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On a Conjecture on Common Fixed Points

By Brian FISHER

In a recent paper, see [1], the following conjecture was made:

Conjecture. *Suppose* S *is a mapping and* T *is a continuous mapping of a complete metric space* (X, d) *into itself satisfying the inequality* $d(Sx, T^2y) \leq c \cdot \max\{d(x, Ty), d(x, Sx), d(Ty, T^2y), d(x, T^2y), d(Ty, Sx)\}$ *of all* x, y *in* X, *where* $0 \leq c < 1$, *then* T *has a unique fixed point* z. *Further* z *is the unique common fixed point of* S *and* T.

This conjecture is false as it stands. This is easily seen by letting $X = \{0, 1, 2, \cdots, n, \cdots\}$ with metric d defined by

$$d(0, n) = 1 \quad \text{for} \quad n = 1, 2, \cdots,$$

$$d(1, n) = 1 \quad \text{for} \quad n = 2, 3, \cdots,$$

$$d(r, n) = 2 \quad \text{for} \quad r \neq n \quad \text{and} \quad r, n = 2, 3, \cdots.$$

Define continuous mappings S and T on X by

$$S(0) = 1, \; Sn = 0 \quad \text{for} \quad n = 1, 2, \cdots,$$

$$Tn = n + 2 \quad \text{for} \quad n = 0, 1, 2, \cdots.$$

Then $d(Sr, T^2n) = 1$ for $r, n = 0, 1, 2, \cdots$, $d(Tn, T^2n) = 2$ for $n = 0, 1, 2, \cdots$, and so the inequality of the conjecture holds for all r, n in X. However, neither S nor T has a fixed point.

In this example the metric space X is bounded and so the conjecture

will still be false if we restrict X to being bounded. However, in this example S and T do not commute but when X is bounded and S and T do commute we have the following theorem:

Theorem 1. *Suppose S is a mapping and T is a continuous mapping of a complete and bounded metric space (X, d) into itself satisfying the inequality*

$$d(Sx, T^2y) \leq c\cdot\max\{d(x, Ty), d(x, Sx), d(Ty, T^2y), d(x, T^2y), d(Ty, Sx)\}$$

for all x, y in X, where $0 \leq c < 1$. Suppose further that $ST = TS$. Then T has a unique fixed point z. Further, z is the unique fixed point of S and T.

Proof. Since X is bounded put $M = \sup\{d(x, y) : x, y \in X\} < \infty$. We now prove by induction that

$$d(S^nT^nx, T^{n+1}S^ny) \leq c^nM \tag{1}$$

for all x, y in X and $n = 1, 2, \cdots$. We have

$$\begin{aligned} d(STx, T^2Sy) &\leq c\cdot\max\{d(Tx, TSy), d(Tx, STx), d(TSy, T^2Sy), \\ &\qquad d(Tx, T^2Sy), d(TSy, STx)\} \\ &\leq cM, \end{aligned}$$

by definition of M and so inequality (1) holds for all x, y in X when $n = 1$. Now assume that inequality (1) holds for all x, y in X and some n. Then since $ST = TS$,

$$\begin{aligned} &d(S^{n+1}T^{n+1}x, T^{n+2}S^{n+1}y) \\ &\leq c\cdot\max\{d(S^nT^{n+1}x, T^{n+1}S^{n+1}y), d(S^nT^{n+1}x, S^{n+1}T^{n+1}x), \\ &\quad d(T^{n+1}S^{n+1}y, T^{n+2}S^{n+1}y), d(S^nT^{n+1}x, T^{n+2}S^{n+1}y), d(T^{n+1}S^{n+1}y, S^{n+1}T^{n+1}x)\} \\ &= c\cdot\max\{d(S^nT^n(Tx), T^{n+1}S^n(Sy)), d(S^nT^n(Tx), T^{n+1}S^n(Sx)), \\ &\qquad d(S^nT^n(STy), T^{n+1}S^n(TSy)), d(S^nT^n(Tx), T^{n+1}S^n(TSy)), \end{aligned}$$

$$d(S^nT^n(STx),\ T^{n+1}S^n(Sy))\}$$

$\leq c^{n+1}M$

by our assumption and so inequality (1) follows by induction.

Now let x be an arbitrary point in X. Then since $ST = TS$,

$$d((ST)^n,\ T(ST)^nx) = d(S^nT^nx,\ T^{n+1}S^nx) \leq c^nM,$$

on using inequality (1), for $n = 1, 2, \cdots$. Similarly we have

$$d(T(ST)^nx,\ (ST)^{n+1}x) \leq c^nM$$

for $n = 1, 2, \cdots$. It follows that since $c < 1$, the sequence $\{x, Tx, STx, \cdots, (ST)^nx, T(ST)^nx, \cdots\}$ is a Cauchy sequence in the complete metric space X and so has a limit z in X. Since $\lim_{n\to\infty} T(ST)^nx = \lim_{n\to\infty} (ST)^nx = z$ and since T is continuous, we have $Tz = z$, and then $\lim_{n\to\infty} T^2(ST)^nx = z$. Further

$$d(Sz,\ T^2(ST)^nx) \leq c\cdot\max\{d(z,\ T(ST)^nx),\ d(z,\ Sz),\ d(T(ST)^nx,\ T^2(ST)^nx),$$
$$d(z,\ T^2(ST)^nx),\ d(T(ST)^nx,\ Sz)\}$$

and on letting n tend to infinity we have $d(Sz, z) \leq cd(Sz, z)$. Since $c < 1$ it follows that $Sz = z$ and so z is a common fixed point of S and T. The uniqueness of z follows easily. This completes the proof of the theorem.

It is possible that the boundedness of X is not necessary in the proof of this theorem. In the particular case $S = T$ it is not necessary for X to be bounded, see [2].

We finally prove a theorem for compact metric spaces.

Theorem 2. *Suppose S and T are continuous mappings of a compact metric space (X, d) into itself satisfying the inequality*

$$d(Sx,\ T^2y) < \max\{d(x,\ Ty),\ d(x,\ Sx),\ d(Ty,\ T^2y),\ d(x,\ T^2y),\ d(Ty,\ Sx)\}$$

for all x, y *in* X *for which* $\max\{d(x, Ty), d(x, Sx), d(Ty, T^2y)\} > 0$. *Suppose further that* $ST = TS$. *Then* T *has a unique fixed point* z. *Further,* z *is the unique fixed point of* S *and* T.

Proof. Suppose first of all that there exists c, with $0 \leq c < 1$, such that

$$d(Sx, T^2y) \leq c \cdot \max\{d(x, Ty), d(x, Sx), d(Ty, T^2y), d(x, T^2y), d(Ty, Sx)\}$$

for all x, y in X. Since a compact metric space is bounded, the result in this case follows from Theorem 1. If no such c exists, then there exist a sequence of numbers $\{c_n\}$ converging to 1 and sequences $\{x_n\}$ and $\{y_n\}$ in X converging to x and y respectively such that

$$d(Sx_n, T^2y_n) > c_n \cdot \max\{d(x_n, Ty_n), d(x_n, Sx_n), d(Ty_n, T^2y_n), d(x_n, T^2y_n), d(Ty_n, Sx_n)\}$$

for $n = 1, 2, \cdots$ (cf. [1]). Letting n tend to infinity we have

$$d(Sx, T^2y) \geq \max\{d(x, Ty), d(x, Sx), d(Ty, T^2y), d(x, T^2y), d(Ty, Sx)\}.$$

This implies that $\max\{d(x, Ty), d(x, Sx), d(Ty, T^2y)\} = 0$, and so

$$x = Ty = Sx = T^2y.$$

It follows that x is a common fixed point of S and T. The uniqueness of the fixed point follows easily. This completes the proof of Theorem 2.

References

[1] B. Fisher: Results and a conjecture on common fixed points, these NOTES, 5 (1977), 367-376.

[2] B. Fisher: Results and a conjecture on fixed points, Atti Accad. Naz. Lincei Rend. Cl. Sci. Fis. Mat. Natur., to appear.

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