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A Collection of Contractive Definitions

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Numerous fixed point theorems have been established for mappings satisfying contractive definitions which are satisfied for all points x, z in a complete metric space X. In a recent paper [3] Fisher has proved several fixed point theorems for contractive definitions which are satisfied only for points x, $z = Ty$, $y \in X$.

In this paper, we observe that many such contractive definitions exist, and we state theorems which extend those listed in [1]-[3].

In [5], 125 contractive definitions were listed, and the most general fixed point theorems were either stated or proved. Adding the number 500 to each of those, to denote the corresponding contractive definitions involving x and Ty, one obtains another 125 contractive definitions. For example, (524) would read: there exists a number h, $0 \leq h < 1$ such that, for all $x, y \in X$,

$$d(Tx, T^2y) \leq h \cdot \max\{d(x, Ty),\ d(x, Tx),\ d(Ty, T^2y),\ d(x, T^2y),\ d(Ty, Tx)\}.$$

Each of the definitions (501)-(625) is a generalization of the corresponding definition from the list (1)-(125). If T is surjective the definitions are equivalent.

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There are 125 additional definitions, which shall be numbered (626)-(750), for pairs of mappings.

The analogues of Theorems 1-19 and 27-29 of [5] are true. Their proofs are immediate, since the substitution of $y = Tx$ or $T^p x$ for some value of p, yields the first line of the proof of the corresponding theorem. The rest of the proofs are identical, except for one or two minor, obvious changes. For brevity only the lemma and two of the theorems are sated here.

Lemma. *Let* T *satisfy* (624) *or* (625). *If* T *has a fixed point, then it is unique.*

Theorem 1. *Let* T *be a self mapping of a complete metric space* X *such that there exist positive integers* p, q *and a number* h, $0 \le h < 1$ *such that for all* $x, y \in X$,

(574) $$d(T^p x, T^{q+p} y) \le h \max\{d(x, T^p y),\ d(x, T^p x),\ d(T^p y, T^{q+p} y),\ d(x, T^{q+p} y),\ d(T^p y, T^p x)\}.$$

Then T *has a unique fixed point* z *and* $T^n x_0 \to z$ *for each* $x_0 \in X$.

The proof of Theorem 1 is similar to that of [5, Theorem 11]. Let $x_0 \in X$ and define the sequence $\{x_n\}$ by $x_{n+1} = Tx_n$, $n \ge 0$. Let j, k be positive integers satisfying $j > k \ge p + q$. By the division algorithm there exist unique integers m, n, r, s satisfying $0 \le r, s < p + q$, such that $j = m(p + q) + r$ and $k = n(p + q) + s$. Since T satisfies (574),

(1) $$d(x_j, x_k) = d(T^p x_{(m-1)(p+q)+q+r}, T^{q+p} x_{(n-1)(p+q)+s})$$
$$\le h \max\{d(x_{(m-1)(p+q)+q+r}, x_{(n-1)(p+q)p+s}),$$

$$d(x_{(m-1)(p+q)+q+r}, x_{m(p+q)+r}), d(x_{(n-1)(p+q)+p+s}, x_{n(p+q)+s}),$$

$$d(x_{(m-1)(p+q)+q+r}, x_{n(p+q)+s}), d(x_{(n-1)(p+q)+p+s}, x_{m(p+q)+r})\}.$$

Again using (574), we have

$$(2)\quad d(x_{(m-1)(p+q)+q+r}, x_{(n-1)(p+q)+p+s}) = d(T^p x_{(n-1)(p+q)+s}, T^{q+p} x_{(m-2)(p+q)+q+r})$$

$$\leq h \max\{d(x_{(n-1)(p+q)+s}, x_{(m-1)(p+q)+r}),$$

$$d(x_{(n-1)(p+q)+s}, x_{(n-1)(p+q)+p+s}), d(x_{(m-1)(p+q)+r}, x_{(m-1)(p+q)+q+r}),$$

$$d(x_{(n-1)(p+q)+s}, x_{(m-1)(p+q)+q+r}), d(x_{(m-1)(p+q)+r}, x_{(n-1)(p+q)+p+s})\},$$

$$(3)\quad d(x_{(m-1)(p+q)+q+r}, x_{m(p+q)+r}) = d(T^p x_{(m-1)(p+q)+q+r}, T^{q+p} x_{(m-2)(p+q)+q+r})$$

$$\leq h \max\{d(x_{(m-1)(p+q)+q+r}, x_{(m-1)(p+q)+r}), d(x_{(m-1)(p+q)+q+r}, x_{m(p+q)+r}),$$

$$d(x_{(m-1)(p+q)+r}, x_{(m-1)(p+q)+q+r}), 0, d(x_{(m-1)(p+q)+r}, x_{m(p+q)+r})\},$$

$$(4)\quad d(x_{(n-1)(p+q)+p+s}, x_{n(p+q)+s}) = d(T^p x_{(n-1)(p+q)+s}, T^{q+p} x_{(n-1)(p+q)+s})$$

$$\leq h \max\{d(x_{(n-1)(p+q)+s}, x_{(n-1)(p+q)+p+s}), d(x_{(n-1)(p+q)+s}, x_{(n-1)(p+q)+p+s}),$$

$$d(x_{(n-1)(p+q)+p+s}, x_{n(p+q)+s}), d(x_{(n-1)(p+q)+s}, x_{n(p+q)+s}), 0\},$$

$$(5)\quad d(x_{(m-1)(p+q)+q+r}, x_{n(p+q)+s}) = d(T^p x_{(n-1)(p+q)+q+s}, T^{q+p} x_{(m-2)(p+q)+q+r})$$

$$\leq h \max\{d(x_{(n-1)(p+q)+q+s}, x_{(m-1)(p+q)+r}),$$

$$d(x_{(n-1)(p+q)+q+s}, x_{n(p+q)+s}), d(x_{(m-1)(p+q)+r}, x_{(m-1)(p+q)+q+r}),$$

$$d(x_{(n-1)(p+q)+q+s}, x_{(m-1)(p+q)+q+r}), d(x_{(m-1)(p+q)+r}, x_{n(p+q)+s})\},$$

(6) $d(x_{(n-1)(p+q)+p+s}, x_{m(p+q)+r}) = d(T^p x_{(n-1)(p+q)+s}, T^{q+p} x_{(m-1)(p+q)+r})$

$$\leq h \max \{d(x_{(n-1)(p+q)+s}, x_{(m-1)(p+q)+p+r}),$$

$$d(x_{(n-1)(p+q)+s}, x_{(n-1)(p+q)+p+s}), d(x_{(m-1)(p+q)+p+r}, x_{m(p+q)+r}),$$

$$d(x_{(n-1)(p+q)+s}, x_{m(p+q)+s}), d(x_{(m-1)(p+q)+p+r}, x_{(n-1)(p+q)+p+s})\},$$

Let $0(x, n) = \{x, Tx, \cdots, T^n x\}$, and, for any set A, $\delta(A) = \sup \{d(x, y) \mid x, y \in A\}$. An examination of the right hand sides of inequalities (2)-(6) shows that the smallest and largest subscripts present are $(n - 1)(p + q) + s$ and $m(p + q) + r$, respectively. Therefore (1) can be written

$$d(x_j, x_k) \leq h^2 \delta(0(x_{(n-1)(p+q)+s}, (m-n+1)(p+q)+r-s)).$$

There exist integers a, b satisfying $(n-1)(p+q)+s \leq a < b \leq m(p+q)+r$, such that $\delta(0(x_{(n-1)(p+q)+s}, (m-n+1)(p+q)+r-s)) = d(x_a, x_b)$, where $a = t(p+q)+u$, $b = v(p+q)+w$.

Repeating the argument which developed inequalities (1)-(6) one obtains

$$d(x_a, x_b) = d(x_{t(p+q)+u}, x_{v(p+q)+w}) = d(x_{v(p+q)+w}, x_{t(p+q)+u})$$

$$\leq h^2 \delta(0(x_{(t-1)(p+q)+u}, (v-t+1)(p+q)+w-u)).$$

Since $(t-1)(p+q)+u \geq (n-2)(p+q)+s$ and $(v-t+1)(p+q)+w-u \leq (m-n+2)(p+q)+r-s$, $d(x_a, x_b) \leq h^2 \delta(0(x_{(n-2)(p+q)+s}, (m-n+2)(p+q)+r-s))$, it follows that $\delta(0(x_{(n-1)(p+q)+s}, (m-n+1)(p+q)+r-s)) \leq h^2 \delta(0(x_{(n-2)(p+q)+s}, (m-n+2)(p+q)+r-s))$, and hence that

(7) $d(x_j, x_k) \leq h^2 (h^2)^{n-1} \delta(0(x_s, m(p+q)+r-s))$

$$\leq h^{2n} \delta(0(x_0, (m+1)(p+q)).$$

For any integer n there exist integers c and d, satisfying $0 \leq c < d \leq n$, such that $\delta(0(x_0, n)) = d(x_c, x_d)$.

Suppose $c \geq p + q$, and write $c = e(p+q)+f$, $d = g(p+q)+h$, where $e \leq g$ and $0 \leq f,\ h < p + q$.

$$\begin{aligned} d(x_c, x_d) &= d(x_{e(p+q)+f}, x_{g(p+q)+h}) \\ &= d(T^p x_{(e-1)(p+q)+q+f}, T^{q+p} x_{(g-1)(p+q)+h}) \\ &\leq h^2 \delta(0(x_{(e-1)(p+q)+f}, (g-e+1)(p+q)+h-f)) \\ &\leq h^{2e} \delta(0(x_f, g(p+q)+h+f)). \end{aligned}$$

Therefore $\delta(0(x_0, n)) = d(x_c, x_d)$ for some value of c satisfying $0 \leq c < p + q$.

Suppose $p + q < d$. Then $d(x_c, x_d) \leq d(x_c, x_{p+q}) + d(x_{p+q}, x_d)$, and from (574),

$$\begin{aligned} d(x_{p+q}, x_d) &= d(T^p x_q, T^{q+p} x_{d-(p+q)}) \\ &\leq h \max \{d(x_q, x_{d-q}), d(x_q, x_{p+q}), d(x_{d-q}, x_d), \\ &\qquad d(x_q, x_d), d(x_{d-q}, x_{p+q})\} \\ &\leq h\delta(0(x_0, d)) \leq h\delta(0(x_0, n)) = hd(x_c, x_d). \end{aligned}$$

Therefore

$$d(x_c, x_d) \leq \frac{hd(x_c, x_{p+q})}{1 - h},$$

so that, for each $n \geq p + q$,

$$(8) \qquad \delta(0(x_0, n) \leq \frac{hd(x_c, x_{p+q})}{1 - h},$$

where c satisfies $0 \leq c < p + q$.

Using (8) in (7) it is clear that $\{x_n\}$ is Cauchy, hence convergent. Call the limit z. Using the triangular inequality.

(9) $d(T^p z, z) \leq d(T^p z, T^{q+p} x_{n(p+q)}) + d(x_{(n-1)(p+q)}, z)$.

From (574),

(10) $d(T^p z, T^{q+p} x_{n(p+q)}) \leq h \max \{d(z, x_{n(p+q)+p}, d(z, T^p z),$

$$d(x_{n(p+q)+p}, x_{(n+1)(p+q)}), d(z, x_{(n+1)(p+q)}), d(x_{n(p+q)+p}, T^pz)\}.$$

Taking the limit as $n \to \infty$ in (10) and (9), it follows that $d(T^pz, z) \le hd(z, T^pz)$, which implies $T^pz = z$. Using this fact and (574), we have

$$\begin{aligned} d(z, T^qz) &= d(T^pz, T^{q+p}z) \\ &\le h\max\{d(z, T^pz), d(z, T^pz), d(T^pz, T^{q+p}z), d(z, T^{q+p}z)\} \\ &= hd(z, T^qz), \end{aligned}$$

and z is also a fixed point of T^q.

Let z and w be two fixed points of T^p. Then they are also fixed points of T^q. Using (574),

$$\begin{aligned} d(z, w) &= d(T^pz, T^qw) = d(T^pz, T^{q+p}w) \\ &\le h\max\{d(z, T^pq), d(z, T^pz), d(T^pw, T^{q+p}w), \\ &\qquad d(z, T^{q+p}w), d(T^pw, T^pz)\} \\ &= hd(z, w), \end{aligned}$$

which implies $z = w$, and the fixed point of T^p is unique. But $T^p(Tz) = T(T^pz) = Tz$, so that Tz is also a fixed point of T^p. Uniqueness now follows from the Lemma.

Theorem 2. *Let S, T be self mappings of a complete metric space X, $x_0 \in X$, satisfying: there exists a number h, $0 \le h < 1$ such that, for all $x, y \in X$,*

$$(64)\quad d(Sx, TSy) \le h\max\{d(x, Sy), d(x, Sx), d(Sy, TSy), [d(x, TSy) + d(Sy, Sx)]/2\}.$$

Then S and T have a common unique fixed point z.

Theorem 2 is Corollary 1 of [4].

Remark 1. Theorem 1 includes the two theorems of [3] and Theorem 7 of [2] as special cases.

Remark 2. Theorem 1, with the roles of x and y interchanged, contains the four theorems of [1] as special cases.

Remark 3. Theorem 2 includes Theorems 3-6 of [2] as special cases. Also the hypothesis of continuity can be deleted from Theorems 3-6 of [2].

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