congruence relation

corresponding to I . Then A/θ is a seminormal autometrized algebra in which (1.7) holds (by Theorem 4) and the mapping $f: A \rightarrow A/\theta$ defined by $af = \bar{a}$, for $a \in A$ is clearly onto. Also for any a, b in A , $(a + b)f = a + b = \bar{a} + \bar{b} = af + bf$ and $(a * b)f = a * b = \bar{a} * \bar{b} = af * bf$. Let $a \leq b$. Then there exists an element $x \geq 0$ such that $a + x = b$. Hence $a + x \equiv b \pmod{\theta}$ which implies $\bar{a} \leq \bar{b}$, i.e., $af \leq bf$. Hence f is a homomorphism so that f is an epimorphism from A to A/θ . Also $x \in \ker f$ if and only if $xf = \bar{0}$ which is equivalent to $\bar{x} = \bar{0}$ or $x \equiv 0 \pmod{\theta}$. It is clear that $x \equiv 0 \pmod{\theta}$ is equivalent to $x \in I$. Hence $\ker f = I$. This completes the proof of the theorem.

3. In [3] we introduced the notion of a semiregular autometrized algebra as follows:

Definition 7. An autometrized algebra $A = (A, +, \leq, *)$ is called *semiregular* if and only if $a \geq 0$ implies $a * 0 = a$.

Throughout this section $A = (A, +, \leq, *)$ denotes a semiregular seminormal lattice ordered autometrized algebra.

The following is an example of a semiregular seminormal lattice ordered autometrized algebra which is not a D. R. ℓ . semigroup.

Example 5. Let A be the set consisting of elements $0, a, b, c, d, e$. Define $0 < a < b < c < d < e$. Define $+$ and $*$ by the tables at right.

$+$	0	a	b	c	d	e
0	0	a	b	c	d	e
a	a	b	b	e	e	e
b	b	b	b	e	e	e
c	c	e	e	e	e	e
d	d	e	e	e	e	e
e	e	e	e	e	e	e

$*$	0	a	b	c	d	e
0	0	a	b	c	d	e
a	a	0	a	c	c	c
b	b	a	0	c	c	c
c	c	c	c	0	a	a
d	d	c	c	a	0	a
e	e	c	c	a	a	0

It is routine to verify that $A = (A, +, \leq, *)$ is a semiregular (in fact regular) seminormal lattice ordered autometrized algebra. If there is a binary operation $-$ such that $A = (A, +, \leq, -)$ is a D. R. ℓ . semigroup, then the operation $-$ must be as follows:

Note that $(d - a) \cup 0 + a = (e \cup 0) + a = e + a = e$ whereas $d \cup a = d \neq e$. Hence the condition

" $(x - y) \cup 0 + y = x \cup y$ for any x, y " of a D. R. ℓ . semigroup fails. Thus A is not a D. R. ℓ . semigroup.

$-$	0	a	b	c	d	e
0	0	0	0	0	0	0
a	a	0	0	0	0	0
b	b	a	0	0	0	0
c	c	c	c	0	0	0
d	d	c	c	a	0	0
e	e	c	c	a	a	0

We now have the following fundamental theorem of homomorphisms for semiregular seminormal autometrized algebras.

Theorem 6. *Let f be an epimorphism from a semiregular seminormal autometrized algebra $S = (S, +, \leq, *)$ in which (1.7) holds to a semiregular seminormal autometrized algebra $T = (T, +, \leq, *)$ in which (1.7) holds. Then $S/\ker f$ is isomorphic with T .*

Proof. From Theorem 5, $\ker f$ is a normal ideal of S . Let θ be the congruence relation corresponding to $\ker f$. Define $g: S/\theta \rightarrow T$ by $\bar{a}g = af$ for any $\bar{a}$ in S/θ . Now since $\bar{a} = \bar{b}$ is clearly equivalent to the fact that $(a * b)f = \bar{0}$, $\bar{a} = \bar{b}$ is equivalent to $(af) * (bf) = \bar{0}$ or to $af = bf$. Hence g is well defined. Obviously g is onto. On the other hand $(\overline{a + b})g = (a + b)f = af + bf = \bar{a}g + \bar{b}g$ (since f is a homomorphism) and $(\overline{a * b})g = (a * b)f = (af) * (bf) = \bar{a}g * \bar{b}g$. Now let $\bar{a} \leq \bar{b}$. Then there exists an $x \geq 0$ such that $a + x \equiv b (\theta)$. Hence $\overline{a + x} = \bar{b}$ so that $(\overline{a + x})g = bg$. Hence $af + xf = (a + x)f = bf$. So $af \leq bf$ (since

$0f \leq xf$). Therefore $\bar{a} \leq \bar{b}$ implies $\bar{a}g \leq \bar{b}g$. Hence g is a homomorphism from S/θ onto T . To show g is an isomorphism, we have to show $g^{-1}: T \rightarrow S/\theta$ is also a homomorphism. Now $(af + bf)g^{-1} = ((a + b)f)g^{-1} = \overline{a + b} = \bar{a} + \bar{b} = (af)g^{-1} + (bf)g^{-1}$ and similarly $(af * bf)g^{-1} = \overline{a * b} = \bar{a} * \bar{b} = (af)g^{-1} * (bf)g^{-1}$. Now let $af \leq bf$. Since (1.7) holds in T and f is onto, there exists an element c in S such that $cf \geq 0f$ and $(af) + (cf) = bf$. Now $[a + (c * 0)]f = af + (c * 0)f = af + [(cf) + (0f)] = (af) + (cf) = bf$ (since T is semiregular). Hence $c * 0$ is an element of S such that $a + (c * 0) \equiv b \pmod{\theta}$ (since as we observed earlier $\bar{a} = \bar{b}$ if and only if $af = bf$). Hence $\bar{a} \leq \bar{b}$. Thus g^{-1} is also a homomorphism from T to S/θ . Hence g is an isomorphism from S/θ to T . This completes the proof.

Now we prove the following:

Lemma 9. *The lattice of ideals of a semiregular seminormal lattice ordered autometrized algebra $A = (A, +, \leq, *)$ is a distributive lattice.*

Proof. It is enough to show that $I \wedge (J \vee K) \subset (I \wedge J) \vee (I \wedge K)$ for any ideals I, J, K of A . Let $x \in I \wedge (J \vee K)$. Then $x \in I$ and $x * 0 \leq (y * 0) + (z * 0)$ for some $y \in J$ and $z \in K$. Put $x * 0 = a$, $y * 0 = b$ and $z * 0 = c$. Now $(a \cap b) + (a \cap c) = ((a \cap b) + a) \cap ((a \cap b) + c) = (a + a) \cap (b + a) \cap (a + c) \cap (b + c) \geq a \cap (b + c) = a$ (since $a \geq 0$, $b \geq 0$ and $c \geq 0$). Now $a \geq 0$ and $b \geq 0$ imply $a \cap b \geq 0$ so that $(a \cap b) * 0 = a \cap b \leq b = y * 0$. Since $y \in J$ it follows that $a \cap b \in J$. Since $x \in I$ and $(a \cap b) * 0 = a \cap b \leq a = x * 0$, it follows that $a \cap b \in I$. Hence $a \cap b$ belongs to $I \wedge J$. $(a \cap c) * 0 = a \cap c \leq a = x * 0$, $x \in I$ and $(a \cap c) * 0 =$

$a \cap c \leq c = z * 0$, $z \in K$ imply that $a \cap c \in I \wedge K$. Now $x * 0 = a \leq (a \cap b) + (a \cap c) = [(a \cap b) * 0] + [(a \cap c) * 0]$ where $a \cap b \in I \wedge J$ and $a \cap c \in I \wedge K$. Hence $x \in (I \wedge J) \vee (I \wedge K)$. Thus $I \wedge (J \vee K) \subset (I \wedge J) \vee (I \wedge K)$, proving the lemma.

According to Epstein and Horn [2], a *Heyting algebra* L is a lattice in which for any given elements a and b , the set of all $x \in L$ such that $a \wedge x \leq b$ contains a greatest element. Thus Heyting algebras are just those what Birkhoff calls Brouwerian lattices (see page 45 of [1]).

Now we have the following:

Theorem 7. *The lattice of ideals of a semiregular seminormal lattice ordered autometrized algebra A is a complete algebraic Heyting algebra.*

Proof. By Theorem 1, the lattice of ideals of A is a complete algebraic lattice. To show that the lattice of ideals of A is a Heyting algebra, it suffices to prove that $I \wedge (\bigvee I_\alpha) = \bigvee (I \wedge I_\alpha)$ where I and I_α 's are ideals of A (see page 128 of [1]). Clearly $\bigvee (I \wedge I_\alpha) \subset I \wedge (\bigvee I_\alpha)$. Let $x \in I \wedge (\bigvee I_\alpha)$. Then $x \in I$ and $x * 0 \leq (a_1 * 0) + \dots + (a_k * 0)$ where $a_1 \in I_{\alpha_1}$, $\dots$, $a_k \in I_{\alpha_k}$ for some $\alpha_1, \dots, \alpha_k$. Hence $x \in I_{\alpha_1} \vee \dots \vee I_{\alpha_k}$ so that $x \in I \wedge (I_{\alpha_1} \vee \dots \vee I_{\alpha_k}) = (I \wedge I_{\alpha_1}) \vee \dots \vee (I \wedge I_{\alpha_k}) \subset \bigvee (I \wedge I_\alpha)$ (by the above lemma). Hence $x \in \bigvee (I \wedge I_\alpha)$. This completes the proof.

Corollary 7. *The lattice of ideals of a semiregular normal lattice ordered autometrized algebra is a complete algebraic Heyting algebra.*

We now recall the following:

Definition 8. For elements a and b in a lattice ordered automet-

rized algebra, we write $a \perp b$ to mean $(a * 0) \cap (b * 0) = 0$.

We observe the following:

Lemma 10. *If J is an ideal of a semiregular seminormal lattice ordered autometrized algebra A , then $J^* = \{x \in A \mid x \perp a \text{ for all } a \text{ in } J\}$ is an ideal of A and J^* is the pseudocomplement of J .*

Proof. Since $0 \perp a$ for all a in J , $0 \in J^*$. Let $x \in J^*$ and $y \in J^*$. Then $(x * 0) \cap (a * 0) = 0$ and $(y * 0) \cap (a * 0) = 0$ for all a in J .

Now

$$\begin{aligned} 0 &\leq [(x + y) * 0] \cap (a * 0) \leq [(x * 0) + (y * 0)] \cap (a * 0) \\ &\leq [(x * 0) + (y * 0)] \cap [(x * 0) + (a * 0)] \cap \\ &\quad [(y * 0) + (a * 0)] \cap [(a * 0) + (a * 0)] \\ &= [(x * 0) \cap (a * 0)] + [(y * 0) \cap (a * 0)] = 0 + 0 = 0. \end{aligned}$$

Hence $[(x + y) * 0] \cap (a * 0) = 0$ for all a in J so that $x + y \in J^*$.

Let $x \in J^*$ and $y * 0 \leq x * 0$. $x \in J^*$ implies that $(x * 0) \cap (a * 0) = 0$ for all a in J . Now $(y * 0) \cap (a * 0) = (y * 0) \cap (x * 0) \cap (a * 0) = (y * 0) \cap 0 = 0$ for all a in J . Hence $y \in J^*$. Therefore J^* is an ideal of A .

Let $x \in J \cap J^*$. Then $x \in J$ and $x \in J^*$ which implies $(x * 0) \cap (x * 0) = 0$. Hence $x * 0 = 0$ so that $x = 0$. Hence $J \cap J^* = \{0\}$. Let I be an ideal of A such that $J \cap I = \{0\}$. Let $x \in I$. Take any $a \in J$. Since $(x * 0) \cap (a * 0) \geq 0$, we have $[(x * 0) \cap (a * 0)] * 0 = (x * 0) \cap (a * 0) \leq x * 0$. Since $x \in I$, it follows that $(x * 0) \cap (a * 0) \in I$. Since $a \in J$ and $[(x * 0) \cap (a * 0)] * 0 = (x * 0) \cap (a * 0) \leq a * 0$, it follows that $(x * 0) \cap (a * 0) \in J$. Hence $(x * 0) \cap (a * 0) \in I \cap J = \{0\}$ so that

$$(x * 0) \cap (a * 0) = 0.$$

Hence $x \in J^*$ which implies $I \subset J^*$. Therefore J^* is the pseudocomplement of J . This completes the proof.

We now have the following

Theorem 8. *If $\Theta(A)$ is the lattice of all ideals of a semiregular seminormal lattice ordered autometrized algebra A then the mapping $J \rightarrow J^{**}$ is a closure operation on $\Theta(A)$ and a lattice epimorphism of $\Theta(A)$ on to the Boolean lattice of closed elements of $\Theta(A)$.*

Proof. This follows from Theorem 7, Lemma 10 and Glivenko theorem.

Corollary 8. *If $\Theta(A)$ is the lattice of all ideals of a semiregular normal lattice ordered autometrized algebra A then the mapping $J \rightarrow J^{**}$ is a closure operation on $\Theta(A)$ and a lattice epimorphism of $\Theta(A)$ onto the Boolean lattice of closed elements of $\Theta(A)$.*

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