



Remarks on Commutative BCK-Algebras

Thaheem, A. B.

(Citation)

Mathematics Seminar Notes, 6(2):237-244

(Issue Date)

1978

(Resource Type)

journal article

(Version)

Version of Record

(JaLCD0I)

<https://doi.org/10.24546/E0001410>

(URL)

<https://hdl.handle.net/20.500.14094/E0001410>



Remarks on Commutative BCK-Algebras

By A. B. THAHEEM*

1. Introduction and preliminaries. K. Iséki [3] introduced the notion of BCK-algebra and in his various papers he has developed the theory of these algebras (see for example [4], [5], [8], and [9]). Let X be a BCK-algebra and x, y be any elements of X . Put $x \wedge y = y * (y * x)$. If $x \wedge y = y \wedge x$ then X is called a *commutative* BCK-algebra (see [11]). If there is a greatest element 1 in X then X is called *bounded* BCK-algebra. For any x in a bounded BCK-algebra X , set $Nx = 1 * x$. Define $x \vee y = N(Nx \wedge Ny)$. K. Iséki [4] proved that if X is a bounded commutative BCK-algebra then $(X, \wedge, \vee)$ is a lattice. Recently, K. Iséki and S. Tanaka [9] have proved another important result: If X is a bounded commutative implicative BCK-algebra then $(X, \wedge, \vee)$ turns out to be a distributive lattice. In this paper we present another proof of this result, and in this process we explore some further properties of these algebras. Our proof combines lattice theoretic techniques, properties of BCK-algebras and Griss algebras - another class of algebras introduced by Y. Imai and K. Iséki [2].

We shall give here only those concepts of the theory of BCK-algebras which are essential for the proof of results in section 2, and for details

* Department of Mathematics, Faculty of Science, University of Garyounis
P. O. Box 9480, Benghazi, Libya.

we refer to [4], [5], [8], and [9].

A BCK-algebra X is a general algebra $\langle X, *, 0 \rangle$ of type $\langle 2, 0 \rangle$ satisfying the following conditions:

- BCK.1. $(x * y) * (x * z) \leq z * y$,
- BCK 2. $x * (x * y) \leq y$,
- BCK 3. $x \leq x$,
- BCK 4. $x \leq y$ and $y \leq x$ imply $x = y$,
- BCK 5. $0 \leq x$
- BCK 6. $x \leq y$ if and only if $x * y = 0$.

We summarize some properties of a BCK-algebra in the following proposition and for the proof we refer to [4].

Proposition 1.1. *Let X be a BCK-algebra and x, y, z be any elements of X , then*

- i) $(x * y) * (z * y) \leq x * z$
- ii) $(x * y) * z = (x * z) * y$
- iii) $x \leq y$ implies $z * y \leq z * x$
- iv) $x \leq y$ implies $x * z \leq y * z$
- v) $x * y \leq x$

If x is a bounded BCK-algebra with greatest element 1 then $x \leq 1$ for all x in X . The following inequality can easily be verified:

$$Nx * Ny \leq y * x.$$

A BCK-algebra X is called *implicative* if $(x * y) * y = x * y$ for all $x, y \in X$ (see [5], [6]).

In 1966, Y. Imai and K. Iséki [2] introduced the notion of a Griss

algebra. A system $G = G, \vee, *, 0$ is called a Griss algebra if and only if:

G1. $G, \vee$ is an upper semilattice with the least element 0,

G2. $(x \vee y) * (x \vee z) \leq y \vee z$,

G3. $x * y \leq (x * z) \vee (z * y)$,

G4. $x * y = 0 \stackrel{+}{\Leftrightarrow} x \leq y \stackrel{+}{\Leftrightarrow} x \vee y = y$.

N. P. Rao [10] replaced G2 by

G2'. $(x \vee y) * z = (x * z) \vee (y * z)$

Recently, K. Iséki [7] has replaced G2 by another condition

G2''. $(x \vee y) * (x \vee z) \leq y * z$.

Hence, there are three kinds of Griss-like algebras. The following theorem is due to K. Iséki [7]:

Theorem 1.2. *If each of the above three Griss algebras is a BCK-algebra with respect to $*$ then these notions coincide.*

2. Results. We begin with the following

Lemma 2.1. *Any bounded commutative implicative BCK-algebra X is a Griss algebra.*

Proof. G1. For any $x, y \in X$, $x \vee y = N(Nx \wedge Ny)$ defines an upper semilattice (see [4, Theorem 3]).

G4. is immediate from the definition of BCK-algebra and the fact that $(X, \wedge, \vee)$ is a lattice.

$$\begin{aligned} (x \vee y) * (x \vee z) &= N(Nx \wedge Ny) * N(Nx \wedge Nz) \\ &\leq (Nx \wedge Nz) * (Nx \wedge Ny) \\ &= (Nx * (Nx * Nz)) * (Nx * (Nx * Ny)) \end{aligned}$$

$$\leq (Nx * Ny) * (Nx * Nz) \quad (\text{By BCK 1})$$

$$\leq Nz * Ny \quad (\text{By BCK 1})$$

$$\leq y * z$$

which is G2" (and hence G2 follows from Theorem 1.2.). We shall now proceed to prove G3. By definition of upper semilattice, $z * y \leq (x * z) \vee (z * y)$. Hence

$$\begin{aligned} (x * y) * ((x * z) \vee (z * y)) &\leq (x * y) * (z * y), \quad (\text{By Prop. 1}) \\ &\leq x * z \quad (\text{By Prop. 1}). \end{aligned}$$

Again

$$\begin{aligned} ((x * y) * ((x * z) \vee (z * y))) * ((x * z) \vee (z * y)) \\ \leq (x * z) * ((x * z) \vee (z * y)) \\ = 0 \end{aligned}$$

Therefore,

$$((x * y) * (x * z) \vee (z * y))) * ((x * z) \vee (z * y)) = 0$$

Since X is implicative, we obtain

$$((x * y) * ((x * z) \vee (z * y))) = 0$$

Or

$$(x * y) \leq (x * z) \vee (z * y)$$

which proves G3 and the proof of the lemma is completed.

Definition 2.2. [1] A lattice L is said to be *modular* if and only if $x \wedge (y \vee z) = (x \wedge y) \vee z$, for all $x, y, z \in L$ such that $z \leq x$.

Lemma 2.3. Any bounded commutative implicative BCK-algebra X is a modular lattice with respect to $\wedge, \vee$.

Proof. $(X, \wedge, \vee)$ is a lattice [4]. To prove the modularity of $(X, \wedge, \vee)$, let $x, y, z \in X$ such that $z \leq x$. Then

$$\begin{aligned}
x \wedge (y \vee z) &= (y \vee z) * ((y \vee z) * x) \\
&= (y \vee z) * ((y * x) \vee (z * x)) \quad (\text{By G2'}) \\
&= (y \vee z) * (y * x) \quad (\text{because } z * x = 0) \\
&= (y * (y * x)) \vee (z * (y * x)) \quad (\text{By G2'}) \\
&= (x \wedge y) \vee (z * (y * x))
\end{aligned}$$

Since $z * (y * x) \leq z$, (By $\vee$) of Proposition 1.1) we have

$$(A) \quad x \wedge (y \wedge z) \leq (x \wedge y) \vee z.$$

Conversely, $x \wedge y \leq x$ and $x \wedge y \leq y \leq y \vee z$. Thus

$$(B) \quad x \wedge y \leq x \wedge (y \vee z)$$

Also $z \leq x$ and $z \leq y \vee z$. Thus

$$(C) \quad z \leq x \wedge (y \vee z)$$

(B) and (C) imply

$$(D) \quad (x \wedge y) \vee z \leq x \wedge (y \vee z)$$

From (A) and (D), we obtain

$$x \wedge (y \vee z) = (x \wedge y) \vee z.$$

Notation. We call a lattice L as $\wedge$ -*semidistributive* if it possesses the following property:

(SD $_{\wedge}$) For any $x, y, z \in L$, $x \wedge y = x \wedge z = s$, say implies

$$x \wedge (y \vee z) = s.$$

Lemma 2.4. Any bounded commutative implicative BCK-algebra possesses (SD $_{\wedge}$) property.

Proof. Let X be a bounded implicative BCK-algebra. Let $x, y, z \in L$ and suppose $x \wedge y = x \wedge z = s$.

Now $x \leq x$ and $y \leq y \vee z$. This implies

$$(A) \quad s = x \wedge y \leq x \wedge (y \vee z)$$

Conversely,

$$\begin{aligned}
 \text{(B)} \quad (x \wedge (y \vee z)) * s &= ((y \vee z) * ((y \vee z) * x)) * s \\
 &= ((y \vee z) * s) * ((y \vee z) * x) \text{ (By Proposition 1.1)} \\
 &= ((y * s) \vee (z * s)) * ((y * x) \vee (z * x)) \text{ (By G2')}
 \end{aligned}$$

Now consider

$$\begin{aligned}
 (y * s) * (y * x) &= (y * (y * x)) * s \\
 &= (x \wedge y) * s = 0.
 \end{aligned}$$

This means

$$\text{(C)} \quad y * s \leq y * x$$

Again $s \leq x$. This implies

$$\text{(D)} \quad y * x \leq y * s \quad \text{(By Proposition 1.1)}$$

Thus from (C) and (D), we obtain

$$\text{(E)} \quad y * s = y * x.$$

Similarly

$$\text{(F)} \quad z * s = z * x.$$

$$\text{(B)}, \text{(E)} \text{ and } \text{(F)} \text{ imply } (x \wedge (y \vee z)) * s = 0 \text{ or}$$

$$\text{(G)} \quad x \wedge (y \vee z) \leq s$$

(A) and (G) imply $x \wedge (y \vee z) = s$. This completes the proof of the lemma.

Definition 2.5. A lattice is called *distributive* if for all $x, y, z \in L$,

$$x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z).$$

The following lemma appears as an exercise in [1, p.79].

Lemma 2.6. *Let L be a modular lattice. Then L is distributive if and only if L has $(SD_{\wedge})$.*

Proof. Suppose $x, y, z \in L$. Put

$$x \wedge y = x \wedge z = s.$$

Then the distributivity of L implies $(SD_{\wedge})$ as:

$$x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z) = s \vee s = s.$$

Conversely, suppose L possesses $(SD_{\wedge})$. We shall prove that L is distributive. Let $x, y, z \in L$ and suppose $x \wedge y = x \wedge z = s$. Then

$(x \wedge y) \vee (x \wedge z) = s$. Now $x \wedge z \leq x$. By modularrrity of L , we have

$$x \wedge (y \vee (x \wedge z)) = (x \wedge y) \vee (x \wedge z) = s.$$

Similarly,

$$x \wedge (z \vee (x \wedge y)) = (x \wedge z) \vee (x \wedge y) = s.$$

Hence using $(SD_{\wedge})$, we obtain

$$(A) \quad x \wedge ((y \vee (x \wedge z)) \vee (z \vee (x \wedge y))) = s = (x \wedge y) \vee (x \wedge z)$$

But

$$(B) \quad (y \vee (x \wedge z)) \vee (z \vee (x \wedge y)) = y \vee z.$$

Therefore, from (A) and (B), we obtain

$$x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z)$$

and the distributivity of L is established.

The following theorem is a consequence of the above lemmas:

Theorem 2.7. *Any bounded commutative implicative BCK-algebra is a distributive lattice with respect to $\wedge, \vee$.*

In conclusion, I would like to thank Professor W. Poguntke for his suggestions in proving Lemma 2.6.

References

- [1] P. M. Cohn: Universal algebra, Harper and Row, New York, 1965.
- [2] Y. Imai and K. Iséki: On axiom systems of propositional calculi, Proc. Japan Acad., 42 (1966), 19-22.
- [3] K. Iséki: An algebra related with propositional calculus, Proc. Japan Acad., 42 (1966), 26-29.
- [4] K. Iséki: Some properties of BCK-algebras, Math. Seminar Notes, 2 (1974), 193-201.
- [5] K. Iséki: Remarks on BCK-algebras, Math. Seminar Notes, 3 (1975), 45-54.
- [6] K. Iséki: On implicative BCK-algebras, Math. Seminar Notes, 4 (1976), 9-10.
- [7] K. Iséki: On special Griss algebras, Math. Seminar Notes, 5 (1977), 273-278.
- [8] K. Iséki and S. Tanaka: Ideal theory of BCK-algebras, Math. Japonica, 21 (1976), 351-366.
- [9] K. Iséki and S. Tanaka: An introduction to the theory of BCK-algebras, Math. Japonica, 23 (1978), 1-26.
- [10] N. P. Rao: Autometrized Griss algebras, Math. Seminar Notes, 5 (1977), 1-21.
- [11] S. Tanaka: A new class of algebras, Math. Seminar Notes, 3 (1975), 37-43.